1.

$$u(t+k) = u(t) + ku'(t) + \frac{k^2}{2}u''(t) + \mathcal{O}(k^3)$$
$$u(t+k) = u(t) + kf(u) + \frac{k^2}{2}f'(u)f(u) + \mathcal{O}(k^3)$$
$$u(t+k) = u(t) + \frac{k}{2}f(u) + \frac{k}{2}f(u) + \frac{k^2}{2}f'(u)f(u) + \mathcal{O}(k^3)$$
$$u(t+k) = u(t) + \frac{k}{2}f(u) + \frac{k}{2}f(u)[1 + kf'(t)] + \mathcal{O}(k^3)$$
$$u(t+k) = u(t) + \frac{k}{2}f(u) + \frac{k}{2}f(u + kf(u)) + \mathcal{O}(k^3)$$
$$U^{n+1} = U^n + \frac{k}{2}f(U^n) + \frac{k}{2}f(U^n + kf(U^n)) + \mathcal{O}(k^3)$$
$$U^{n+1} = U^n + k[\frac{1}{2}f(U^n) + \frac{1}{2}f(U^n + kf(U^n))] + \mathcal{O}(k^3)$$
$$U^{n+1} = U^n + k[\frac{1}{2}V_1 + \frac{1}{2}f(U^n + kV_1)] + \mathcal{O}(k^3)$$
$$U^{n+1} = U^n + k(\frac{1}{2}V_1 + \frac{1}{2}V_2) + \mathcal{O}(k^3)$$

2. (a)

$$a_{ij} = 0, i \leq j$$
$$a_{1,1} = 0, a_{1,2} = 0$$
$$a_{2,1} \in \mathbb{R}, a_{2,2} = 0$$
$$\sum_{j=1}^{r} b_j = 1$$
$$b_1 + b_2 = \frac{1}{4} + \frac{3}{4}$$
$$b_1 + b_2 = 1$$
$$\sum_{j=1}^{r} a_{ij} = c_i$$
$$a_{1,1} + a_{1,2} = c_1$$
$$0 + 0 = c_1$$
$$a_{2,1} + a_{2,2} = c_2$$
$$\frac{2}{3} + 0 = c_2$$

(b)

$$\tau^n = \frac{2}{3} * \frac{3}{4}\mathcal{O}(h^2) = \frac{1}{2}\mathcal{O}(h^2), c_2 = \frac{2}{3}$$
$$\tau^n = \frac{1}{2} * \frac{1}{2}\mathcal{O}(h^2) = \frac{1}{4}\mathcal{O}(h^2), c_2 = \frac{1}{2}$$

The error is smaller for the midpoint method.

3.

$$\begin{aligned}
u'(t) &= f(t,u(t)), r = 1 \\
u(t_{n+1}) - u(t_n) &= \int_{t_n}^{t_{n+1}} f(\tau, u(\tau))d\tau \\
u(t_{n+1}) - u(t_n) &\approx \int_{t_n}^{t_{n+1}} P(\tau)d\tau \\
&= \int_{t_n}^{t_{n+1}} \sum_{i=0}^{0} L_i(\tau) f(t_{n+i}, u(t_{n+i}))d\tau \\
&= \int_{t_n}^{t_{n+1}} L_0(\tau) f(t_{n+1}, u(t_{n+1}))d\tau \\
&= f(t_{n+1}, u(t_{n+1})) \int_{t_n}^{t_{n+1}} L_0(\tau)d\tau \\
u(t_{n+1}) - u(t_n) &= f(t_{n+1}, u(t_{n+1}))(t_{n+1} - t_n) \\
u(t_{n+1}) &= u(t_n) + f(t_{n+1}, u(t_{n+1}))(t_{n+1} - t_n) \\
U^{n+1} &= U^n + kf_{n+1}
\end{aligned}$$

4. (a)

$$\begin{aligned}
r = 2, \alpha_0 = 2, \alpha_1 = -3, \alpha_2 = 1 \\
\beta_0 = \frac{-5}{12}, \beta_1 = \frac{-20}{12}, \beta_2 = \frac{13}{12} \\
\sum_{j=0}^{2} \alpha_j = 0 \\
\sum_{j=0}^{2} j\alpha_j = \sum_{j=0}^{2} \beta_j \\
-1 = -1
\end{aligned}$$

(b)

$$\begin{aligned}
U^{n+2} &= 3U^{n+1} - 2U^n + \frac{k}{12}[-5f(t_n, U^n) - 20f(t_{n+1}, U^{n+1}) + 12f(t_{n+2}, U^{n+2})], U^n = c_1 + c_2 2^n \\
c_1 + c_2 2^{n+2} &= 3c_1 + c_2 2^{n+1} - 2c_1 + -2c_2 2^n \\
c_1 &= c_1 = 1 \\
c_2 &= 0
\end{aligned}$$

(c) We do not converge to the true values in this case.

(d) Convergence → Consistency

5. (a)

$$r = 2, \alpha_0 = \frac{1}{3}, \alpha_1 = \frac{-4}{3}, \alpha_2 = 1, \beta_2 = \frac{2}{3}$$
$$[\frac{1}{k}\sum_{j=0}^{r} \alpha_j]u(t_n) = 0$$
$$[\sum_{j=0}^{r} j\alpha_j - \beta_j]u'(t_n) = 0$$
$$[k\sum_{j=0}^{r} \frac{j^2}{2}\alpha_j - j\beta_j]u''(t) = 0$$
$$\tau(t_{n+2}) = \mathcal{O}(h^2)$$

(b)

$$r = 3, \alpha_1 = -1, \alpha_3 = 1, \beta_0 = \frac{1}{3}, \beta_1 = \frac{-2}{3}, \beta_2 = \frac{7}{3}$$
$$[\frac{1}{k}\sum_{j=0}^{r} \alpha_j]u(t_n) = 0$$
$$[\sum_{j=0}^{r} j\alpha_j - \beta_j]u'(t_n) = 0$$
$$[k\sum_{j=0}^{r} \frac{j^2}{2}\alpha_j - j\beta_j]u''(t) = 0$$
$$\tau(t_{n+2}) = \mathcal{O}(h^2)$$

6. (a) (i) is not consistent.
(ii) is not consistent.

(b) i

$$p(\xi) = 0 + -1\xi + 1\xi^2$$
$$\xi = 0, 1$$

Zero stable

ii

$$p(\xi) = -2 + -1\xi + 1\xi^2$$
$$\xi = 2, -1$$

Not zero stable

(c)

(d)

(e)

7. (a)

(b) The errors trend towards $\mathcal{O}(h^2)$