

1.

$$u(t+k) = u(t) + ku'(t) + \frac{k^2}{2}u''(t) + \mathcal{O}(k^3)$$

$$u(t+k) = u(t) + kf(u) + \frac{k^2}{2}f'(u)f(u) + \mathcal{O}(k^3)$$

$$u(t+k) = u(t) + \frac{k}{2}f(u) + \frac{k}{2}f(u) + \frac{k^2}{2}f'(u)f(u) + \mathcal{O}(k^3)$$

$$u(t+k) = u(t) + \frac{k}{2}f(u) + \frac{k}{2}f(u)[1 + kf'(t)] + \mathcal{O}(k^3)$$

$$u(t+k) = u(t) + \frac{k}{2}f(u) + \frac{k}{2}f(u + kf(u)) + \mathcal{O}(k^3)$$

$$U^{n+1} = U^n + \frac{k}{2}f(U^n) + \frac{k}{2}f(U^n + kf(U^n)) + \mathcal{O}(k^3)$$

$$U^{n+1} = U^n + k[\frac{1}{2}f(U^n) + \frac{1}{2}f(U^n + kf(U^n))] + \mathcal{O}(k^3)$$

$$U^{n+1} = U^n + k[\frac{1}{2}V_1 + \frac{1}{2}f(U^n + kV_1)] + \mathcal{O}(k^3)$$

$$U^{n+1} = U^n + k(\frac{1}{2}V_1 + \frac{1}{2}V_2) + \mathcal{O}(k^3)$$

2. (a)

$$a_{ij} = 0, i \leq j$$

$$a_{1,1} = 0, a_{1,2} = 0$$

$$a_{2,1} \in \mathbb{R}, a_{2,2} = 0$$

$$\sum_{j=1}^r b_j = 1$$

$$b_1 + b_2 = \frac{1}{4} + \frac{3}{4}$$

$$b_1 + b_2 = 1$$

$$\sum_{j=1}^r a_{ij} = c_i$$

$$a_{1,1} + a_{1,2} = c_1$$

$$0 + 0 = c_1$$

$$a_{2,1} + a_{2,2} = c_2$$

$$\frac{2}{3} + 0 = c_2$$

(b)

$$\begin{aligned}
U^{n+1} &= U^n + \frac{k}{4}V_1 + \frac{3k}{4}V_2 \\
V_1 &= f(t, u) \\
V_2 &= f\left(t + \frac{2k}{3}, u + \frac{2k}{3}f(t, u)\right) \\
\tau^n &= \frac{u(t_{n+1}) - u(t_n)}{k} - f\left(u(t_n) + \frac{k}{2}f(u(t_n))\right) \\
\tau^n &= \frac{1}{4}\mathcal{O}(h^2) + \frac{2}{3}\frac{3}{4}\mathcal{O}(h^2) \\
\tau^n &= \frac{3}{4}\mathcal{O}(h^2), c_2 = \frac{2}{3} \\
U^{n+1} &= U^n + \frac{k}{2}V_1 + \frac{k}{2}V_2 \\
V_1 &= f(t, u) \\
V_2 &= f\left(t + \frac{k}{2}, u + \frac{k}{2}f(t, u)\right) \\
\tau^n &= \frac{u(t_{n+1}) - u(t_n)}{k} - f\left(u(t_n) + \frac{k}{2}f(u(t_n))\right) \\
\tau^n &= \frac{1}{2}\mathcal{O}(h^2) + \frac{1}{2}\frac{1}{2}\mathcal{O}(h^2) \\
\tau^n &= \frac{3}{4}\mathcal{O}(h^2), c_2 = \frac{1}{2}
\end{aligned}$$

The error has the same coefficient.

3.

$$\begin{aligned}
u'(t) &= f(t, u(t)), r = 1 \\
u(t_{n+1}) - u(t_n) &= \int_{t_n}^{t_{n+1}} f(\tau, u(\tau)) d\tau \\
u(t_{n+1}) - u(t_n) &\approx \int_{t_n}^{t_{n+1}} P(\tau) d\tau \\
&= \int_{t_n}^{t_{n+1}} \sum_{i=0}^0 L_i(\tau) f(t_{n+i}, u(t_{n+i})) d\tau \\
&= \int_{t_n}^{t_{n+1}} L_0(\tau) f(t_{n+1}, u(t_{n+1})) d\tau \\
&= f(t_{n+1}, u(t_{n+1})) \int_{t_n}^{t_{n+1}} L_0(\tau) d\tau \\
u(t_{n+1}) - u(t_n) &= f(t_{n+1}, u(t_{n+1}))(t_{n+1} - t_n) \\
u(t_{n+1}) &= u(t_n) + f(t_{n+1}, u(t_{n+1}))(t_{n+1} - t_n) \\
U^{n+1} &= U^n + kf_{n+1}
\end{aligned}$$

4. (a)

$$\begin{aligned}
r = 2, \alpha_0 = 2, \alpha_1 = -3, \alpha_2 = 1 \\
\beta_0 = \frac{-5}{12}, \beta_1 = \frac{-20}{12}, \beta_2 = \frac{13}{12} \\
\sum_{j=0}^2 \alpha_j = 0 \\
\sum_{j=0}^2 j\alpha_j = \sum_{j=0}^2 \beta_j \\
-1 = -1
\end{aligned}$$

(b)

$$\begin{aligned}
U^{n+2} &= 3U^{n+1} - 2U^n + \frac{k}{12}[-5f(t_n, U^n) - 20f(t_{n+1}, U^{n+1}) + 12f(t_{n+2}, U^{n+2})], U^n = c_1 + c_2 2^n \\
c_1 + c_2 2^{n+2} &= 3c_1 + c_2 2^{n+1} - 2c_1 + -2c_2 2^n \\
c_1 &= c_1 = 1 \\
c_2 &= 0
\end{aligned}$$

(c) We do not converge to the true values in this case.

(d) Convergence  $\rightarrow$  Consistency

5. (a)

$$\begin{aligned}
r = 2, \alpha_0 = \frac{1}{3}, \alpha_1 = \frac{-4}{3}, \alpha_2 = 1, \beta_2 = \frac{2}{3} \\
[\frac{1}{k} \sum_{j=0}^r \alpha_j] u(t_n) &= 0 \\
[\sum_{j=0}^r j\alpha_j - \beta_j] u'(t_n) &= 0 \\
[k \sum_{j=0}^r \frac{j^2}{2} \alpha_j - j\beta_j] u''(t) &= 0 \\
\tau(t_{n+2}) &= \mathcal{O}(h^2)
\end{aligned}$$

(b)

$$\begin{aligned}
 r = 3, \alpha_1 = -1, \alpha_3 = 1, \beta_0 = \frac{1}{3}, \beta_1 = \frac{-2}{3}, \beta_2 = \frac{7}{3} \\
 \left[\frac{1}{k} \sum_{j=0}^r \alpha_j\right] u(t_n) = 0 \\
 \left[\sum_{j=0}^r j \alpha_j - \beta_j\right] u'(t_n) = 0 \\
 \left[k \sum_{j=0}^r \frac{j^2}{2} \alpha_j - j \beta_j\right] u''(t) = 0 \\
 \tau(t_{n+2}) = \mathcal{O}(h^2)
 \end{aligned}$$

6. (a) (i) is not consistent.  
 (ii) is not consistent.

(b) i

$$\begin{aligned}
 p(\xi) &= 0 + -1\xi + 1\xi^2 \\
 \xi &= 0, 1
 \end{aligned}$$

Zero stable  
 ii

$$\begin{aligned}
 p(\xi) &= -2 + -1\xi + 1\xi^2 \\
 \xi &= 2, -1
 \end{aligned}$$

Not zero stable

(c)

(d)

(e)

7. (a)

(b) The errors trend towards  $\frac{3}{4}\mathcal{O}(h^2)$