Integral Equations Notes: [457]

[Drew Remmenga]

July 11, 2024

Definition 1. *An integral equation is any equyation in which the unknown function is inside the integral sign.*

Example 1.

$$\int_a^b k(x,y)u(y)dy = f(x), x \in (a,b)$$

Here k and f are given and u(x) is the unknown function.

Example 2.

$$\int_0^\infty e^{-xt}u(t)dt = f(x), x \in (0,\infty)$$

f(x) is the laplace transform of u.

Remark 1. *There is an inversion formula giving u from f but it involves an integral of f(x) over complex values of x. In practice we may only know f(x) for real values of x.*

Example 3.

$$\int_{-1}^1 u(t)dt = 1$$

One solution

$$u(t) = \frac{1}{2}, t \in (-1,1)$$

Clearly there are lots of solutions, e.g.

$$u(t) = \frac{1}{2} + \textit{any odd funtion of } t.$$

or

$$u(t) = \frac{1}{2} + \sum_{n=1}^\infty \{a_n cos(nt\pi) + b_n sin(nt\pi)\}$$

For any reasonable a_n and b_n.

Example 4.

$$\int_{-1}^1 u(t)dt = f(x), x \in (-1,1)$$

Now there are no solutions unless f(x) is constant.

Example 5.

$$\alpha u(x) + \beta \int_{-1}^{1} u(t)dt = f(x), x \in (-1, 1) \tag{1}$$

Where α and β are given constants with $\alpha \neq 0$ or example 4 put $\int_{-1}^{1} u(t)dt = c$ a known constant. Then (1) gives

$$u(x) = \frac{1}{\alpha}\{f(x) - \beta C\} \tag{2}$$

So we will have solved (1) and know C. Integrate (2) gives

$$C = \int_{-1}^{1} u(x)dx = \frac{1}{\alpha}\int_{-1}^{1} f(x)dx - \frac{\beta C}{\alpha}\int_{-1}^{1} dx$$

$$(1 + \frac{2\beta}{\alpha})C = \frac{1}{\alpha}\int_{-1}^{1} f(x)dx \tag{3}$$

$$C = \frac{1}{\alpha + 2\beta}\int_{-1}^{1} f(x)dx$$

then (2) gives unique solution of (1) real provided $\alpha + 2\beta \neq 0$. What happens if $\alpha + 2\beta = 0$? As before we get to (3) which reduces to

$$0 = \int_{-1}^{1} f(x)dx \tag{4}$$

If f(x) satisfies (4) eg f cpould be odd then (2) solves (1) for any choice of the constant C: we have existence but not uniqueness. If f(x) does not satisfy (4) eg f(x) could by x^2 then (1) does not have any solutions $\alpha + 2\beta = 0$. All this is reminicent of linear algebra (solving $Ax = \underset{\sim}{b}$).

Exercise 1. *Put $\frac{\beta}{\alpha} = -\lambda$ and $f = 0$ giving*

$$u(x) = \int_{-1}^{1} u(t)dt = 0$$

A homogenous integral equation. Are there values of λ for which this integral equation has non trivial solution $u \neq 0$?

Example 6.

$$\int_{0}^{t} u(\tau)d\tau = f(t), t > 0 \tag{5}$$

Differentiate with respect to x(t):

$$u(x) = f'(x)$$

Two anxieties. 1. (5) suggests that f(0)=0 in $\lim_{x\to 0}$ (5) what happens if $f(0) \neq 0$ eg $f(x) = 1, \forall x$ 2. What happens if f is not differentiable? Try using laplace transform. Let $U(s) = \mathscr{L}\{u\}$

$$\int_{0}^{\infty} u(t)e^{-st}dt$$

We have:

$$\begin{aligned}
\mathscr{L}\{\int_0^t u(\tau)d\tau\} &= \int_{t=0}^{\infty} e^{-st}\int_0^{\tau=t} u(\tau)d\tau \\
&= \int_0^{\infty} u(\tau)\int_{\tau}^{\infty} e^{-st}dtd\tau \\
&= \frac{e^{st}}{s}\|_{\tau}^{infty} = \frac{e^{-s\tau}}{s} \\
&= \int_0^{\infty} u(\tau)\frac{e^{-s\tau}}{s}d\tau \\
&= \frac{1}{s}\int_0^{\infty} u(\tau)e^{-s\tau}d\tau \\
&= \frac{1}{s}\mathscr{L}\{u\} \\
\mathscr{L}\{(5)\} &= \frac{1}{s}U(s) = F(s) \\
&\implies \\
u(s) &= sF(s) \\
\mathscr{L}\{f'\} &= sF(s) - f(0) \\
\mathscr{L}\{u\} &= U = sF(s) + sf(x) - f(0) \\
&= \mathscr{L}\{f\} + f(0)
\end{aligned}$$

Question is is there a g(t) such that $\mathscr{L}\{g\} = 1$ *Answer: It depends on what you mean by a function.*

$$LHS = \mathscr{L}\{g\} = \int_0^{\infty} g(t)e^{st}dt = G(s)$$

Fact: If g is an ordinary integrable function then we must not have $G(s) \to 0$ *as* $s \to \infty$ *As we want G(s) = 1. g can't be ordinary. Fact:* $\mathscr{L}\{\delta\} = 1$ *Wwhere* δ *= dirac delta function. SO solution of (5).*

$$u(x) = f'(x) + f(0)dx$$

In particular if $f(0) \neq 0$ *then (5) has no conbtinuous solutions for* $x \geq 0$*. Second Anxiety: Consider (5) where f is not a continous function.*

Example 7.

$$\begin{aligned}
f(x) &= \begin{cases} 0, & 0 \leq x < c \\ x - c, & x \geq c \end{cases} \\
&= (x-c)H(x-c)
\end{aligned}$$

H(x) is the divided unit function. Then

$$\begin{aligned}
\mathscr{L}\{(5)\} = \frac{1}{s}u(s) = \mathscr{L}\{f\} &= \mathscr{L}\{x-c)H(x-c)\} \\
&= \int_0^{\infty}(x-c)H(x-c)e^{-sx}dx \\
&= \int_0^{\infty}(x-c)e^{sx}dx, (x-c) = y \\
&= \int_0^{\infty} ye^{-s(y+c)}dy \\
&= e^{-sc}\int_0^{\infty} yce^{-sy}dy \\
&= e^{-sc}\mathscr{L}\{t\} = \frac{e^{-sc}}{s^2} \\
&\implies U(s) = se^{-sc} \\
u(x) = H(x-c)
\end{aligned}$$

Example 8.

$$u(x) = \frac{\alpha}{\pi}\int_{-1}^{1}\frac{u(t)dt}{(x-2)^2+\alpha^2} = 1, x\in(-1,1)$$

The integral contains a parameter α. It turns up in at least two distinct physical applications. One in electro statics. whgere it is called Love's equation and in quantum phases where it is called Lieb's equation. It looks simple byt no explicit solution is known. Soi what is known is that u(x) exists, its continuous, it's unique, and it's even. Numerical solutions are easy to compute unless α is very small.

1 Abel's Equation

A bead starts froim rest at P and slides down a smooth wire riching 0 at time t_0 Find t_p. Motion is under gravity in a vertical plane put point P at (x,y) and Q at (ξ,δ). Let s denote the length from 0 to Q. Energy at P = Energy at Q. m = mass of the bead.

$$mgy = mg\delta + \frac{1}{2}m(\frac{ds}{dt})^2$$
$$\frac{ds}{dt} = -\sqrt{2gz(y-\delta)}$$

integrating

$$t_p = \int_0^{t_p} dt = \int_{\delta=y}^{\delta=0}\frac{ds}{\sqrt{2g(y-\delta)}}$$

Next specify the shape (equation) of the wire. Its convienient to suppost that $s = S(\delta)$ for some function S. Thus $ds = S(\gamma)d\gamma$

$$t_p = \int_y^0 \frac{s'(\gamma)d\gamma}{\sqrt{2g(y-\gamma)}}$$

This gives t_p if we know the shape of the wire. Abel asked given t_p can we find S? Put $u(\gamma) = S'(\gamma)$ and $f(y) = \sqrt{2g}t_p$ giving:

$$\int_0^y \frac{u(\gamma)d\gamma}{\sqrt{y-\gamma}} = f(y), y>0 \tag{6}$$

This is Abel's integral equation. LHS of (6) is a laplace convolution. Recall:

$$(f*g)(t) = \int_0^t f(\gamma)g(t-\gamma)d\gamma$$
$$\mathscr{L}\{f*g\} = \mathscr{L}\{f\}\mathscr{L}\{g\}$$

Proof.

$$\begin{aligned}\mathscr{L}\{f*g\} &= \int_0^\infty e^{-st}(f*g)(t)dt \\ &= \int_0^\infty e^{-st}\int_0^t f(\gamma)g(t-\gamma)d\gamma dt \\ &= \int_0^\infty f(\gamma)\int_\delta^\infty e^{-st}g(t-\gamma)dtd\gamma \\ &= \int_0^\infty f(\gamma)\int_{x=0}^\infty e^{-s(\gamma+x)}g(x)dxd\gamma \\ &= \mathscr{L}\{f\}\mathscr{L}\{g\}\end{aligned}$$

□

$$\mathscr{L}\{(6)\} : U(s)\mathscr{L}\{\frac{1}{\sqrt{t}}\} = F(s) \tag{7}$$

Now:

$$\begin{aligned}
\mathscr{L}\{t^\nu\} &= \int_0^\infty t^\nu e^{-st} dt \\
&= \int_0^\infty (\frac{x}{s})^\nu e^{-x} \frac{dx}{s} \\
\text{Put } st = x & \\
&= \frac{1}{st^{\nu+1}} \int_0^\infty x^\nu e^{-x} dx, \nu > -1
\end{aligned}$$

The remaining integral is denoted by $\Gamma'(\nu+1)$ where Γ is the gamma function. We want $\int_0^\infty x^{-1/2} e^{-x} dx = \Gamma(1/2) = 2\int_0^\infty e^{-w^2} dw = \sqrt{\pi}$. SOo $\mathscr{L}\{t^{-1/2}\} = \sqrt{\frac{\pi}{6}}$. Returning to (7), we get

$$U(s) = \sqrt{\frac{s}{\pi}} F(s) \tag{8}$$

This looks like $\mathscr{L}\{\text{Convolution}\}$ but its not because $\sqrt{s}$ is not the LT of an ordinary function. We know $\mathscr{L}\{\text{Ordinary Function}\} \to 0$ as $s \to \infty$ So we use a trick:

$$\begin{aligned}
U(s) &= \frac{1}{\sqrt{\pi s}} sF(s) \\
&= \frac{1}{\pi}\sqrt{\frac{\pi}{s}}\{sF(s) - f(0) + f(0)\} \\
&= \frac{1}{\pi}\mathscr{L}\{t^{-1/2}\}\{\mathscr{L}\{f'\} + f(0)\} \\
u(t) &= \frac{1}{\pi}\{t^{-1/2} * f' + \frac{f(0)}{\sqrt{t}}\}
\end{aligned}$$

$$= \frac{1}{\pi}\{\int_0^t (t-\delta)^{-1/2} f'(\delta) d\delta + f(0)\frac{1}{\sqrt{t}}\} \tag{9}$$

ie

$$S'(y) = \frac{1}{\pi}\{\frac{f()}{\sqrt{y}} + \int_0^y \frac{f'(\delta)}{\sqrt{y-\delta}} d\delta$$

Note for Arbel's problem f(0) =0. First Alternative method. We write (8) as $\frac{1}{s}U(s) = \frac{1}{\pi s}F(s)$. Invert:

$$\int_0^t u(x)dx = \frac{1}{\pi} t^{-1/2} * f \tag{10}$$

Differentiate (legal?)

$$u(t) = \frac{1}{\pi}\frac{d}{dt}\int_0^t (t-\delta)^{-1/2} f(\delta) d\delta$$

This is the same as (9). Second Alternative Method: Special and Sneaky
Multiply (6) by $\frac{1}{\sqrt{x-y}}$ and integrate from 0 to x with resepct to y.

$$\begin{aligned}
\int_0^x f(y)\frac{1}{\sqrt{x-y}} &= \int_0^x \frac{1}{\sqrt{x-y}}\int_0^y \frac{u(\delta)d\delta}{\sqrt{y-\delta}}dy \\
&= \int_0^x u(\delta)\int_\delta^x \frac{dy}{\sqrt{x-y}\sqrt{y-\delta}}d\delta \\
y &= \delta + (x-\delta)\sin^2(\theta)
\end{aligned}$$

This gets to (10).

2 ODEs

Ordinary Differential Equations (odes) Problems inbvolving odes can often be rewritten as integral equations. This can be useful for tracking theoretical questions such as existence.

Example 9. *1st order inital value problem (ivp).*

$$\frac{dy}{dt} = f(t, y(t)) \text{ with } y(0) = 0$$

Integrate:

$$\begin{aligned}
\int_0^x \frac{dy}{dt} &= \int_0^x f(t, y(t))dt \\
y(x) - y_0 &= \int_0^x f(t, y(t))dt
\end{aligned}$$

In general this is a nonlinear integral equation because of the dependence of f on y.

,

Example 10. *2nd order IVP*

$$y''(t) = f(t, y(t)), y(0) = y_0 \text{ and } y'(0) = y_1$$

Integrate:

$$y'(x) - y_1 = \int_0^x f(t, y(t))dt$$

Integrate:

$$\begin{aligned}
\int_0^s (y'(x) - y_1)ds &= \int_0^s \int_0^x f(t, y(t))dtdx \\
&\implies \\
y(s) - y_0 - y_1 s &= \int_0^s f(t, y(t)) \int_t^s dxdt \\
&= \int_0^s (s-t)f(t, y(t))dt, s > 0 \ (1)
\end{aligned}$$

Example 11. *2nd order boundary value problem (BVP)*

$$y''(t) = f(t, y(t)), 0 < t < L \text{ with } y(0) = y_0, y(L) = y_L$$

We can still use (1) except we don't know $y_1 = y'(0)$. Put s=L in (1) we receive:

$$y_L - y_0 - y_1 L = \int_0^L (L-t) f(t, y(t)) dt$$

Write this as $y_1 = ...$ and instert back into (1) gives:

$$\begin{aligned} y(s) &= y_0 + \frac{s}{L}\{y_L - y_0 - \int_0^L (L-t) f(t,y(t))dt + \int_0^s (s-t) f(t,y(t))dt\} \\ y(s) &= y_0(s - \frac{s}{L}) + y_L \frac{s}{L} - \frac{s}{L}\int_s^L (L-t) f(t,y(t))dt - \int_0^S \{\frac{s}{L}(L-t) - (s-t)\} f(t,y(t))dt \\ &= y_0(1 - \frac{s}{L}) + y_L \frac{s}{L} - \int_0^L G(s,t) f(t,y(t))dt \end{aligned}$$

Where

$$G(s,t) = \begin{cases} \frac{s}{L}(L-t), t > s \\ \frac{t}{L}(L-s), t < s \end{cases}$$

G is known as a Green's Function. For linear eigenvalue problems (a Sturm-Liouville Problem) e.g. f(t,y(t)) but if its linear $\lambda y(t), y_0 = 0, y_L = 0$ We get the integral equation:

$$y(s) = -\lambda \int_0^L G(s,t) y(t) dt, 0 < s < L(2)$$

This eignevalue problem $[y'' = \lambda y, y(0) = 0, y(L) = 0]$ is easy to solve: $\lambda = (\frac{n\pi}{L})^2, y(t) = \sin(\frac{n\pi t}{L}), n \in \mathbb{N}$

3 Classification

$\int_a^b K(x,y)u(y)dy = f(x), a \leq x \leq b$ This is called a Fredholm Integral Equation of the first kind. (FIE1). K is called the kernal, a and b are constants with b ¿ a. Next f(x) is given for $a \leq x \leq b$. Often we may have f(x) for a¡x¡b, e.g. f(a) may not exist or $a = -\infty$ or $b = \infty$. K(x,y) is a given function of two variables defined in a square in the x,y plane. $\Omega = \{(x,y) : a \leq x \leq b, a \leq y \leq b$. If f(x) is not defined $\forall x \in [a,b]$ then the same applies to K. Also th eintegral must make sense, so k could have singularities of various types, e.g. k(x,x) might be undefined. If FIE1 holds for another interval of x values, e.g.

$$\int_a^y \tilde{k}(x', y) u(y) dy = \tilde{f}(x'), c \leq x' \leq d$$

We can make a simple substituition - $x' = \frac{(d-c)x + bc - ad}{b-a}$ which then gives back FIE1. Next

$$u(x) - \int_a^b k(x,y)u(y)dy = f(x), x \in [a,b] FIE2$$

Fredhold integral equation of the second kind. FIE2 holds for any $x \in [a,b]$ which is the same as the range of integration. IF we were given FIE2 $x \in [a,b]$ we could not use the linear substituition that we use for FEI1- this is because u appears in two places. There are analogou sintegral equations with variable limits:

$$\int_a^x k(x,y)u(y)dy = f(x), x \geq a VIE1$$

Volterra integral equation of the first kind.

$$u(x) - \int_a^x k(x,y)u(y)dy = f(x), x \geq aVIE2$$

Volterra integral equation of the second kind. Note: If we suppose k(x,y) =0 for y ¿x in FIE1 or 2 we get VIE1 or 2. So in some sense VIE1 and 2 are special cases of FIE1 and 2. However VIE1 and 2 do have special properties.