Integral Equations Notes: [457]

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Definition 1. *An integral equation is any equyation in which the unknown function is inside the integral sign.*

Example 1.

$$\int_a^b k(x,y)u(y)dy = f(x), x \in (a,b)$$

Here k and f are given and u(x) is the unknown function.

Example 2.

$$\int_0^\infty e^{-xt}u(t)dt = f(x), x \in (0,\infty)$$

f(x) is the laplace transform of u.

Remark 1. *There is an inversion formula giving u from f but it involves an integhrral of f(x) over complex values of x. In practice we may only know f(x) for real values of x.*

Example 3.

$$\int_{-1}^1 u(t)dt = 1$$

One solution

$$u(t) = \frac{1}{2}, t \in (-1,1)$$

Clearly there are lots of solutions, e.g.

$$u(t) = \frac{1}{2} +$$

any odd funtion of t.

or

$$u(t) = \frac{1}{2} + \sum_{n=1}^{\infty}\{a_n cos(nt\pi) + b_n sin(nt\pi)\}$$

For any reasonable a_n and b_n.

Example 4.

$$\int_{-1}^1 u(t)dt = f(x), x \in (-1,1)$$

Now there are no solutions unless f(x) is constant.

Example 5.

$$\alpha u(x) + \beta \int_{-1}^{1} u(t)dt = f(x), x \in (-1, 1) \tag{1}$$

Where α *and* β *are given constants with* $\alpha \neq 0$ *or example 4 put* $\int_{-1}^{1} u(t)dt = c$ *a known constant. Then (1) gives*

$$u(x) = \frac{1}{\alpha}\{f(x) - \beta C\} \tag{2}$$

So we will have solved (1) and know C. Integrate (2) gives

$$C = \int_{-1}^{1} u(x)dx = \frac{1}{\alpha}\int_{-1}^{1} f(x)dx - \frac{\beta C}{\alpha}\int_{-1}^{1} dx$$

$$(1 + \frac{2\beta}{\alpha})C = \frac{1}{\alpha}\int_{-1}^{1} f(x)dx \tag{3}$$

$$C = \frac{1}{\alpha + 2\beta}\int_{-1}^{1} f(x)dx$$

then (2) gives unique solution of (1) real provided $\alpha + 2\beta \neq 0$*. What happens if* $\alpha + 2\beta = 0$*? As before we ghet to (3) which reduces to*

$$0\int_{-1}^{1} f(x)dx \tag{4}$$

If f(x) satisfies (4) eg f cpould be odd then (2) solves (1) for any choice of the constant C: we have existence but not uniqueness. If f(x) does not satisfy (4) eg f(x) could by x^2 *then (1) does not have any solutions* $\alpha + 2\beta = 0$*. All this is reminicent of linear algebra (solving* $Ax = \underset{\sim}{b}$*).*

Exercise 1. *Put* $\frac{\beta}{\alpha} = -\lambda$ *and* $f = 0$ *giving*

$$u(x) = \int_{-1}^{1} u(t)dt = 0$$

A homogenous integral equation. Are there values of λ *for which this integral equation has non trivial solution* $u \neq 0$*?*

Example 6.

$$\int_{0}^{t} u(\tau)d\tau = f(t), t > 0 \tag{5}$$

Differentiate with respect to x(t):

$$u(x) = f'(x)$$

Two anxieties. 1. (5) suggests that f(0)=0 in $\lim_{x\to 0}$ *(5) what happens if* $f(0) \neq 0$ *eg* $f(x) = 1, \forall x$ *2. What happens if f is not differentiable? Try using laplace transform. Let* $U(s) = \mathscr{L}\{u\}$

$$\int_{0}^{\infty} u(t)e^{-st}dt$$

We have:

$$
\begin{aligned}
\mathscr{L}\{\int_0^t u(\tau)d\tau\} &= \int_{t=0}^{\infty} e^{-st}\int_0^{\tau=t} u(\tau)d\tau \\
&= \int_0^{\infty} u(\tau)\int_{\tau}^{\infty} e^{-st}dtd\tau \\
&= \frac{e^{st}}{s}\|_{\tau}^{infty} = \frac{e^{-s\tau}}{s} \\
&= \int_0^{\infty} u(\tau)\frac{e^{-s\tau}}{s}d\tau \\
&= \frac{1}{s}\int_0^{\infty} u(\tau)e^{-s\tau}d\tau \\
&= \frac{1}{s}\mathscr{L}\{u\} \\
\mathscr{L}\{(5)\} &= \frac{1}{s}U(s) = F(s) \\
&\implies \\
u(s) &= sF(s) \\
\mathscr{L}\{f'\} &= sF(s) - f(0) \\
\mathscr{L}\{u\} &= U = sF(s) + sf(x) - f(0) \\
&= \mathscr{L}\{f\} + f(0)
\end{aligned}
$$

Question is is there a g(t) such that $\mathscr{L}\{g\} = 1$