Homework 4

1. (Problem from the Reading Quiz)

   Lagrange's Theorem. Let G be a finite group. If H is a subgroup of G, then $|H|$ divides $|G|$.

   The following example in the textbook is given to show the converse of Lagrange's Theorem is false.

   > **EXAMPLE 5 The Converse of Lagrange's Theorem Is False.**† The group A_4 of order 12 has no subgroups of order 6. To verify this, recall that A_4 has eight elements of order 3 (α_5 through α_{12}, in the notation of Table 5.1) and suppose that H is a subgroup of order 6. Let a be any element of order 3 in A_4. If a is not in H, then $A_4 = H \cup aH$. But then a^2 is in H or a^2 is in aH. If a^2 is in H then so is $(a^2)^2 = a^4 = a$, so this case is ruled out. If a^2 is in aH, then $a^2 = ah$ for some h in H, but this also implies that a is in H. This argument shows that any subgroup of A_4 of order 6 must contain all eight elements of A_4 of order 3, which is absurd. ∎

   The author left out a lot of the details in the example (as usual). Fill in the missing details to write a more complete explanation.

   (a) State the converse of Lagrange's Theorem.

   (b) A_4 has eight elements of order 3 (α_5 through α_{12} in the Table 5.1 in text). Explain why those elements and no others have order 3. (Do not compute the orders directly.)

   (c) Define the notation used.

      i. Let H be ...

      ii. Let a be ...

   (d) Suppose $a \notin H$. Then $A_4 = H \cup aH$. Why?

   (e) Then $a^2 \in H$ or $a^2 \in aH$. Why?

   (f) If $a^2 \in H$, then $a \in H$. Why? And why is "this case ruled out"?

   (g) If $a^2 \in aH$, then $a^2 = ah$ for some $h \in H$. So $a \in H$. Why?

   (h) So any subgroup of A_4 of order 6 size must contain all elements of A_4 of order 3. Why? And why is this "absurd"?

   (i) What initial assumption must be false given this contradiction?

   (j) How does this example show the converse of Lagrange's Theorem is false (refer to your answer in part (a))?

2. Let H be a subgroup of the D_n with odd order. Show that H must be cyclic. (Recall: D_n can be defined as $D_n = \langle r, f \rangle$ where $|r| = n$, $|f| = 2$, and $r^k f = f r^{-k}$.)

3. Let H be a subgroup of S_n. Use properties of cosets to prove that either every member of H is an even permutation or exactly half of the members is even.

4. Prove that an abelian group of order 15 is cyclic. Do not use Cauchy's Theorem for Abelian Groups.