1. (Problem from the Reading Quiz)

   Lagrange's Theorem. Let G be a finite group. If H is a subgroup of G, then $|H|$ divides $|G|$.

   The following example in the textbook is given to show the converse of Lagrange's Theorem is false. The author left out a lot of the details in the example (as usual). Fill in the missing details to write a more complete explanation.

   (a) State the converse of Lagrange's Theorem. Given a group G for any non empty subset H of G if $|H|$ divides $|G|$ then H is a subgroup of G.

   (b) A_4 has eight elements of order 3 (α_5 through α_{12} in the Table 5.1 in text). Explain why those elements and no others have order 3. (Do not compute the orders directly.) $A_4 = \{I, (1,2,3), (1,3,2), (1,2,4), (1,4,2), (1,3,4), (1,4,3), (2,3,4), (2,4,3), (1,2)(3,4), (1,3)(2,4), (1,4)(2,3)\}$ Clearly A_4 has 8 elements of three cycles so we can directly say it has 8 elements of order 3.

   (c) Define the notation used.

      i. Let H be a subgroup of A_4 of order 6.

      ii. Let a be an element of order 3 in A_4.

   (d) Suppose $a \notin H$. Then $A_4 = H \cup aH$. Why? Given that $A \notin H$. Then to show that $A_4 = H \cup aH$. As a be an element of order 3 $a \in A_4$. Then H be a subgroup of A_4 So $e \in H$ so $ae \in aH$. Then as $a \notin H$ but $a \in aH \implies a \in H \cup aH$. Now let $x \in H \cup aH \implies$ either $x \in H$ or $x \in aH$. If $x \in H$ and H is a subgroup of A_4 order 6. Clearly $H \subseteq A_4$ due to subgroup definition. So $x \in A_4 \implies H \cup aH \subseteq A_4$. This implies $A_4 = H \cup aH$.

   (e) Then $a^2 \in H$ or $a^2 \in aH$. Why? As $a \in A_4 \implies a^3 = e$ because it is element of order 3 in A_4. And also given H is a subgroup of order 6 it means it has at least one lement of order 6. Now as $a^3 = e \implies (a^3)^2 = a^6 = e^2 = e$. Thus $a^2 \in H$. Or if $a^2 \notin H$ then $a^2 \in aH$ is trivially true.

   (f) If $a^2 \in H$, then $a \in H$. Why? And why is "this case ruled out"? If given $a^2 \in H$. Then as H is a subgroup of A_4 of order 6 so $(a^2)^6 = e = a^{12}$. By right cancellation law we obtain $a^3 = e \implies a \in H$. Now this is ruled out because it is given that $a \notin H$.

   (g) If $a^2 \in aH$, then $a^2 = ah$ for some $h \in H$. So $a \in H$. Why? For $a^2 \in aH \implies a^2 = ah$ for some $h \in H$. Now by left cancellation law $a^2 = aa = ah$ Multiply by a^{-1} on both sides and we get $a = h$. So $a \in H$.

   (h) So any subgroup of A_4 of order 6 size must contain all elements of A_4 of order 3. Why? And why is this "absurd"? As given H is a subgroup of order 6. That is it has only 6 elements in numbers how is possible that all the eight elements of order 3 within itself. So this statement is absurd in that it should contain all eight elements of order 3.

   (i) What initial assumption must be false given this contradiction? The inital statement let H be a subgroup of A_4 of order 6 is contradictory.

   (j) How does this example show the converse of Lagrange's Theorem is false (refer to your answer in part (a))? Now as the order of A_4 is 12 and we also have taken the subset of given group of order 6. 6 divides 12 but it is not true that the subset becomes a subgroup.

2. Let H be a subgroup of the D_n with odd order. Show that H must be cyclic. (Recall: D_n can be defined as $D_n = \langle r, f \rangle$ where $|r| = n$, $|f| = 2$, and $r^k f = f r^{-k}$.) Let H denote reflections. Then

we have $a = r^i f$ and $b = r^j f$ both $\in H$. Then $r^i f(r^j f)^{-1} = (r^i f)^{-1} r^j f = f r^{-i} r^j = f^2 r^{i-j}$. Then we have $r^{i-j} \in H$ which is a contradiction. Hence the set of reflections can not form a subgroup. Hence H must be with only rotations. So $|H|$ = n = odd and $H = \langle r^k \rangle$. Where $k \| n$. As set of rotations forms a group and $r^n = e$ and $r^i rj - r^j r^i$ ie comutative. Hence H is a cyclic group of odd order generated by r.

3. Let H be a subgroup of S_n. Use properties of cosets to prove that either every member of H is an even permutation or exactly half of the members is even. Let H be a subgroup of S_n. If H contains no odd permutations, then H contains only even permutations, and we're done. Otherwise, let $o \in H$ be an odd permutation and consider the function $f : H \to H$ that multiplies each element by o. Note that this function is bijective: it's injective, with an inverse function that multiplies each element by o^{-1}, and it's surjective, because for every element $h \in H$, we can find an element $h\dot{o}^{-1}$ that f maps into it. Note that multiplying by an odd permutation changes odd permutations into even permutations and vice versa. It follows that f is a bijection that perfectly pairs up the odd permutations in H with the even permutations. Hence there are exactly as many odd permutations as even permutations in H.

4. Prove that an abelian group of order 15 is cyclic. Do not use Cauchy's Theorem for Abelian Groups. Let G be a group of order 15. Since 15 is divisible by two prime numbers by the theoreom of abelian groups of semi prime order being cyclic all abelian groups of order 15 are cyclic.