1. Find a subgroup of $\mathbb{Z}_{20} \oplus U(16)$ that is isomorphic to $\mathbb{Z}_4 \oplus \mathbb{Z}_5$. Provide the isomorphism (but you do not need to prove your map works). Let $H = \{0, 4, 8, 12, 16\}$ which is a subgroup of $\mathbb{Z}_{20}$ of order 5. Let $K = \{1, 5, 9, 13\}$ which is a subgroup of U(16) of order 4. Now $H \oplus K$ is a subgroup of $G = \mathbb{Z}_{20} \oplus U(16)$. Now define isomophism $\phi : \mathbb{Z}_4 \oplus \mathbb{Z}_5 \to K \oplus H$ as $\phi(x, y) = (5^x, 4y)$. Then ϕ is an ismorphism.

2. Consider the group $\mathbb{Z}_{90} \oplus \mathbb{Z}_{36}$.

   (a) Without computing all of them, determine how many elements of order 15 are there in $\mathbb{Z}_{90} \oplus \mathbb{Z}_{36}$? 8 choices for $\mathbb{Z}_{90}$ 3 choices for $\mathbb{Z}_{36}$ plus 4 choices for $\mathbb{Z}_{90}$ and 2 choices for $\mathbb{Z}_{36}$ leads to a total of 32 elements of order 15.

   (b) Determine the number of cyclic groups of order 15 in $\mathbb{Z}_{90} \oplus \mathbb{Z}_{36}$. ~~Provide a generator for each of the subgroups.~~ We know that each group of order fifteen has a unique generator so 32/8= 4 cyclic groups.

3. In this problem, we show why the operation we defined on cosets only makes sense when the subgroup is normal.

   (a) Let H be a subgroup of a group G with the property that for all $a, b \in G$, $aHbH = abH$. Prove that H must be normal. Consider any element $g \in G$. Then gH and $g^{-1}H$ are two left cosets of H in G. This $gHg^{-1}H$ is also a left coset of H in G. Also $gHg^{-1}H = gg^{-1}H = eH$. Now $eH = geg^{-1}H \in gHg^{-1}H \implies e \in gHg^{-1}H$. Also $e \in H$ as H is a subgroup of G. So we get H and $gHg^{-1}H$ are two left cosets having one common elemeent e. We know from the property of equivalence classes that two left cosets are either equal or have no elements in common. Therefore $H = gHg^{-1}H$. Now for all $h_1, h_2 \in H$ and $g \in G$, $gh_1g^{-1}h_2 = gHg^{-1}H \implies gh_1g^{-1}h_2 \in H \implies gh_1g^{-1}h_2h_2^{-1} = Hh_2^{-1} \implies gh_1g^{-1} \in H$ for all $h_1 \in H$ and all $g \in G$ implies H is normal in G.

   (b) Give an example of a group G and subgroup K such that $aKbK \neq abK$ for some $a, b \in G$. $G = \mathbb{R}^*$ $K = 2^x$ for all integers.

4. Let H be a normal subgroup of a finite group G and let $x \in G$. If $\gcd(|x|, |G/H|) = 1$, show that $x \in H$. Let $|x| = a$ and $|G/H| = b$. We are given that $\gcd(|x|, |G/H|) = 1$ so we know there exists $m, n \in \mathbb{Z}$ such that $ma + nb = 1$. Now consider the coset xH. $(xH)^a = eH = H \implies x^a \in H$. Likewise we have $(xH)^b = H \implies x^b \in H$. Now we can write $x = x^1 = x^{ma+nb} = x^{am}x^{bn} \in H$. Since $x^a \in H$ $(x^a)^m \in H$. Similarly $(x^b)^n \in H$ so $x \in H$.

5. The following theorem and proof is presented in the textbook (Theorem 9.7 in the 9th edition).

   Theorem. Let G be group of order p^2 where p is prime. Then G is isomorphic to $\mathbb{Z}_{p^2}$ or $\mathbb{Z}_p \oplus \mathbb{Z}_p$.

   Proof. Let G be a group of order p^2, where p is a prime. If G has an element of order p^2, then G is isomorphic to $\mathbb{Z}_{p^2}$. **(1) So, by Corollary 2 of Lagrange's Theorem, we may assume that every nonidentity element of G has order p.** First we show that for any element a, the subgroup $\langle a \rangle$ is normal in G. **(2) If this is not the case, then there is an element b in G such that bab^{-1} is not in $\langle a \rangle$. (3) Then $\langle a \rangle$ and $\langle bab^{-1} \rangle$ are distinct subgroups of order p.** Since $\langle a \rangle \cap \langle bab^{-1} \rangle$ is a subgroup of both $\langle a \rangle$ and $\langle bab^{-1} \rangle$, we have that $\langle a \rangle \cap \langle bab^{-1} \rangle = \{e\}$. **(4) From this it follows that the distinct left cosets of $\langle bab^{-1} \rangle$ are $\langle bab^{-1} \rangle$, $a\langle bab^{-1} \rangle$, $a^2\langle bab^{-1} \rangle$, . . . , $a^{p-1}\langle bab^{-1} \rangle$.** Since b^{-1} must lie in one of these cosets, we may write b^{-1} in the form $b^{-1} = a^i(bab^{-1})^j = a^iba^jb^{-1}$ for some i and j. Canceling the b^{-1} terms, we obtain $e = a^iba^j$ and therefore $b = a^{-i-j} \in \langle a \rangle$. **(5) This contradiction verifies our assertion that every subgroup of the form $\langle a \rangle$ is normal in G.** To complete the proof, let x be any nonidentity element in G and y be any element of G not in $\langle x \rangle$. **(6) Then, by comparing orders and using Theorem 9.6, we see that $G = \langle x \rangle \times \langle y \rangle \cong \mathbb{Z}_p \oplus \mathbb{Z}_p$.**

The author left out a lot of the details in the proof (as usual). Most sentences could use more explanation, but the 6 sentences in bold in particular require more justification. Replace those sentences with the missing details to write a more detailed proof (and add some paragraph spacing to make it more readable). In a finite group the order of each element divides the order of the group. Assume the subgroup $\langle a \rangle$ is normal in G. The subgroup is normal iff $gag^{-1} = \langle a \rangle$. For all $g \in G$. That won't turn out to be correct because we willl have an elemenet b in $\langle a \rangle$ such that it has order p. So $bab^{-1} \notin \langle a \rangle$ as it will not satify the condition of a normal subgroup. Now as $bab^{-1} \notin \langle a \rangle$ we have two subgroups of order p namely $\langle bab^{-1} \rangle$ and $\langle a \rangle$ because the order couldn't be one and the only other choice is p. Since the intersection of two subgroups is always a subgroup we are left with these two subgroups having an order of one namely it is the identity. b^{-1} must lie in one of these two cosets. This contradicts the assumption that $b \in \langle a \rangle$ and $\langle bab^{-1} \rangle = \langle a \rangle$. Hence every subgroup of the form $\langle a \rangle$ is normal in G. Hence satifying the condition of the theorem. $G = \langle x \rangle \oplus \langle b \rangle \cong \mathbb{Z}_p \oplus \mathbb{Z}_p$