1. Find a subgroup of $\mathbb{Z}_{20} \oplus U(16)$ that is isomorphic to $\mathbb{Z}_4 \oplus \mathbb{Z}_5$. Provide the isomorphism (but you do not need to prove your map works). Let $H = \{0, 4, 8, 12, 16\}$ which is a subgroup of $\mathbb{Z}_{20}$ of order 5. Let $K = \{1, 5, 9, 13\}$ which is a subgroup of U(16) of order 4. Now $H \oplus K$ is a subgroup of $G = \mathbb{Z}_{20} \oplus U(16)$. Now define isomophism $\phi : \mathbb{Z}_4 \oplus \mathbb{Z}_5 \to K \oplus H$ as $\phi(x, y) = (5^x, 4y)$. Then $\phi$ is an ismorphism.

2. Consider the group $\mathbb{Z}_{90} \oplus \mathbb{Z}_{36}$.

   (a) Without computing all of them, determine how many elements of order 15 are there in $\mathbb{Z}_{90} \oplus \mathbb{Z}_{36}$? 32 elements of order 15.

   (b) Determine the number of cyclic groups of order 15 in $\mathbb{Z}_{90} \oplus \mathbb{Z}_{36}$. ~~Provide a generator for each of the subgroups.~~ 4 cyclic groups.

3. In this problem, we show why the operation we defined on cosets only makes sense when the subgroup is normal.

   (a) Let $H$ be a subgroup of a group $G$ with the property that for all $a, b \in G$, $aHbH = abH$. Prove that $H$ must be normal. Consider any element $g \in G$. Then $gH$ and $g^{-1}H$ are two left cosets of H in G. This $gHg^{-1}H$ is also a left coset of $H$ in $G$. Also $gHg^{-1}H = gg^{-1}H = eH$. Now $eH = geg^{-1}H \in gHg^{-1}H \implies e \in gHg^{-1}H$. Also $e \in H$ as $H$ is a subgroup of $G$. So we get $H$ and $gHg^{-1}H$ are two left cosets having one common elemeent $e$. We know from the property of equivalence classes that two left cosets are either equal or have no elements in common. Therefore $H = gHg^{-1}H$. Now for all $h_1, h_2 \in H$ and $g \in G$, $gh_1g^{-1}h_2 = gHg^{-1}H \implies gh_1g^{-1}h_2 \in H \implies gh_1g^{-1}h_2h_2^{-1} = Hh_2^{-1} \implies gh_1g^{-1} \in H$ for all $h_1 \in H$ and all $g \in G$ implies $H$ is normal in $G$.

   (b) Give an example of a group $G$ and subgroup $K$ such that $aKbK \neq abK$ for some $a, b \in G$. $G = \mathbb{R}^*$ $K = 2^x$ for all integers.

4. Let $H$ be a normal subgroup of a finite group $G$ and let $x \in G$. If $\gcd(|x|, |G/H|) = 1$, show that $x \in H$. Let $|x| = a$ and $|G/H| = b$. We are given that $\gcd(|x|, |G/H|) = 1$ so we know there exists $m, n \in \mathbb{Z}$ such that $ma + nb = 1$. Now consider the coset $xH$. $(xH)^a = eH = H \implies x^a \in H$. Likewise we have $(xH)^b = H \implies x^b \in H$. Now we can write $x = x^1 = x^{ma+nb} = x^{am}x^{bn} \in H$. Since $x^a \in H$ $(x^a)^m \in H$. Similarly $(x^b)^n \in H$ so $x \in H$.

5. The following theorem and proof is presented in the textbook (Theorem 9.7 in the 9th edition).

   *Theorem.* Let $G$ be group of order $p^2$ where $p$ is prime. Then $G$ is isomorphic to $\mathbb{Z}_{p^2}$ or $\mathbb{Z}_p \oplus \mathbb{Z}_p$.

   *Proof.* Let $G$ be a group of order $p^2$, where $p$ is a prime. If $G$ has an element of order $p^2$, then $G$ is isomorphic to $\mathbb{Z}_{p^2}$. **(1) So, by Corollary 2 of Lagrange's Theorem, we may assume that every nonidentity element of $G$ has order $p$.** First we show that for any element $a$, the subgroup $\langle a \rangle$ is normal in $G$. **(2) If this is not the case, then there is an element $b$ in $G$ such that $bab^{-1}$ is not in $\langle a \rangle$. (3) Then $\langle a \rangle$ and $\langle bab^{-1} \rangle$ are distinct subgroups of order $p$.** Since $\langle a \rangle \cap \langle bab^{-1} \rangle$ is a subgroup of both $\langle a \rangle$ and $\langle bab^{-1} \rangle$, we have that $\langle a \rangle \cap \langle bab^{-1} \rangle = \{e\}$. **(4) From this it follows that the distinct left cosets of $\langle bab^{-1} \rangle$ are $\langle bab^{-1} \rangle$, $a\langle bab^{-1} \rangle$, $a^2\langle bab^{-1} \rangle$ , . . . , $a^{p-1}\langle bab^{-1} \rangle$.** Since $b^{-1}$ must lie in one of these cosets, we may write $b^{-1}$ in the form $b^{-1} = a^i(bab^{-1})^j = a^iba^jb^{-1}$ for some $i$ and $j$. Canceling the $b^{-1}$ terms, we obtain $e = a^iba^j$ and therefore $b = a^{-i-j} \in \langle a \rangle$. **(5) This contradiction verifies our assertion that every subgroup of the form $\langle a \rangle$ is normal in $G$.** To complete the proof, let $x$ be any nonidentity element in

$G$ and $y$ be any element of $G$ not in $\langle x \rangle$. **(6) Then, by comparing orders and using Theorem 9.6, we see that $G = \langle x \rangle \times \langle y \rangle \cong \mathbb{Z}_p \oplus \mathbb{Z}_p$.**

The author left out a lot of the details in the proof (as usual). Most sentences could use more explanation, but the 6 sentences in bold in particular require more justification. Replace those sentences with the missing details to write a more detailed proof (and add some paragraph spacing to make it more readable). In a finite group the order of each element divides the order of the group. Assume the subgroup $\langle a \rangle$ is normal in $G$. The subgroup is normal iff $gag^{-1} = \langle a \rangle$. For all $g \in G$. That won't turn out to be correct because we willl have an elemenet $b$ in $\langle a \rangle$ such that it has order p. So $bab^{-1} \notin \langle a \rangle$ as it will not satify the condition of a normal subgroup. Now as $bab^{-1} \notin \langle a \rangle$ we have two subgroups of order p namely $\langle bab^{-1} \rangle$ and $\langle a \rangle$ because the order couldn't be one and the only other choice is p. Since the intersection of two subgroups is always a subgroup we are left with these two subgroups having an order of one namely it is the identity. $b^{-1}$ must lie in one of these two cosets. This contradicts the assumption that $b \in \langle a \rangle$ and $\langle bab^{-1} \rangle = \langle a \rangle$. Hence every subgroup of the form $\langle a \rangle$ is normal in $G$. Hence satifying the condition of the theorem. $G = \langle x \rangle \oplus \langle b \rangle \cong \mathbb{Z}_p \oplus \mathbb{Z}_p$