Homework 6

1. Find a subgroup of $\mathbb{Z}_{20} \oplus U(16)$ that is isomorphic to $\mathbb{Z}_4 \oplus \mathbb{Z}_5$. Provide the isomorphism (but you do not need to prove your map works).

2. Consider the group $\mathbb{Z}_{90} \oplus \mathbb{Z}_{36}$.

   (a) Without computing all of them, determine how many elements of order 15 are there in $\mathbb{Z}_{90} \oplus \mathbb{Z}_{36}$? (Hint: Use Theorem 2.29 from the Module 2 Notes.)

   (b) Determine the number of cyclic groups of order 15 in $\mathbb{Z}_{90} \oplus \mathbb{Z}_{36}$. ~~Provide a generator for each of the subgroups.~~

3. In this problem, we show why the operation we defined on cosets only makes sense when the subgroup is normal.

   (a) Let H be a subgroup of a group G with the property that for all $a, b \in G$, $aHbH = abH$. Prove that H must be normal.

   (b) Give an example of a group G and subgroup K such that $aKbK \neq abK$ for some $a, b \in G$.

4. Let H be a normal subgroup of a finite group G and let $x \in G$. If $\gcd(|x|, |G/H|) = 1$, show that $x \in H$.

5. The following theorem and proof is presented in the textbook (Theorem 9.7 in the 9th edition).

   Theorem. Let G be group of order p^2 where p is prime. Then G is isomorphic to $\mathbb{Z}_{p^2}$ or $\mathbb{Z}_p \oplus \mathbb{Z}_p$.

   Proof. Let G be a group of order p^2, where p is a prime. If G has an element of order p^2, then G is isomorphic to $\mathbb{Z}_{p^2}$. **(1) So, by Corollary 2 of Lagrange's Theorem, we may assume that every nonidentity element of G has order p.** First we show that for any element a, the subgroup $\langle a \rangle$ is normal in G. **(2) If this is not the case, then there is an element b in G such that bab^{-1} is not in $\langle a \rangle$. (3) Then $\langle a \rangle$ and $\langle bab^{-1} \rangle$ are distinct subgroups of order p.** Since $\langle a \rangle \cap \langle bab^{-1} \rangle$ is a subgroup of both $\langle a \rangle$ and $\langle bab^{-1} \rangle$, we have that $\langle a \rangle \cap \langle bab^{-1} \rangle = \{e\}$. **(4) From this it follows that the distinct left cosets of $\langle bab^{-1} \rangle$ are $\langle bab^{-1} \rangle$, $a\langle bab^{-1} \rangle$, $a^2\langle bab^{-1} \rangle$, . . . , $a^{p-1}\langle bab^{-1} \rangle$.** Since b^{-1} must lie in one of these cosets, we may write b^{-1} in the form $b^{-1} = a^i(bab^{-1})^j = a^iba^jb^{-1}$ for some i and j. Canceling the b^{-1} terms, we obtain $e = a^iba^j$ and therefore $b = a^{-i-j} \in \langle a \rangle$. **(5) This contradiction verifies our assertion that every subgroup of the form $\langle a \rangle$ is normal in G.** To complete the proof, let x be any nonidentity element in G and y be any element of G not in $\langle x \rangle$. **(6) Then, by comparing orders and using Theorem 9.6, we see that $G = \langle x \rangle \times \langle y \rangle \cong \mathbb{Z}_p \oplus \mathbb{Z}_p$.**

   The author left out a lot of the details in the proof (as usual). Most sentences could use more explanation, but the 6 sentences in bold in particular require more justification. Replace those sentences with the missing details to write a more detailed proof (and add some paragraph spacing to make it more readable).