A Practical Application of the Banach-Tarski Paradox to Fluid Dynamics?

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Abstract

We postulate on a connection between blow up times of Navier-Stokes equations and the Banach-Tarski Paradox.

Observations

If we plug our sink, turn on the faucet to a critical value, and watch the fluid evolve away from a single body given by a manifold in the reservoir M_r. It evolves through the tap and dribbles into multiple manifolds $M_1 \cdots M_i$ and into the sink where the submanifolds merge M_s. We turn off the sink and the fluid rests in the reservoir and in the sink. $M_r \to M_r + M_1 + \cdots + M_n + M_s \to M_r + M_s$. As topology this looks like nonsense. We will try to identify a way to talk about this phenomenon.

Existing Methods

Navier-Stokes

Navier-Stokes equations blow up in finite time [4] and have lots of restrictions which I will not repeat but can be found here [4]. This blow up is not a bug but a feature, more on that later. I'm only going to make the problem worse with more restrictions. Let's denote the set of solutions to these equations on a manifold M as $F(M)$.

Volume Preserving Diffeomorphism

The group of volume preserving diffeomorphism of smoothness k between homeotopic manifolds is denoted $\text{Diff}^k_\omega(M)$. As a group it is perfect and as a Lie Algebra it is infinite dimensional [1]. We tend to favor the infinitely smooth diffeomorphisms $k \to \infty$. Additionally, instead of volume form ω preserving diffeomorphisms in this case we want the mass preserving ones denoted with an m. m is defined the same as in the Navier-Stokes Equations. This group denoted by $\text{Diff}^k_m(M)$ is probably isomorphic to $\text{Diff}^k_\omega(M)$. [1] We use mass because mass is conserved even in compressible fluids.

Banach Tarski

The Banach-Tarski paradox states that as a consequence of the axiom of choice one can disassemble a unit ball into discrete subsets and reassemble these subsets into two new unit balls [3]. We know that simples - molecules - in a fluid can be interchanged. Fundamentally two water molecules are isomorphic, the water in our reservoir M_r can probably be considered well mixed. And we can select a molecule of water and move it into the new subsets in general. We denote the *split* of a manifold M using the Banach-Tarski result as $M|^\wedge_+$ yielding two new manifolds M_1, M_2. We can impose an additional restriction on the manifolds such that mass[2] is conserved. $(M_1, m_1)|^{wedge}_+ = (M_2, m_2) \oplus (M_3, m_3)$. Where $m_1 = m_2 + m_3$. In further notation we will drop the little m's and take this result as part of the operation $|^\wedge_+$. As far as I am aware there is no set theory result to reverse this operation, but we will denote it $|^\wedge_-$. Which takes in two manifolds M_1 and M_2 and yields a third manifold M_3 where mass is once again conserved. There may be something clever one can do with Ricci flow here to dissolve one of the two smaller manifolds as time evolves. [2] As a whole we refer to all these manifolds under these operations $\mathfrak{M} = (M, |^\wedge)$

[1] I very badly want to put a citation here.
[2] Volume form may work just as .

Set of Solutions

So if we want to bound $F(M)$ we can take the intersections of $\mathfrak{F}(M) = F(M) \cap \mathrm{Diff}_m^\infty(M)$ [3] Furthermore we want our manifolds M to *split* and *merge*. We want to talk about the total space X with its own topology $\mathscr{O}$ with the submanifolds embedded within that topology $\mathfrak{M}$. Then the totality of fluid solutions can be written $(X, \mathscr{O}, \mathfrak{F}(\mathfrak{M}))$. As solutions $F(M)$ become unbounded at time t it may be possible to drop the intersection with $\mathrm{Diff}_m^\infty(M)$ and link the time t to a *split*.

Further Research

Couple of open questions:

$$\mathrm{Diff}_m^k(M) \cong \mathrm{Diff}_\omega^k(M) \tag{1}$$

In set theory:

$$\exists|_{-}^{\wedge} \tag{2}$$

A critical point in time t:

$$\text{Ricci Flow}(t) \propto |_{-}^{wedge}(t) \tag{3}$$

$$\sup(|F(m)|)(t) \propto |_{+}^{wedge}(t) \tag{4}$$

Finally: is the total topology $(X, \mathscr{O})$ well defined?

References

[1] Augustin Banyaga. *The Structure of Classical Diffeomorphism Groups*, volume 400 of *Mathematics and Its Applications*. Springer, Dordrecht, 1997.

[2] Panagiotis Gianniotis. The ricci flow on manifolds with boundary, 2015.

[3] Terence Tao. *An Introduction to Measure Theory*, volume 126 of *Graduate Studies in Mathematics*. American Mathematical Society, 2011.

[4] Terence Tao. Finite time blowup for an averaged three-dimensional navier-stokes equation, 2015.

[3] It may also work to intersect solutions to Navier-Stokes with C^k functions.