

Notes on Fluid Dynamics

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Abstract

We postulate on a connection between blow up times of Navier-Stokes equations and the Banach-Tarski Paradox.

Observations

If we plug our sink, turn on the faucet to a critical value and watch the fluid evolve away from a single body given by a manifold in the reservoir M_r . It evolves through the tap and dribbles into multiple manifolds $M_1 \cdots M_i$ and into the sink where the submanifolds merge M_s . We turn off the sink and the fluid rests in the reservoir and in the sink. $M_r \rightarrow M_r + M_1 + \cdots + M_n + M_s \cot M_r + M_s$. As topology this looks like nonsense. We will try to identify a way to talk about this phenomenon.

Existing Methods

Navier-Stokes

Navier-Stokes equations blow up in finite time [4] and have lots of restrictions which I will not repeat but can be found here [4]. This blow up is not a bug but a feature, more on that later. I'm only going to make the problem worse with more restrictions. Let's denote the set of solutions to these equations on a manifold M as $F(M)$.

Volume Preserving Diffeomorphism

The group of volume preserving diffeomorphism of smoothness k between homeotopic manifolds is denoted $\text{Diff}_\omega^k(M)$. As a group it is perfect and as a Lie Algebra it is infinite dimensional [1]. We tend to favor the infinitely smooth diffeomorphisms $k \rightarrow \infty$. Additionally, instead of volume form ω preserving diffeomorphisms in this case we want the mass preserving ones denoted with an m . m is defined the same as in the Navier-Stokes Equations. This group denoted by $\text{Diff}_m^k(M)$ is probably isomorphic to $\text{Diff}_\omega^k(M)$.¹ We use mass because mass is conserved even in compressible fluids.

Banach Tarski

The Banach-Tarski paradox states that as a consequence of the axiom of choice one can disassemble a unit ball into discrete subsets and reassemble these subsets into two new unit balls [3]. We know that simples - molecules - in a fluid can be interchanged. Fundamentally two water molecules are isomorphic, the water in our reservoir M_r can probably be considered well mixed. And we can select a molecule of water and move it into the new subsets in general. We denote the *split* of a manifold M using the Banach-Tarski result as $M|_+^\wedge$ yielding two new manifolds M_1, M_2 . We can impose an additional restriction on the manifolds such that mass² is conserved. $(M_1, m_1)|_+^{wedge} = (M_2, m_2) \oplus (M_3, m_3)$. Where $m_1 = m_2 + m_3$. In further notation we will drop the little m 's and take this result as part of the operation $|_+^\wedge$. As far as I am aware there is no set theory result to reverse this operation, but we will denote it $|_-^\wedge$. Which takes in two manifolds M_1 and M_2 and yields a third manifold M_3 where mass is once again conserved. There may be something clever one can do with Ricci flow here to dissolve one of the two smaller manifolds as time evolves. [2] As a whole we refer to all these manifolds under these operations $\mathfrak{M} = (M, |^\wedge)$

¹I very badly want to put a citation here.

²Volume form may work just as well

Set of Solutions

So if we want to bound $F(M)$ we can take the intersections of $\mathfrak{F}(M) = F(M) \cap \text{Diff}_m^\infty(M)$. Furthermore we want our manifolds M to *split* and *merge*. We want to talk about the total space X with its own topology \mathcal{O} with the submanifolds embedded within that topology \mathfrak{M} . Then the totality of fluid solutions can be written $(X, \mathcal{O}, \mathfrak{F}(\mathfrak{M}))$. As solutions $F(M)$ become unbounded at time t it may be possible to drop the intersection with $\text{Diff}_m^\infty(M)$ and link the time t to a *split*.

Further Research

Couple of open questions:

$$\text{Diff}_m^k(M) \cong \text{Diff}_\omega^k(M) \tag{1}$$

In set theory:

$$\exists |_-^\wedge \tag{2}$$

A critical point in time t :

$$\text{Ricci Flow}(t) \propto |_-^{wedge}(t) \tag{3}$$

$$\sup(|F(m)|)(t) \propto |_+^{wedge}(t) \tag{4}$$

Finally: is the total topology (X, \mathcal{O}) well defined?

References

- [1] Augustin Banyaga. *The Structure of Classical Diffeomorphism Groups*, volume 400 of *Mathematics and Its Applications*. Springer, Dordrecht, 1997.
- [2] Panagiotis Gianniotis. The ricci flow on manifolds with boundary, 2015.
- [3] Terence Tao. *An Introduction to Measure Theory*, volume 126 of *Graduate Studies in Mathematics*. American Mathematical Society, 2011.
- [4] Terence Tao. Finite time blowup for an averaged three-dimensional navier-stokes equation, 2015.