From Schrödinger to Riemann: A Quantum-Theoretic Conjecture

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Abstract

We explore a conjectural link between the time-independent Schrödinger equation and the non-trivial zeros of the Riemann zeta function. By reinterpreting the Basel problem and fractional calculus, we propose a framework where the eigenvalues of a quantum system correspond to the critical zeros of $\zeta(s)$. This work is inspired by Hilbert's vision of unifying quantum mechanics and number theory.

1 Schrödinger's Equation and Eigenvalues

The time-independent Schrödinger equation in one dimension (with $\hbar = 1$) is [1]:

$$-\frac{1}{2}\frac{d^2\psi}{dx^2} = E\psi(x), \quad \psi(0) = \psi(\pi) = 0. \tag{1}$$

The solutions are sinusoidal with eigenvalues $E_n = \frac{n^2}{2}$ for $n \in \mathbb{N}$. The general solution is:

$$\psi(x) = \sum_{n=1}^{\infty} A_n \sin(nx). \tag{2}$$

2 Basel Problem and Zeta Connection

The Fourier series of $f(x) = x^2$ on $[-\pi, \pi]$ yields:

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}, \tag{3}$$

revealing a deep link between $\zeta(s)$ and harmonic analysis. We generalize this to $\zeta(s)$ for $\Re(s) > 1$.

3 Fractional Calculus and Zeta Zeros

The Riemann-Liouville fractional integral for $\Re(s) > 0$ is:

$${}_0D_x^{-s} f(x) = \frac{1}{\Gamma(s)} \int_0^x (x-t)^{s-1} f(t)\, dt. \tag{4}$$

Applying this to $|\psi(x)|^2$ and demanding unit normalization suggests:

$$\sum_{n=1}^{\infty} \frac{|A_n|^2}{n^{2\sigma}} = 1, \quad \sigma = \Re(s). \tag{5}$$

4 Conjecture: Quantum Zeta Correspondence

Conjecture 1 (Quantum-Theoretic Riemann Hypothesis). *Let $\psi(x)$ be a solution to (1) with eigenvalues $E_n = \frac{n^2}{2}$. If the coefficients A_n are chosen such that:*

$$\sum_{n=1}^{\infty} \frac{|A_n|^2}{n^s} = 0, \tag{6}$$

then the non-trivial solutions s satisfy $\Re(s) = \frac{1}{2}$. This implies an isomorphism between the energy spectrum of $\psi(x)$ and the critical zeros of $\zeta(s)$.

5 Discussion

The conjecture posits that:

- The normalization condition (5) mirrors the analytic continuation of $\zeta(s)$.
- The critical line $\Re(s) = \frac{1}{2}$ emerges from the symmetry of the quantum system.
- A violation would require a non-unitary or asymmetric $\psi(x)$, akin to a "phase transition" in the zeta zeros.

References

[1] Jamal Nazrul Islam. The Schrödinger equation in quantum field theory. *Foundations of Physics*, 24(5):593–630, May 1994.