From Basel to Schrodinger: The Grand Riemann Conjecture

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Abstract

Hilbert dreamed of a theory of Quantum gravity with eigenvalues given by the Riemann Zeta Function. We derive a quantum theory of spin and connect it to eigenvalues of Dirichlet L-Functions.

1 Solving the Schrodinger Equation

Schroginger introduced his wave equation [5] which in a field with zero potential we write:

$$i\frac{\partial\psi(x,t)}{\partial t} = \frac{\partial^2\psi(x,t)}{\partial x^2} \tag{1}$$

This is idealized and set in natural units and in one dimension but works for our purposes. We begin with the separation of variables.

$$\begin{aligned} \psi(x,t) &= T(t)X(x) \\ iT'(t)X(x) &= X''(x)T(t) \\ i\frac{T'(t)}{T(t)} &= \frac{X''(x)}{X(x)} = \lambda \end{aligned}$$

Where λ is some constant. Furthermore we write:

$$\begin{aligned} iT'(t) &= \lambda T(t) \\ T(t) &= Ae^{i\lambda t} \end{aligned}$$

And:

$$\begin{aligned} X''(x) &= \lambda X(x) \\ X(x) &= c_1e^{\sqrt{\lambda}x} + c_2e^{-\sqrt{\lambda}x} \end{aligned}$$

In idealized cases of the wave equation we fix the origin to be zero and a distance L to be zero. For our purposes we shall set $L = \pi$. If we take $\lambda > 0$ we get the trivial solution. If we set $\lambda = 0$ we once again arrive at the trivial solution. We take $\lambda < 0$ and write:

$$0 = c_2\sin(\mu_n x)$$

Without loss of generality in the constant c_2. The constant $\mu_n = n\frac{\pi}{L}$. In the time dependent Schrodinger equation linear combinations of solutions are solutions so we need $n \in \mathbb{N}$ and $\lambda_n = -\mu_n^2$. We arrive at:

$$\psi(x,t) = \sum_{n=0}^{\infty} A_n e^{itn^2}\sin(nx) \tag{2}$$

Traditionally in quantum mechanics we renormalize the equation to find the missing constant A_n.

2 Basel Problem and Fourier

Taking the fourier series of $f(x) = x^2$ over the interval $[-\pi, \pi]$ results in a derivation of a closed form solution of the Basel Problem or $\zeta(2)$. By Fourier we know [2] [3]:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\frac{2\pi nx}{L}) + b_n \sin(\frac{2\pi nx}{L}) \tag{3}$$

With a_n and b_n in 3 given by:

$$\begin{aligned} a_n &= \frac{2}{L}\int_L f(x)\cos(\frac{2\pi nx}{L}) \\ b_n &= \frac{2}{L}\int_L f(x)\sin(\frac{2\pi nx}{L}) \end{aligned}$$

x^2 is an even function so $b_n = 0, \forall n \in \mathbb{N}$. Solving for a_n in 3 yields:

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 dx \\ &= \frac{1}{\pi}\frac{1}{3}x^3|_{x=-\pi}^{\pi} \\ &= \frac{2\pi}{3} \end{aligned}$$

And in the $n \neq 0$ case we yield:

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos(nx) dx \\ &= (-1)^n \frac{4}{n^2} \end{aligned}$$

Putting a_n in 3 and setting $x = \pi$ we get our solution to the Basel problem.

$$\begin{aligned} x^2 &= \frac{1}{3}\pi^2 + \sum_{n=1}^{\infty}(-1)^n \frac{4}{n^2}\cos(nx) \\ \pi^2 &= \frac{1}{3}\pi^2 + \sum_{n=1}^{\infty}\frac{4}{n^2} \\ \frac{\pi^2}{6} &= \sum_{n=1}^{\infty}\frac{1}{n^2} \end{aligned}$$

We know by this there is an intimate connection between the Zeta Function and fourier series.

3 Grand Riemann Conjecture

Here we take the notion of fractional integrals and generalize it to a complex number s. The fraction is given by the variable α.

$${}_aD_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)}\int_t^b (\tau - t)^{\alpha-1} f(\tau) d\tau \tag{4}$$

This 4 is the Riemann-Liouville fractional integral [4]. Here we take $\alpha = s$ to be any complex number. Now we apply it to 2 and renormalize the constants in A_n.

$$
\begin{aligned}
{}_aD_x^{-s}|\psi(x,t)|^2 &= 1 \\
\frac{1}{\Gamma(s)}\int_0^\pi |(\pi - x)^{s-1}\psi(x,t)|^2 dx &= 1 \\
\frac{1}{\Gamma(s)}\int_0^\pi |(\pi - x)^{s-1}\sum_{n=1}^{\infty} A_n \sin(nx)e^{itn^2}|^2 dx &= 1 \\
\frac{1}{\Gamma(s)}|\sum_{n=1}^{\infty}\frac{A_n}{2(-n)^s(-1)^{\frac{3s}{2}}}[-i(-1)^n(\Gamma(s,-in(x-\pi)) - (-1)^s\Gamma(s,in(x-\pi)))]_{x=0}^{x=\pi}\psi(t)|^2 &= 1 \\
\frac{1}{\Gamma(s)}|\sum_{n=1}^{\infty}\frac{A_n}{2(-n)^s(-1)^{\frac{3s}{2}}}[i(-1)^n(\Gamma(s,0)) - (-1)^s\Gamma(s,0) - (-1)\Gamma(s,0) + i(-1)^n\Gamma(s,-in\pi) + (-1)^s\Gamma(s,in\pi)\psi(t)]|^2 &= 1
\end{aligned}
$$

This is cool but we just want the Zeta function. We can absorb the constants in Γ into A_n without losing generality. As well as constants in s.

$$\sum_{n=1}^{\infty}\frac{A_n}{n^s}\frac{\bar{A_n}}{\bar{n^s}}e^{in^2t}e^{\bar{in^2t}} = 1$$

$$\sum_{n=1}^{\infty}\frac{A_n\bar{A_n}}{n^s\bar{n^s}}e^{itn^2}e^{\bar{itn^2}} = 1 \tag{5}$$

We know the Riemann Hypothesis over finite fields is true. [6] [7] But more work must be done to prove it over the field of complex numbers. There exists a map between solutions of 5 to the set of L-Functions by setting A_n appropriately and multiplying by a constant. Products of solutions to 5 are also solutions. We perform a product integral over the domain. This is reminiscent of what is described in the field with one unit here [7]. All solutions across time are solutions. We write:

$$\int_{-\infty}^{\infty}\sum_{n=1}^{\infty}\frac{A_n\bar{A_n}}{n^s\bar{n^s}}|e^{itn^2}|^{2dt} = 1$$

$\psi(t)$ cycles through complex numbers in the circle. We can absorb without loss of generality into A_n and $\bar{A_n}$ since we're finding the magnitude so this result about the Riemann Zeta function is stronger than the generalized Riemann Hypothesis. We can rewrite this as a contour integral through the entire complex plane. We can absorb A_n and $\bar{A_n}$ into a new real number which we can call $\chi(n)$. We can also rewrite s to $\frac{s}{2}$ without loss of generality since we are integrating over the entire complex plane. The general Riemann Hypothesis reads: [1]

$$L(\chi, s) = \sum_{n=1}^{\infty}\frac{\chi(n)}{n^s} \tag{6}$$

In our formulation we can once again integrate over the entire complex plane. In fact we can do this infinitely many times.

$$\sum_{n=1}^{\infty}\frac{\chi(n)}{n^s}\frac{\bar{\chi(n)}}{\bar{n^s}} = 1 \tag{7}$$

$$\tag{8}$$

We obtain the Grand Riemann Zeta hypothesis as described here [7] and here [1]. This function clearly has eigenvalues at the negative even integers, the so called 'trivial solutions'. And assuming the variable s isn't equal to one of those values it can be an eigenvalue if the real part of s is equal to $\frac{1}{2}$. Suppose there exists a solution not on the real line or not on the critical line. This would fundamentally break the symmetry between subtraction and division demonstrated between the trivial and non-trivial zeros.

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