

From Schrödinger to Riemann: A Quantum-Theoretic Conjecture

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Abstract

We explore a conjectural link between the time-independent Schrödinger equation and the non-trivial zeros of the Riemann zeta function. By reinterpreting the Basel problem and fractional calculus, we propose a framework where the eigenvalues of a quantum system correspond to the critical zeros of $\zeta(s)$. This work is inspired by Hilbert's vision of unifying quantum mechanics and number theory.

1 Schrödinger's Equation and Eigenvalues

The time-independent Schrödinger equation in one dimension (with $\hbar = 1$) is [3]:

$$-\frac{1}{2} \frac{d^2 \psi}{dx^2} = E \psi(x), \quad \psi(0) = \psi(\pi) = 0. \quad (1)$$

The solutions are sinusoidal with eigenvalues $E_n = \frac{n^2}{2}$ for $n \in \mathbb{N}$. The general solution is:

$$\psi(x) = \sum_{n=1}^{\infty} A_n \sin(nx). \quad (2)$$

2 Basel Problem and Zeta Connection

The Fourier series of $f(x) = x^2$ on $[-\pi, \pi]$ yields:

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}, \quad (3)$$

revealing a deep link between $\zeta(s)$ and harmonic analysis. We generalize this to $\zeta(s)$ for $\Re(s) > 1$.

3 Fractional Calculus and Zeta Zeros

The Riemann-Liouville fractional integral for $\Re(s) > 0$ is [1] [2]:

$${}_0D_x^{-s} f(x) = \frac{1}{\Gamma(s)} \int_0^x (x-t)^{s-1} f(t) dt. \quad (4)$$

Applying this to $|\psi(x)|^2$ and demanding unit normalization suggests:

$$\left| \sum_{n=1}^{\infty} \frac{A_n}{n^{\sigma}} \right|^2 = 1, \quad \sigma = \Re(s). \quad (5)$$

4 Conjecture: Quantum Zeta Correspondence

Conjecture 1 (Quantum-Theoretic Riemann Hypothesis). *Let $\psi(x)$ be a solution to (1) with eigenvalues $E_n = \frac{n^2}{2}$. If the coefficients A_n are chosen such that:*

$$\left| \sum_{n=1}^{\infty} \frac{A_n}{n^s} \right|^2 = 0, \quad (6)$$

then the non-trivial solutions s satisfy $\Re(s) = \frac{1}{2}$. This implies an isomorphism between the energy spectrum of $\psi(x)$ and the critical zeros of $\zeta(s)$.

5 Discussion

The conjecture posits that:

- The normalization condition (5) mirrors the analytic continuation of $\zeta(s)$.
- The critical line $\Re(s) = \frac{1}{2}$ emerges from the symmetry of the quantum system.
- A violation would require a non-unitary or asymmetric $\psi(x)$, akin to a "phase transition" in the zeta zeros.
- The half integer spin of the electron (all fermions) proves the Grand Riemann Conjecture.

References

- [1] J. Hadamard. Essai sur l'étude des fonctions données par leur développement de Taylor. *Journal de Mathématiques Pures et Appliquées*, 4(8):101–186, 1892.
- [2] Richard Hermann. *Fractional Calculus: An Introduction for Physicists*. World Scientific Publishing, New Jersey, 2nd edition, 2014.
- [3] Jamal Nazrul Islam. The Schrödinger equation in quantum field theory. *Foundations of Physics*, 24(5):593–630, May 1994.