

Notes on the Structure C^k Functions

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Abstract

C^k Smooth Functions

A function with k continuous derivatives on a domain X is denoted by $C^k(X)$ [4] There is an ordering:

$$C^0 \supset C^1 \supset \dots \supset C^\omega \supset C^\infty$$

Each C^k satisfies the axioms of a ring.

Proof. We take: $f, g, h \in C^k$. Closure under addition: $(f + g) \in C^k$ because differentiation is linear:

$$\partial^\alpha (f + g) = \partial^\alpha f + \partial^\alpha g$$

Closure under multiplication: $f \cdot g \in C^k$ by Leibniz rule we know:

$$\partial^\alpha (f \cdot g) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} (\partial^\beta f) (\partial^{\alpha-\beta} g)$$

which remains C^k continuous. Associativity under Addition and Multiplication:

$$(f + g) + h = f + (g + h), (f \cdot g) \cdot h = f \cdot (g \cdot h)$$

Commutativity of Addition and Multiplication:

$$f + g = g + f, f \cdot g = g \cdot f$$

Additive identity: given by the zero function $0(X) = 0$, $0 \in C^k$ and satisfies:

$$f + 0 = f, \forall f \in C^k$$

Additive inverses: $\forall f \in C^k$ the function $\exists -f \in C^k$ and satisfies:

$$f + (-f) = 0$$

Multiplicative identity: the constant function $1(X) = 1 \in C^k$ and satisfies:

$$1 \cdot f = f, \forall f \in C^k$$

Distributivity of Multiplication over Addition:

$$f \cdot (g + h) = f \cdot g + f \cdot h$$

□

There is an additional structure of C^k functions. Namely we can compose them with each other. Denote \circ composition of functions $f(g(X))$ for $f, g \in C^k$ when the domain $X = \mathbb{R}^d$ then this is a semigroup under this operation.

Proof. Take $f, g \in C^k$ as before. The space is closed under the operation: follows from the chain rule and the fact that derivatives of f and g up to k are continuous. Associativity: inherited from function composition.

$$(f \circ g) \circ h = f \circ (g \circ h)$$

An identity element $\exists \text{id}(x) = x \in C^k$ and satisfies $f \circ \text{id} = \text{id} \circ f = f$.

□

Invertible C^k Functions

If we restrict C^k to the set of invertible functions we can satisfy all the group axioms under \circ . Take Diff^k as the set of invertible C^k functions. Then there is an ordering:

$$\begin{array}{ccccccc} C^0 & \supset & C^1 & \supset & \dots & \supset & C^\infty \\ \cup & & \cup & & & & \cup \\ \text{Diff}^0 & \supset & \text{Diff}^1 & \supset & \dots & \supset & \text{Diff}^\infty \end{array}$$

Then the only missing property to satisfy the group axioms was the existence of inverses and by construction now for a function f we have the necessary inverse f^{-1} such that:

$$f \circ f^{-1} = \text{id}$$

Indeed Diff^k is an infinite dimensional Lie Algebra and it is perfect. [1]

Res and Ind

We can imagine a forgetful functor inspired by representation theory and the $\text{Res}_H^G(V)$ or the restriction from a group G to a subgroup H over a representation V . Define the functor $\text{Res} : C^k \rightarrow \text{Diff}^k$. We can imagine a left adjoint functor to rebuild C^k from Diff^k through the operations $(+, \cdot, \circ)$ on elements $f_1, \dots, f_n \in \text{Diff}^k$. Specify the functor $\text{Ind} : \text{Diff}^k \rightarrow C^k$. Take an arbitrary element $h \in C^k$ we would like to construct h with a looser countable Schauder basis [3] using the ring and semi-group operations of elements in Diff^k . Let $A := \text{span}_{+, \circ} \{f_1, \dots, f_n | f_i \in \text{Diff}^k\}^1$. Then by Stone-Weierstrass A is dense in C^k at least for $X = \mathbb{R}^d$ [2]. So we can generate h with the topological closure of A given by \overline{A} . However this may not be true for an arbitrary smooth manifold X (hairy ball theorem?) [2].

References

- [1] Augustin Banyaga. *The Structure of Classical Diffeomorphism Groups*, volume 400 of *Mathematics and Its Applications*. Springer, Dordrecht, 1997. First systematic treatment of the algebraic structure of diffeomorphism groups.
- [2] Jean Dieudonné. *Foundations of Modern Analysis*, volume 1 of *Treatise on Analysis*. Academic Press, 1969. Covers both Stone-Weierstrass (§7.3) and Hairy Ball Theorem via Poincaré-Hopf.
- [3] Christopher Heil. *Basis Theory in Banach Spaces*. Springer, 2018.
- [4] Serge Lang. *Fundamentals of Differential Geometry*. Graduate Texts in Mathematics. Springer, 1999.

¹Often it is necessary in a Schauder basis our functions given by f_i be orthonormal by an inner-product $\langle f_i, f_j \rangle = \delta_{i,j}$ and we don't specify that same condition here.