Notes on the Structure C^k Functions

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Abstract

C^k Smooth Functions

A function with k continuous derivatives on a domain X is denoted by $C^k(X)$ [2] There is an ordering:

$$C^0 \subset C^1 \subset \cdots \subset C^\omega \subset C^\infty$$

Each C^k satisfies the axioms of a ring.

Proof. We take: $f, g, h \in C^k$. Closure under addition: $(f+g) \in C^k$ because differentiation is linear:

$$\partial^\alpha (f+g) = \partial^\alpha f + \partial^\alpha g$$

Closure under multiplication: $f \cdot g \in C^k$ by Leibniz rule we know:

$$\partial^\alpha = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} (\partial^\beta f)(\partial^{\alpha-\beta} g)$$

which remains C^k continuous. Associativity under Addition and Multiplication:

$$(f+g)+h = f+(g+h), (f \cdot g) \cdot h = f \cdot (g \cdot h)$$

Commutativity of Addition and Multiplication:

$$f+g = g+f, f \cdot g = g \cdot h$$

Additive identity: given by the zero function $0(X) = 0$, $0 \in C^k$ and satisfies:

$$f + 0 = f, \forall f \in C^k$$

Additive inverses: $\forall f \in C^k$ the function $-f \in C^k$ and satisfies:

$$f + (-f) = 0$$

Multiplicative identity: the constant function $1(X) = 1 \in C^k$ and satisfies:

$$1 \cdot f = f, \forall f \in C^k$$

Distributivity of Multiplication over Addition:

$$f \cdot (g+h) = f \cdot g + f \cdot h$$

□

There is an additional structure of C^k functions. Namely we can compose them with each other. Denote $\circ$ composition of functions $f(g(X))$ for $f, g \in C^k$ when the domain $X = \mathbb{R}^d$ This is a semigroup under this operation.

Proof. Take $f, g \in C^k$ as before. The space is closed under the operation: follows from the chain rule and the fact that derivatives of f and g up to k are continuous. Associativity: inherited from function composition.

$$(f \circ g) \circ h = f \circ (g \circ h)$$

An identity element exists $\mathrm{id}(x) = x \in C^k$ and satisfies $f \circ \mathrm{id} = \mathrm{id} \circ f = f$. □

Invertible C^k Functions

If we restrict C^k to the set of invertible functions we can satisfy all the group axioms under $\circ$. Take Diff^k as the set of invertible C^k functions. Then there is an ordering:

$$\begin{array}{ccccccc} C^0 & \supset & C^1 & \supset & \cdots & \supset C^\infty \\ \cup & & \cup & & & \cup \\ \text{Diff}^0 & \supset & \text{Diff}^1 & \supset & \cdots & \supset \text{Diff}^\infty \end{array}$$

Then the only missing property to satisfy the group axioms was the existence of inverses and by construction now for a function f we have the necessary inverse f^{-1} such that:

$$f \circ f^{-1} = \text{id}$$

Indeed Diff^k is an infinite dimensional Lie Algebra and it is perfect. [1]. We can imagine a forgetful functor inspired by representation theory and the Res^G_H or the restriction from a group G to a subgroup H. Define the functor $F : C^k \to \text{Diff}^k$. We can imagine a left adjoint functor to rebuild C^k from Diff^k through the $+$ and $\cdot$ operations on elements $f, g \in \text{Diff}^k$. Take the functor $G : \text{Diff}^k \to C^k$. Indeed it may be possible to build the functors F and G out of elements $f_1 \dots f_n$ in the respective algebras using the ring and semi-group operations; at least within $\mathbb{R}^d$.

References

[1] Augustin Banyaga. *The Structure of Classical Diffeomorphism Groups*, volume 400 of *Mathematics and Its Applications*. Springer, Dordrecht, 1997. First systematic treatment of the algebraic structure of diffeomorphism groups.

[2] Serge Lang. *Fundamentals of Differential Geometry*. Graduate Texts in Mathematics. Springer, 1999.