Solutions to the Schrodinger Equation As a Ring

[Drew Remmenga]

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Abstract

Hilbert dreamed of a theory of Quantum gravity with eigenvalues given by the Riemann Zeta Function. We derive a quantum theory of spin and connect it to eigenvalues of L-Functions.

1 Solving the Schrodinger Equation

Schroginger introduced his wave equation [3] which in a field with zero potential we write:

$$i\frac{\partial\psi(x,t)}{\partial t} = \frac{\partial^2\psi(x,t)}{\partial x^2} \tag{1}$$

This is idealized and set in natural units and in one dimension but works for our purposes. We begin with the separation of variables.

$$\begin{aligned}\psi(x,t) &= T(t)X(x)\\ iT'(t)X(x) &= X''(x)T(t)\\ i\frac{T'(t)}{T(t)} &= \frac{X''(x)}{X(x)} = \lambda\end{aligned}$$

Where λ is some constant. Furthermore we write:

$$\begin{aligned}iT'(t) &= \lambda T(t)\\ T(t) &= Ae^{i\lambda t}\end{aligned}$$

And:

$$\begin{aligned}X''(x) &= \lambda X(x)\\ X(x) &= c_1 e^{\sqrt{\lambda}x} + c_2 e^{-\sqrt{\lambda}x}\end{aligned}$$

In idealized cases of the wave equation we fix the origin to be zero and a distance L to be zero. For our purposes we shall set $L = \pi$. If we take $\lambda > 0$ we get the trivial solution. If we set $\lambda = 0$ we once again arrive at the trivial solution. We take $\lambda < 0$ and write:

$$0 = c_2 \sin(\mu_n x)$$

Without loss of generality in the constant c_2. The constant $\mu_n = n\frac{\pi}{L}$. In the time dependent Schrodinger equation linear combinations of solutions are solutions so we need $n \in \mathbb{N}$ and $\lambda_n = -\mu_n^2$. We arrive at:

$$\psi(x,t) = \sum_{n=0}^{\infty} A_n e^{itn^2} \sin(nx)$$

Traditionally in quantum mechanics we renormalize the equation to find the missing constant A_n. Here we take the notion of fractional integrals. The fraction is given by the variable α.

$${}_aD_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (\tau - t)^{\alpha-1} f(\tau) d\tau \tag{2}$$

This 2 is the Riemann-Liouville fractional integral [2]. Here we take $\alpha = s$ to be any complex number.

$$ {}_aD_x^{-s}|\psi(x,t)|^2 = 1 $$

$$ \frac{1}{\Gamma(s)}\int_0^\pi |(\pi - x)^{s-1}\psi(x,t)|^2 dx = 1 $$

$$ \frac{1}{\Gamma(s)}\int_0^\pi |(\pi - x)^{s-1}\sum_{n=1}^{\infty} A_n \sin(nx)e^{itn^2}|^2 dx = 1 $$

$$ \frac{1}{\Gamma(s)}|\sum_{n=1}^{\infty}\frac{A_n}{2(-n)^s(-1)^{\frac{3s}{2}}}[-i(-1)^n(\Gamma(s,-in(x-\pi)) - (-1)^s\Gamma(s,in(x-\pi)))]_{x=0}^{x=\pi}\psi(t)|^2 = 1 $$

$$ \frac{1}{\Gamma(s)}|\sum_{n=1}^{\infty}\frac{A_n}{2(-n)^s(-1)^{\frac{3s}{2}}}[i(-1)^n(\Gamma(s,0)) - (-1)^s\Gamma(s,0) - (-1)\Gamma(s,0) + i(-1)^n\Gamma(s,-in\pi) + (-1)^s\Gamma(s,in\pi)\psi(t)]|^2 = 1 $$

This is cool but we just want the Zeta function. We can absorb the constants in Γ into A_n without losing generality. As well as constants in s.

$$ \sum_{n=1}^{\infty}\frac{A_n}{n^s}\frac{\bar{A}_n}{\bar{n^s}}e^{in^2t}e^{\bar{in^2t}} = 1 $$

Now we take the natural log of both sides.

$$ \sum_{n=1}^{\infty}\frac{A_n\bar{A}_n}{n^s\bar{n^s}}e^{itn^2}e^{\bar{itn^2}} = 1 \tag{3} $$

$$ ln[\sum_{n=1}^{\infty}\frac{A_n}{n^s}\frac{\bar{A}_n}{\bar{n^s}}e^{itn^2}e^{\bar{itn^2}}] = 2m\pi \tag{4} $$

2 Proof Solutions form a Field

We shall take $m = 0$ for our purposes.

Remark 1. *This forms a ring.*

Proof. Let a, b, and c be solutions to 4. Additions of these equations are associative.

$$ (a+b)+c = a+(b+c) $$

Multiplication is associative.

$$ (ab)c = a(bc) $$

Addition is commutative.

$$ a+b = b+a $$

Multiplication is commutative.

$$ (ab) = (ba) $$

Addition is distributive:

$$ a(b+c) = ab+ac $$

There exists a zero identity for addition. The trivial solution. There exists an identity for multiplication the A_n which sets the wave equal to 1. There exists additive inverse elements. If a is a solution to 3 then $-a$ is also a solution. ☐

Remark 2. *This forms a group isomorphic to $\mathbb{S}^{\aleph_0}$. This is the circle in countably infinite dimensions. Take an element of 4. There exists a map from the A_n terms to each dimension of the circle by renormalizing. This map is clearly bijective.*

3 The Field of L-Functions

We know the Riemann Hypothesis over finite fields is true. [4] [5] But more work must be done to prove it over the field of complex numbers. There exists a map between solutions of 3 to the set of L-Functions by setting A_n appropriately and multiplying by a constant. Products of solutions to 3 are also solutions. We perform a product integral over the domain. This is reminiscent of what is described in the field with one unit here [5]. All solutions across time are solutions. We write:

$$\int_{-\infty}^{\infty} \sum_{n=1}^{\infty} \frac{A_n \bar{A_n}}{n^s \bar{n^s}} |e^{itn^2}|^{2dt} = 1$$

$\psi(t)$ cycles through complex numbers in the circle. We can absorb without loss of generality into A_n and $\bar{A_n}$ since we're finding the magnitude so this result about the Riemann Zeta function is stronger than the generalized Riemann Hypothesis. We can rewrite this as a contour integral through the entire complex plane. We can absorb A_n and $\bar{A_n}$ into a new real number which we can call $\chi(n)$. We can also rewrite s to $\frac{s}{2}$ without loss of generality since we are integrating over the entire complex plane. The general Riemann Hypothesis reads: [1]

$$L(\chi, s) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} \tag{5}$$

In our formulation we can once again integrate over the entire complex plane. In fact we can do this infinitely many times.

$$\sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} \frac{\bar{\chi(n)}}{\bar{n^s}} = 1 \tag{6}$$

$$\tag{7}$$

We obtain the Grand Riemann Zeta hypothesis as described here [5] and here [1]. This function clearly has eigenvalues at the negative even integers, the so called 'trivial solutions'. And assuming the variable s isn't equal to one of those values it can be an eigenvalue if the real part of s is equal to $\frac{1}{2}$. Suppose there exists a solution not on the real line or not on the critical line. This would fundamentally break the symmetry between subtraction and division demonstrated between the trivial and non-trivial zeros.

References

[1] Harold Davenport. *Multiplicative Number Theory*, volume 74 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, third edition, 2000. Revised and with a preface by Hugh L. Montgomery.

[2] Richard Hermann. *Fractional Calculus: An Introduction for Physicists*. World Scientific Publishing, New Jersey, 2nd edition, 2014.

[3] Jamal Nazrul Islam. The Schrödinger equation in quantum field theory. *Foundations of Physics*, 24(5):593–630, May 1994.

[4] James Milne. The riemann hypothesis over finite fields: From weil to the present day, 2015.

[5] PeakMath. The rosetta stone (rh saga s1e12). YouTube, March 2025.