Notes From Fredric Schuller's Lectures on Relativity

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Abstract

Notes on Fredric Schuller's Relativity lectures. "Spacetime is a four-dimensional topolgical manifold with a smooth atlas carrying a torsion free connection compatible with a Lorentzian metric and time orientation satisfying the Einstein Equations."

Lecture 1: Topological Spaces

@ the coursest level spacetime is a set. This is not enough to talk about conintuity of maps. In classical physics there are no jumps. Sets are not enough alone to talk about continuity. We are interested in establishing the weakest possible structure on a set to talk about continuity of maps. The mathematician knows the weakest structurenecessary to do this is a topology.

Definition 1. *Let $\mathscr{M}$ be a set. A topology $\mathscr{O}$ is a subset of the powerset of $\mathscr{M}$ denoted $\mathscr{O} \subseteq \mathscr{P}(\mathscr{M})$ satisfying three axioms:*

- $\emptyset \subset \mathscr{O}$, $\mathscr{M} \subset \mathscr{O}$
- *For arbitrary* $U, V \in \mathscr{O} \implies U \cap V \in \mathscr{O}$
- *For arbitrary* $U_\alpha \in \mathscr{O} \implies (\bigcap_{\alpha \in A} U_\alpha) \in \mathscr{O}$ *where* α *is an index of the set* A.

Example 1.
- *Let* $\mathscr{M} = \{1, 2, 3\}$
  - *Let* $\mathscr{O}_1 = \{\emptyset, \{1, 2, 3\}\}$ *then* $\mathscr{O}_1$ *is a topology for* $\mathscr{M}$.
  - *Let* $\mathscr{O}_2 = \{\emptyset, \{1\}, \{2\}, \{1, 2, 3\}\}$ *then* $\mathscr{O}_2$ *is not a topology for* $\mathscr{M}$ *because* $\{1, 2\} \notin \mathscr{O}_2$.
- *Let* $\mathscr{M}$ *be any set.*
  - $\mathscr{O}_{chaotic} = \{\emptyset, \mathscr{M}\}$ *is a topology*
  - $\mathscr{O}_{discrete} = \mathscr{P}(\mathscr{M})$ *is a topology*
- $\mathscr{M} = \mathbb{R}^d$ *(tuples of dimenions d from* $\mathbb{R}$*) then* $\mathscr{O}_{standard} \subseteq \mathscr{P}(\mathscr{M})$ *is a topology for* $\mathscr{M}$ *defined as follows.*

  Definition 2. $\mathscr{O}_{standard}$ *defined in two steps.*
  - $B_r(p) := \{q_1, \cdots, q_d\} | \sum_{i=1}^{d} (q_i - p)^2 < r, r \in \mathbb{R}^+, q_i \in \mathbb{R}, p \in \mathbb{R}^d$
  - $\mathscr{U} \in \mathscr{O}_{standard} \iff \forall p \in \mathscr{U}, \exists r \in \mathbb{R}^+ : B_r(p) \subseteq \mathscr{U}$

We want $\mathscr{M}$ to be spacetime and we want to equip it with an apropriate topology $\mathscr{O}$ to be able to talk about it. We want to make the implicit assumptions of spacetime to be explicit.

Terminology 1. *Let* $\mathscr{M}$ *be a set defined from (ZFC) then* $\mathscr{O}$ *is the topology on* $\mathscr{M}$ *and it a collection of open sets.* $(\mathscr{M}, \mathscr{O})$ *is a topological space.*

- $\mathscr{U} \in \mathscr{O} \iff \mathscr{U} \subseteq \mathscr{M}$ *is an open set.*
- $M/A \in \mathscr{O} \iff A \subseteq \mathscr{M}$ *is closed.*

$\not$*open* $\not\Longrightarrow$ *closed and* $\not$*closed* $\not\Longrightarrow$ *open*

Definition 3. *A map $f : M \to N$ takes all elements $m \in M$ to an element $n \in N$. M is the domain. N is the target. If $\exists m_1, m_2 \in M$ such that $f(m_1) = f(m_2) = n \in N$ then f is $\not$injective. If $\exists n \in N$ such that $\forall m \in M : f(m) \neq n$ then f is not surjective. Is a map f continuous? Depends by definition on topologies $\mathscr{O}_M$ on M and $\mathscr{O}_N$ on N.*

Definition 4. *A map f is called continuous between $(M, \mathscr{O}_M)$ and $(N, \mathscr{O}_N)$. Then a map f is called continuous with respect to these topologies if for every open $V \in \mathscr{O}_N$ the preimage of V is open in $\mathscr{O}_M$. $\forall V \in O_N : preim_f(V) \in \mathscr{O}_M$, $preim_f(V) := \{m \in M\}|f(m) \in V$.*

Example 2. $M = 1, 2, \mathscr{O}_M = \emptyset, 1, 2, 1, 2$
$N = 1, 2, \mathscr{O}_N = \emptyset, 1, 2$
$f : M \to N|f(1) = 2, f(2) = 1$
Is f continuous?
$preim_f(\emptyset) = \emptyset \in \mathscr{O}_M$
$preim_f(1, 2) = M \in \mathscr{O}_M$
Therefore f is continuous.

Example 3. *$g : N \to M$ or f^{-1} then $preim_g(1) = 2 \notin \mathscr{O}_N$ so g is not continuous.*

Composition of Maps

$M \xrightarrow{f} N \xrightarrow{g} P$ so $g \circ f : M \to P$ by $m \to (g \circ f)(m) := g(f(m))$.

Theorem 1. *Composition of continuous maps is continuous.*

Proof. Let $V \in \mathscr{O}_P$ then

$$\begin{aligned} \text{preim}_{g \circ f}(V) &:= m \in M|(g \circ f)(m) \in V \\ &= m \in M|f(m) \in \text{preim}_g(V) \\ &= \text{preim}_f(\text{preim}_g(V) \in \mathscr{O}_N) \in \mathscr{O}_M \end{aligned}$$

□

Inheritance of a Topology

Many useful ways to inherit a topology from another topological space or set of topological spaces. Of particular importance for spacetime physics is $S \subseteq M$ where M has topology $\mathscr{O}_M$. Can we construct a topology $\mathscr{O}_S$ from $\mathscr{O}_M$. Yes.

Definition 5. $\mathscr{O}|_S \subseteq \mathscr{P}(S)$
$\mathscr{O}|_S := \mathscr{U} \cap S|\mathscr{U} \in \mathscr{O}_M$
This is a topology called the subset topology.

Use of this specific way to inherit a topology from a super set. Let it be easy to say a map f is continuous. Then a subset S of a set M with a inherited topology then the restriction of the map $f|_S : S \to N$ it can be easy to show this restriction is continuous.

Lecture 2: Topological Manifolds

$\exists$ too many topological spaces to classify. Too many topological spaces exist which have no known connection to the study of spacetime. For spacetime physics we may focus on topological spaces $(\mathscr{M}, \mathscr{O})$ which can be charted analogously to how the surface of the Earth is charted in an atlas.

Topological Manifolds

Definition 6. *A topological space $(\mathscr{M},)$ is called a d-dimenional topological manifold if $\forall p \in \mathscr{M} : \exists p \in \mathscr{U} \in \mathscr{O} : \exists x : \mathscr{U} \to x(\mathscr{U}) \subseteq \mathbb{R}^d$ with $\mathbb{R}^d$ equiped with $\mathscr{O}_{standard}$such that:*

- *x is invertible: $x^{-1} : x(\mathscr{U}) \to \mathscr{U}$*

- *x is continous.*
- *x^{-1} is continous.*

Example 4.
The surface of a torus $:= \mathscr{M} \subseteq \mathbb{R}^3$. This is a $d = 2$ dimensional topolgical manifold.
A mobius strip $:= \mathscr{M} \subseteq (\mathbb{R}^3, \mathscr{O}_{st})$ then this is a $d = 1$ manifold.
A bifurcating line $:= \mathscr{M} \subset (\mathbb{R}^2, \mathscr{O}_{st})$ then $(\mathscr{M}, \mathscr{O}_{st}|_{\mathscr{M}})$ is a topology but the point of bifurcation is not invertible so it is not a manifold.

Terminology 2. *1. A pair $(\mathscr{U}, x)$ is called a chart.*

2. *A set $\mathscr{A} = (\mathscr{U}_\alpha, x_\alpha)|\alpha \in A$ is called an atlas of the topoligcal manifold $\mathscr{M}$ if $\mathscr{M} = \bigcup_{\alpha \in A} \mathscr{U}_\alpha$.*
3. *$x : \mathscr{U} \to x(\mathscr{U}) \subseteq \mathbb{R}^d$ is called the chart map.*
4. *$x^i \mathscr{U} \to \mathbb{R}$ is called the coordinate maps.*
5. *$p \in \mathscr{U}$ then x^i is the i-th coordinate of the point p with respect to the chosen chart $(\mathscr{U}, x)$.*

Example 5. •

$$\mathscr{M} = \mathbb{R}^2$$
$$\mathscr{U} = \mathbb{R}^2/0, 0$$
$$x : \mathscr{U} \to \mathbb{R}^2 : x(m, n) \to (-m, -n)$$

- *We can take another chart map on $\mathscr{U}$*
  $(m, n) \to (\sqrt{m^2 + n^2}, arctan(\frac{n}{m}))$

Chart Transition Maps

Imagine two charts $(\mathscr{U}, x)$ and $(\mathscr{V}, y)$ with $\mathscr{U} \cap \mathscr{V} = \mathscr{A} \neq \emptyset$. The same point $p \in \mathscr{A}$ can be mapped via x and y to two different charts. Because these maps are continuously invertible we can smoothly transition between pages of our atlas: $(y \circ x^{-1})(p) = y(x^{-1}(p))$. This map is called the chart tranisition map. Informally the chart transition map contains the instructions how to *glue* together the charts of our atlas.

Manifold Philosophy

Often it is desirable (or indeed the way) to define properties ("contiuityt") of real-world objects ("$\mathbb{R} \xrightarrow{\gamma} \mathscr{M}$) by judging suitible condition not on the real-world object iself but on the chart-representation of the real-world object. Advantages: You can define continuity in this way. Disadvantages: The chart map x is a 'fantasy,' x may be ill defined because the chart chosen is arbitrary. Solution: The property must be maintained regardless of chart.

$$\mathbb{R} \xrightarrow{y \circ \gamma} y(\mathscr{U})$$
$$\mathbb{R} \xrightarrow{x \circ \gamma} x(\mathscr{U})$$
$$\mathbb{R} \xrightarrow{\gamma} \mathscr{U}$$

We are interested in properties of γ but must talk about it with respect to x and y.

Lecture 3: Multilinear Algebra

We will not equip space(time) with a vector space structure. There is no such thing as five times Paris or Paris plus Vienna. However, the tangent spaces of $T_p\mathscr{M}$ of smooth manifolds carry a vector space structure. It is beneficial to first study vecotr spaces abstractly for two reasons.

- For construction of $T_p\mathscr{M}$ one needs an intermediate vector space $C^\infty(\mathscr{M})$.
- Tensor teqniques are best understoond in an abstract setting.

Vecotr Spaces

Definition 7. *A vector space* $(V, +, \cdot)$ *is*

- *a set* V
- $+ : V \times V \to V$
- $\cdot \mathbb{R} \times V \to V$

Which satisfies CANI and ADDU axioms. For $w, v, u \in V$ *and* $\lambda, \mu \in \mathbb{R}$

- $C^+ : v + w = w + v$
- $A^+ : (u + v) + w = u + (v + w)$
- $N^+ : \exists 0 \in V : \forall v \in V : v + 0 = v$
- $I^+ : \forall v \in V : \exists(-v) \in V : v + (-v) = 0$
- $A : \lambda \cdot (\mu \cdot v) = (\lambda \cdot \mu) \cdot v$
- $D : (\lambda + \mu) \cdot v = \lambda \cdot v + \mu \cdot v$
- $D : \lambda \cdot v + \lambda \cdot w = \lambda \cdot (v + w)$
- $U : 1 \cdot v = v$

Terminology 3. *An element of a vector space is often referred to (informally) as a vector.*

Example 6.

Definition 8. *Set* $P := p : (-1, +1) \to \mathbb{R}$ *of polynomials of (fixed) degree.* $p(x) = \sum_{n=1}^{N} p_n x^n$. *Is* $\square(x) = x^2$ *a vector? No.* $\square \in P$.

Definition 9. *Define the operations in the space:*
$+ : P \times P \to P : (p, q) \rightarrowtail p + q$
$\cdot : \mathbb{R} \times P \to P : (\lambda, p) \rightarrowtail \lambda \cdot p$
Is $\square \in (P, + \cdot)$ *a vector in a vector space? Yes!*

Linear Maps

We want to study maps which preserve (vector space) structure. On vector spaces these are called linear maps.

Definition 10. *Let* $(V, +, \cdot)$ *and* $(W, +, \cdot)$ *be vector spaces. Then a map* $\phi V \to W$ *is called linear if:*

- $\phi(v + \tilde{v}) = \phi(v) + \phi(\tilde{v})$
- $\phi(\lambda \cdot v) = lambda \cdot \phi(v)$

Example 7. *Take* P *as before. Take* $\delta : P \to P :\to \delta(p) : p'$. $\delta(p, q) = (p + q)' = \delta(p) + \delta(q)$ *and* $\delta(\lambda p) = (\lambda p)' = \lambda p'$

Notation: $\phi : V \to W$ linear $\iff \phi : V \xrightarrow{\sim} W$

Vector space of Homomorphisms

Take $(V, +, \cdot)$ and $(W, +, \cdot)$ vector spaces.

Definition 11. *Hom*$(V, W := \phi : V \xrightarrow{\sim} V)$ *as a set. This is a vectorspace:*

- $\oplus : \mathit{Hom}(V, W) \times \mathit{Hom}(V, W) \to \mathit{Hom}(V, W) : (\phi, \psi) \rightarrowtail \phi \oplus \psi$ *where* $(\phi \oplus \psi)(v) := \phi(v) + \psi(v)$
- $\otimes$ *defined similarly.*

$(\mathit{Hom}(V, W), \oplus, \otimes)$ *is a vector space.*

Example 8. *Take* P *as before. Then* δ *is a vector space similarly.*

Dual Vector Space

$(V, +, \cdot)$ as a vector space.

Definition 12. *$V^* := \psi : V \xrightarrow{\sim} \mathbb{R} = Hom(V, \mathbb{R})$ is also a vector space called the dual vector space of V.*

Terminology 4. *$\phi \in V^*$ is called (informally) a covector.*

Example 9. *$I : P \xrightarrow{\sim} \mathbb{R}$ so $I \in P^*$*

Definition 13. *$I(p) := \int_0^1 dxp(x)$*

I is clearly linear.

Tensors

Definition 14. *Take a vector space V. Then a (r, s) tensor T over V is a multilinear map: $T : \bigotimes_{i=1}^{r} V^* \otimes \bigotimes_{j=1}^{s} V \xrightarrow{\sim(r+s)} \mathbb{R}$*

Example 10. *$T(1,1)-$ tensor. $T(\phi+\psi, v) = T(\phi, v) + T(\psi, v), T(\lambda\psi, v) = \lambda \cdot T(\phi, v)T(\psi, v+w) = T(\psi, v) + T(\psi, w), T(\psi, \lambda \cdot v) = \lambda \cdot T(\psi, v)$ Linear in both entries. Hence, multi-linear.*

Example 11. *$g : P \times P \xrightarrow{\sim} \mathbb{R}, (p, q) \rightarrow \int_0^1 dxp(x)q(x)$ is a $(0, 2)$ - temnor over example of P.*

Vectors and Covectors

Theorem 2. *$\phi \in V^* \iff \phi : V \xrightarrow{\sim} \mathbb{R} \iff \phi(0,1)-$ tensor.*

Theorem 3. *$dim(V) < \infty \implies v \in V = (V^*)^* \iff v : V^* \xrightarrow{\sim} \mathbb{R} \iff v$ is a $(1,0)-$ tensor.*

Bases

Definition 15. *Take a vector space V. A subset $B \subset V$ is called a basis if $\forall v \in V \exists F = \{f_1, ..., f_n\} \in B : \exists! \{v^1, ..., v^n\} \in R : v = v^1 f_1 + ... + v^n f_n$.*

Definition 16. *If $\exists$ basis B with finite many elements (d many elements) then we call $d =: dim(V)$.*

Remark 1. *Let V be finite dimensional vector space. Choose a basis $e_1, ..., e_n$ of V. We may uniquely associate a vector $v \in V$ with $v \rightarrowtail (v^1, ..., v^n)$ called the components of $v = v^1 e_1 + ... + v^n e_n$ with respect to the chosen basis. It is mor eeconomical to require your basis on V once chosen such that $\epsilon^a(e_b) = \delta^a_b$. This uniquely determines choice of vector components from a choice of basis.*

Definition 17. *If a basis $\epsilon^1, ..., \epsilon^n$ of V^* satisfies these axioms it is called the dual basis.*

Example 12. *Take P with $(N = 3)$. Then $e_0, ..., e_3$ is a basis if $e_a(x) := x^a$ is a basis of P. Then dual basis is given by $\epsilon^a := \frac{1}{a!}\partial^a|_{x=0}$.*

Components of Tensors

Definition 18. *Let T be an $(r, s)-$ tensor over a finite dimensional vectors space V. Then define the $(r+s)^{dim(V)}$ many real numbers $i_1, ..., i_r, j_1, ..., j_s \in \{1, ..., dim(V)\}, T^i_j \in R := T(\epsilon^i, e_j)$. The T^i_j elements are called the components of the tensor with respect to the chosen basis. Knowing the components and basis one can reconstruct the entire tensor.*

Example 13. *$T(1,1,)$ - tensor with $T^i_j = T(\epsilon^i, e_j)$ reconstruct*

$$\begin{aligned} T(\phi, v) &= T(\sum_{i=1}^{dim(V)} \phi_i \epsilon^i, \sum_{j=1}^{dim(V)} v^j e_j) \\ &= \sum_{i=1}^{dim(V)} \sum_{j=1}^{dim(V)} \phi_1 v^j T(\epsilon^i, e_j) \end{aligned}$$

with $\phi_i, v^j \in \mathbb{R}$.

Terminology 5. *If we agree to label $T^i_j = T(\epsilon^i, e_j)$ with the up and down components then Einstein sumation convention of tensors $:= \phi_i v^j T^i_j$ where we drop $\sum$. This only works over multilinear maps.*

Lecture 4: Differentiable Manifolds

So far we have topolgical manifolds $(\mathscr{M}, \mathscr{O})$ dim$(\mathscr{M})=d$. We want to talk about velocitcy vectors on them but this structure is insufficient to do so. Pick a dimension $d \neq 4$ for your manifold. In that dimension the choices of topology are countable, so presumably we can do experiments to discern which one coorisponds to our reality. In $d = 4$ our choices of topology are suddenly uncountable. We need additional sturucte to talk about differentiable curves on manifolds and between manifolds.

1. Curves: $\mathbb{R} \to \mathscr{M}$
2. Functions: $\mathscr{M} \to \mathbb{R}$
3. Maps: $\mathscr{M}_1 \to \mathscr{M}_2$

Strategy

$\gamma : \mathbb{R} \to \mathscr{U}$ and choose a chart $(\mathscr{U}, x)$. $\mathscr{U} \xrightarrow{x} x(\mathscr{U}) \subseteq \mathbb{R}^d$. Try and 'lift' the undergraduate notion of differentiable of a curve in $\mathbb{R}^d$ to a notion of differentiability on a curve on $\mathscr{M}$. Problem: Is this well defined under a change of chart? We don't want it to depend on our taste.

$$\gamma : \mathbb{R} \to \mathscr{U} \cap \mathscr{V} \neq 0 \tag{1}$$
$$y \circ \gamma : \mathbb{R} \to y(\mathscr{U} \cap \mathscr{V}) \subseteq \mathbb{R}^d \tag{2}$$
$$x \circ \gamma : \mathbb{R} \to x(\mathscr{U} \cap \mathscr{V}) \subseteq \mathbb{R}^d \tag{3}$$
$$y \circ \gamma = (y \circ x^{-1}) \circ (x \circ \gamma) = y \circ (x^{-1} \circ x) \circ \gamma \tag{4}$$

If $(x \circ \gamma)$ differentiable and $(y \circ x^{-1})$ continuous is $y \circ \gamma$ guarenteed to be continuously differentiable? No.

Compatible Charts

In previous we took a chart on the topological manifold. Our atlases $\mathscr{V}$ and $\mathscr{U}$ were elements of the maximal atlas of the manifold.

Definition 19. *Two charts $(\mathscr{U}, x)$ and $(\mathscr{V}, y)$ are called $\square$ compatible if either:*

1. $\mathscr{U} \cap \mathscr{V} = \emptyset$
2. $y \circ x^{-1} : x(\mathscr{U} \cap \mathscr{V}) \to y(\mathscr{U} \cap \mathscr{V})$ *and* $x \circ y^{-1} : y(\mathscr{U} \cap \mathscr{V}) \to x(\mathscr{U} \cap \mathscr{V})$ *are differentiable.*

Philosophy of $\square$:

Definition 20. *An atlas $\mathscr{A}_\square$ is a $\square$ - compatible atlas if any two charts in $\mathscr{A}_\square$ are $\square$ - compatible.*

Definition 21. *A $\square$ - manifold is a triple $(\mathscr{M}, \mathscr{O}, \mathscr{A}_\square)$*

Undergraduate $\square$. C^0: $C^0(\mathbb{R}^d \to \mathbb{R}^d)$ Continuous maps. C^1: $\mathbb{C}^1(\mathbb{R}^d \to \mathbb{R}^d)$ Differentiable Once. C^k: k-times continuous differentiable. D^k: k-times differentiable. C^∞: $\mathbb{C}^\infty(\mathbb{R}^d \to \mathbb{R}^d)$ infinitly continously differentiable. C^ω: $\exists$ multidimensional Taylor Expansion. $\mathbb{C}^\infty$: Satisfies Cauchy-Riemann Equations.

Theorem 4. *Any $\mathbb{C}^{k \geq 1}$ - atlas $\mathscr{A}$ of a topolgical manifold contains a C^∞ - atlas.*

Thus we will consider C^∞ manifolds or "smoothmanifolds" unless we wish to define taylor expantions or complex differentiable.

Definition 22. *A smooth manifold is a tripple $(\mathscr{M}, \mathscr{O}, \mathscr{A})$ where $\mathscr{A}$ is a C^∞ - atlas.*

Diffeororphisms

Take a map: $\mathscr{M} \xrightarrow{\phi} \mathscr{N}$. If $\mathscr{M}$ and $\mathscr{N}$ are naked sets then the structure preserviung maps are the bijections. If $\exists \phi$ is a bijection $\mathscr{M} \cong \mathscr{N}$.

Example 14.

$$\mathbb{N} \cong \mathbb{Z}$$
$$\mathbb{N} \cong \mathbb{Q}$$
$$\mathbb{N} \ncong \mathbb{R}$$

Linear bijections between vector spaces are the structure preserving maps between vector spaces.

Definition 23. *Two C^∞ manifolds are said to be diffeomorphic* $\iff \exists$ *bijection* $\phi : \mathscr{M} \to \mathscr{N}$ *such that* ϕ, ϕ^{-1} *are both C^∞ maps.*

Theorem 5. *The number of C^∞ manifolds that can be contructed from a given C^0 - manifold up to diffeomorphism by the More-Radon Theorms in $d = 1, 2, 3$ dimension are 1. In $d > 4$ the number is finitie. In $d = 4 \exists$ uncountably infinitely such manifolds.*