

# Notes From Fredric Schuller's Lectures on Relativity

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## Abstract

Notes on Fredric Schuller's Relativity lectures. "Spacetime is a four-dimensional topological manifold with a smooth atlas carrying a torsion free connection compatible with a Lorentzian metric and time orientation satisfying the Einstein Equations."

## 1 Lecture 1: Topological Spaces [1]

At the coarsest level spacetime is a set. This is not enough to talk about continuity of maps. In classical physics there are no jumps. Sets are not enough alone to talk about continuity. We are interested in establishing the weakest possible structure on a set to talk about continuity of maps. The mathematician knows the weakest structure necessary to do this is a topology.

**Definition 1.** Let  $M$  be a set. A topology  $\mathcal{O}$  is a subset of the powerset of  $M$  denoted  $\mathcal{O} \subseteq \mathcal{P}(M)$  satisfying three axioms:

- $\emptyset \in \mathcal{O}, M \in \mathcal{O}$
- For arbitrary  $U, V \in \mathcal{O} \implies U \cap V \in \mathcal{O}$
- For arbitrary  $U_\alpha \in \mathcal{O} \implies (\bigcap_{\alpha \in A} U_\alpha) \in \mathcal{O}$  where  $\alpha$  is an index of the set  $A$ .

**Example 1.** • Let  $M = \{1, 2, 3\}$

- Let  $\mathcal{O}_1 = \{\emptyset, \{1, 2, 3\}\}$  then  $\mathcal{O}_1$  is a topology for  $M$ .
- Let  $\mathcal{O}_2 = \{\emptyset, \{1\}, \{2\}, \{1, 2, 3\}\}$  then  $\mathcal{O}_2$  is not a topology for  $M$  because  $\{1, 2\} \notin \mathcal{O}_2$ .

- Let  $M$  be any set.
  - $\mathcal{O}_{chaotic} = \{\emptyset, M\}$  is a topology
  - $\mathcal{O}_{discrete} = \mathcal{P}(M)$  is a topology
- $M = \mathbb{R}^d$  (tuples of dimensions  $d$  from  $\mathbb{R}$ ) then  $\mathcal{O}_{standard} \subseteq \mathcal{P}(M)$  is a topology for  $M$  defined as follows.

**Definition 2.**  $\mathcal{O}_{standard}$  defined in two steps.

- $B_r(p) := \{q_1, \dots, q_d \mid \sum_{i=1}^d (q_i - p)^2 < r, r \in \mathbb{R}^+, q_i \in \mathbb{R}, p \in \mathbb{R}^d$
- $\mathcal{U} \in \mathcal{O}_{standard} \iff \forall p \in \mathcal{U}, \exists r \in \mathbb{R}^+ : B_r(p) \subseteq \mathcal{U}$

We want  $M$  to be spacetime and we want to equip it with an appropriate topology  $\mathcal{O}$  to be able to talk about it. We want to make the implicit assumptions of spacetime to be explicit.

**Terminology 1.** Let  $M$  be a set defined from (ZFC) then  $\mathcal{O}$  is the topology on  $M$  and it is a collection of open sets.  $(M, \mathcal{O})$  is a topological space.

- $\mathcal{U} \in \mathcal{O} \iff \mathcal{U} \subseteq M$  is an open set.
- $M/A \in \mathcal{O} \iff A \subseteq M$  is closed.

$\text{not open} \not\implies \text{closed}$  and  $\text{not closed} \not\implies \text{open}$

**Definition 3.** A map  $f : M \rightarrow N$  takes all elements  $m \in M$  to an element  $n \in N$ .  $M$  is the domain.  $N$  is the target. If  $\exists m_1, m_2 \in M$  such that  $f(m_1) = f(m_2) = n \in N$  then  $f$  is *injective*. If  $\exists n \in N$  such that  $\forall m \in M : f(m) \neq n$  then  $f$  is not surjective. Is a map  $f$  continuous? Depends by definition on topologies  $\mathcal{O}_M$  on  $M$  and  $\mathcal{O}_N$  on  $N$ .

**Definition 4.** A map  $f$  is called continuous between  $(M, \mathcal{O}_M)$  and  $(N, \mathcal{O}_N)$ . Then a map  $f$  is called continuous with respect to these topologies if for every open  $V \in \mathcal{O}_N$  the preimage of  $V$  is open in  $\mathcal{O}_M$ .  $\forall V \in \mathcal{O}_N : \text{preim}_f(V) \in \mathcal{O}_M$ ,  $\text{preim}_f(V) := \{m \in M \mid f(m) \in V\}$ .

**Example 2.**  $M = 1, 2, \mathcal{O}_M = \emptyset, 1, 2, 1, 2$   
 $N = 1, 2, \mathcal{O}_N = \emptyset, 1, 2$   
 $f : M \rightarrow N \mid f(1) = 2, f(2) = 1$   
 Is  $f$  continuous?  
 $\text{preim}_f(\emptyset) = \emptyset \in \mathcal{O}_M$   
 $\text{preim}_f(1, 2) = M \in \mathcal{O}_M$   
 Therefore  $f$  is continuous.

**Example 3.**  $g : N \rightarrow M$  or  $f^{-1}$  then  $\text{preim}_g(1) = 2 \notin \mathcal{O}_N$  so  $g$  is not continuous.

## 1.1 Composition of Maps

$M \xrightarrow{f} N \xrightarrow{g} P$  so  $g \circ f : M \rightarrow P$  by  $m \rightarrow (g \circ f)(m) := g(f(m))$ .

**Theorem 1.** Composition of continuous maps is continuous.

*Proof.* Let  $V \in \mathcal{O}_P$  then

$$\begin{aligned} \text{preim}_{g \circ f}(V) &:= m \in M \mid (g \circ f)(m) \in V \\ &= m \in M \mid f(m) \in \text{preim}_g(V) \\ &= \text{preim}_f(\text{preim}_g(V) \in \mathcal{O}_N) \in \mathcal{O}_M \end{aligned}$$

□

## 1.2 Inheritance of a Topology

Many useful ways to inherit a topology from another topological space or set of topological spaces. Of particular importance for spacetime physics is  $S \subseteq M$  where  $M$  has topology  $\mathcal{O}_M$ . Can we construct a topology  $\mathcal{O}_S$  from  $\mathcal{O}_M$ . Yes.

**Definition 5.**  $\mathcal{O}|_S \subseteq \mathcal{P}(S)$

$\mathcal{O}|_S := \mathcal{U} \cap S \mid \mathcal{U} \in \mathcal{O}_M$

This is a topology called the subset topology.

Use of this specific way to inherit a topology from a super set. Let it be easy to say a map  $f$  is continuous. Then a subset  $S$  of a set  $M$  with an inherited topology then the restriction of the map  $f|_S : S \rightarrow N$  it can be easy to show this restriction is continuous.

# 2 Lecture 2: Topological Manifolds [6]

$\exists$  too many topological spaces to classify. Too many topological spaces exist which have no known connection to the study of spacetime. For spacetime physics we may focus on topological spaces  $(M, \mathcal{O})$  which can be charted analogously to how the surface of the Earth is charted in an atlas.

## 2.1 Topological Manifolds

**Definition 6.** A topological space  $(M)$  is called a  $d$ -dimensional topological manifold if  $\forall p \in M : \exists p \in \mathcal{U} \in \mathcal{O} : \exists x : \mathcal{U} \rightarrow x(\mathcal{U}) \subseteq \mathbb{R}^d$  with  $\mathbb{R}^d$  equipped with  $\mathcal{O}_{\text{standard}}$  such that:

- $x$  is invertible:  $x^{-1} : x(\mathcal{U}) \rightarrow \mathcal{U}$

- $x$  is continuous.
- $x^{-1}$  is continuous.

**Example 4.**

The surface of a torus  $:= M \subseteq \mathbb{R}^3$ . This is a  $d = 2$  dimensional topological manifold.

A mobius strip  $:= M \subseteq (\mathbb{R}^3, \mathcal{O}_{st})$  then this is a  $d = 1$  manifold.

A bifurcating line  $:= M \subset (\mathbb{R}^2, \mathcal{O}_{st})$  then  $(M, \mathcal{O}_{st}|_M)$  is a topology but the point of bifurcation is not invertible so it is not a manifold.

**Terminology 2.** 1. A pair  $(\mathcal{U}, x)$  is called a chart.

2. A set  $\mathcal{A} = (\mathcal{U}_\alpha, x_\alpha) | \alpha \in A$  is called an atlas of the topological manifold  $M$  if  $M = \bigcup_{\alpha \in A} \mathcal{U}_\alpha$ .

3.  $x : \mathcal{U} \rightarrow x(\mathcal{U}) \subseteq \mathbb{R}^d$  is called the chart map.

4.  $x^i \mathcal{U} \rightarrow \mathbb{R}$  is called the coordinate maps.

5.  $p \in \mathcal{U}$  then  $x^i$  is the  $i$ -th coordinate of the point  $p$  with respect to the chosen chart  $(\mathcal{U}, x)$ .

**Example 5.** •

$$\begin{aligned} M &= \mathbb{R}^2 \\ \mathcal{U} &= \mathbb{R}^2 / 0, 0 \\ x : \mathcal{U} &\rightarrow \mathbb{R}^2 : x(m, n) \rightarrow (-m, -n) \end{aligned}$$

- We can take another chart map on  $\mathcal{U}$   
 $(m, n) \rightarrow (\sqrt{m^2 + n^2}, \arctan(\frac{n}{m}))$

## 2.2 Chart Transition Maps

Imagine two charts  $(\mathcal{U}, x)$  and  $(\mathcal{V}, y)$  with  $\mathcal{U} \cap \mathcal{V} = \mathcal{A} \neq \emptyset$ . The same point  $p \in \mathcal{A}$  can be mapped via  $x$  and  $y$  to two different charts. Because these maps are continuously invertible we can smoothly transition between pages of our atlas:  $(y \circ x^{-1})(p) = y(x^{-1}(p))$ . This map is called the chart transition map. Informally the chart transition map contains the instructions how to *glue* together the charts of our atlas.

## 2.3 Manifold Philosophy

Often it is desirable (or indeed the way) to define properties ("continuity") of real-world objects ( $\mathbb{R} \xrightarrow{\gamma} M$ ) by judging suitable condition not on the real-world object itself but on the chart-representation of the real-world object. Advantages: You can define continuity in this way. Disadvantages: The chart map  $x$  is a 'fantasy,'  $x$  may be ill defined because the chart chosen is arbitrary. Solution: The property must be maintained regardless of the chart.

$$\begin{aligned} \mathbb{R} &\xrightarrow{y \circ \gamma} y(\mathcal{U}) \\ \mathbb{R} &\xrightarrow{x \circ \gamma} x(\mathcal{U}) \\ \mathbb{R} &\xrightarrow{\gamma} \mathcal{U} \end{aligned}$$

We are interested in properties of  $\gamma$  but must talk about it with respect to  $x$  and  $y$ .

## 3 Lecture 3: Multilinear Algebra [7]

We will not equip space(time) with a vector space structure. There is no such thing as five times Paris or Paris plus Vienna. However, the tangent spaces of  $T_p M$  of smooth manifolds carry a vector space structure. It is beneficial to first study vector spaces abstractly for two reasons.

- For construction of  $T_p M$  one needs an intermediate vector space  $C^\infty(M)$ .
- Tensor techniques are best understood in an abstract setting.

### 3.1 Vector Spaces

**Definition 7.** A vector space  $(V, +, \cdot)$  is

- a set  $V$
- $+: V \times V \rightarrow V$
- $\cdot: \mathbb{R} \times V \rightarrow V$

Which satisfies CANI and ADDU axioms. For  $w, v, u \in V$  and  $\lambda, \mu \in \mathbb{R}$

- $C^+ : v + w = w + v$
- $A^+ : (u + v) + w = u + (v + w)$
- $N^+ : \exists 0 \in V : \forall v \in V : v + 0 = v$
- $I^+ : \forall v \in V : \exists (-v) \in V : v + (-v) = 0$
- $A : \lambda \cdot (\mu \cdot v) = (\lambda \cdot \mu) \cdot v$
- $D : (\lambda + \mu) \cdot v = \lambda \cdot v + \mu \cdot v$
- $D : \lambda \cdot v + \lambda \cdot w = \lambda \cdot (v + w)$
- $U : 1 \cdot v = v$

**Terminology 3.** An element of a vector space is often referred to (informally) as a vector.

**Example 6.**

**Definition 8.** Set  $P := p : (-1, +1) \rightarrow \mathbb{R}$  of polynomials of (fixed) degree.  $p(x) = \sum_{n=1}^N p_n x^n$ . Is  $\square(x) = x^2$  a vector? No.  $\square \in P$ .

**Definition 9.** Define the operations in the space:

$$+ : P \times P \rightarrow P : (p, q) \mapsto p + q$$

$$\cdot : \mathbb{R} \times P \rightarrow P : (\lambda, p) \mapsto \lambda \cdot p$$

Is  $\square \in (P, +, \cdot)$  a vector in a vector space? Yes!

### 3.2 Linear Maps

We want to study maps which preserve (vector space) structure. On vector spaces these are called linear maps.

**Definition 10.** Let  $(V, +, \cdot)$  and  $(W, +, \cdot)$  be vector spaces. Then a map  $\phi : V \rightarrow W$  is called linear if:

- $\phi(v + \tilde{v}) = \phi(v) + \phi(\tilde{v})$
- $\phi(\lambda \cdot v) = \lambda \cdot \phi(v)$

**Example 7.** Take  $P$  as before. Take  $\delta : P \rightarrow P : \delta(p) := p'$ .  $\delta(p, q) = (p + q)' = \delta(p) + \delta(q)$  and  $\delta(\lambda p) = (\lambda p)' = \lambda p'$

Notation:  $\phi : V \rightarrow W$  linear  $\iff \phi : V \xrightarrow{\sim} W$

### 3.3 Vector space of Homomorphisms

Take  $(V, +, \cdot)$  and  $(W, +, \cdot)$  vector spaces.

**Definition 11.**  $\text{Hom}(V, W) := \phi : V \xrightarrow{\sim} W$  as a set. This is a vectorspace:

- $\oplus : \text{Hom}(V, W) \times \text{Hom}(V, W) \rightarrow \text{Hom}(V, W) : (\phi, \psi) \mapsto \phi \oplus \psi$  where  $(\phi \oplus \psi)(v) := \phi(v) + \psi(v)$
- $\otimes$  defined similarly.

$(\text{Hom}(V, W), \oplus, \otimes)$  is a vector space.

**Example 8.** Take  $P$  as before. Then  $\delta$  is a vector space similarly.

### 3.4 Dual Vector Space

$(V, +, \cdot)$  as a vector space.

**Definition 12.**  $V^* := \psi : V \xrightarrow{\sim} \mathbb{R} = \text{Hom}(V, \mathbb{R})$  is also a vector space called the dual vector space of  $V$ .

**Terminology 4.**  $\phi \in V^*$  is called (informally) a covector.

**Example 9.**  $I : P \xrightarrow{\sim} \mathbb{R}$  so  $I \in P^*$

**Definition 13.**  $I(p) := \int_0^1 dx p(x)$

$I$  is clearly linear.

### 3.5 Tensors

**Definition 14.** Take a vector space  $V$ . Then a  $(r, s)$  tensor  $T$  over  $V$  is a multilinear map:  $T : \bigotimes_{i=1}^r V^* \otimes \bigotimes_{j=1}^s V \xrightarrow{\sim(r+s)} \mathbb{R}$

**Example 10.**  $T(1, 1)$ - tensor.  $T(\phi + \psi, v) = T(\phi, v) + T(\psi, v)$ ,  $T(\lambda\psi, v) = \lambda \cdot T(\psi, v)$ ,  $T(\psi, v + w) = T(\psi, v) + T(\psi, w)$ ,  $T(\psi, \lambda \cdot v) = \lambda \cdot T(\psi, v)$  Linear in both entries. Hence, multi-linear.

**Example 11.**  $g : P \times P \xrightarrow{\sim} \mathbb{R}$ ,  $(p, q) \rightarrow \int_0^1 dx p(x)q(x)$  is a  $(0, 2)$  - tensor over example of  $P$ .

### 3.6 Vectors and Covectors

**Theorem 2.**  $\phi \in V^* \iff \phi : V \xrightarrow{\sim} \mathbb{R} \iff \phi(0, 1)$ - tensor.

**Theorem 3.**  $\dim(V) < \infty \implies v \in V = (V^*)^* \iff v : V^* \xrightarrow{\sim} \mathbb{R} \iff v$  is a  $(1, 0)$ - tensor.

### 3.7 Bases

**Definition 15.** Take a vector space  $V$ . A subset  $B \subset V$  is called a basis if  $\forall v \in V \exists F = \{f_1, \dots, f_n\} \in B : \exists ! \{v^1, \dots, v^n\} \in \mathbb{R} : v = v^1 f_1 + \dots + v^n f_n$ .

**Definition 16.** If  $\exists$  basis  $B$  with finite many elements ( $d$  many elements) then we call  $d =: \dim(V)$ .

**Remark 1.** Let  $V$  be finite dimensional vector space. Choose a basis  $e_1, \dots, e_n$  of  $V$ . We may uniquely associate a vector  $v \in V$  with  $v \mapsto (v^1, \dots, v^n)$  called the components of  $v = v^1 e_1 + \dots + v^n e_n$  with respect to the chosen basis. It is more economical to require your basis on  $V$  once chosen such that  $\epsilon^a(e_b) = \delta_b^a$ . This uniquely determines the choice of vector components from a choice of basis.

**Definition 17.** If a basis  $\epsilon^1, \dots, \epsilon^n$  of  $V^*$  satisfies these axioms it is called the dual basis.

**Example 12.** Take  $P$  with  $(N = 3)$ . Then  $e_0, \dots, e_3$  is a basis if  $e_a(x) := x^a$  is a basis of  $P$ . Then dual basis is given by  $\epsilon^a := \frac{1}{a!} \partial^a|_{x=0}$ .

### 3.8 Components of Tensors

**Definition 18.** Let  $T$  be an  $(r, s)$ - tensor over a finite dimensional vector space  $V$ . Then define the  $(r + s)^{\dim(V)}$  many real numbers  $i_1, \dots, i_r, j_1, \dots, j_s \in \{1, \dots, \dim(V)\}$ ,  $T_j^i \in \mathbb{R} := T(\epsilon^i, e_j)$ . The  $T_j^i$  elements are called the components of the tensor with respect to the chosen basis. Knowing the components and basis one can reconstruct the entire tensor.

**Example 13.**  $T(1, 1, )$  - tensor with  $T_j^i = T(\epsilon^i, e_j)$  reconstruct

$$\begin{aligned} T(\phi, v) &= T\left(\sum_{i=1}^{\dim(V)} \phi_i \epsilon^i, \sum_{j=1}^{\dim(V)} v^j e_j\right) \\ &= \sum_{i=1}^{\dim(V)} \sum_{j=1}^{\dim(V)} \phi_i v^j T(\epsilon^i, e_j) \end{aligned}$$

with  $\phi_i, v^j \in \mathbb{R}$ .

**Terminology 5.** If we agree to label  $T_j^i = T(\epsilon^i, e_j)$  with the up and down components then Einstein summation convention of tensors  $:= \phi_i v^j T_j^i$  where we drop  $\sum$ . This only works over multilinear maps.

## 4 Lecture 4: Differentiable Manifolds [8]

So far we have topological manifolds  $(M, \mathcal{O})$   $\dim(M)=d$ . We want to talk about velocity vectors on them but this structure is insufficient to do so. Pick a dimension  $d \neq 4$  for your manifold. In that dimension the choices of topology are countable, so presumably we can do experiments to discern which one corresponds to our reality. In  $d = 4$  our choices of topology are suddenly uncountable. We need additional structure to talk about differentiable curves on manifolds and between manifolds.

1. Curves:  $\mathbb{R} \rightarrow M$
2. Functions:  $M \rightarrow \mathbb{R}$
3. Maps:  $M_1 \rightarrow M_2$

### 4.1 Strategy

$\gamma : \mathbb{R} \rightarrow \mathcal{U}$  and choose a chart  $(\mathcal{U}, x)$ .  $\mathcal{U} \xrightarrow{x} x(\mathcal{U}) \subseteq \mathbb{R}^d$ . Try and 'lift' the undergraduate notion of differentiability of a curve in  $\mathbb{R}^d$  to a notion of differentiability on a curve on  $M$ . Problem: Is this well defined under a change of chart? We don't want it to depend on our taste.

$$\gamma : \mathbb{R} \rightarrow \mathcal{U} \cap \mathcal{V} \neq \emptyset \quad (1)$$

$$y \circ \gamma : \mathbb{R} \rightarrow y(\mathcal{U} \cap \mathcal{V}) \subseteq \mathbb{R}^d \quad (2)$$

$$x \circ \gamma : \mathbb{R} \rightarrow x(\mathcal{U} \cap \mathcal{V}) \subseteq \mathbb{R}^d \quad (3)$$

$$y \circ \gamma = (y \circ x^{-1}) \circ (x \circ \gamma) = y \circ (x^{-1} \circ x) \circ \gamma \quad (4)$$

If  $(x \circ \gamma)$  differentiable and  $(y \circ x^{-1})$  continuous is  $y \circ \gamma$  guaranteed to be continuously differentiable? No.

### 4.2 Compatible Charts

In previous we took a chart on the topological manifold. Our atlases  $\mathcal{V}$  and  $\mathcal{U}$  were elements of the maximal atlas of the manifold.

**Definition 19.** Two charts  $(\mathcal{U}, x)$  and  $(\mathcal{V}, y)$  are called  $\square$  compatible if either:

1.  $\mathcal{U} \cap \mathcal{V} = \emptyset$
2.  $y \circ x^{-1} : x(\mathcal{U} \cap \mathcal{V}) \rightarrow y(\mathcal{U} \cap \mathcal{V})$  and  $x \circ y^{-1} : y(\mathcal{U} \cap \mathcal{V}) \rightarrow x(\mathcal{U} \cap \mathcal{V})$  are differentiable.

Philosophy of  $\square$ :

**Definition 20.** An atlas  $\mathcal{A}_\square$  is a  $\square$  - compatible atlas if any two charts in  $\mathcal{A}_\square$  are  $\square$  - compatible.

**Definition 21.** A  $\square$  - manifold is a triple  $(M, \mathcal{O}, \mathcal{A}_\square)$

Undergraduate  $\square$ .  $C^0$ :  $C^0(\mathbb{R}^d \rightarrow \mathbb{R}^d)$  Continuous maps.  $C^1$ :  $C^1(\mathbb{R}^d \rightarrow \mathbb{R}^d)$  Differentiable Once.  $C^k$ :  $k$ -times continuous differentiable.  $D^k$ :  $k$ -times differentiable.  $C^\infty$ :  $C^\infty(\mathbb{R}^d \rightarrow \mathbb{R}^d)$  infinitely continuously infinitely continuously differentiable.  $C^\omega$ :  $\exists$  multidimensional Taylor Expansion.  $\mathbb{C}^\infty$ : Satisfies Cauchy-Riemann Equations.

**Theorem 4.** Any  $C^{k \geq 1}$  - atlas  $\mathcal{A}$  of a topological manifold contains a  $C^\infty$  - atlas.

Thus we will consider  $C^\infty$  manifolds or "smooth manifolds" unless we wish to define Taylor expansions or complex differentiable.

**Definition 22.** A smooth manifold is a tripple  $(M, \mathcal{O}, \mathcal{A})$  where  $\mathcal{A}$  is a  $C^\infty$  - atlas.

### 4.3 Diffeomorphisms

Take a map:  $M \xrightarrow{\phi} N$ . If  $M$  and  $N$  are naked sets then the structure preserving maps are the bijections. If  $\exists \phi$  is a bijection  $M \cong N$ .

**Example 14.**

$$\mathbb{N} \cong \mathbb{Z}$$

$$\mathbb{N} \cong \mathbb{Q}$$

$$\mathbb{N} \not\cong \mathbb{R}$$

Linear bijections between vector spaces are the structure preserving maps between vector spaces.

**Definition 23.** Two  $C^\infty$  manifolds are said to be diffeomorphic  $\iff \exists$  bijection  $\phi : M \rightarrow N$  such that  $\phi, \phi^{-1}$  are both  $C^\infty$  maps.

**Theorem 5.** The number of  $C^\infty$  manifolds that can be constructed from a given  $C^0$  - manifold up to diffeomorphism by the More-Radon Theorems in  $d = 1, 2, 3$  dimension are 1. In  $d > 4$  the number is finite. In  $d = 4$  uncountably infinitely many such manifolds.

## 5 Lecture 5: Tangent Spaces [9]

Lead question: "What is the velocity of a curve  $\gamma$  at a point  $p$ ?"

### 5.1 Velocities

**Definition 24.** Take a  $(M, \mathcal{O}, \mathcal{A})$  smooth manifold. Take a curve:  $\gamma : \mathbb{R} \rightarrow M$  of at least  $C^1$ . Suppose  $\gamma(\lambda_0) = p$ . The velocity of  $\gamma$  at  $p$  is the linear map:  $v_{\gamma,p} : C^\infty(M) \xrightarrow{\sim} \mathbb{R}$ .  $C^\infty(M) := \{f : M \rightarrow \mathbb{R} \mid f \text{ smooth function} \}$  equipped with  $(f \oplus g)(p) := f(p) +_{\mathbb{R}} g(p)$ ,  $(\lambda \oplus g)(p) := \lambda \cdot g(p)$ . So velocity is a linear map:  $f \mapsto v_{\gamma,p}(f) := (f \circ \gamma)'(\lambda_0)$ .

### 5.2 Tangent Vector Space

**Definition 25.** For each point  $p \in M$  we define the set "tangent space to  $M$  at  $p$ " as  $T_p M := \{v_{\gamma,p} \mid \gamma \text{ smooth curve} \}$ .

Observe that  $T_p M$  can be made into a vector space.

**Definition 26.** We need to define  $\oplus$  and  $\otimes$ .

$$\begin{aligned} \oplus : T_p M \times T_p M &\rightarrow \text{Hom}(C^\infty(M), \mathbb{R}) \\ (v_{\gamma,p} \oplus v_{\delta,p})(f) &:= v_{\gamma,p}(f) +_{\mathbb{R}} v_{\delta,p}(f), f \in C^\infty(M) \\ \otimes : \mathbb{R} \times T_p &\rightarrow \text{Hom}(C^\infty(M), \mathbb{R}) \\ (\alpha \otimes v_{\gamma,p})(f) &:= \alpha \cdot_{\mathbb{R}} v_{\gamma,p}(f), \alpha \in \mathbb{R} \end{aligned}$$

Remains to be shown that:

1.  $\exists \sigma$  curve:  $v_{\gamma,p} \oplus v_{\delta,p} = \sigma$ ,
2.  $\exists \tau$  curve:  $\lambda \otimes v_{\gamma,p} = v_{\tau,p}$

*Proof.* Construct:

$$\begin{aligned} \tau : \mathbb{R} &\rightarrow M \\ &\mapsto \tau(\lambda) := \gamma(\alpha \lambda \cdot \lambda_0) = (\gamma \circ \mu_\alpha)(\lambda) \\ \mu_\alpha : \mathbb{R} &\rightarrow \mathbb{R} \\ \triangle &\mapsto \triangle \cdot \alpha \end{aligned}$$

Then:

$$\begin{aligned} \tau(0) &= \gamma(\lambda_0) = p \\ v_{\tau,p} &:= (f \circ \tau)'(0) = (f \circ \gamma \mu_\alpha)'(0) \\ &= (f \circ \gamma)'(\lambda_0) \cdot \alpha \\ &= \alpha v_{\gamma,p} \end{aligned}$$

And for addition:

$$v_{\gamma,p} \oplus v_{\delta,p} \stackrel{?}{=} \sigma,$$

Make a choice of chart  $(\mathcal{U}, x)$  with  $p \in \mathcal{U}$  and define:

$$\begin{aligned} \sigma_x : \mathbb{R} &\rightarrow M \\ \sigma_x(\lambda) &:= x^{-1}((x \circ \gamma)(\lambda_0 + \lambda) + (x \circ \gamma)(\lambda_0)) \end{aligned}$$

Then:

$$\begin{aligned} \sigma_x(0) &= x^{-1}((x \circ \gamma)(\lambda_0) + (x \circ \delta)(\lambda_0) - (x \circ \gamma)(\lambda_0)) \\ &= \delta(\lambda_0) = 0 \end{aligned}$$

Now:

$$\begin{aligned} v_{\sigma,p} &:= (f \circ \sigma_x)'(0) \\ &= (f \circ x^{-1}(x \circ \sigma_x))'(\gamma) \\ &= (x \circ \sigma_x)^i(0) \cdot (\partial_i(f \circ x^{-1}))(x(\sigma_x(0))) (x \circ \gamma)^i(\lambda_0) + (x \circ \delta)^i(\lambda_0) \\ &= (x \circ \gamma)^i(\lambda_0) (\partial_i(f \circ x^{-1}))(x(p)) + (x \circ \delta)^i(\lambda_0) (\partial_i(f \circ x^{-1}))(x(p)) \\ &= (f \circ \gamma)'(\lambda_0) + (f \circ \delta)'(\lambda_0) \\ &= v_{\gamma,p}(f) + v_{\delta,p}(f), \forall f \in C^\infty(M) \\ v_{\sigma,p} &= v_{\gamma,p} \oplus v_{\delta,p} \end{aligned}$$

And we are done. □

### 5.3 Components of a Vector With Respect to a Chart

**Definition 27.** Let  $(\mathcal{U}, x) \in \mathcal{A}_{smooth}$  and let  $\gamma : \mathbb{R} \rightarrow \mathcal{U}, \gamma(0) = p$ . Calculate

$$\begin{aligned} v_{\gamma,p}(f) &:= (f \circ \gamma)'(0) = ((f \circ x^{-1}) \circ (x \circ \gamma))'(0) \\ &= (x \circ \gamma)^i(0) \cdot (\partial_i(f \circ x^{-1}))(x(p)) \\ &= \dot{\gamma}_x^i(0) \cdot \left(\frac{\partial}{\partial x^i}\right)_p f, \forall f \in C^\infty(M) \end{aligned}$$

This is the components of the velocity vector from the chart induced basis of the vector from  $T_p M$ .

### 5.4 Chart-induced Basis

**Definition 28.** Take  $(\mathcal{U}, x) \in \mathcal{A}_{smooth}$ . Then

$$\left(\frac{\partial}{\partial x^1}\right)_p, \dots, \left(\frac{\partial}{\partial x^d}\right)_p \in T_p M$$

Constitute a basis of  $T_p \mathcal{U}$ .

*Proof.* Linear independence remains to be shown.

$$\begin{aligned} \lambda^i \left(\frac{\partial}{\partial x^i}\right)_p &= \\ \lambda^i \left(\frac{\partial}{\partial x^i}\right)_p (x^i) &= \\ \lambda^i \partial_i (x^j \circ x^{-1})(x(p)) &= \\ \lambda^i \delta_i^j &= \lambda^j \end{aligned}$$

Corollary  $\dim T_p M = d = \dim(M)$ . □

**Terminology 6.**  $X \in T_p M \implies \exists \gamma : \mathbb{R} \rightarrow M : X = v_{\gamma,p}$  and  $\exists x^1, \dots, x^d : X = x^i \left(\frac{\partial}{\partial x^i}\right)_p$



## 5.5 Change of Vector Components Under a Change of Chart

The physical vector in the real world doesn't change under change of chart. The vector components change under change of chart. Let  $(\mathcal{U}, x), (\mathcal{V}, y) \in \mathcal{A}_{smooth}, p \in \mathcal{U} \cap \mathcal{V} \neq \emptyset$ . Let  $X \in T_p M$  then

$$X_{(y)}^i \cdot \left(\frac{\partial}{\partial y^i}\right)_p =_{(\mathcal{V}, y)} X =_{(\mathcal{U}, x)} \left(\frac{\partial}{\partial x^i}\right)_p$$

To study change of components formula take:

$$\begin{aligned} \left(\frac{\partial}{\partial x^i}\right)_p f &= \partial_i(f \circ x^{-1})(x(p)) \\ &= \partial_i((f \circ y^{-1}) \circ (y \circ x^{-1})(x(p))) \\ &= (\partial_i(y^j \circ x^{-1})(x(p)) \cdot (\partial_j(f \circ y^{-1}))(y(p))) \\ &= \left(\frac{\partial y^j}{\partial x^i}\right)_p \cdot \left(\frac{\partial f}{\partial y^j}\right)_p \\ &= X_{(x)}^i \cdot \left(\frac{\partial y^j}{\partial x^i}\right)_p \cdot \left(\frac{\partial}{\partial y^j}\right)_p \\ &= X_{(y)}^i \left(\frac{\partial}{\partial y^j}\right)_p \\ &\implies \\ X_{(y)}^j &= \left(\frac{\partial y^j}{\partial x^i}\right)_p X_{(x)}^i \end{aligned}$$

## 5.6 Cotangent Spaces

Trivial  $(T_p^* M) := \{\phi : T_p M \xrightarrow{\sim} \mathbb{R}\}$ .

**Example 15.**  $f \in C^\infty(M)$

$$\begin{aligned} (df)_p : T_p M &\xrightarrow{\sim} \mathbb{R} \\ X &\mapsto (df)_p(X) := Xf \end{aligned}$$

So  $(df)_p$  is called the gradient of  $f$  at  $p \in M$ . Calculate the components of the gradient.  $(0,1)$ - tensor of the vector space with respect to a chart  $(\mathcal{U}, x)$ .

$$\begin{aligned} ((df)_p)_j &:= (df)_p \left(\frac{\partial}{\partial x^j}\right)_p \\ &= \left(\frac{\partial f}{\partial x^j}\right)_p \\ &= \partial_j(f \circ x^{-1})(x(p)) \end{aligned}$$

**Theorem 6.** Consider a chart  $(\mathcal{U}, x) \implies x^i : \mathcal{U} \rightarrow \mathbb{R}$ . Claim:  $(dx^i)_p$  is the basis of  $T_p^* M$  or dual basis.  $(dx^a)_p \left(\frac{\partial}{\partial x^b}\right)_p = \delta_b^a$

## 5.7 Change of Components of A Covector with Respect to a Chart

$\omega \in T_p^* M$  then  $\omega_{((x)i)}(dx^i)_p \implies \omega_{(y)j} = \left(\frac{\partial x^j}{\partial y^i}\right) \omega_{(x)i}$ .

## 6 Lecture 6: Fields [10]

We want to assign to every point on a manifold a vector. To do this we need Theory of Bundles.

### 6.1 Bundles

**Definition 29.** A bundle is a triple  $E \xrightarrow{\pi} M$  where  $E$  is the 'total space' and is a smooth manifold.  $\pi$  is a smooth surjective map called the 'projective map'. And  $M$  is our smooth manifold called the 'base space.'

**Definition 30.** Take bundle with  $p \in M$  we define the Fibre over  $p$  as the  $\text{preim}_\pi(\{p\})$ .

**Definition 31.** A section  $\sigma$  of a bundle is the  $\text{preim}_\pi(M)$  where we require  $\pi \circ \sigma = \text{id}_M$ .

As an aside in quantum mechanics we take a wave function of a position space  $\psi : M \rightarrow \mathbb{C}$ . They are actually sections of  $\mathbb{C}$ -line bundle of the phase-space.

## 6.2 Tangent Bundle of a Smooth Manifold

Take a smooth manifold  $(M, \mathcal{O}, \mathcal{A})$ .

- The tangent bundle is a set:  $TM := \bigcup_{p \in M} T_p M$ .
- $\pi : TM \rightarrow M : X \mapsto p$  where  $p$  is the unique point in  $M$  such that  $X \in T_p M$ .
- We want the coarsest topology such that  $\pi$  is continuous. This is called the 'initial topology.'  $\mathcal{O}_{TM} := \{\text{preim}_\pi(\mathcal{U}) \mid \mathcal{U} \in \mathcal{O}\}$ .

Construction of a  $C^\infty$  atlas on  $TM$  from the  $C^\infty$  atlas on  $M$ .  $\mathcal{A}_{TM} := \{(T\mathcal{U}, \psi_x) \mid (\mathcal{U}, x) \in \mathcal{A}\}$  with  $\psi_x : T\mathcal{U} \rightarrow \mathbb{R}^{2 \cdot \dim(M)} : X \in T_{\pi(x)} M$  with  $X \mapsto ((x^1 \circ \pi)(X), \dots, (x^d \circ \pi)(X), (dx^1)_{\pi(x)}(X), \dots, (dx^d)_{\pi(x)}(X))$ . Note we need to reconstruct  $\psi^{-1}$  from our data. So we have  $(\alpha^i, \beta^j) \in \mathbb{R}^{2d}$  such that we have  $\beta^i (\frac{\partial}{\partial x^i})_{x^{-1}(\alpha^j)}$ . Alphas are the point and the betas are the components. Check that this map is smooth by change of atlas!

## 6.3 Vector Fields

**Definition 32.** A smooth vector field is a  $\chi$  is a smooth map that is a section.

## 6.4 The $C^\infty(M)$ - module $\Gamma(TM)$

$(C^\infty(M), +, \cdot)$  This satisfies the notion of a ring. It is the ring of smooth function on a manifold.  $\Gamma(TM) = \{\chi : TM \rightarrow M \mid \text{smooth sections}\}$ . This is a vector space module over the ring. The set of all smooth vector fields can be made into such a module structure. This is important because we cannot guarantee that there exists a basis. There is no way to generate a basis for a vector field over a sphere (Harry Ball Theorem).

## 6.5 Tensor Fields

**Definition 33.** An  $(r, s)$  - tensor field  $T$  is a  $C^\infty(M)$  multilinear map between modules.

**Example 16.** Define  $df := \Gamma(TM) \rightarrow C^\infty(M) : \chi \mapsto df(\chi) := \chi(f)$  with  $(\chi f)(p) := \chi(p)f$  with  $p \in M$  and  $\chi(p) \in T_p M$ . This is a covector field.

# 7 Lecture 7: Connections [11]

So far we saw that a vector field  $X$  can be used to provide a directional derivative.

$$\nabla_X f := Xf$$

of a  $f \in C^\infty(M)$ .

**Remark 2.** From now on we will mostly consider vector fields. Seems like notational overkill?  $\nabla_X f = Xf = (df)(X)$ . Not quite the same thing.

$$\begin{aligned} X &: C^\infty(M) \rightarrow C^\infty(M) \\ df &: \Gamma(TM) \rightarrow C^\infty(M) \\ \nabla_X &: C^\infty(M) \rightarrow C^\infty(M) \end{aligned}$$

But we want to extend  $\nabla_X$  to  $(p, q) \rightarrow (p, q)$  -tensor fields. This is not free.

## 7.1 Directional Derivatives of Tensor Fields

We formulate a wish list of properties which the  $\nabla_X$  acting on a tensor field should have. Any remaining freedom in choice  $\nabla$  will need to be provided as additional structure beyond our smooth manifold  $(M, \mathcal{O}, \mathcal{A})$ .

**Definition 34.** A connection  $\nabla$  on a smooth manifold is a map that takes a pair consisting of a vector (field)  $X$  and a  $(p, q)$  - tensor (field)  $T$  and yields a  $(p, q)$  - tensor (field)  $\nabla_X T$  satisfying four properties:

1.  $\nabla_X f = Xf, \forall f \in C^\infty(M)$
2.  $\nabla_X(T + S) = \nabla_X T + \nabla_X S$
3.  $\nabla_X(T(\omega, Y)) = (\nabla_X T)(\omega, Y) + T(\nabla_X \omega, Y) + T(\omega, \nabla_X Y)$  (Liebnitz rule)<sup>1</sup>
4.  $\nabla_{fX+gY} T = f\nabla_X T + g\nabla_Y T, \forall f, g \in C^\infty(M)$

A manifold with connection is a quadruple of structures  $(M, \mathcal{O}, \mathcal{A}, \nabla)$ .

**Remark 3.**  $\nabla_X$  is the extension of  $X$ . So  $\nabla_\cdot$  is the extension of  $d$ .

## 7.2 New Structure on $(M, \mathcal{O}, \mathcal{A})$ Required to Fix $\nabla$

Consider  $X, Y$  vector fields.

$$\begin{aligned} \nabla_X Y &=_{(\mathcal{U}, x)} \nabla_{X^i \frac{\partial}{\partial x^i}} (Y^m \cdot \frac{\partial}{\partial x^m}) \\ &=_{(iii)} X^i (\nabla_{\frac{\partial}{\partial x^i}} Y^m) \cdot \frac{\partial}{\partial x^m} + X^i Y^m \cdot \frac{\partial}{\partial x^i} (\frac{\partial}{\partial x^m}) \\ &=_{(i)} X^i (\frac{\partial}{\partial x^i} Y^m) \cdot \frac{\partial}{\partial x^m} X^i Y^m \cdot \Gamma_{mi}^q \frac{\partial}{\partial x^q} \end{aligned}$$

$\Gamma$  is the connection coefficient functions (on  $M$ ) of  $\nabla$  with respect to the chart  $(\mathcal{U}, x)$

**Definition 35.**  $(M, \mathcal{O}, \mathcal{A}, \nabla), (\mathcal{U}, x) \in \mathcal{A}$  then the connection coefficient functions ("Ts) with respect to our chart  $(\mathcal{U}, x)$  are the  $(\dim(M))^3$  many functions:

$$\begin{aligned} \Gamma(x)_{jk}^i : \mathcal{U} &\rightarrow \mathbb{R} \\ p &\mapsto (dx^i)(\nabla_{\frac{\partial}{\partial x^k}} \frac{\partial}{\partial x^j})(p) \end{aligned}$$

$$\text{Thus } (\nabla_X Y)^i = X^m (\frac{\partial}{\partial x^m} Y^j) + \Gamma_{nm}^i \cdot Y^n \cdot X^m$$

**Remark 4.** On a chart domain  $\mathcal{U}$  the choice of the  $(\dim(M))^3$  functions  $\Gamma_{nm}^i$  functions suffices to fix the action  $\nabla$  on a vector field. Those same functions fix the action of  $\nabla$  on any tensor field.

**Remark 5.**  $\nabla_{\frac{\partial}{\partial x^m}} (dx^i) = \Sigma_{jm}^i dx^j$ . But now

$$\begin{aligned} \nabla_{\frac{\partial}{\partial x^m}} (dx^i (\frac{\partial}{\partial x^j})) &= \frac{\partial}{\partial x^m} \delta_j^i \\ &= (\nabla_{\frac{\partial}{\partial x^m}} dx^i) + dx^i (\nabla_{\frac{\partial}{\partial x^m}} \frac{\partial}{\partial x^j}) \\ &\implies \\ (\nabla_{\frac{\partial}{\partial x^m}} dx^i)_j &= -\Gamma_{jm}^i \end{aligned}$$

In summary  $(\nabla_X Y)^i = X(Y^i) + \Gamma_{jm}^i Y^j X^m$  and  $(\nabla_X \omega)_i = X(\omega_i) - \Gamma_{im}^j \omega_j X^m$ .<sup>3</sup> So:  $(\nabla_X T)_{jk}^i = X(T_{jk}^i) + \Gamma_{sm}^i T_{jk}^s X^m - \Gamma_{jm}^s T_{sk}^i X^m - \Gamma_{km}^s T_{js}^i X^m$  where  $T$  is a  $(1, 2)$  - tensor field.

<sup>1</sup>For  $(1, 1)$  tensor field  $T$ , but analogously for any  $(p, q)$  - TF

<sup>2</sup>On vector fields

<sup>3</sup>On covector fields

### 7.3 Change of $\Gamma$ s Under Change of Chart

$(\mathcal{U}, x), (\mathcal{V}, y) \in \mathcal{A}$  with nonzero intersection.

$$\begin{aligned}
\Gamma_{(y)jk}^i &:= dy^i \left( \nabla_{\frac{\partial}{\partial y^k}} \frac{\partial}{\partial y^j} \right) \\
&= \frac{\partial y^i}{\partial x^q} dx^q \left( \nabla_{\frac{\partial}{\partial y^k}} \frac{\partial x^s}{\partial x^p} \frac{\partial}{\partial x^j} \cdot \frac{\partial}{\partial x^s} \right) \\
&= \frac{\partial y^i}{\partial x^q} dx^q \left( \frac{\partial x^p}{\partial y^k} \right) \left[ \left( \nabla_{\frac{\partial}{\partial x^p}} \frac{\partial x^s}{\partial y^j} \right) \frac{\partial}{\partial x^s} + \frac{\partial x^s}{\partial y^j} \left( \nabla_{\frac{\partial x^p}{\partial x^s}} \right) \right] \\
&= \frac{\partial y^i}{\partial x^q} \frac{\partial x^p}{\partial y^k} \frac{\partial}{\partial x^p} \left( \frac{\partial x^s}{\partial y^j} \right) \delta_s^q + \frac{\partial y^i}{\partial x^q} \frac{\partial x^p}{\partial y^k} \frac{\partial x^s}{\partial y^j} \frac{\partial x^p}{\partial y^k} \Gamma_{(x),sp}^q
\end{aligned}$$

### 7.4 Normal Coordinates

Let  $p \in M$  of  $(M, \mathcal{O}, \mathcal{A}, \nabla)$ . Then one can construct a chart  $(\mathcal{U}, x)$  with  $p \in \mathcal{U}$  such that  $\Gamma_{(x)jk}^i(p) = 0$  at  $p$ . Not necessarily in any neighborhood.

*Proof.* Let  $(\mathcal{V}, y)$  be any chart with  $p \in \text{mathcal{V}}$ . This in general the  $\gamma_{(y)jk}^i \neq 0$ . Then consider the new chart  $(\mathcal{U}, x)$  to which one transitions.

$$(x \circ y^{-1})^i(\alpha^d) := \alpha^i \pm \frac{1}{2} \Gamma_{(y)jk}^i(p) \alpha^j \alpha^k$$

Then:

$$\begin{aligned}
\frac{\partial x^i}{\partial y^j} &= \partial_j (x^i \circ y^{-1}) \\
&= \delta_j^i \pm \Gamma_{(y),mj}^i(p) \alpha^m \\
\frac{\partial x^i}{\partial y^k \partial y^j} &= \pm \Gamma_{(y)kj}^i(p) \\
&\implies \Gamma_{(x)(jk)}^i(p) = \Gamma_{(y)jk}^i(p) - \Gamma_{(y)(kj)}^i(p) \\
&= 0
\end{aligned}$$

□

## 8 Lecture 8: Parallel Transport [12]

### 8.1 Parallellity of Vector Fields

$(M, \mathcal{O}, \mathcal{A}, \nabla)$  manifold with connection.

**Definition 36.** A vector field  $X$  on  $M$  is said to be parallelly transported along a smooth curve  $\gamma : \mathbb{R} \rightarrow M$  if  $\nabla_{v_\gamma} X = 0$  i.e.  $(\nabla_{v_\gamma, \gamma(\lambda)} X)_{\gamma(\lambda)}$

**Definition 37.** A slightly weaker condition then parallelly transported is parallel:  $(\nabla_{v_\gamma, \gamma(\lambda)} X)_{\gamma(\lambda)} = \mu(\lambda) X_{\gamma(\lambda)}$  for  $\mu : \mathbb{R} \rightarrow \mathbb{R}$

### 8.2 (self) Autoparallely Transported Curves

**Definition 38.** A curve  $\gamma : \mathbb{R} \rightarrow M$  is called autoparallely transported if  $\nabla_{v_\gamma} v_\gamma = 0 \iff \nabla_{v_\gamma, \gamma(\lambda)} v_{\gamma(\lambda)} = 0$ .

**Remark 6.** Define the notion of an autoparalel curve  $\nabla_{v_\gamma} v_\gamma = \mu v_\gamma$ .

### 8.3 Autoparallel Equation

Autoparallely transported curve  $\gamma$  then consider the portion of the curve that lies in  $\mathcal{U}$  when  $(\mathcal{U}, x) \in \mathcal{A}$ . Express  $\nabla_{v_\gamma} v_\gamma = 0$  in terms of chart representations.

$$\begin{aligned} (\nabla_{v_\gamma} v_\gamma)^i &= (\nabla_{\gamma_{(x)}^m (\frac{\partial}{\partial x^m})_\gamma} \gamma_{(x)}^n) \cdot \frac{\partial}{\partial x^n} \\ v_{\gamma, \gamma} &= \dot{\gamma}_{(x)} \cdot (\frac{\partial}{\partial x^m})_\gamma \\ \gamma_{(x)}^m &:= x^m \circ \gamma \\ &= \dot{\gamma}^m \frac{\partial \dot{\gamma}^q}{\partial x^m} \cdot \frac{\partial}{\partial x^q} + \gamma^m \dot{\gamma}^n \cdot \Gamma_{nm}^q \frac{\partial}{\partial x^q} \end{aligned}$$

In summary  $\ddot{\gamma}_{(x)}^m + \Gamma_{ab}^m(\gamma(\lambda)) \dot{\gamma}^a(\lambda) \dot{\gamma}^b(\lambda) = 0$  Chart expression of the condition that  $\gamma$  be autoparallely transported.

**Example 17.** • Euclidean plane,  $\mathcal{U} = \mathbb{R}^2, x = id_{\mathbb{R}^2}, \Gamma_{(x)jk}^i = 0 \implies \ddot{\gamma}_{(x)}^m = 0$   
 $\implies \gamma_{(x)}^m(\lambda) = a^m \lambda + b^m, a, b \in \mathbb{R}^d$ .

• Round sphere  $(S^2, \mathcal{O}, \mathcal{A}, \nabla_{round})$ . Consider a chart:  $x(p) = (\theta, \phi), \theta \in (0, 2\pi), \phi \in (0, 2\pi)$   
 $\implies \Gamma_{(x),22}^1(x^{-1}(\theta, \phi)) := \sin(\theta) \cos(\theta), \Gamma_{(x)21}^2 = \Gamma_{(x)12}^2 := \cot(\theta)$

### 8.4 Torsion

Question: Can we use  $\nabla$  to define tensors on  $(M, \mathcal{O}, \mathcal{A}, \nabla)$ ?

**Definition 39.** The torsion of a connection  $\nabla$  is the  $(1,2)$ -tensor field. Use lie algebra commutators.  $T(\omega, X, Y) := \omega(\nabla_X Y - \nabla_Y X - [X, Y])$ .

*Proof.* Check  $T$  is  $C^\infty$  linear in each entry.

$$\begin{aligned} T(f \cdot \omega, X, Y) &= f \cdot \omega(\dots) = fT(\omega, X, Y) \\ T(\omega + \psi, X, Y) &= \dots = T(\omega, X, Y) + T(\psi, X, Y) \\ T(\omega, fX, Y) &= \omega(\nabla_{fX} Y - \nabla_Y(fX) - [fX, Y]) \\ &= \omega(f\nabla_X Y - (Yf)X - f\nabla_Y X - f[X, Y] + (Yf)X) \\ &= fT(\omega, X, Y) = -fT(\omega, Y, X) \end{aligned}$$

□

**Definition 40.** A  $(M, \mathcal{O}, \mathcal{A}, \nabla)$  is called torsion-free if  $T = 0$ . In a chart:  $T_{ab}^i := T(dx^i, \frac{\partial}{\partial x^a}, \frac{\partial}{\partial x^b}) = dx^i(\dots)$ .

### 8.5 Curvature

**Definition 41.** The Riemann curvature of a connection  $\nabla$  is the  $(1,3)$ -tensor field  $Riem(\omega, Z, XY) := \omega(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z)$

Algebraic relevance of Riem is given by  $(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z) = R(\cdot, X, Y, Z) + \nabla_{[X,Y]} Z$ . In one chart  $(\mathcal{U}, x)$ :

$$(\nabla_a \nabla_b Z) - (\nabla_b \nabla_a Z) = Riem_{nab}^m Z^n + \nabla_{[\frac{\partial}{\partial x^a}, \frac{\partial}{\partial x^b}]} Z$$

Geometric Significance of Riem: if  $[X, Y] \neq 0$  or commutes then the geometry is not closed.

## 9 Lecture 9: Newtonian Spacetime is Curved [13]

- Newton I: A body on which no force acts moves uniformly along a straight line.
- Newton II: Deviation of a body's motion from such uniform straight motion is effected by a force, reduced by a factor of the body's reciprocal mass.

**Remark 7.** • 1st axiom - in order to be relevant - must be read as a measurement prescription for the geometry of space.

- Since gravity universally acts on every particle, in a universe with at least two particles, gravity must not be considered a force if Newton I is supposed to remain applicable.

## 10 Laplace's Question

Q: "Can gravity be encoded in a curvature of space, such that its effects show if particles under the influence of (no other) force are postulated to move along straight lines in this curved space?" A: No!

*Proof.* Take gravity as a force (point of view):

$$\begin{aligned} m\ddot{x}^\alpha(t) &= F^\alpha(x(t)) \\ -\partial_\alpha F^\alpha &= 4\pi G\rho \end{aligned}$$

True? Experiment decides. Yes! Called "weak equivalence principle."

$$\ddot{\cdot} \cdot x^\alpha - f^\alpha(x(t)) = 0$$

Can we (Laplace asks) this to:

$$\ddot{x}^\alpha(t) + \Gamma_{\beta\gamma}^\alpha(x(t))\dot{x}^\beta(t)\dot{x}^\gamma(t) = 0$$

The two velocity factors are independent of  $\Gamma$ ! We conclude we cannot find  $\Gamma$ 's to formulate Newton's axioms in the form of an autoparallel.  $\square$

## 11 The Full Wisdom of Newton I

Use also the information from Newton's first law that particles (no force) move uniformly. Introduce the appropriate setting (data structure) to talk about the difference between 'straight motion' and 'uniform motion' easily. Insight: in spacetime uniform and straight motion is simply straight motion. So let's try in spacetime. Let  $x : \mathbb{R} \rightarrow \mathbb{R}^3$  be a particle's trajectory in space. We can construct the worldline (history) of the particle.  $X : \mathbb{R} \rightarrow \mathbb{R}^4, t \mapsto (t, x^1(t), x^2(t), x^3(t))$ . Let's assume  $\ddot{x}^\alpha = f^\alpha(x(t))$ . Trivial rewritings  $\dot{X}^0 = 1$  then:

$$\begin{aligned} \ddot{X}^0 &= 0 \\ \ddot{X}^\alpha - f^\alpha(X(T)) \cdot \dot{X}^0 \cdot \dot{X}^0 &= 0 \end{aligned}$$

$$\alpha = 0, 1, 2, 3$$

$\iff$

$$\ddot{X}^a + \Gamma_{bc}^a \dot{X}^b \dot{X}^c = 0$$

$a = 0, 1, 2, 3$  Yes choosing  $\Gamma_{ab}^0 = 0, \Gamma_{\beta\gamma}^\alpha = 0 = \Gamma_{0\beta}^{alpha} = \Gamma_{\beta 0}^\alpha$  then only:  $\Gamma_{00}^\alpha = -f^\alpha$ . Question: Is this a coordinate choice artifact. No. Since we can calculate  $R_{0\beta 0}^\alpha = -\frac{\partial}{\partial x^\beta} f^\alpha \implies R_{00} = R_{0m0}^m = -\partial_\alpha f^\alpha = 4\pi G\rho$ . Writing  $T_{00} = \frac{1}{P^2}\rho \implies R_{00} = 8\pi G T_{00}$ . Conclusion: Laplace's idea works on (Newtonian) spacetime.

**Remark 8.**  $\Gamma_{00}^\alpha = -f^\alpha$   
 $R_{\beta\gamma\delta}^\alpha = 0, \alpha, \beta, \gamma, \delta = 0, 1, 2, 3$  with  $R_{00} = 4\pi G\rho$ .

Question: What about the transformation behavior of LHS of  $\ddot{X}^a + \Gamma_{bc}^a \dot{X}^b \dot{X}^c = 0 = (\nabla_{v_x} v_x)^a := \text{acceleration}^a$ .

### 11.1 The Foundations of the Geometric FORMulation of Newton's Axioms

**Definition 42.** A Newtonian spacetime is a quintuple of structures  $(M, \mathcal{O}, \mathcal{A}, \nabla, t)$  where  $(M, \mathcal{O}, \mathcal{A})$  is a 4-dimensional manifold. And absolute time  $t$  is a smooth function  $t : M \rightarrow \mathbb{R}$ . Satisfying:

- "There is an absolute space"  $(dt)_p \neq 0, \forall p \in M \implies S_\tau := p \in M | t(p) = \tau \implies M = \bigcup_{dt \neq 0} S_\tau$
- "(absolute) Time flows uniformly"  $\nabla dt = 0, \forall p \in M$

**Definition 43.** A vector  $X \in T_p M$  is called:

- Future-directed if  $dt(x) > 0$

- *Spatial* if  $dt(X) = 0$
- *Past-Directed* if  $dt(X) < 0$

**Remark 9.** *Newton I: The worldline of a particle under the influence of no force (gravity isn't a force in this context) is a future-directed Autoparallel*

**Remark 10.** *Newton II:  $\nabla_{v_x} v_x = F/m$  where  $F$  is a spatial vector field:  $dt(F) = 0$ .*

Convention: Restrict the attention to atlases  $\mathcal{A}_{\text{stratified}}$  where charts  $(\mathcal{U}, x)$  have the property that  $x^0 = t|_{\mathcal{U}}$ .  
 $0 = \nabla dt$  Lets evaluate in a chart  $(\mathcal{U}, x)$  of a stratified atlas  $\mathcal{A}_{\text{strat}}$ .

$$\nabla_{v_x} v_x = \frac{F}{m}$$

In a chart<sup>4</sup>:

$$\begin{aligned} X^{0''} + \Gamma_{cd}^0 X^{a'} X^{b'} &= 0 \\ X^{\alpha''} + \Gamma_{\gamma\delta}^\alpha X^{\gamma'} X^{\delta'} + \Gamma_{00}^\alpha X^{0'} X^{0'} + 2\Gamma_{\gamma 0}^\alpha X^{\gamma'} X^{0'} &= \frac{F^\alpha}{m} \end{aligned}$$

## 12 Lecture 10: Metric Manifold [2]

We establish a structure on a smooth manifold that allows us to assign vectors in each tangent space a length and an angle between vectors in the same tangent space. From this structure, one can then define a notion of length of a curve. Then we can look at shortest curves (Geodesics). Requiring then that the shortest curves coincide with the straightest curves (with respect to  $\nabla$ ) will result in  $\nabla$  being determined by the metric structure.

### 12.1 Metrics

**Definition 44.** *A metric  $g$  on a smooth manifold  $(M, \mathcal{O}, \mathcal{A})$  is a  $(0,2)$ - tensor field satisfying:*

- *Symmetry*  $g(X, Y) = g(Y, X)$
- *Non-degeneracy:* the musical map  $\flat: \Gamma(TM) \rightarrow \Gamma(T^*M), X \rightarrow \flat(X)$  where  $\flat(X)(Y) := g(X, Y)$  is a  $C^\infty$  isomorphism. In otherwords it is invertible.

**Definition 45.** *The  $(2,0)$  - tensor field  $g^{-1}$  with respect to a metric  $g$  is the symmetric  $g^{-1}: \Gamma(T^*M) \times \Gamma(T^*M) \rightarrow \mathbb{C}^\infty(M)(\omega, \sigma) \rightarrow \omega(\flat^{-1}(\sigma))$ .*

**Example 18.** *The chart  $(S^2, \mathcal{O}, \mathcal{A})$  with chart  $(\mathcal{U}, x): \phi \in (0, 2\pi), \theta \in (0, \pi)$  has metric tensor components:*

$$g_{ij}(x^{-1}(\theta, \phi)) = \begin{pmatrix} R^2 & 0 \\ 0 & R^2 \sin^2(\theta) \end{pmatrix}_{ij}, \quad R \in \mathbb{R}^+$$

*This represents the metric of the round sphere of radius  $R$ .*

### 12.2 Signature

In linear algebra, we have  $A^a v^m = \lambda v^a$  and  $g_{am} v^m = \lambda \cdot v^a$ . A  $(1,1)$ -tensor has eigenvalues, while a  $(0,2)$ -tensor has signature.

**Definition 46.** *A metric is called Riemannian if its signature is either all positive or all negative.*

**Definition 47.** *A metric is called Lorentzian (or pseudo-Riemannian) if its signature has exactly one positive or one negative eigenvalue.*

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<sup>4</sup>We choose  $\nabla$  so it is torsion free

## 12.3 Length of a Curve

Let  $\gamma$  be a smooth curve with velocity  $v_{\gamma, \gamma(\lambda)}$  at each  $\gamma(\lambda) \in M$ .

**Definition 48.** On a Riemannian manifold  $(M, \mathcal{O}, \mathcal{A}, g)$ , the speed of a curve  $\gamma(\lambda)$  is:

$$s(\lambda) = \sqrt{g(v_\gamma, v_\gamma)_{\gamma(\lambda)}}$$

**Definition 49.** Let  $\gamma: (0, 1) \rightarrow M$  be a smooth curve. The length of  $\gamma$  is:

$$L[\gamma] := \int_0^1 d\lambda s(\lambda) = \int_0^1 d\lambda \sqrt{g(v_\gamma, v_\gamma)_{\gamma(\lambda)}} \in \mathbb{R}$$

**Example 19.** Reconsider the round sphere of radius  $R$ . Consider its equator:  $\theta(\lambda) = (x' \circ \gamma)(\lambda) := \frac{\pi}{2}, \phi(\lambda) := (x^2 \circ \gamma)(\lambda) = 2\pi\lambda^3$   
then

$$\begin{aligned} L[\gamma] &= \int_0^1 d\lambda \sqrt{g_{ij}(x^{-1}(\theta(\lambda), \phi(\lambda)))(x^i \circ \gamma)'(\lambda)(x^j \circ \gamma)'(\lambda)} \\ &= \int_0^1 d\lambda \sqrt{R^2 \cdot 0 + R^2 \sin^2(\theta(\lambda)) 36\pi^2 \lambda^4} \\ &= 6\pi R \int_0^1 d\lambda \lambda^2 \\ &= 6\pi R \left[ \frac{1}{3} \lambda^3 \right]_0^1 \\ &= 2\pi R \end{aligned}$$

**Theorem 7.**  $\gamma: (0, 1) \rightarrow M$  and  $\sigma: (0, 1) \rightarrow (0, 1)$  smooth bijection and increasing then the "reparameterization" implies  $L[\gamma] = L[\gamma \circ \sigma]$ .

## 12.4 Geodesics

**Definition 50.** A curve  $\gamma: (0, 1) \rightarrow M$  is called a geodesic on a Riemannian manifold  $(M, \mathcal{O}, \mathcal{A}, g)$  if it is a stationary curve with respect to the length functional  $L$ .

**Theorem 8.**  $\gamma$  is called a geodesic iff it satisfies the Euler-Lagrange equations for the lagrangian  $\mathcal{L}: TM \rightarrow \mathbb{R}$  to  $\sqrt{g(X, X)'} in a chart the Euler-Lagrange equations take the form:$

$$\left( \frac{\partial \mathcal{L}}{\partial \dot{X}^m} \right)' - \frac{\partial \mathcal{L}}{\partial x^m} = 0$$

Here  $\mathcal{L}(\gamma^i, \dot{\gamma}^i) = \sqrt{g_{ij}(\gamma^i(\lambda)) \dot{\gamma}^i(\lambda) \dot{\gamma}^j(\lambda)}$

Euler-Lagrange equations:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \dot{\gamma}^m} &= \frac{1}{\sqrt{\dots}} g_{mj}(\gamma(\lambda)) \dot{\gamma}^j(\lambda) \\ \left( \frac{\partial \mathcal{L}}{\partial \dot{\gamma}^m} \right)' &= \left( \frac{\dots}{\sqrt{\dots}} g_{mj}(\gamma(\lambda)) \right)' \cdot \dot{\gamma}^j(\lambda) + \frac{1}{\sqrt{\dots}} (g_{mj}(\gamma(\lambda)) \ddot{\gamma}^j(\lambda) + \dot{\gamma}^s (\partial_s g_{mj}) \dot{\gamma}^j(\lambda)) \\ \frac{\partial \mathcal{L}}{\partial \gamma^m} &= \frac{1}{2} \partial_m g_{ij}(\gamma(\lambda)) \dot{\gamma}^i(\lambda) \dot{\gamma}^j(\lambda) \end{aligned}$$

Reparameterize  $g(\dot{\gamma}, \dot{\gamma}) = 1$ .

$$\begin{aligned} g_{mj} \ddot{\gamma}^j + \partial_s g_{mj} \dot{\gamma}^s \dot{\gamma}^j - \frac{1}{2} \partial_m g_{ij} \dot{\gamma}^i \dot{\gamma}^j &= 0 \quad ||(g^{-1})^{qm} \\ \ddot{\gamma}^q + (g^{-1})^{qm} \left( \partial_i g_{mj} - \frac{1}{2} \partial_m g_{ij} \right) \dot{\gamma}^i \dot{\gamma}^j &= 0 \\ \ddot{\gamma}^q + (g^{-1})^{qm} \frac{1}{2} (\partial_i g_{mj} + \partial_j g_{mi} - \partial_m g_{ij}) \dot{\gamma}^i \dot{\gamma}^j &= 0 \end{aligned}$$

Geodesic equation for  $\gamma$  in a chart.



**Definition 51.**  ${}^{L.C}\Gamma$  are the connection coefficient functions of the so called Levi-Civita connection  ${}^{L.C}\nabla$

We generally make this choice of  $\nabla$  when  $g$  is given.  $(M, \mathcal{O}, \mathcal{A}, g) \rightarrow (M, \mathcal{O}, \mathcal{A}, g, {}^{L.C}\nabla)$  abstract way:  $\nabla g = 0$  and  $T = 0$  (form)  $\implies \nabla = {}^{L.C}\nabla$

**Definition 52.** • The Riemann-Christoffel curvature is defined by:  $R_{abcd} := g_{am} R_{bcd}^m$

- Ricci  $R_{ab} = R_{amb}^m$  defined by  ${}^{L.C}\nabla$
- Ricci scalar curvature:  $R = (g^{-1})^{ab} R_{ab}$  defined by  ${}^{L.C}\nabla$ .

**Definition 53.** Einstein curvature:  $G_{ab} := R_{ab} - \frac{1}{2}g_{ab}R$   
Convention:  $g^{ab} := (g^{-1})^{ab}$

## 13 Lecture 11: Symmetry [3]

Recall your feeling that the round sphere  $(S^2, \mathcal{O}, \mathcal{A}, g^{\text{round}})$  has rotational symmetry while the potato  $(S^2, \mathcal{O}, \mathcal{A}, g^{\text{potato}})$  does not. The metric defines the symmetry. How do we describe symmetries of a metric?

### 13.1 Push-Forward Map

Let  $\phi M \rightarrow N$  both be smooth.

**Definition 54.** The push-forward  $\phi_*$  is the map  $TM \xrightarrow{\phi_*} TN$  defined by  $X \rightarrow \phi_*(X)$  where  $\phi_*(X)f := X(f \circ \phi)$  for any  $f \in C^\infty(N)$ .

Note:  $\phi_*(T_p M) \subseteq T_{\phi(p)} N$ , vectors are pushed forward. Components of  $\phi_*$  with respect to two charts  $(\mathcal{U}, x) \in \mathcal{A}_M$  and  $(\mathcal{V}, y) \in \mathcal{A}_N$ .

$$dy^a(\phi_*((\frac{\partial}{\partial x^i})_p)) := \phi_*((\frac{\partial}{\partial x^i})_p)y^a = (\frac{\partial}{\partial x^i})_p(y \circ \phi)^a = (\frac{\partial \hat{\phi}^a}{\partial x^i})_p$$

Claim:  $\phi_*(v_{\gamma,p}) = v_{\phi \circ \gamma, \phi(p)}$  particularly if  $\phi : N \rightarrow M$  is injective.

*Proof.*

$$\begin{aligned} \phi_*(v_{\gamma,p})f &= v_{\gamma,p}(f \circ \phi) = ((f \circ \phi) \circ \gamma)'(x_0) \\ &= (f \circ (\phi \circ \gamma))'(x_0) \\ &= v_{\phi \circ \gamma, \phi(p)}f \end{aligned}$$

□

### 13.2 Pull-Back

**Definition 55.** Take  $\phi : M \rightarrow N$  smooth. Define:  $\phi^* : T^*N \rightarrow T^*M : \omega \rightarrow \phi^*(\omega)$  where  $\phi^*(\omega)(X) := \omega(\phi_*(X))$ .

Components of pull-back with respect to charts.

$$\begin{aligned} \phi^{*a} &:= \phi^*((dy^a)_{\phi(p)})(\frac{\partial}{\partial x^i})_p \\ &= dy^a(\phi_*(\frac{\partial}{\partial x^i})_p) = \phi_*i^a \\ (\phi_*(X))^a &:= \phi_{*i}^a X^i \\ (\phi^*(\omega))_i &:= \phi_i^{*a} \omega_a \end{aligned}$$

Covectors are pulled back. Important application. Suppose  $\phi$  is injective. And on  $N$  lives a metric  $g/$

**Example 20.**  $N = (\mathbb{R}^3, \mathcal{O}_{st}, \mathcal{A}), M = (S^2, \mathcal{O}, \mathcal{A}) \xrightarrow{\phi} (\mathbb{R}^3, g_E)$  then the induced metric  $g(X, Y) := g(\phi_*(X), \phi_*(Y))$

### 13.3 Flow of a Complete Vector Field

Let  $(M, \mathcal{O}_M)$  be smooth. With  $X$  as a vector field on  $M$ .

**Definition 56.** A curve  $\gamma : I \rightarrow M$  is called an integral curve of  $X$  with  $v_{\gamma, \gamma(p)} = X_{\gamma(p)}$

**Definition 57.** A vector field  $X$  is called complete if all integral curves take  $I = \mathbb{R}$

**Theorem 9.** A compactly supported smooth vector field is complete.

**Definition 58.** The flow of a complete vector field  $X$  is a one-parameter family  $h^X : \mathbb{R} \times M \rightarrow M : (\lambda, p) \rightarrow \gamma_p(\lambda)$  where  $\gamma_p : \mathbb{R} \rightarrow M$  is the integral curve of  $X$  with  $\gamma(0) = p$ . Thus for fixed  $\lambda \in \mathbb{R}$  with  $h_\lambda^X$  smooth.

Thus for fixed  $\lambda \in \mathbb{R}$  we receive  $h_\lambda^X : M \rightarrow M$  smooth (a particular case of our pushforward map  $\phi$ ).

### 13.4 Lie Subalgebras of the Lie Algebra $(\Gamma(TM), [\cdot, \cdot])$ of Vector Fields

- $\Gamma(TM)$  = Set of all vector fields  $\implies [X, Y] \in \Gamma(TM)$  with  $[X, Y]F := X(Yf) - Y(Xf)$ . Must be antisymmetric and satisfy Jacobi identity. Any algebra satisfying this is a Lie Algebra.
- Let  $X_1, \dots, X_s$  be  $s$  many vector fields on  $M$  such that  $i, j \in 1 \dots s$  with  $[X_i, X_j] = C_{ij}^K X_K \in \mathbb{R}$

**Definition 59.**  $(\text{span} X_1, \dots, X_s, [\cdot, \cdot]) := L$  is a lie subalgebra.

**Example 21.** In  $S^2$  with  $[X_1, X_2] = X_3, [X_2, X_3] = X_1, [X_3, X_1] = X_2$  defines  $\mathfrak{so}(3)$  when spanned over  $\mathbb{R}$ .

### 13.5 Symmetry

**Definition 60.** An  $s$ -dim lie subalgebra is said to be a symmetry of a metric tensor field  $g$  if  $\forall X \in L$   $g((h_\lambda^X)_*)(X), (h_\lambda^X)_*(Y) = g(X, Y) \iff (h_\lambda^X)^*g = g \forall \lambda \in \mathbb{R}$ .

### 13.6 Lie Derivative

$X \in L$  if  $\mathcal{L}_X g := \lim_{\lambda \rightarrow 0} \frac{(h_\lambda^X)^*g - g}{\lambda}$ .

**Definition 61.** The Lie derivative  $\mathcal{L}$  on a smooth manifold sends a pair of a vector field  $X$  and a  $(p, q)$ -tensor fields  $T$  to as  $(p, q)$ -tensor field such that:

- $\mathcal{L}_X f = Xf$
- $\mathcal{L}_X Y = [X, Y]$
- $\mathcal{L}_X(T + S) = \mathcal{L}_X T + \mathcal{L}_X S$
- $\mathcal{L}_X(T\omega, Y) = (\mathcal{L}_X T)(\omega, Y) + T(\mathcal{L}_X \omega, Y) + T(\omega, \mathcal{L}_X Y)$
- $\mathcal{L}_{X+Y} T = \mathcal{L}_X T + \mathcal{L}_Y T$

## 14 Lecture 12: Integration [4]

Completion of our "lift" of analysis on the charts to the manifold level. We want to integrate a function  $f$  over the manifold. This requires additional structure, namely a volume form and restriction of the atlas to an orientation.

### 14.1 Review of Integration on $\mathbb{R}^d$

- $F : \mathbb{R} \rightarrow \mathbb{R}$  assume a notion of integration is known.  $\int_{(a,b)} F := \int_a^b dx F(x)$ .
- $F : \mathbb{R}^d \rightarrow \mathbb{R}$  on a box shaped domain.  $(a, b) \times (c, d) \times \dots \times (u, v) \subseteq \mathbb{R}^d$   
 $\int_{(a,b) \times \dots \times (u,v)} F := \int_a^b dx^1 \dots \int_u^v dx^d F(x^1, \dots, x^d)$ .
- Other domains  $G \subseteq \mathbb{R}^d$

**Definition 62.**  $M_G : \mathbb{R}^d \rightarrow \mathbb{R} : x \mapsto 1$  if  $x \in G, 0$  otherwise then we define  $\int_G d^d x F(x) := \int_{-\infty}^{\infty} dx^1 \dots \int_{-\infty}^{\infty} dx^d \mu_g(x) \cdot F(x)$

All of this is assuming existence.

**Theorem 10.**  $\int_G d^d x F(x) = \int_{\text{preimage}_\phi(G)} d^d y |det(\partial\phi)(y)| (F \circ \phi)(y)$

## 14.2 Integration on one Chart

Take our smooth manifold. With  $f : M \rightarrow \mathbb{R}$  choose two charts in our atlas  $\mathcal{A}$ . Consider

$$\begin{aligned} \int_{\mathcal{V}} f &:= \int_{y(\mathcal{V}) \subseteq \mathbb{R}^d} d^d y f_{(y)}(\alpha) = \int_{x(\mathcal{U})} d^d \alpha |det(\partial_a(y^b \circ x^{-1}))(\alpha)| (f_{(y)} \circ (y \circ x^{-1})) \\ &= \int_{x(\mathcal{U})} d^d \alpha |det(\frac{\partial y^b}{\partial x^a})_{x^{-1}(\alpha)}| (f \circ y^{-1} \circ y \circ x^{-1})(\alpha) \\ &\neq \int_{x(\mathcal{U})} d^d \alpha f_{(x)}(\alpha) \end{aligned}$$

Observe: An attempt to define  $\int_{\mathcal{U}} f$  this way fails.

## 14.3 Volume forms

**Definition 63.** On a smooth manifold a  $(0, \dim M)$  - tensor field  $\Omega$  is called a volume form if:

- $\Omega$  vanishes nowhere.
- Totally antisymmetric.

**Example 22.** Given a Riemannian metric manifold we can construct a volume form  $\Omega$  from  $g$  in any chart.

$$\Omega_{i_1, \dots, i_d} := \sqrt{\det(g_{(x)ij})} \epsilon_{i_1 \dots i_d}$$

So we require that our chart transition maps  $y \circ x^{-1}$  have positive determinant jacobians. Such a restricted atlas is called orientated.

**Definition 64.** Let  $\Omega$  be a volume form on a smooth manifold with an orientated atlas. Consider a chart, then  $\Omega_{(x)} := \Omega_{i_1 \dots i_d}(p) \epsilon^{i_1 \dots i_d}, p \in M$

## 14.4 Integration on one chart domain $\mathcal{U}$

**Definition 65.**

$$\int_{\mathcal{U}} f := \int_{x(\mathcal{U})} d^d \alpha \omega_{(x)}(x^{-1}(\alpha)) f_{(x)}(\alpha)$$

This is well defined under change of chart.

$$\int_{\mathcal{U}} f := \int_{x(\mathcal{U})} d^d \alpha \sqrt{\det(g_{(x),ij})(x^{-1}(\alpha))} f_{(x)}(\alpha)$$

## 14.5 Integration on the entire Manifold

Idea: Require that the manifold admit a so-called partition of unity. Roughly: For any finite subatlas of our orientated atlas  $\exists$  continuous functions  $\rho_i : \mathcal{U}_i \rightarrow \mathbb{R}$  such that

$$\forall p \in M : \sum_{p \in \mathcal{U}_i} \rho_i(p) = 1$$

**Definition 66.** Then we define the integral over the full manifold as:

$$\int_M f := \sum_{i=1}^N \int_{\mathcal{U}_i} (\rho_i \cdot f)$$

## 15 Lecture 13: Relativistic spacetime [5]

Recall from lecture 9 the definition of Newtonian spacetime  $(M, \mathcal{O}, \mathcal{A}, \nabla, t)$  where  $\nabla$  is a torsion free connection,  $t \in C^\infty(M)$ ,  $dt \neq 0$ ,  $\nabla dt = 0$ . The definition of relativistic spacetime also has  $g$  a Lorentzian metric, the atlas is orientated, and a time-orientation  $T$ .

## 15.1 Time orientation

**Definition 67.** Take a Lorentzian manifold  $(M, \mathcal{O}, \mathcal{A}^\uparrow, g)$ . Then a time-orientation is given by a vector field  $T$  that:

- Does not vanish anywhere
- $g(T, T) > 0$

This definition of spacetime has been made to enable the following postulates.

- The worldline  $\gamma$  of a massive particle satisfies:

$$\begin{aligned} - g(v_{\gamma, \gamma(\lambda)}, v_{\gamma, \gamma(\lambda)}) &> 0 \\ - g_{\gamma(\lambda)}(T, v_{\gamma, \gamma(\lambda)}) &> 0 \end{aligned}$$

- Worldline of massless particles satisfy:

$$\begin{aligned} - g(v_{\gamma, \gamma(\lambda)}, v_{\gamma, \gamma(\lambda)}) &= 0 \\ - g_{\gamma(\lambda)}(T_{\gamma(\lambda)}, v_{\gamma, \gamma(\lambda)}) &> 0 \end{aligned}$$

## 15.2 Observers

**Definition 68.** An observer in  $(M, \mathcal{O}, \mathcal{A}^\uparrow, \nabla, g, T)$  is a worldline  $\gamma$  with  $g(v_\gamma, v_\gamma) > 0, g(T, v_\gamma) > 0$  together with a choice of basis:  $e_0(\lambda), e_1(\lambda), e_2(\lambda), e_3(\lambda)$  of each  $T_{\gamma(\lambda)}M$  where the observer worldline passes if  $g(e_a(\lambda), e_b(\lambda)) = \eta_{ab} = \text{diag}(1, -1, -1, -1)$ . Observer is a smooth curve in the frame bundle  $LM$  over  $M$ .

Two additional physical postulates. A clock carried by a specific observer  $(\gamma, e)$  will measure a time:

$$\tau := \int_{\lambda_0}^{\lambda_1} d\lambda \sqrt{g_{\gamma(\lambda)}(v_{\gamma, \gamma(\lambda)}, v_{\gamma, \gamma(\lambda)})}$$

between the two "events"  $\gamma(\lambda_0)$  "start the clock" and  $\gamma(\lambda_1)$  "stop the clock".<sup>5</sup>

**Example 23.**  $M = \mathbb{R}^4, \mathcal{O} = \mathcal{O}_{st}, \mathcal{A}^\uparrow = (\mathbb{R}^4, id_{\mathbb{R}^4}), g : g_{(x), ij} = \mu_{ij}, T_{(x)}^i = (1, 0, 0, 0)^i$   
 $\implies \Gamma_{(x), jk}^o = 0$  everywhere.  $\implies$  this space-time is flat. Consider two observers  $\gamma : (0, 1) \rightarrow M, \gamma_{(x)}^i = (\lambda, 0, 0, 0)^i$  and  $\delta : (0, 1) \rightarrow M, \alpha \in (0, 1) : \delta_{(x)}^i = (\lambda, \alpha\lambda, 0, 0)^i, \lambda \leq \frac{1}{2}, (\lambda, (1 - \alpha)\lambda, 0, 0)^i, \lambda > \frac{1}{2}$ . Twin paradox. We calculate:

$$\begin{aligned} \tau_\gamma &:= \int_0^1 d\lambda \sqrt{g_{(x)ij} \dot{\gamma}_{(x)}^i \dot{\gamma}_{(x)}^j} = \int_0^1 d\lambda 1 = 1 \\ \tau_\delta &:= \int_0^{\frac{1}{2}} d\lambda \sqrt{1 - \alpha^2} + \int_{\frac{1}{2}}^1 d\lambda \sqrt{1 - (-\alpha)^2} = \int_0^1 d\lambda \sqrt{1 - \alpha^2} = \sqrt{1 - \alpha^2} \end{aligned}$$

Taking this postulate seriously we need a clock design that takes this into account. Another postulate: Let  $(\gamma, e)$  be an observer and  $\delta$  be a massive particle worldline that is parameterized such that  $g(v_\delta, v_\gamma) = 1$ . Suppose the observer and the particle meet at some space-time.  $\delta(\tau_2) = p = \gamma(\tau_1)$ . This observer measures the 4-velocity (spatial velocity) of this particle as

$$v_{\delta, \delta(\tau_2)} := \epsilon^\alpha(v_{\delta, \delta(\tau_2)})e_\alpha, \alpha = 1, 2, 3$$

Where  $\epsilon$  are the dual basis of  $e$ . Consequence: An observer  $(\gamma, e)$  will extract quantities measurable in his laboratory from objective space-time quantities always like this.

## 15.3 Role of the Lorentz Transformations

Lorentz transformations emerge as follows: Let  $(\gamma, e)$  and  $(\tilde{\delta}, \tilde{e})$  be observers with  $\gamma(0) = \tilde{\delta}(0)$  thus  $\tilde{e}_a = \Lambda_a^b e_b$  now:  $\eta_{ab} = g(\tilde{e}_a, \tilde{e}_b) = g(\Lambda_a^m e_m, \Lambda_b^n e_n) = \Lambda_a^m \Lambda_b^n g(e_m, e_n)$ . Lorentz transformers relate the frames of any two observers at the same point.

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<sup>5</sup>We call  $\tau$  the proper time.

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