

Notes From Fredric Schuller's Lectures on Relativity

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Abstract

Notes on Fredric Schuller's Relativity lectures. "Spacetime is a four-dimensional topological manifold with a smooth atlas carrying a torsion free connection compatible with a Lorentzian metric and time orientation satisfying the Einstein Equations."

Lecture 1: Topological Spaces

@ the coarsest level spacetime is a set. This is not enough to talk about continuity of maps. In classical physics there are no jumps. Sets are not enough alone to talk about continuity. We are interested in establishing the weakest possible structure on a set to talk about continuity of maps. The mathematician knows the weakest structure necessary to do this is a topology.

Definition 1. Let \mathcal{M} be a set. A topology \mathcal{O} is a subset of the powerset of \mathcal{M} denoted $\mathcal{O} \subseteq \mathcal{P}(\mathcal{M})$ satisfying three axioms:

- $\emptyset \subset \mathcal{O}, \mathcal{M} \subset \mathcal{O}$
- For arbitrary $U, V \in \mathcal{O} \implies U \cap V \in \mathcal{O}$
- For arbitrary $U_\alpha \in \mathcal{O} \implies (\bigcap_{\alpha \in A} U_\alpha) \in \mathcal{O}$ where α is an index of the set A .

Example 1. • Let $\mathcal{M} = \{1, 2, 3\}$

- Let $\mathcal{O}_1 = \{\emptyset, \{1, 2, 3\}\}$ then \mathcal{O}_1 is a topology for \mathcal{M} .
- Let $\mathcal{O}_2 = \{\emptyset, \{1\}, \{2\}, \{1, 2, 3\}\}$ then \mathcal{O}_2 is not a topology for \mathcal{M} because $\{1, 2\} \notin \mathcal{O}_2$.
- Let \mathcal{M} be any set.
 - $\mathcal{O}_{chaotic} = \{\emptyset, \mathcal{M}\}$ is a topology
 - $\mathcal{O}_{discrete} = \mathcal{P}(\mathcal{M})$ is a topology
- $\mathcal{M} = \mathbb{R}^d$ (tuples of dimensions d from \mathbb{R}) then $\mathcal{O}_{standard} \subseteq \mathcal{P}(\mathcal{M})$ is a topology for \mathcal{M} defined as follows.

Definition 2. $\mathcal{O}_{standard}$ defined in two steps.

- $B_r(p) := \{q_1, \dots, q_d \mid \sum_{i=1}^d (q_i - p)^2 < r, r \in \mathbb{R}^+, q_i \in \mathbb{R}, p \in \mathbb{R}^d$
- $\mathcal{U} \in \mathcal{O}_{standard} \iff \forall p \in \mathcal{U}, \exists r \in \mathbb{R}^+ : B_r(p) \subseteq \mathcal{U}$

We want \mathcal{M} to be spacetime and we want to equip it with an appropriate topology \mathcal{O} to be able to talk about it. We want to make the implicit assumptions of spacetime to be explicit.

Terminology 1. Let \mathcal{M} be a set defined from (ZFC) then \mathcal{O} is the topology on \mathcal{M} and it a collection of open sets. $(\mathcal{M}, \mathcal{O})$ is a topological space.

- $\mathcal{U} \in \mathcal{O} \iff \mathcal{U} \subseteq \mathcal{M}$ is an open set.
- $M/A \in \mathcal{O} \iff A \subseteq \mathcal{M}$ is closed.

$\text{not open} \not\implies \text{closed}$ and $\text{not closed} \not\implies \text{open}$

Definition 3. A map $f : M \rightarrow N$ takes all elements $m \in M$ to an element $n \in N$. M is the domain. N is the target. If $\exists m_1, m_2 \in M$ such that $f(m_1) = f(m_2) = n \in N$ then f is not injective. If $\exists n \in N$ such that $\forall m \in M : f(m) \neq n$ then f is not surjective. Is a map f continuous? Depends by definition on topologies \mathcal{O}_M on M and \mathcal{O}_N on N .

Definition 4. A map f is called continuous between (M, \mathcal{O}_M) and (N, \mathcal{O}_N) . Then a map f is called continuous with respect to these topologies if for every open $V \in \mathcal{O}_N$ the preimage of V is open in \mathcal{O}_M . $\forall V \in \mathcal{O}_N : \text{preim}_f(V) \in \mathcal{O}_M$, $\text{preim}_f(V) := \{m \in M \mid f(m) \in V\}$.

Example 2. $M = 1, 2, \mathcal{O}_M = \emptyset, 1, 2, 1, 2$

$N = 1, 2, \mathcal{O}_N = \emptyset, 1, 2$

$f : M \rightarrow N \mid f(1) = 2, f(2) = 1$

Is f continuous?

$\text{preim}_f(\emptyset) = \emptyset \in \mathcal{O}_M$

$\text{preim}_f(1, 2) = M \in \mathcal{O}_M$

Therefore f is continuous.

Example 3. $g : N \rightarrow M$ or f^{-1} then $\text{preim}_g(1) = 2 \notin \mathcal{O}_N$ so g is not continuous.

Composition of Maps

$M \xrightarrow{f} N \xrightarrow{g} P$ so $g \circ f : M \rightarrow P$ by $m \rightarrow (g \circ f)(m) := g(f(m))$.

Theorem 1. Composition of continuous maps is continuous.

Proof. Let $V \in \mathcal{O}_P$ then

$$\begin{aligned} \text{preim}_{g \circ f}(V) &:= m \in M \mid (g \circ f)(m) \in V \\ &= m \in M \mid f(m) \in \text{preim}_g(V) \\ &= \text{preim}_f(\text{preim}_g(V) \in \mathcal{O}_N) \in \mathcal{O}_M \end{aligned}$$

□

Inheritance of a Topology

Many useful ways to inherit a topology from another topological space or set of topological spaces. Of particular importance for spacetime physics is $S \subseteq M$ where M has topology \mathcal{O}_M . Can we construct a topology \mathcal{O}_S from \mathcal{O}_M . Yes.

Definition 5. $\mathcal{O}|_S \subseteq \mathcal{P}(S)$

$\mathcal{O}|_S := \mathcal{U} \cap S \mid \mathcal{U} \in \mathcal{O}_M$

This is a topology called the subset topology.

Use of this specific way to inherit a topology from a super set. Let it be easy to say a map f is continuous. Then a subset S of a set M with a inherited topology then the restriction of the map $f|_S : S \rightarrow N$ it can be easy to show this restriction is continuous.

Lecture 2: Topological Manifolds

\exists too many topological spaces to classify. Too many topological spaces exist which have no known connection to the study of spacetime. For spacetime physics we may focus on topological spaces $(\mathcal{M}, \mathcal{O})$ which can be charted analogously to how the surface of the Earth is charted in an atlas.

Topological Manifolds

Definition 6. A topological space $(\mathcal{M}, \mathcal{O})$ is called a d -dimensional topological manifold if $\forall p \in \mathcal{M} : \exists \mathcal{U} \in \mathcal{O} : \exists x : \mathcal{U} \rightarrow x(\mathcal{U}) \subseteq \mathbb{R}^d$ with \mathbb{R}^d equipped with $\mathcal{O}_{\text{standard}}$ such that:

- x is invertible: $x^{-1} : x(\mathcal{U}) \rightarrow \mathcal{U}$

- x is continuous.
- x^{-1} is continuous.

Example 4.

The surface of a torus $:= \mathcal{M} \subseteq \mathbb{R}^3$. This is a $d = 2$ dimensional topological manifold.

A mobius strip $:= \mathcal{M} \subseteq (\mathbb{R}^3, \mathcal{O}_{st})$ then this is a $d = 1$ manifold.

A bifurcating line $:= \mathcal{M} \subset (\mathbb{R}^2, \mathcal{O}_{st})$ then $(\mathcal{M}, \mathcal{O}_{st}|_{\mathcal{M}})$ is a topology but the point of bifurcation is not invertible so it is not a manifold.

Terminology 2. 1. A pair (\mathcal{U}, x) is called a chart.

2. A set $\mathcal{A} = (\mathcal{U}_\alpha, x_\alpha) | \alpha \in A$ is called an atlas of the topological manifold \mathcal{M} if $\mathcal{M} = \bigcup_{\alpha \in A} \mathcal{U}_\alpha$.

3. $x : \mathcal{U} \rightarrow x(\mathcal{U}) \subseteq \mathbb{R}^d$ is called the chart map.

4. $x^i \mathcal{U} \rightarrow \mathbb{R}$ is called the coordinate maps.

5. $p \in \mathcal{U}$ then x^i is the i -th coordinate of the point p with respect to the chosen chart (\mathcal{U}, x) .

Example 5. •

$$\begin{aligned}\mathcal{M} &= \mathbb{R}^2 \\ \mathcal{U} &= \mathbb{R}^2 / 0, 0 \\ x : \mathcal{U} &\rightarrow \mathbb{R}^2 : x(m, n) \rightarrow (-m, -n)\end{aligned}$$

- We can take another chart map on \mathcal{U}
 $(m, n) \rightarrow (\sqrt{m^2 + n^2}, \arctan(\frac{n}{m}))$

Chart Transition Maps

Imagine two charts (\mathcal{U}, x) and (\mathcal{V}, y) with $\mathcal{U} \cap \mathcal{V} = \mathcal{A} \neq \emptyset$. The same point $p \in \mathcal{A}$ can be mapped via x and y to two different charts. Because these maps are continuously invertible we can smoothly transition between pages of our atlas: $(y \circ x^{-1})(p) = y(x^{-1}(p))$. This map is called the chart transition map. Informally the chart transition map contains the instructions how to *glue* together the charts of our atlas.

Manifold Philosophy

Often it is desirable (or indeed the way) to define properties ("continuity") of real-world objects ($\mathbb{R} \xrightarrow{\gamma} \mathcal{M}$) by judging suitable condition not on the real-world object itself but on the chart-representation of the real-world object. Advantages: You can define continuity in this way. Disadvantages: The chart map x is a 'fantasy,' x may be ill defined because the chart chosen is arbitrary. Solution: The property must be maintained regardless of chart.

$$\begin{aligned}\mathbb{R} &\xrightarrow{y \circ \gamma} y(\mathcal{U}) \\ \mathbb{R} &\xrightarrow{x \circ \gamma} x(\mathcal{U}) \\ \mathbb{R} &\xrightarrow{\gamma} \mathcal{U}\end{aligned}$$

We are interested in properties of γ but must talk about it with respect to x and y .

Lecture 3: Multilinear Algebra

We will not equip space(time) with a vector space structure. There is no such thing as five times Paris or Paris plus Vienna. However, the tangent spaces of $T_p \mathcal{M}$ of smooth manifolds carry a vector space structure. It is beneficial to first study vector spaces abstractly for two reasons.

- For construction of $T_p \mathcal{M}$ one needs an intermediate vector space $C^\infty(\mathcal{M})$.
- Tensor techniques are best understood in an abstract setting.

Vecotr Spaces

Definition 7. A vector space $(V, +, \cdot)$ is

- a set V
- $+: V \times V \rightarrow V$
- $\cdot: \mathbb{R} \times V \rightarrow V$

Which satisfies CANI and ADDU axioms. For $w, v, u \in V$ and $\lambda, \mu \in \mathbb{R}$

- $C^+: v + w = w + v$
- $A^+: (u + v) + w = u + (v + w)$
- $N^+: \exists 0 \in V : \forall v \in V : v + 0 = v$
- $I^+: \forall v \in V : \exists (-v) \in V : v + (-v) = 0$
- $A: \lambda \cdot (\mu \cdot v) = (\lambda \cdot \mu) \cdot v$
- $D: (\lambda + \mu) \cdot v = \lambda \cdot v + \mu \cdot v$
- $D: \lambda \cdot v + \lambda \cdot w = \lambda \cdot (v + w)$
- $U: 1 \cdot v = v$

Terminology 3. An element of a vector space is often referred to (informally) as a vector.

Example 6.

Definition 8. Set $P := p: (-1, +1) \rightarrow \mathbb{R}$ of polynomials of (fixed) degree. $p(x) = \sum_{n=1}^N p_n x^n$. Is $\square(x) = x^2$ a vector? No. $\square \in P$.

Definition 9. Define the operations in the space:

$$+: P \times P \rightarrow P : (p, q) \mapsto p + q$$

$$\cdot: \mathbb{R} \times P \rightarrow P : (\lambda, p) \mapsto \lambda \cdot p$$

Is $\square \in (P, +, \cdot)$ a vector in a vector space? Yes!

Linear Maps

We want to study maps which preserve (vector space) structure. On vector spaces these are called linear maps.

Definition 10. Let $(V, +, \cdot)$ and $(W, +, \cdot)$ be vector spaces. Then a map $\phi: V \rightarrow W$ is called linear if:

- $\phi(v + \tilde{v}) = \phi(v) + \phi(\tilde{v})$
- $\phi(\lambda \cdot v) = \lambda \cdot \phi(v)$

Example 7. Take P as before. Take $\delta: P \rightarrow P : \delta(p) = p'$. $\delta(p, q) = (p + q)' = \delta(p) + \delta(q)$ and $\delta(\lambda p) = (\lambda p)' = \lambda p'$

Notation: $\phi: V \rightarrow W$ linear $\iff \phi: V \xrightarrow{\sim} W$

Vector space of Homomorphisms

Take $(V, +, \cdot)$ and $(W, +, \cdot)$ vector spaces.

Definition 11. $\text{Hom}(V, W) := \{ \phi: V \xrightarrow{\sim} W \}$ as a set. This is a vectorspace:

- $\oplus: \text{Hom}(V, W) \times \text{Hom}(V, W) \rightarrow \text{Hom}(V, W) : (\phi, \psi) \mapsto \phi \oplus \psi$ where $(\phi \oplus \psi)(v) := \phi(v) + \psi(v)$
- \otimes defined dimilarly.

$(\text{Hom}(V, W), \oplus, \otimes)$ is a vector space.

Example 8. Take P as before. Then δ is a vector space similarly.

Dual Vector Space

$(V, +, \cdot)$ as a vector space.

Definition 12. $V^* := \psi : V \xrightarrow{\sim} \mathbb{R} = \text{Hom}(V, \mathbb{R})$ is also a vector space called the dual vector space of V .

Terminology 4. $\phi \in V^*$ is called (informally) a covector.

Example 9. $I : P \xrightarrow{\sim} \mathbb{R}$ so $I \in P^*$

Definition 13. $I(p) := \int_0^1 dx p(x)$

I is clearly linear.

Tensors

Definition 14. Take a vector space V . Then a (r, s) tensor T over V is a multilinear map: $T : \bigotimes_{i=1}^r V^* \otimes \bigotimes_{j=1}^s V \xrightarrow{\sim(r+s)} \mathbb{R}$

Example 10. $T(1, 1)$ - tensor. $T(\phi + \psi, v) = T(\phi, v) + T(\psi, v), T(\lambda\psi, v) = \lambda \cdot T(\psi, v)$
 $T(\psi, v + w) = T(\psi, v) + T(\psi, w), T(\psi, \lambda \cdot v) = \lambda \cdot T(\psi, v)$ Linear in both entries. Hence, multi-linear.