

Notes From Fredric Schuller's Lectures on Relativity

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July 12, 2025

Abstract

Notes on Fredric Schuller's Relativity lectures. "Spacetime is a four-dimensional topological manifold with a smooth atlas carrying a torsion free connection compatible with a Lorentzian metric and time orientation satisfying the Einstein Equations."

Lecture 1: Topological Spaces

@ the coarsest level spacetime is a set. This is not enough to talk about continuity of maps. In classical physics there are no jumps. Sets are not enough alone to talk about continuity. We are interested in establishing the weakest possible structure on a set to talk about continuity of maps. The mathematician knows the weakest structure necessary to do this is a topology.

Definition 1. Let \mathcal{M} be a set. A topology \mathcal{O} is a subset of the powerset of \mathcal{M} denoted $\mathcal{O} \subseteq \mathcal{P}(\mathcal{M})$ satisfying three axioms:

- $\emptyset \subset \mathcal{O}, \mathcal{M} \subset \mathcal{O}$
- For arbitrary $U, V \in \mathcal{O} \implies U \cap V \in \mathcal{O}$
- For arbitrary $U_\alpha \in \mathcal{O} \implies (\bigcap_{\alpha \in A} U_\alpha) \in \mathcal{O}$ where α is an index of the set A .

Example 1. • Let $\mathcal{M} = \{1, 2, 3\}$

- Let $\mathcal{O}_1 = \{\emptyset, \{1, 2, 3\}\}$ then \mathcal{O}_1 is a topology for \mathcal{M} .
- Let $\mathcal{O}_2 = \{\emptyset, \{1\}, \{2\}, \{1, 2, 3\}\}$ then \mathcal{O}_2 is not a topology for \mathcal{M} because $\{1, 2\} \notin \mathcal{O}_2$.
- Let \mathcal{M} be any set.
 - $\mathcal{O}_{chaotic} = \{\emptyset, \mathcal{M}\}$ is a topology
 - $\mathcal{O}_{discrete} = \mathcal{P}(\mathcal{M})$ is a topology
- $\mathcal{M} = \mathbb{R}^d$ (tuples of dimensions d from \mathbb{R}) then $\mathcal{O}_{standard} \subseteq \mathcal{P}(\mathcal{M})$ is a topology for \mathcal{M} defined as follows.

Definition 2. $\mathcal{O}_{standard}$ defined in two steps.

- $B_r(p) := \{q_1, \dots, q_d \mid \sum_{i=1}^d (q_i - p)^2 < r, r \in \mathbb{R}^+, q_i \in \mathbb{R}, p \in \mathbb{R}^d$
- $\mathcal{U} \in \mathcal{O}_{standard} \iff \forall p \in \mathcal{U}, \exists r \in \mathbb{R}^+ : B_r(p) \subseteq \mathcal{U}$

We want \mathcal{M} to be spacetime and we want to equip it with an appropriate topology \mathcal{O} to be able to talk about it. We want to make the implicit assumptions of spacetime to be explicit.

Terminology 1. Let \mathcal{M} be a set defined from (ZFC) then \mathcal{O} is the topology on \mathcal{M} and it a collection of open sets. $(\mathcal{M}, \mathcal{O})$ is a topological space.

- $\mathcal{U} \in \mathcal{O} \iff \mathcal{U} \subseteq \mathcal{M}$ is an open set.
- $M/A \in \mathcal{O} \iff A \subseteq \mathcal{M}$ is closed.

$\text{not open} \not\implies \text{closed}$ and $\text{not closed} \not\implies \text{open}$

Definition 3. A map $f : M \rightarrow N$ takes all elements $m \in M$ to an element $n \in N$. M is the domain. N is the target. If $\exists m_1, m_2 \in M$ such that $f(m_1) = f(m_2) = n \in N$ then f is *injective*. If $\exists n \in N$ such that $\forall m \in M : f(m) \neq n$ then f is not surjective. Is a map f continuous? Depends by definition on topologies \mathcal{O}_M on M and \mathcal{O}_N on N .

Definition 4. A map f is called continuous between (M, \mathcal{O}_M) and (N, \mathcal{O}_N) . Then a map f is called continuous with respect to these topologies if for every open $V \in \mathcal{O}_N$ the preimage of V is open in \mathcal{O}_M . $\forall V \in \mathcal{O}_N : \text{preim}_f(V) \in \mathcal{O}_M$, $\text{preim}_f(V) := \{m \in M \mid f(m) \in V\}$.

Example 2. $M = 1, 2, \mathcal{O}_M = \emptyset, 1, 2, 1, 2$

$N = 1, 2, \mathcal{O}_N = \emptyset, 1, 2$

$f : M \rightarrow N \mid f(1) = 2, f(2) = 1$

Is f continuous?

$\text{preim}_f(\emptyset) = \emptyset \in \mathcal{O}_M$

$\text{preim}_f(1, 2) = M \in \mathcal{O}_M$

Therefore f is continuous.

Example 3. $g : N \rightarrow M$ or f^{-1} then $\text{preim}_g(1) = 2 \notin \mathcal{O}_N$ so g is not continuous.

Composition of Maps

$M \xrightarrow{f} N \xrightarrow{g} P$ so $g \circ f : M \rightarrow P$ by $m \rightarrow (g \circ f)(m) := g(f(m))$.

Theorem 1. Composition of continuous maps is continuous.

Proof. Let $V \in \mathcal{O}_P$ then

$$\begin{aligned} \text{preim}_{g \circ f}(V) &:= m \in M \mid (g \circ f)(m) \in V \\ &= m \in M \mid f(m) \in \text{preim}_g(V) \\ &= \text{preim}_f(\text{preim}_g(V) \in \mathcal{O}_N) \in \mathcal{O}_M \end{aligned}$$

□

Inheritance of a Topology

Many useful ways to inherit a topology from another topological space or set of topological spaces. Of particular importance for spacetime physics is $S \subseteq M$ where M has topology \mathcal{O}_M . Can we construct a topology \mathcal{O}_S from \mathcal{O}_M . Yes.

Definition 5. $\mathcal{O}|_S \subseteq \mathcal{P}(S)$

$\mathcal{O}|_S := \mathcal{U} \cap S \mid \mathcal{U} \in \mathcal{O}_M$

This is a topology called the subset topology.

Use of this specific way to inherit a topology from a super set. Let it be easy to say a map f is continuous. Then a subset S of a set M with a inherited topology then the restriction of the map $f|_S : S \rightarrow N$ it can be easy to show this restriction is continuous.

Lecture 2: Topological Manifolds

\exists too many topological spaces to classify. Too many topological spaces exist which have no known connection to the study of spacetime. For spacetime physics we may focus on topological spaces $(\mathcal{M}, \mathcal{O})$ which can be charted analogously to how the surface of the Earth is charted in an atlas.

Topological Manifolds

Definition 6. A topological space $(\mathcal{M}, \mathcal{O})$ is called a d -dimensional topological manifold if $\forall p \in \mathcal{M} : \exists p \in \mathcal{U} \in \mathcal{O} : \exists x : \mathcal{U} \rightarrow x(\mathcal{U}) \subseteq \mathbb{R}^d$ with \mathbb{R}^d equipped with $\mathcal{O}_{\text{standard}}$ such that:

- x is invertible: $x^{-1} : x(\mathcal{U}) \rightarrow \mathcal{U}$

- x is continuous.
- x^{-1} is continuous.

Example 4.

The surface of a torus $:= \mathcal{M} \subseteq \mathbb{R}^3$. This is a $d = 2$ dimensional topological manifold.

A mobius strip $:= \mathcal{M} \subseteq (\mathbb{R}^3, \mathcal{O}_{st})$ then this is a $d = 1$ manifold.

A bifurcating line $:= \mathcal{M} \subset (\mathbb{R}^2, \mathcal{O}_{st})$ then $(\mathcal{M}, \mathcal{O}_{st}|_{\mathcal{M}})$ is a topology but the point of bifurcation is not invertible so it is not a manifold.

Terminology 2. 1. A pair (\mathcal{U}, x) is called a chart.

2. A set $\mathcal{A} = (\mathcal{U}_\alpha, x_\alpha) | \alpha \in A$ is called an atlas of the topological manifold \mathcal{M} if $\mathcal{M} = \bigcup_{\alpha \in A} \mathcal{U}_\alpha$.

3. $x : \mathcal{U} \rightarrow x(\mathcal{U}) \subseteq \mathbb{R}^d$ is called the chart map.

4. $x^i \mathcal{U} \rightarrow \mathbb{R}$ is called the coordinate maps.

5. $p \in \mathcal{U}$ then x^i is the i -th coordinate of the point p with respect to the chosen chart (\mathcal{U}, x) .

Example 5. •

$$\begin{aligned}\mathcal{M} &= \mathbb{R}^2 \\ \mathcal{U} &= \mathbb{R}^2 / 0, 0 \\ x : \mathcal{U} &\rightarrow \mathbb{R}^2 : x(m, n) \rightarrow (-m, -n)\end{aligned}$$

- We can take another chart map on \mathcal{U}
 $(m, n) \rightarrow (\sqrt{m^2 + n^2}, \arctan(\frac{n}{m}))$

Chart Transition Maps

Imagine two charts (\mathcal{U}, x) and (\mathcal{V}, y) with $\mathcal{U} \cap \mathcal{V} = \mathcal{A} \neq \emptyset$. The same point $p \in \mathcal{A}$ can be mapped via x and y to two different charts. Because these maps are continuously invertible we can smoothly transition between pages of our atlas: $(y \circ x^{-1})(p) = y(x^{-1}(p))$. This map is called the chart transition map. Informally the chart transition map contains the instructions how to *glue* together the charts of our atlas.

Manifold Philosophy

Often it is desirable (or indeed the way) to define properties ("continuity") of real-world objects ($\mathbb{R} \xrightarrow{\gamma} \mathcal{M}$) by judging suitable condition not on the real-world object itself but on the chart-representation of the real-world object. Advantages: You can define continuity in this way. Disadvantages: The chart map x is a 'fantasy,' x may be ill defined because the chart chosen is arbitrary. Solution: The property must be maintained regardless of chart.

$$\begin{aligned}\mathbb{R} &\xrightarrow{y \circ \gamma} y(\mathcal{U}) \\ \mathbb{R} &\xrightarrow{x \circ \gamma} x(\mathcal{U}) \\ \mathbb{R} &\xrightarrow{\gamma} \mathcal{U}\end{aligned}$$

We are interested in properties of γ but must talk about it with respect to x and y .

Lecture 3: Multilinear Algebra

We will not equip space(time) with a vector space structure. There is no such thing as five times Paris or Paris plus Vienna. However, the tangent spaces of $T_p \mathcal{M}$ of smooth manifolds carry a vector space structure. It is beneficial to first study vector spaces abstractly for two reasons.

- For construction of $T_p \mathcal{M}$ one needs an intermediate vector space $C^\infty(\mathcal{M})$.
- Tensor techniques are best understood in an abstract setting.

Vecotr Spaces

Definition 7. A vector space $(V, +, \cdot)$ is

- a set V
- $+: V \times V \rightarrow V$
- $\cdot: \mathbb{R} \times V \rightarrow V$

Which satisfies CANI and ADDU axioms. For $w, v, u \in V$ and $\lambda, \mu \in \mathbb{R}$

- $C^+ : v + w = w + v$
- $A^+ : (u + v) + w = u + (v + w)$
- $N^+ : \exists 0 \in V : \forall v \in V : v + 0 = v$
- $I^+ : \forall v \in V : \exists (-v) \in V : v + (-v) = 0$
- $A : \lambda \cdot (\mu \cdot v) = (\lambda \cdot \mu) \cdot v$
- $D : (\lambda + \mu) \cdot v = \lambda \cdot v + \mu \cdot v$
- $D : \lambda \cdot v + \lambda \cdot w = \lambda \cdot (v + w)$
- $U : 1 \cdot v = v$

Terminology 3. An element of a vector space is often referred to (informally) as a vector.

Example 6.

Definition 8. Set $P := p : (-1, +1) \rightarrow \mathbb{R}$ of polynomials of (fixed) degree. $p(x) = \sum_{n=1}^N p_n x^n$. Is $\square(x) = x^2$ a vector? No. $\square \in P$.

Definition 9. Define the operations in the space:

$$+ : P \times P \rightarrow P : (p, q) \mapsto p + q$$

$$\cdot : \mathbb{R} \times P \rightarrow P : (\lambda, p) \mapsto \lambda \cdot p$$

Is $\square \in (P, +, \cdot)$ a vector in a vector space? Yes!

Linear Maps

We want to study maps which preserve (vector space) structure. On vector spaces these are called linear maps.

Definition 10. Let $(V, +, \cdot)$ and $(W, +, \cdot)$ be vector spaces. Then a map $\phi : V \rightarrow W$ is called linear if:

- $\phi(v + \tilde{v}) = \phi(v) + \phi(\tilde{v})$
- $\phi(\lambda \cdot v) = \lambda \cdot \phi(v)$

Example 7. Take P as before. Take $\delta : P \rightarrow P : \delta(p) = p'$. $\delta(p, q) = (p + q)' = \delta(p) + \delta(q)$ and $\delta(\lambda p) = (\lambda p)' = \lambda p'$

Notation: $\phi : V \rightarrow W$ linear $\iff \phi : V \xrightarrow{\sim} W$

Vector space of Homomorphisms

Take $(V, +, \cdot)$ and $(W, +, \cdot)$ vector spaces.

Definition 11. $\text{Hom}(V, W) := \phi : V \xrightarrow{\sim} W$ as a set. This is a vectorspace:

- $\oplus : \text{Hom}(V, W) \times \text{Hom}(V, W) \rightarrow \text{Hom}(V, W) : (\phi, \psi) \mapsto \phi \oplus \psi$ where $(\phi \oplus \psi)(v) := \phi(v) + \psi(v)$
- \otimes defined similarly.

$(\text{Hom}(V, W), \oplus, \otimes)$ is a vector space.

Example 8. Take P as before. Then δ is a vector space similarly.

Dual Vector Space

$(V, +, \cdot)$ as a vector space.

Definition 12. $V^* := \psi : V \xrightarrow{\sim} \mathbb{R} = \text{Hom}(V, \mathbb{R})$ is also a vector space called the dual vector space of V .

Terminology 4. $\phi \in V^*$ is called (informally) a covector.

Example 9. $I : P \xrightarrow{\sim} \mathbb{R}$ so $I \in P^*$

Definition 13. $I(p) := \int_0^1 dx p(x)$

I is clearly linear.

Tensors

Definition 14. Take a vector space V . Then a (r, s) tensor T over V is a multilinear map: $T : \bigotimes_{i=1}^r V^* \otimes \bigotimes_{j=1}^s V \xrightarrow{\sim(r+s)} \mathbb{R}$

Example 10. $T(1, 1)$ - tensor. $T(\phi + \psi, v) = T(\phi, v) + T(\psi, v)$, $T(\lambda\psi, v) = \lambda \cdot T(\psi, v)$, $T(\psi, v + w) = T(\psi, v) + T(\psi, w)$, $T(\psi, \lambda \cdot v) = \lambda \cdot T(\psi, v)$ Linear in both entries. Hence, multi-linear.

Example 11. $g : P \times P \xrightarrow{\sim} \mathbb{R}$, $(p, q) \rightarrow \int_0^1 dx p(x)q(x)$ is a $(0, 2)$ - tensor over example of P .

Vectors and Covectors

Theorem 2. $\phi \in V^* \iff \phi : V \xrightarrow{\sim} \mathbb{R} \iff \phi(0, 1)$ - tensor.

Theorem 3. $\dim(V) < \infty \implies v \in V = (V^*)^* \iff v : V^* \xrightarrow{\sim} \mathbb{R} \iff v$ is a $(1, 0)$ - tensor.

Bases

Definition 15. Take a vector space V . A subset $B \subset V$ is called a basis if $\forall v \in V \exists F = \{f_1, \dots, f_n\} \in B : \exists ! \{v^1, \dots, v^n\} \in \mathbb{R} : v = v^1 f_1 + \dots + v^n f_n$.

Definition 16. If \exists basis B with finite many elements (d many elements) then we call $d =: \dim(V)$.

Remark 1. Let V be finite dimensional vector space. Choose a basis e_1, \dots, e_n of V . We may uniquely associate a vector $v \in V$ with $v \mapsto (v^1, \dots, v^n)$ called the components of $v = v^1 e_1 + \dots + v^n e_n$ with respect to the chosen basis. It is more economical to require your basis on V once chosen such that $\epsilon^a(e_b) = \delta_b^a$. This uniquely determines choice of vector components from a choice of basis.

Definition 17. If a basis $\epsilon^1, \dots, \epsilon^n$ of V^* satisfies these axioms it is called the dual basis.

Example 12. Take P with $(N = 3)$. Then e_0, \dots, e_3 is a basis if $e_a(x) := x^a$ is a basis of P . Then dual basis is given by $\epsilon^a := \frac{1}{a!} \partial^a|_{x=0}$.

Components of Tensors

Definition 18. Let T be an (r, s) - tensor over a finite dimensional vectors space V . Then define the $(r + s)^{\dim(V)}$ many real numbers $i_1, \dots, i_r, j_1, \dots, j_s \in \{1, \dots, \dim(V)\}$, $T_j^i \in \mathbb{R} := T(\epsilon^i, e_j)$. The T_j^i elements are called the components of the tensor with respect to the chosen basis. Knowing the components and basis one can reconstruct the entire tensor.

Example 13. $T(1, 1)$ - tensor with $T_j^i = T(\epsilon^i, e_j)$ reconstruct

$$\begin{aligned} T(\phi, v) &= T\left(\sum_{i=1}^{\dim(V)} \phi_i \epsilon^i, \sum_{j=1}^{\dim(V)} v^j e_j\right) \\ &= \sum_{i=1}^{\dim(V)} \sum_{j=1}^{\dim(V)} \phi_i v^j T(\epsilon^i, e_j) \end{aligned}$$

with $\phi_i, v^j \in \mathbb{R}$.

Terminology 5. If we agree to label $T_j^i = T(\epsilon^i, e_j)$ with the up and down components then Einstein summation convention of tensors $:= \phi_i v^j T_j^i$ where we drop \sum . This only works over multilinear maps.

Lecture 4: Differentiable Manifolds

So far we have topological manifolds $(\mathcal{M}, \mathcal{O})$ $\dim(\mathcal{M})=d$. We want to talk about velocity vectors on them but this structure is insufficient to do so. Pick a dimension $d \neq 4$ for your manifold. In that dimension the choices of topology are countable, so presumably we can do experiments to discern which one corresponds to our reality. In $d = 4$ our choices of topology are suddenly uncountable. We need additional structure to talk about differentiable curves on manifolds and between manifolds.

1. Curves: $\mathbb{R} \rightarrow \mathcal{M}$
2. Functions: $\mathcal{M} \rightarrow \mathbb{R}$
3. Maps: $\mathcal{M}_1 \rightarrow \mathcal{M}_2$

Strategy

$\gamma : \mathbb{R} \rightarrow \mathcal{U}$ and choose a chart (\mathcal{U}, x) . $\mathcal{U} \xrightarrow{x} x(\mathcal{U}) \subseteq \mathbb{R}^d$. Try and 'lift' the undergraduate notion of differentiable of a curve in \mathbb{R}^d to a notion of differentiability on a curve on \mathcal{M} . Problem: Is this well defined under a change of chart? We don't want it to depend on our taste.

$$\gamma : \mathbb{R} \rightarrow \mathcal{U} \cap \mathcal{V} \neq \emptyset \quad (1)$$

$$y \circ \gamma : \mathbb{R} \rightarrow y(\mathcal{U} \cap \mathcal{V}) \subseteq \mathbb{R}^d \quad (2)$$

$$x \circ \gamma : \mathbb{R} \rightarrow x(\mathcal{U} \cap \mathcal{V}) \subseteq \mathbb{R}^d \quad (3)$$

$$y \circ \gamma = (y \circ x^{-1}) \circ (x \circ \gamma) = y \circ (x^{-1} \circ x) \circ \gamma \quad (4)$$

If $(x \circ \gamma)$ differentiable and $(y \circ x^{-1})$ continuous is $y \circ \gamma$ guaranteed to be continuously differentiable? No.

Compatible Charts

In previous we took a chart on the topological manifold. Our atlases \mathcal{V} and \mathcal{U} were elements of the maximal atlas of the manifold.

Definition 19. Two charts (\mathcal{U}, x) and (\mathcal{V}, y) are called \square compatible if either:

1. $\mathcal{U} \cap \mathcal{V} = \emptyset$
2. $y \circ x^{-1} : x(\mathcal{U} \cap \mathcal{V}) \rightarrow y(\mathcal{U} \cap \mathcal{V})$ and $x \circ y^{-1} : y(\mathcal{U} \cap \mathcal{V}) \rightarrow x(\mathcal{U} \cap \mathcal{V})$ are differentiable.

Philosophy of \square :

Definition 20. An atlas \mathcal{A}_\square is a \square - compatible atlas if any two charts in \mathcal{A}_\square are \square - compatible.

Definition 21. A \square - manifold is a triple $(\mathcal{M}, \mathcal{O}, \mathcal{A}_\square)$

Undergraduate \square . C^0 : $C^0(\mathbb{R}^d \rightarrow \mathbb{R}^d)$ Continuous maps. C^1 : $C^1(\mathbb{R}^d \rightarrow \mathbb{R}^d)$ Differentiable Once. C^k : k -times continuous differentiable. D^k : k -times differentiable. C^∞ : $C^\infty(\mathbb{R}^d \rightarrow \mathbb{R}^d)$ infinitely continuously differentiable. C^ω : \exists multidimensional Taylor Expansion. \mathbb{C}^∞ : Satisfies Cauchy-Riemann Equations.

Theorem 4. Any $\mathbb{C}^{k \geq 1}$ - atlas \mathcal{A} of a topological manifold contains a C^∞ - atlas.

Thus we will consider C^∞ manifolds or "smoothmanifolds" unless we wish to define Taylor expansions or complex differentiable.

Definition 22. A smooth manifold is a triple $(\mathcal{M}, \mathcal{O}, \mathcal{A})$ where \mathcal{A} is a C^∞ - atlas.

Diffeomorphisms

Take a map: $\mathcal{M} \xrightarrow{\phi} \mathcal{N}$. If \mathcal{M} and \mathcal{N} are naked sets then the structure preserving maps are the bijections. If $\exists \phi$ is a bijection $\mathcal{M} \cong \mathcal{N}$.

Example 14.

$$\mathbb{N} \cong \mathbb{Z}$$

$$\mathbb{N} \cong \mathbb{Q}$$

$$\mathbb{N} \not\cong \mathbb{R}$$

Linear bijections between vector spaces are the structure preserving maps between vector spaces.

Definition 23. Two C^∞ manifolds are said to be diffeomorphic $\iff \exists$ bijection $\phi : \mathcal{M} \rightarrow \mathcal{N}$ such that ϕ, ϕ^{-1} are both C^∞ maps.

Theorem 5. The number of C^∞ manifolds that can be constructed from a given C^0 - manifold up to diffeomorphism by the More-Radon Theorems in $d = 1, 2, 3$ dimension are 1. In $d > 4$ the number is finite. In $d = 4 \exists$ uncountably infinitely such manifolds.

Lecture 5: Tangent Spaces

Lead question: "What is the velocity of a curve γ at a point p ?"

Velocities

Definition 24. Take a $(\mathcal{M}, \mathcal{O}, \mathcal{A})$ smooth manifold. Take a curve: $\gamma : \mathbb{R} \rightarrow \mathcal{M}$ of at least C^1 . Suppose $\gamma(\lambda_0) = p$. The velocity of γ at p is the linear map: $v_{\gamma,p} : C^\infty(\mathcal{M}) \xrightarrow{\sim} \mathbb{R}$. $C^\infty(\mathcal{M}) := \{f : \mathcal{M} \rightarrow \mathbb{R} | f \text{ smooth function} \}$ equipped with $(f \oplus g)(p) := f(p) +_{\mathbb{R}} g(p)$, $(\lambda \oplus g)(p) := \lambda \cdot g(p)$. So velocity is a linear map: $f \mapsto v_{\gamma,p}(f) := (f \circ \gamma)'(\lambda_0)$.

Tangent Vector Space

Definition 25. For each point $p \in \mathcal{M}$ we define the set "tangent space to \mathcal{M} at p " as $T_p\mathcal{M} := \{v_{\gamma,p} | \gamma \text{ smooth curve} \}$.

Observe that $T_p\mathcal{M}$ can be made into a vector space.

Definition 26. We need to define \oplus and \otimes .

$$\begin{aligned} \oplus : T_p\mathcal{M} \times T_p\mathcal{M} &\rightarrow \text{Hom}(C^\infty(\mathcal{M}), \mathbb{R}) \\ (v_{\gamma,p} \oplus v_{\delta,p})(f) &:= v_{\gamma,p}(f) +_{\mathbb{R}} v_{\delta,p}(f), f \in C^\infty(\mathcal{M}) \end{aligned}$$

$$\begin{aligned} \otimes : \mathbb{R} \times T_p\mathcal{M} &\rightarrow \text{Hom}(C^\infty(\mathcal{M}), \mathbb{R}) \\ (\alpha \otimes v_{\gamma,p})(f) &:= \alpha \cdot_{\mathbb{R}} v_{\gamma,p}(f), \alpha \in \mathbb{R} \end{aligned}$$

Remains to be shown that:

1. $\exists \sigma$ curve: $v_{\gamma,p} \oplus v_{\delta,p} = \sigma$,
2. $\exists \tau$ curve: $\lambda \otimes v_{\gamma,p} = v_{\tau,p}$

Proof. Construct:

$$\begin{aligned} \tau : \mathbb{R} &\rightarrow \mathcal{M} \\ &\mapsto \tau(\lambda) := \gamma(\alpha \lambda \cdot \lambda_0) = (\gamma \circ \mu_\alpha)(\lambda) \\ \mu_\alpha : \mathbb{R} &\rightarrow \mathbb{R} \\ \Delta &\mapsto \Delta \cdot \alpha \end{aligned}$$

Then:

$$\begin{aligned} \tau(0) &= \gamma(\lambda_0) = p \\ v_{\tau,p} &:= (f \circ \tau)'(0) = (f \circ \gamma \mu_\alpha)'(0) \\ &= (f \circ \gamma)'(\lambda_0) \cdot \alpha \\ &= \alpha v_{\gamma,p} \end{aligned}$$

And for addition:

$$v_{\gamma,p} \oplus v_{\delta,p} \stackrel{?}{=} \sigma,$$

Make a choice of chart (\mathcal{U}, x) with $p \in \mathcal{U}$ and define:

$$\begin{aligned} \sigma_x : \mathbb{R} &\rightarrow \mathcal{M} \\ \sigma_x(\lambda) &:= x^{-1}((x \circ \gamma)(\lambda_0 + \lambda) + (x \circ \gamma)(\lambda_0)) \end{aligned}$$

Then:

$$\begin{aligned} \sigma_x(0) &= x^{-1}((x \circ \gamma)(\lambda_0) + (x \circ \delta)(\lambda_0) - (x \circ \gamma)(\lambda_0)) \\ &= \delta(\lambda_0) = 0 \end{aligned}$$

Now:

$$\begin{aligned} v_{\sigma,p} &:= (f \circ \sigma_x)'(0) \\ &= (f \circ x^{-1}(x \circ \sigma_x))'(\gamma) \\ &= (x \circ \sigma_x)^i(0) \cdot (\partial_i(f \circ x^{-1}))(x(\sigma_x(0)))(x \circ \gamma)^i(\lambda_0) + (x \circ \delta)^i(\lambda_0) \\ &= (x \circ \gamma)^i(\lambda_0)(\partial_i(f \circ x^{-1}))(x(p)) + (x \circ \delta)^i(\lambda_0)(\partial_i(f \circ x^{-1}))(x(p)) \\ &= (f \circ \gamma)'(\lambda_0) + (f \circ \delta)'(\lambda_0) \\ &= v_{\gamma,p}(f) + v_{\delta,p}(f), \forall f \in C^\infty(\mathcal{M}) \\ v_{\sigma,p} &= v_{\gamma,p} \oplus v_{\delta,p} \end{aligned}$$

And we are done. □

Components of a Vector With Respect to a Chart

Definition 27. Let $(\mathcal{U}, x) \in \mathcal{A}_{smooth}$ and let $\gamma : \mathbb{R} \rightarrow \mathcal{U}, \gamma(0) = p$. Calculate

$$\begin{aligned} v_{\gamma,p}(f) &:= (f \circ \gamma)'(0) = ((f \circ x^{-1}) \circ (x \circ \gamma))'(0) \\ &= (x \circ \gamma)^i(0) \cdot (\partial_i(f \circ x^{-1}))(x(p)) \\ &= \dot{\gamma}_x^i(0) \cdot \left(\frac{\partial}{\partial x^i}\right)_p f, \forall f \in C^\infty(\mathcal{M}) \end{aligned}$$

This is the components of velocity vector from the chart induced basis of the vector from $T_p\mathcal{M}$.

Chart-induced Basis

Definition 28. Take $(\mathcal{U}, x) \in \mathcal{A}_{smooth}$. Then

$$\left(\frac{\partial}{\partial x^1}\right)_p, \dots, \left(\frac{\partial}{\partial x^d}\right)_p \in T_p\mathcal{M}$$

Constitute a basis of $T_p\mathcal{U}$.

Proof. Linear independence remains to be shown.

$$\begin{aligned} \lambda^i \left(\frac{\partial}{\partial x^i}\right)_p &= \\ \lambda^i \left(\frac{\partial}{\partial x^i}\right)_p (x^i) &= \\ \lambda^i \partial_i (x^j \circ x^{-1})(x(p)) &= \\ \lambda^i \delta_i^j &= \lambda^j \end{aligned}$$

Corollary $\dim T_p\mathcal{M} = d = \dim(\mathcal{M})$. □

Terminology 6. $X \in T_p\mathcal{M} \implies \exists \gamma : \mathbb{R} \rightarrow \mathcal{M} : X = v_{\gamma,p}$ and $\exists x^1, \dots, x^d : X = x^i \left(\frac{\partial}{\partial x^i}\right)_p$

Change of Vector Components Under a Change of Chart

The physical vector in the real world doesn't change under change of chart. The vector components change under change of chart. Let $(\mathcal{U}, x), (\mathcal{V}, y) \in \mathcal{A}_{smooth}, p \in \mathcal{U} \cap \mathcal{V} \neq \emptyset$. Let $X \in T_p \mathcal{M}$ then

$$X_{(y)}^i \cdot \left(\frac{\partial}{\partial y^i} \right)_p =_{(\mathcal{V}, y)} X =_{(\mathcal{U}, x)} \left(\frac{\partial}{\partial x^i} \right)_p$$

To study change of components formula take:

$$\begin{aligned} \left(\frac{\partial}{\partial x^i} \right)_p f &= \partial_i (f \circ x^{-1})(x(p)) \\ &= \partial_i ((f \circ y^{-1}) \circ (y \circ x^{-1}))(x(p)) \\ &= (\partial_i (y^j \circ x^{-1}))(x(p)) \cdot (\partial_j (f \circ y^{-1}))(y(p)) \\ &= \left(\frac{\partial y^j}{\partial x^i} \right)_p \cdot \left(\frac{\partial f}{\partial y^j} \right)_p \\ &= X_{(x)}^i \cdot \left(\frac{\partial y^j}{\partial x^i} \right)_p \cdot \left(\frac{\partial}{\partial y^j} \right)_p \\ &= X_{(y)}^i \left(\frac{\partial}{\partial y^j} \right)_p \\ &\implies \\ X_{(y)}^j &= \left(\frac{\partial y^j}{\partial x^i} \right)_p X_{(x)}^i \end{aligned}$$

Cotangent Spaces

Trivial $(T_p^* \mathcal{M}) := \{\phi : T_p \mathcal{M} \xrightarrow{\sim} \mathbb{R}\}$.

Example 15. $f \in C^\infty(\mathcal{M})$

$$\begin{aligned} (df)_p : T_p \mathcal{M} &\xrightarrow{\sim} \mathbb{R} \\ X &\mapsto (df)_p(X) := Xf \end{aligned}$$

So $(df)_p$ is called the gradient of f at $p \in \mathcal{M}$. Calculate the components of the gradient. $(0,1)$ - tensor of the vector space with respect to a chart (\mathcal{U}, x) .

$$\begin{aligned} ((df)_p)_j &:= (df)_p \left(\frac{\partial}{\partial x^j} \right)_p \\ &= \left(\frac{\partial f}{\partial x^j} \right)_p \\ &= \partial_j (f \circ x^{-1})(x(p)) \end{aligned}$$

Theorem 6. Consider a chart $(\mathcal{U}, x) \implies x^i : \mathcal{U} \rightarrow \mathbb{R}$. Claim: $(dx^i)_p$ is the basis of $T_p^* \mathcal{M}$ or dual basis. $(dx^a)_p \left(\frac{\partial}{\partial x^b} \right)_p = \delta_b^a$

0.1 Change of Components of A Covector with Respect to a Chart

$\omega \in T_p^* \mathcal{M}$ then $\omega_{((x)_i)}(dx^i)_p \implies \omega_{(y)_j} = \left(\frac{\partial x^i}{\partial y^j} \right) \omega_{(x)_i}$.