

Notes From Fredric Schuller's Lectures on Relativity

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Abstract

Notes on Fredric Schuller's Relativity lectures. "Spacetime is a four-dimensional topological manifold with a smooth atlas carrying a torsion free connection compatible with a Lorentzian metric and time orientation satisfying the Einstein Equations."

Lecture 1: Topological Spaces

At the coarsest level spacetime is a set. This is not enough to talk about continuity of maps. In classical physics there are no jumps. Sets are not enough alone to talk about continuity. We are interested in establishing the weakest possible structure on a set to talk about continuity of maps. The mathematician knows the weakest structure necessary to do this is a topology.

Definition 1. Let \mathcal{M} be a set. A topology \mathcal{O} is a subset of the powerset of \mathcal{M} denoted $\mathcal{O} \subseteq \mathcal{P}(\mathcal{M})$ satisfying three axioms:

- $\emptyset \subset \mathcal{O}, \mathcal{M} \subset \mathcal{O}$
- For arbitrary $U, V \in \mathcal{O} \implies U \cap V \in \mathcal{O}$
- For arbitrary $U_\alpha \in \mathcal{O} \implies (\bigcap_{\alpha \in A} U_\alpha) \in \mathcal{O}$ where α is an index of the set A .

Example 1. • Let $\mathcal{M} = \{1, 2, 3\}$

- Let $\mathcal{O}_1 = \{\emptyset, \{1, 2, 3\}\}$ then \mathcal{O}_1 is a topology for \mathcal{M} .
- Let $\mathcal{O}_2 = \{\emptyset, \{1\}, \{2\}, \{1, 2, 3\}\}$ then \mathcal{O}_2 is not a topology for \mathcal{M} because $\{1, 2\} \notin \mathcal{O}_2$.
- Let \mathcal{M} be any set.
 - $\mathcal{O}_{chaotic} = \{\emptyset, \mathcal{M}\}$ is a topology
 - $\mathcal{O}_{discrete} = \mathcal{P}(\mathcal{M})$ is a topology
- $\mathcal{M} = \mathbb{R}^d$ (tuples of dimensions d from \mathbb{R}) then $\mathcal{O}_{standard} \subseteq \mathcal{P}(\mathcal{M})$ is a topology for \mathcal{M} defined as follows.

Definition 2. $\mathcal{O}_{standard}$ defined in two steps.

- $B_r(p) := \{q_1, \dots, q_d \mid \sum_{i=1}^d (q_i - p)^2 < r, r \in \mathbb{R}^+, q_i \in \mathbb{R}, p \in \mathbb{R}^d$
- $\mathcal{U} \in \mathcal{O}_{standard} \iff \forall p \in \mathcal{U}, \exists r \in \mathbb{R}^+ : B_r(p) \subseteq \mathcal{U}$

We want \mathcal{M} to be spacetime and we want to equip it with an appropriate topology \mathcal{O} to be able to talk about it. We want to make the implicit assumptions of spacetime to be explicit.

Terminology 1. Let \mathcal{M} be a set defined from (ZFC) then \mathcal{O} is the topology on \mathcal{M} and it is a collection of open sets. $(\mathcal{M}, \mathcal{O})$ is a topological space.

- $\mathcal{U} \in \mathcal{O} \iff \mathcal{U} \subseteq \mathcal{M}$ is an open set.
- $M/A \in \mathcal{O} \iff A \subseteq \mathcal{M}$ is closed.

$\text{not open} \not\implies \text{closed}$ and $\text{not closed} \not\implies \text{open}$

Definition 3. A map $f : M \rightarrow N$ takes all elements $m \in M$ to an element $n \in N$. M is the domain. N is the target. If $\exists m_1, m_2 \in M$ such that $f(m_1) = f(m_2) = n \in N$ then f is *injective*. If $\exists n \in N$ such that $\forall m \in M : f(m) \neq n$ then f is not surjective. Is a map f continuous? Depends by definition on topologies \mathcal{O}_M on M and \mathcal{O}_N on N .

Definition 4. A map f is called continuous between (M, \mathcal{O}_M) and (N, \mathcal{O}_N) . Then a map f is called continuous with respect to these topologies if for every open $V \in \mathcal{O}_N$ the preimage of V is open in \mathcal{O}_M . $\forall V \in \mathcal{O}_N : \text{preim}_f(V) \in \mathcal{O}_M$, $\text{preim}_f(V) := \{m \in M \mid f(m) \in V\}$.

Example 2. $M = 1, 2, \mathcal{O}_M = \emptyset, 1, 2, 1, 2$

$N = 1, 2, \mathcal{O}_N = \emptyset, 1, 2$

$f : M \rightarrow N \mid f(1) = 2, f(2) = 1$

Is f continuous?

$\text{preim}_f(\emptyset) = \emptyset \in \mathcal{O}_M$

$\text{preim}_f(1, 2) = M \in \mathcal{O}_M$

Therefore f is continuous.

Example 3. $g : N \rightarrow M$ or f^{-1} then $\text{preim}_g(1) = 2 \notin \mathcal{O}_N$ so g is not continuous.

Composition of Maps

$M \xrightarrow{f} N \xrightarrow{g} P$ so $g \circ f : M \rightarrow P$ by $m \rightarrow (g \circ f)(m) := g(f(m))$.

Theorem 1. Composition of continuous maps is continuous.

Proof. Let $V \in \mathcal{O}_P$ then

$$\begin{aligned} \text{preim}_{g \circ f}(V) &:= m \in M \mid (g \circ f)(m) \in V \\ &= m \in M \mid f(m) \in \text{preim}_g(V) \\ &= \text{preim}_f(\text{preim}_g(V) \in \mathcal{O}_N) \in \mathcal{O}_M \end{aligned}$$

□

Inheritance of a Topology

Many useful ways to inherit a topology from another topological space or set of topological spaces. Of particular importance for spacetime physics is $S \subseteq M$ where M has topology \mathcal{O}_M . Can we construct a topology \mathcal{O}_S from \mathcal{O}_M . Yes.

Definition 5. $\mathcal{O}|_S \subseteq \mathcal{P}(S)$

$\mathcal{O}|_S := \mathcal{U} \cap S \mid \mathcal{U} \in \mathcal{O}_M$

This is a topology called the subset topology.

Use of this specific way to inherit a topology from a super set. Let it be easy to say a map f is continuous. Then a subset S of a set M with an inherited topology then the restriction of the map $f|_S : S \rightarrow N$ it can be easy to show this restriction is continuous.

Lecture 2: Topological Manifolds

\exists too many topological spaces to classify. Too many topological spaces exist which have no known connection to the study of spacetime. For spacetime physics we may focus on topological spaces $(\mathcal{M}, \mathcal{O})$ which can be charted analogously to how the surface of the Earth is charted in an atlas.

Topological Manifolds

Definition 6. A topological space $(\mathcal{M}, \mathcal{O})$ is called a d -dimensional topological manifold if $\forall p \in \mathcal{M} : \exists p \in \mathcal{U} \in \mathcal{O} : \exists x : \mathcal{U} \rightarrow x(\mathcal{U}) \subseteq \mathbb{R}^d$ with \mathbb{R}^d equipped with $\mathcal{O}_{\text{standard}}$ such that:

- x is invertible: $x^{-1} : x(\mathcal{U}) \rightarrow \mathcal{U}$

- x is continuous.
- x^{-1} is continuous.

Example 4.

The surface of a torus $:= \mathcal{M} \subseteq \mathbb{R}^3$. This is a $d = 2$ dimensional topological manifold.

A mobius strip $:= \mathcal{M} \subseteq (\mathbb{R}^3, \mathcal{O}_{st})$ then this is a $d = 1$ manifold.

A bifurcating line $:= \mathcal{M} \subset (\mathbb{R}^2, \mathcal{O}_{st})$ then $(\mathcal{M}, \mathcal{O}_{st}|_{\mathcal{M}})$ is a topology but the point of bifurcation is not invertible so it is not a manifold.

Terminology 2. 1. A pair (\mathcal{U}, x) is called a chart.

2. A set $\mathcal{A} = (\mathcal{U}_\alpha, x_\alpha) | \alpha \in A$ is called an atlas of the topological manifold \mathcal{M} if $\mathcal{M} = \bigcup_{\alpha \in A} \mathcal{U}_\alpha$.

3. $x : \mathcal{U} \rightarrow x(\mathcal{U}) \subseteq \mathbb{R}^d$ is called the chart map.

4. $x^i \mathcal{U} \rightarrow \mathbb{R}$ is called the coordinate maps.

5. $p \in \mathcal{U}$ then x^i is the i -th coordinate of the point p with respect to the chosen chart (\mathcal{U}, x) .

Example 5. •

$$\begin{aligned}\mathcal{M} &= \mathbb{R}^2 \\ \mathcal{U} &= \mathbb{R}^2 / 0, 0 \\ x : \mathcal{U} &\rightarrow \mathbb{R}^2 : x(m, n) \rightarrow (-m, -n)\end{aligned}$$

- We can take another chart map on \mathcal{U}
 $(m, n) \rightarrow (\sqrt{m^2 + n^2}, \arctan(\frac{n}{m}))$

Chart Transition Maps

Imagine two charts (\mathcal{U}, x) and (\mathcal{V}, y) with $\mathcal{U} \cap \mathcal{V} = \mathcal{A} \neq \emptyset$. The same point $p \in \mathcal{A}$ can be mapped via x and y to two different charts. Because these maps are continuously invertible we can smoothly transition between pages of our atlas: $(y \circ x^{-1})(p) = y(x^{-1}(p))$. This map is called the chart transition map. Informally the chart transition map contains the instructions how to *glue* together the charts of our atlas.

Manifold Philosophy

Often it is desirable (or indeed the way) to define properties ("continuity") of real-world objects ($\mathbb{R} \xrightarrow{\gamma} \mathcal{M}$) by judging suitable condition not on the real-world object itself but on the chart-representation of the real-world object. Advantages: You can define continuity in this way. Disadvantages: The chart map x is a 'fantasy,' x may be ill defined because the chart chosen is arbitrary. Solution: The property must be maintained regardless of the chart.

$$\begin{aligned}\mathbb{R} &\xrightarrow{y \circ \gamma} y(\mathcal{U}) \\ \mathbb{R} &\xrightarrow{x \circ \gamma} x(\mathcal{U}) \\ \mathbb{R} &\xrightarrow{\gamma} \mathcal{U}\end{aligned}$$

We are interested in properties of γ but must talk about it with respect to x and y .

Lecture 3: Multilinear Algebra

We will not equip space(time) with a vector space structure. There is no such thing as five times Paris or Paris plus Vienna. However, the tangent spaces of $T_p \mathcal{M}$ of smooth manifolds carry a vector space structure. It is beneficial to first study vector spaces abstractly for two reasons.

- For construction of $T_p \mathcal{M}$ one needs an intermediate vector space $C^\infty(\mathcal{M})$.
- Tensor techniques are best understood in an abstract setting.

Vector Spaces

Definition 7. A vector space $(V, +, \cdot)$ is

- a set V
- $+: V \times V \rightarrow V$
- $\cdot: \mathbb{R} \times V \rightarrow V$

Which satisfies CANI and ADDU axioms. For $w, v, u \in V$ and $\lambda, \mu \in \mathbb{R}$

- $C^+ : v + w = w + v$
- $A^+ : (u + v) + w = u + (v + w)$
- $N^+ : \exists 0 \in V : \forall v \in V : v + 0 = v$
- $I^+ : \forall v \in V : \exists (-v) \in V : v + (-v) = 0$
- $A : \lambda \cdot (\mu \cdot v) = (\lambda \cdot \mu) \cdot v$
- $D : (\lambda + \mu) \cdot v = \lambda \cdot v + \mu \cdot v$
- $D : \lambda \cdot v + \lambda \cdot w = \lambda \cdot (v + w)$
- $U : 1 \cdot v = v$

Terminology 3. An element of a vector space is often referred to (informally) as a vector.

Example 6.

Definition 8. Set $P := p : (-1, +1) \rightarrow \mathbb{R}$ of polynomials of (fixed) degree. $p(x) = \sum_{n=1}^N p_n x^n$. Is $\square(x) = x^2$ a vector? No. $\square \in P$.

Definition 9. Define the operations in the space:

$$+ : P \times P \rightarrow P : (p, q) \mapsto p + q$$

$$\cdot : \mathbb{R} \times P \rightarrow P : (\lambda, p) \mapsto \lambda \cdot p$$

Is $\square \in (P, +, \cdot)$ a vector in a vector space? Yes!

Linear Maps

We want to study maps which preserve (vector space) structure. On vector spaces these are called linear maps.

Definition 10. Let $(V, +, \cdot)$ and $(W, +, \cdot)$ be vector spaces. Then a map $\phi : V \rightarrow W$ is called linear if:

- $\phi(v + \tilde{v}) = \phi(v) + \phi(\tilde{v})$
- $\phi(\lambda \cdot v) = \lambda \cdot \phi(v)$

Example 7. Take P as before. Take $\delta : P \rightarrow P : \delta(p) := p'$. $\delta(p, q) = (p + q)' = \delta(p) + \delta(q)$ and $\delta(\lambda p) = (\lambda p)' = \lambda p'$

Notation: $\phi : V \rightarrow W$ linear $\iff \phi : V \xrightarrow{\sim} W$

Vector space of Homomorphisms

Take $(V, +, \cdot)$ and $(W, +, \cdot)$ vector spaces.

Definition 11. $\text{Hom}(V, W) := \phi : V \xrightarrow{\sim} W$ as a set. This is a vectorspace:

- $\oplus : \text{Hom}(V, W) \times \text{Hom}(V, W) \rightarrow \text{Hom}(V, W) : (\phi, \psi) \mapsto \phi \oplus \psi$ where $(\phi \oplus \psi)(v) := \phi(v) + \psi(v)$
- \otimes defined similarly.

$(\text{Hom}(V, W), \oplus, \otimes)$ is a vector space.

Example 8. Take P as before. Then δ is a vector space similarly.

Dual Vector Space

$(V, +, \cdot)$ as a vector space.

Definition 12. $V^* := \psi : V \xrightarrow{\sim} \mathbb{R} = \text{Hom}(V, \mathbb{R})$ is also a vector space called the dual vector space of V .

Terminology 4. $\phi \in V^*$ is called (informally) a covector.

Example 9. $I : P \xrightarrow{\sim} \mathbb{R}$ so $I \in P^*$

Definition 13. $I(p) := \int_0^1 dx p(x)$

I is clearly linear.

Tensors

Definition 14. Take a vector space V . Then a (r, s) tensor T over V is a multilinear map: $T : \bigotimes_{i=1}^r V^* \otimes \bigotimes_{j=1}^s V \xrightarrow{\sim(r+s)} \mathbb{R}$

Example 10. $T(1, 1)$ - tensor. $T(\phi + \psi, v) = T(\phi, v) + T(\psi, v)$, $T(\lambda\psi, v) = \lambda \cdot T(\psi, v)$, $T(\psi, v + w) = T(\psi, v) + T(\psi, w)$, $T(\psi, \lambda \cdot v) = \lambda \cdot T(\psi, v)$ Linear in both entries. Hence, multi-linear.

Example 11. $g : P \times P \xrightarrow{\sim} \mathbb{R}$, $(p, q) \rightarrow \int_0^1 dx p(x)q(x)$ is a $(0, 2)$ - tensor over example of P .

Vectors and Covectors

Theorem 2. $\phi \in V^* \iff \phi : V \xrightarrow{\sim} \mathbb{R} \iff \phi(0, 1)$ - tensor.

Theorem 3. $\dim(V) < \infty \implies v \in V = (V^*)^* \iff v : V^* \xrightarrow{\sim} \mathbb{R} \iff v$ is a $(1, 0)$ - tensor.

Bases

Definition 15. Take a vector space V . A subset $B \subset V$ is called a basis if $\forall v \in V \exists F = \{f_1, \dots, f_n\} \in B : \exists ! \{v^1, \dots, v^n\} \in \mathbb{R} : v = v^1 f_1 + \dots + v^n f_n$.

Definition 16. If \exists basis B with finite many elements (d many elements) then we call $d =: \dim(V)$.

Remark 1. Let V be finite dimensional vector space. Choose a basis e_1, \dots, e_n of V . We may uniquely associate a vector $v \in V$ with $v \mapsto (v^1, \dots, v^n)$ called the components of $v = v^1 e_1 + \dots + v^n e_n$ with respect to the chosen basis. It is more economical to require your basis on V once chosen such that $\epsilon^a(e_b) = \delta_b^a$. This uniquely determines the choice of vector components from a choice of basis.

Definition 17. If a basis $\epsilon^1, \dots, \epsilon^n$ of V^* satisfies these axioms it is called the dual basis.

Example 12. Take P with $(N = 3)$. Then e_0, \dots, e_3 is a basis if $e_a(x) := x^a$ is a basis of P . Then dual basis is given by $\epsilon^a := \frac{1}{a!} \partial^a|_{x=0}$.

Components of Tensors

Definition 18. Let T be an (r, s) - tensor over a finite dimensional vector space V . Then define the $(r + s)^{\dim(V)}$ many real numbers $i_1, \dots, i_r, j_1, \dots, j_s \in \{1, \dots, \dim(V)\}$, $T_j^i \in \mathbb{R} := T(\epsilon^i, e_j)$. The T_j^i elements are called the components of the tensor with respect to the chosen basis. Knowing the components and basis one can reconstruct the entire tensor.

Example 13. $T(1, 1,)$ - tensor with $T_j^i = T(\epsilon^i, e_j)$ reconstruct

$$\begin{aligned} T(\phi, v) &= T\left(\sum_{i=1}^{\dim(V)} \phi_i \epsilon^i, \sum_{j=1}^{\dim(V)} v^j e_j\right) \\ &= \sum_{i=1}^{\dim(V)} \sum_{j=1}^{\dim(V)} \phi_i v^j T(\epsilon^i, e_j) \end{aligned}$$

with $\phi_i, v^j \in \mathbb{R}$.

Terminology 5. If we agree to label $T_j^i = T(\epsilon^i, e_j)$ with the up and down components then Einstein summation convention of tensors $:= \phi_i v^j T_j^i$ where we drop \sum . This only works over multilinear maps.

Lecture 4: Differentiable Manifolds

So far we have topological manifolds $(\mathcal{M}, \mathcal{O})$ $\dim(\mathcal{M})=d$. We want to talk about velocity vectors on them but this structure is insufficient to do so. Pick a dimension $d \neq 4$ for your manifold. In that dimension the choices of topology are countable, so presumably we can do experiments to discern which one corresponds to our reality. In $d = 4$ our choices of topology are suddenly uncountable. We need additional structure to talk about differentiable curves on manifolds and between manifolds.

1. Curves: $\mathbb{R} \rightarrow \mathcal{M}$
2. Functions: $\mathcal{M} \rightarrow \mathbb{R}$
3. Maps: $\mathcal{M}_1 \rightarrow \mathcal{M}_2$

Strategy

$\gamma: \mathbb{R} \rightarrow \mathcal{U}$ and choose a chart (\mathcal{U}, x) . $\mathcal{U} \xrightarrow{x} x(\mathcal{U}) \subseteq \mathbb{R}^d$. Try and 'lift' the undergraduate notion of differentiability of a curve in \mathbb{R}^d to a notion of differentiability on a curve on \mathcal{M} . Problem: Is this well defined under a change of chart? We don't want it to depend on our taste.

$$\gamma: \mathbb{R} \rightarrow \mathcal{U} \cap \mathcal{V} \neq \emptyset \quad (1)$$

$$y \circ \gamma: \mathbb{R} \rightarrow y(\mathcal{U} \cap \mathcal{V}) \subseteq \mathbb{R}^d \quad (2)$$

$$x \circ \gamma: \mathbb{R} \rightarrow x(\mathcal{U} \cap \mathcal{V}) \subseteq \mathbb{R}^d \quad (3)$$

$$y \circ \gamma = (y \circ x^{-1}) \circ (x \circ \gamma) = y \circ (x^{-1} \circ x) \circ \gamma \quad (4)$$

If $(x \circ \gamma)$ differentiable and $(y \circ x^{-1})$ continuous is $y \circ \gamma$ guaranteed to be continuously differentiable? No.

Compatible Charts

In previous we took a chart on the topological manifold. Our atlases \mathcal{V} and \mathcal{U} were elements of the maximal atlas of the manifold.

Definition 19. Two charts (\mathcal{U}, x) and (\mathcal{V}, y) are called \square compatible if either:

1. $\mathcal{U} \cap \mathcal{V} = \emptyset$
2. $y \circ x^{-1}: x(\mathcal{U} \cap \mathcal{V}) \rightarrow y(\mathcal{U} \cap \mathcal{V})$ and $x \circ y^{-1}: y(\mathcal{U} \cap \mathcal{V}) \rightarrow x(\mathcal{U} \cap \mathcal{V})$ are differentiable.

Philosophy of \square :

Definition 20. An atlas \mathcal{A}_\square is a \square - compatible atlas if any two charts in \mathcal{A}_\square are \square - compatible.

Definition 21. A \square - manifold is a triple $(\mathcal{M}, \mathcal{O}, \mathcal{A}_\square)$

Undergraduate \square . C^0 : $C^0(\mathbb{R}^d \rightarrow \mathbb{R}^d)$ Continuous maps. C^1 : $C^1(\mathbb{R}^d \rightarrow \mathbb{R}^d)$ Differentiable Once. C^k : k -times continuous differentiable. D^k : k -times differentiable. C^∞ : $C^\infty(\mathbb{R}^d \rightarrow \mathbb{R}^d)$ infinitely continuously infinitely continuously differentiable. C^ω : \exists multidimensional Taylor Expansion. \mathbb{C}^∞ : Satisfies Cauchy-Riemann Equations.

Theorem 4. Any $C^{k \geq 1}$ - atlas \mathcal{A} of a topological manifold contains a C^∞ - atlas.

Thus we will consider C^∞ manifolds or "smooth manifolds" unless we wish to define Taylor expansions or complex differentiable.

Definition 22. A smooth manifold is a tripple $(\mathcal{M}, \mathcal{O}, \mathcal{A})$ where \mathcal{A} is a C^∞ - atlas.

Diffeomorphisms

Take a map: $\mathcal{M} \xrightarrow{\phi} \mathcal{N}$. If \mathcal{M} and \mathcal{N} are naked sets then the structure preserving maps are the bijections. If $\exists \phi$ is a bijection $\mathcal{M} \cong \mathcal{N}$.

Example 14.

$$\mathbb{N} \cong \mathbb{Z}$$

$$\mathbb{N} \cong \mathbb{Q}$$

$$\mathbb{N} \not\cong \mathbb{R}$$

Linear bijections between vector spaces are the structure preserving maps between vector spaces.

Definition 23. Two C^∞ manifolds are said to be diffeomorphic $\iff \exists$ bijection $\phi : \mathcal{M} \rightarrow \mathcal{N}$ such that ϕ, ϕ^{-1} are both C^∞ maps.

Theorem 5. The number of C^∞ manifolds that can be constructed from a given C^0 - manifold up to diffeomorphism by the More-Radon Theorems in $d = 1, 2, 3$ dimension are 1. In $d > 4$ the number is finite. In $d = 4 \exists$ uncountably infinitely many such manifolds.

Lecture 5: Tangent Spaces

Lead question: "What is the velocity of a curve γ at a point p ?"

Velocities

Definition 24. Take a $(\mathcal{M}, \mathcal{O}, \mathcal{A})$ smooth manifold. Take a curve: $\gamma : \mathbb{R} \rightarrow \mathcal{M}$ of at least C^1 . Suppose $\gamma(\lambda_0) = p$. The velocity of γ at p is the linear map: $v_{\gamma,p} : C^\infty(\mathcal{M}) \xrightarrow{\sim} \mathbb{R}$. $C^\infty(\mathcal{M}) := \{f : \mathcal{M} \rightarrow \mathbb{R} | f \text{ smooth function} \}$ equipped with $(f \oplus g)(p) := f(p) +_{\mathbb{R}} g(p)$, $(\lambda \oplus g)(p) := \lambda \cdot g(p)$. So velocity is a linear map: $f \mapsto v_{\gamma,p}(f) := (f \circ \gamma)'(\lambda_0)$.

Tangent Vector Space

Definition 25. For each point $p \in \mathcal{M}$ we define the set "tangent space to \mathcal{M} at p " as $T_p\mathcal{M} := \{v_{\gamma,p} | \gamma \text{ smooth curve} \}$.

Observe that $T_p\mathcal{M}$ can be made into a vector space.

Definition 26. We need to define \oplus and \otimes .

$$\begin{aligned} \oplus : T_p\mathcal{M} \times T_p\mathcal{M} &\rightarrow \text{Hom}(C^\infty(\mathcal{M}), \mathbb{R}) \\ (v_{\gamma,p} \oplus v_{\delta,p})(f) &:= v_{\gamma,p}(f) +_{\mathbb{R}} v_{\delta,p}(f), f \in C^\infty(\mathcal{M}) \end{aligned}$$

$$\begin{aligned} \otimes : \mathbb{R} \times T_p\mathcal{M} &\rightarrow \text{Hom}(C^\infty(\mathcal{M}), \mathbb{R}) \\ (\alpha \otimes v_{\gamma,p})(f) &:= \alpha \cdot_{\mathbb{R}} v_{\gamma,p}(f), \alpha \in \mathbb{R} \end{aligned}$$

Remains to be shown that:

1. $\exists \sigma$ curve: $v_{\gamma,p} \oplus v_{\delta,p} = \sigma$,
2. $\exists \tau$ curve: $\lambda \otimes v_{\gamma,p} = v_{\tau,p}$

Proof. Construct:

$$\begin{aligned} \tau : \mathbb{R} &\rightarrow \mathcal{M} \\ &\mapsto \tau(\lambda) := \gamma(\alpha\lambda \cdot \lambda_0) = (\gamma \circ \mu_\alpha)(\lambda) \\ \mu_\alpha : \mathbb{R} &\rightarrow \mathbb{R} \\ \Delta &\mapsto \Delta \cdot \alpha \end{aligned}$$

Then:

$$\begin{aligned} \tau(0) &= \gamma(\lambda_0) = p \\ v_{\tau,p} &:= (f \circ \tau)'(0) = (f \circ \gamma \mu_\alpha)'(0) \\ &= (f \circ \gamma)'(\lambda_0) \cdot \alpha \\ &= \alpha v_{\gamma,p} \end{aligned}$$

And for addition:

$$v_{\gamma,p} \oplus v_{\delta,p} \stackrel{?}{=} \sigma,$$

Make a choice of chart (\mathcal{U}, x) with $p \in \mathcal{U}$ and define:

$$\begin{aligned} \sigma_x : \mathbb{R} &\rightarrow \mathcal{M} \\ \sigma_x(\lambda) &:= x^{-1}((x \circ \gamma)(\lambda_0 + \lambda) + (x \circ \gamma)(\lambda_0)) \end{aligned}$$

Then:

$$\begin{aligned} \sigma_x(0) &= x^{-1}((x \circ \gamma)(\lambda_0) + (x \circ \delta)(\lambda_0) - (x \circ \gamma)(\lambda_0)) \\ &= \delta(\lambda_0) = 0 \end{aligned}$$

Now:

$$\begin{aligned} v_{\sigma,p} &:= (f \circ \sigma_x)'(0) \\ &= (f \circ x^{-1}(x \circ \sigma_x))'(\gamma) \\ &= (x \circ \sigma_x)^i(0) \cdot (\partial_i(f \circ x^{-1}))(x(\sigma_x(0)))(x \circ \gamma)^i(\lambda_0) + (x \circ \delta)^i(\lambda_0) \\ &= (x \circ \gamma)^i(\lambda_0)(\partial_i(f \circ x^{-1}))(x(p)) + (x \circ \delta)^i(\lambda_0)(\partial_i(f \circ x^{-1}))(x(p)) \\ &= (f \circ \gamma)'(\lambda_0) + (f \circ \delta)'(\lambda_0) \\ &= v_{\gamma,p}(f) + v_{\delta,p}(f), \forall f \in C^\infty(\mathcal{M}) \\ v_{\sigma,p} &= v_{\gamma,p} \oplus v_{\delta,p} \end{aligned}$$

And we are done. □

Components of a Vector With Respect to a Chart

Definition 27. Let $(\mathcal{U}, x) \in \mathcal{A}_{smooth}$ and let $\gamma : \mathbb{R} \rightarrow \mathcal{U}, \gamma(0) = p$. Calculate

$$\begin{aligned} v_{\gamma,p}(f) &:= (f \circ \gamma)'(0) = ((f \circ x^{-1}) \circ (x \circ \gamma))'(0) \\ &= (x \circ \gamma)^i(0) \cdot (\partial_i(f \circ x^{-1}))(x(p)) \\ &= \dot{\gamma}_x^i(0) \cdot \left(\frac{\partial}{\partial x^i}\right)_p f, \forall f \in C^\infty(\mathcal{M}) \end{aligned}$$

This is the components of the velocity vector from the chart induced basis of the vector from $T_p\mathcal{M}$.

Chart-induced Basis

Definition 28. Take $(\mathcal{U}, x) \in \mathcal{A}_{smooth}$. Then

$$\left(\frac{\partial}{\partial x^1}\right)_p, \dots, \left(\frac{\partial}{\partial x^d}\right)_p \in T_p\mathcal{M}$$

Constitute a basis of $T_p\mathcal{U}$.

Proof. Linear independence remains to be shown.

$$\begin{aligned} \lambda^i \left(\frac{\partial}{\partial x^i}\right)_p &= \\ \lambda^i \left(\frac{\partial}{\partial x^i}\right)_p (x^i) &= \\ \lambda^i \partial_i(x^j \circ x^{-1})(x(p)) &= \\ \lambda^i \delta_i^j &= \lambda^j \end{aligned}$$

Corollary $\dim T_p\mathcal{M} = d = \dim(\mathcal{M})$. □

Terminology 6. $X \in T_p\mathcal{M} \implies \exists \gamma : \mathbb{R} \rightarrow \mathcal{M} : X = v_{\gamma,p}$ and $\exists x^1, \dots, x^d : X = x^i \left(\frac{\partial}{\partial x^i}\right)_p$

Change of Vector Components Under a Change of Chart

The physical vector in the real world doesn't change under change of chart. The vector components change under change of chart. Let $(\mathcal{U}, x), (\mathcal{V}, y) \in \mathcal{A}_{smooth}, p \in \mathcal{U} \cap \mathcal{V} \neq \emptyset$. Let $X \in T_p \mathcal{M}$ then

$$X_{(y)}^i \cdot \left(\frac{\partial}{\partial y^i} \right)_p =_{(\mathcal{V}, y)} X =_{(\mathcal{U}, x)} \left(\frac{\partial}{\partial x^i} \right)_p$$

To study change of components formula take:

$$\begin{aligned} \left(\frac{\partial}{\partial x^i} \right)_p f &= \partial_i (f \circ x^{-1})(x(p)) \\ &= \partial_i ((f \circ y^{-1}) \circ (y \circ x^{-1})(x(p))) \\ &= (\partial_i (y^j \circ x^{-1}))(x(p)) \cdot (\partial_j (f \circ y^{-1}))(y(p)) \\ &= \left(\frac{\partial y^j}{\partial x^i} \right)_p \cdot \left(\frac{\partial f}{\partial y^j} \right)_p \\ &= X_{(x)}^i \cdot \left(\frac{\partial y^j}{\partial x^i} \right)_p \cdot \left(\frac{\partial}{\partial y^j} \right)_p \\ &= X_{(y)}^i \left(\frac{\partial}{\partial y^j} \right)_p \\ &\implies \\ X_{(y)}^j &= \left(\frac{\partial y^j}{\partial x^i} \right)_p X_{(x)}^i \end{aligned}$$

Cotangent Spaces

Trivial $(T_p^* \mathcal{M}) := \{\phi : T_p \mathcal{M} \xrightarrow{\sim} \mathbb{R}\}$.

Example 15. $f \in C^\infty(\mathcal{M})$

$$\begin{aligned} (df)_p : T_p \mathcal{M} &\xrightarrow{\sim} \mathbb{R} \\ X &\mapsto (df)_p(X) := Xf \end{aligned}$$

So $(df)_p$ is called the gradient of f at $p \in \mathcal{M}$. Calculate the components of the gradient. $(0,1)$ - tensor of the vector space with respect to a chart (\mathcal{U}, x) .

$$\begin{aligned} ((df)_p)_j &:= (df)_p \left(\frac{\partial}{\partial x^j} \right)_p \\ &= \left(\frac{\partial f}{\partial x^j} \right)_p \\ &= \partial_j (f \circ x^{-1})(x(p)) \end{aligned}$$

Theorem 6. Consider a chart $(\mathcal{U}, x) \implies x^i : \mathcal{U} \rightarrow \mathbb{R}$. Claim: $(dx^i)_p$ is the basis of $T_p^* \mathcal{M}$ or dual basis. $(dx^a)_p \left(\frac{\partial}{\partial x^b} \right)_p = \delta_b^a$

0.1 Change of Components of A Covector with Respect to a Chart

$\omega \in T_p^* \mathcal{M}$ then $\omega_{((x)_i)}(dx^i)_p \implies \omega_{(y)_j} = \left(\frac{\partial x^i}{\partial y^j} \right) \omega_{(x)_i}$.

Lecture 6: Fields

We want to assign to every point on a manifold a vector. To do this we need Theory of Bundles.

Bundles

Definition 29. A bundle is a triple $E \xrightarrow{\pi} \mathcal{M}$ where E is the 'total space' and is a smooth manifold. π is a smooth surjective map called the 'projective map'. And \mathcal{M} is our smooth manifold called the 'base space.'

Definition 30. Take bundle with $p \in \mathcal{M}$ we define the Fibre over p as the $\text{preim}_\pi(\{p\})$.

Definition 31. A section σ of a bundle is the $\text{preim}_\pi(\mathcal{M})$ where we require $\pi \circ \sigma = \text{id}_\mathcal{M}$.

As an aside in quantum mechanics we take a wave function of a position space $\psi : \mathcal{M} \rightarrow \mathbb{C}$. They are actually sections of \mathbb{C} -line bundle of the phase-space.

Tangent Bundle of a Smooth Manifold

Take a smooth manifold $(\mathcal{M}, \mathcal{O}, \mathcal{A})$.

- The tangent bundle is a set: $TM := \bigcup_{p \in \mathcal{M}} T_p \mathcal{M}$.
- $\pi : T\mathcal{M} \rightarrow \mathcal{M} : X \mapsto p$ where p is the unique point in \mathcal{M} such that $X \in T_p \mathcal{M}$.
- We want the coarsest topology such that π is continuous. This is called the 'initial topology.' $\mathcal{O}_{T\mathcal{M}} := \{\text{preim}_\pi(\mathcal{U}) \mid \mathcal{U} \in \mathcal{O}\}$.

Construction of a C^∞ atlas on $T\mathcal{M}$ from the C^∞ atlas on \mathcal{M} . $\mathcal{A}_{T\mathcal{M}} := \{(T\mathcal{U}, \psi_x) \mid (\mathcal{U}, x) \in \mathcal{A}\}$ with $\psi_x : T\mathcal{U} \rightarrow \mathbb{R}^{2 \cdot \dim(\mathcal{M})} : X \in T_{\pi(x)} \mathcal{M}$ with $X \mapsto ((x^1 \circ \pi)(X), \dots, (x^d \circ \pi)(X), (dx^1)_{\pi(x)}(X), \dots, (dx^d)_{\pi(x)}(X))$. Note we need to reconstruct ψ^{-1} from our data. So we have $(\alpha^i, \beta^j) \in \mathbb{R}^{2d}$ such that we have $\beta^i (\frac{\partial}{\partial x^i})_{x^{-1}(\alpha^j)}$. Alphas are the point and the betas are the components. Check that this map is smooth by change of atlas!

Vector Fields

Definition 32. A smooth vector field is a χ is a smooth map that is a section.

The $C^\infty(\mathcal{M})$ - module $\Gamma(T\mathcal{M})$

$(C^\infty(\mathcal{M}), +, \cdot)$ This satisfies the notion of a ring. It is the ring of smooth function on a manifold. $\Gamma(T\mathcal{M}) = \{\chi : T\mathcal{M} \rightarrow \mathcal{M} \mid \text{smooth sections}\}$. This is a vector space module over the ring. The set of all smooth vector fields can be made into such a module structure. This is important because we cannot guarantee that there exists a basis. There is no way to generate a basis for a vector field over a sphere (Harry Ball Theorem).

Tensor Fields

Definition 33. An (r, s) - tensor field T is a $C^\infty(\mathcal{M})$ multilinear map between modules.

Example 16. Define $df := \Gamma(T\mathcal{M}) \rightarrow C^\infty(\mathcal{M}) : \chi \mapsto df(\chi) := \chi(f)$ with $(\chi f)(p) := \chi(p)f$ with $p \in \mathcal{M}$ and $\chi(p) \in T_p \mathcal{M}$. This is a covector field.

Lecture 7: Connections

So far we saw that a vector field X can be used to provide a directional derivative.

$$\nabla_X f := Xf$$

of a $f \in C^\infty(\mathcal{M})$.

Remark 2. From now on we will mostly consider vector fields. Seems like notational overkill? $\nabla_X f = Xf = (df)(X)$. Not quite the same thing.

$$\begin{aligned} X &: C^\infty(\mathcal{M}) \rightarrow C^\infty(\mathcal{M}) \\ df &: \Gamma(T\mathcal{M}) \rightarrow C^\infty(\mathcal{M}) \\ \nabla_X &: C^\infty(\mathcal{M}) \rightarrow C^\infty(\mathcal{M}) \end{aligned}$$

But we want to extend ∇_X to $(p, q) \rightarrow (p, q)$ -tensor fields. This is not free.

Directional Derivatives of Tensor Fields

We formulate a wish list of properties which the ∇_X acting on a tensor field should have. Any remaining freedom in choice ∇ will need to be provided as additional structure beyond our smooth manifold $(\mathcal{M}, \mathcal{O}, \mathcal{A})$.

Definition 34. A connection ∇ on a smooth manifold is a map that takes a pair consisting of a vector (field) X and a (p, q) - tensor (field) T and yields a (p, q) - tensor (field) $\nabla_X T$ satisfying four properties:

1. $\nabla_X f = Xf, \forall f \in C^\infty(\mathcal{M})$
2. $\nabla_X(T + S) = \nabla_X T + \nabla_X S$
3. $\nabla_X(T(\omega, Y)) = (\nabla_X T)(\omega, Y) + T(\nabla_X \omega, Y) + T(\omega, \nabla_X Y)$ (Liebnitz rule)¹
4. $\nabla_{fX+gT} = f\nabla_X T + g\nabla_T, \forall f \in C^\infty(\mathcal{M})$

A manifold with connection is a quadruple of structures $(\mathcal{M}, \mathcal{O}, \mathcal{A}, \nabla)$.

Remark 3. ∇_X is the extension of X . So ∇_\cdot is the extension of d .

New Structure on $(\mathcal{M}, \mathcal{O}, \mathcal{A})$ Required to Fix ∇

Consider X, Y vector fields.

$$\begin{aligned} \nabla_X Y &=_{(\mathcal{U}, x)} \nabla_{X^i \frac{\partial}{\partial x^i}} (Y^m \cdot \frac{\partial}{\partial x^m}) \\ &=_{(iii)} X^i (\nabla_{\frac{\partial}{\partial x^i}} Y^m) \cdot \frac{\partial}{\partial x^m} + X^i Y^m \cdot \frac{\partial}{\partial x^i} (\frac{\partial}{\partial x^m}) \\ &=_{(i)} X^i (\frac{\partial}{\partial x^i} Y^m) \cdot \frac{\partial}{\partial x^m} X^i Y^m \cdot \Gamma_{mi}^q \frac{\partial}{\partial x^q} \end{aligned}$$

Γ is the connection coefficient functions (on \mathcal{M}) of ∇ with respect to the chart (\mathcal{U}, x)

Definition 35. $(\mathcal{M}, \mathcal{O}, \mathcal{A}, \nabla), (\mathcal{U}, x) \in \mathcal{A}$ then the connection coefficient functions ("Ts) with respect to our chart (\mathcal{U}, x) are the $(\dim(\mathcal{M}))^3$ many functions:

$$\begin{aligned} \Gamma(x)_{jk}^i : \mathcal{U} &\rightarrow \mathbb{R} \\ p &\mapsto (dx^i)(\nabla_{\frac{\partial}{\partial x^k}} \frac{\partial}{\partial x^j})(p) \end{aligned}$$

$$\text{Thus } (\nabla_X Y)^i = X^m (\frac{\partial}{\partial x^m} Y^j) + \Gamma_{nm}^i \cdot Y^n \cdot X^m$$

Remark 4. On a chart domain \mathcal{U} the choice of the $(\dim(\mathcal{M}))^3$ functions Γ_{nm}^i functions suffices to fix the action ∇ on a vector field. Those same functions fix the action of ∇ on any tensor field.

Remark 5. $\nabla_{\frac{\partial}{\partial x^m}} (dx^i) = \Sigma_{jm}^i dx^j$. But now

$$\begin{aligned} \nabla_{\frac{\partial}{\partial x^m}} (dx^i (\frac{\partial}{\partial x^j})) &= \frac{\partial}{\partial x^m} \delta_j^i \\ &= (\nabla_{\frac{\partial}{\partial x^m}} dx^i) + dx^i (\nabla_{\frac{\partial}{\partial x^m}} \frac{\partial}{\partial x^j}) \\ &\implies \\ (\nabla_{\frac{\partial}{\partial x^m}} dx^i)_j &= -\Gamma_{jm}^i \end{aligned}$$

In summary $(\nabla_X Y)^i = X(Y^i) + \Gamma_{jm}^i Y^j X^m$ and $(\nabla_X \omega)_i = X(\omega_i) - \Gamma_{im}^j \omega_j X^m$.³ So: $(\nabla_X T)_{jk}^i = X(T_{jk}^i) + \Gamma_{sm}^i T_{jk}^s X^m - \Gamma_{jm}^s T_{sk}^i X^m - \Gamma_{km}^s T_{js}^i X^m$ where T is a $(1, 2)$ - tensor field.

¹For $(1, 1)$ tensor field T , but analogously for any (p, q) - TF

²On vector fields

³On Covector fields