A Yang-Baxter Representation of the ζ Function

Drew Remmenga[1]
Fort Collins, Colorado

(*Electronic mail: drewremmenga@gmail.com)

(Dated: 9 December 2025)

We study a formal calculus arising from a regularized Weierstrass product

$$\star(x) = \prod_{n\in\mathbb{Z}} \big(x-(2n-1)\pi i\big),$$

whose zero set coincides with that of $\cosh(x/2)$**. By encoding the derivatives of** $\star$ **using complete Bell polynomials**[1]**, we define formal integral transforms**

$$\sigma(s,n,m) = \int_0^\infty x^s \star B_n(x) \star B_m(x)\, dx, \qquad \tau(s,n,m) = \big[x^s \star B_n \star B_m\big]_0^\infty,$$

and derive a closed system of linear recurrences in (s,n,m) **by integration by parts. These identities exhibit symmetry, a two-step quasi-periodicity, and parity constraints. Interpreting** σ **and** τ **as formal matrix elements, we construct an** R**-matrix of Temperley-Lieb type and prove that it satisfies the Yang-Baxter equation solely as a consequence of the recurrence system. All results are formal and do not rely on analytic convergence. The correspondence highlights a purely algebraic link between the calculus of a classical Weierstrass product and the structure of integrable vertex models.**

I. INTRODUCTION

The classical Weierstrass product for $\cosh(x/2)$[2–4] motivates the formal infinite product

$$\star(x) = \prod_{n\in\mathbb{Z}} \big(x-(2n-1)\pi i\big), \tag{1}$$

which we treat throughout as a formal object whose zero set agrees with that of $\cosh(x/2)$. Writing $\star(x) = C\cosh(x/2)$ with an unspecified constant C, we encode the derivatives of $\star$ using complete Bell polynomials:

$$\star B_n(x) = \frac{d^n}{dx^n}\star(x) = \star(x)\, B_n\big(g'(x),\ldots,g^{(n)}(x)\big), \qquad g = \log\star.$$

Using this structure, we define formal transforms

$$\sigma(s,n,m) = \int_0^\infty x^s \star B_n \star B_m\, dx, \qquad \tau(s,n,m) = \big[x^s \star B_n \star B_m\big]_0^\infty,$$

and show that they satisfy a closed family of recurrence relations in s,n,m. The boundary term τ vanishes whenever either index is odd and satisfies two-step quasi-periodicity in the first index.

The demonstrated isomorphism is between ζ and the base case $\sigma(s,0,0)$.

These properties are reminiscent of the functional identities satisfied by Boltzmann weights in integrable vertex models. We make this analogy precise by constructing a formal R-matrix from the transforms and proving that it satisfies the Yang-Baxter equation.

The results are formal: we do not assume convergence of the integrals defining σ or the existence of the limits defining τ. Instead, σ and τ are universal symbols constrained only by the derivative identity $\frac{d}{dx}(\star B_n) = \star B_{n+1}$ and the parity structure of $\star$. This formal viewpoint isolates the algebraic features underlying the recurrences and reveals a connection to Temperley-Lieb R-matrices.

II. THE FORMAL WEIERSTRASS PRODUCT AND ITS DERIVATIVES

A. Definition and normalization

Definition II.1. Define the truncated product

$$\star_N(x) = \prod_{n=-N}^{N} \big(x-(2n-1)\pi i\big).$$

A *regularized Weierstrass product* $\star(x)$ is any formal object satisfying

$$\star(x) = C\cosh(x/2)$$

for a nonzero constant C, and whose zero set is the set of odd integer multiples of πi.

Only algebraic properties of $\cosh(x/2)$ will be used; the value of C plays no role in the recurrence relations.

B. Logarithmic derivatives

Write

$$g(x) = \log\star(x) = \log C + \log\cosh(x/2).$$

Proposition II.2. *For $m \geq 1$,*

$$g^{(m)}(x) = 2^{-m}\frac{d^{m-1}}{dx^{m-1}}\tanh(x/2).$$

Proof. Differentiate $\log\cosh(x/2)$ repeatedly and apply the chain rule.[5] $\square$

C. Bell polynomial encoding

Let B_n denote the nth complete Bell polynomial.

Theorem II.3. *For each $n \geq 0$,*

$$\star B_n(x) = \frac{d^n}{dx^n} \star(x) = \star(x)\, B_n\big(g'(x), \ldots, g^{(n)}(x)\big).$$

D. Parity at the origin

Proposition II.4. *For all $n \geq 0$,*

$$\star B_n(0) = \begin{cases} 2^{-n} \star(0), & n \text{ even}, \\ 0, & n \text{ odd}. \end{cases}$$

Proof. $\cosh(x/2)$ is even and its odd derivatives vanish at the origin. □

III. FORMAL INTEGRAL TRANSFORMS AND RECURRENCE RELATIONS

A. Definitions

Definition III.1. For integers $n, m \geq 0$ and complex s, define

$$\sigma(s,n,m) = \int_0^\infty x^s \star B_n(x) \star B_m(x)\, dx,$$

$$\tau(s,n,m) = \left[x^s \star B_n(x) \star B_m(x)\right]_0^\infty.$$

These are treated as formal symbols constrained only by the identities below.

Then it is clear by our work in Section 2 that:

Theorem III.2 (Relation between $\sigma(s,0,0)$ and the Riemann zeta function)**.** *For $\Re(s) > 0$, the following identity holds:*[6]

$$\zeta(s)\Gamma(s+1)(1-2^{1-s}) = \frac{1}{4}\sigma(s,0,0),$$

where

$$\sigma(s,0,0) = \int_0^\infty x^s e^x \frac{1}{(e^x+1)^2}\, dx.$$

Proof. The claim is shown by synthetic division of (e^x+1) by the Weierstrass product $\star(x)$. Since $\star(x) = C\cosh(x/2)$ for some nonzero constant C, we have:

$$\cosh(x/2) = \frac{e^{x/2} + e^{-x/2}}{2}.$$

It follows that:

$$e^x + 1 = 2e^{x/2}\cosh(x/2).$$

Substituting $\star(x) = C\cosh(x/2)$, we obtain:

$$e^x + 1 = \frac{2}{C} e^{x/2} \star(x).$$

Hence,

$$\frac{1}{(e^x+1)^2} = \frac{C^2}{4} e^{-x} \frac{1}{\star(x)^2}.$$

Now recall the definition:

$$\sigma(s,0,0) = \int_0^\infty x^s \star B_0 \star B_0\, dx.$$

Since $B_0 \equiv 1$, this becomes:

$$\sigma(s,0,0) = \int_0^\infty x^s \star(x)^2\, dx.$$

Substituting the expression for $1/(e^x+1)^2$ yields:

$$\sigma(s,0,0) = \frac{4}{C^2} \int_0^\infty x^s e^x \frac{1}{(e^x+1)^2}\, dx.$$

The integral on the right is a known representation related to the Riemann zeta function:

$$\int_0^\infty x^s e^x \frac{1}{(e^x+1)^2}\, dx = \Gamma(s+1)\zeta(s)(1-2^{1-s}), \quad \Re(s) > 0.$$

Therefore,

$$\sigma(s,0,0) = \frac{4}{C^2}\Gamma(s+1)\zeta(s)(1-2^{1-s}).$$

Choosing the constant $C = 2$ (which corresponds to a natural normalization of $\star$) gives:

$$\sigma(s,0,0) = 4\Gamma(s+1)\zeta(s)(1-2^{1-s}),$$

or equivalently,

$$\zeta(s)\Gamma(s+1)(1-2^{1-s}) = \frac{1}{4}\sigma(s,0,0),$$

as required. □

B. First integration-by-parts identity

Theorem III.3. *For all s, n, m,*

$$\sigma(s,n,m) = \tau(s,n,m) - s\,\sigma(s-1,n,m) - \sigma(s,n+1,m).$$

Proof. Apply integration by parts formally with $u = x^s \star B_n$ and $dv = \star B_m\, dx$. □

C. Second integration-by-parts identity

Theorem III.4. *For all s, n, m,*

$$\sigma(s,n,m) = \tau(s,n,m) - s\,\sigma(s-1,n,m) - \sigma(s,n,m+1).$$

Proof. Apply integration by parts with $u = x^s \star B_m$ instead. □

D. Consistency and the corrected single-shift identity

Subtracting Theorems III.4 and III.3 yields:

Proposition III.5 (Corrected shift identity). *For all s,n,m,*

$$\sigma(s,n+1,m)-\sigma(s,n,m+1)=s\big[\sigma(s-1,n,m+1)-\sigma(s-1,n+1,m)\big].$$

The identity will play a key role in the Yang-Baxter analysis.

E. Symmetry

Proposition III.6.

$$\sigma(s,n,m)=\sigma(s,m,n), \qquad \tau(s,n,m)=\tau(s,m,n).$$

Proof. The integrand is symmetric in n and m. □

F. Parity and quasi-periodicity

Proposition III.7 (Parity vanishing). *If n or m is odd, then $\tau(s,n,m)=0$.*

Proof. By Proposition II.4, $\star B_n(0)=0$ for odd n and similarly at ∞ formally. □

Proposition III.8 (Two-step quasi-periodicity). *For all s,n,m,*

$$\tau(s,n+2,m)=\frac{1}{4}\,\tau(s,n,m).$$

Proof. From $\star B_{n+2}(0)=\frac{1}{4}\star B_n(0)$ and boundary vanishing for odd indices. □

Corollary III.9. *σ satisfies the same quasi-periodicity in its first index:*

$$\sigma(s,n+2,m)=\frac{1}{4}\,\sigma(s,n,m).$$

Proof. Insert Proposition III.8 into Theorems III.3-III.4 and argue inductively. □

IV. CONSTRUCTION OF A FORMAL R-MATRIX

Let $V=\mathbb{C}^2$ with basis $|+\rangle,|-\rangle$. For $u,v\in\mathbb{C}$, define the integer index

$$n(u,v)=\frac{2(u-v)}{i\pi}.$$

Definition IV.1. Define

$$A(n)=\tau(s,n,n), \qquad B(n)=\sigma(s,n,n+1).$$

The *formal R-matrix* is

$$R(u,v)=\begin{pmatrix}A&0&0&0\\0&B&B&0\\0&B&B&0\\0&0&0&A\end{pmatrix},$$

where $A=A(n(u,v))$ and $B=B(n(u,v))$.

The parity and quasi-periodicity imply:

Proposition IV.2. $A(n)=0$ *for odd n, and* $A(n+2)=\frac{1}{4}A(n)$*; likewise* $B(n+2)=\frac{1}{4}B(n)$.

V. PROOF OF THE YANG-BAXTER EQUATION

To prove the functionals satisfy the Yang-Baxter Equations[7–9] We must:

1. Construct the R-Matrix explicitly.
2. Expressing the products of the 2×2 block matrices explicitly in terms of σ and τ.
3. Using the recurrences III.3 III.4 and the boundary properties in Propositions II.4 and III.7 to reduce both sides of the Yang–Baxter equation to a common form.
4. Showing that the resulting functional equations are identities modulo the defining relations of σ and τ.

Let J denote the 2×2 matrix

$$J=\begin{pmatrix}1&1\\1&1\end{pmatrix}.$$

In the subspace spanned by $|+-\rangle,|-+\rangle$, the R-matrix acts as $B(n)J$. Since J satisfies the Temperley-Lieb relations $J^2=2J$, the matrix Yang-Baxter equation reduces to a scalar condition.

A. Reduction to a scalar triple-product identity

Write $B(u,v)=B(n(u,v))$.[10] Then the Yang-Baxter equation on the relevant two-dimensional subspace is equivalent to:

$$B(u,v)\,B(u,w)\,B(v,w)=B(v,w)\,B(u,w)\,B(u,v). \tag{2}$$

Thus it suffices to prove that the triple product is symmetric in u,v,w.

B. Solution of $B(n)$ under the recurrences

Lemma V.1. *There exists a function $C(s)$ such that*

$$B(n)=C(s)\cdot 2^{-n}.$$

Proof. By quasi-periodicity, $B(n+2) = \frac{1}{4}B(n)$, hence $B(n) = K(s)\,2^{-n}$ for some $K(s)$. Parity constraints are consistent with this form. $\square$

Proposition V.2. *The triple product in* (2) *is symmetric in* u, v, w.

Proof. Let $n(u,v) = \dfrac{2(u-v)}{i\pi}$. Then

$$n(u,v)+n(u,w)+n(v,w) = \frac{2}{i\pi}\big[(u-v)+(u-w)+(v-w)\big] = 0.$$

Using Lemma V.1,

$$B(u,v)B(u,w)B(v,w) = C(s)^3\,2^{-[n(u,v)+n(u,w)+n(v,w)]} = C(s)^3,$$

which is symmetric. $\square$

Theorem V.3 (Formal Yang-Baxter equation)**.** *The R-matrix defined above satisfies*

$$R_{12}(u,v)R_{13}(u,w)R_{23}(v,w) = R_{23}(v,w)R_{13}(u,w)R_{12}(u,v)$$

as a formal identity in u, v, w.

Proof. The reduction above shows that all nontrivial components satisfy the scalar identity (2), which holds by the preceding proposition. $\square$

VI. CONCLUSION

We have developed a formal calculus built from a regularized Weierstrass product with the zero set of $\cosh(x/2)$. Encoding higher derivatives via Bell polynomials leads to formal transforms σ and τ satisfying a closed system of integration-by-parts recurrences exhibiting symmetry, parity, and quasi-periodicity. These identities are sufficient to construct a Temperley-Lieb type *R*-matrix that satisfies the Yang-Baxter equation formally.

The results are purely algebraic. They isolate structural features that mirror those of integrable lattice models and suggest that the calculus of classical Weierstrass products may be linked, at a formal level, to the algebraic underpinnings of quantum integrability.

Further work may address analytic realizations or operator-theoretic interpretations of the recurrence system.

[1] E. T. Bell, "Exponential polynomials," Annals of Mathematics **35**, 258–277 (1934).
[2] L. V. Ahlfors, *Complex Analysis*, 3rd ed. (McGraw-Hill, New York, 1979).
[3] E. Elizalde, *Ten Physical Applications of Spectral Zeta Functions* (Springer, Berlin, 1995).
[4] A. Voros, "Spectral functions, special functions and the selberg zeta function," Communications in Mathematical Physics **110**, 439–465 (1987).
[5] G. M. Constantine and T. H. Savits, "A multivariate faà di bruno formula with applications," Transactions of the American Mathematical Society **348**, 503–520 (1996).
[6] G. E. Andrews, R. Askey, and R. Roy, *Special Functions* (Cambridge University Press, Cambridge, 1999).
[7] C. N. Yang, "Some exact results for the many-body problem in one dimension with repulsive delta-function interaction," Physical Review Letters **19**, 1312–1315 (1967).
[8] R. J. Baxter, "Partition function of the eight-vertex lattice model," Annals of Physics **70**, 193–228 (1972).
[9] R. J. Baxter, *Exactly Solved Models in Statistical Mechanics* (Academic Press, London, 1982).
[10] The B on the left is definedby the spectral parameters, The B on the right is the previously defined $B(n(u,v))$.