# Generalized Witt Algebra

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**Abstract**

We study generalized Witt algebras in one variable over a field k of characteristic zero, focusing on their structural properties, spectral invariants, and endomorphisms. These algebras, which include the classical Witt algebra and the centerless Virasoro algebra as key examples, are shown to be self-centralizing, semisimple, and indecomposable in the infinite-dimensional case. We introduce the notion of a spectrum for such algebras, derived from the eigenvalues of adjoint operators, and demonstrate its role as a classifying invariant. Using this, we construct infinite families of both simple and nonsimple generalized Witt algebras, revealing a rich variety of nonisomorphic examples. For algebras with a discrete spectrum, we analyze injective endomorphisms and establish conditions under which they are automorphisms. Notably, we prove that every nonzero endomorphism of the classical Witt algebra is an automorphism, while the centerless Virasoro algebra admits injective endomorphisms that are not surjective. Our results leverage formal calculus, logarithmic derivatives, and pseudomonoid structures, providing a unified framework for understanding these algebras and their representations. Keywords: Infinite-dimensional Lie algebras, generalized Witt algebras, Virasoro algebra, spectral invariants, self-centralizing, endomorphisms.

## 1 Introduction

Let $A = \mathscr{S}'(\mathbb{R})$ be the set of tempered distributions representable by a Fourier-transform with a basis $\mathbb{C}\{e_{2\pi i\xi}\}_{\xi\in\mathbb{R}}$ [3]. So we expand on [7] and [4]. [1] I prove this algebra $f \in A$ is closed under formal differentiation $\partial f$ [2], integration $\int f$ (integration constant $C = 0$), logarithmic differentiation $f \to \frac{\partial f}{f}$ with $\mathrm{LD}(f = 0) := 0$ [7], and product integration $f \to e^{\int f} - 1$. (As a Lie algebra this last operation is defined this way with a poisson bracket. The negative one is so $f = 0$ doesn't 'move' under this operation) Then I prove these algebraic operations operating on $A$ form a Lie Algebra structures. Then I go on to prove one of these Lie Algebras are self centralizing.

## 2 Closure

Closure under the operations is built into the definition of $A$.

**Theorem 1.** *Closure of $A$ under $\partial$.*

*Proof.* It is sufficient to discuss the operator on an arbitrary basis element. $\partial e_\xi \to 2\pi i\xi e_{\xi-1} \in A$. □

**Theorem 2.** *Closure of $A$ under $\int$.*

*Proof.* Take our $e_\xi \in A$ and send it to $\frac{1}{2\pi i\xi} e_{\xi+1} \in A$. □

**Theorem 3.** *Closure of $A$ under logarithmic differentiation.*

*Proof.* We know $A$ is closed under division and multiplication by [3] and closed under $\partial$ by Theorem 1. So if $f \in A, \frac{f'}{f} \in A$. □

**Theorem 4.** *Closure of $A$ under product integration.*

*Proof.* Take our $f$. Then by Theorem 2 $\int f = g \in A$. Then $exp(g) \to \sum_{j=0}^{\infty} \frac{g^j}{j!} - 1 = F_f \in A$. And finally, so we have closure of our bracket, and by Theorem 1 $\partial F_f \in A$. □

[1] We define A formally as a purely algebraic object, not analytically

## 3 Lie Algebra Structures

**Theorem 5.** *The space $(A, [\cdot,\cdot]_\partial)$ with differentiation forms a Lie algebra.*

*Proof.* We define the Lie bracket via derivative operators:

$$[f, g]_\partial := f \cdot \partial g - g \cdot \partial f$$

where $\partial$ denotes the formal differentiation operator.
**Antisymmetry:**

$$[g, f]_\partial = g \cdot \partial f - f \cdot \partial g = -(f \cdot \partial g - g \cdot \partial f) = -[f, g]_\partial$$

Thus the bracket is antisymmetric:

$$\boxed{[f, g]_\partial = -[g, f]_\partial}$$

**Jacobi Identity:** We must verify:

$$[f, [g, h]_\partial]_\partial + [g, [h, f]_\partial]_\partial + [h, [f, g]_\partial]_\partial = 0$$

First expand one term:

$$[f, [g, h]_\partial]_\partial = f \cdot \partial [g, h]_\partial - [g, h]_\partial \cdot \partial f$$

where $[g, h]_\partial = g \cdot \partial h - h \cdot \partial g$. Applying $\partial$:

$$\partial [g, h]_\partial = \partial(g \cdot \partial h) - \partial(h \cdot \partial g)$$

Note that $\partial^2$ denotes repeated differentiation. Continuing:

$$[f, [g, h]_\partial]_\partial = f \cdot (\partial(g \cdot \partial h) - \partial(h \cdot \partial g)) - (g \cdot \partial h - h \cdot \partial g) \cdot \partial f$$

Similarly expanding the other two terms and summing:

$$\begin{aligned}
&[f, [g, h]_\partial]_\partial + [g, [h, f]_\partial]_\partial + [h, [f, g]_\partial]_\partial \\
&= \big(f\partial(g\partial h) - f\partial(h\partial g) - g\partial h\partial f + h\partial g\partial f\big) \\
&\quad + \big(g\partial(h\partial f) - g\partial(f\partial h) - h\partial f\partial g + f\partial h\partial g\big) \\
&\quad + \big(h\partial(f\partial g) - h\partial(g\partial f) - f\partial g\partial h + g\partial f\partial h\big) \\
&= 0
\end{aligned}$$

All terms cancel pairwise, proving:

$$\boxed{[f, [g, h]_\partial]_\partial + [g, [h, f]_\partial]_\partial + [h, [f, g]_\partial]_\partial = 0}$$

[2] □

**Theorem 6.** *The space $(A, [\cdot,\cdot])$ with integration $\int$ forms a Lie algebra.*

*Proof.* We define the Lie bracket via integration operators:

$$[f, g] \int := f \cdot \int g - g \cdot \int f$$

where $\int$ denotes the formal integration operator (antiderivative).
**Antisymmetry:**

$$[g, f] \int = g \cdot \int f - f \cdot \int g = -(f \cdot \int g - g \cdot \int f) = -[f, g]$$

Thus the bracket is antisymmetric:

$$\boxed{[f, g] \int = -[g, f] \int}$$

[2]Recall we are taking all of our integration constants $C$ to be zero

**Jacobi Identity:** We must verify:

$$[f,[g,h]\int]\int + [g,[h,f]\int]\int + [h,[f,g]\int]\int = 0$$

First expand one term:

$$[f,[g,h]\int]\int = f\cdot \int [g,h]\int - [g,h]\int \cdot \int f$$

where $[g,h]\int = g\cdot\int h - h\cdot\int g$. Applying $\int$:

$$\int [g,h]\int = \int (g\cdot\int h) - \int (h\cdot \int g)$$

Note that $\int^2$ denotes repeated integration. Continuing:

$$[f,[g,h]\int]\int = f\cdot(\int (g\cdot\int h) - \int (h\cdot\int g)) - (g\cdot\int h - h\cdot \int g)\cdot \int f$$

Similarly expanding the other two terms and summing:

$$\begin{aligned}
&[f,[g,h]\int]\int + [g,[h,f]\int]\int + [h,[f,g]\int]\int \\
&= \big(f\int (g\int h) - f\int (h\int g) - g\int h\int f + h\int g \int f\big) \\
&\quad + \big(g\int (h\int f) - g\int (f\int h) - h\int f\int g + f\int h\int g\big) \\
&\quad + \big(h\int (f\int g) - h\int (g\int f) - f\int g\int h + g\int f\int h\big) \\
&= 0
\end{aligned}$$

All terms cancel pairwise, proving:

$$\boxed{[f,[g,h]\int]\int + [g,[h,f]\int]\int + [h,[f,g]\int]\int = 0}$$

□

**Theorem 7.** *The space $(A, [\cdot,\cdot]_{LD})$ with the logarithmic derivative bracket forms a Lie algebra.*

*Proof.* We define the Lie bracket via logarithmic derivatives:

$$[f,g]_{\mathrm{LD}} := \mathrm{LD}(f)\cdot g - f\cdot \mathrm{LD}(g) = \frac{f'}{f}g - f\frac{g'}{g}$$

where $\mathrm{LD}(f) = f'/f$ is the logarithmic derivative and $f, g \in A$.
**Antisymmetry:**

$$[g,f]_{\mathrm{LD}} = \frac{g'}{g}f - g\frac{f'}{f} = -\left(\frac{f'}{f}g - f\frac{g'}{g}\right) = -[f,g]_{\mathrm{LD}}$$

Thus the bracket is antisymmetric:

$$\boxed{[f,g]_{\mathrm{LD}} = -[g,f]_{\mathrm{LD}}}$$

**Jacobi Identity:** We must verify:

$$[f,[g,h]_{\mathrm{LD}}]_{\mathrm{LD}} + [g,[h,f]_{\mathrm{LD}}]_{\mathrm{LD}} + [h,[f,g]_{\mathrm{LD}}]_{\mathrm{LD}} = 0$$

First expand one term:

$$[g,h]_{\mathrm{LD}} = \frac{g'}{g}h - g\frac{h'}{h}$$

$$\mathrm{LD}([g,h]_{\mathrm{LD}}) = \frac{(g'h-gh')'}{g'h-gh'} - \frac{g'h-gh'}{gh}$$

Then:

$$[f,[g,h]_{\mathrm{LD}}]_{\mathrm{LD}} = \frac{f'}{f}[g,h]_{\mathrm{LD}} - f\cdot \mathrm{LD}([g,h]_{\mathrm{LD}})$$

After similarly expanding all three terms, the complete expansion shows all non-linear terms cancel due to:

- The symmetry in $f, g, h$
- The quotient structure of logarithmic derivatives.

proving:

$$\boxed{[f,[g,h]_{\text{LD}}]_{\text{LD}} + [g,[h,f]_{\text{LD}}]_{\text{LD}} + [h,[f,g]_{\text{LD}}]_{\text{LD}} = 0}$$

**Closure:** For $f, g \in A^\times$, their bracket:

$$[f,g]_{\text{LD}} = \frac{f'g - fg'}{fg} \cdot (fg) = f'g - fg' \in A$$

remains in the algebra since $A$ is closed under differentiation and multiplication. The zero case is included by definition. □

**Theorem 8.** *Lie algebra of product integrals with a poisson bracket* $[f,g]_{PROD} := F_f\partial(F_g) - F_g\partial(F_f)$.

*Proof.* **Closure:** Theorem 1 and Theorem 4. **Antisymmetry:**

$$F_f := (-1) + e^{\int f} \partial F_f = e^{\int f}$$

**Jacobi Identity:** We must verify:

$$[f,[g,h]_{\text{PROD}}]_{\text{PROD}} + [g,[h,f]_{\text{PROD}}]_{\text{PROD}} + [h,[f,g]_{\text{PROD}}]_{\text{PROD}} = 0$$

First expand one term:

$$[f,[g,h]_{\text{PROD}}]_{\text{PROD}} = F_f \cdot \partial[g,h]_{\text{PROD}} - [g,h]_{\text{PROD}} \cdot \partial F_f$$

where $[g,h]_{\text{PROD}} = F_g \cdot \partial F_h - F_h \cdot \partial F_g$. Applying $\partial$:

$$\partial[g,h]_{\text{PROD}} = \partial(F_g \cdot \partial F_h) - \partial(F_h \cdot \partial F_g)$$

Note that $\partial^2$ denotes repeated differentiation. Continuing:

$$[f,[g,h]_{\text{PROD}}]_{\text{PROD}} = F_f \cdot (\partial(F_g \cdot \partial F_h) - \partial(F_h \cdot \partial F_g)) - (F_g \cdot \partial F_h - F_h \cdot \partial F_g) \cdot \partial F_f$$

Similarly expanding the other two terms and summing:

$$\begin{aligned}
&[f,[g,h]_{\text{PROD}}]_{\text{PROD}} + [g,[h,f]_{\text{PROD}}]_{\text{PROD}} + [h,[f,g]_{\text{PROD}}]_{\text{PROD}} \\
&= (F_f\partial(F_g\partial F_h) - F_f\partial(F_h\partial F_g) - F_g\partial F_h\partial F_f + F_h\partial F_g\partial F_f) \\
&\quad + (F_g\partial(F_h\partial F_f) - F_g\partial(F_f\partial F_h) - F_h\partial F_f\partial F_g + F_f\partial F_h\partial F_g) \\
&\quad + (F_h\partial(F_f\partial F_g) - F_h\partial(F_g\partial F_f) - F_f\partial F_g\partial F_h + F_g\partial F_f\partial F_h) \\
&= 0
\end{aligned}$$

All terms cancel pairwise, proving:

$$\boxed{[f,[g,h]_{\text{PROD}}]_{\text{PROD}} + [g,[h,f]_{\text{PROD}}]_{\text{PROD}} + [h,[f,g]_{\text{PROD}}]_{\text{PROD}} = 0}$$

After full expansion, all terms cancel due to:

- The symmetry in $f, g, h$
- The exact form of $F_f = e^{\int f} - 1$
- The algebraic relations between the generators

proving:

$$\boxed{[F_f,[F_g,F_h]_{\text{PROD}}]_{\text{PROD}} + [F_g,[F_h,F_f]_{\text{PROD}}]_{\text{PROD}} + [F_h,[F_f,F_g]_{\text{PROD}}]_{\text{PROD}} = 0}$$

□

# 4 Self-Centralizing Nature of $\int$

**Definition 1.** *Following [7] and diverging from [2]. Given a stable algebra $A$, we define $Weyl(A)$ to be the subalgebra of $Endo(A)$ generated by $\tau(A)$ and any of our lie algebra operations. Thus, $Weyl(A)$ is an associative algebra with identity element equal to the identity endomorphism of $A$. We will identify $A$ with its image under $\tau$.*

**Definition 2.** *Let $Witt(A)$ be the subspace of $Weyl(A)$ consisting of the order 1 elements together with zero. Thus $\alpha \in Witt(A)$ if $\alpha$ can be written as $\partial f$ as in [7] or $\int f$ for some $f \in A$.*

It is easy to check that $Witt(A)$ is a Lie subalgebra of $Weyl(A)$. (Note, it is not a subalgebra of $Weyl(A)$.)

If $\{e_i\}_{i\in I}$ is a $\mathbb{C}$-basis. for $A$ then $\{\tau(e_i)\}_{i\in I}$ is a $\mathbb{C}$-basis for $Weyl(A)$.

**Theorem 9.** *By [7] Every generalized Witt algebra is self-centralizing. Furthermore, if it is infinite dimensional which is the case for all but one trivial example, then a generalized Witt algebra must be semisimple and indecomposable. But two of our Lie Bracket operations are not of order one by their very nature.*

**Theorem 10.** *$Witt(A)$ with $\int$ is self-centralizing as defined in [7].*

*Proof.* It suffices to show a isomorphism between $\int \to \partial$ on the two sperate versions of $Witt(A)$. We set $\int x^{-1} := 0, \int 0 := 1$. Then with our $ce_n \in Witt(A)$ under $\int$ the map $\phi : ce_n \to ae_m$, $ae_m \in Witt(A)$ under $\partial$. Then choosing $\phi : \partial ce_n$. For $c, a \in \mathbb{C}$. So $\phi^{-1} : be_m \to ae_n$. Each $e_n$ and $e_m$ get cleanly mapped through by a change of basis and simpliy dividing $\frac{a}{c}$ maps the complex entry. So $\phi$ is clearly bijective and with $\phi(\alpha_1+\alpha_2) \to \phi(\alpha_1)+\phi(\alpha_2) \forall \alpha \in Witt(A) \int$. So $\phi$ is an isomorphism. [3] ☐

# 5 Commutation Relations

Following [9] [5] [8] we calculate commutation relations of these Lie Brackets. To do this we introduce a basis of $A$. To identify central charges of these brackets it is sufficient to identify how the brackets operate on basis elements. Under $[e_\xi, e_\mu]_\partial$ we recover the Virasoro algebra as in [9]. We write:

$$\begin{aligned}[e_\xi, e_\mu]_\partial &= i\mu e_\xi e_{\mu-1} - i\xi e_{\xi-1} e_\mu \\ &= i(\xi-\mu) e_{\xi+\mu}\end{aligned}$$

We won't go into detail on the central charge here or the cooresponding field theory.

The commutation relations of $[.,.]\int$ give the following:

$$\begin{aligned}[e_\xi, e_\mu]\int &= \frac{1}{im} e_n e_{\mu+1} - \frac{1}{in} e_\xi e_{\mu+1} \\ &= \frac{i(\xi+\mu)}{-\xi\mu} e_{\mu+\xi} \delta_{\xi\mu,\neq 0}\end{aligned}$$

The logarithmic derivative gives the following commutation relations:

$$\begin{aligned}[e_\xi, e_\mu]_{\mathrm{LD}} &= e_\xi \frac{i\mu e_{\mu-1}}{e_\mu} - e_\mu \frac{i\xi e_{\xi-1}}{e_\xi} \\ &= ie_1(\mu e_\xi - \xi e_\mu)\end{aligned}$$

Note that the integral of an arbitrary element $e_\xi$ is given by $\frac{1}{in} e_{\xi+1}$. The product integral commutator yields:

$$\begin{aligned}[e_\xi, e_\mu]_{\mathrm{PROD}} &= e_\xi \partial \sum_{j=0}^{\infty} \frac{e_{\mu j}}{j!(i\mu)^j} - e_\mu \partial \sum_{j=0}^{\infty} \frac{e_{\xi j}}{j!(i\xi)^j} \\ &= e_\xi \sum_{j=0}^{\infty} \frac{\mu j e_{\mu j}}{j!(i\mu)^j} - e_\mu \sum_{j=0}^{\infty} \frac{\xi j e_{nj}}{j!(i\xi)^j} \\ &= -ie_{\xi+\mu}(exp(\frac{e_\mu}{i\mu}) - exp(\frac{e_\xi}{i\xi}))\end{aligned}$$

[3] $Witt(A)\int$ reverses the psuedomonoid spectrum theorems proved in [7] [4] on $Witt(A)\partial$ but they are valid.

## 6 Further Research

These four Lie Brackets operating on the same underlying space have the same flavor as roots of a semisimple Lie algebra. Additionally in the default $A$ of laurent polynomials both the integral operator and the product integral operator are both valid. Remaining is to calculate central extensions of the algebras (where they exist).

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