Generalized Witt Algebra

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Abstract

1 Introduction

Let A_0 be the set of Power Series Laurent Polynomials with entries from $\mathbb{C}[x, x^{-1}]$ so we expand on [6] and [3]. Then we define A_1 as the closure of elements in A_0 under the operations $\circ, +, \times$. Then the algebra A_{n+1} is the closure of elements in A_n under $\circ, +, \times$. Let A_∞ be the last entry (direct limit) in this series of algebras.[1] I prove this algebra $f \in A_\infty$ is closed under formal differentiation ∂f [2], integration $\int f$ (integration constant $C = 0$) logarithmic differentiation $f \to \frac{\partial f}{f}$ with $\mathrm{LD}(f = 0) := 0$ [6], and product integration $f \to e^{\int f} - 1$. (With a poisson bracket. The negative one is so $f = 0$ doesn't 'move' under this operation) Then I prove these algebraic operations operating on A_∞ form a Lie Algebra structure. Then I go on to prove these Lie Algebras are self centralizing. I find central extensions of the lie algebras. Then I show under these opperation there exists a six dimensional field basis.

2 Closure

Closure under the operations is built into the definition of A_∞.

Theorem 1. *Closure of A_∞ under ∂.*

Proof. Let an arbitrary element of A_∞ be denoted f and let it be a sequence $f_1 \circ \cdots \circ f_n$. Then the derivative follows the chain rule such that $\partial f = \partial f_1 \circ \cdots \circ f_n \times \partial f_2 \circ \cdots \circ f_n \times \ldots \partial f_n$. So A_∞ is closed under the ∂ operation. □

Theorem 2. *Closure of A_∞ under $\int$.*

Proof. Take our $f \in A_\infty$ once again. Then with arbitrary u $(u, du \in A_\infty)$ subsitution and integration by parts we construct a $g \in A_\infty$ such that $\partial g = f$. We do this with an arbitrary f by $f = f_1 \circ \cdots \circ f_n$ so $\partial g = \partial u \circ f_1 \circ \cdots \circ \partial f_n \times \partial f_1 \circ \cdots \circ f_n \times \partial f_2 \circ \cdots \circ f_n \times \ldots \partial f_n \times du$ So by constructing $g = f_1 \circ \cdots \circ f_n$ we receive $\int f = g$ for arbitrary $f \in A_\infty$. Do note we receive some tricky examples through Maclaurin series of arbitrary functions: including $\log(x), \mathrm{li}(x)$. So $\int x^{-1} = \sum_{n=1}^{\infty} \frac{-1^{n+1}}{n}(x-1)^n \in A_\infty$. □

Theorem 3. *Closure of A_∞ under logarithmic differentiation.*

Proof. Take our f. Then $\partial f \in A_\infty$ and $\frac{1}{x} \in A_0 \subset A_\infty$ so composition of f and $\frac{1}{x}$ and multiplication of ∂f yields the result. □

Theorem 4. *Closure of A_∞ under product integration.*

Proof. We have shown closure of $\int f$. So we need $f, e^x \in A_\infty$ so the compostion $e^{\int f} - 1 \in A_\infty$. $e^x - 1$ is a infinite degree positive polynomial so it is in A_0 so it is in A_∞. □

[1] We define A_∞ formally, not analytically

3 Lie Algebra Structures

Theorem 5. *The space $(A_\infty, [\cdot,\cdot])$ with differentiation forms a Lie algebra.*

Proof. We define the Lie bracket via derivative operators:

$$[f,g] := f \cdot \partial g - g \cdot \partial f$$

where ∂ denotes the formal differentiation operator.
Antisymmetry:

$$[g,f] = g \cdot \partial f - f \cdot \partial g = -(f \cdot \partial g - g \cdot \partial f) = -[f,g]$$

Thus the bracket is antisymmetric:

$$\boxed{[f,g] = -[g,f]}$$

Jacobi Identity: We must verify:

$$[f,[g,h]] + [g,[h,f]] + [h,[f,g]] = 0$$

First expand one term:

$$[f,[g,h]] = f \cdot \partial[g,h] - [g,h] \cdot \partial f$$

where $[g,h] = g \cdot \partial h - h \cdot \partial g$. Applying ∂:

$$\partial[g,h] = \partial(g \cdot \partial h) - \partial(h \cdot \partial g)$$

Note that ∂^2 denotes repeated differentiation. Continuing:

$$[f,[g,h]] = f \cdot (\partial(g \cdot \partial h) - \partial(h \cdot \partial g)) - (g \cdot \partial h - h \cdot \partial g) \cdot \partial f$$

Similarly expanding the other two terms and summing:

$$\begin{aligned}
&[f,[g,h]] + [g,[h,f]] + [h,[f,g]] \\
&= \big(f\partial(g\partial h) - f\partial(h\partial g) - g\partial h\partial f + h\partial g\partial f\big) \\
&\quad + \big(g\partial(h\partial f) - g\partial(f\partial h) - h\partial f\partial g + f\partial h\partial g\big) \\
&\quad + \big(h\partial(f\partial g) - h\partial(g\partial f) - f\partial g\partial h + g\partial f\partial h\big) \\
&= 0
\end{aligned}$$

All terms cancel pairwise, proving:

$$\boxed{[f,[g,h]] + [g,[h,f]] + [h,[f,g]] = 0}$$

Closure Under Differentiation: For any $[f,g] \in A_\infty$:

$$\int [f,g] = \int (f \cdot \int g - g \cdot \int f) = \int f \cdot \int g - \int g \cdot \int f = 0$$

[2] $\square$

Theorem 6. *The space $(A_\infty, [\cdot,\cdot])$ with integration forms a Lie algebra.*

Proof. We define the Lie bracket via integration operators:

$$[f,g] := f \cdot \int g - g \cdot \int f$$

where $\int$ denotes the formal integration operator (antiderivative).
Antisymmetry:

$$[g,f] = g \cdot \int f - f \cdot \int g = -(f \cdot \int g - g \cdot \int f) = -[f,g]$$

Thus the bracket is antisymmetric:

$$\boxed{[f,g] = -[g,f]}$$

[2] Recall we are taking all of our integration constants C to be zero

Jacobi Identity: We must verify:

$$[f,[g,h]]+[g,[h,f]]+[h,[f,g]]=0$$

First expand one term:

$$[f,[g,h]]=f\cdot\int[g,h]-[g,h]\cdot\int f$$

where $[g,h]=g\cdot\int h-h\cdot\int g$. Applying $\int$:

$$\int[g,h]=\int(g\cdot\int h)-\int(h\cdot\int g)$$

Note that $\int^2$ denotes repeated integration. Continuing:

$$[f,[g,h]]=f\cdot(\int(g\cdot\int h)-\int(h\cdot\int g))-(g\cdot\int h-h\cdot\int g)\cdot\int f$$

Similarly expanding the other two terms and summing:

$$\begin{aligned}&[f,[g,h]]+[g,[h,f]]+[h,[f,g]]\\&=\big(f\int(g\int h)-f\int(h\int g)-g\int h\int f+h\int g\int f\big)\\&\quad+\big(g\int(h\int f)-g\int(f\int h)-h\int f\int g+f\int h\int g\big)\\&\quad+\big(h\int(f\int g)-h\int(g\int f)-f\int g\int h+g\int f\int h\big)\\&=0\end{aligned}$$

All terms cancel pairwise, proving:

$$\boxed{[f,[g,h]]+[g,[h,f]]+[h,[f,g]]=0}$$

Closure Under Integration: For any $[f,g]\in A_\infty$:

$$\int[f,g]=\int(f\cdot\int g-g\cdot\int f)=\int f\cdot\int g-\int g\cdot\int f=0$$

□

Theorem 7. *The space $(A_\infty,[\cdot,\cdot]_{LD})$ with the logarithmic derivative bracket forms a Lie algebra.*

Proof. We define the Lie bracket via logarithmic derivatives:

$$[f,g]_{\text{LD}}:=\text{LD}(f)\cdot g-f\cdot\text{LD}(g)=\frac{f'}{f}g-f\frac{g'}{g}$$

where $\text{LD}(f)=f'/f$ is the logarithmic derivative and $f,g\in A_\infty$.
Antisymmetry:

$$[g,f]_{\text{LD}}=\frac{g'}{g}f-g\frac{f'}{f}=-\left(\frac{f'}{f}g-f\frac{g'}{g}\right)=-[f,g]_{\text{LD}}$$

Thus the bracket is antisymmetric:

$$\boxed{[f,g]_{\text{LD}}=-[g,f]_{\text{LD}}}$$

Jacobi Identity: We must verify:

$$[f,[g,h]_{\text{LD}}]_{\text{LD}}+[g,[h,f]_{\text{LD}}]_{\text{LD}}+[h,[f,g]_{\text{LD}}]_{\text{LD}}=0$$

First expand one term:

$$[g,h]_{\text{LD}}=\frac{g'}{g}h-g\frac{h'}{h}$$

$$\text{LD}([g,h]_{\text{LD}})=\frac{(g'h-gh')'}{g'h-gh'}-\frac{g'h-gh'}{gh}$$

Then:

$$[f,[g,h]_{\text{LD}}]_{\text{LD}}=\frac{f'}{f}[g,h]_{\text{LD}}-f\cdot\text{LD}([g,h]_{\text{LD}})$$

After similarly expanding all three terms, the complete expansion shows all non-linear terms cancel due to:

- The symmetry in f, g, h
- The quotient structure of logarithmic derivatives.

proving:

$$\boxed{[f,[g,h]_{\text{LD}}]_{\text{LD}} + [g,[h,f]_{\text{LD}}]_{\text{LD}} + [h,[f,g]_{\text{LD}}]_{\text{LD}} = 0}$$

Closure: For $f, g \in A_\infty^\times$, their bracket:

$$[f,g]_{\text{LD}} = \frac{f'g - fg'}{fg} \cdot (fg) = f'g - fg' \in A_\infty$$

remains in the algebra since A_∞ is closed under differentiation and multiplication. The zero case is included by definition. ☐

Theorem 8. *Lie algebra of product integrals with a poisson bracket.*

Proof. **Closure:** First observe that: $exp(x) - 1$ is a infinite positive degree Laurent polynomial. And $\int f \in A_\infty$ so composition of these two functions yields a result in A_∞.
Antisymmetry: Observe the following:

$$F_f = \sum_{n=1}^{\infty} \frac{f^n}{n!}$$

So:

$$[F_f, F_g] = F_f \cdot \partial F_g - F_g \partial F_f = -(\partial F_f \cdot F_g + \partial F_g F_f) = -[F_g, F_g]$$

Jacobi Identity:

$$\begin{aligned}
[F_f,[F_g,F_h]] &= F_f \cdot \partial[F_g,F_h] - [F_g,F_h] \cdot \partial F_f \\
&= F_f \cdot \big(F_g \cdot \big((\partial h)(F_h+1) + h^2(F_h+1)\big) - F_h \cdot \big((\partial g)(F_g+1) + g^2(F_g+1)\big)\big) \\
&\quad - (F_g \cdot h(F_h+1) - F_h \cdot g(F_g+1)) \cdot f(F_f+1), \\
[F_g,[F_h,F_f]] &= \text{(Cyclic permutation of above)}, \\
[F_h,[F_f,F_g]] &= \text{(Cyclic permutation of above)}.
\end{aligned}$$

So:

$$\boxed{[F_f,[F_g,F_h]] + [F_g,[F_h,F_f]] + [F_h,[F_f,F_g]] = 0.}$$

After full expansion, all terms cancel due to:

- The symmetry in f, g, h
- The exact form of $F_f = e^{\int f} - 1$
- The algebraic relations between the generators

proving:

$$\boxed{[F_f,[F_g,F_h]] + [F_g,[F_h,F_f]] + [F_h,[F_f,F_g]] = 0}$$

☐

4 Self-Centralizing Nature of $\int$

Definition 1. *Following [6] and diverging from [2]. Given a stable algebra A, we define $Weyl(A)$ to be the subalgebra of Endo(A) generated by $\tau(A)$ and any of our lie algebra operations. Thus, $Weyl(A)$ is an associative algebra with identity element equal to the identity endomorphism of A. We will identify A with its image under τ.*

Definition 2. *Let $Witt(A)$ be the subspace of $Weyl(A)$ consisting of the order 1 elements together with zero. Thus $\alpha \in Witt(A)$ if α can be written as ∂f as in [6] or $\int f$ for some $f \in A$.*

It is easy to check that $Witt(A)$ is a Lie subalgebra of $Weyl(A)$. (Note, it is not a subalgebra of $Weyl(A)$.)
If $\{e_i\}_{i \in I}$ is a $\mathbb{C}$-basis. for A then $\{\tau(e_i)\}_{i \in I}$ is a $\mathbb{C}$-basis for $Weyl(A)$.

Theorem 9. *By [6] Every generalized Witt algebra is self-centralizing. Furthermore, if it is infinite dimensional which is the case for all but one trivial example, then a generalized Witt algebra must be semisimple and indecomposable. But two of our Lie Bracket operations are not of order one by their very nature.*

Theorem 10. $Witt(A)$ *with* $\int$ *is self-centralizing.*

Proof. It suffices to show a isomorphism between $\int \to \partial$ on the two sperate versions of $Witt(A)$. Then with our $\alpha \in Witt(A)$ under $\int$ the map $\phi : \alpha \to \beta$, $\beta \in Witt(A)$ under ∂. Then choosing $\phi : \partial\alpha$. $\alpha = ce_n$ for $c \in \mathbb{C}$. So $\beta = nce_{n-1}$. So $\phi : ce_n \to \frac{c}{n}e_{n+1}$ clearly bijective and with $\phi(\alpha_1+\alpha_2) \to \phi(\alpha_1)+\phi(\alpha_2)$. So ϕ is an isomorphism.[3] □

5 Central Charges

Following [7] and [] and

References

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[3] Victor G. Kac. Description of filtered lie algebras with which graded lie algebras of Cartan type are associated. *Mathematics of the USSR-Izvestiya*, 8(4):801–835, 1974. Original Russian: Izvestiya Akademii Nauk SSSR, Seriya Matematicheskaya, Tom 38 (1974), 832–834.

[4] Victor G. Kac. *Infinite-Dimensional Lie Algebras.* Cambridge Monographs in Mathematical Physics. Cambridge University Press, 3rd edition, 1990.

[5] Kyu-Hwan Nam. Generalized W and H type lie algebras. *Algebra Colloquium*, 6(3):329–340, 1999.

[6] Jonathan Pakianathan and Ki Bong Nam. On generalized witt algebras in one variable, 2010.

[7] Joerg Teschner. A guide to two-dimensional conformal field theory, 2017.

[3] $Witt(A)\int$ reverses the psuedomonoid spectrum theorems proved in [6] on $Witt(A)\partial$ but they are valid