Complex Deformations of the Witt Algebra via Weyl and q-Calculus

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Abstract. This paper explores extensions of the Witt algebra to fractional and complex parameters using Weyl derivatives and q-calculus in order to understand a broader class of representations and meromorphic conformal functions. We generalize the classical Witt algebra by introducing complex-valued generators and structure functions, and we establish an analytic continuation of the q-Pochhammer symbol and the complex parameter Weyl derivatives. A fractional Leibniz rule is extended to complex parameters, and a complexified q-Witt algebra is constructed. The results unify and extend earlier work by La Nave-Phillips and Purohit, providing a framework for infinite-dimensional Lie algebras with complex structure constants.
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1. Introduction

The Witt algebra is a fundamental object in the theory of infinite-dimensional Lie algebras, with deep connections to conformal field theory, integrable systems, and mathematical physics. Classically, it is defined over the complex numbers with generators $\{L_n : n \in \mathbb{Z}\}$ satisfying the bracket relation

$$[L_n, L_m] = (n - m)L_{n+m}.$$

Extensions to complex parameters are motivated by the need for analytic continuation in scattering amplitudes, connections with quantum groups where q is a root of unity, or the study of meromorphic conformal field theory. In recent years, there has been growing interest in extending such structures to fractional and complex parameters, motivated by connections to q-analysis, fractional calculus, and meromorphic deformation theory.

We construct three distinct pathways to a complex-parameter Witt algebra: W' extends the La Nave-Phillips model by complexification, W'' builds a new algebra from the ground up using q-calculus, and W''' which serves as a canonical middle ground to compare the structures of the former two. Our main contributions include:

- A generalization of the q-Pochhammer symbol and its connection to the dilogarithm and q-Gamma function.
- A complexified fractional Leibniz rule and its application to q-deformed Witt algebras.
- The construction of three complex-parameter Witt-like algebras with meromorphic structure functions.

This work lays the groundwork for further study of deformed Virasoro algebras and their representations in the complex setting.

1.1. q-Calculus q-calculus provides a deformation of classical calculus that interpolates between discrete and continuous mathematics. The q-derivative is defined as:

$$D_q f(z) = \frac{f(qz) - f(z)}{(q-1)z}, \quad q \neq 1.$$

Central to q-calculus is the q-Pochhammer symbol:

$$(a; q)_n := \prod_{k=0}^{n-1} (1 - aq^k), \quad n \in \mathbb{N},$$

which we extend here to complex parameters. This extension connects to special functions through the q-Gamma function and the dilogarithm.

2. Witt Algebra

The Witt Algebra is the infinite dimensional Lie algebra of Conformal Field theory. It is defined as follows [6] [3]:

$$W := \mathbb{C}\{L_n : n \in \mathbb{Z}\}$$
$$L_n := -ie^{-in\theta} \frac{d}{d\theta}$$

Acting on Fourier representable functions:

$$f(\theta) \in C^\infty(\mathbb{S}, \mathbb{C})$$
$$f(\theta) = \sum_{n=-\infty}^{\infty} c_n e^{in\theta}$$

So our Lie bracket is:

$$\begin{aligned}[L_n, L_m]f &= L_n L_m f - L_m L_n f \\ &= ((1-m) - (1-n))(-e^{-in\theta - im\theta})\frac{d}{d\theta} f(\theta) \\ &= (n-m)L_{n+m} f(\theta)\end{aligned}$$

Or an equivalent definition:

$$W := \mathbb{C}\{L_n : n \in \mathbb{N}\}$$
$$L_n := -z^{n+1}\frac{d}{dz}$$

Acting again on Fourier representable functions:

$$f(z) = \sum_{k=0}^{\infty} c_k z^k$$
$$z \in \mathbb{C}$$

With the same bracket.

3. The Combinatorics of $(a; q)_z$

In [1] they extend the k-Pochhammer symbol from $R \times R \times \mathbb{N} \to R$ to a more general formulation. We follow to analytically extend the q-Pochhammer symbol. We have the standard q-Pochhammer symbol:

$$(a; q)_n := \prod_{k=0}^{n-1}(1 - aq^k), n \in \mathbb{N}$$

With this we derive a more general q-Pochhammer expression which is necessary for extending the q-derivative to complex values in q.

$$(a; q)_z := \frac{(a; q)_\infty}{(aq^z; q)_\infty}, z \in \mathbb{C}$$

By noting that: $\Gamma_q(z) := \frac{(q;q)_\infty}{(q^z;q)_\infty}(1-q)^{1-z}$ we obtain:

$$(a;q)_z = \exp(-\frac{\mathrm{Li}_2(aq^z) - \mathrm{Li}_2(a)}{\ln(q)})$$

Where Li_2 is the dilogarithm. Or equivalently:

$$(a;q)_z = \exp(\int_0^z \ln(1-aq^t)dt)$$

■We suffer the same restrictions as [4] namely that both a and q must be in the unit disk.

4. The Weyl and q Derivatives

4.1. Weyl Derivatives The (holomorphic) Weyl derivative ∂^s [5] acts on Fourier representable functions $f(\theta) = \sum_{n=-\infty}^{\infty} a_n e^{in\theta}, a_0 = 0$.[1] by:

$$\partial^s f(\theta) = \sum_{n=-\infty}^{\infty} (in)^s a_n e^{in\theta}$$

Then we have:

$$\begin{aligned}\sum_{n=-\infty}^{\infty} (in)^s a_n e^{in\theta} &= \sum_{n=1}^{\infty} (in)^s a_n e^{in\theta} + (-in)^s a_{-n} e^{-in\theta} \\ &= i^s \sum_{n=1}^{\infty} (n)^s a_n e^{in\theta} + (-n)^s a_{-n} e^{-in\theta}\end{aligned}$$

We can include constant zero modes in θ by mapping them to zero under strictly negative s. It is clear that s can take on any value in $\mathbb{C}$.

4.2. Fractional Leibniz In [4] they construct a Leibniz rule on an equivalent expression of Weyl derivatives. We define the right handed q-Weyl-derivative:

$${}_zD^\alpha_{q,\infty}\{f(z)\} := \frac{q^{-\alpha(1+\alpha)/2}}{\Gamma_q(-\alpha)} \int_0^\infty (t-z)_{-\alpha-1} f(tq^{1+\alpha}) d(t;q)$$

Where Γ_q is once again the q-Gamma function.[2] Where we define a q-integral:

$$\int_z^\infty f(t)d(t;q) := z(1-q)\sum_{k=1}^{\infty} q^{-k} f(zq^{-k})$$

[1] In [5] they define the zero mode to be zero to avoid division by zero.
[2] Note the meromorphicity of Γ_q restricting the domain.

Lemma 4.1. *And in the particular case $f(z)=z^{-p}$ we adopt from [4]:*

$$ {}_zD^{\alpha}_{q,\infty}\{z^{-p}\} = \frac{\Gamma_q(p+\alpha)}{\Gamma_q(p)} q^{-\alpha p+\alpha(1-\alpha)/2} z^{-p-\alpha} $$

Lemma 4.2. *So in the particular case of a monomial in z^p we have:*

$$ {}_zD^{\alpha}_{q,\infty}\{z^{p}\} = \frac{\Gamma_q(-p+\alpha)}{\Gamma_q(-p)} q^{\alpha p+\alpha(1-\alpha)/2} z^{p-\alpha} $$

Lemma 4.3. *They prove a fractional Leibniz Rule over an integer α:*

$$ {}_zD^{\alpha}_{q,\infty}\{U(z)V(z)\} = \sum_{r=0}^{\alpha} \frac{(-1)^r q^{r(r+1)/2}(q^{-\alpha};q)_r}{(q;q)_r} {}_zD^{\alpha-r}_{q,\infty}\{U(z)\}{}_zD^{\alpha}_{q,\infty}\{V(zq^{\alpha-r})\} $$

By our work in Section 3 and Section 4.1 we can now extend to any complex α with a classic contour integral in our disks in α, r, and q.We take q-calculus to leverage r as a continuous, fractional, and complex parameter:

$$ {}_zD^{\alpha}_{q,\infty}\{U(z)V(z)\} = \int_0^{\alpha} \frac{(-1)^r q^{r(r+1)/2}(q^{-\alpha};q)_r}{(q;q)_r} {}_zD^{\alpha-r}_{q,\infty}\{U(z)\}{}_zD^{\alpha}_{q,\infty}\{V(zq^{\alpha-r})\}dr $$

$$ A(r) = \frac{(-1)^r q^{r(r+1)/2}(q^{-\alpha};q)_r}{(q;q)_r} \tag{1} $$

[3]

5. The Complex Parameter Witt Algebras

5.1. La Nave and Phillips' Work In [2] they build a fractional Witt Algebra:

$$ L'^a_n := -ie^{-ia(n+1)\theta}\partial^a, a \in \mathbb{R} $$
$$ \Gamma_p(s) := \frac{\Gamma(a(s+p)+1)}{\Gamma(a(s+p-1)+1)}, p \in \mathbb{Z} $$
$$ A_{p,q} := \Gamma_p(s) - \Gamma_q(s) $$

Where Γ is the Gamma function. We have bracket:

$$ [L'^a_n, L'^a_m] = A_{m,n}(s) \otimes L'^a_{n+m} $$

[3] Note the convergence of 1 with respect to r within our α, q disks.

s is a complex parameter of the representation.
We observe that $\Gamma(z)$ is defined for all complex $z \notin -\mathbb{N}$. Furthermore in Section 4 the Weyl derivative can be taken from $\mathbb{C}$. Consequently the parameter a in our generators can also be generalized to $\mathbb{C}$. Therefore we have the extension of this algebra from $a \in \mathbb{R}$ to $a \in \mathbb{C}$ and $s \in \mathbb{C}$ and therefore of $\Gamma_p \in \mathbb{R}$ to $\Gamma_p \in \mathbb{C}$. This extends the representation structure constraints in $A_{p,q}(s) \in \mathbb{R}$ to a value in $\mathbb{C}$. However, p, q (n, m) remain $\in \mathbb{Z}$. ■

5.2. Complexified q-Witt Relations in Purohit's Construction We define:

$$L_n''^{\alpha} := -z^{n+1}_z D^{\alpha}_{q,\infty}, \alpha \in \mathbb{C}, q \in \mathbb{C}$$
$$W'' := \mathbb{C}\{L_n''^{\alpha}, n \in \mathbb{Z}\}$$

We now compute the bracket on an arbitrary $V(z)$ using our fractional Leibniz rule in Section 4.2.

$$V(z) = \sum_{k=0}^{\infty} z^k C_k$$

We apply Lemma 4.2 to $V(z)$:

$$_zD^{\alpha}_{q,\infty} V(z) = \sum_{k=0}^{\infty} C_k \frac{\Gamma_q(-k+\alpha)}{-k} q^{\alpha k + \alpha(1-\alpha)/2} z^{k-\alpha}$$

We calculate the derivative operator again on $V(zq^{\alpha-r})$ so we can apply our fractional Leibniz in Lemma 4.3:

$$_zD^{\alpha}_{q,\infty} V(zq^{\alpha-r}) = \sum_{k=0}^{\infty} C_k \frac{\Gamma_q(-k+\alpha)}{\Gamma_q(-k)} q^{\alpha k + \alpha(1-\alpha)/2} z^{k-\alpha} q^{(k-\alpha)(\alpha-r)}$$

We need Lemma 4.2 for another derivative:

$$(_zD^{\alpha}_{q,\infty})_z D^{\alpha}_{q,\infty} V(zq^{\alpha-r}) = \sum_{k=0}^{\infty} C_k \frac{\Gamma_q(-k+2\alpha)}{\Gamma_q(-k+\alpha)} \frac{\Gamma_q(-k+\alpha)}{\Gamma_q(-k)}$$
$$q^{\alpha k + \alpha(1-\alpha)/2} q^{\alpha(k-\alpha)+\alpha(1-\alpha)/2} q^{(k-\alpha)(\alpha-r)} z^{k-2\alpha}$$
$$g(r) = \sum_{k=0}^{\infty} C_k \frac{\Gamma_q(-k+2\alpha)}{\Gamma_q(-k)} q^{3\alpha k - 2\alpha^2 + \alpha(1-\alpha) - rk + r\alpha} z^{k-2\alpha} \qquad (2)$$

We need the $\alpha - r$ derivative on $-z^{m+1}$ so we use Lemma 4.2 again:

$$f(r,m) =_z D^{\alpha-r}_{q,\infty}(-z^{m+1}) = \frac{\Gamma_q(-m-1+\alpha-r)}{\Gamma_q(-m-1)} q^{(\alpha-r)m + (\alpha-r)(1-\alpha+r)/2} z^{m+1-\alpha+r} \qquad (3)$$

Then by combining 3, 2, and 1 we have 'half-bracket':

$$L_n^{\prime\prime\alpha} L_m^{\prime\prime\alpha} V(z) = -z^{n+1} \int_0^\alpha A(r)g(r)f(r,m)dr \tag{4}$$

■

Conjecture 5.1. Clearly the bracket is anti-symmetric but it remain the subject of future work if it satisfies the Jacobi identity.

5.3. A Middle Ground Deformation We examine a Witt-like algebra of complex differential operators on f. We define:

$$L_n^{\prime\prime\prime\alpha} := -z^{n+1}\partial^\alpha, \alpha \in \mathbb{C}$$
$$W''' := \mathbb{C}\{L_n^{\prime\prime\prime\alpha}, n \in \mathbb{Z}\}$$
$$f(z) = \sum_{k=0}^{\infty} C_k z^k$$

With:

$$\begin{aligned} L_n^{\prime\prime\prime\alpha} L_m^{\prime\prime\prime\alpha} f(z) &= -z^{n+1}\partial^\alpha(-z^{m+1})\partial^\alpha \sum_{k=0}^{\infty} C_k z^k \\ &= \sum_{k=0}^{\infty} -(k+m+1-\alpha)^\alpha z^{-\alpha} L_{n+m}^{\prime\prime\prime\alpha} f(z) \end{aligned}$$

So we have relations:

$$[L^{\prime\prime\prime\alpha}, L_m^{\prime\prime\prime\alpha}]f(z) = \sum_{k=0}^{\infty} \left((k+n+1-\alpha)^\alpha - (k+m+1-\alpha)^\alpha\right) z^{-\alpha} L_{n+m}^\alpha f(z)$$

■

This isn't quite a Lie-Algebra structure as it depends on k but it is a related algebra of deformed Fourier representations on the circle. Indeed, it is the algebra of Witt operators on the integers (a spectrum on $k \in \mathbb{Z}$). The most precise description for W''' is that it is an example of a non-linear Lie algebra or, more specifically, an infinite-dimensional Lie algebra of differential operators with functional coefficients that fails to form a basis in the standard Lie algebra sense.

6. Conclusion

We have constructed several generalizations of the Witt algebra by introducing complex and fractional parameters into its defining structure. Using the Weyl derivative and q-calculus, we extended the classical Witt algebra to complex-valued generators and structure functions, leading to a family of Lie algebras whose brackets are governed by meromorphic functions. Remaining and of most importance is to check the Jacobi identity of W''. Ideally, if we have right handed q-Weyl derivatives equivalent to our base Weyl derivative then $W'' \cong W'''$. The most significant contributions are the complexifications in W' and the examination of the relations in W'''. The complexification of the La Nave-Phillips and Purohit constructions allows for a unified treatment of fractional and q-deformed Witt algebras. Our results suggest natural directions for future research, including the study of central extensions, representation theory, and applications to conformal field theory and integrable systems with complex parameters.

Conflict of Interest

On behalf of all authors, the corresponding author states that there is no conflict of interest.

Data Availability Statement

This manuscript contains no external data libraries.

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