A Note on the Complexified Weyl Derivative

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Abstract. This note explores the complexified Weyl derivative, ∂^s acting on Fourier series. We demonstrate that its action can be decomposed into a sum involving two generalized zeta functions. A conjecture is presented regarding the non-trivial zeros of these zeta functions, suggesting that they are too "special" for the Weyl derivative to annihilate a function unless the Fourier coefficients exhibit a specific symmetry.
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1. Introduction

The study of infinite-dimensional Lie algebras, such as the Witt and Virasoro algebras, is a cornerstone of modern mathematical physics, with profound applications in conformal field theory and string theory. Central to these structures are the derivations that act on the underlying spaces of functions. In this note, we focus on a fundamental operator in this context: the complexified Weyl derivative, ∂^s [2]. Our primary result is a reformulation of this action, showing that $\partial^s f$ can be expressed as a linear combination of two generalized zeta functions, $L(-s, \chi(n)), L(-s, \chi(-n))$, where $\chi(n) = a_n e^{in\theta}$. This decomposition connects the spectral properties of the derivative to the analytic number-theoretic properties of zeta functions. Based on this connection, we posit a conjecture (Conjecture 3.1) that the annihilation of a function by imposes a strong symmetry condition on its Fourier coefficients, as the non-trivial zeros of the associated zeta functions are generically insufficient to cause cancellation.

2. Generalized Zeta Functions

We can generalize a zeta function in the numerator of each summnation term by:

$$L(s, \chi(n)) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$$

Where $\chi(n)$ depends only on n but could be complex.

3. The Weyl Derivative as the sum of two generalized Zeta functions

The Weyl derivative ∂^s [2] acts of fourier representable functions $f(\theta) = \sum_{n=-\infty}^{\infty} a_n e^{in\theta}, a_0 = 0$ by:

$$\partial^s f(\theta) = \sum_{n=-\infty}^{\infty} (in)^s a_n e^{in\theta}$$

Then we have:

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} (in)^s a_n e^{in\theta} &= \sum_{n=1}^{\infty} (in)^s a_n e^{in\theta} + (-in)^s a_{-n} e^{-in\theta} \\
&= i^s \sum_{n=1}^{\infty} (n)^s a_n e^{in\theta} + (-n)^s a_{-n} e^{-in\theta} \\
\chi(n) &= a_n e^{in\theta} \\
&= i^s L(-s, \chi(n)) + i^s e^{i\pi s} L(-s, \chi(-n))
\end{aligned}$$

Conjecture 3.1. Non-trivial zeros of generalized zeta functions are too 'special' for ∂^s to ever annihilate f unless $\chi(n)$ is symmetric in n that is $\chi(n) = \chi(-n)$.

Furthermore, when $\chi(n)$ satisfies the conditions of a Dirichlet character we achieve $\partial^s f$ as the sum of two distinct Dirichlet-L functions. [1]

4. Evidence for Conjecture 3.1

Putting f representable by finite positive degree polynomial then ∂^s annihilates f for trivial s given by $s = n \in \mathbb{Z}, n > \deg(f) + 1$.

Conflict of Interest

On behalf of all authors, the corresponding author states that there is no conflict of interest.

Data Availability Statement

This manuscript contains no external data libraries.

References

[1] Harold Davenport. *Multiplicative Number Theory*. Springer, New York, 3rd edition, 2000. Revised by Hugh L. Montgomery.

[2] Ravinder Raina and C.L. Koul. On weyl fractional calculus. *Proceedings of the American Mathematical Society*, 73:188–192, 02 1979.