Journal of *
Volume **0** (2025)

The Boundary of the Virasoro Algebra

[Drew Remmenga drewremmenga@gmail.com]

Abstract.
Mathematics Subject Classification 2010: 17B68,16S32,17B66,17B40,81R10.
Key Words and Phrases: Infinite-dimensional Lie algebras, generalized Witt algebras, Virasoro algebra, spectral invariants, self-centralizing, endomorphisms.

1. Introduction

The Witt Algebra is defined as follows [9] [7]:

$$W := \mathbb{C}\{L_n : n \in \mathbb{Z}\}$$
$$L_n := -ie^{-in\theta}\frac{d}{d\theta}$$

Acting on fourier representable functions:

$$f(\theta) \in C^\infty(\mathbb{S}, \mathbb{C})$$
$$f(\theta) = \sum_{n=-\infty}^{\infty} c_n e^{in\theta}$$

So we have a bracket:

$$\begin{aligned}
[L_n, L_m]f &= L_n L_m f - L_m L_n f(\theta) \\
&= ((1-m) - (1-n))(-e^{-in\theta - im\theta})\frac{d}{d\theta}f(\theta) \\
&= (n-m)L_{n+m}f(\theta)
\end{aligned}$$

The conformal group in $\mathbb{R}^{1,1}$ is isomorphic to [9]:

$$\mathrm{Diff}(\mathbb{S})_+ \times \mathrm{Diff}_+(\mathbb{S})$$

But the boundary of this algebra yields antother algebra given by:

$$W_* := \mathbb{C}\{M_n : n \in \mathbb{Z}\}$$
$$M_n := -ie^{-in\theta} \int d\theta$$

Acting on those same fourier representable functions f.
Then we have a bracket:

$$\begin{aligned}
[M_n, M_m]f &= -ie^{-in\theta} \int d\theta e^{-im\theta} \int d\theta f - (-i)e^{-im\theta} \int d\theta (-i) e^{-in\theta} \int d\theta f \\
&= (\frac{1}{m} - \frac{1}{n}) M_{n+m} f
\end{aligned}$$

The zero element interacts with arbitrary n-labeled element by:

$$\begin{aligned}
[M_n, M_0]f &= -ie^{-in\theta} \int d\theta e^{-i0\theta} \int d\theta f - (-i)e^{-i0\theta} \int d\theta (-i) e^{-in\theta} \int d\theta f \\
&= -ie^{-in\theta} \int d\theta (-i) \int d\theta f + i \int d\theta (-i) e^{-in\theta} \int d\theta f \\
&= -e^{-in\theta} \int \int d\theta d\theta f + \frac{1}{in} e^{-in\theta} \int \int d\theta d\theta f \\
&= (\frac{i}{n} - 1) e^{-in\theta} \int \int d\theta d\theta f \\
&= (-i - \frac{1}{n}) M_n f
\end{aligned}$$

This leads to a naturall extension of both W and W_* to the Gaussian Integers $\mathbb{Z}[i, 1]$. So we have:

$$W = \mathbb{C} L_n : n \in \mathbb{Z}[i, 1]$$
$$W_* = \mathbb{C} M_n : n \in \mathbb{Z}[i, 1]$$

Anti-holomorphic $\bar{L_n}$ have bracket with L_m identically zero. Indeed anti-holomorphic $\bar{M_n}$ bracket with holomorphic M_m is also zero.
Composition of L_n with M_m is given by:

$$\begin{aligned}
[L_n, M_m] &= -ie^{-in\theta} \frac{d}{d\theta} (-ie^{-im\theta}) \int d\theta f - (-ie^{-im\theta}) \int d\theta (-ie^{-in\theta}) \frac{d}{d\theta} f \\
&= (im + \frac{i}{n}) e^{-i\theta(m+n)} f
\end{aligned}$$

Conflict of Interest

On behalf of all authors, the corresponding author states that there is no conflict of interest.

Data Availability Statement

This manuscript contains no external data libraries.

References

[1] R. K. Amayo and Ian Stewart. *Infinite-dimensional Lie Algebras*. Mathematical Monographs. Noordhoff International Publishing, Leyden, Netherlands, 1974.

[2] Dragomir Ž. Doković and Kaiming Zhao. Derivations, isomorphisms and second cohomology of generalized Witt algebras. *Transactions of the American Mathematical Society*, 350(2):643–664, 1998. Actual page range: 643-664 per AMS records.

[3] Lars Hormander. On the division of distributions by polynomials. *Acta Math.*, 97:239–309, 1958.

[4] Victor G. Kac. Description of filtered lie algebras with which graded lie algebras of Cartan type are associated. *Mathematics of the USSR-Izvestiya*, 8(4):801–835, 1974. Original Russian: Izvestiya Akademii Nauk SSSR, Seriya Matematicheskaya, Tom 38 (1974), 832–834.

[5] Victor G. Kac. *Infinite-Dimensional Lie Algebras*. Cambridge Monographs in Mathematical Physics. Cambridge University Press, 3rd edition, 1990.

[6] Douglas Lundholm. The virasoro algebra and its representations in physics. Course report or lecture notes. F01, KTH (Royal Institute of Technology), Stockholm, Sweden, January 2005.

[7] Kyu-Hwan Nam. Generalized W and H type lie algebras. *Algebra Colloquium*, 6(3):329–340, 1999.

[8] Jonathan Pakianathan and Ki Bong Nam. On generalized witt algebras in one variable, 2010.

[9] Martin Schottenloher. *A Mathematical Introduction to Conformal Field Theory*. Springer, 1997.

[10] Joerg Teschner. A guide to two-dimensional conformal field theory, 2017.