# Objectives and motivation of the project

In the aeronautical industry, it is of vital importance the study of the fluid’s behaviour in different conditions as the air behaves as a fluid and in order to design an aircraft (airfoil, sustaining surfaces, etc ) it is necessary these study with the aim of being able to perform their functions at all times.

In fluid dynamics, the study of the fluid behaviour has a great complexity; several manipulations of complex equations are required and some simplifications are needed to simplify them and attempt achieving an analytical solution.

However, apart from these simplifications, nowadays we have help of computers and computational fluids dynamics to facilitate even more the process of obtaining the solutions of these problems related with the behaviour study of fluids, giving more visual results and using numerical resolution techniques instead of complex analytical solutions.

This project is orientated to the simulation of one of the process that occurs inside the supersonic air flux, in concrete, the study of an expansive wave, also known as Prandtl-Meyer expansion.

# Description of the physic problem and its relevance



Figure 1. Prandtl-Meyer expansive wave [1]

The Prandtl-Meyer expansion or expansive wave is the process that occurs inside a supersonic flux when it expands over a convex corner (that forms an angle θ with the horizontal), creating infinity of Mach waves.

This expansion is caused by the drastic change in the direction of the geometry of flux, as it is a convex corner. This phenomenon affects the fluid properties in a gradual and continuous way, making the temperature, pressure and density decrease at the same time that velocity increases.

It has to be empathized that the increase in Mach number and velocity is soft and, as the variations of temperature, pressure and density are also infinitesimal, we can consider the flux as isentropic.

This flux has a leading edge with which creates an angle and a trailing edge with which the final flux forms an angle . These angles are the Mach angles and, as their name already indicates, they are related with the Mach numbers at the beginning () and final () with the following expressions:

This problem is of great interest to us in the aeronautical industry for the design of wing profiles, since practically all aircraft move within supersonic flows (due to the high speeds they reach) where the wing of an aircraft in question acts in a similar to the convex corner of the Prandtl-Meyer model. The air flow will maintain its direction parallel to the wing surface at the leading edge and it will be when, upon reaching the trailing edge, the flow direction will change producing an expansion wave and accelerating the air behind the wing (at the same time temperature, pressure and density decrease).

# Description of the math’s involved

The objective of this project is to simulate and obtain a numerical solution of a flow over a Prandtl-Meyer expansion corner.

In this simulation, we are going to suppose that the flow that moves on the surface is two-dimensional, supersonic and invisible and we are going to establish a series of initial conditions in order to simplify the problem and obtain the 5 properties of the flow that interests us to study its behavior: velocity (both vertical and horizontal), Mach number, density, temperature and pressure. For that purpose, we will solve the problem numerically using the MacCormack’s predictor-corrector explicit finite-difference method solution technique.

## Physical equations

All fluids are governed by the equations of conservation of mass (continuity equation), conservation of momentum, and conservation of energy. These equations are:

* Continuity equation:
* X-momentum conservation equation:
* Y-momentum conservation equation:
* Energy conservation equation:

Let us now analyze all the variables involved in these equations.

F and G are the flux variables, two columns of vectors that contain different values ​​of the properties that we want to study to describe the behavior of the fluid, defined as follows:

Where are the density, horizontal speed, pressure, vertical speed and module of the speed respectively.

Assuming a calorically perfect gas, we can use the next relation between the energy and  
the temperature of the gas in favor of the pressure and density of the fluid:

, where is the specific heat at constant volume, a  
constant that measures the amount of heat that a gas can release or absorb during a  
change on temperature keeping constant the volume.

The vectors F and G are related to each other thanks to the Euler equation for a stationary and two-dimensional flow as follows:

So we can assume that if we know the flow variables at a point as a function of y (initial data line), the y derivative of G is known along this entire line and therefore the x derivative of F is known (from where we can extract F). Therefore, the problem can be solved by making Δx jumps along the x-direction of the flow and calculating the next line above at each step.



Figure 2. Model for the downstream solution [1]

To obtain the so-called primitive variables (, u, v, p, T) we must perform a decoding task of the flow variables in the following way:

And knowing the Fs and all the primitive variables we can obtain the Gs in order to calculate the next point (at ):

To end-up with this section, note that, since the the grid where the problem develops is not completely rectangular, we must transform it into a fully rectangular computational plane. That is why we introduce the variables and . In a plane, denotes horizontal step and denotes vertical step. The conversion from x,y coordinates to coordinates is as follows:

Where is the y location of the lower surface and h(x) is the local height from the lower to the upper boundary in the physical plane. So, now transforming the Euler equation into coordinates:

From where we obtain the aforementioned equations that govern the flow.

## MacCormack’s technique

As we said before, we will use the finite-difference method of the MacCormack’s technique in order to numerically solve the problem. This method is represented in the following general equation: , where we discretize A with respect to b. In our case, A will be both of the flow variables (F and G) and b will be the or components of the computational plane.

From the previous conservation equations we can set up an explicit finite-difference solution for the two-dimensional flow by the MacCormack technique from the space derivatives isolated from the conservation equations of mass, momentum and energy written in terms of forward differences:

Next, with the previous equations and following the same explicit method in space, we can obtain the predicted values ​​of the fluid properties, which can be identified with a bar above each property:

## Predictor-Corrector method

In the previous point, what we have obtained are predicted values ​​since they do not have an accuracy of second order. Therefore, to improve the accuracy of our results, we must implement a predictor-corrector method in our calculations with which we will obtain the final value of each one of the flow properties.

The Predictor-Corrector method uses the same general equation mentioned from the MacCormack’s technique with explicit finite-differences but using an average derivative that corrects the predicted values of the used method. The name "average derivative" comes from the fact that the derivative in question comes from averaging two derivatives, that of point i, j of the mesh with that of point i + 1, j of the same:

Where the derivatives on the right-hand side of the equations are, of course, known numbers. In this equation we are considering the space derivative of the conservation equations in a point and the time derivative of the equations by using the MacCormack’s technique in the next interval of space (point). Finally, we obtain the following equations:

# Bibliography

[1] J.D. Anderson, Chapter 8: “Numerical solution of a two-dimensional supersonic flow: Prandtl-Meyer Expansion Wave” from Computational Fluid Mechanics, 1995. [Book].