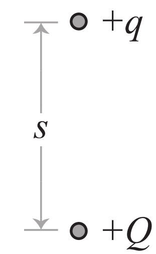
Studio Week 2
Charge Distributions

Picture credit: *Batman and Robin* (1998).

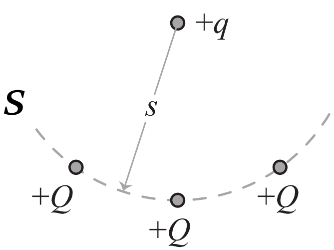
Principles for Success

- **Treat everyone with respect.**
- **Learn by doing and questioning.**
- **Everything should make sense.**

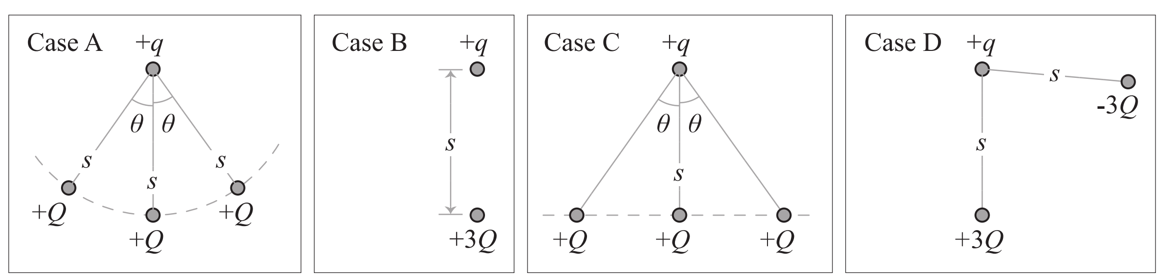
Activity 2-1 – Point Charges



1. Two positive point charges +q and +Q are held in place a distance s apart.
   a. Indicate the direction of the electric force exerted on each charge.
   b. Compare the magnitudes of the forces.



2. Two more +Q charges are held in place the same distance s away from the +q charge as shown.
   a. What, if anything, can be said about how the magnitude of the net electric force on the +q charge changes when the two +Q charges are added?
3. Rank the four cases below according to the magnitude of the net electric force on the +q charge.

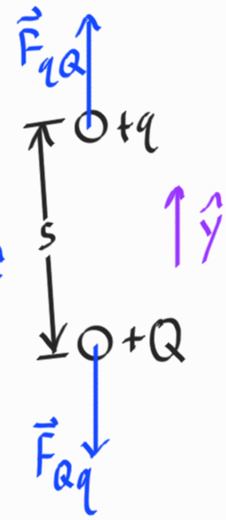


2-1 Point Charges

1)

$\vec{F}^E = q\vec{E}$

$\vec{E} = \frac{kq_{source}}{r^2}\hat{r}$

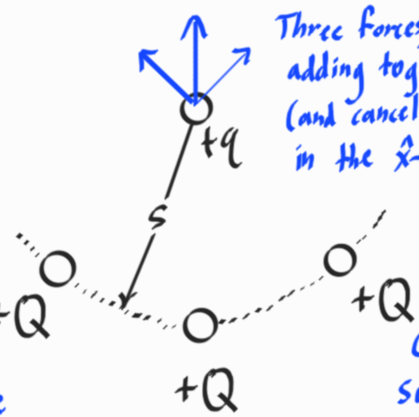


At q, $\vec{E}_Q = \frac{kQ}{s^2}\hat{y}$, so $\vec{F}_{qQ} = q\vec{E}_Q = \frac{kqQ}{s^2}\hat{y}$

$|\vec{F}_{qQ}| = |\vec{F}_{Qq}|$, agreeing with Newton's 3rd law!

At Q, $\vec{E}_q = \frac{kq}{s^2}(-\hat{y})$, so $\vec{F}_{Qq} = Q\vec{E}_q = -\frac{kqQ}{s^2}\hat{y}$

2)



Three forces, same magnitude, adding together in the $\hat{y}$-direction (and cancelling or nearly cancelling in the $\hat{x}$-direction).

The magnitude of the force on +q will increase!

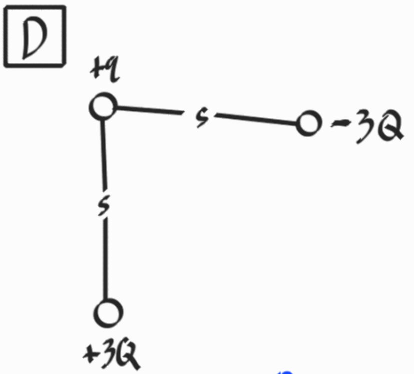
Creates $\vec{E}$ of same magnitude (b/c same distance), but angled left.

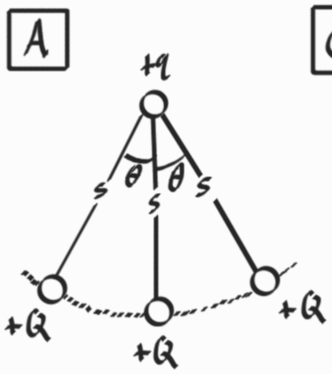
Creates same $\vec{E}_Q$ at site of +q as before.

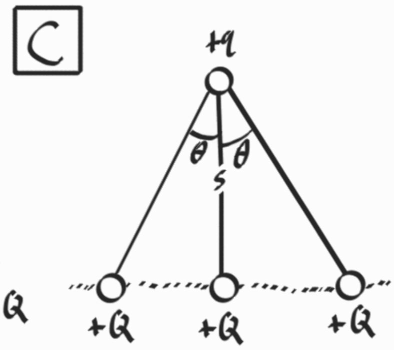
Creates $\vec{E}$ of same magnitude (b/c same distance), but angled left.

3)

$$\boxed{F_D \quad > \quad F_B \quad > \quad F_A \quad > \quad F_C}$$







D vs. B

The +3Q charge exerts the same force as in B, but now the -3Q charge exerts a force of the same magnitude toward itself, almost directly to the right. At this angle, the two forces will add to make an even larger force.

B vs. A

B is the limit of A when $\theta \to 0$. When we have all three forces of all three +Q charges pointing in the same direction, they add perfectly to create a stronger net force. When they are not aligned, the forces cancel somewhat.

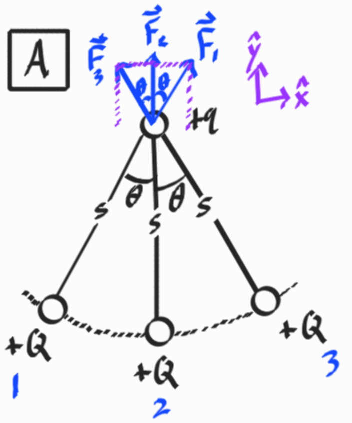
A vs. C

The three forces from the +Q charges have the same directions in A and C, but the charges on the right and left are farther away, so they exert weaker forces that contribute less to the net force.

$\boxed{B}$

$$\vec{F}^{net} = 3\frac{kqQ}{s^2}\hat{y} \qquad |\vec{F}^{net}| = 3\frac{kqQ}{s^2}$$

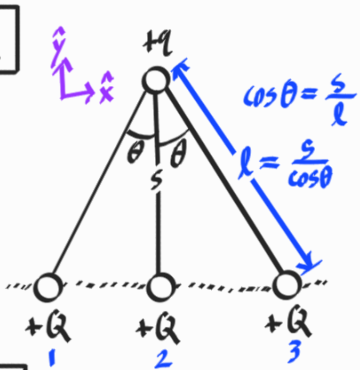
$\boxed{A}$



$$\vec{F}_1 = \frac{kqQ}{s^2}(\sin\theta\,\hat{x} + \cos\theta\,\hat{y}) \qquad \vec{F}_2 = \frac{kqQ}{s^2}\hat{y} \qquad \vec{F}_3 = \frac{kqQ}{s^2}(-\sin\theta\,\hat{x} + \cos\theta\,\hat{y})$$

$$\vec{F}^{net} = (1+2\cos\theta)\frac{kqQ}{s^2}\hat{y} \qquad |\vec{F}^{net}| = |1+2\cos\theta|\frac{kqQ}{s^2}$$

$\boxed{C}$



$$\vec{F}_1 = \frac{kqQ}{l^2}(\sin\theta\,\hat{x} + \cos\theta\,\hat{y}) = \cos^2\theta\frac{kqQ}{s^2}(\sin\theta\,\hat{x} + \cos\theta\,\hat{y})$$

$$\vec{F}_2 = \frac{kqQ}{s^2}\hat{y}$$

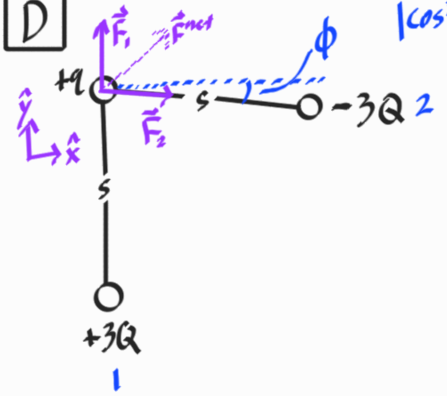
$\theta \in [0°, 90°)$ or the model breaks.

$$\vec{F}_3 = \frac{kqQ}{l^2}(-\sin\theta\,\hat{x} + \cos\theta\,\hat{y}) = \cos^2\theta\frac{kqQ}{s^2}(-\sin\theta\,\hat{x} + \cos\theta\,\hat{y})$$

$$\vec{F}^{net} = (1+2\cos^3\theta)\frac{kqQ}{s^2} \qquad |\vec{F}^{net}| = (1+2\cos^3\theta)\frac{kqQ}{s^2}$$

$|\cos^3\theta| \le |\cos\theta|$, $\cos^3\theta = \cos\theta$ only when $\cos\theta = 0, \pm 1$

$\boxed{D}$



$$\vec{F}_1 = 3\frac{kqQ}{s^2}\hat{y} \qquad \vec{F}_2 = 3\frac{kqQ}{s^2}(\cos\phi\,\hat{x} - \sin\phi\,\hat{y})$$

Repulsive — Attractive

Points away from 1 — Points toward 2

$$\vec{F}^{net} = 3\frac{kqQ}{s^2}\left[\cos\phi\,\hat{x} + (1-\sin\phi)\hat{y}\right]$$

$$|\vec{F}^{net}| = \sqrt{\cos^2\phi + (1-\sin\phi)^2}\;3\frac{kqQ}{s^2}$$

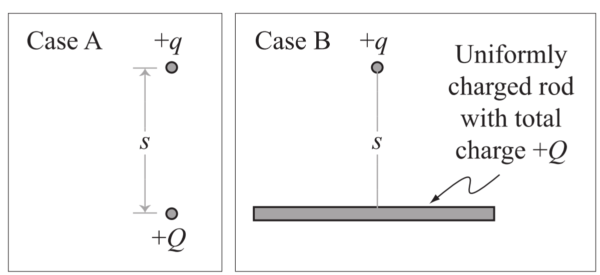
$$= 3\underbrace{\sqrt{2}\sqrt{1-\sin\phi}}_{>1 \text{ if } \sin\phi < \frac{1}{2},\ \text{so we need } \phi < 30°}\frac{kqQ}{s^2}$$

b/c $(1-\sin\phi)^2 = 1 - 2\sin\phi + \sin^2\phi$ and $\cos^2\phi + \sin^2\phi = 1$

Activity 2-2 – Charge Debate

- In case A below, a point charge $+q$ is a distance s from the center of a small ball with charge $+Q$.
- In case B the $+q$ charge is a distance s from the center of an acrylic rod with a total charge $+Q$.
- Consider the following dialogue between two students, neither of whom are completely correct:
  - Student 1: *"The charged rod and the charged ball have the same charge, +Q, and are the same distance from the point charge, +q. So the force on +q will be the same in both cases."*
  - Student 2: *"No, in case B there are charges spread all over the rod. The charge directly below the point charge will exert the same force on +q as the ball in case A. The rest of the charge on the rod will make the force in case B bigger."*

1. Discuss with your group the errors made by each student.
2. Write a correct description of how the forces compare.
3. Suppose you were given the values of q, Q, and s. Discuss possible methods you might use to calculate the electric force on $+q$. (You do not have to calculate this force.)



2-1 Charge Debate

Student 1

Same charge on ball as rod
$\Rightarrow$ same force on point

1) Not quite. Both the ball and the rod have the same charge, but some of the charge on the rod is farther from the point, so those bits of charge will exert less force (not to mention exert force in different directions).

Student 2

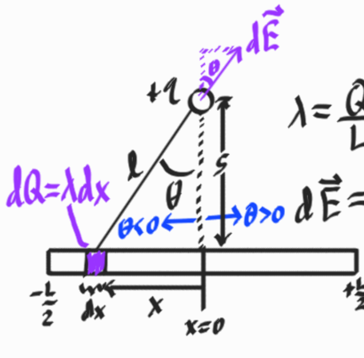
Charges on rod are spread out. Charge in center of rod exerts same force as ball, plus we add the force from the rest of the rod.
$\Rightarrow$ Rod exerts more force on point

This student is considering the shape, but not correctly. Since the charge is spread across the rod, its center has less than $+Q$ charge, and will not exert the same force on the point.

2) The force on the point from the rod will be weaker than the force on the point from the ball. By distributing the charge along the rod, some of the charge will be farther away (and so will exert less force) and will exert force out of alignment with the forces of the rest of the charges in the rod, leading to cancellations that further reduce the net force exerted by the rod.

3) Case A is a simple point charge with $\vec{F}_{qQ} = \frac{kqQ}{s^2}\hat{y}$.

In case B, you can use Q and the length of the rod (call it L) to find the charge density (λ) in the rod. In any infinitesimal segment of the rod (with length dx), there is a little bit of charge ($dQ = \lambda dx$), which exerts its own small electric field ($d\vec{E}$) on the point. These fields can be added up (by integrating) to get the electric field of the entire rod.



$$\lambda = \frac{Q}{L} = \frac{dQ}{dx} \Rightarrow dQ = \lambda dx = \frac{Q}{L}dx$$

$$d\vec{E} = \frac{k\,dQ}{\ell^2}(-\sin\theta\,\hat{x} + \cos\theta\,\hat{y})$$

$$\ell^2 = s^2 + x^2$$

$$\sin\theta = \frac{x}{\sqrt{s^2+x^2}} \qquad \cos\theta = \frac{s}{\sqrt{s^2+x^2}}$$

$$\vec{E} = \int d\vec{E} = \int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{kQ}{L(s^2+x^2)}\left(-\frac{x}{\sqrt{s^2+x^2}}\hat{x} + \frac{s}{\sqrt{s^2+x^2}}\hat{y}\right)dx$$

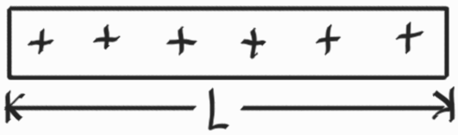
Activity 2-3 – Electric Field

1. A charged wire (length L) is oriented horizontally. The wire has a positive uniform charge density (the charge is spread out uniformly along the one-dimensional wire).
   a. Make a sketch of the wire that shows the location of the charges on the wire.
2. Divide your wire into six equal, point-like pieces. Use superposition to sketch the net electric field at each of the following points.
   a. A horizontal distance L from one end of the wire.
   b. A vertical distance L from the center of the wire.
   c. A vertical distance L from one end of the wire.
3. Use the simulation to check if your sketch is correct by creating a similar distribution using point-like charges: https://phet.colorado.edu/en/simulation/charges-and-fields.

2-3 Electric Field

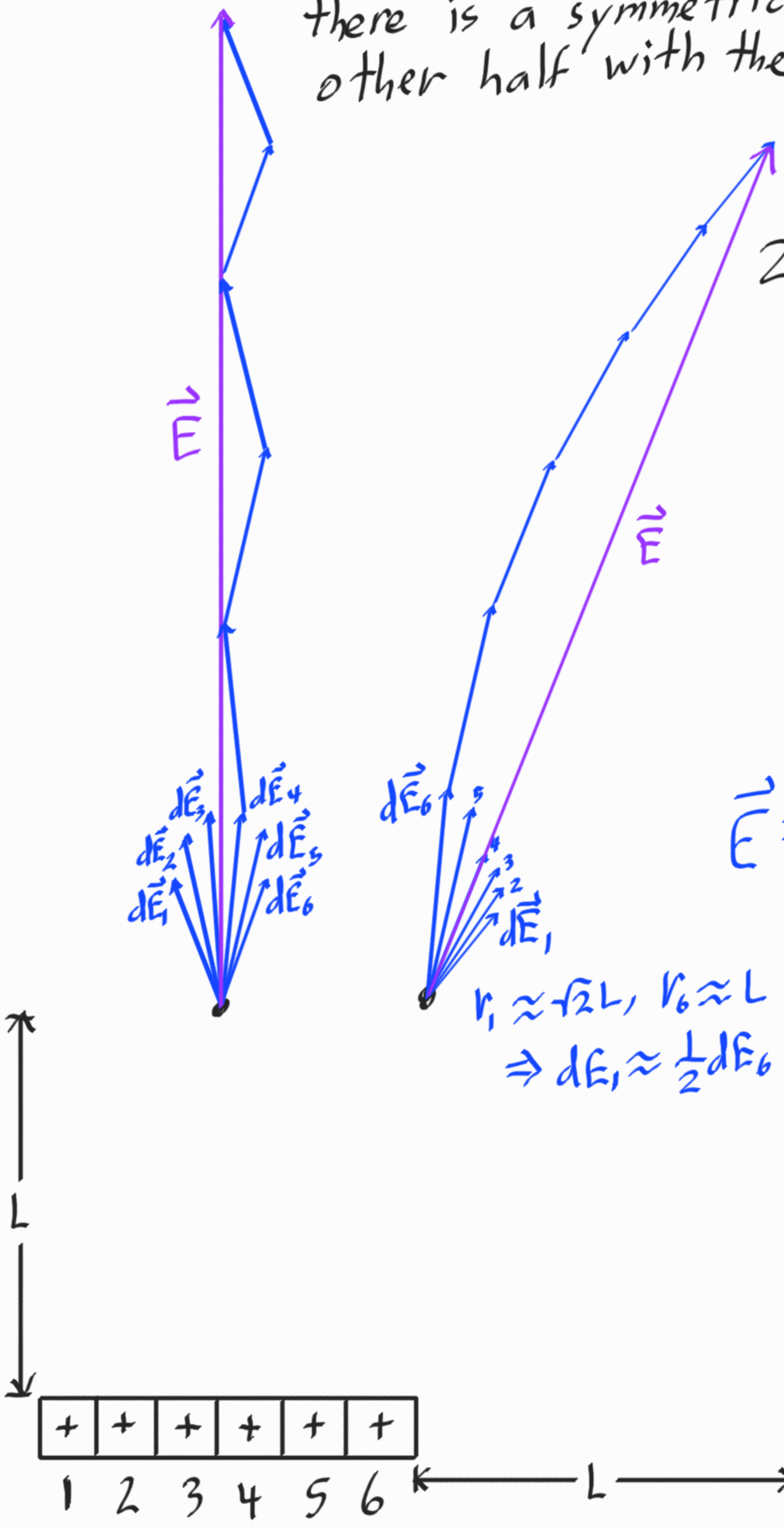
Uniform Charge Density

All charge symbols in the diagram are evenly spaced.



2b) For each $d\vec{E}$ from a dq in one half of the wire, there is a symmetric dE from a dq in the other half with the opposite x-component.

2c) Each $d\vec{E}$ points up and to the right at a different angle.



$$\vec{E} = \frac{kq}{r^2}\hat{r}$$

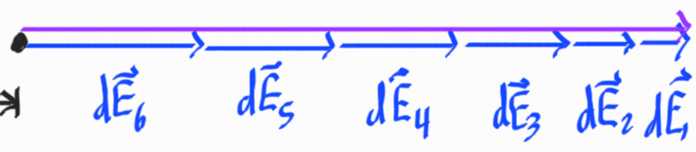
$r_1 \approx \sqrt{2}L$, $r_6 \approx L$

$\Rightarrow dE_1 \approx \frac{1}{2} dE_6$

2a) Each $d\vec{E}$ from each segment of the wire points in the same direction.

$r_1 \approx 2L$, $r_6 \approx L$

$dE_1 \approx \frac{1}{4} dE_6$



Activity 2-4 – Semi-Circular Wire

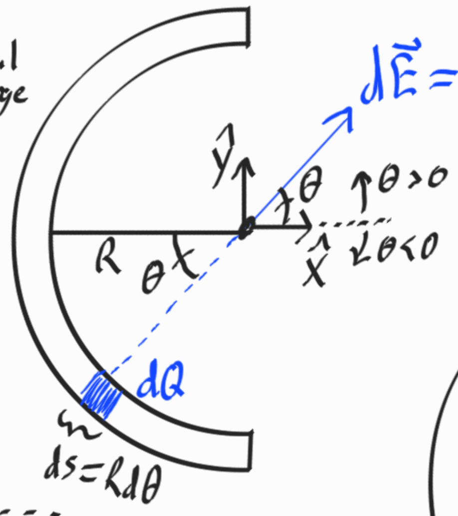
1. A positively-charged semi-circular wire (radius R) has a uniform charge density λ. Determine the electric field at the center of the semi-circle using the following method:
    a. Divide the wire into at least 6 point-like charges
    b. Find a symbolic expression for the electric field at the center of the semi-circle due to each of these point-like charges.
    c. Add together the electric field due to each of these six charges.
    d. Make sense of both the magnitude and the direction of your electric field.

2-4 Semi-Circular Wire

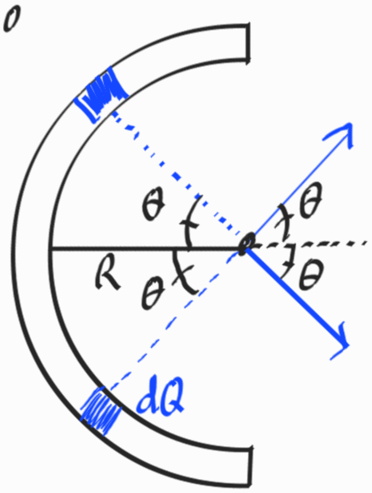
Uniform

$\lambda = \frac{Q}{\pi R}$ total charge

$dQ = \lambda R d\theta$
$= \frac{Q}{\pi} d\theta$



$$d\vec{E} = \frac{k\,dQ}{R^2}(\cos\theta\,\hat{x} + \sin\theta\,\hat{y})$$



For every dQ in the bottom half, there is an equal dQ in the top half which creates a $d\vec{E}$ with the same magnitude and a negative y-component, so we should see $\vec{E}$ be entirely along $\hat{x}$.

Integral Version

$$\vec{E} = \int_{-\pi/2}^{\pi/2} \frac{kQ}{\pi R^2}(\cos\theta\,\hat{x} + \sin\theta\,\hat{y})\,d\theta$$

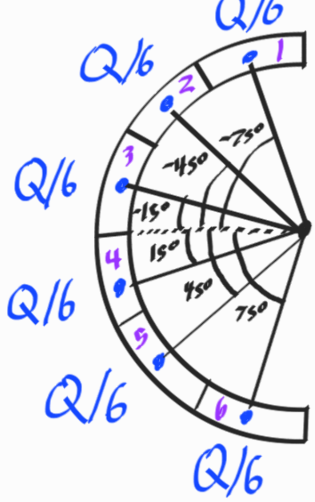
$$= \frac{kQ}{\pi R^2}\left[\underbrace{\int_{-\pi/2}^{\pi/2}\cos\theta\,d\theta}_{\sin\theta\big|_{-\pi/2}^{\pi/2} = 2}\,\hat{x} + \underbrace{\int_{-\pi/2}^{\pi/2}\sin\theta\,d\theta}_{-\cos\theta\big|_{-\pi/2}^{\pi/2} = 0}\,\hat{y}\right]$$

$$= \frac{2}{\pi}\frac{kQ}{R^2}\hat{x}$$

Dividing Into Six Segments

Charges go in the centers of their segments.

Placing the charges in these spots is easier ($\theta = 0°, \pm 30°, \pm 60°, \pm 90°$), but less accurate with regard to the uniform distribution.



$\frac{Q}{6} = \frac{\lambda \pi R}{6}$

$$\vec{E} = \frac{kQ}{6R^2}\Big[\big(\cos(-75°) + \cos(-45°) + \cos(-15°) + \cos(15°) + \cos(45°) + \cos(75°)\big)\hat{x}$$
$$+\big(\sin(-75°) + \sin(-45°) + \sin(-15°) + \sin(15°) + \sin(45°) + \sin(75°)\big)\hat{y}\Big]$$

$\cos(-75°) = \cos(75°)$, $\cos(-45°) = \cos(45°)$, $\cos(-15°) = \cos(15°)$

$\sin(-75°) = -\sin(75°)$, $\sin(-45°) = -\sin(45°)$, $\sin(-15°) = -\sin(15°)$

$\updownarrow \vec{E}_1$ $\updownarrow \vec{E}_2$ $\updownarrow \vec{E}_3$ $\updownarrow \vec{E}_4$ $\updownarrow \vec{E}_5$ $\updownarrow \vec{E}_6$

$$= \frac{2}{6}\frac{kQ}{R^2}\big(\underbrace{\cos(15°)}_{\approx 0.97} + \underbrace{\cos(45°)}_{\approx 0.71} + \underbrace{\cos(75°)}_{\approx 0.26}\big)\hat{x}$$

$$\approx \frac{1.9}{3}\frac{kQ}{R^2}\hat{x}$$

$\frac{1.9}{3} \approx \frac{2}{\pi}$ from integral

If we had a single point charge Q a distance R away from the point, we would have $\vec{E} = \frac{kQ}{R^2}\hat{x}$, so we should expect a smaller field from a spread out charge.