Lecture 4: Using Integrals in Physics

Warm-Up Activity

How is acceleration symbolically related to velocity?

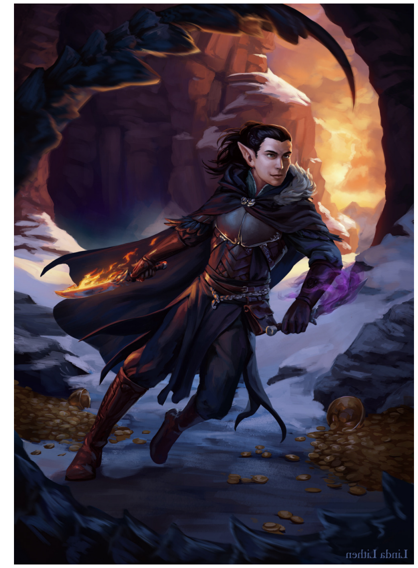
(A) Velocity is acceleration times t.

(B) Acceleration is velocity times t.

(C) Acceleration is the derivative of velocity.

(D) Velocity is the derivative of acceleration.

L4-1: Vax'ildan's Acceleration



- Vax'ildan Vessar is initially located at position x_i, running to the right with initial speed v_i.
- At $t = 0$, Vax clicks his *boots of haste*, which provide an acceleration:
$$\vec{a}(t) = a_0 \left(1 - \frac{t}{T}\right) \hat{x}$$
- Our goals are:
  - Find how much time it takes for Vax to return to his initial velocity.
  - Find Vax's position at this time.

Solving an ARCS Problem

1. Analyze and Represent

1a. **Understand the problem** – identify quantities by symbol and number.
1b. **Identify Assumptions** – identify important simplifications and assumptions.
1c. **Represent physically** – draw and label one or more appropriate diagrams and/or graphs that might help you solve the problem.

2. Calculate

2a. **Represent principles** – identify relevant concepts, laws, or definitions.
2b. **Find unknown(s) symbolically** – without numbers, find any unknown(s) in terms of symbols representing known quantities.
2c. **Plug in numbers** – plug numbers (with units) into your symbolic answer!

3. Sensemake

3a. **Units** – check that the units of your answer agree with the units you expect
3b. **Numbers** – compare your answer to other numbers in the problem or in the everyday world; if relevant, check the sign or direction.
3c. **Symbols** – use a strategy like covariation or special cases to check that your answer makes physical sense.

We need $1 - \frac{t}{T}$ to be unitless, because 1 is unitless, so t/T must also be. Since t is in seconds, it follows that T must also be.

Note that, when we plug in $t = T$, we find that $\vec{a}(T) = 0\hat{x}$, so the acceleration burst stops after T passes, at which point the acceleration changes direction to bring Vax back to his initial velocity. As such, T can be thought of as the duration of the acceleration burst.

Since $\left(1 - \frac{t}{T}\right)\hat{x}$ is unitless, the overall units of the right hand side come from a_0. We need the right hand side to be an acceleration to match the left, so a_0 has units of $\mathrm{m/s^2}$.

Note that, when we plug in $t = 0$ s, we find that $\vec{a}(0\text{ s}) = a_0\hat{x}$, so a_0 is the magnitude of the initial acceleration.

The unit vector $\hat{x}$ carries all of the direction information. It tells us that the acceleration is in the x-direction (though left or right depends on the sign).

Understand and Plan

Knowns

- Initial Position: $x_i = 0$ m (simplifying assumption)
- Initial Velocity: $v_i = 2$ m/s (a reasonable speed to estimate for a half-elf)
- $T = 6$ s (rounds in *Dungeons & Dragons* last six seconds)
- $a_0 = 0.5\ \mathrm{m/s^2}$ (significantly less than free-fall acceleration)

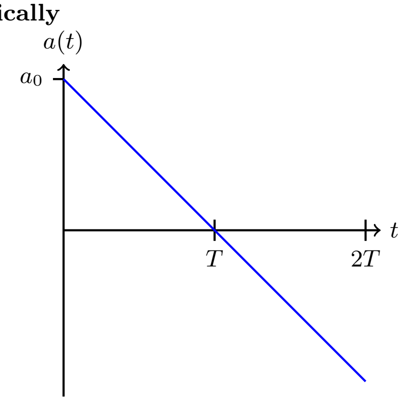
Unknowns

- When Vax returns to his initial velocity: t_f
- Vax's final position: x_f
- Equations of motion for velocity and position: $\vec{v}(t)$ and $\vec{x}(t)$

Identify Assumptions

- Particle Model
  - We do not wish to handle the complexities of how Vax's arms, legs, and wings move as he runs, so we will treat him as a point mass.
- 1-D Motion
  - We will assume Vax is travelling over relatively level ground so we do not have to consider Vax's vertical motion.
- Vax is not obstructed in his movement.
  - If Vax were to bump into something, that brief interaction would alter his motion in additional ways that the acceleration of the boots does not account for.

Represent Physically



L4-1: Vax'ildan's Acceleration – Calculate

- At $t = 0$, Vax clicks his *boots of haste*, which provide an acceleration:

$$\vec{a}(t) = a_0\left(1 - \frac{t}{T}\right)\hat{x}$$

  - First find a symbolic expression for Vax's velocity as a function of time.
  - Use your expression to find when Vax's velocity is equal to v_i.
  - Estimate any quantities to find numerical answers.

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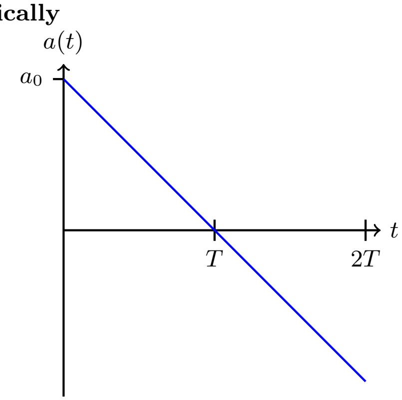

L4-1: Vax'ildan's Acceleration – Calculate

- At $t=0$, Vax clicks his *boots of haste*, which provide an acceleration:

$$\vec{a}(t)=a_0\left(1-\frac{t}{T}\right)\hat{x}$$

- His velocity as a function of time is

$$\vec{v}(t)=\left[v_i+a_0\left(t-\frac{t^2}{2T}\right)\right]\hat{x},$$

and he returns to his initial velocity at $t_f=2T$.

  - Now find a symbolic expression for Vax's position as a function of time and use it to find Vax's position at t_f.

We need $1-\frac{t}{T}$ to be unitless, because 1 is unitless, so t/T must also be. Since t is in seconds, it follows that T must also be.

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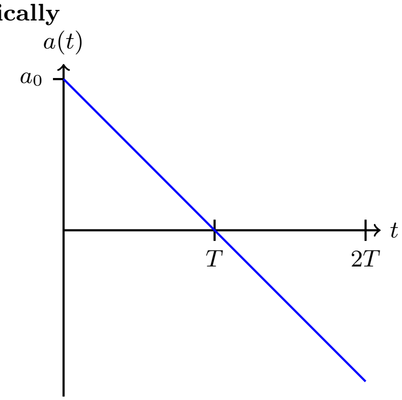

L4-1: Vax'ildan's Acceleration – Sensemake

- At $t=0$, Vax clicks his *boots of haste*, which provide an acceleration:

$$\vec{a}(t) = a_0\left(1-\frac{t}{T}\right)\hat{x}$$

  - How can we make sense of these equations?

$$\vec{v}(t) = \left[v_i + a_0\left(t-\frac{t^2}{2T}\right)\right]\hat{x} \quad \vec{x}(t) = \left[x_i + v_i t + a_0\left(\frac{t^2}{2}-\frac{t^3}{6T}\right)\right]\hat{x}$$

$$t_f = 2T \qquad \vec{x}_f = \left[x_i + 2v_iT + \frac{2}{3}a_0T^2\right]\hat{x}$$

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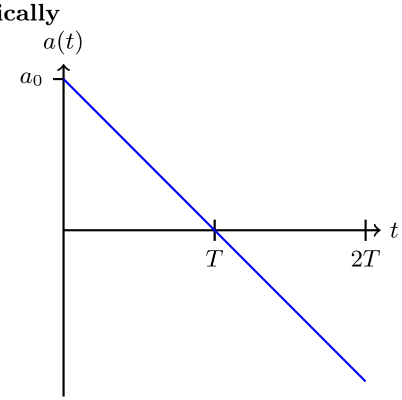

L4-1: Vax'ildan's Acceleration – Sensemake

- How can we make sense of these equations?

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$$t_f = 2T \qquad \vec{x}_f = \left[x_i + 2v_iT + \frac{2}{3}a_0T^2\right]\hat{x}$$

  - Are the units correct?
  - Which things are vectors?
  - What do the graphs of $\vec{v}(t)$ and $\vec{x}(t)$ look like?
  - What happens if you change a_0 or T?
  - Try plugging in some reasonable numbers: $v_i = 2$ m/s, $T = 8$ s, $a_0 = 0.5\ \mathrm{m/s}^2, x_i = 15$ m.

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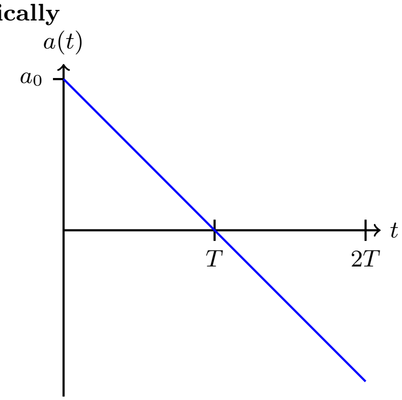

L4-2: Constant Acceleration

- What if Vax's acceleration had been constant?

$$\vec{a}(t) = a\hat{x}$$

Main Ideas

- If we know the acceleration of an object as a function of time, we can determine the velocity as a function of time.
- If we know the velocity as a function of time, we can determine the position as a function of time.