PH 222 Activity 4

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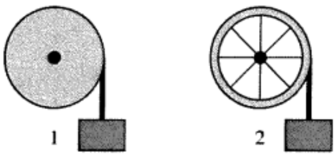
These problems are borrowed/adapted from Chapter 12 of the *Student Workbook* for *Physics for Scientists and Engineers.*

Activity 1

A solid cylinder and a cylindrical shell have the same mass, same radius, and turn on frictionless, horizontal axles. (The cylindrical shell has light-weight spokes connecting the shell to the axle.) A rope is wrapped around each cylinder and tied to a block. The blocks have the same mass and are held the same height above the ground. Both blocks are released simultaneously. The ropes do not slip. Which block hits the ground first? Or is it a tie? Explain.



The block attached to the solid cylinder hits the ground first. The solid cylinder has a smaller moment of inertia (less of its mass is far from the axle), so it has less resistance to angular acceleration ($\alpha = \frac{\tau_{net}}{I}$).

We can look at this more mathematically. Let T_1 be the tension in the rope attached to the solid cylinder, and let T_2 be the tension in the rope attached to the shell. Let m be the mass of each block, M be the mass of each cylinder, and R be the radius of each cylinder. The magnitude of the net force on the block in setup 1 is $F_{net,1} = mg - T_1$, while for setup 2 it is $F_{net,2} = mg - T_2$. The magnitude of net torque on the cylinder in setup 1 is $\tau_{net,1} = T_1 R$, while for setup 2 it is $\tau_{net,2} = T_2 R$. Since $F_{net} = ma$ and $\tau_{net} = I\alpha$, we know that

$$\begin{aligned} a_1 &= g - \frac{T_1}{m}, \\ a_2 &= g - \frac{T_2}{m}, \\ \alpha_1 &= \frac{T_1 R}{I_1}, \\ \alpha_2 &= \frac{T_2 R}{I_2}. \end{aligned}$$

For each cylinder, tangential and angular acceleration are related by $a_t = \alpha R$. Furthermore, since the blocks are connected to the cylinders by ropes that do not slip, the tangential acceleration of each cylinder is equal to the downward acceleration of its attached block. As such,

$$\begin{aligned} a_1 = \alpha_1 R = \frac{T_1 R^2}{I_1}, \\ a_2 = \alpha_2 R = \frac{T_2 R^2}{I_2}. \end{aligned}$$

For the cylindrical shell, $I_2 = MR^2$, and for the solid cylinder, $I_1 = \frac{1}{2}MR^2$. It follows that,

$$a_1 = \frac{2T_1}{M},$$
$$a_2 = \frac{T_2}{M}.$$

Substituting this into the acceleration equations for the blocks, we get

$$a_1 = g - \frac{M}{2m}a_1 \implies a_1 = \frac{g}{1+\frac{M}{2m}},$$
$$a_2 = g - \frac{M}{m}a_2 \implies a_2 = \frac{g}{1+\frac{M}{m}}.$$

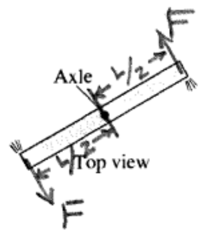
As we can see, $a_1 > a_2$, so the first block will fall faster. If we solve instead for tension, we find

$$T_1 = \frac{Ma_1}{2} = \frac{Mg}{2+\frac{M}{m}},$$
$$T_2 = Ma_2 = \frac{Mg}{1+\frac{M}{m}}.$$

Interestingly enough, the tension pulling on the solid cylinder (and thus the net torque) is smaller than the tension pulling on the shell. This is necessary, as if the two tensions were equal, the two boxes (which have the same force of gravity upon them) would fall at the same rate.

Activity 2

A metal bar of mass M and length L can rotate in a horizontal plane about a vertical, frictionless axle through its center. A hollow channel down the bar allows compressed air (fed in at the axle) to spray out of two small holes at the ends of the bar, as shown. The bar is found to speed up to angular velocity ω in a time interval Δt, starting from rest. What force does each escaping jet of air exert on the bar?



a) On the figure, draw vectors to show the forces exerted on the bar. Then label the radial distance from the axle to the point of application of each force.

b) The forces in your drawing exert a torque about the axles. Write an expression for each torque, and then add them to get the net torque. Your expression should be in terms of the unknown force F and "known" quantities such as M, L, g, etc.

Each individual torque has magnitude $\tau = rF\sin(90°) = (L/2)F$. They point in the same direction (encouraging counterclockwise rotation), so they add to $\tau_{net} = LF$.

c) What is the moment of inertia of this bar about the axle?

The linear density of the rod is $\lambda = m/L$. We can view this linear density as $\lambda = \frac{dm}{dr_\perp}$, where $r_\perp$ is the radial coordinate along the rod measured from the axle. As such, $dm = \frac{m}{L}dr_\perp$, and

$$I = \int r_\perp^2\, dm = \int_{-L/2}^{L/2} \frac{mr_\perp^2}{L} dr_\perp = \frac{m}{3L}\left[r_\perp^3\right]_{r_\perp=-L/2}^{L/2} = \frac{m}{3L}\left[\left(\frac{L}{2}\right)^3 - \left(-\frac{L}{2}\right)^3\right] = \frac{1}{12}mL^2.$$

d) According to Newton's second law, the torque causes the bar to undergo an angular acceleration. Use your results from parts (b) and (c) to write an expression for the angular acceleration. Simplify the expression as much as possible.

We know that $\tau_{net} = I\alpha$. Rearranging this, we find

$$\alpha = \frac{\tau_{net}}{I} = \frac{LF}{\frac{1}{12}mL^2} = \frac{12F}{mL}$$

e) You can now use rotational kinematics to write an expression for the bar's angular velocity after time Δt has elapsed. Do so.

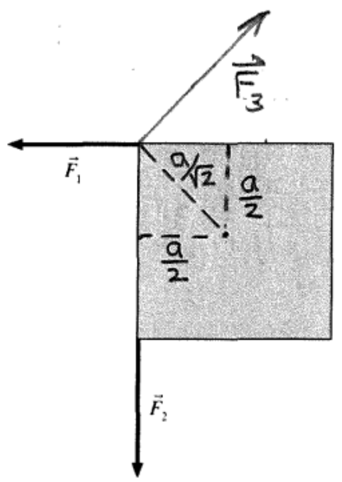
$$\omega = \omega_0 + \alpha\Delta t = \alpha\Delta t = \frac{12F}{mL}\Delta t$$

f) Finally, solve your equation in part (e) for the unknown force.

$$F = \frac{mL\omega}{12\Delta t}$$

Activity 3

Forces $\vec{F}_1$ and $\vec{F}_2$ to the comers of a square plate. Is there a single force $\vec{F}_3$ that, if applied to the appropriate point on the plate, will cause the plate to be in total equilibrium? If so, draw it, making sure it has the right position, orientation, and length. If not, explain why not.



Since $\vec{F}_1 + \vec{F}_2$ is the current net force on the plate, we know that $\vec{F}_3 = -(\vec{F}_1 + \vec{F}_2)$ to keep the plate in translational equilibrium with zero net force. What remains is to find if we can position the force to counteract the current net torque.

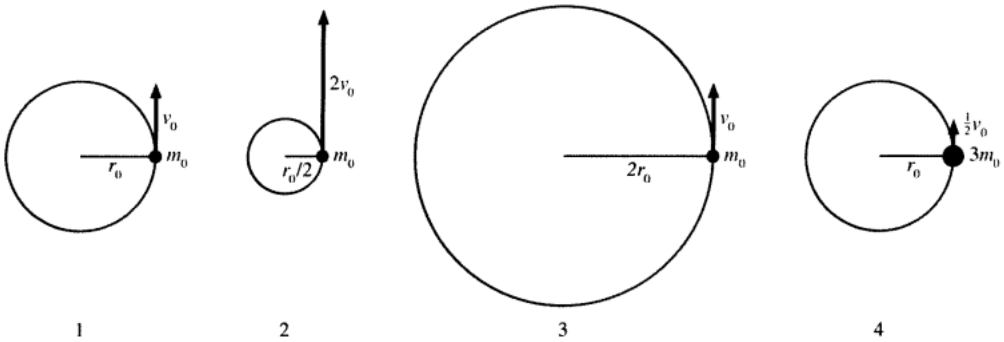
If we let a be the side length of the plate, then both $\vec{F}_1$ and $\vec{F}_2$ are applied at a distance $a/\sqrt{2}$ from the center of mass, with an angle of 45° between the position and force vectors in each case. The forces are both trying to turn the plate counterclockwise, so the torques add together:

$$\tau_1 + \tau_2 = \frac{a}{\sqrt{2}}F\sin(45°) + \frac{a}{\sqrt{2}}F\sin(45°) = \frac{a}{2}F + \frac{a}{2}F = aF.$$

We need $\tau_3 = aF$ in the opposite direction. Let $F = F_1 = F_2$, therefore $F_3 = \sqrt{2}F$. Applied to the upper left corner of the plate, $\vec{F}_3$ is applied at a distance $a/\sqrt{2}$ from the center of mass, with an angle of 90° between the position and force vectors. That means $\tau_3 = \frac{a}{\sqrt{2}}(\sqrt{2}F) = aF$, as we desire, and the force acts to turn the plate clockwise, opposing the other two.

Activity 4

Rank in order, from largest to smallest, the angular momenta L_1 to L_4.



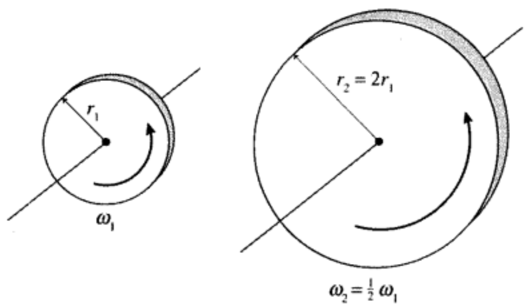
Order: $L_3 > L_4 > L_2 = L_1$

Explanation: $L = |\vec{r} \times \vec{p}| = rp\sin\theta$, and when the position vector for the point of application and the momentum are at right angles, the magnitude is just $L = rp = rmv$.

$$L_1 = r_0 m_0 v_0, \quad L_2 = \frac{r_0}{2} m_0 (2v_0),$$
$$L_3 = 2r_0 m_0 v_0, \quad L_4 = r_0 (3m_0) \frac{v_0}{2}.$$

Activity 5

Disks 1 and 2 have equal mass. Is the angular momentum of disk 2 larger than, smaller than, or equal to the angular momentum of disk 1? Explain.



As it happens, $L_2 > L_1$. Let $m = m_1 = m_2$. We know that $L = I\omega = \frac{1}{2}mr^2\omega$ for a solid disk. As such, $L_1 = \frac{1}{2}mr_1^2\omega_1$ and $L_2 = \frac{1}{2}m(2r_1)^2(\frac{1}{2}\omega_1) = mr_1^2\omega_1 = 2L_1$.