PH 221 Week 2

Benjamin Bauml

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The first activity is Problem 22 of Chapter 2 from the *Student Workbook* for *Physics for Scientists and Engineers*.

R2-1: Toy Rocket Launch

A toy rocket is launched straight up with constant acceleration a. It runs out of fuel at time t.

(a) Is the rocket at its maximum height the instant it runs out of fuel? Explain briefly.

No. It has been gaining speed since it launched, and now it still has upward velocity to carry it further.

(b) What assumptions would you make in order to solve this problem?

First, we will be modeling the rocket as a point mass for simplicity. We are given the assumption that the rocket is launched perfectly straight up, so we don't have to worry about horizontal motion and may leave the problem completely one-dimensional. We should also assume that the rocket is near the Earth's surface (thus gravity does not change appreciably with elevation). We should ignore air resistance, which further helps us to assume that the acceleration of the object is constant as it falls (after the fuel runs out).

> Air resistance is only one aspect of moving through a fluid. Students have suggested other such assumptions as lack of wind (no flow in the fluid medium) and constant density atmosphere (which is good for keeping drag and buoyancy uniform with respect to height; buoyancy itself can be ignored as an assumption).

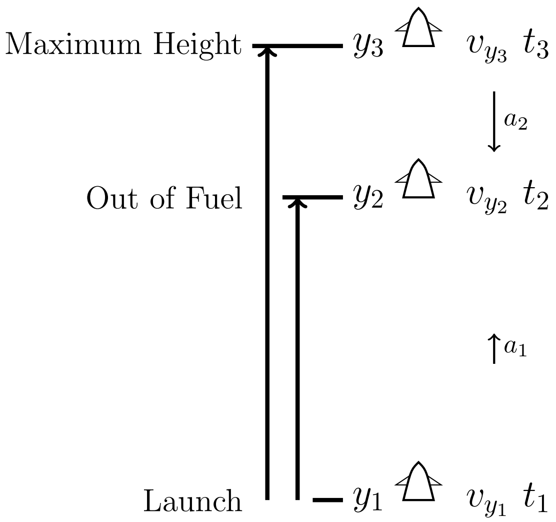
(c) What is the name of motion under the influence of only gravity? How would you write the vector for acceleration due to gravity?

Motion under the influence of only gravity is called *free-fall* (even if the object is moving upward during the motion; it does not need to be going down to be in free-fall).

Assuming that $\hat{y}$ points upward (a common and reasonable choice of coordinate system), then the vector for acceleration due to gravity would be $\vec{a} = -g\hat{y}$. Note that the negative sign comes from our coordinate system; the constant $g \approx 9.8$ m/s is a positive quantity.

(d) Draw a motion diagram for this problem. You should have three identified points in the motion: launch, out of fuel, maximum height. We'll call these points 1, 2, and 3 to have consistent definitions.

- Using subscripts, define quantities: y, v_y, and t at each of the three points,
- Describe the acceleration a_1 during the interval from point 1 to point 2, and acceleration a_2 during the interval from point 2 to point 3.
- Identify each quantity as Unknown or Known:
  - Was it given numerically?
  - Was it given symbolically?
  - Can we reason that it must be zero?
  - Be careful with signs!



Knowns
$y_1 = 0$ by choice
$t_1 = 0$ by choice
$v_{y_1} = 0$
$a_1 = a$ as given
$t_2 = t$ as given
$v_{y_3} = 0$
$a_2 = -g$

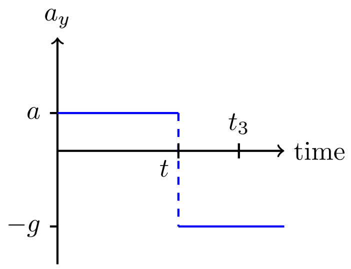
Unknowns
y_3, v_{y_2}, t_3, y_2

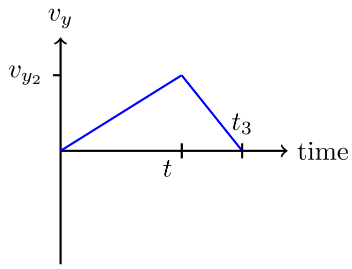
We can set the starting height y_1 of the rocket to zero, and we can always choose to start counting time at the launch (therefore making $t_1 = 0$ s). The final velocity v_{y_3} is zero by definition, as the rocket has no speed when it is at its maximum height. We can reason that the rocket starts from rest on the ground, so $v_{y_1} = 0$ m/s. These are all of our numerical knowns.

Symbolically, we are given the out-of-fuel time, $t_2 = t$, and the upward acceleration during the launch, $a_2 = a$. Note that this is the overall acceleration, which accounts for both the downward pull of gravity and the stronger upward thrust of the rocket engine. We also know that $a_2 = -g$, as the object is in free-fall after it runs out of fuel. It is negative, as the acceleration is downward and we are choosing up to be the positive direction.

That leaves y_2, v_{y_2}, t_3, and y_3 as unknowns.

(e) Draw qualitatively accurate graphs of the acceleration vs. time and the velocity vs. time.





Having constant acceleration means that the acceleration will consist of (piecewise) horizontal lines, and thus the velocity graph will be (piecewise) linear. Since the acceleration graph has a jump discontinuity at t, the velocity graph will have a corner (a place where the slopes do not match) at the same time.

These graphs were drawn to match the picture in part (d). If the rocket is assumed to cover less distance after the fuel runs out than before, then the area under the velocity graph must be smaller from t to t_3 than from 0 to t. This also means that the slope is steeper from t to t_3, so the acceleration due to gravity must have a greater magnitude than a.

Suppose we want to know the maximum height of the rocket.

(f) This is a two-part problem. Use your knowledge of calculus and motion to write *kinematic equations* for the first part of the motion. Use the given (symbolic) values for a_1 and t_2 to determine—again symbolically—the two unknown quantities at point 2.

We know the time of flight, acceleration, and initial velocity, and we want to know v_{y_2}, so the following equation is most relevant:

$$v_{y_2} = v_{y_1} + a_1(t_2 - t_1) = 0 + a(t - 0) = at.$$

We know the inital position and velocity, the acceleration, and the time of flight, and we want to find y_2, so the following equation is most relevant:

$$\begin{aligned} y_2 &= y_1 + v_{y_1}(t_2 - t_1) + \frac{1}{2}a_1(t_2 - t_1)^2 \\ &= 0 + 0 + \frac{1}{2}a(t - 0)^2 = \frac{1}{2}at^2. \end{aligned}$$

(g) Use your knowledge of calculus and motion to write similar *kinematic equations* for the second interval of the motion. Just write the equations; don't solve it yet.

By this point, we know the acceleration (solely gravitational) and the final velocity $v_{y_3} = 0$, and we would know y_2 and v_{y_2} if we solved the above equations. We want to know the maximum height y_3, so it would make sense to take the equation

$$\begin{aligned} y_3 &= y_2 + v_{y_2}(t_3 - t_2) + \frac{1}{2}a_2(t_3 - t_2)^2 \\ &= y_2 + v_{y_2}(t_3 - t) - \frac{1}{2}g(t_3 - t)^2. \end{aligned}$$

However, we need t_3 to solve this, so we can bring in the kinematic equation for velocity:

$$\begin{aligned} v_{y_3} &= v_{y_2} + a_2(t_3 - t_2) \\ 0 &= v_{y_2} - g(t_3 - t). \end{aligned}$$

I will go ahead and combine these equations now in order to eliminate t_3. If I replace $t_3 - t$ in the first equation with $\frac{v_{y_2}}{g}$, I get

$$y_3 = y_2 + \frac{v_{y_2}^2}{g} - \frac{v_{y_2}^2}{2g} = y_2 + \frac{v_{y_2}^2}{2g}.$$

If you happen to know the third kinematic equation,

$$v_f^2 = v_i^2 + 2a\Delta y,$$

then you can skip the step of substituting for time and just obtain the prior result directly.

Students may not have seen the third equation in class, and may be interested in seeing the derivation. Start with

$$v_f = v_i + a\Delta t,$$

and square it to get

$$v_f^2 = v_i^2 + 2v_i a\Delta t + a^2(\Delta t)^2.$$

Now take

$$\Delta y = v_i\Delta t + \frac{1}{2}a(\Delta t)^2,$$

and multiply it by $2a$ to get

$$2a\Delta y = 2v_i a\Delta t + a^2(\Delta t)^2.$$

This substitutes cleanly into the prior equation to give

$$v_f^2 = v_i^2 + 2a\Delta y.$$

(h) Now, substitute the known information from previous parts of the question into your equations from part (g). Find y_3 in terms of quantities given in the problem and the constant g.

We begin with

$$y_3 = y_2 + \frac{v_{y_2}^2}{2g}.$$

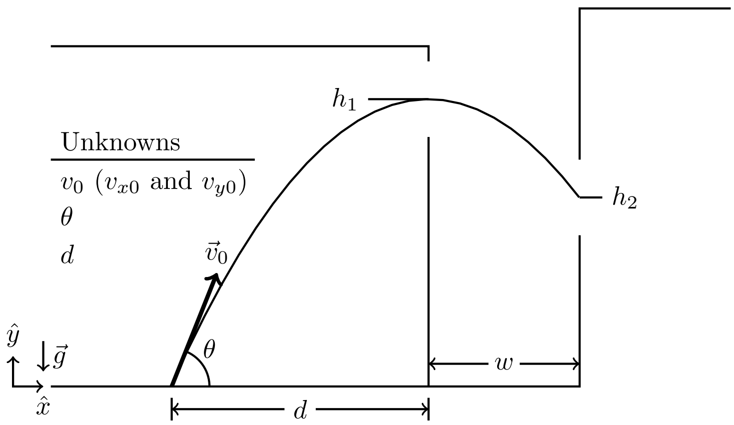
We know that $v_{y_2} = at$ and $y_2 = \frac{1}{2}at^2$, so we get

$$y_3 = \frac{1}{2}at^2 + \frac{a^2t^2}{2g} = \frac{(ga + a^2)t^2}{2g}.$$

R2-2: Two Window Toss

You are standing in a large building, and there is a single window right up by the ceiling. This window opens on an alleyway, and the building on the other side has an open window below the level of your building's window. You want to launch something from your building into the other, right through the center of each window. You know the heights of the windows and the width of the alley, so what do you need to do to throw it correctly?

(1) Draw a physical representation of the situation. On this drawing, indicate the height of the center of the first window h_1, the height of center of the second window h_2, the width of the alley w, the initial velocity $\vec{v}_0$, the launch angle θ, and your distance to the wall of the first window d. Add a coordinate system (indicate the directions of $+x$ and $+y$), and indicate the direction of gravitational acceleration $\vec{g}$. What are your unknowns?



(2) Since the window is right by the ceiling, you cannot arc your projectile too high. Suppose for simplicity that you want it to reach its maximum height right as it goes through the first window. What must its initial velocity be in the y direction for this to happen?

We know the final y-velocity at the peak of motion will be zero, and we know the object will have traveled $\Delta y = h_1$, so we can use the kinematics equation which does not involve time of flight:

$$\begin{aligned} v_y^2 &= v_{y0}^2 + 2a_y \Delta y \\ 0 &= v_{y0}^2 - 2gh_1 \\ v_{y0} &= \sqrt{2gh_1}. \end{aligned} \tag{1}$$

(3) Exiting the first window, the object only has velocity in the x direction. How long does it take to fall to the height of the second window? How fast must it be going to cross the alley in this time and enter the second window?

The horizontal velocity has no impact on the vertical motion of the object. It begins at $y_i = h_1$ and free-falls to $y_f = h_2$, starting from $v_{yi} = 0$. The change in position kinematics equation gives us

$$\begin{aligned} y_f &= y_i + v_{yi}\Delta t + \frac{a_y}{2}(\Delta t)^2 \\ h_2 &= h_1 + 0 - \frac{g}{2}(\Delta t_{h_1\to h_2})^2 \\ h_1 - h_2 &= \frac{g}{2}(\Delta t_{h_1\to h_2})^2 \\ \Delta t_{h_1\to h_2} &= \sqrt{\frac{2(h_1-h_2)}{g}}. \end{aligned} \tag{2}$$

The velocity v_x does not change during flight, as there is no horizontal component to acceleration. As such, it must be that the object has total velocity $\vec{v} = v_{x0}\hat{\imath}$ as it leaves the first window. It must travel across $\Delta x = w$ over time $\Delta t_{h_1\to h_2}$ to enter the other window, which means

$$\begin{aligned} \Delta x &= v_{x0}\Delta t + \frac{a_x}{2}(\Delta t)^2 \\ w &= v_{x0}\Delta t_{h_1\to h_2} + 0 \\ v_{x0} &= \frac{w}{\Delta t_{h_1\to h_2}} = w\sqrt{\frac{g}{2(h_1-h_2)}}. \end{aligned} \tag{3}$$

(4) Given the horizontal and vertical components of initial velocity, how far back from the first window do you have to stand to get the object through it?

The link between the x and y kinematics equations is Δt, which is shared between them. As such, we must find the time it takes for the object to reach $\Delta y = h_1$, then we must find how far $\Delta x = d$ it travels horizontally in that time. We begin with

$$\begin{aligned} \Delta y &= v_{y0}\Delta t + \frac{a}{2}(\Delta t)^2 \\ 0 &= \Delta y - v_{y0}\Delta t - \frac{a}{2}(\Delta t)^2 \\ &= h_1 - v_{y0}\Delta t + \frac{g}{2}(\Delta t)^2. \end{aligned} \tag{4}$$

Now we have a form to which we can apply the quadratic formula, which gives

$$\begin{aligned} \Delta t &= \frac{-(-v_{y0}) \pm \sqrt{(-v_{y0})^2 - 4(g/2)(h_1)}}{2(g/2)} \\ &= \frac{v_{y0} \pm \sqrt{v_{y0}^2 - 2gh_1}}{g}. \end{aligned} \tag{5}$$

Inserting our expression from part (2), we get

$$\begin{aligned} \Delta t &= \frac{\sqrt{2gh_1} \pm \sqrt{2gh_1 - 2gh_1}}{g} \\ &= \frac{\sqrt{2gh_1}}{g} \\ &= \sqrt{\frac{2h_1}{g}}. \end{aligned} \tag{6}$$

Without acceleration, we know that $\Delta x = v_{x0}\Delta t$. With our new expression for Δt, plus our expression for v_{x0} from part (3), we find

$$d = \Delta x = w\sqrt{\frac{g}{2(h_1 - h_2)}}\sqrt{\frac{2h_1}{g}} = w\sqrt{\frac{h_1}{h_1 - h_2}}. \tag{7}$$

(5) Given the components of initial velocity, what is the magnitude of initial velocity? At what angle with respect to the floor must the object be launched?

The magnitude of velocity is simply the result of the Pythagorean theorem:

$$v_0 = \sqrt{v_{x0}^2 + v_{y0}^2} = \sqrt{\frac{w^2 g}{2(h_1 - h_2)} + 2gh_1}. \tag{8}$$

The angle requires an application of trigonometry. If v_0 is the hypotenuse of a right triangle with legs v_{x0} and v_{y0}, then the angle θ is opposite v_{y0} and adjacent to v_{x0}. This means

$$\tan\theta = \frac{v_{y0}}{v_{x0}}. \tag{9}$$

Taking the inverse tangent of both sides will recover θ. A good sensemaking technique with recovering angles in this way is to think about the relative sizes of the triangle's legs. If $v_{y0} > v_{x0}$, then we are guaranteed that $\theta > 45°$.