Rigid-Body Pendulum

A ball of mass *m* and radius *R* is attached to the end of a rod with length *L* and mass *M*. The rod hangs from the ceiling. The ball and rod system are pulled to some initial angle θ_0.

1. Determine the moment of inertia of the rod-ball system. (Hint: you can look up the moments of inertia for the rod and the ball. How do we know which one to choose?)
2. Determine the center of mass of the rod-ball system.
3. Draw an extended free-body diagram for this system.
4. Determine the period of this pendulum.
5. How can you make sense of your answer?

1) For the rod, we have a fairly straightforward decision. Presumably, as it hangs from the ceiling, it will rotate about one of its ends, therefore we choose

$$I_r = \frac{1}{3}ML^2$$

For the ball, we know it is spherical in shape, which suggests using

$$I_{sphere} = \frac{2}{5}mR^2$$

However, that is for a sphere rotating around an axis going through its center of mass. This ball is fixed to one end of the rod, and is rotating with the rod about the *opposite* end of the rod, a distance $L+R$ away from the ball's center of mass. We can use the parallel axis theorem to shift the ball's axis of rotation to the correct place:

$$\text{Parallel Axis Thm: } I_B = I_{cm} + md^2 \qquad d = L+R$$
$$= \frac{2}{5}mR^2 + m(L+R)^2$$

When we attach objects so they rotate together about the same axis, we can add their individual moments of inertia to get their combined moment of inertia:

$$\underset{\text{(pendulum)}}{I_P} = I_r + I_B = \frac{1}{3}ML^2 + \frac{2}{5}mR^2 + m(L+R)^2$$

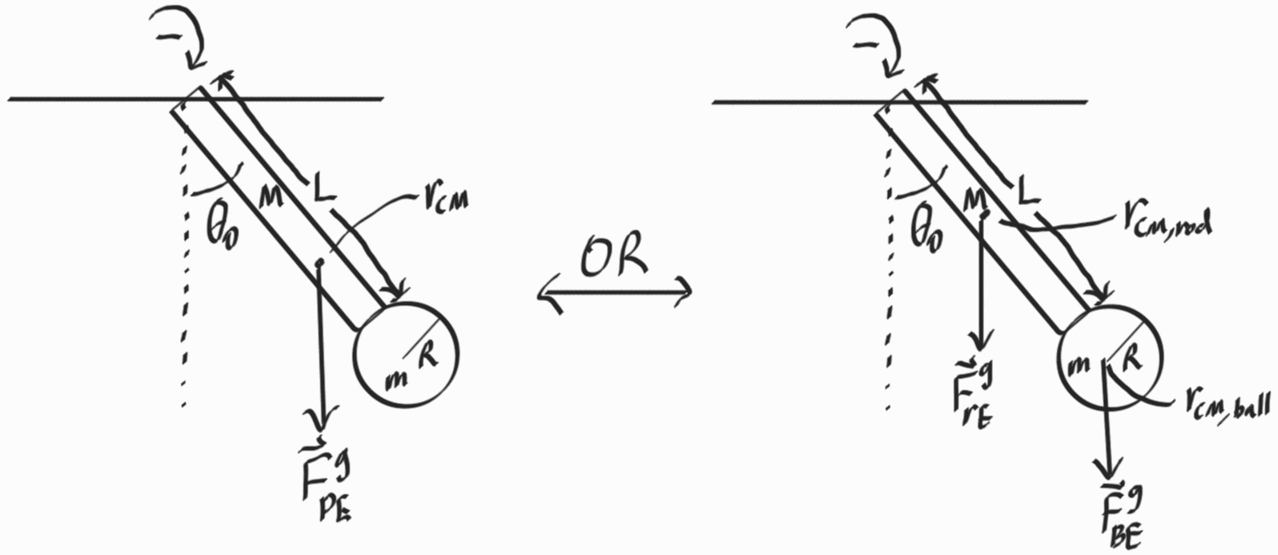
2) To find the center of mass of the pendulum, we can weight the centers of mass of the individual pieces by their masses and divide by the total mass.

center of ball is L+R from origin at top of rod

$$\text{Total Mass: } M_T = m + M$$

$$r_{cm} = \frac{1}{M_T}\left(r_{cm,rod}M + r_{cm,ball}m\right) = \frac{1}{M_T}\left[\frac{L}{2}M + (L+R)m\right] = \frac{1}{M_T}\left[\frac{LM + 2(L+R)m}{2}\right]$$

3)



4) To determine the period, we need to know about how the pendulum moves under the influence of gravity. Therefore, we start with the torque:

$$\begin{aligned}\tau_{net} &= -r_{cm} F^G_{PE} \sin\theta \\ &= -\frac{1}{\cancel{M_T}}\left(\frac{LM+2(L+R)m}{2}\right)\cdot \cancel{M_T} g \sin\theta \\ &= -\left(\frac{L}{2}Mg + (L+R)mg\right)\sin\theta = -r_{cm,rod}F^g_{rE}\sin\theta - r_{cm,ball}F^g_{BE}\sin\theta\end{aligned}$$

Note here that calculating the torque of gravity at the center of mass of the pendulum is equivalent to calculating the individual gravitational torques on the rod and the ball, so either approach is valid. Using this net torque, we can solve for the angular acceleration:

$$\vec{\tau}_{net} = I\vec{\alpha}$$

$$-\left(\frac{L}{2}Mg + (L+R)mg\right)\sin\theta = \left(\frac{1}{3}ML^2 + \frac{2}{5}mR^2 + m(L+R)^2\right)\alpha$$

$$\Rightarrow \alpha = -\frac{\frac{L}{2}Mg + (L+R)mg}{\frac{1}{3}ML^2 + \frac{2}{5}mR^2 + m(L+R)^2}\sin\theta$$

We know that angular acceleration is the second derivative of angular position, and if the angle is small, we can approximate the sine of the angle with the angle. This gives us the differential equation

$$\frac{d^2\theta}{dt^2} = -\frac{\frac{L}{2}Mg + (L+R)mg}{\frac{1}{3}ML^2 + \frac{2}{5}mR^2 + m(L+R)^2}\theta \qquad \text{SHM: } \frac{d^2x}{dt^2} = -\omega^2 x$$

This is the form we always see for simple harmonic motion, so we know that the fraction in front of the angle is the square of the angular frequency:

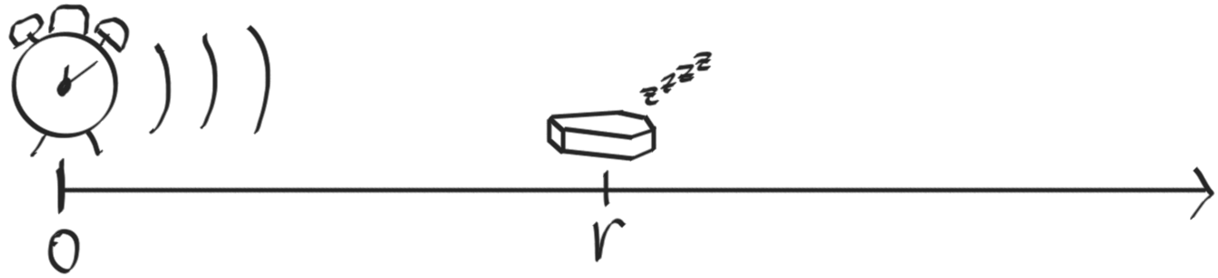
$$\omega^2 = \frac{\frac{L}{2}M + (L+R)m}{\frac{1}{3}ML^2 + \frac{2}{5}mR^2 + m(L+R)^2}g \quad \overset{\omega = \frac{2\pi}{T}}{\Rightarrow} \quad T^2 = \frac{\frac{1}{3}ML^2 + \frac{2}{5}mR^2 + m(L+R)^2}{g\left(\frac{L}{2}M + (L+R)m\right)}(2\pi)^2$$

5) This is an excellent place to use special cases sensemaking. If the mass of the rod goes to zero, and the radius of the ball goes to zero (making the ball a point mass), then we end up in the physical situation of the simple pendulum, which has $\omega^2 = g/L$

$$\omega^2 = \frac{\frac{L}{2}\cancel{M}^{\to 0} + (L+\cancel{R}^{\to 0})m}{\frac{1}{3}\cancel{M}^{\to 0}L^2 + \frac{2}{5}m\cancel{R}^{2 \to 0} + m(L+\cancel{R}^{\to 0})^2} g = \frac{\cancel{L}\cancel{m}}{\cancel{m}L^{\cancel{2}}} g = \frac{g}{L} \quad \checkmark$$

The Alarm

You are a heavy sleeper and need a sound level of 80 dB to wake up. To further incentivize you to get out of bed, you place your alarm a distance *r* away from you. Determine the power of the source.



Intensity is translated into decibels by the equation

$$\beta = \beta_0 \log\left(\frac{I}{I_0}\right) \quad (\text{where } \beta_0 = 10\,\text{dB}) \quad \Rightarrow I = I_0 10^{\beta/\beta_0} \quad \text{and } \frac{\beta}{\beta_0} = 8$$

The intensity at your location can be thought of as being that of a spherically spreading wave that has traveled a distance r from its source at your alarm clock. Let its power be P, and the intensity is

$$I = \frac{P}{4\pi r^2}$$

$$\Rightarrow \frac{P}{4\pi r^2} = I_0\, 10^{\beta/\beta_0}$$

$$\Rightarrow P = 4\pi I_0 r^2 10^8$$

Note that this solution, which incorporates data about how loud you need your alarm clock to be, is still partially symbolic, and can handle any chosen distance you decide to place the alarm clock at. If you wanted your solution to be more broadly applicable (say a different person requires a different loudness level), then you might not want to replace β/β_0 with a number.

Two Speakers

Two water-proof speakers sit along the x-axis underwater. The left speaker sits at x = -3m and the right speaker sits at x = 9m. They both produce a sound at a frequency of 1000 Hz.

1. Determine the wavelength of the sound (Hint: the speed of sound in water is 1500 m/s).
2. If both speakers produce a sound that is in phase with the other, what kind of interference occurs if you are standing at a distance 250 m away from the speaker on the right? What is the phase difference?
3. If the speakers produced a sound that was exactly out of phase, what is the phase difference if you are standing at a distance 250 m away from the speaker on the right? What does this mean for what kind of interference occurs at that spot?

$d_1 = 3m$ $\quad V = 1500\, m/s$

$d_2 = 9m$ $\quad f = 1000\, Hz$

1) The wavelength is related to wave speed and frequency by

$$V = \lambda f \Rightarrow \lambda = \frac{V}{f} = \frac{1500\, m/\cancel{s}}{1000\, \cancel{Hz}} = 1.5\, m \quad or \quad \frac{3}{2}\, m$$

2 & 3) If you are 250 meters away from the right hand speaker along the x-axis (in either direction), then the path length difference is the distance between the speakers:

$$\Delta D = d_2 + d_1 = 9m + 3m = 12m = \left(\frac{3}{2}\, m\right) \cdot 8 = 8\lambda$$

whole integer!

2) If the two speakers are in phase, then this path length difference leads to **constructive** interference, as the phase difference between the speakers is only determined by the path length difference:

$$\Delta \phi_D = \frac{\Delta D}{\lambda} 2\pi = \frac{8\lambda}{\lambda} 2\pi = 16\pi$$

3) When we add some inherent phase difference between the speakers (specifically, $\Delta\phi_0 = \pi$ to put them completely out of phase with each other), this adds to the distance-induced phase difference to give the total phase difference

$$\Delta\phi = \Delta\phi_D + \Delta\phi_0 = 16\pi + \pi = 17\pi$$

This phase difference between the waves leads to **destructive** interference.

Activities adopted from Grant Sherer, Benjamin Bauml, Danielle Skinner & Rebecka Tumblin

Diffraction

Light with a wavelength of $\lambda = 777$ nm is incident on a single slit of width $a = 7.27\ \mu$m.

1. At what angle does the second diffraction minimum occur?
2. What is the full width of the central maximum in degrees?

Both of these questions require the use of the equation which determines the location of diffraction minima:

$$a \sin\theta_p = p\lambda$$

This can be rearranged to solve for the angle:

$$\theta_p = \arcsin\left(\frac{p\lambda}{a}\right)$$ (arcsin is another way to say $\sin^{-1}$)

1) The second diffraction minima occur at the angle

$$\theta_2 = \arcsin\left(\frac{2\lambda}{a}\right) = \arcsin\left(\frac{2 \cdot 777\cancel{nm}}{7270\cancel{nm}}\right) \approx 12.34^\circ$$

2) The full width of the central maximum is the angular distance between the first order dark fringes, which bound the central maximum. The angle corresponding to these fringes is

$$\theta_1 = \arcsin\left(\frac{\lambda}{a}\right) \approx 6.14^\circ$$

Therefore, the angular width of the central maximum is $W = 2\theta_1 \approx 12.27^\circ$