Lecture 5: Constant Acceleration

<u>Warm-Up Activity</u>

Which of these equations are valid for an object moving with constant acceleration? Choose all that apply, and WRITE BIG!

(A) $\vec{a}(t) = a\hat{x}$

(B) $\vec{v}(t) = \left[v_i + a_0\left(t - \frac{t^2}{T}\right)\right]\hat{x}$

(C) $v_f^2 = v_i^2 + 2a\Delta x$

(D) $\vec{x}(t) = \left[x_i + v_i t + \frac{1}{2}at^2\right]\hat{x}$

(E) None of the above.

A Model for Motion

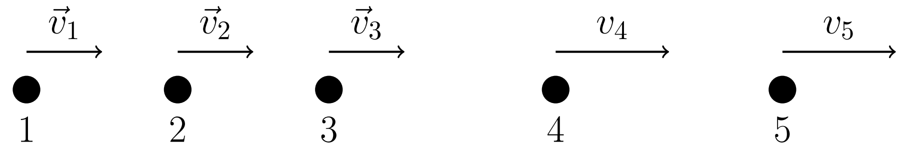
Quantities

- Position: $\vec{r}$
- Velocity: $\vec{v} = \frac{d\vec{r}}{dt}$
- Acceleration: $\vec{a} = \frac{d\vec{v}}{dt}$

Assumptions

- Use the Particle Model

Motion Diagram



To smoothly derive the third kinematics equation, start with

$$v_f = v_i + at,$$

and square it to get

$$v_f^2 = v_i^2 + 2v_i at + a^2 t^2.$$

You can factor out $2a$ from the last two terms to get

$$v_f^2 = v_i^2 + 2a\left(v_i t + \frac{1}{2}at^2\right).$$

Now you can substitute

$$\Delta y = v_i t + \frac{1}{2}at^2$$

cleanly into the prior equation to give

$$v_f^2 = v_i^2 + 2a\Delta y.$$

Constant Acceleration

The *constant-acceleration* equations (kinematics):

- $a(t) = a$
- $v(t) = v_i + at$
- $x(t) = x_i + v_i t + \frac{1}{2}at^2$

L5-1: A Falling Rock

You drop a rock from a bridge that is 45 m above the water.

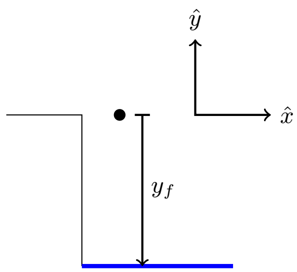
- What assumptions should we make about the motion?
- What interesting quantities can we ask about?

Assumptions

- Particle Model
- 1-D Motion
  - We will not complicate our calculations by worrying about minor amounts of horizontal motion, instead assuming all motion is vertical.
- Near Earth
  - Near the surface of Earth, acceleration due to gravity is approximately constant, so we can use the constant acceleration equations to describe the motion.

Interesting Quantities

- How long does it take to hit the water?
- How fast is the rock moving when it hits the water?



Identify Quantities by Symbol and Number

$$y_i = 0 \text{ m} \qquad v_i = 0 \text{ m} \qquad \vec{a}(t) = -g\hat{y}$$
$$y_f = -45 \text{ m} \qquad v_f = ? \qquad g = 9.81 \text{ m/s}^2 \approx 10 \text{ m/s}^2$$

Symbolic Solution

First, note that the kinematics equation for $y(t)$ simplifies quite a lot, as y_i and v_i are both zero:

$$y(t) = -\frac{g}{2}t^2.$$

At t_f, we can rearrange this to find that

$$t_f = \sqrt{-\frac{2y_f}{g}}$$

This time is the only thing we are missing from the velocity kinematics equation:

$$v_f = v_i - gt_f = -g\sqrt{-\frac{2y_f}{g}} = -\sqrt{-2gy_f}.$$

Plug in Numbers

$$v_f = -\sqrt{-2(10 \text{ m/s}^2)(-45 \text{ m})} = -\sqrt{900 \text{ m}^2/\text{s}^2} = -30 \text{ m/s}$$
$$t_f = \sqrt{\frac{-2(-45 \text{ m})}{10 \text{ m/s}^2}} = \sqrt{9 \text{ s}^2} = 3 \text{ s}$$

What if the acceleration in one direction is equal to zero? For constant acceleration, we can always turn our coordinate axes such that one axis (usually the y-axis) points along the acceleration vector, reducing our equations to:

$$
\begin{aligned}
a_x(t) &= 0 & a_y(t) &= a_y \\
v_x(t) &= v_{ix} & v_y(t) &= v_{iy} + a_y t \\
x(t) &= x_i + v_{ix} t & y(t) &= y_i + v_{iy} t + \frac{1}{2} a_y t^2
\end{aligned}
$$

What if the acceleration in the y-direction is only due to gravity?

$$
\begin{aligned}
a_x(t) &= 0 & a_y(t) &= -g \\
v_x(t) &= v_{ix} & v_y(t) &= v_{iy} - gt \\
x(t) &= x_i + v_{ix} t & y(t) &= y_i + v_{iy} t - \frac{1}{2} g t^2
\end{aligned}
$$

This assumes that $\hat{y}$ points upward, away from the surface of the Earth. We could choose a coordinate system such that down is positive, which would make $a_y(t) = g$.
NOTE: $g \approx 9.81\ \mathrm{m/s^2}$. The negative sign comes from the coordinate system, not the definition of the constant!

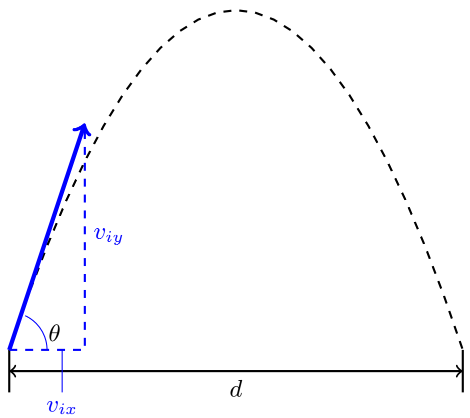
Acceleration in Two Dimensions

- What if we have an object that moves (with constant acceleration) in two directions?
- We can treat the motion in each direction independently!

$$
\begin{aligned}
a_x(t) &= a_x & a_y(t) &= a_y \\
v_x(t) &= v_{ix} + a_x t & v_y(t) &= v_{iy} + a_y t \\
x(t) &= x_i + v_{ix} t + \frac{1}{2} a_x t^2 & y(t) &= y_i + v_{iy} t + \frac{1}{2} a_y t^2
\end{aligned}
$$

- What if the acceleration in one direction is equal to zero?
- What if the acceleration in the y-direction is only due to gravity?

(1)



From the picture, we can determine that $v_{ix} = v\cos\theta$ and $v_{iy} = v\sin\theta$.

Identify Quantities by Symbol and Number

$$\begin{array}{lll} v_i = v & t_i = 0 & x_i = 0 \\ v_f = ? & t_f = ? & x_f = d \end{array}$$

(2)

First, if we make the simplifying assumption that the rock starts and lands at the same height (ignoring the initial height from the person), then we can use $y_i = y_f$ write

$$\begin{aligned} y_f &= y_i + v_{iy}t - \frac{1}{2}gt^2 \\ 0 &= 0 + v\sin\theta t - \frac{1}{2}gt^2 \\ \frac{1}{2}gt^2 &= v\sin\theta t \end{aligned}$$

Thus, the time of flight is

$$t_{\text{ToF}} = \frac{2v\sin\theta}{g}.$$

We also know that the rock travels a distance d over the course of the time of flight, so

$$\begin{aligned} x_f &= x_i + v_{ix}t \\ d &= v\cos\theta t_{\text{ToF}}. \end{aligned}$$

This gives us a second equation for time of flight,

$$t_{\text{ToF}} = \frac{d}{v\cos\theta},$$

which together with our previous expression gives us

$$\frac{2v\sin\theta}{g} = \frac{d}{v\cos\theta},$$

which may be useful in relating many different quantities.

(3)

At the highest point, $v_y = 0$ (and $v_x = v$). At the time it takes to reach the peak, we have

$$\begin{aligned} v_y(t_{\text{peak}}) &= v_{iy} - gt_{\text{peak}} \\ 0 &= v\sin\theta - gt_{\text{peak}} \\ t_{\text{peak}} &= \frac{v}{g}\sin\theta. \end{aligned}$$

The peak occurs half way through the motion ($t_{\text{peak}} = \frac{1}{2}t_{\text{ToF}}$)!

L5-2: A Thrown Rock

- You throw a rock across a flat field.
- The rock's initial speed is v, thrown at an angle θ above the horizontal.
- The rock lands a distance d away from you.

(1) Sketch a diagram to help you find the x- and y-components of the rock's velocity in terms of v and θ.

(2) Write an equation that would allow you to find the amount of time that the ball is in the air.

(3) How is this time related to the amount of time the ball takes to reach its highest point above the ground?

Main Ideas

- We can use the kinematics equations to solve for any quantity of interest when the acceleration is constant.
- Motion in 2 dimensions can be broken down into independent motion in each dimension.