Studio Week 3
Field and Flux

Picture credit: *Back to the Future* (1985).

Principles for Success

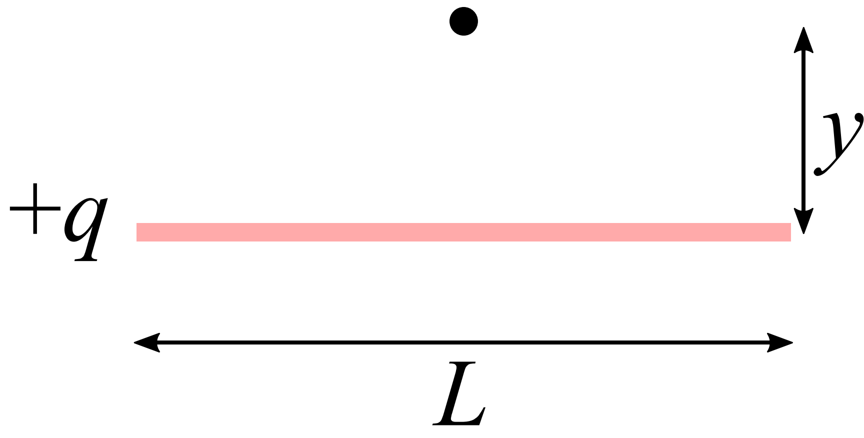
- **Treat everyone with respect.**
- **Learn by doing and questioning.**
- **Everything should make sense.**

Integrals

- Find the electric field due to any charge distribution:
  - **Chop** up the charged object into tiny pieces and find dq on each one.
  - **Multiply** dq and other quantities to find dE due to that part of the charge (don't forget direction!).
  - **Add** together all the dE's using an integral to get the net electric field.

Activity 3-1 – The Wire

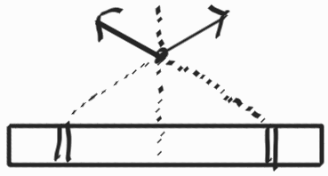
A total charge $+q$ is spread uniformly over a wire of length L.



1. Predict the electric field at the indicated point
   a. What direction do you expect E to point?
   b. Do you expect it to be bigger, smaller, or equal to kq/y^2?
2. Calculate the field
   a. **Chop** to find dq
   b. **Multiply** to find dE (don't forget direction!)
   c. **Add** to find E.
3. Make sense of your answer!

Take a picture!

3-1 The Wire



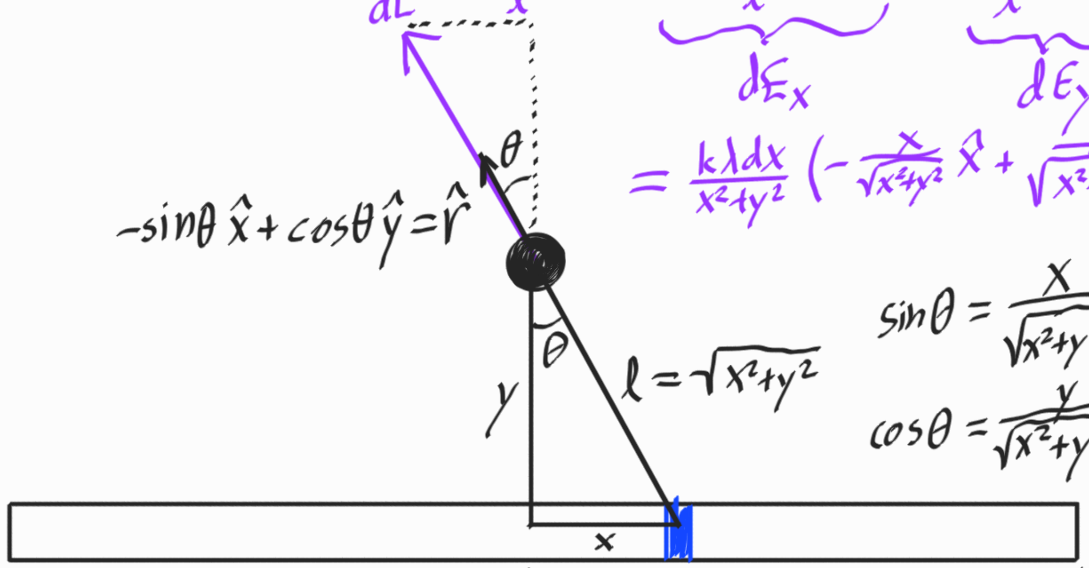
Predict

$\vec{E}$ points up — x-components cancel due to symmetry

$|\vec{E}| < \frac{kq}{y^2}$ (point charge) — charge is distributed, with some farther from the point than in the point charge case (and vector cancellations will weaken it further)

$$d\vec{E} = \frac{k\,dq}{l^2}\hat{r} = \underbrace{-\frac{k\,dq}{l^2}\sin\theta}_{dE_x}\hat{x} + \underbrace{\frac{k\,dq}{l^2}\cos\theta}_{dE_y}\hat{y}$$

$$= \frac{k\lambda dx}{x^2+y^2}\left(-\frac{x}{\sqrt{x^2+y^2}}\hat{x} + \frac{y}{\sqrt{x^2+y^3}}\hat{y}\right)$$



$\sin\theta = \frac{x}{\sqrt{x^2+y^2}}$

$\cos\theta = \frac{y}{\sqrt{x^2+y^2}}$

$dq = \lambda dx$

$+q$ spread uniformly: $\lambda = \frac{q}{L}$

$$\vec{E} = \int d\vec{E} = \int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{kq}{L}\left(\frac{-x}{(x^2+y^2)^{3/2}}\hat{x} + \frac{y}{(x^2+y^2)^{3/2}}\hat{y}\right)dx = \frac{kq}{L}\left(-\hat{x}\int_{-\frac{L}{2}}^{\frac{L}{2}}\frac{x}{(x^2+y^2)^{3/2}}dx + \hat{y}\int_{-\frac{L}{2}}^{\frac{L}{2}}\frac{y}{(x^2+y^2)^{3/2}}dx\right)$$

$\frac{x}{(x^2+y^2)^{3/2}}$ is an odd function, so it integrates to zero:

$$\int_{-\frac{L}{2}}^{\frac{L}{2}}\frac{x}{(x^2+y^2)^{3/2}}dx = \int_0^{\frac{L}{2}}\frac{x}{(x^2+y^2)^{3/2}}dx + \int_{-\frac{L}{2}}^{0}\frac{x}{(x^2+y^2)^{3/2}}dx = \int_0^{\frac{L}{2}}\frac{x}{(x^2+y^2)^{3/2}}dx + \int_{L/2}^{0}\frac{(-\bar{x})}{(\bar{x}^2+y^2)^{3/2}}(-d\bar{x})$$

$\bar{x} = -x$

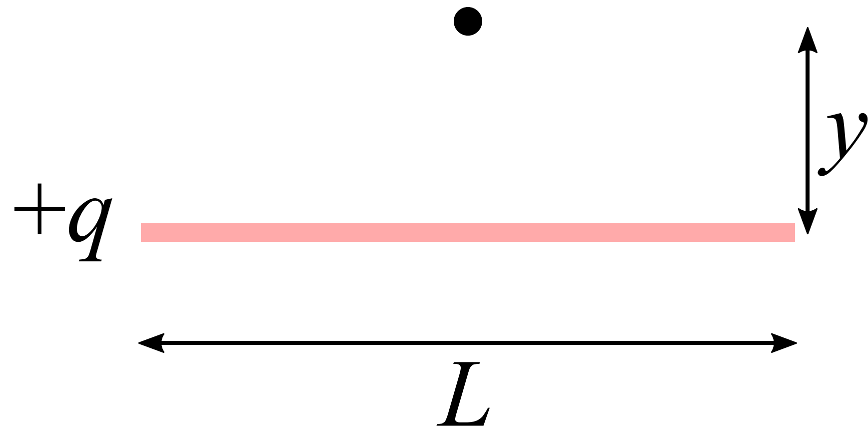
$d\bar{x} = -dx$

$$= \int_0^{\frac{L}{2}}\frac{x}{(x^2+y^2)^{3/2}}dx - \int_0^{\frac{L}{2}}\frac{\bar{x}}{(\bar{x}^2+y^2)^{3/2}}d\bar{x} = 0$$

$$\vec{E} = \frac{kqy}{L}\hat{y}\int_{-\frac{L}{2}}^{\frac{L}{2}}\frac{dx}{(x^2+y^2)^{3/2}} = \frac{kq\cancel{y}}{\cancel{L}}\frac{2\cancel{L}}{y^{\cancel{2}}\sqrt{L^2+4y^2}}\hat{y} = \frac{kq}{y\sqrt{\left(\frac{L}{2}\right)^2+y^2}}\hat{y}$$

trig substitution or computer

Activity 3-1 – The Wire



$$\vec{E} = \frac{kqy}{L}\int_{-L/2}^{L/2}\frac{dx}{(x^2+y^2)^{3/2}}\,\hat{y} = \frac{kq}{y\sqrt{\left(L/_2\right)^2+y^2}}\,\hat{y}$$

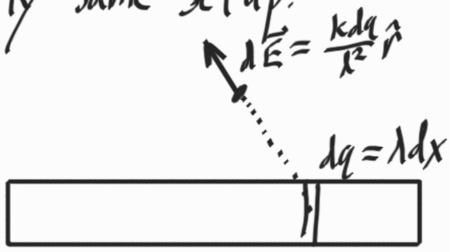
Activity 3-2a – Nonuniform Wire I

1. A wire of length $2L$ lies along the x-axis and has its center at the origin. The non-uniform charge density of this wire is known to be $\lambda(x) = \frac{+Q_o}{L^2}|x|$. We are interested in finding the electric field created by this wire at a point along the y-axis.
   a. Draw a graph of the charge density. Use your graph to draw a physical picture of the wire showing the location of charges in the wire.
   b. Predict the direction of the electric field at the indicated point.
   c. Choose a small section of the wire and write a symbolic expression for the charge on that section (dq). Label your figure with all relevant distances.
   d. Find the electric field created by your dq at a point along the y-axis.
   e. Find the electric field created by the entire wire at a point along the y-axis.

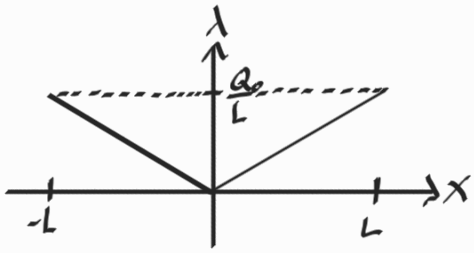
Take a picture!

3-2a Nonuniform Wire I

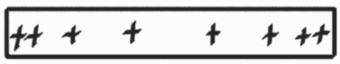
Nearly same setup:



but now $\lambda = \frac{Q_0}{L}|x|$

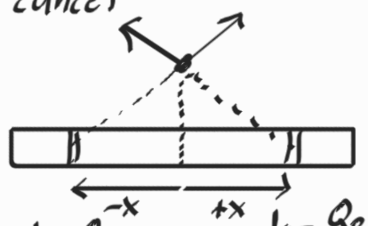


Charge Diagram



Predict:

$\vec{E}$ still points up ($\hat{y}$), as the x-components will still cancel



$dq = \frac{Q_0}{L^2}|-x|dx = \frac{Q_0}{L^2}|x|dx$ $\quad$ $dq = \frac{Q_0}{L^2}|+x|dx = \frac{Q_0}{L^2}|x|dx$

Same charge, same distance $\Rightarrow$ same $|d\vec{E}|$

$$d\vec{E} = \frac{k\lambda dx}{x^2+y^2}\Big(-\underbrace{\frac{x}{\sqrt{x^2+y^2}}}\hat{x} + \frac{y}{\sqrt{x^2+y^2}}\hat{y}\Big)$$

$\lambda = \frac{Q_0}{L^2}|x|$ $\quad$ $\frac{x|x|}{(x^2+y^2)^{3/2}}$ is an odd function, and will integrate to zero.

That leaves: $\vec{E} = \frac{kQ_0 y}{L^2}\hat{y}\int_{-L}^{L}\frac{|x|}{(x^2+y^2)^{3/2}}dx$

$$\int_{-L}^{L}\frac{|x|}{(x^2+y^2)^{3/2}}dx = \int_{-L}^{0}\frac{-x}{(x^2+y^2)^{3/2}}dx + \int_{0}^{L}\frac{x}{(x^2+y^2)^{3/2}}dx = 2\int_{0}^{L}\frac{x}{(x^2+y^2)^{3/2}}dx$$

$$= 2\Big[-(x^2+y^2)^{-\frac{1}{2}}\Big]_{x=0}^{L} = \frac{2}{y} - \frac{2}{\sqrt{L^2+y^2}}$$

Can transform one integral into the other

$x \to -x$ (pick up $-$ sign) $\quad$ (also makes $\int_{-L}^{0} \to \int_{L}^{0}$)

$dx \to -dx$ (pick up $-$ sign)

$\int_{L}^{0} \to \int_{0}^{L}$ (pick up $-$ sign)

$$\vec{E} = \frac{kQ_0 y}{L^2}\underbrace{\Big(\frac{2}{y} - \frac{2}{\sqrt{L^2+y^2}}\Big)}_{2\frac{\sqrt{L^2+y^2}-y}{y\sqrt{L^2+y^2}}}\hat{y} = \frac{2kQ_0}{L^2}\frac{\sqrt{L^2+y^2}-y}{\sqrt{L^2+y^2}}\hat{y}$$

Activity 3-2b – Nonuniform Wire II

- A wire of length $2L$ lies along the x-axis and has its center at the origin. The non-uniform charge density of this wire is known to be $\lambda(x) = \frac{+Q_o}{L^2}|x|$. We just found the electric field created by this wire at a point along the y-axis is

$$\vec{E} = \frac{2kQ_o}{L^2}\left(\frac{\sqrt{L^2+y^2}-y}{\sqrt{L^2+y^2}}\right)\hat{y}$$

1. Make sense of this answer using the following strategies:
   a. Check that the units are correct for electric field!
   b. Check that it points in the correct direction.
   c. Evaluate the answer in the special case that y goes to ∞. (Remember to compare the actual answer to your expectation!)
   d. Sketch a graph of the electric field along the y-axis. Does your sketch have the right qualitative behavior?

Take a picture!

3-2b Nonuniform Wire II

$$\vec{E} = \frac{2kQ_0}{L^2}\,\frac{\sqrt{L^2+y^2}-y}{\sqrt{L^2+y^2}}\,\hat{y}$$

points up as predicted

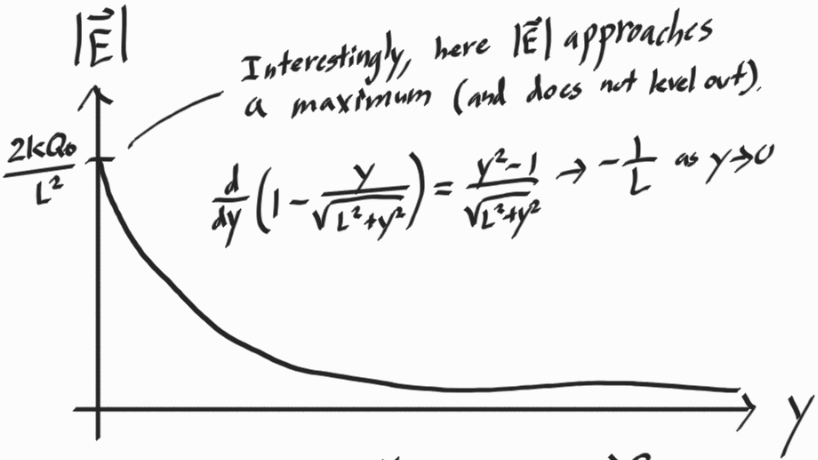
Units $\quad \dfrac{N}{C} = \dfrac{\frac{N\cancel{m^2}}{C^{\cancel{2}}}\,\cancel{C}}{\cancel{m^2}}\;\underbrace{\dfrac{\sqrt{m^2+m^2}-m}{\sqrt{m^2+m^2}}}$

$$\frac{\sqrt{m^2}-m}{\sqrt{m^2}} = \frac{m-m}{m} = \frac{m}{m} = \text{unitless}$$

$\underline{y\to\infty}$

$$\frac{\sqrt{L^2+y^2}-y}{\sqrt{L^2+y^2}} = 1 - \frac{y}{\underbrace{\sqrt{L^2+y^2}}_{\to 1 \text{ as } y\to\infty}} \to 0 \text{ as } y\to\infty$$

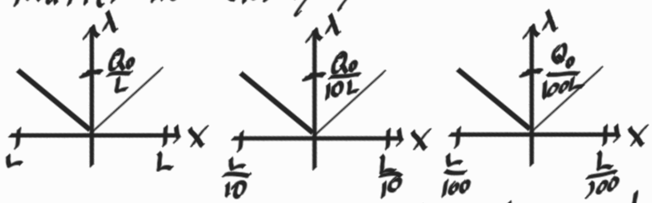
$\vec{E}\to 0$ as $y\to\infty$, which we expect to happen when the charge gets far away.



Interestingly, here $|\vec{E}|$ approaches a maximum (and does not level out).

$$\frac{d}{dy}\left(1-\frac{y}{\sqrt{L^2+y^2}}\right) = \frac{y^2-1}{\sqrt{L^2+y^2}} \to -\frac{1}{L} \text{ as } y\to 0$$

In 3-1, $|\vec{E}|\to\infty$ as $y\to 0$.
Here, there is no charge right at $x=0$ on the wire, so we aren't trying to stack one charge directly onto another as $y\to 0$ (hence $|\vec{E}|\not\to\infty$).
However, there is always a tiny bit of charge close to the center, no matter how closely you zoom in.

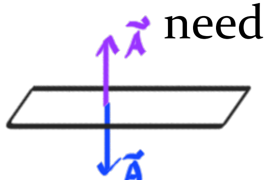


As $y\to 0$, the point is getting closer and closer to these charges, where their electric fields are stronger. The charges close enough to exert most of their field in the y-direction are fewer and fewer, though, so the lessening of the effective charge and the strengthening of its field balances to give a finite maximum as $y\to 0$. At $y=0$, of course, there are no y-components, so $\vec{E}=0$ (but we built our model on the idea that the point was above the wire, so this violates our assumption).

Activity 3-3 – Area Vectors

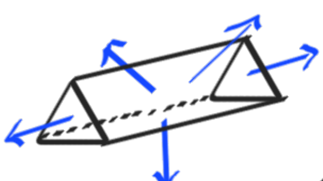
1. Take a single rectangular sheet of paper and hold it up.
   a. Indicate the area vector for your piece of paper. How many area vectors do you need?
2. Fold your paper twice so that it makes a triangular prism.
   a. Indicate the area vector for your piece of paper. How many area vectors do you need?
3. Find another piece of paper and curl it into a cylinder.
   a. Indicate the area vector for the cylinder. How many area vectors do you need?

1)



One area vector, but you are free to choose if it points one way or the other.

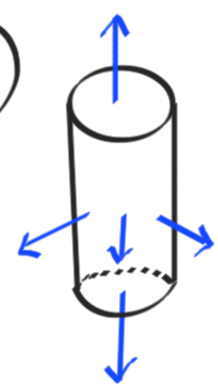
2)



Five area vectors (if you close the triangular ends).

By convention, area vectors of a closed surface point out.

3)



One area vector on each end, and perhaps infinitely many on the curved side?

Technically, any surface (even flat) has infinitely many area vectors (because these vectors are local to each spot on the surface), but we can often find a way to express them concisely. Here, using $\hat{s}$ (a unit vector from cylindrical coordinates) can help to characterize the vectors on the curved side in brief.

Take a picture!

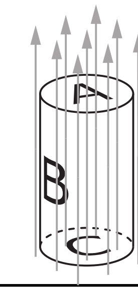
In the following questions, an imaginary cylinder with radius *a* and length *l* is placed in various electric fields. The end caps are labeled A and C and the side surface is labeled B. In each case, *base your answer about the net flux only on qualitative arguments about the absolute value of the flux through the end caps and side surface.*

The Gaussian cylinder is in a uniform electric field of magnitude E_o aligned with the cylinder axis.

- Find the sign and absolute value of the flux through:

  Surface A: Surface B: Surface C:

- Is the net flux through the Gaussian surface *positive, negative,* or *zero?*

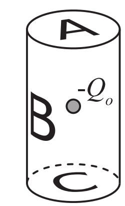


The Gaussian cylinder encloses a negative charge. (The field from part A is removed.)

- Find the sign of the flux through:

  Surface A: Surface B: Surface C:

- Is the net flux through the Gaussian surface *positive, negative,* or *zero?*

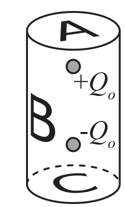


The Gaussian cylinder encloses opposite charges of equal magnitude. The charges are on the axis of the cylinder and equidistant from the center.

- Find the sign of the flux through:

  Surface A: Surface B: Surface C:

- Is the net flux through the Gaussian surface *positive, negative,* or *zero?*

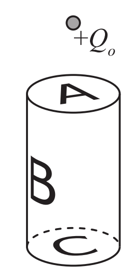


A positive charge is located above the Gaussian cylinder.

- Find the sign of the flux through:

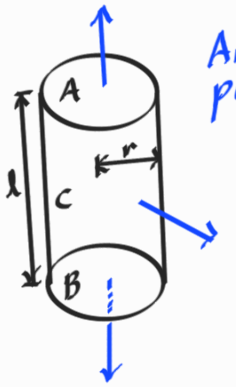
  Surface A: Surface B: Surface C:

- Can you tell whether the net flux through the Gaussian surface is *positive, negative,* or *zero?* Explain.



Take a picture!

3-4 Gaussian Cylinders

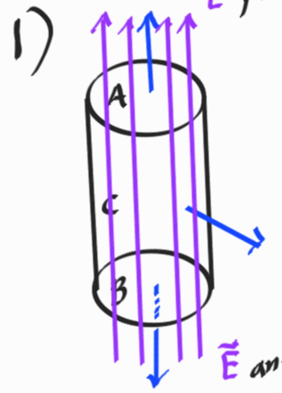


Area vectors point out.

Area $A, C = \pi r^2$

Area $B = 2\pi r l$

1)

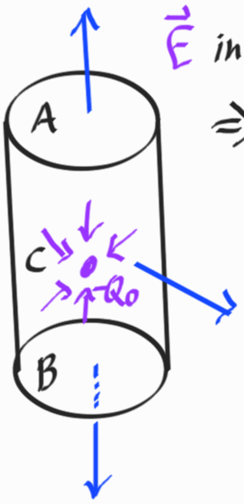


$\vec{E}$ parallel to $\vec{A}$: positive flux $\vec{E}\cdot\vec{A} = E_0 \pi r^2$

$\vec{E}$ perpendicular to $\vec{A}$: zero flux

$\vec{E}$ antiparallel to $\vec{A}$: negative flux $\vec{E}\cdot\vec{A} = -E_0 \pi r^2$

2)

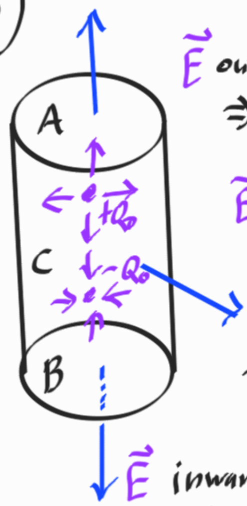


$\vec{E}$ inward, $\vec{A}$ outward

$\Rightarrow$ flux negative through all three surfaces

$\Rightarrow$ total flux is zero

3)



$\vec{E}$ outward through A

$\Rightarrow$ flux is positive

$\vec{E}$ out through top half of C, in through bottom half

$\Rightarrow$ flux positive in top half, negative in bottom half

symmetry $\Rightarrow$ total flux zero (halves have equal and opposite flux)

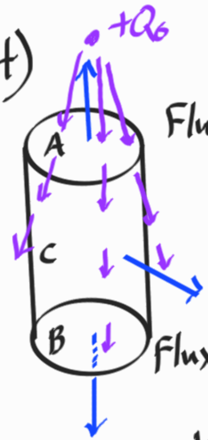
$\vec{E}$ inward through B

$\Rightarrow$ flux is negative

symmetry $\Rightarrow$ same magnitude as flux through A

Total flux is zero.

4)



Flux through A is strong and negative.

$\vec{E}$ points slightly outward through C, so flux is positive.

Flux through B is weak and positive.

We can't tell directly from the picture, but Gauss' law will allow us to conclude that the net flux must be zero (thus the negative flux through A cancels with the positive flux through B and C).