Studio Week 7
Work

Picture credit: Snow White and the Seven Dwarfs (1939).

Principles for Success

- **Treat everyone with respect.**
- **Learn by doing and questioning.**
- **Make sense of everything.**

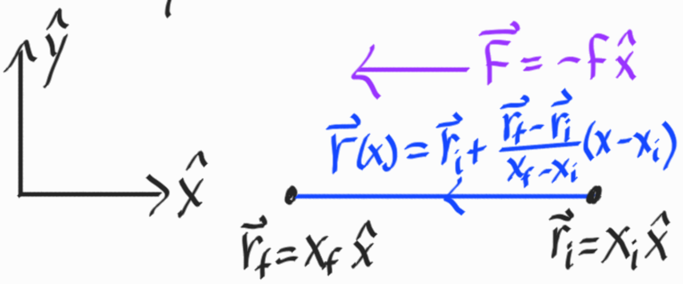
Work and Kinetic Energy

- Definitions
  - Work $W = \int_{x_i}^{x_f} \vec{F} \cdot d\vec{x}$
  - Kinetic energy $K = \frac{1}{2}mv^2$

Work and Kinetic Energy

Coordinate System Care

What if my path from the initial point to the final point goes in the negative direction of my coordinate system?



If force and displacement are in the same direction, the work should be positive, but how does this show up in the math?

$$W = \int_{r_i}^{r_f} \vec{F}\cdot d\vec{r} = \int_{x_i}^{x_f} \vec{F}\cdot \frac{d\vec{r}}{dx}\,dx = -\int_{x_i}^{x_f} F\,dx = \int_{x_f}^{x_i} F\,dx > 0$$

$$\frac{d\vec{r}}{dx} = \frac{\vec{r}_f - \vec{r}_i}{x_f - x_i} = \frac{x_f\hat{x} - x_i\hat{x}}{x_f - x_i} = \frac{x_f - x_i}{x_f - x_i}\hat{x} = \hat{x}$$

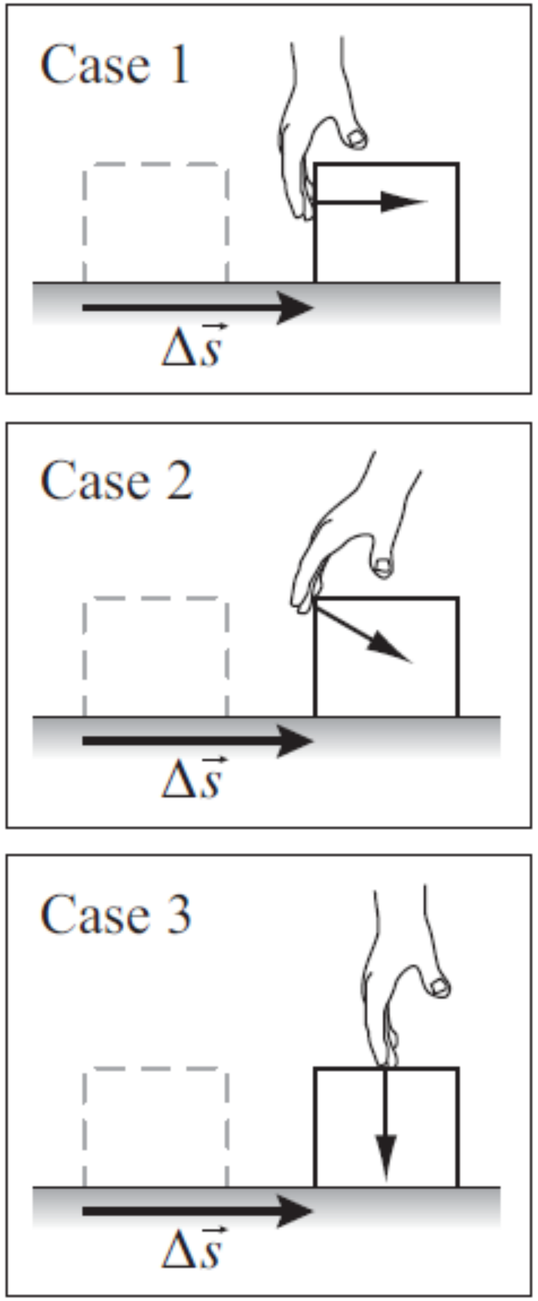
Because our path $\vec{r}(x)$ has the same length scale as our parameterization x, the derivative just gives us direction.

An easy mistake is to look at $W = \int_{x_i}^{x_f} \vec{F}\cdot d\vec{x}$, plug in $W = \int_{x_i}^{x_f} (-F)(-dx) = \int_{x_i}^{x_f} F\,dx = -\int_{x_f}^{x_i} F\,dx$ and get a negative work by accidentally introducing an extra negative sign. It is important to make a clear distinction between the path $\vec{r}(x)$ and the parameterization x.

Warm-up

A block is moving to the right (displacement Δs) while a hand exerts a force of magnitude F_o on the block.

- In each case, is the work done by the hand *positive, negative,* or *zero?*
- How does the absolute value of the work compare in the three cases?

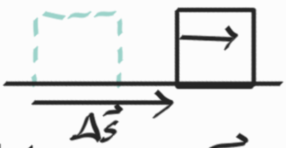


Warm-Up: Pushing Blocks Along

Since the force is constant, and the angle between the force and the displacement is constant, the work simplifies:

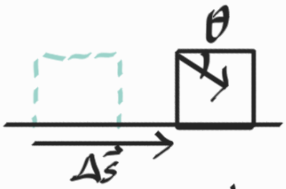
$$W = \int_{x_i}^{x_f} \vec{F}_0 \cdot d\vec{x} = \int_{x_i}^{x_f} F_0 \, dx \cos\theta = F_0 \cos\theta \int_{x_i}^{x_f} dx = F_0 \Delta s \cos\theta$$
$$= \vec{F}_0 \cdot \Delta \vec{s}$$

Case 1



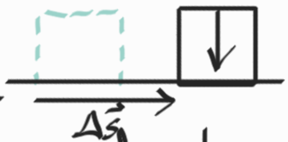
$\vec{F}_0$ is parallel to $\Delta\vec{s}$, so $W_1 = \vec{F}_0 \cdot \Delta\vec{s} = F_0 \Delta s > 0$

Case 2



$\vec{F}_0$ makes an acute angle θ with $\Delta\vec{s}$, so $W_2 = F_0 \Delta s \cos\theta$, where $0 < \cos\theta < 1$, so $W_2 > 0$.

Case 3



$\vec{F}_0$ is perpendicular to $\Delta\vec{s}$, so $W_3 = \vec{F}_0 \cdot \Delta\vec{s} = 0$.

Since all of the works are nonnegative, they are equal to their absolute values, and we can rank them as follows:

$$W_1 > W_2 > W_3 = 0$$

Activity 7-1

- A block is initially moving to the right on a level, frictionless surface. The system consists of only the block.
- A hand exerts a constant horizontal force on the block that causes the block to slow down (stage 1), then move in the opposite direction while speeding up (stage 2).
- For each stage:
  - Draw a free-body diagram for the block
  - Determine whether the work done by each force is positive, negative, or zero
  - Fill in the table below

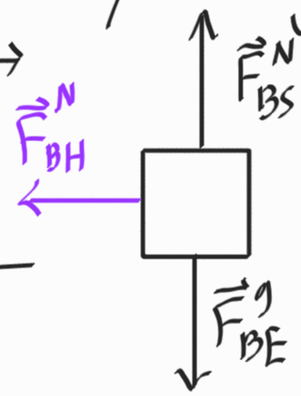
	Displacement		Force by hand		$W_{by\ hand}$	$W_{net,ext}$
	Direction	Sign	Direction	Sign	Sign	Sign
Stage 1						
Stage 2						

7-1 Reversing a Block

Free-Body Diagram

Stage 1: $\Delta\vec{s} \longrightarrow$
Slowing Down

Stage 2: $\Delta\vec{s} \longleftarrow$
Speeding Up



$\vec{F}^N_{BS} \perp \Delta\vec{s} \Rightarrow W_{by\ surface} = 0$

$$\boxed{W_{net,ext} = W_{by\ hand}} + \cancel{W_{by\ surface}} + \cancel{W_{by\ gravity}}$$

$\vec{F}^g_{BS} \perp \Delta\vec{s} \Rightarrow W_{by\ gravity} = 0$

Coordinate System: $\longrightarrow +X$

	Displacement		Force by Hand		$W_{by\ hand}$	$W_{net,ext}$
	Direction	Sign	Direction	Sign	Sign	Sign
Stage 1	$\rightarrow$	+	$\leftarrow$	−	−	−
Stage 2	$\leftarrow$	−	$\leftarrow$	−	+	+

Coordinate System: $\longleftarrow +X$

	Displacement		Force by Hand		$W_{by\ hand}$	$W_{net,ext}$
	Direction	Sign	Direction	Sign	Sign	Sign
Stage 1	$\rightarrow$	−	$\leftarrow$	+	−	−
Stage 2	$\leftarrow$	+	$\leftarrow$	+	+	+

The work done by a force does not depend on your choice of coordinate system.

The work done is negative in stage 1, where the force is opposite the displacement. Energy is subtracted from the system, and the speed of the block decreases.

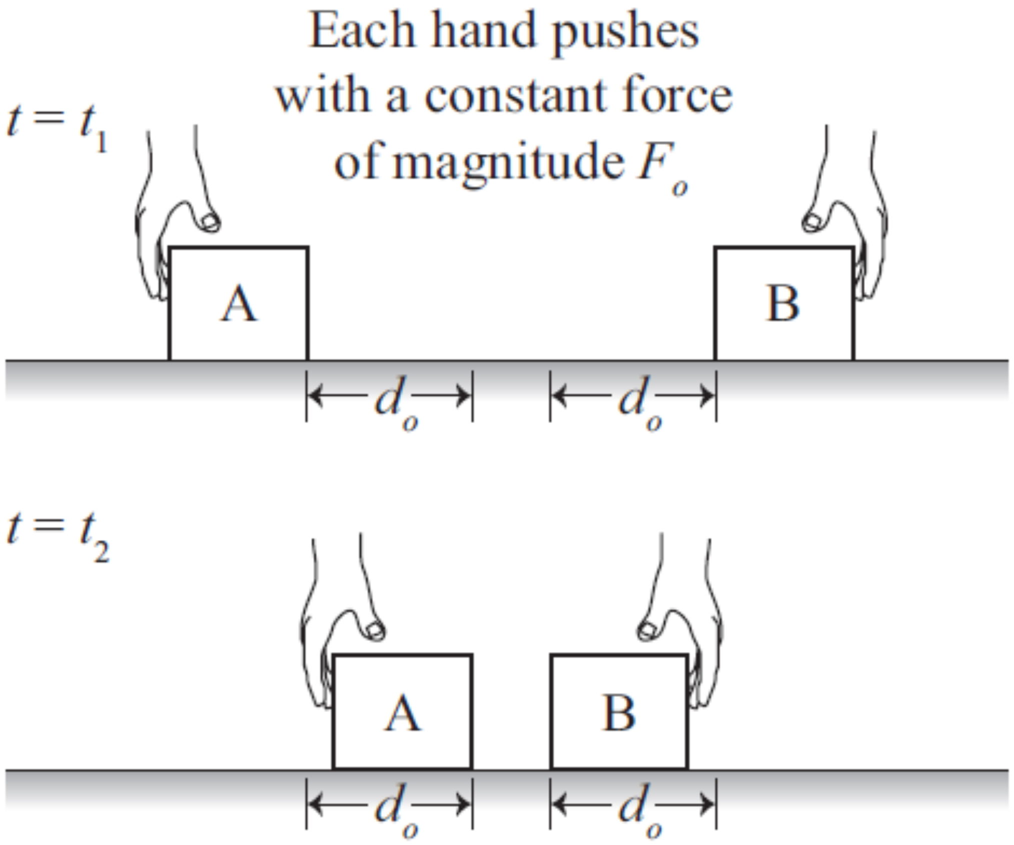
The work done is positive in stage 2, where the force is in the same direction as the displacement. Energy is added from the system, and the speed of the block decreases.

The Work-Energy Theorem

$$W_{\text{net,ext}} = \Delta E_{\text{total}}$$

Activity 7-2

- Two identical blocks, A and B, are initially at rest on a level, frictionless surface. System AB consists of **both blocks.**
- At time $t = t_1$, hands begin to push the blocks toward each other as shown at right.
- Each hand exerts a constant horizontal force of magnitude F_o.
- At time $t = t_2$, each block has moved a distance d_o from its initial position.
  1. determine if the **change in kinetic energy** of system AB is *positive, negative,* or *zero*
  2. determine if the work done by **each force** is *positive, negative,* or *zero*
  3. determine if the **net work** on system AB is *positive, negative,* or *zero*
  4. check to see if your answers are consistent with the work-energy theorem



7-2 Pushing Boxes Together

Pushed inward by the two forces, both blocks will accelerate, so they each have a nonzero final speed. They began at rest, so $K_i = 0\,J$, and their final kinetic energy is $K_f = \frac{1}{2} m_A v_A + \frac{1}{2} m_B v_B$, therefore the change in kinetic energy of the system is positive ($\Delta K > 0$).

The force of the hand on A points right, and its point of application (where the hand is touching the block) is displaced to the right, so the work done by the hand is positive ($W_A^{hand} > 0$).

For block B, the force of the hand and the displacement are both to the left, so the work is still positive ($W_B^{hand} > 0$).

Note that W_B^{hand} is not negative due to being "in the opposite direction." The sign of work is determined by the direction of the force relative to the direction of the displacement of it's point of contact.

The work done by gravity and the work done by the normal force on both blocks are zero ($W_A^g = W_B^g = W_A^N = W_B^N = 0$). Their points of contact are displacing, but the forces are perpendicular to the displacement.

The net work is the sum of the individual works:

A	B
$W_A^{hand} > 0$	$W_B^{hand} > 0$
$+ W_A^g = 0$	$+ W_B^g = 0$
$+ W_A^N = 0$	$+ W_B^N = 0$
$= W_A^{net} > 0$	$= W_B^{net} > 0$

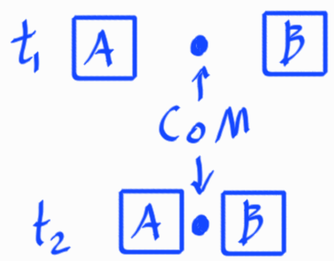
$$W_{AB}^{net} = W_A^{net} + W_B^{net} > 0$$

The net work on the system is positive.

Note that the net work is not generally equal to the integral of the net force dotted with the displacement of the center of mass:

$$W_{net} \neq \int \vec{F}^{net} \cdot d\vec{s}_{CoM}$$

This would give the wrong answer here, as the net force on AB is $\vec{F}_A + \vec{F}_B = F_0 \hat{x} + F_0(-\hat{x}) = \vec{0}$, and the center of mass doesn't move.



Work - Energy Theorem

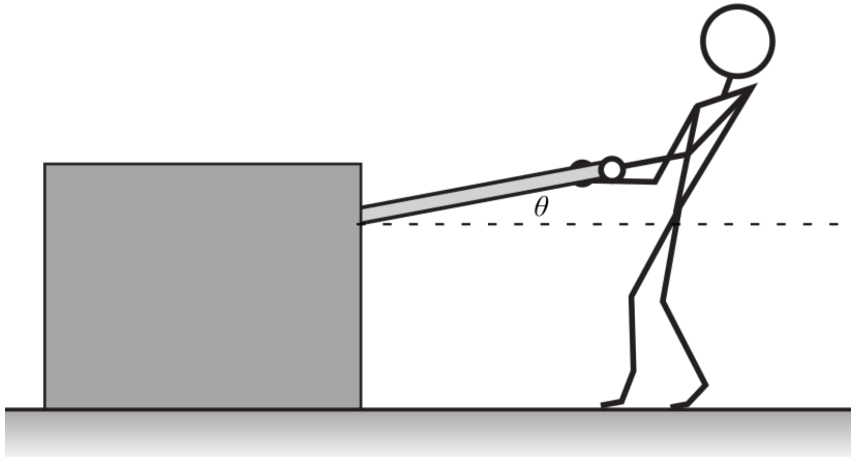
$$W_{net} = \Delta E_{total}$$

$$W_{AB}^{net} > 0 \quad \Big| \quad \Delta K > 0$$

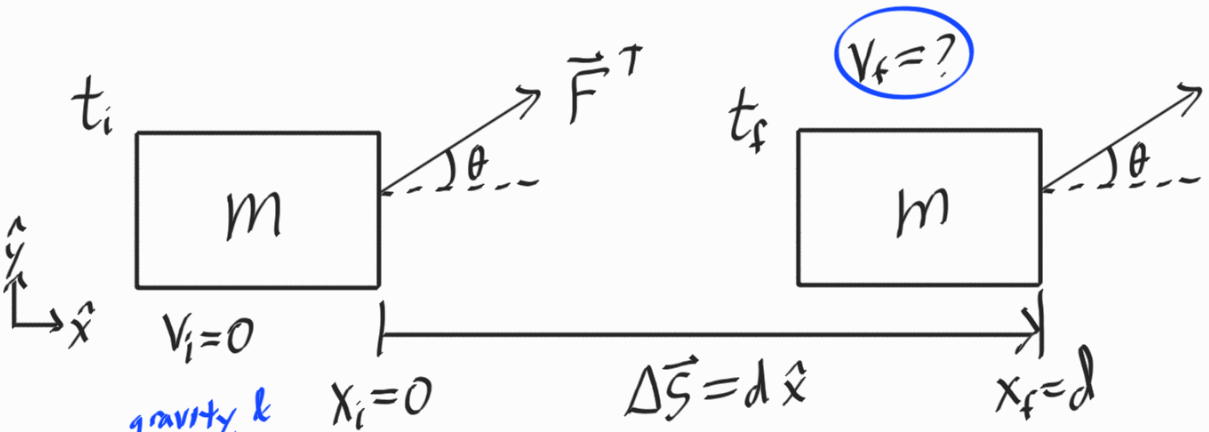
Both are positive, so our results are consistent.

Activity 7-3

- You are pulling a box with known mass m at the angle shown below. The box moves from $x = 0$ to $x = d$. Find the work done by the rope in each of the following situations:
  A. The tension is constant.
  B. The tension decreases linearly from $3T_o$ to T_o.
- If the box began at rest, find the final speed of the box in each of the situations above.
- Don't forget to make sense of your answers!



7-3 Dragging a Box



gravity & normal do no work here

$$W^{net} = W_T = \int_0^d \vec{F}^T \cdot d\vec{x} = \int_0^d F^T \cos\theta \, dx = \cos\theta \int_0^d F^T dx$$

$$\left.\begin{array}{l} K_i = 0 \\ K_f = \frac{1}{2} m v_f^2 \end{array}\right\} \Delta K = \frac{1}{2} m v_f^2 = W^{net} \Rightarrow V_f = \sqrt{\frac{2W^{net}}{m}}$$

A) $F^T = T_0$ (constant)

$$W_T = T_0 \cos\theta \int_0^d dx = T_0 d \cos\theta$$

$$V_f = \sqrt{2\frac{T_0 d}{m}\cos\theta}$$

Unit Check:

2 & $\cos\theta$ are unitless

$$\left[\sqrt{\frac{T_0 d}{m}}\right] = \sqrt{\frac{[T_0][d]}{[m]}} = \sqrt{\frac{kg \cdot m \cdot s^{-2} \cdot m}{kg}} = \sqrt{\frac{m^2}{s^2}} = \frac{m}{s} = [V_f] \checkmark$$

Covariation

- T_0 or d increases $\Rightarrow V_f$ increases.
  This makes sense, as pushing the block harder or pulling it farther will impart more kinetic energy, causing it to end up going faster.
- θ or m increases $\Rightarrow V_f$ decreases.
  This makes sense, as pulling at a larger angle to the displacement is less efficient for doing work, and a more massive object resists accelerating more.

B) $F^T(x) = 3T_0 - \frac{2T_0}{d}x$

(linear in x)

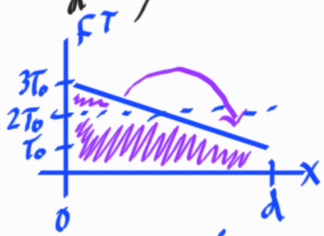
$$W_T = T_0 \cos\theta \int_0^d \left(3 - \frac{2}{d}x\right) dx$$

$$= T_0 \cos\theta \left[3x - \frac{1}{d}x^2\right]_{x=0}^{d}$$

$$= T_0 \cos\theta \left(3d - \frac{1}{d}d^2\right)$$

$$= 2T_0 d \cos\theta$$

$$V_f = 2\sqrt{\frac{T_0 d}{m}\cos\theta}$$



We can make sense of seeing $2T_0 d$ appear by observing how the area under this graph is the same as the area of a rectangle of height $2T_0$ and width d.

Activity 7-4

- For each situation to the right:
  1. identify all external forces acting on the system
  2. draw an energy system diagram
  3. determine if the work done by **each force** is positive, negative, or zero
  4. determine if the **net work** is positive, negative, or zero
  5. determine if the **kinetic energy** is increasing, decreasing, or remaining constant

A. An elevator moves downward at constant speed (system = elevator)

B. A ball has been thrown and travels upward in a straight line (system = ball)

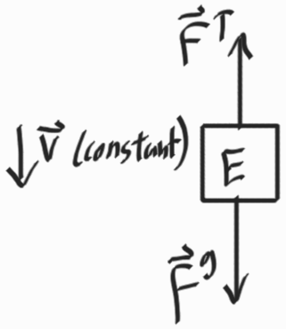
C. A sled slides to a stop while traveling up a rocky slope (system = sled)

7-4 Three Cases of the Work-Energy Theorem

Elevator

F^T – tension from cable

F^g – force of gravity from Earth



$W_T < 0$

The work done by tension is negative, as the force points up, and the displacement points down.

$W_g > 0$

The work done by gravity is positive, as even though $\vec{F}^g$ points down, the displacement also points down.

$W_{net} = W_T + W_g = 0$

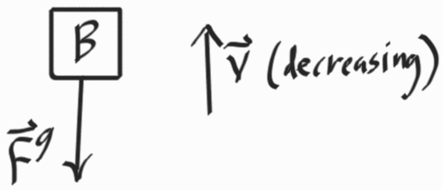
The forces of tension and gravity are equal in magnitude over the same displacement, so $|W_g| = |W_T|$.

$\Delta K = 0$

Velocity is constant, and no work is done.

Ball Tossed Up

F^g – force of gravity from Earth



$W_g < 0$

The work done by gravity is negative, as the force points down while the displacement points up.

$W_{net} = W_g < 0$

$\Delta K < 0$

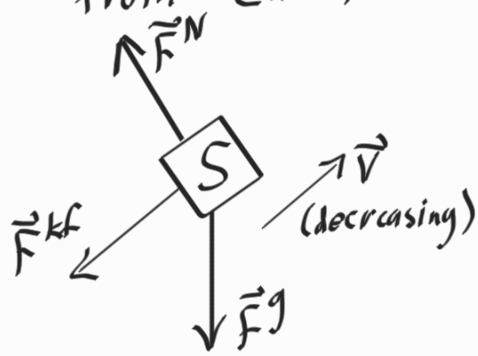
The kinetic energy is decreasing, as the work takes energy out of the system. We can also see this in the decreasing speed of the ball.

Slowing Sled

F^N – normal force from slope

F^{kf} – kinetic friction from slope

F^g – force of gravity from Earth



$W_N = 0$

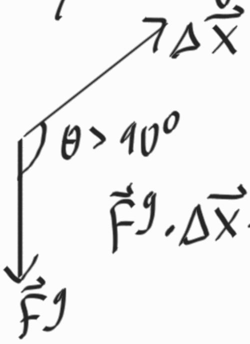
The work done by the normal force is zero, as the force is perpendicular to the displacement.

"W_{kf}" < 0

The "work" done by friction is negative, as the force is opposite the displacement.

$W_g < 0$

The work done by gravity is negative:



$\vec{F}^g \cdot \Delta\vec{X} < 0$

$W_{net} = W_N +$ "W_{kf}" $+ W_g < 0$

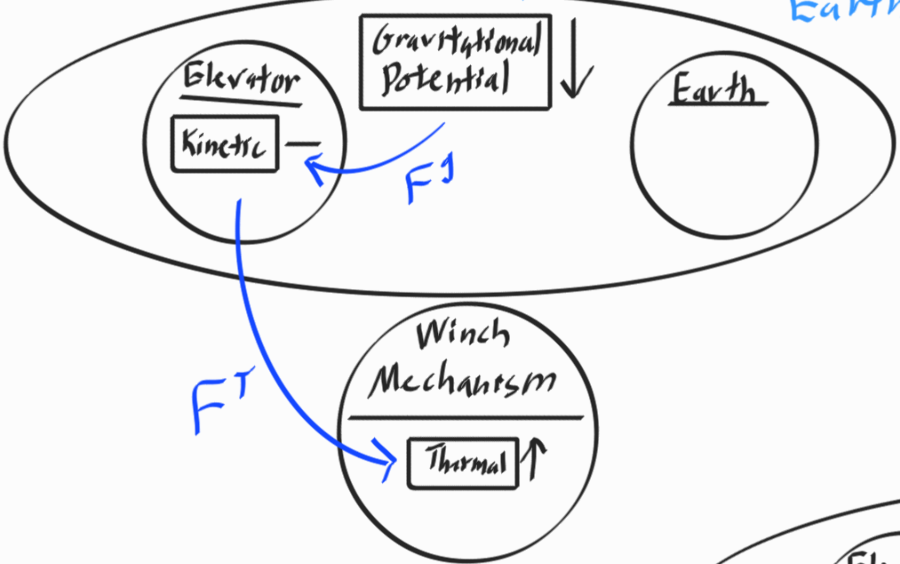
$\Delta K < 0$

The kinetic energy is decreasing, which we can see from $W_{net} < 0$ or from velocity decreasing.

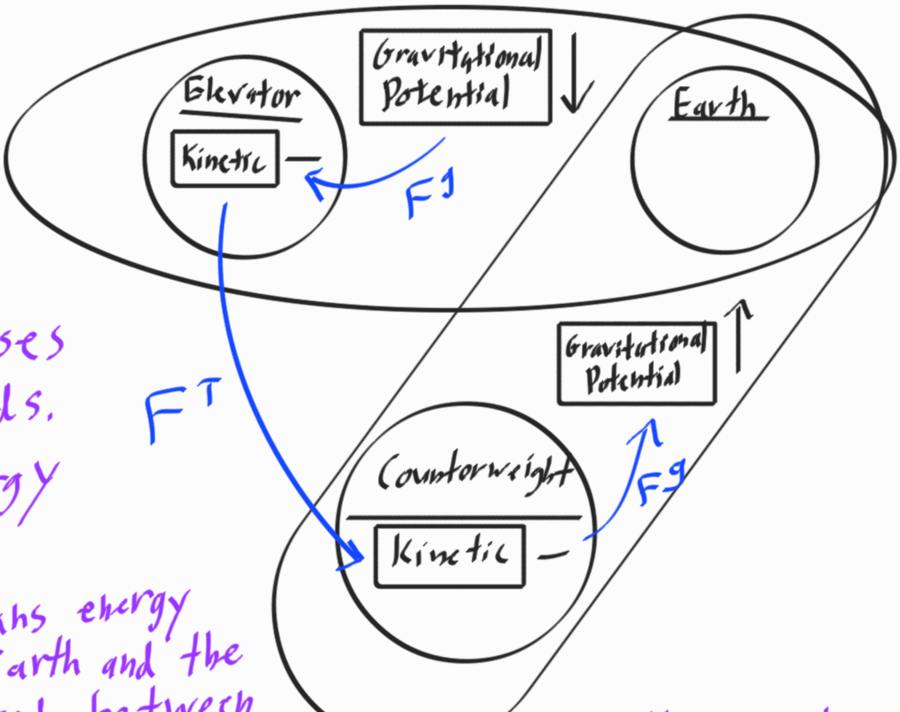
Energy System Diagrams

Elevator

Note that not just the elevator has gravitational potential energy. That energy is shared between Earth and the elevator.

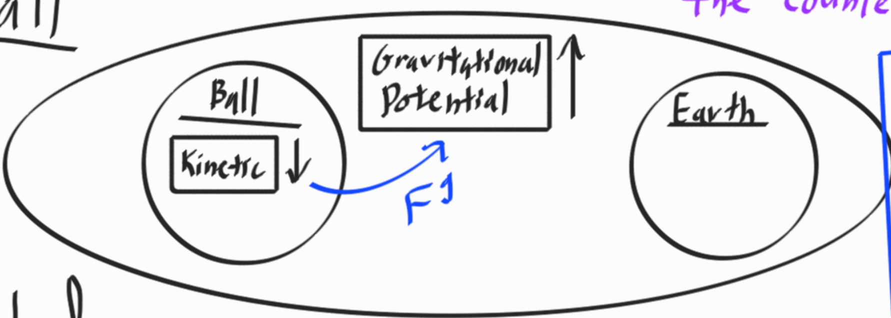


In this first interpretation of the elevator, some winch is used to lower the elevator, and keeps it from speeding up using some sort of friction brake that dissipates the energy as heat.

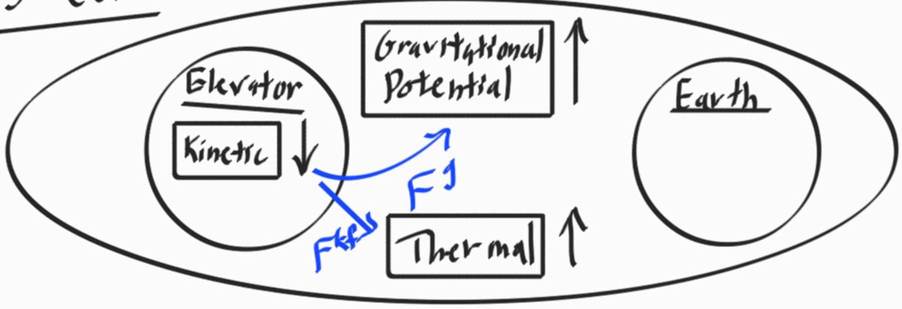


In this second interpretation, the cable connects to a counterweight that rises as the elevator descends. The cable transfers energy from the elevator to the counterweight, which means energy once shared between the Earth and the elevator is now shared between the counterweight and the elevator.

Ball



Sled

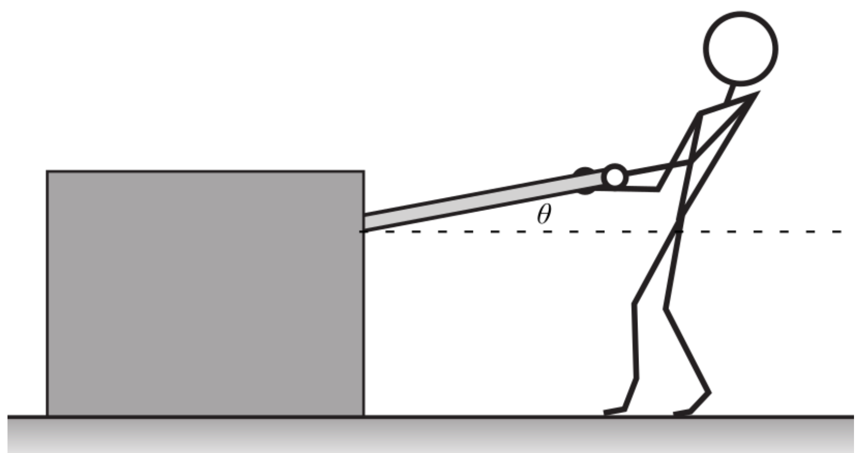


"W_{kf}"

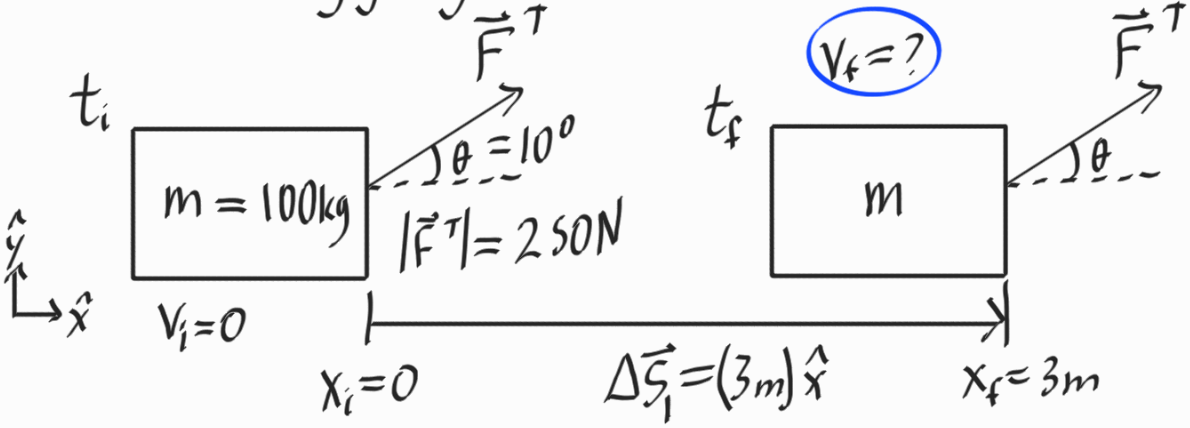
Technically, the energy lost due to friction ΔE_{th} is not "work" (hence my earlier quotation marks). The reason for this is kind of techical, and you can calculate this energy loss just like work, so you don't need to worry.

Activity 7-~~4~~

- You are pulling a box with mass 100 kg with a tension force of 250 N at an angle of 10°, as shown below.
  - How much work (in J) have you done after the box has slid 3 m along the floor?
  - How fast is the box moving at this instant?
- You now want to bring the box back to rest by pushing on it with a force of 200 N.
  - How far will the box move before it returns to rest?



7-X Dragging a Box II



$$W_T = \vec{F}^T \cdot \Delta\vec{s}_1 = |\vec{F}^T||\Delta\vec{s}_1|\cos\theta = (250\,N)(3\,m)\cos(10^\circ)$$
$$= 738.6\,J$$

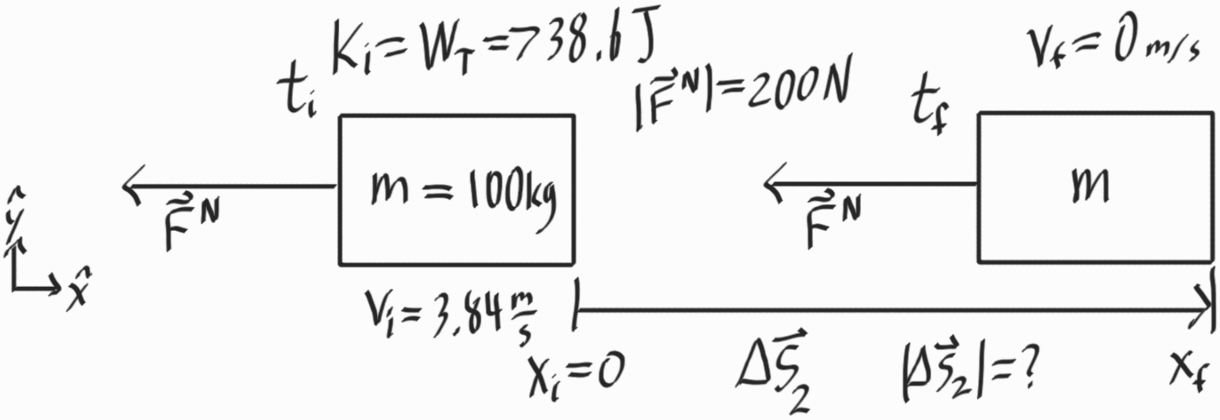
$$K_f - \cancel{K_i}^{\,0} = \Delta K = W_T$$
$$\hookrightarrow = \tfrac{1}{2} m v_f^2$$

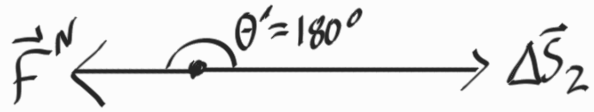
Symbolic Solution

$$v_f = \sqrt{\frac{2W_T}{m}} = \sqrt{\frac{2F^T \Delta s_1 \cos\theta}{m}}$$
$$= \sqrt{\frac{2(738.6\,J)}{100\,kg}}$$
$$= 3.84\,m/s$$

Units: $J = \frac{kg\,m^2}{s^2}$

$$\sqrt{\frac{J}{kg}} = \sqrt{\frac{m^2}{s^2}} = \frac{m}{s}$$





$$W_N = \vec{F}^N \cdot \Delta\vec{s}_2 = |\vec{F}^N||\Delta\vec{s}_2|\cos\theta' = -|\vec{F}^N||\Delta\vec{s}_2|$$

$$W_N = \Delta K = \cancel{K_f}^{\,0} - K_i = -W_T$$

Symbolic Solution

$$|\Delta\vec{s}_2| = \frac{W_T}{|\vec{F}^N|} = \frac{F^T \Delta s_1 \cos\theta}{F^N}$$
$$= \frac{738.6\,J}{200\,N} = 3.69\,m$$

Sensemaking

F^N is 50 N weaker than F^T, and only at a slightly more direct angle, so F^N should be able to stop the box over somewhat more distance than it took F^T to speed it up. This agrees with our numbers, as $\Delta s_2 = 3.69$ is moderately bigger than $\Delta s_1 = 3\,m$.