(C) is true, as this condition makes the net impulse on the object zero. If the net force is nonzero, but very weak, or lasting only for a very short while, then we can also say that the impulse is approximately equal.

Studio 7: Impulse and Conservation of Momentum

(and some energy too)

Warm-Up Activity

Momentum is conserved when…

(A) The kinetic energy is zero.

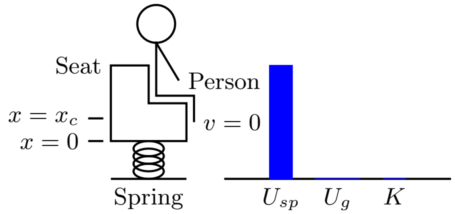
(B) The net external work is zero.

(C) The net force is zero.

(D) There is no change in potential energy.

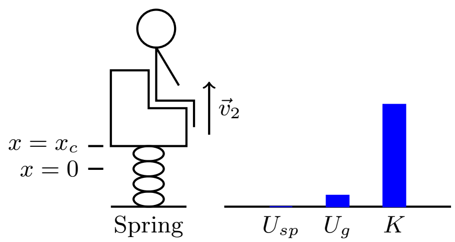
We want to include the Earth and the spring in our system with the seat and passenger in order to use conservation of energy. If we don't, then we have to go to the trouble of calculating work done by gravity and the spring.

At t_1, the person and seat are stationary, and the spring is compressed beneath them. All energy is spring potential energy.



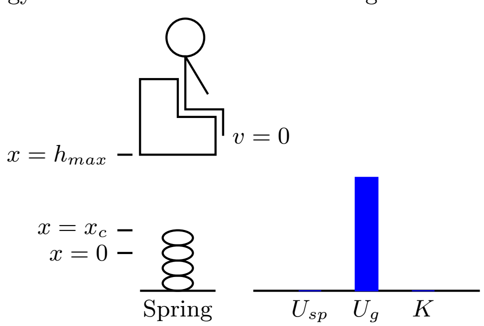
Here, I define the amount by which the spring was compressed as x_c, and I set $x = 0$ at the top of the spring (so the spring is compressed to $x = 0$, and its equilibrium is at $x = x_c$).

At t_2, the spring is at its equilibrium length, and the person and seat have been lifted up slightly and now have some speed. The spring potential energy has all been transformed into kinetic energy and a little bit of gravitational potential energy.



I use the label v_2 fr the speed at this moment. It is not v_{max}, which would actually happen before the spring reaches equilibrium.

At t_3, the person and seat stop momentarily at their maximum height. The kinetic energy has all been transformed into gravitational potential energy.



	U_{sp}	U_g	K	E_{total}
t_1	$\frac{1}{2}kx_c^2$	0	0	$\frac{1}{2}kx_c^2$
t_2	0	mgx_c	$\frac{1}{2}mv_2^2$	$mgx_c + \frac{1}{2}mv_2^2$
t_3	0	mgh_{max}	0	mgh_{max}

S7-1: Ejector Seat

- A stationary stunt car driver has an ejector seat that rests on a compressed vertical spring.
- When the spring is released, the seat with its passenger is launched out of the car and into the air.
  - Explain why you might want to include the Earth and the spring in your system.
- Consider three instants in time: (1) just before the seat is released, (2) when the spring is at equilibrium, and (3) the person reaches maximum height.
  - Write a qualitative description of how the energy of the system transforms through these three instants.
- For each instant:
  - Draw a physical diagram.
  - Construct an energy bar chart.
  - Write each energy symbolically.
- How high does the person go?
- What is the person's speed when the ejector seat leaves the spring?
  - Don't forget to make sense of your answers!

	U_{sp}	U_g	K	E_{total}
t_1	$\frac{1}{2}kx_c^2$	0	0	$\frac{1}{2}kx_c^2$
t_2	0	mgx_c	$\frac{1}{2}mv_2^2$	$mgx_c + \frac{1}{2}mv_2^2$
t_3	0	mgh_{max}	0	mgh_{max}

If there is no external work between any two times, then the E_{total} entries for those times are equal!

$$E_{\text{total},1} = E_{\text{total},3} \implies \frac{1}{2}kx_c^2 = mgh_{max} \implies h_{max} = \frac{kx_c^2}{2mg}$$

$$E_{\text{total},1} = E_{\text{total},2} \implies \frac{1}{2}kx_c^2 = mghx_c + \frac{1}{2}mv_2^2 \implies \left(\frac{k}{m}x_c - 2g\right)x_c = v_2^2$$

$$\implies v_2 = \sqrt{\left(\sqrt{\frac{k}{m}} - 2g\right)x_c}$$

Sensemaking

- $h_{max} = \frac{kx_c^2}{2mg}$
  - Unit Check: $[h_{max}] = \frac{[k][x_c]^2}{[mg]} = \frac{\frac{\text{N}}{\text{m}}\cdot\text{m}^2}{\text{N}} = m$
  - Covariation
    * k increases or x_c increases $\implies$ h_{max} increases.
      A stiffer spring, (higher k) stores more energy for the same compression, and a more compressed spring (higher x_c) stores more energy. Imparting more kinetic energy to the seat and person will allow them to go higher.
    * m increases or g increases $\implies$ h_{max} decreases.
      A more massive object will store more gravitational potential energy in less of an elevation change, as would an object in stronger gravity. As such, the object would transform its kinetic energy entirely at a smaller maximum height.
- $v_2 = \sqrt{\left(\frac{k}{m}x_c - 2g\right)x_c}$
  - Unit Check:

$$[v_2] = \sqrt{\left(\frac{[k]}{[m]}[x_c] - [g]\right)[x_c]} = \sqrt{\left(\frac{\text{N/m}}{\text{kg}}\cdot\text{m} - \frac{\text{m}}{\text{s}^2}\right)\cdot\text{m}}$$
$$= \sqrt{\frac{\text{N}\cdot\text{m}}{\text{kg}} - \frac{\text{m}^2}{\text{s}^2}} = \sqrt{\frac{\text{m}^2}{\text{s}^2} - \frac{\text{m}^2}{\text{s}^2}} = \frac{\text{m}}{\text{s}}$$

  - Special Cases
    * What if $\frac{k}{m}x_c = 2g$?
      The equation would say $v_2 = 0$, so the seat does not leave the spring. Rearranging, we can see that $x_c = \frac{2mg}{k}$, and when we plug this into our equation for h_{max}, we get $h_{max} = x_c$, so our equations are consistent with each other.
    * What if $\frac{k}{m}x_c < 2g$?
      Then v_2 is not real (we are talking about the square root of an imaginary number).
      Physically, if $x_c < \frac{2mg}{k}$, then the spring never fully decompresses, and v_2 is defined based on the assumption that it does, so it makes sense that we would get an invalid answer.

S7-1: Ejector Seat

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Impulse and Momentum

We rewrote Newton's 2nd law in a new way, which we will call the impulse-momentum theorem.

$$\text{Impulse} = \text{Change in Momentum}$$
$$\vec{J}_{net} = \Delta\vec{p}$$
$$\int_{t_i}^{t_f} \vec{F}^{net} dt = m\vec{v}_f - m\vec{v}_i$$

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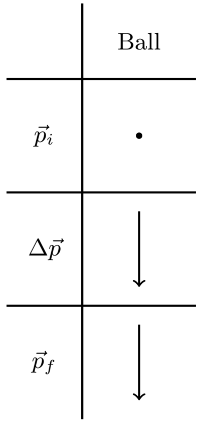
A Deeper Model for Interactions

- Quantities
  - Energy E
  - Work $W = \int_{r_i}^{r_f} \vec{F} \cdot d\vec{r}$
  - Kinetic Energy $K = \frac{1}{2}mv^2$
  - Potential Energy $U =$ depends on interaction
    You have to tell everyone where zero PE is!
    - Gravity $U_g = mgy$
    - Spring $U_{sp} = \frac{1}{2}kx^2$
  - Momentum $\vec{p} = m\vec{v}$
  - Impulse $\vec{J}_{net} = \int_{t_i}^{t_f} \vec{F}^{net} dt$
- Laws
  - Work-energy theorem $W_{\text{net,ext}} = \Delta E_{\text{total}}$
  - Impulse-momentum theorem $\vec{J}_{net} = \Delta\vec{p}$

S7-2: The Ball and the Ground I

- You drop a 0.05 kg tennis ball from rest and it takes 0.5 s to hit the ground.
- Use the impulse-momentum theorem to find the velocity of the ball just before impact with the ground.



This problem gives us the simple momentum vector diagram above. We know that $\vec{F}^{net} = \vec{F}^{g}$ during the fall, so the impulse is

$$\begin{aligned}\vec{J}_{net} = \int_{t_i}^{t_f} \vec{F}^{net} dt &= m\vec{v}_f - m\vec{v}_i \\ -mg\Delta t &= mv_f \\ -g\Delta t &= v_f.\end{aligned}$$

Plugging in numbers ($g \approx 10$ m/s^2 and $\Delta t = 0.5$ s) gives us $v_f = -5$m/s

S7-2: The Ball and the Ground II

- You drop a 0.05 kg tennis ball from rest and it takes 0.5 s to hit the ground.
- The tennis ball rebounds from the ground with the same speed as impact. The collision takes 0.01 s.
- Find the average net force on the tennis ball.

	Ball
$\vec{p}_i$	$\downarrow$
$\Delta\vec{p}$	$\uparrow$
$\vec{p}_f$	$\uparrow$

Let the velocities before and after the bounce be $\vec{v}_i = (-5\text{ m/s})\hat{y}$ and $\vec{v}_f = (5\text{ m/s})\hat{y}$. The change in momentum is

$$\Delta\vec{p} = m\vec{v}_f - m\vec{v}_i = m\left(\vec{v}_f - \vec{v}_i\right) = (0.05\text{ kg})(5\text{ m/s}-(-5\text{ m/s}))\hat{y} = 0.5\text{kg m/s}^2\hat{y}.$$

We can simplify the impulse integral by writing it in terms of the net force:

$$\int_{t_i}^{t_f} \vec{F}^{net} dt = \vec{F}^{net}_{avg}\Delta t.$$

As such,

$$\vec{F}^{net}_{avg}\Delta t = \Delta\vec{p} \implies \vec{F}^{net}_{avg} = \frac{m}{\Delta t}\left(\vec{v}_f - \vec{v}_i\right).$$

With numbers, this becomes

$$\vec{F}^{net}_{avg} = \frac{0.05\text{ kg}}{0.01\text{ s}}\left(5\text{ m/s} - (-5\text{ m/s})\right)\hat{y} = (50\text{ N})\hat{y}.$$

If we subtract off the force of gravity (-0.5 N $\hat{y}$), we can find out the average normal force (50.5 N).

Conservation of Momentum

If the net force on a system is zero, then the impulse is zero.

$$\vec{J} = \Delta\vec{p}$$
$$\vec{0} = \Delta\vec{p}$$

Under this condition, we say that the momentum of the system is *conserved*—it does not change!

S7-3: Two Rocks Collide

- A small rock (mass m) is moving to the right on a frictionless table with speed v.
- It hits a second rock (mass M) that is initially at rest on the table. The rocks do not stick together.
  - Is momentum conserved? For what system?
  - Is energy conserved? For what system?
- Our goal is to find the final speed of each rock, but don't try to solve it yet.
  - Instead, what special cases do you want to think about for this situation? What makes these special cases easier to think about than the general problem?

Momentum is conserved for the system of both rocks together. There is no net external impulse, as the only forces (those of the collision itself) are internal to the system.

Energy is conserved for this system as well, since a collision without sticking is perfectly elastic, and there is no external work being done (all external forces are perpendicular to the rocks' displacements).

Case 1: $M \gg m$

If the second rock is very massive, it should basically not budge after being struck by the first rock.

Case 2: $M = m$

If the two rocks are of equal mass, then the first will stop after the collision and the second will be moving at speed v. This is exactly like a Newton's cradle desk toy.

Solution:

First, I shall orient myself with a momentum vector diagram. I have assumed that m will still have rightward momentum after the collision, so if I solve for its final speed v_m and get a negative number, I know that the rock actually gets turned backward by the collision.

	m	M	m & M
$\vec{p}_i$	$\longrightarrow$	$\bullet$	$\longrightarrow$
$\Delta\vec{p}$	$\longleftarrow$	$\longrightarrow$	$\bullet$
$\vec{p}_f$	$\rightarrow$	$\longrightarrow$	$\longrightarrow$

The momentum conservation equation (choosing right as the positive direction) is

$$mv = mv_m + Mv_M.$$

Since the collision is perfectly elastic, we also have conservation of energy:

$$\frac{1}{2}mv^2 = \frac{1}{2}mv_m^2 + \frac{1}{2}Mv_M^2.$$

We can use these two equations to solve for both unknown final velocities.

First, let us simplify both expressions by dividing through by m (and multiplying the energy equation by 2):

$$v = v_m + \frac{M}{m}v_M,$$
$$v^2 = v_m^2 + \frac{M}{m}v_M^2.$$

We can write the first equation as $v_m = v - \frac{M}{m}v_M$ and substitute it into the second to get

$$v^2 = \left(v - \frac{M}{m}v_M\right)^2 + \frac{M}{m}v_M^2$$
$$v^2 = v^2 - 2\frac{M}{m}vv_M + \frac{M^2}{m^2}v_M^2 + \frac{M}{m}v_M^2$$
$$0 = -2\frac{M}{m}vv_M + \frac{M^2}{m^2}v_M^2 + \frac{M}{m}v_M^2.$$

Dividing through by $\frac{M}{m}v_M$ gives us

$$0 = -2v + \frac{M}{m}v_M + v_M$$
$$\left(1 + \frac{M}{m}\right)v_M = 2v$$
$$v_M = \frac{2m}{m+M}v.$$

Subsituting this into our simplified momentum expression gives us

$$v_m = v - \frac{M}{m}\frac{2m}{m+M}v$$
$$= \left(1 - \frac{2M}{m+M}\right)v$$
$$= \frac{m-M}{m+M}v.$$

Sensemaking:

If $M \gg m$, then v_M approaches zero, as predicted. We also see that $v_m \approx -v$, so the small rock reflects back at is original speed.

If $M = m$, then $v_M = v$ and $v_m = 0$, which is the Newton's cradle behavior that we predicted.

We didn't talk about $M \ll m$, but it is interesting. In this case, $v_m \approx v$ (the more massive object doesn't slow down after the collision) and $v_M \approx 2v$ (the stationary object gets launched forward at twice the massive object's speed). In the reference frame of the massive object, the smaller object is leftbound at speed v and reflects to the right at the same speed, much like the previous special case.

S7-3: Two Rocks Collide

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  - Instead, what special cases do you want to think about for this situation? What makes these special cases easier to think about than the general problem?

Energy and Collisions

- Collisions where kinetic energy is conserved are known as *elastic collisions*.
- Collisions where the energy of the system decreases are known as *inelastic collisions*.
  - When two things stick together, this is a *perfectly inelastic collision*.
- Collisions where the energy of the system increases are known as *superelastic collisions*.
  - Think explosions.

Main Ideas

- Momentum and impulse are useful quantities for solving dynamics problems.
- The impulse is always equal to the change in momentum for a system.
- When the impulse is zero (because the net force is zero), the momentum of the system is constant—it is *conserved*.