PH 221 Week 9

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L9-1: Colliding Rocks

A small rock (mass m) is moving to the right on a frictionless table with speed v.

It hits a second rock (mass M) that is initially at rest on the table. The rocks do not stick together.
(i) Is momentum conserved? For what system?
(ii) Is energy conserved? For what system?

Momentum is conserved for the system of both rocks together. There is no net external impulse, as the only forces (those of the collision itself) are internal to the system.

Energy is conserved for this system as well, since a collision without sticking is perfectly elastic, and there is no external work being done (all external forces are perpendicular to the rocks' displacements).

Our goal is to find the final speed of each rock, but don't try to solve it yet.
Instead, what special cases do you want to think about for this situation? What makes these special cases easier to think about than the general problem?

Case 1: $M \gg m$

If the second rock is very massive, it should basically not budge after being struck by the first rock.

Case 2: $M = m$

If the two rocks are of equal mass, then the first will stop after the collision and the second will be moving at speed v. This is exactly like a Newton's cradle desk toy.

Solution:

First, I shall orient myself with a momentum vector diagram. I have assumed that m will still have rightward momentum after the collision, so if I solve for its final speed v_m and get a negative number, I know that the rock actually gets turned backward by the collision.

	m	M	m & M
$\vec{p}_i$	$\longrightarrow$	$\bullet$	$\longrightarrow$
$\Delta\vec{p}$	$\longleftarrow$	$\longrightarrow$	$\bullet$
$\vec{p}_f$	$\rightarrow$	$\longrightarrow$	$\longrightarrow$

The momentum conservation equation (choosing right as the positive direction) is

$$mv = mv_m + Mv_M.$$

Since the collision is perfectly elastic, we also have conservation of energy:

$$\frac{1}{2}mv^2 = \frac{1}{2}mv_m^2 + \frac{1}{2}Mv_M^2.$$

We can use these two equations to solve for both unknown final velocities.

First, let us simplify both expressions by dividing through by m (and multiplying the energy equation by 2):

$$\begin{aligned} v &= v_m + \frac{M}{m}v_M, \\ v^2 &= v_m^2 + \frac{M}{m}v_M^2. \end{aligned}$$

We can write the first equation as $v_m = v - \frac{M}{m}v_M$ and substitute it into the second to get

$$\begin{aligned} v^2 &= \left(v - \frac{M}{m}v_M\right)^2 + \frac{M}{m}v_M^2 \\ v^2 &= v^2 - 2\frac{M}{m}vv_M + \frac{M^2}{m^2}v_M^2 + \frac{M}{m}v_M^2 \\ 0 &= -2\frac{M}{m}vv_M + \frac{M^2}{m^2}v_M^2 + \frac{M}{m}v_M^2. \end{aligned}$$

Dividing through by $\frac{M}{m}v_M$ gives us

$$\begin{aligned} 0 &= -2v + \frac{M}{m}v_M + v_M \\ \left(1 + \frac{M}{m}\right)v_M &= 2v \\ v_M &= \frac{2m}{m+M}v. \end{aligned}$$

Subsituting this into our simplified momentum expression gives us

$$\begin{aligned} v_m &= v - \frac{M}{m}\frac{2m}{m+M}v \\ &= \left(1 - \frac{2M}{m+M}\right)v \\ &= \frac{m-M}{m+M}v. \end{aligned}$$

Sensemaking:

If $M \gg m$, then v_M approaches zero, as predicted. We also see that $v_m \approx -v$, so the small rock reflects back at is original speed.

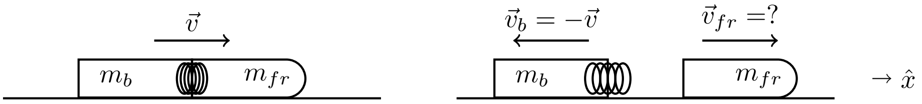
If $M = m$, then $v_M = v$ and $v_m = 0$, which is the Newton's cradle behavior that we predicted.

We didn't talk about $M \ll m$, but it is interesting. In this case, $v_m \approx v$ (the more massive object doesn't slow down after the collision) and $v_M \approx 2v$ (the stationary object gets launched forward at twice the massive object's speed). In the reference frame of the massive object, the smaller object is leftbound at speed v and reflects to the right at the same speed, much like the previous special case.

L9-2: Springloaded Sled

You are designing a sled with a compressed spring inside, which can be released to separate the sled into two pieces of equal mass ($m/2$). You are racing the sled across level snow at speed v when you trigger the separation.

Right after the two halves push apart, the back end of the sled is moving backward with speed v. What is the velocity of the other piece? How much kinetic energy did the system gain?



When setting up a motion vector diagram for this problem, I know the initial momentum of the combined sled system, and I know both parts of the sled have half of this momentum. I also know that the final momentum of the back half is reversed, and the total momentum of the system is unchanged, so I can infer the rest of the table from there.

	Back	Front	Both
$\vec{p}_i$	$\rightarrow$	$\rightarrow$	$\longrightarrow$
$\Delta\vec{p}$	$\longleftarrow$	$\longrightarrow$	$\bullet$
$\vec{p}_f$	$\leftarrow$	$\longrightarrow$	$\longrightarrow$

The initial momentum of the system is $mv\hat{x}$, and the final momentum is $(\frac{m}{2}v_{fr} - \frac{m}{2}v)\hat{x}$. Since the momentum is conserved (no friction, and the normal force and gravitational force are in balance, so no other net impulse), we know that

$$\begin{aligned} mv\hat{x} &= \left(\frac{m}{2}v_{fr} - \frac{m}{2}v\right)\hat{x} \\ mv &= \frac{m}{2}v_{fr} - \frac{m}{2}v \\ 2v &= v_{fr} - v \\ v_{fr} &= 3v. \end{aligned}$$

The front half of the sled gets launched forward at triple its original speed!

As for kinetic energy, the system started with $K_i = \frac{1}{2}mv^2$, and now it has

$$K_f = \frac{1}{2}\frac{m}{2}v^2 + \frac{1}{2}\frac{m}{2}(3v)^2 = \frac{5}{2}mv^2,$$

therefore the change in kinetic energy is

$$\Delta K = K_f - K_i = 2mv^2.$$

This came from the spring. If the spring constant is k and the spring was compressed by a length Δx, then we have

$$\begin{aligned} \frac{1}{2}k\Delta x^2 &= 2mv^2 \\ k\Delta x^2 &= 4mv^2. \end{aligned}$$

This has some interesting design implications. For example, say each half of our sled is 100 kg (say that accounts for the machinery and the load of a single passenger on each half) and its initial speed was a lazy 1 m/s. That would mean the spring has to store 400 J of energy. If $\Delta x = 0.5$ m (which may be too much of a compression for a reasonable use of Hooke's law), then $k = 3200$ N/m (or 32 N/cm), which is a pretty stiff spring. If we cannot get a spring this stiff, then we need more compression, but if we cannot obtain a spring that compresses far enough without permanently deforming, then we need it stiffer. The key will be finding the perfect middle ground (and those of you who are doing experiments with elasticity in your capstone lab might have a better idea than using a single spring).