Lecture 4: Using Integrals in Physics

Office Hours

- Drop-In: 4:00 p.m. – 5:00 p.m. Wednesdays & Thursdays
- Appointments: 11:00 a.m. – 11:45 a.m. (W)/12:00 p.m. (Th)

Warm-Up Activity

How is acceleration symbolically related to velocity? WRITE BIG!

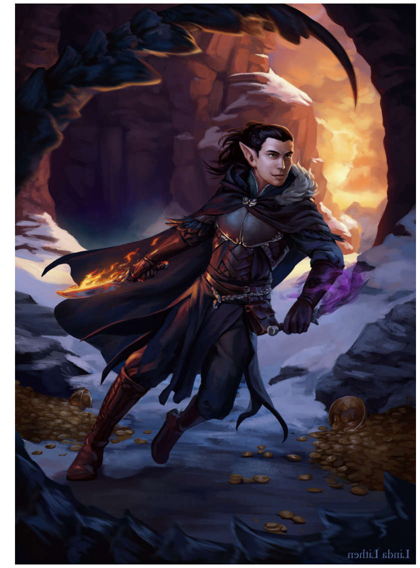
(A) Velocity is acceleration times t.

(B) Acceleration is velocity times t.

(C) Acceleration is the derivative of velocity.

(D) Velocity is the derivative of acceleration.

L4-1: Vax'ildan's Acceleration



- Vax'ildan Vessar is initially located at position x_i, running to the right with initial speed v_i.
- At $t = 0$, Vax clicks his *boots of haste*, which provide an acceleration:
$$\vec{a}(t) = a_0 \left(1 - \frac{t}{T}\right) \hat{x}$$
- Our goals are:
  - Find how much time it takes for Vax to return to his initial velocity.
  - Find Vax's position at this time.

Solving an ARCS Problem

1. Analyze and Represent

1a. **Understand the problem** – identify quantities by symbol and number.
1b. **Identify Assumptions** – identify important simplifications and assumptions.
1c. **Represent physically** – draw and label one or more appropriate diagrams and/or graphs that might help you solve the problem.

2. Calculate

2a. **Represent principles** – identify relevant concepts, laws, or definitions.
2b. **Find unknown(s) symbolically** – without numbers, find any unknown(s) in terms of symbols representing known quantities.
2c. **Plug in numbers** – plug numbers (with units) into your symbolic answer!

3. Sensemake

3a. **Units** – check that the units of your answer agree with the units you expect
3b. **Numbers** – compare your answer to other numbers in the problem or in the everyday world; if relevant, check the sign or direction.
3c. **Symbols** – use a strategy like covariation or special cases to check that your answer makes physical sense.

Calculate

Represent Principles

Definition of acceleration:

$$\vec{a}(t) = \frac{d\vec{v}}{dt}$$

Find Unknowns Symbolically

From the definition of acceleration, we know that

$$d\vec{v} = \vec{a}(t)dt.$$

We want to add up all of the tiny changes in velocity $d\vec{v}$ in order to find the total change $\Delta\vec{v} = \vec{v}(t) - \vec{v}_i$. To do so, we integrate the left side from v_i to $v(t)$ and the right side from the initial time $t_i = 0$ to the final time $t_f = t$:

$$\begin{aligned}
\int_{v_i}^{v(t)} d\vec{v} &= \int_0^t \vec{a}(t)dt \\
\vec{v}(t) - \vec{v}_i &= \int_0^t a_0\left(1 - \frac{t}{T}\right)\hat{x} \\
&= a_0\left(t - \frac{t^2}{2T}\right)\hat{x} \\
\vec{v}(t) &= \left[v_i + a_0\left(t - \frac{t^2}{2T}\right)\right]\hat{x}
\end{aligned}$$

To determine t_f, we take our final condition, $\vec{v}(t_f) = v_i\hat{x}$ and rearrange the expression to get

$$\begin{aligned}
v_i &= v_i + a_0\left(t_f - \frac{t_f^2}{2T}\right) \\
0 &= a_0 t_f\left(1 - \frac{t_f}{2T}\right) \\
0 &= 1 - \frac{t_f}{2T} \\
t_f &= 2T.
\end{aligned}$$

Plug in Numbers

Plugging in our numbers from before ($v_i = 2$ m/s, $T = 6$ s, $a_0 = 0.5$ m/s^2), we obtain

$$\begin{aligned}
\vec{v}(t) &= \left[2\text{ m/s} + (0.5\text{ m/s}^2)\left(t - \frac{t^2}{12\text{ s}}\right)\right]\hat{x}, \\
t_f &= 12\text{ s}.
\end{aligned}$$

L4-1: Vax'ildan's Acceleration – Calculate

- At $t = 0$, Vax clicks his *boots of haste*, which provide an acceleration:

$$\vec{a}(t) = a_0\left(1 - \frac{t}{T}\right)\hat{x}$$

- **Represent Principles**
  - Identify relevant concepts, laws, or definitions.
- **Find Unknown(s) Symbolically**
  - First find a symbolic expression for Vax's velocity as a function of time.
  - Use your expression to find when Vax's velocity is equal to v_i.
- **Plug in Numbers**
  - Estimate any quantities to find numerical answers.

Calculate

Represent Principles

Definition of velocity:

$$\vec{v}(t) = \frac{d\vec{x}}{dt}$$

Find Unknowns Symbolically

Just like before, we rearrange the definition of velocity and integrate from the initial position at the initial time to the final position at the final time:

$$\begin{aligned} d\vec{x} &= \vec{v}(t)dt \\ \int_{x_i}^{x(t)} d\vec{x} &= \int_0^t \vec{v}(t)dt \\ \vec{x}(t) - \vec{x}_i &= \int_0^t \left[v_i + a_0\left(t - \frac{t^2}{2T}\right)\right]\hat{x}dt \\ &= \left[v_i t + a_0\left(\frac{t^2}{2} - \frac{t^3}{6T}\right)\right]\hat{x} \\ \vec{x}(t) &= \left[x_i + v_i t + a_0\left(\frac{t^2}{2} - \frac{t^3}{6T}\right)\right]\hat{x}. \end{aligned}$$

Plugging in the final time gives us Vax's final position:

$$\begin{aligned} \vec{x}_f &= \left[x_i + 2v_i T + a_0\left(2T^2 - \frac{8T^3}{6T}\right)\right]\hat{x} \\ &= \left[x_i + 2v_i T + \frac{2}{3}a_0 T^2\right]\hat{x} \end{aligned}$$

Plug in Numbers

Plugging in our earlier numbers (now including $x_i = 0$ m) gives us

$$\begin{aligned} \vec{x}(t) &= \left[(2\text{ m/s})t + (0.5\text{ m/s}^2)\left(\frac{t^2}{2} - \frac{t^3}{36\text{ s}}\right)\right]\hat{x}, \\ \vec{x}_f &= \left[2(2\text{ m/s})(6\text{ s}) + \frac{2}{3}(0.5\text{ m/s}^2)(6\text{ s})^2\right]\hat{x} \\ &= [24\text{ m} + 12\text{ m}] \\ &= (36\text{ m})\hat{x}. \end{aligned}$$

L4-1: Vax'ildan's Acceleration – Calculate

- At $t = 0$, Vax clicks his *boots of haste*, which provide an acceleration:

$$\vec{a}(t) = a_0\left(1 - \frac{t}{T}\right)\hat{x}$$

- His velocity as a function of time is

$$\vec{v}(t) = \left[v_i + a_0\left(t - \frac{t^2}{2T}\right)\right]\hat{x},$$

  and he returns to his initial velocity at $t_f = 2T$.

- Now, find a symbolic expression for Vax's position as a function of time and use it to find Vax's position at t_f.

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Sensemake

Units

$$\vec{v}(t) = \left[v_i + a_0\left(t - \frac{t^2}{2T}\right)\right]\hat{x}$$

$$\begin{aligned}\frac{\mathrm{m}}{\mathrm{s}} &= \frac{\mathrm{m}}{\mathrm{s}} + \left(\frac{\mathrm{m}}{\mathrm{s}^2}\right)\left(\mathrm{s} - \frac{\mathrm{s}^2}{\mathrm{s}}\right)\\ &= \frac{\mathrm{m}}{\mathrm{s}} + \left(\frac{\mathrm{m}}{\mathrm{s}^2}\right)(\mathrm{s} - \mathrm{s})\\ &= \frac{\mathrm{m}}{\mathrm{s}} + \frac{\mathrm{m}}{\mathrm{s}}\\ &= \frac{\mathrm{m}}{\mathrm{s}}.\end{aligned}$$

$$\vec{x}(t) = \left[x_i + v_i t + a_0\left(\frac{t^2}{2} - \frac{t^3}{6T}\right)\right]\hat{x}$$

$$\begin{aligned}\mathrm{m} &= \mathrm{m} + \left(\frac{\mathrm{m}}{\mathrm{s}}\right)(\mathrm{s}) + \left(\frac{\mathrm{m}}{\mathrm{s}^2}\right)\left(\mathrm{s}^2 - \frac{\mathrm{s}^3}{\mathrm{s}}\right)\\ &= \mathrm{m} + \mathrm{m} + (\mathrm{m}/\mathrm{s}^2)(\mathrm{s}^2)\\ &= \mathrm{m}\end{aligned}$$

We are indeed obtaining m/s for our velocity equation and meters for our position equation.

Numbers

Our velocity and position equations (as well as our final position expression) are all vector quantities, as they should be. The final time is a scalar.

We found that Vax returned to his original speed after 12 seconds, and traveled 36 meters in that time. This is half again as far as he would have traveled had he stayed at $v_i = 2$ m/s (he would have gone 24 m). According to 5th edition rules, the *haste* spell should actually double the distance the target can move, so this is actually rather low. It is possible that we underestimated a_0 for the *boots of haste*.

Symbols

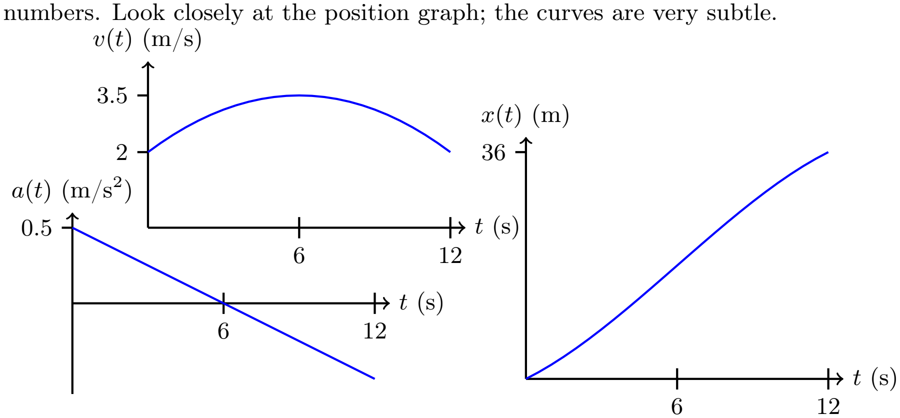
I will focus on the final time and position equations for this part.

First, changing a_0 has no effect on the final time. Increasing a_0 increases both the initial positive acceleration and the final negative acceleration, so even though Vax will attain a higher maximum speed in the course of his motion, he will reach that maximum and slow back to his original speed just as quickly due to the increased magnitude of acceleration.

Increasing a_0 or increasing T will increase the final distance Vax travels. Logically, accelerating more means attaining a higher maximum speed, as does accelerating for a longer period of time, so Vax will travel farther with that higher speed. Increasing T also just means being on the move for longer, so Vax will certainly travel a greater distance if he is moving for a longer period of time.

Graphs

These graphs are made with the numerical results of the proposed reasonable numbers. Look closely at the position graph; the curves are very subtle.



L4-1: Vax'ildan's Acceleration – Sensemake

- At $t = 0$, Vax clicks his *boots of haste*, which provide an acceleration:

$$\vec{a}(t) = a_0\left(1 - \frac{t}{T}\right)\hat{x}$$

- How can we make sense of these equations?

$$\vec{v}(t) = \left[v_i + a_0\left(t - \frac{t^2}{2T}\right)\right]\hat{x} \quad \vec{x}(t) = \left[x_i + v_i t + a_0\left(\frac{t^2}{2} - \frac{t^3}{6T}\right)\right]\hat{x}$$

$$t_f = 2T \qquad \vec{x}_f = \left[x_i + 2v_iT + \frac{2}{3}a_0T^2\right]\hat{x}$$

- **Units**
- **Numbers**
  - Which things are vectors?
  - Try plugging in some reasonable numbers.
- **Symbols**
  - What happens if you change a_0 or T?
- What do the graphs of $\vec{v}(t)$ and $\vec{a}(t)$ look like?

Constant Acceleration
The process for deriving the equations of motion is the same:

$$\frac{d\vec{v}}{dt} = a\hat{x} \qquad \frac{d\vec{x}}{dt} = \vec{v}(t)$$

$$d\vec{v} = adt\hat{x} \qquad d\vec{x} = \vec{v}(t)dt$$

$$\int_{v_i}^{v(t)} d\vec{v} = \int_0^t adt\hat{x} \qquad \int_{x_i}^{x(t)} d\vec{x} = \int_0^t [v_i + at]\, dt\hat{x}$$

$$\vec{v}(t) - \vec{v}_i = at\hat{x} \qquad \vec{x}(t) - \vec{x}_i = \left[v_i t + \frac{1}{2}at^2\right]\hat{x}$$

$$\vec{v}(t) = [v_i + at]\,\hat{x} \qquad \vec{x}(t) = \left[x_i + v_i t + \frac{1}{2}at^2\right]\hat{x}$$

These equations are often called the **kinematics equations**. There is a third kinematics equation often bundled with these that you can derive in Get-Ready #4 (in §2.15 of the online textbook).

L4-2: Constant Acceleration

- What if Vax's acceleration had been constant?

$$\vec{a}(t) = a\hat{x}$$

Main Ideas

- If we know the acceleration of an object as a function of time, we can determine the velocity as a function of time.
- If we know the velocity as a function of time, we can determine the position as a function of time.