

# Cliffside Midair Collision

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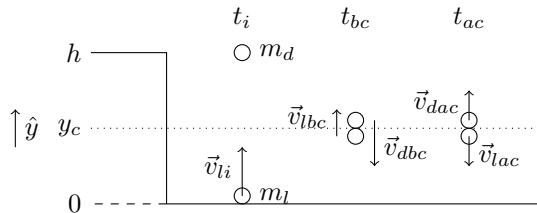
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## XX-1: Cliffside Midair Collision

You are standing at the bottom of a cliff, your friend is standing at the top, and each of you is in possession of a ball. Your friend drops the ball off of the edge, and at the exact same time, you throw your ball up toward it such that they collide in midair.

This problem is achievable for students, as they have been exposed to all of the necessary concepts. However, the lack of scaffolding (which is a rather healthy thing for them to experience) and length of the problem make it quite intimidating. For that reason, it may take a more hands-on approach from the instructor. First, go through the diagram with the entire class, asking them to identify the critical pieces of information that should be included based on what is being asked about in the subsequent parts. Part (a) requests a height of collision and a collision speed, so the diagram should at least depict the cliff, establish a coordinate system, and indicate the height of collision, as well as perhaps indicate initial and final speeds. Part (b) requests information about energy before and after the collision, and part (c) requests similar information about momentum, so our diagram should depict the situation (with relevant quantities indicated) at multiple relevant instants (at least the initial time, the time immediately before the collision, and the time immediately after). I chose to not depict the final time, as the exact location of the lower ball is not immediately apparent (though we will pretty quickly find out that it will return to the ground at that time). Be sure to define all of the symbols you devise on the board.

Let  $h$  be the height of the cliff,  $y_c$  be the height of the collision,  $m_d$  be the mass of the ball your friend drops, and  $m_l$  be the mass of the ball you launch upward. At the time of the drop and launch,  $t_i$ , your friend's ball is at rest at height  $h$ , and yours has initial speed  $v_{li}$  at height zero. At the instant right before the collision,  $t_{bc}$ , both balls are at  $y_c$ , with your friend's ball moving at speed  $v_{dbc}$  and your ball moving at speed  $v_{lbc}$ . The instant after the collision,  $t_{ac}$ , your friend's ball has speed  $v_{dac}$ , and your ball has speed  $v_{lac}$ . The time at which your friend's ball reaches the top of the cliff once again will be denoted  $t_f$ .



(a) At what height do the two balls collide? How fast is your ball going when the two balls collide?

Based on kinematics, we know that for the dropped ball,

$$y_c = h - \frac{1}{2}gt_{bc}^2,$$

and for the launched ball,

$$y_c = v_{li}t_{bc} - \frac{1}{2}gt_{bc}^2.$$

Setting these equal, we find that  $h = v_{li}t_{bc}$ , so  $t_{bc} = \frac{h}{v_{li}}$ . It follows that

$$y_c = h - \frac{1}{2}g\frac{h^2}{v_{li}^2}. \quad (1)$$

As for the speed of your ball, we can conclude that

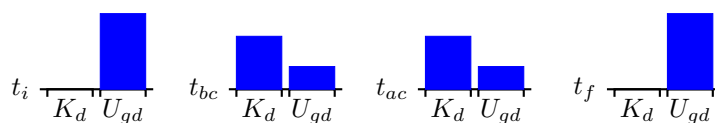
$$v_{lbc} = v_{li} - gt_{bc} = v_{li} - \frac{gh}{v_{li}}. \quad (2)$$

(b) If you want your friend's ball to bounce back to the top of the cliff after the collision, how much kinetic energy must it have after the collision? What is its velocity after the collision?

Consider leading the students through the energy bar charts, asking them to help you set up the columns. They'll get the punchline of the activity, but they can still do the symbolic parts, now with the guidance of knowing the outcome.

We can start by guiding ourselves with a graphical representation of how the energy in the dropped ball is distributed. We know that the height (and therefore the gravitational potential energy) must be the same at  $t_i$  and  $t_f$ . Since the height right before and right after the collision is the same, the potential energies at  $t_{bc}$  and  $t_{ac}$  are equal to each other as well. These energies are smaller than the beginning and ending potential energies, as the collision height is lower than the top of the cliff.

The system we've defined (Earth and the ball) has no net external work from  $t_i$  to  $t_{bc}$ , and no external work from  $t_{ac}$  to  $t_f$ , so  $E_{tot}(t_i) = E_{tot}(t_{bc})$  and  $E_{tot}(t_{ac}) = E_{tot}(t_f)$ . The potential energy decrease from  $t_i$  to  $t_{bc}$  is therefore equal to the kinetic energy at  $t_{bc}$ , and the potential energy increase from  $t_{ac}$  to  $t_f$  is equal to the kinetic energy at  $t_{ac}$ .



However, since that potential energy difference is the same across each of those time intervals, The kinetic energies at  $t_{bc}$  and  $t_{ac}$  must be equal. This is enough to conclude that the speed does not change after the collision, though the velocity does (it reverses direction, since the balls bounce off of each other).

To see this in symbolic form, let us write expressions for the energies of the dropped ball at all four times.

	$K_d$	$U_{gd}$	$E_{tot}$		$K_d$	$U_{gd}$	$E_{tot}$
$t_i$	0	$m_dgh$	$m_dgh$	$t_{ac}$	$\frac{1}{2}m_d v_{dac}^2$	$m_dgy_c$	$\frac{1}{2}m_d v_{dac}^2 + m_dgy_c$
$t_{bc}$	$\frac{1}{2}m_d v_{dbc}^2$	$m_dgy_c$	$\frac{1}{2}m_d v_{dbc}^2 + m_dgy_c$	$t_f$	0	$m_dgh$	$m_dgh$

We know that the kinetic energy of the system of *both* balls is conserved by the collision, but we don't know that the kinetic energy of the dropped ball is conserved from  $t_{bc}$  to  $t_{ac}$ , as some energy

could be transferred between the balls. However, we know that its energy is conserved before the collision, therefore

$$m_dgh = E_{tot}(t_i) = E_{tot}(t_{bc}) = \frac{1}{2}m_d v_{dbc}^2 + m_d g y_c. \quad (3)$$

We also know that its energy is conserved after the collision, therefore

$$m_dgh = E_{tot}(t_f) = E_{tot}(t_{ac}) = \frac{1}{2}m_d v_{dac}^2 + m_d g y_c.$$

Since the ball returns to rest at the same height as it started, we actually can conclude that its energy cannot change during the course of the collision:

$$\frac{1}{2}m_d v_{dbc}^2 + m_d g y_c = \frac{1}{2}m_d v_{dac}^2 + m_d g y_c.$$

Since the kinetic energy before and after the collision is the same, the ball's speed does not change after the collision:  $v_{dbc} = v_{dac}$ . However, if it is going to be returning to its original height, it must be moving up, so

$$\vec{v}_{dac} = -\vec{v}_{dbc} = v_{dbc}\hat{y} = \sqrt{2g(h - y_c)}\hat{y},$$

where  $v_{dbc} = \sqrt{2g(h - y_c)}$  comes from rearranging Eqn. 3. From Eqn. 1, we can conclude that

$$\vec{v}_{dac} = \sqrt{2g\left(\frac{1}{2}g\frac{h^2}{v_{li}^2}\right)}\hat{y} = \frac{gh}{v_{li}}\hat{y}.$$

It is also worth noting that  $\vec{v}_{dac} = -\vec{v}_{dbc}$  implies that  $\vec{p}_{dac} = -\vec{p}_{dbc}$ .

(c) How fast do you have to throw your ball in order for your friend's ball to bounce back to the top of the cliff? Assume the collision is perfectly elastic.

Students need to recognize that, since part (a) gave them  $v_{lbc}$  in terms of  $v_{li}$ , we need to find  $v_{lbc}$  to get our answer. Leading the students through the momentum vector diagrams, asking them to spot your (intentional) mistakes, would be useful, and it would still leave the math for them.

Since kinetic energy is conserved and the dropped ball does not gain or lose energy in the course of the collision, we know that the launched ball cannot gain or lose energy either, so  $v_{lac} = v_{lbc}$ .

As of this moment, we know that the after-collision momentum of the dropped ball is equal in magnitude and opposite in direction to its before-collision momentum, and we know that the before-collision and after-collision momenta of the launched ball must be equal in magnitude. We can conclude that the before-collision and after-collision momenta of the launched ball must also be in opposite directions. Otherwise, the momentum of the launched ball does not change, and the momentum change for the system of both balls is equal to the momentum change of the dropped ball, which we know is nonzero, which would violate conservation of momentum.

We still don't yet know how the magnitudes of the launched ball's momenta compare to those of the dropped ball, so let us make momentum vector diagrams for the case where the before-collision momentum of the launched ball is lesser in magnitude than the before-collision momentum of the dropped ball (the logic is the same for the case in which it is greater, so we will skip it):

	$\vec{p}_{bc}$	$\Delta\vec{p}$	$\vec{p}_{ac}$
dropped	$\downarrow$	$\uparrow$	$\uparrow$
launched	$\uparrow$	$\downarrow$	$\downarrow$
system	$\downarrow$	$\uparrow$	$\uparrow$

This situation conserves energy for each ball, but not the momentum of the whole system (as  $\Delta\vec{p}_{sys}$  is nonzero). We need the change in momentum for the launched ball to be equal and opposite to the change in momentum of the dropped ball, which requires making the momentum of the launched ball equal in magnitude to the momentum of the dropped ball, as shown in this diagram:

	$\vec{p}_{bc}$	$\Delta\vec{p}$	$\vec{p}_{ac}$
dropped	$\downarrow$	$\uparrow$	$\uparrow$
launched	$\uparrow$	$\downarrow$	$\downarrow$
system	$\vec{0}$	$\vec{0}$	$\vec{0}$

In symbolic form, we can equate the initial and final momenta of the system, as momentum is conserved here:

$$\begin{aligned}\vec{p}_{sys,bc} &= \vec{p}_{sys,ac} \\ \vec{p}_{dbc} + \vec{p}_{lbc} &= \vec{p}_{dac} + \vec{p}_{lac} \\ m_d\vec{v}_{dbc} + m_l\vec{v}_{lbc} &= m_d\vec{v}_{dac} + m_l\vec{v}_{lac}.\end{aligned}$$

Since both of the final speeds just reverse direction, this expression becomes

$$\begin{aligned}m_d\vec{v}_{dbc} + m_l\vec{v}_{lbc} &= -m_d\vec{v}_{dbc} - m_l\vec{v}_{lbc} \\ 2m_l\vec{v}_{lbc} &= -2m_d\vec{v}_{dbc} \\ \vec{v}_{lbc} &= -\frac{m_d}{m_l}\vec{v}_{dbc}.\end{aligned}$$

The velocity of the dropped ball before collision was downward, so this can be rewritten as

$$\vec{v}_{lbc} = \frac{m_d}{m_l}v_{dbc}\hat{y},$$

and we expect the lower ball to be traveling upward, so this has the correct direction.

We know  $v_{dbc} = v_{dac} = \frac{gh}{v_{li}}$ , which we found in part (b), so

$$v_{lbc} = \frac{m_d}{m_l} \frac{gh}{v_{li}}.$$

From Eqn. 2, we can see that

$$\begin{aligned}
 v_{li} - \frac{gh}{v_{li}} &= \frac{m_d}{m_l} \frac{gh}{v_{li}} \\
 \implies v_{li} &= \left( \frac{m_d}{m_l} + 1 \right) \frac{gh}{v_{li}} \\
 \implies v_{li} &= \sqrt{\left( \frac{m_d}{m_l} + 1 \right) gh}.
 \end{aligned}$$

(d) Perform sensemaking (use multiple techniques!) on your answer for initial speed of the launched ball.

We can see that  $v_{li}$  increases as  $m_d$ ,  $g$ , or  $h$  increases, which makes sense, as a more massive dropped ball, stronger gravity, or a longer fall all give the dropped ball more momentum right before the collision, which means we have to increase the momentum of the launched ball to successfully turn the dropped ball around. Conversely,  $v_{li}$  decreases as  $m_l$  increases, as we can get the same momentum for less speed by using a more massive ball.

In the special case  $m_l \gg m_d$ , that the launched ball is far more massive than the dropped ball, we have  $v_{li} \approx \sqrt{gh}$ . Note that Eqn. 2 tells us that  $v_{lbc} \approx \sqrt{gh} - \frac{gh}{\sqrt{gh}} = 0$ , so starting any slower would mean that the launched ball would be moving down at the time of the collision, and starting any faster would mean that it would be moving up. We know that, in the limit of striking a massive, stationary object in a perfectly elastic collision, the incoming object will reflect back at the same speed. However, if the incoming object were to strike a massive object that was moving, the reflection speed would either be more (for striking an approaching massive object) or less (for striking a receding massive object).

One way to see why involves reasoning in different reference frames; in the frame of an approaching massive object, the incoming object will strike at a speed greater than in a stationary observer's reference frame, and reflect at this greater speed in the massive object's reference frame. In the observer's reference frame, the reflection speed is even larger (as one must add the speed of the massive object's reference frame). Similar reasoning demonstrates the opposite case for the receding massive object.

Anyway, for a massive launched ball, it must be launched at the perfect speed to strike the dropped ball at the instant the massive ball comes to rest, otherwise the launched ball will bounce at the wrong speed.