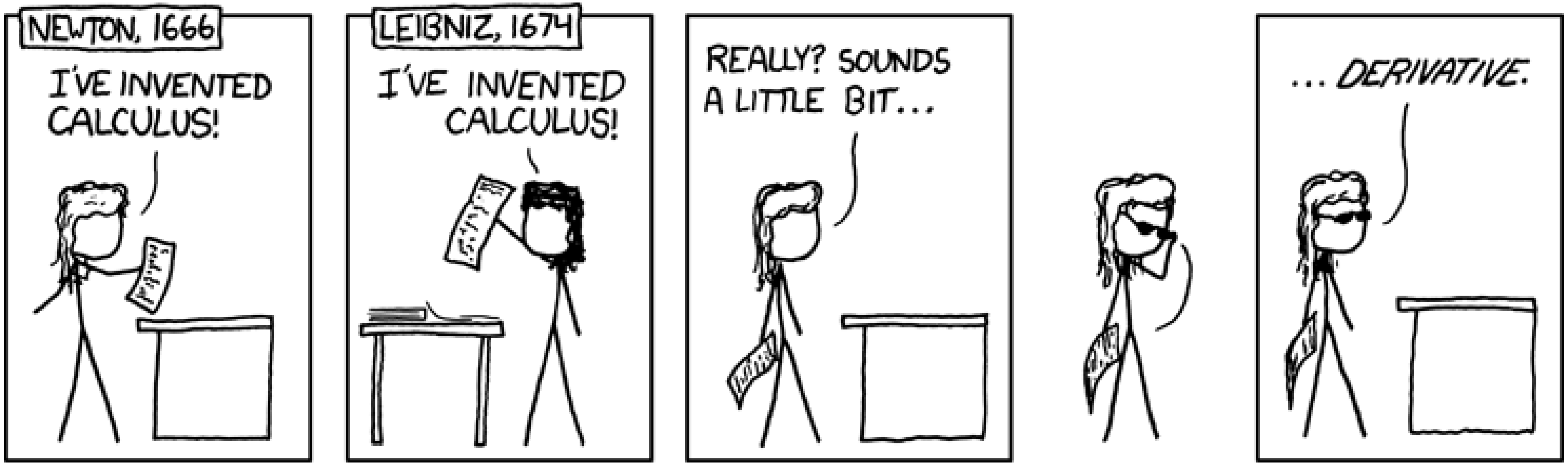
Studio Week 3
Kinematics

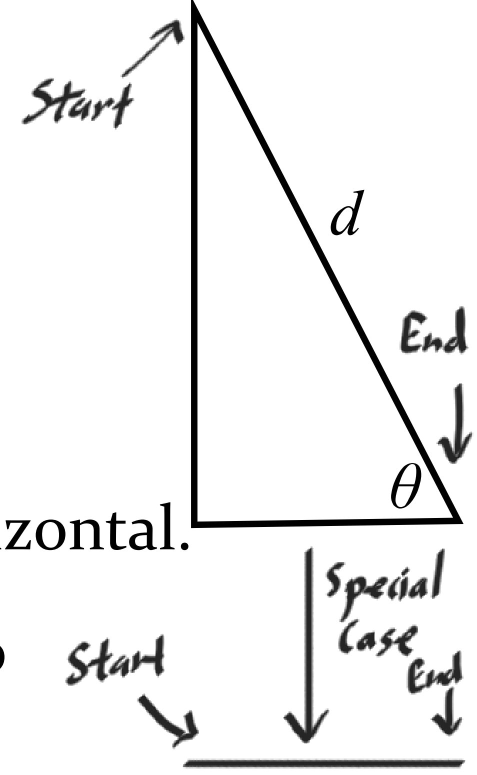


Picture credit: xkcd.com

Principles for Success

- **Treat everyone with respect.**
- **Learn by doing and questioning.**
- **Make sense of everything.**

Warm-up



- You slide down the ramp shown at right, starting from rest.
- You decide to consider the special case where the ramp is made horizontal.
- Which of the following statements about the time it takes for you to reach the end of the ramp in this case is correct?

  A. It would take no time for you to reach the end of the ramp.

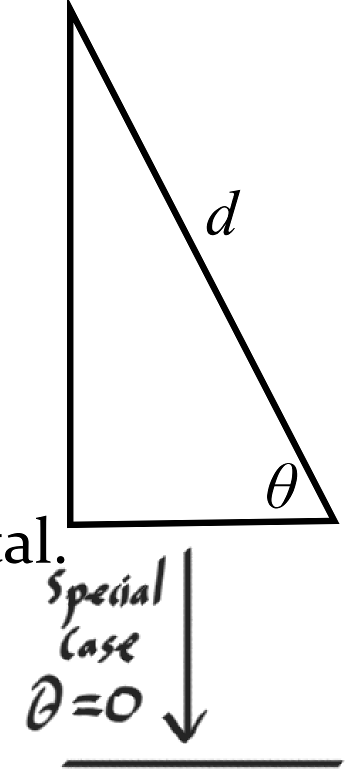
  B. It takes the same amount of time for you to reach the end of the ramp.

  C. It would take an infinite amount of time for you to reach the end of the ramp.

  D. We do not have enough information to determine how much time it will take for you to reach the end of the ramp.

You will not slide naturally across a flat surface, so you will never reach the end.

Warm-up



- You slide down the ramp shown at right, starting from rest.
- You decide to consider the special case where the ramp is made horizontal.
- Which of the following statements about the time it takes for you to reach the end of the ramp in this case is correct?
  - Does the equation below agree with our prediction?

$$t = \sqrt{\frac{2d}{g \sin \theta}}$$

As $\theta \to 0$, $\sin\theta \to 0$, which means the denominator of the fraction will go to zero. This will cause the fraction (and thus the equation for t) to explode to infinity, thus agreeing with our prediction.

Activity 3-1: Archery AR

- You are a long-distance archer who releases an arrow that hits a target 200 m away. Your arrow makes an initial angle with the horizontal of 17°. Following the steps for solving an A*R*C*S problem, find the initial speed of the arrow.
  - **A**nalyze and **R**epresent
    - Choose appropriate symbols for the quantities you **know** and **don't know**.
    - What assumptions are you making and why are you making them?
    - Sketch a diagram showing the known and unknown quantities.

3-1 Archery ARCS

Analyze and Represent

1a) Understand the Problem

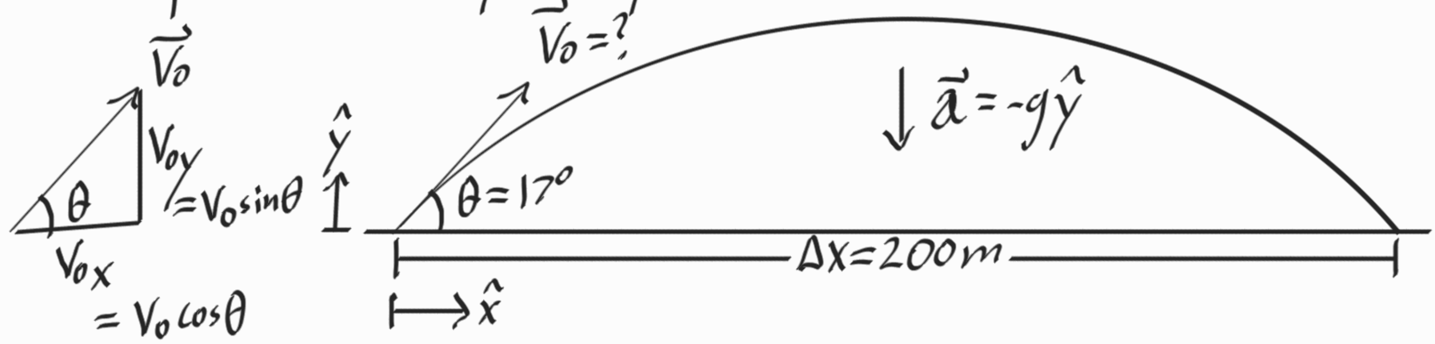
Known: $\Delta X = 200m$, $\theta = 17°$, $\underbrace{\vec{a} = -g\hat{y},\ g \approx 9.8 m/s^2,\ \Delta y = 0m}_{\text{by assumption}}$

Unknown: V_0 (initial speed), t_f (time of flight), V_f (final speed)

1b) Identify Assumptions

- $\Delta y = 0m$
  - Both the arrow (held aloft by the archer) and a typical target are not right at ground level, and archery ranges are typically in flat areas, so for simplicity, it won't be too egregious to assume that the arrow starts and ends at the same height.
- The arrow is not affected by moving through air.
  - We don't really know how to handle air resistance or the effects of wind, so we need to ignore them to solve the problem. Arrows tend to be fast and aerodynamic, so this simplification will still be quite accurate.
- Near Earth's surface ($g \approx 9.8 m/s^2$)
  - So far, all known archery occurs on Earth, and the height an arrow flies is not significant enough to worry about variations in gravity.
- $\vec{a} = -g\hat{y}$
  - Since the object is in free-fall with no wind or air effects, and it isn't traveling far enough for the curvature of the Earth to matter, acceleration is straight down and purely gravitational.

1c) Represent Physically



Activity 3-1: Archery C

- You are a long-distance archer who releases an arrow that hits a target 200 m away. Your arrow makes an initial angle with the horizontal of 17°. Following the steps for solving an A*R*C*S problem, find the initial speed of the arrow.
  - **C**alculate: Find a **symbolic expression** for the initial speed of the arrow.
    - *Hint 1:* **Use the symbols you defined before** to help you choose which kinematics equations to use for the *x*- and *y*-directions.
    - *Hint 2:* Think about which quantity is the same for the *x*- and *y*-directions.
    - Wait until the end to plug in numbers!

3-1 Archery ARCS

Calculate

2a) Represent Principles

2-D Kinematics: Known Unknown Wanted

$$\Delta x = V_{ox} t_f + \tfrac{1}{2} a_x t_f^2 \qquad \Delta y = V_{oy} t_f + \tfrac{1}{2} a_y t_f^2$$

$$\left\{\begin{array}{ll} V_{fx} = V_{ox} + a_x t_f & V_{fy} = V_{oy} + a_y t_f \\ V_{fx}^2 = V_{ox}^2 + 2 a_x \Delta x & V_{fy}^2 = V_{oy}^2 + 2 a_y \Delta y \end{array}\right\}$$

Let's avoid these, since we don't really want V_f.

Trigonometry:

$$V_{ox} = V_0 \cos\theta \qquad V_{oy} = V_0 \sin\theta$$

2b) Find Unknown(s) Symbolically

Between $\Delta x = V_{ox} t_f + \frac{1}{2} a_x t_f^2$ and $\Delta y = V_{oy} t_f + \frac{1}{2} a_y t_f^2$, we have four knowns ($\Delta x, \Delta y, a_x, a_y$) and three unknowns ($t, V_{ox}, V_{oy}$). However, V_{ox} and V_{oy} are components of $\vec{V}_0$, and we know θ, so they are sort of a single shared unknown, as is time. Two equations sharing two unknowns can be solved.

$$\Delta y = V_{oy} t_f + \tfrac{1}{2} a_y t_f^2$$

$$0 = V_{oy} t_f - \tfrac{1}{2} g t_f^2$$

$$= \left(V_{oy} - \tfrac{1}{2} g t_f\right) t_f$$

t_f is not zero (the arrow doesn't teleport), so

$$0 = V_{oy} - \tfrac{1}{2} g t_f$$

$$t_f = \frac{2V_{oy}}{g}$$

$$\Delta x = V_{ox} t_f + \tfrac{1}{2} a_x t_f^2$$

$$\Delta x = V_{ox} \frac{2V_{oy}}{g} + 0$$

$$= \frac{2V_0^2}{g} \cos\theta \sin\theta$$

$$V_0 = \sqrt{\frac{\Delta x\, g}{2\sin\theta\cos\theta}}$$

If you look up trig identities, you will find $2\sin\theta\cos\theta = \sin(2\theta)$

$$\boxed{V_0 = \sqrt{\frac{\Delta x\, g}{\sin(2\theta)}}}$$

2c) Plug in Numbers

$$V_0 = \sqrt{\frac{(200\,m)(9.8\,m/s^2)}{\sin(2\times 17°)}} \approx \sqrt{\frac{19600}{0.559} \frac{m^2}{s^2}} \approx 59.2\,\frac{m}{s}$$

Simplifications
If $g \approx 10\,\frac{m}{s^2}$, then $V_0 \approx 59.8\,m/s$.
If $g \approx 10\,\frac{m}{s^2}$ and $\theta \approx 15°$, then $V_0 \approx 63.2\,m/s$.

Activity 3-1: Archery S

- You are a long-distance archer who releases an arrow that hits a target 200 m away. Your arrow makes an initial angle with the horizontal of 17°. Following the steps for solving an A*R*C*S problem, find the initial speed of the arrow.
  - **S**ensemake
    - Check the units of your answer.
    - Does your number make sense?
    - Give a physical explanation for how the initial speed of the arrow would change if you increased each variable that your answer depends on.

3-1 Archery ARCS

Sensemake

3a) Units

We want $[v_0] = \frac{m}{s}$, and we know $[\Delta x] = m$, $[g] = \frac{m}{s^2}$, and $[\sin(2\theta)] = 1$ (unitless), so

$$[v_0] = \sqrt{\frac{[\Delta x][g]}{[\sin(2\theta)]}} = \sqrt{\frac{m \cdot m/s^2}{1}} = \sqrt{\frac{m^2}{s^2}} = \frac{m}{s} \checkmark$$

Note that I got a bit of a free unit check by plugging in numbers with units in 2c.

3b) Numbers

$v_0 \approx 132$ miles per hour

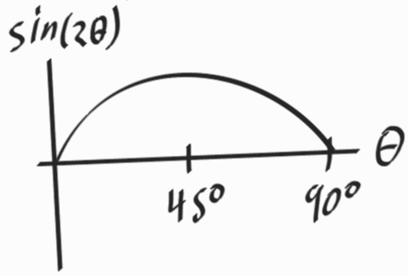
According to a quick internet search, the speed of a recurve bow shot is around 150 mph (around 204 mph for a compound bow), so we are in the right neighborhood.

Side Note: In Olympic archery, the target is set 70m from the archer, so the given $\Delta x = 200m$ is rather large in comparison to modern competition.

3c) Symbols

- Δx increases
  - If the range increases, the arrow must cover a greater distance before it hits the ground, so one would expect it to travel faster.
  - v_0 increases as Δx increases, so our equation matches this prediction.
- g increases
  - If g increases (say we do archery on Jupiter), then the arrow will be dragged to the ground more quickly, so it must go faster to cover the same distance in less time.
  - v_0 increases as g increases, so our equation matches this prediction.

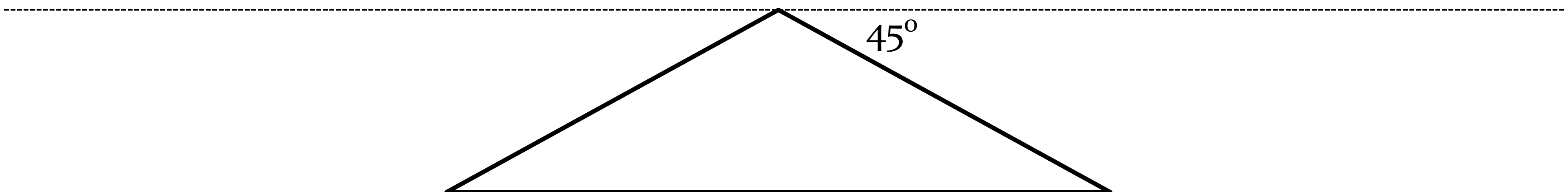
- θ increases
  - This one isn't as easy. Note the graph of $\sin(2\theta)$:



  - $\sin(2\theta)$ increases for $0° \leq \theta < 45°$, so V_0 decreases with increasing θ for shallower angles.
    - At shallow angles, the arrow gains more vertical speed (and thus time of flight) than it loses horizontal speed as θ increases, so tilting up would increase the range if V_0 did not decrease.
  - $\sin(2\theta)$ decreases for $45° < \theta \leq 90°$, so V_0 increases with increasing θ for steeper angles.
    - At steep angles, the arrow loses more horizontal speed than it gains vertical speed (and thus time of flight) as θ increases, so tilting up would shorten the range if V_0 did not increase.

Activity 3-2: Archery II

- You (still a long-distance archer) move to the top of the mountain below. The initial speed of your arrow is now 25.0 m/s, and you release it horizontally to the right. Following the steps for solving A*R*C*S problem, find where the arrow lands.
  - Choose appropriate symbols and do not plug in numbers until you have a symbolic expression for the horizontal distance the arrow travels.
    - *Hint 1:* You will still need to set up separate equations for the *x*- and *y*-directions.
    - *Hint 2:* You can also relate the horizontal and vertical distances the arrow moves before striking the ground.
  - Make sense of your answer:
    - Check the units of your answer.
    - Does your number make sense?
    - Give a physical explanation for how the distance would change if you increased each of the three given variables.



3-2 Archery II ARCS

Analyze and Represent

1a) Understand the Problem

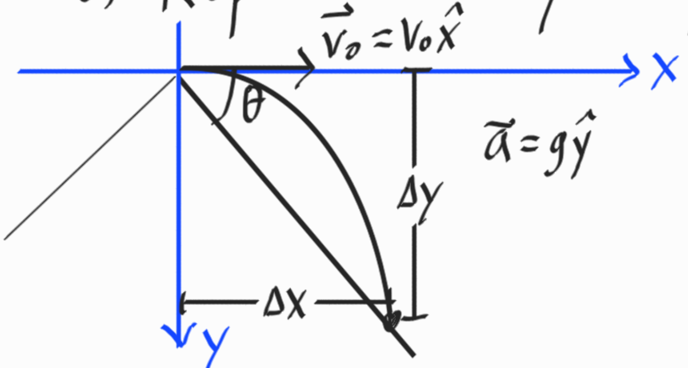
Known: $V_0 = 25 m/s$, $\theta = 45°$, $\underbrace{\vec{a} = g\hat{y},\ g \approx 9.8 m/s^2}_{\text{by assumption}}$

Unknown: Δx (horizontal distance traveled), Δy (vertical distance traveled), t_f (time of flight)

1b) Identify Assumptions

- The mountain has a perfectly smooth slope, and the arrow will not travel far enough to reach the bottom.
  - While mountains are not really this smooth, this assumption reflects the picture, and makes it easier to mathematically describe the slope.
  - Mountains are really tall, and reaching the base of the mountain would complicate the problem.
- The arrow is not affected by moving through air.
  - We don't really know how to handle air resistance or the effects of wind, so we need to ignore them to solve the problem. Arrows tend to be fast and aerodynamic, so this simplification will still be quite accurate.
- Near Earth's surface ($g \approx 9.8 m/s^2$)
  - So far, all known archery occurs on Earth, and though we are farther up on this mountain, the arrow won't fall so far that variations in gravity are significant.
- $\vec{a} = g\hat{y}$
  - Since the object is in free-fall with no wind or air effects, and it isn't traveling far enough for the curvature of the Earth to matter, acceleration is straight down and purely gravitational.

1c) Represent Physically



Choosing $\hat{y}$ pointing down saves me a few negative signs in the process of my solution.

3-2 Archery II ARCS

Calculate

2a) Represent Principles

2-D Kinematics: Known Unknown Wanted

$$\Delta x = V_{0x} t_f + \frac{1}{2} a_x t_f^2 \qquad \Delta y = V_{0y} t_f + \frac{1}{2} a_y t_f^2$$

$$\left\{ \begin{aligned} V_{fx} &= V_{0x} + a_x t_f \qquad & V_{fy} &= V_{0y} + a_y t_f \\ V_{fx}^2 &= V_{0x}^2 + 2 a_x \Delta x \qquad & V_{fy}^2 &= V_{0y}^2 + 2 a_y \Delta y \end{aligned} \right\}$$

Let's avoid these, since we don't really want V_f.

Trigonometry: $\tan\theta = \frac{\Delta y}{\Delta x}$

2b) Find Unknown(s) Symbolically

$$\Delta x = V_{0x} t_f + \frac{1}{2} a_x t_f^2 \qquad \Delta y = V_{0y} t_f + \frac{1}{2} a_y t_f^2$$

$$\Delta x = V_0 t_f \qquad \Delta y = \frac{1}{2} g t_f^2$$

$$\tan\theta = \frac{\Delta y}{\Delta x} = \frac{\frac{1}{2} g t_f^2}{V_0 t_f} = \frac{g}{2V_0} t_f \Rightarrow t_f = \frac{2V_0}{g} \tan\theta$$

$$\boxed{\Delta x = \frac{2V_0^2}{g} \tan\theta} \qquad \left(\text{and } \Delta y = \frac{2V_0^2}{g} \tan^2\theta\right)$$

2c) Plug in Numbers

$$\Delta x \approx \frac{2(25\,m/s)^2}{10\,m/s^2} \underbrace{\tan(45^\circ)}_{=1} = 125\,m$$

$g \approx 10\,m/s^2$ for simplicity

3-2 Archery II ARCS

Sensemake

3a) Units

We want $[\Delta x] = m$, and we know $[V_0] = \frac{m}{s}$, $[g] = \frac{m}{s^2}$, and $[\tan\theta] = 1$ and $[2] = 1$ (unitless), so

$$[\Delta x] = \frac{[2][V_0]^2}{[g]}[\tan\theta] = \frac{m^2/s^2}{m/s^2} = m \checkmark$$

3b) Numbers

Δx and Δy are both 125 m, and $\sqrt{\Delta x^2 + \Delta y^2} = 125\sqrt{2}\ m \approx 176\ m$, which is comparable to the distance we shot in Activity 3-1.

We achieved it at a lower speed here, since our arrow could fall toward its target (rather than having to keep itself above flat ground for the entire flight).

3c) Symbols

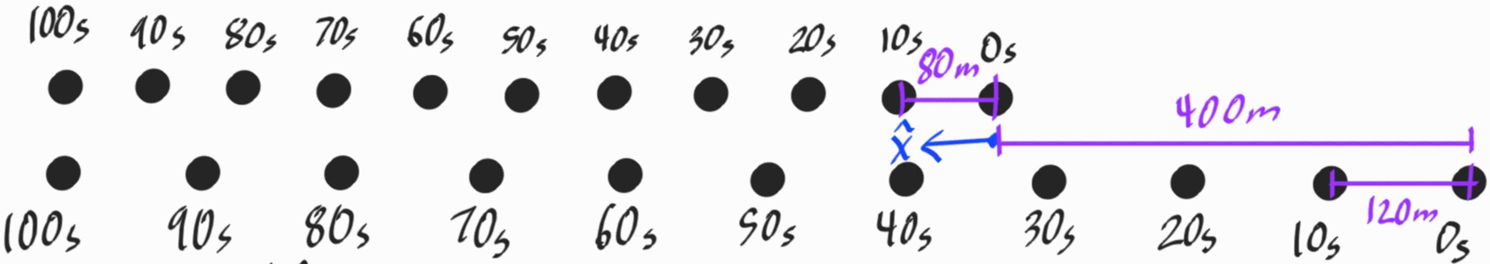
- V_0 increases
  - If it is shot faster, the arrow will have more horizontal speed and have farther to fall (and hence more time in flight), so it should achieve more horizontal distance.
  - Δx increases as V_0 increases, so our equation matches this prediction.
- g increases
  - If g increases (say we do archery on Jupiter), then the arrow will be dragged to the ground more quickly, so it will not travel as far.
  - Δx decreases as g increases, so our equation matches this prediction.
- $\theta = 0$
  - In this special case, there is no mountain, and an arrow launched horizontally at ground level will hit the ground immediately, having no range.
  - $\Delta x = \frac{2V_0^2}{g}\tan(0) = 0 \checkmark$

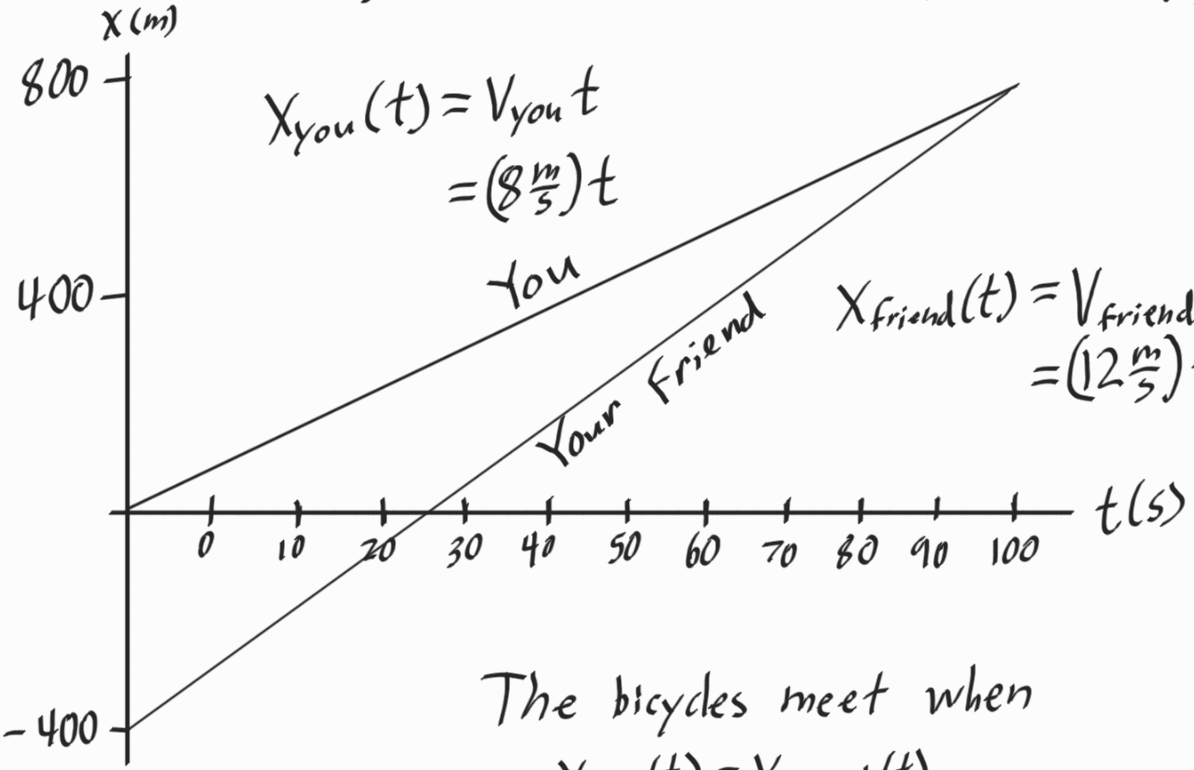
Activity 3-3: Bicycles

- You ride your bike west at a speed of 8.0 m/s. Your friend, 400 m east of you, is riding her bike west at a speed of 12 m/s.
  - Draw a motion diagram for each bike. Include a coordinate system.
  - Draw position *vs.* time graphs for each bicycle.
  - Write equations that describe the position of each bicycle as a function of time.
  - Determine when the bicycles are at the same position.

3-3 Bicycles

Let $\Delta t = 10\,s$ for this motion diagram. I will choose west to be the positive direction, and place the origin at my initial position.





$$X_{you}(t) = V_{you}\,t = \left(8\tfrac{m}{s}\right)t$$

$$X_{friend}(t) = V_{friend}\,t + X_i = \left(12\tfrac{m}{s}\right)t - 400m$$

The bicycles meet when

$$X_{you}(t) = X_{friend}(t)$$

$$V_{you}\,t = V_{friend}\,t + X_i$$

$$(V_{you} - V_{friend})t = X_i$$

$$t = \frac{X_i}{V_{you} - V_{friend}} = \frac{-400m}{8\frac{m}{s} - 12\frac{m}{s}} = \frac{400m}{4\frac{m}{s}} = 100s$$

Cut Activity

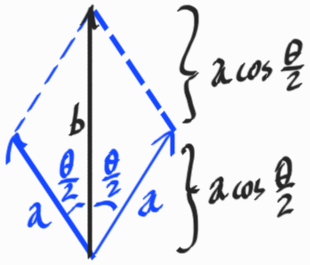
Activity 1-3

- You have two vectors that have the same magnitude *a*, but can point in any direction.
- Sketch the vectors in each case below. Your sketches should be quantitatively accurate (you should know the angle between the vectors).
  - The magnitude of the sum is 0.
  - The magnitude of the sum is $2a$.
  - The magnitude of the sum is $\sqrt{2}a$.
  - The magnitude of the sum is $\sqrt{3}a$.
  - The magnitude of the sum is b.
- Make sense of your answer with the following questions:
  - What is the largest possible sum? What is the smallest?

3-X Adding Two Vectors

Let us do the general case first:

Parallelogram



$$b = 2a\cos\frac{\theta}{2}$$

The largest b can be is $2a$.

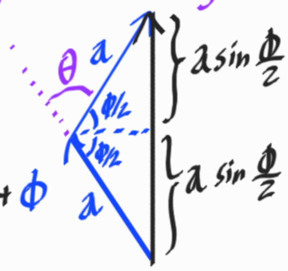
This is expected, as the largest resultant vector happens when the two vectors are parallel, which can only be as long as their combined length.

The smallest b can be is zero (we take θ to be the smallest angle between, so $\frac{\theta}{2} \leq 90°$).

This is expected, as we can put two equal-length vectors antiparallel to get $\vec{0}$. Magnitudes must be positive, so zero is the minimum.

Tip-to-Tail

If you set it up this way, remember that the angle between two vectors is still best thought of as though they were tail-to-tail.



$180° = \theta + \phi$

$$b = 2a\sin\frac{\phi}{2}$$

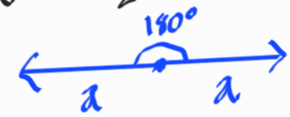
Note that these two answers are equivalent, as $\sin(x+90°) = \cos(x)$ and $\cos(-x) = \cos(x)$.

$$b = 2a\sin\left(\frac{180° - \theta}{2}\right)$$
$$= 2a\sin\left(90° - \frac{\theta}{2}\right)$$
$$= 2a\cos\left(-\frac{\theta}{2}\right)$$
$$= 2a\cos\frac{\theta}{2}$$

This symbolic solution is powerful, as we can use it to find each of the specific cases.

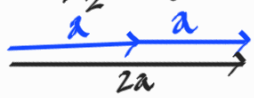
Magnitude of sum is zero

$$b = 0 \Rightarrow 0 = 2a\cos\frac{\theta}{2} \Rightarrow 0 = \cos\frac{\theta}{2} \Rightarrow \theta = 180°$$



Magnitude of sum is 2a

$$b = 2a \Rightarrow 2a = 2a\cos\frac{\theta}{2} \Rightarrow 1 = \cos\frac{\theta}{2} \Rightarrow \theta = 0°$$



Magnitude of sum is $\sqrt{2}a$

$$b = \sqrt{2}a \Rightarrow \sqrt{2}a = 2a\cos\frac{\theta}{2} \Rightarrow \frac{\sqrt{2}}{2} = \cos\frac{\theta}{2} \Rightarrow \theta = 90°$$

Magnitude of sum is $\sqrt{3}a$

$$b = \sqrt{3}a \Rightarrow \sqrt{3}a = 2a\cos\frac{\theta}{2} \Rightarrow \frac{\sqrt{3}}{2} = \cos\frac{\theta}{2} \Rightarrow \theta = 60°$$