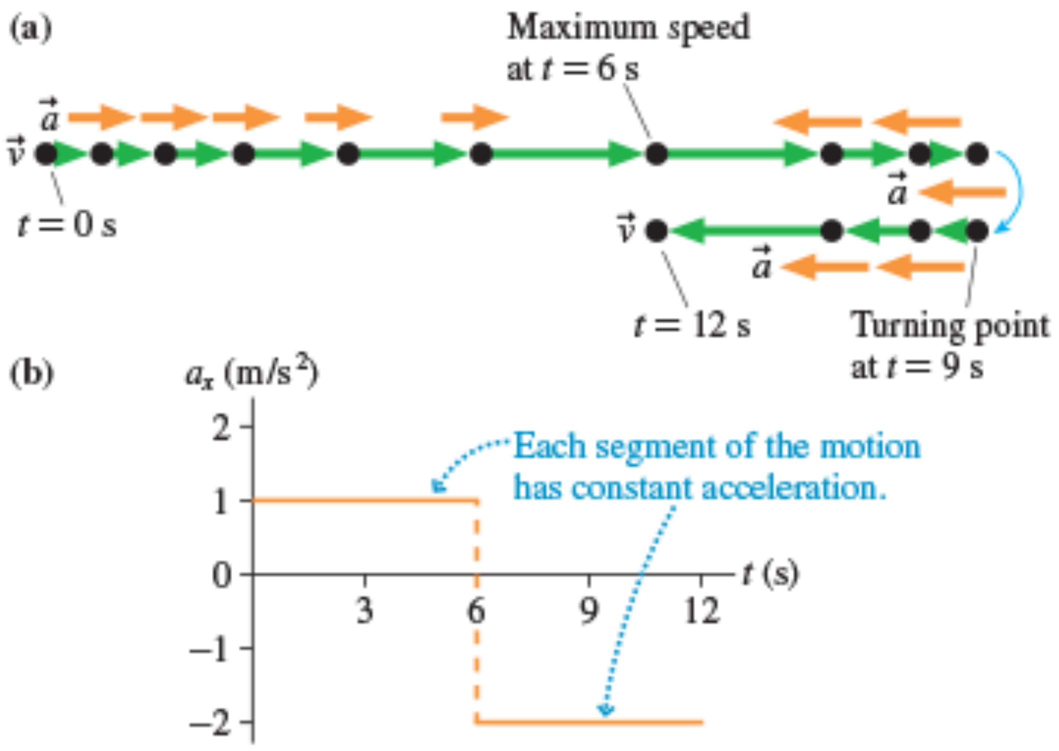
Studio Week 2

Representations of Motion



Picture credit: Textbook (Knight)

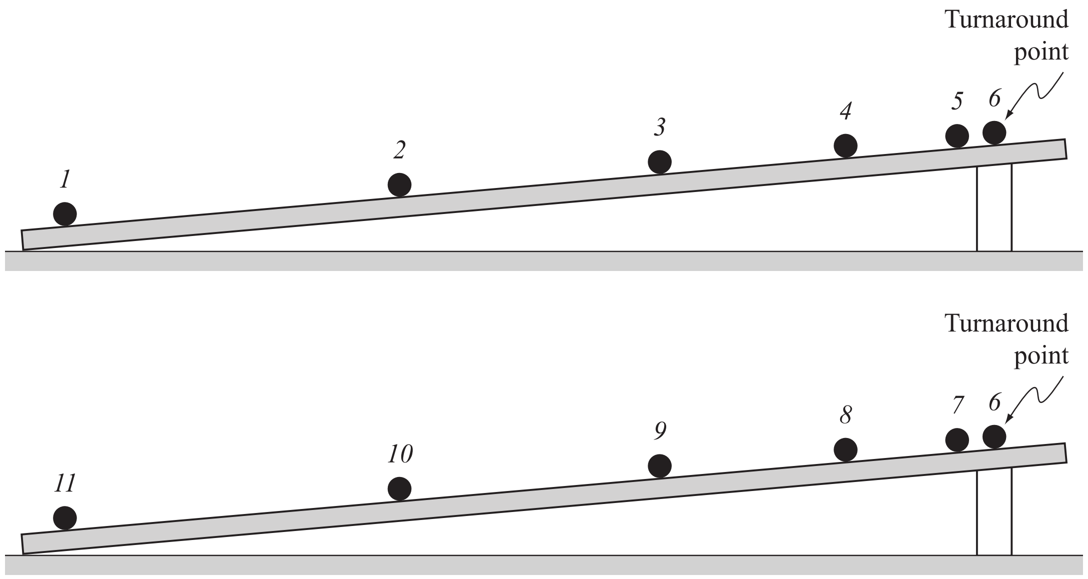
Principles for Success

- **Treat everyone with respect.**
- **Learn by doing and questioning.**
- **Make sense of everything.**

Activity 2-1: Rolling on a Ramp

At right is a motion diagram for a ball that first rolls up and then down a track.

1. Draw velocity vectors that represent the instantaneous velocity of the ball at each instant shown.
2. Draw a vector that represents Δv, the change in velocity between instants 1 and 2.
    - A. How does the direction of Δv compare to the direction of v?
3. Draw a vector that represents the acceleration of the ball between instants 1 and 2.
4. Draw a vector that represents the acceleration of the ball between instants 9 and 10.
5. Devise a procedure that you can use to draw a vector that represents the acceleration at point 6 (the turnaround point).



2-1 Rolling on a Ramp

Adding Velocity Vectors

If you are given a strobe diagram (that is, a motion diagram without vectors on it), you may need to add velocity and acceleration vectors. There are many ways to approximate the instantaneous velocity, including:

- Forward Average (most common)

  Take $\Delta\vec{r}$ from the current point to the next.

  2 $\xrightarrow{\Delta\vec{r}_{23}}$ 3

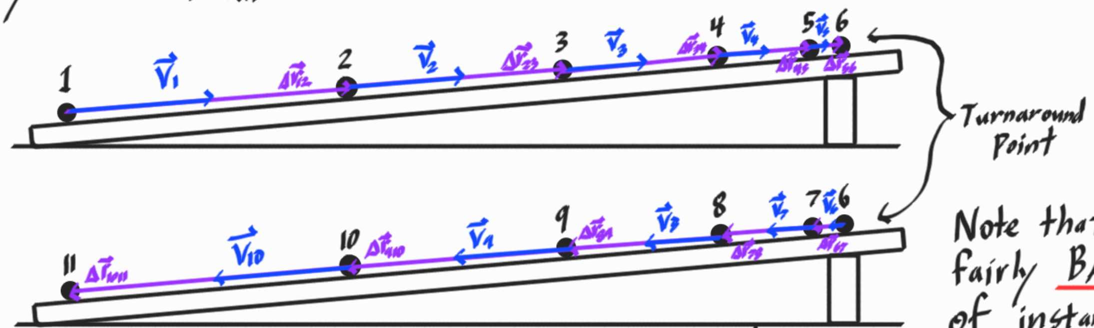
  Since $\Delta\vec{V}_{avg} = \frac{\Delta\vec{r}}{\Delta t}$, this displacement vector is proportional to the average velocity between the two points.

  Some people like to leave the vector connecting the two points and just call it $\vec{V}$, which is nice and easy. Other people insist that, since $\vec{V}_{avg} = \frac{\Delta\vec{r}}{\Delta t}$, the velocity shouldn't be the same length as $\Delta\vec{r}$ (you are dividing by something, after all).

  I personally don't buy that argument completely, as velocity and displacement have different units, so their magnitudes cannot be directly compared. (Is a meter bigger than a meter per second? There is no definitive answer.)

  However, I do still recommend rescaling all of your $\Delta\vec{r}$ vectors before calling them $\vec{V}$, just so you remember that they are not the same.

  As an example, here is our motion diagram for this activity, with my $\vec{v}$ vectors all drawn as half of the $\Delta\vec{r}$ vectors:



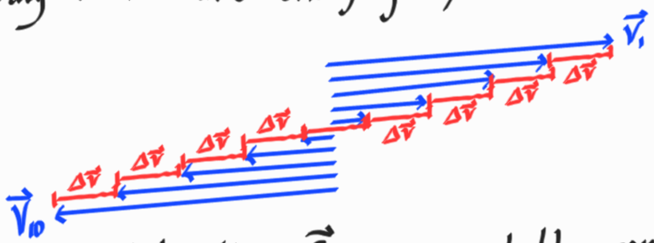
Note that this is a fairly <u>BAD</u> estimate of instantaneous velocity, especially at point 6, where instantaneous velocity should be zero at the turnaround.

This method also doesn't extend well to finding acceleration. Each $\vec{V}_{avg}$ is the instantaneous velocity somewhere between the two points (that is the mean value theorem from calculus), but since it isn't likely to be the instantaneous velocity at either point, using $\frac{\vec{V}_{avg23} - \vec{V}_{avg12}}{\Delta t}$ to approximate $\vec{a}_{avg} = \frac{\vec{v}_3 - \vec{v}_2}{\Delta t}$ introduces error.

Still, this method is sufficient for determining if $\vec{a}_{avg}$ is constant.

$$\vec{a}_{avg} = \frac{\vec{v}(t_f) - \vec{v}(t_i)}{t_f - t_i} = \frac{\frac{d\vec{x}}{dt}(t_f) - \frac{d\vec{x}}{dt}(t_i)}{t_f - t_i} = \frac{d}{dt}\,\frac{\vec{x}(t_f) - \vec{x}(t_i)}{t_f - t_i} = \frac{d}{dt}\vec{v}_{avg}$$

So, if the $\vec{v}_{avg}$ vectors are changing by a constant amount, like so:



then you can conclude that $\vec{a}_{avg}$ is probably constant, and thus it might be alright to assume that the instantaneous acceleration is constant.

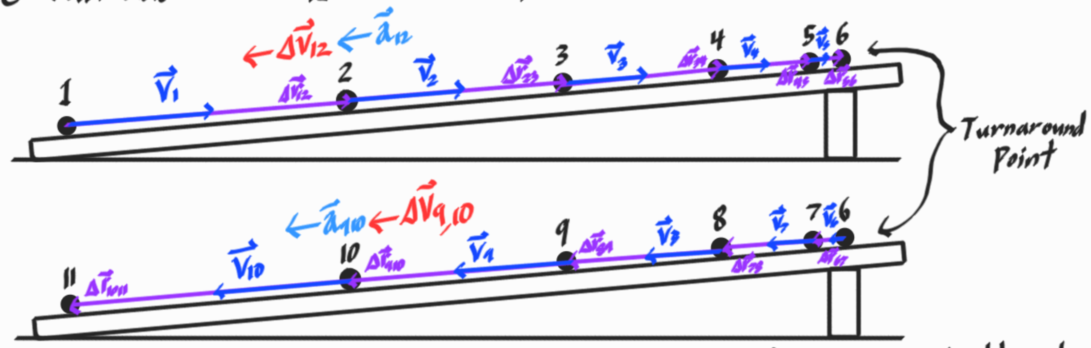
$$\Delta\vec{r} = \vec{v}_i \Delta t + \tfrac{1}{2}\vec{a}\Delta t^2 \Rightarrow \frac{\Delta\vec{r}}{\Delta t^2} = \frac{\vec{v}_i}{\Delta t} + \tfrac{1}{2}\vec{a}$$

If that is true, then $\dfrac{\vec{v}_{avg\,2,3} - \vec{v}_{avg\,1,2}}{\Delta t} = \dfrac{\Delta\vec{r}_{23} - \Delta\vec{r}_{12}}{\Delta t^2} = \left(\dfrac{\vec{v}_2}{\Delta t} + \tfrac{1}{2}\vec{a}\right) - \left(\dfrac{\vec{v}_1}{\Delta t} + \tfrac{1}{2}\vec{a}\right)$

$$= \frac{\vec{v}_2 - \vec{v}_1}{\Delta t} = \vec{a},$$

so the error-prone estimate from before actually gives you the right acceleration! You really should check that $\Delta\vec{v}$ between your averages is constant before you rely on this, however.

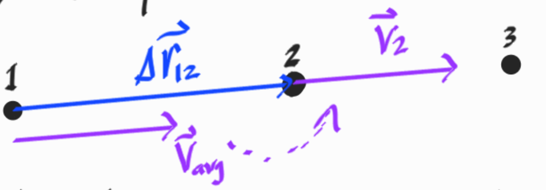
Now, we can add in $\Delta\vec{v}_{12}$ and $\Delta\vec{v}_{910}$, which are proportional to $\vec{a}_{avg} = \frac{\Delta\vec{v}}{\Delta t}$.



Note that $\Delta\vec{v}_{12}$ points opposite $\vec{v}$ as the ball slows down going up the ramp, and $\Delta\vec{v}_{910}$ points with $\vec{v}$ as the ball speeds up goind down the ramp.

● Backward Average (uncommon)

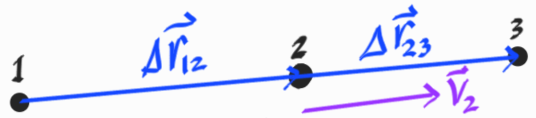
Take $\Delta\vec{r}$ from the previous point to the current point, determine $\vec{v}_{avg}$ from that, then put it at the current point:

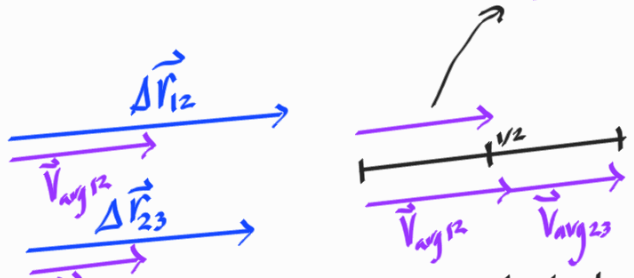


This method has the same accuracy issues as the forward average, plus you can't give the first point a velocity vector.

Balanced Average

Take $\Delta\vec{r}$ from the current point to the next point <u>and</u> from the previous point to the current point. Find $\vec{V}_{avg}$ for each, then find the average of the two $\vec{V}_{avg}$ vectors and use that as the velocity at the current point.



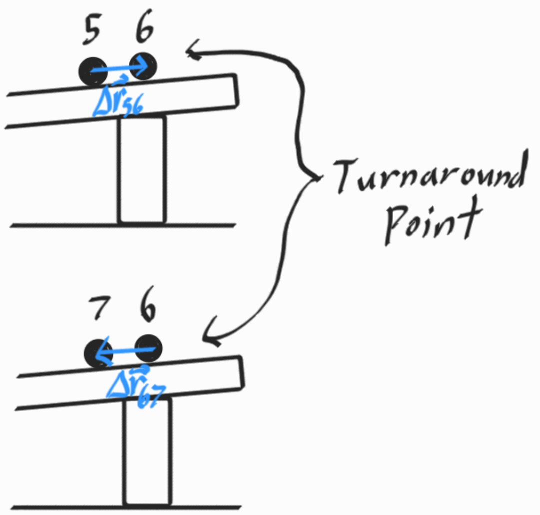


$$\vec{V}_2 \approx \frac{1}{2}\left(\vec{V}_{avg12} + \vec{V}_{avg23}\right) = \frac{1}{2}\left(\frac{\Delta\vec{r}_{12}}{\Delta t} + \frac{\Delta\vec{r}_{23}}{\Delta t}\right) = \frac{\Delta\vec{r}_{12} + \Delta\vec{r}_{23}}{2\Delta t} = \frac{\Delta\vec{r}_{13}}{2\Delta t} = \vec{V}_{avg13}$$

It is equivalent to find $\Delta\vec{r}_{13}$ (displacement from 1 to 3) and find $\vec{V}_{avg13}$.

This method is still an approximation, but it improves the accuracy of the estimate at point 2 by considering both what happened before and what happened after.

In particular, note that it fixes the inaccurate velocity vector at the turning point:

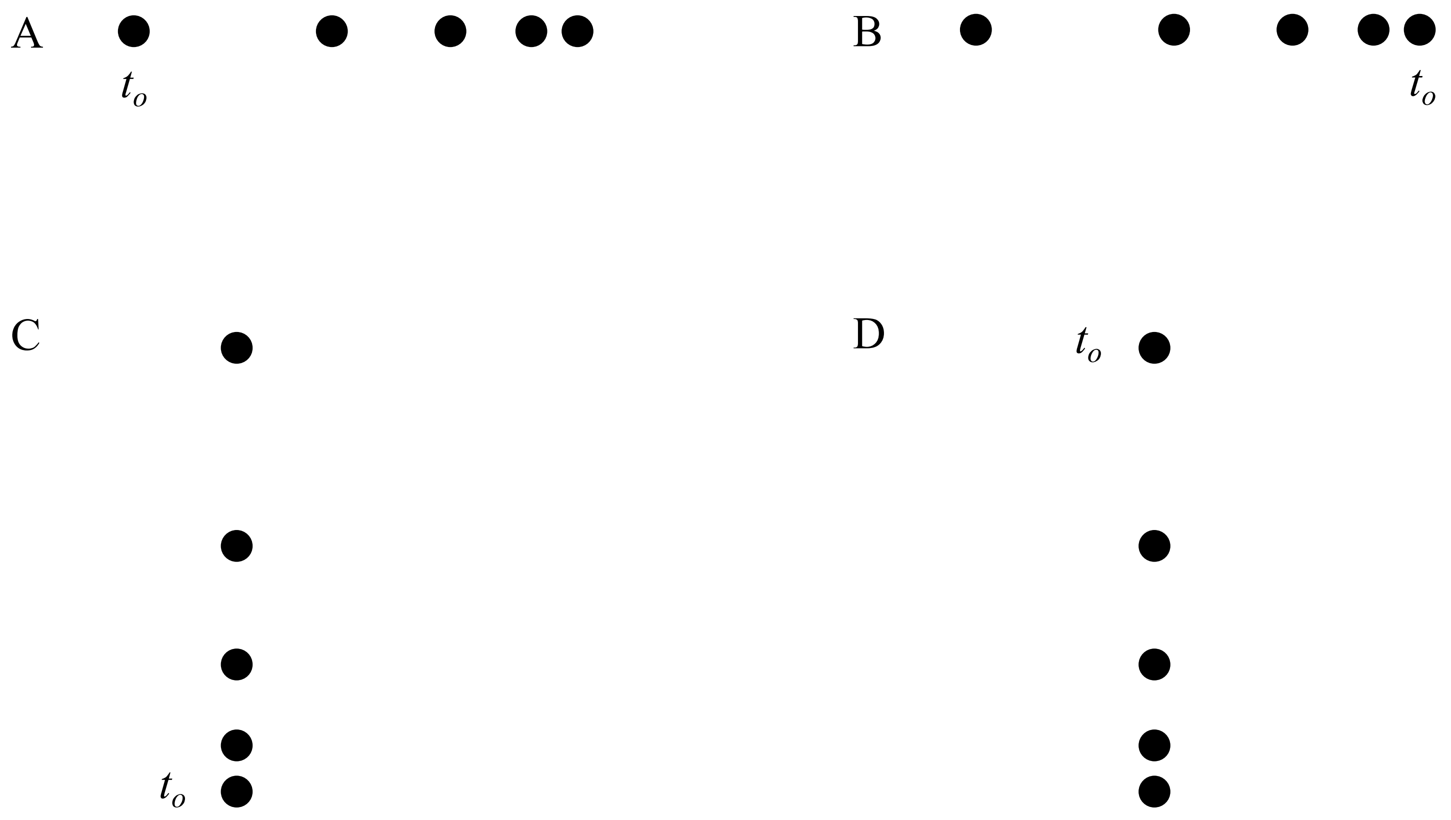


$$\Delta\vec{r}_{56} + \Delta\vec{r}_{67} = \vec{0}$$

$$\vec{V}_6 \approx \frac{\Delta\vec{r}_{56} + \Delta\vec{r}_{67}}{2\Delta t} = \frac{\vec{0}}{2\Delta t} = \vec{0}$$

Activity 2-2: Strobe Diagrams

- The initial position in each strobe diagram to the right is labeled t_o. For each case:
  1. Choose a coordinate system (use the same system for all problems).
  2. Sketch velocity vectors for each point.
  3. Sketch acceleration vectors for each point.
  4. Identify whether each of the following is positive or negative:
     A. position
     B. velocity
     C. acceleration



2-2 Strobe Diagrams and Coordinate Systems

Let us choose a standard coordinate system: $\overset{\hat{y}}{\uparrow}\!\rightarrow\hat{x}$

"Use the same system for all problems" is a little ambiguous.

Let us assume we just need to use the same orientation ($\begin{smallmatrix} y \text{ up} \\ x \text{ right}\end{smallmatrix}$) for all problems, but we can choose a different origin.

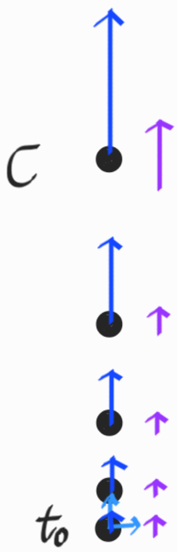
I will indicate the location of the origin with a light blue copy of our coordinate system.

Velocity vectors will be darker blue.
Acceleration vectors will be purple.
} These vectors will be drawn qualitatively.

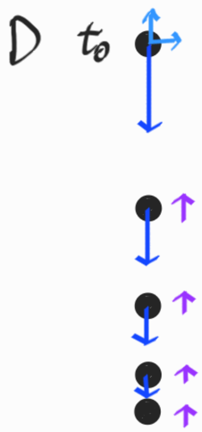
We will say a vector is "positive" or "negative" if it is parallel ($\rightrightarrows$) or antiparallel ($\rightleftarrows$), respectively, to one of the coordinate axes.

Acceleration is negative.
Velocity is positive.
The object is slowing down.

Acceleration is negative.
Velocity is negative.
The object is speeding up.



Acceleration is positive.
Velocity is positive.
The object is speeding up.



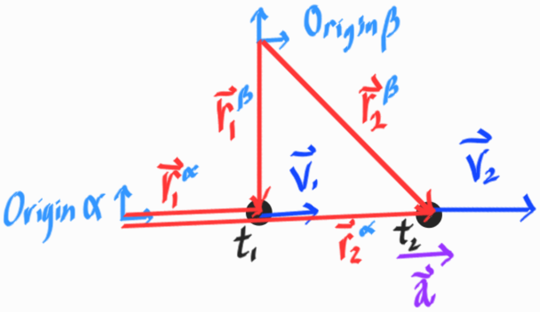
Acceleration is positive.
Velocity is negative.
The object is slowing down.

Big Takeaway: An object gains speed when $\vec{v}$ and $\vec{a}$ point in the same direction (both positive or both negative in 1-D), and it gets slower when $\vec{v}$ and $\vec{a}$ are in opposite directions.

"Negative acceleration" does <u>NOT</u> mean slowing down!

Position and Origin: For the above origins, $\vec{r}$ is positive (or zero) for all points in A, negative for all points in B, positive for all points in C, and negative for all points in D.

Position is the only one of these vectors that depends on the origin.



$\vec{r}_1^{\,\alpha}$ and $\vec{r}_2^{\,\alpha}$ are both positive for Origin α, but $\vec{r}_1^{\,\beta}$ is negative for Origin β, and $\vec{r}_2^{\,\beta}$ is neither positive nor negative (though its x-component is positive and its y-component is negative).

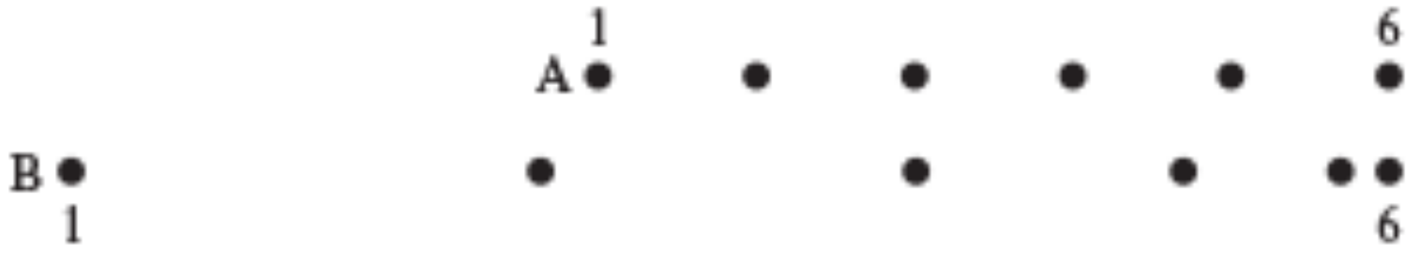
The displacement, $\Delta\vec{r}_{12} = \vec{r}_1 - \vec{r}_2$, is the same in both coordinate systems, and remains positive as we shift the origin.

Shifting the origin does not change that $\vec{v}_1$, $\vec{v}_2$, and $\vec{a}$ are all parallel to $\hat{x}$, so their signs remain the same.

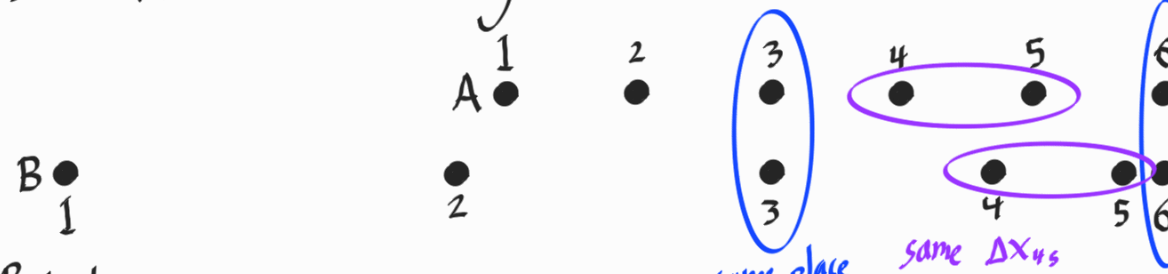
Activity 2-3: Motion Diagrams

Below are motion diagrams for two cars.

1. Draw and label *x vs. t* graphs for A and B on the same set of axes.
2. Draw and label *v vs. t* graphs for A and B on the same set of axes.
3. Do the cars ever have the same position at one instant in time?
4. Do the cars ever have the same velocity at one instant in time?



2-3 Motion Diagrams to Graphs



A moves at a constant velocity (ΔX constant between instants)

B is slowing down and ends going more slowly.

same place same time

same ΔX_{45}
$\Downarrow$
same V_{avg}

same place same time

B starts farther left and faster.

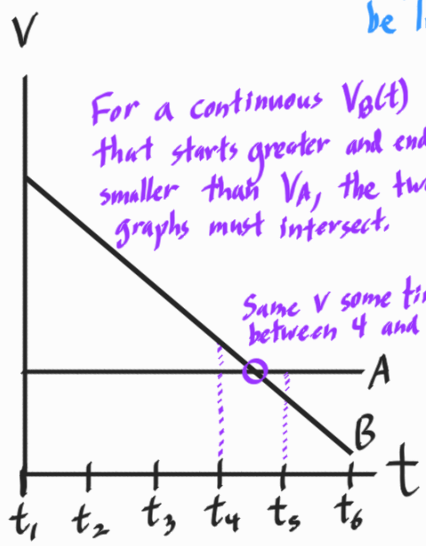
Decrease in ΔX (and thus in V_{avg+}) appears constant, so average acceleration is also constant.

Let's assume instantaneous acceleration is constant.
$\Rightarrow$ Graph of V_B should be linear.

Forward Approximation: $V_{avg+}(t) = \dfrac{X(t+\Delta t) - X(t)}{\Delta t}$

$$\frac{d}{dt} V_{avg+}(t) = \frac{\frac{d}{dt} X(t+\Delta t) - \frac{d}{dt} X(t)}{\Delta t}$$

$$= \frac{V(t+\Delta t) - V(t)}{\Delta t} = a_{avg+}(t)$$



For a continuous $V_B(t)$ that starts greater and ends smaller than V_A, the two graphs must intersect.

$\Rightarrow$ The two cars must have the same velocity at some instant of time. It will happen between t_4 and t_5.

In fact, since ΔX_{45} must equal the area under the V graph between times 4 and 5 for each object, and we assumed V_B was linear, the intersection must be exactly half way between.

$$V_B(t) = V_{B0} - a_B t$$

Let t^* be the time when $V_B = V_A$.

Let $0 \leq m \leq 1$ such that $t^* = t_4 + m\Delta t$
(and note that $t_4 + \Delta t = t_5$, so $t_5 = t^* + (1-m)\Delta t$).

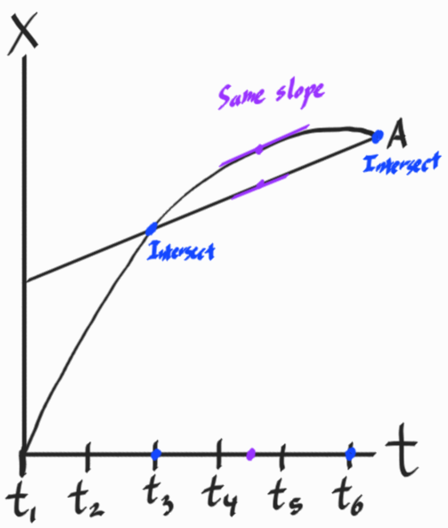
$$V_A \Delta t = \Delta X_{45} = \frac{V(t_4) + V(t_5)}{2} \Delta t \quad \left(\begin{matrix}\text{trapezoid}\\ \text{area}\end{matrix}\right)$$

$$= \frac{V(t^* - m\Delta t) + V(t^* + (1-m)\Delta t)}{2} \Delta t$$

$$V_A = \frac{V_{B0} - a_B(t^* - m\Delta t) + V_{B0} - a_B(t^* + (1-m)\Delta t)}{2}$$

$$= \underbrace{V_{B0} - a_B t^*}_{V_B(t^*) = V_A} - \underbrace{\frac{1-2m}{2} a_B \Delta t}_{\Rightarrow \text{ must be zero}}$$

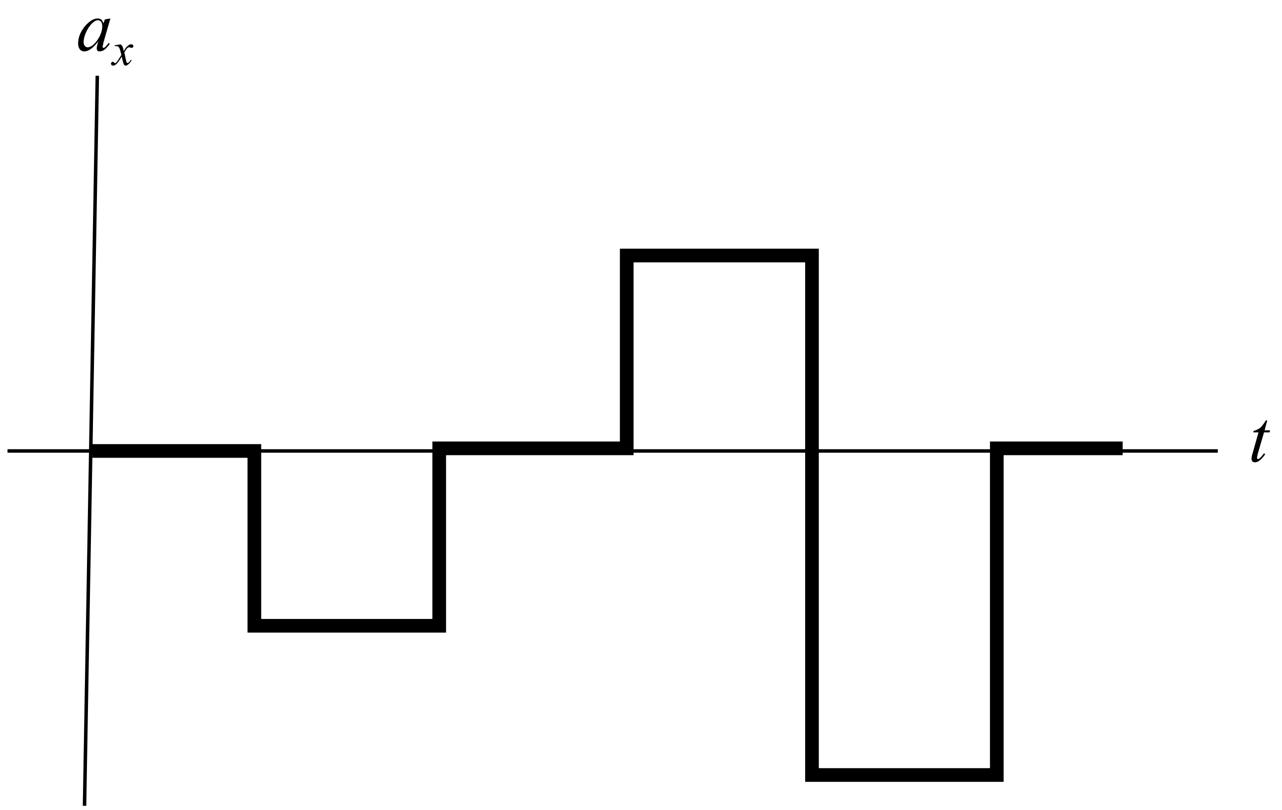
$\Rightarrow m = \frac{1}{2}$



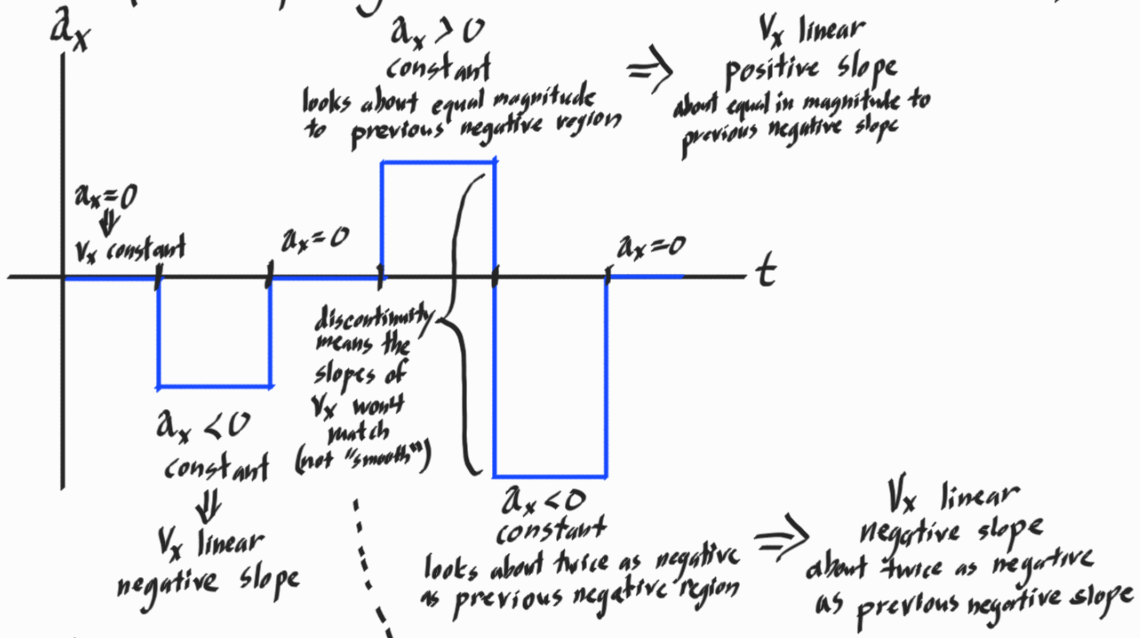
Activity 2-4: Graphing

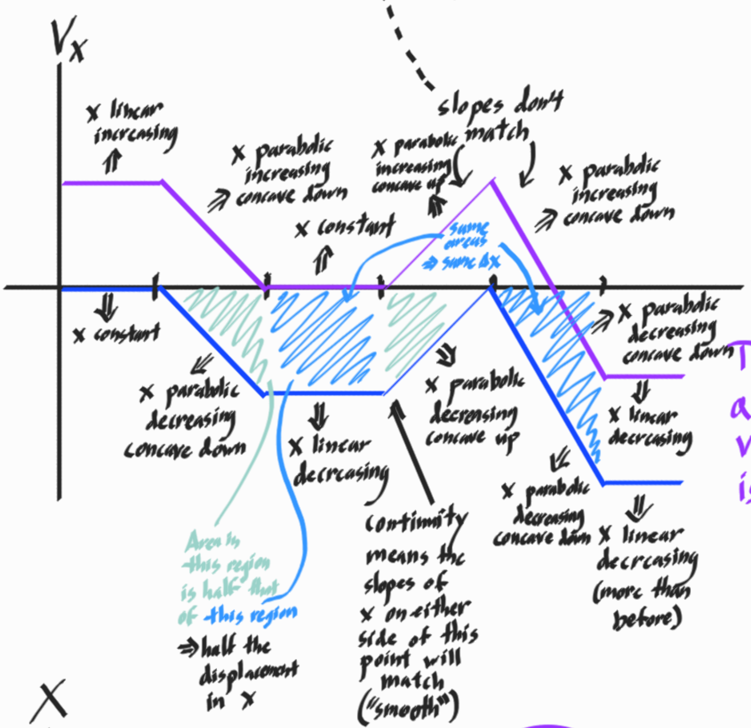
Below is a graph of *a vs. t* for a car.

- Draw and label graphs of *v vs. t* and *x vs. t* for the car.

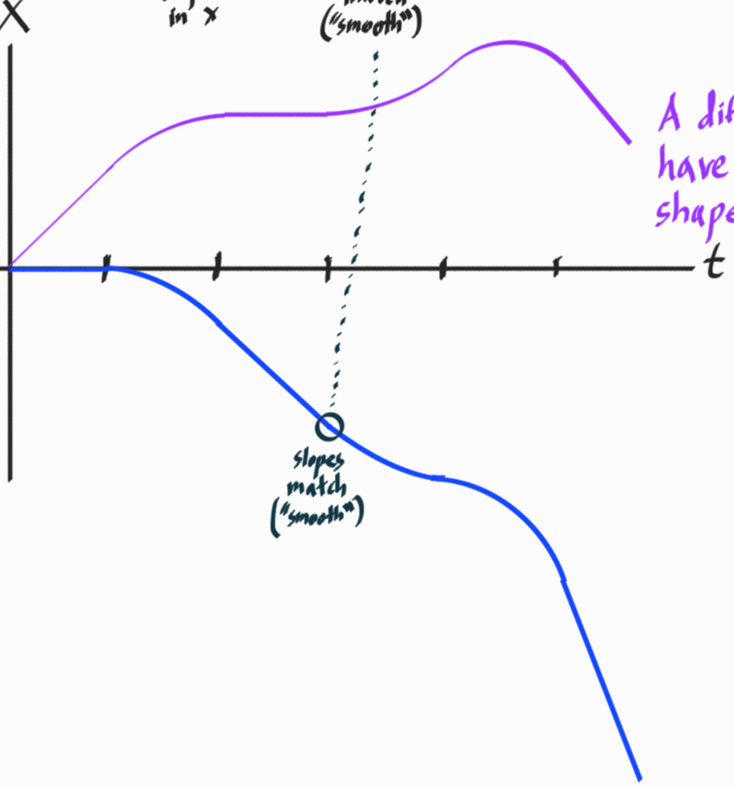


2-4 Graphing: Acceleration to Velocity to Position





The acceleration graph doesn't say anything about initial velocity, so we could start the velocity graph at any initial value. The shape is the same.



A different choice of initial velocity does have a significant impact on the shape of the position graph.

I could have chosen any initial position. The graph would be shifted up or down, but have the same shape.