PH 221 Week 3

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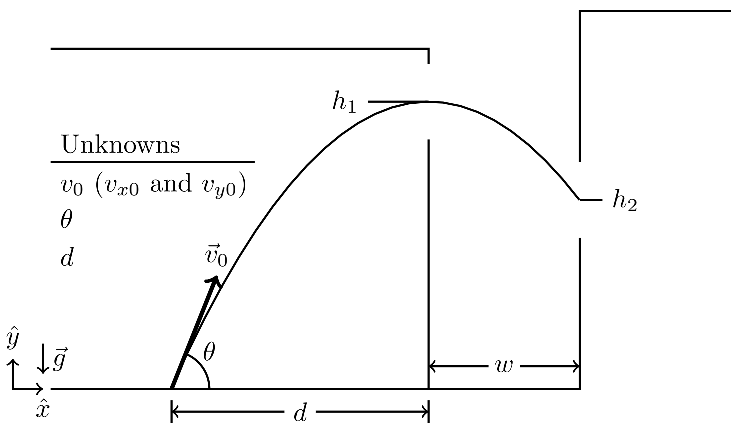
Activity

You are standing in a large building, and there is a single window right up by the ceiling. This window opens on an alleyway, and the building on the other side has an open window below the level of your building's window. You want to launch something from your building into the other, right through the center of each window. You know the heights of the windows and the width of the alley, so what do you need to do to throw it correctly?

(1) Draw a physical representation of the situation. On this drawing, indicate the height of the center of the first window h_1, the height of center of the second window h_2, the width of the alley w, the initial velocity $\vec{v}_0$, the launch angle θ, and your distance to the wall of the first window d. Add a coordinate system (indicate the directions of $+x$ and $+y$), and indicate the direction of gravitational acceleration $\vec{g}$. What are your unknowns?



(2) Since the window is right by the ceiling, you cannot arc your projectile too high. Suppose for simplicity that you want it to reach its maximum height right as it goes through the first window. What must its initial velocity be in the y direction for this to happen?

We know the final y-velocity at the peak of motion will be zero, and we know the object will have traveled $\Delta y = h_1$, so we can use the kinematics equation which does not involve time of flight:

$$\begin{aligned} v_y^2 &= v_{y0}^2 + 2a_y\Delta y \\ 0 &= v_{y0}^2 - 2gh_1 \\ v_{y0} &= \sqrt{2gh_1}. \end{aligned} \tag{1}$$

(3) Exiting the first window, the object only has velocity in the x direction. How long does it take to fall to the height of the second window? How fast must it be going to cross the alley in this time and enter the second window?

The horizontal velocity has no impact on the vertical motion of the object. It begins at $y_i = h_1$ and free-falls to $y_f = h_2$, starting from $v_{yi} = 0$. The change in position kinematics equation gives us

$$\begin{aligned} y_f &= y_i + v_{yi}\Delta t + \frac{a_y}{2}(\Delta t)^2 \\ h_2 &= h_1 + 0 - \frac{g}{2}(\Delta t_{h_1\to h_2})^2 \\ h_1 - h_2 &= \frac{g}{2}(\Delta t_{h_1\to h_2})^2 \\ \Delta t_{h_1\to h_2} &= \sqrt{\frac{2(h_1-h_2)}{g}}. \end{aligned} \tag{2}$$

The velocity v_x does not change during flight, as there is no horizontal component to acceleration. As such, it must be that the object has total velocity $\vec{v} = v_{x0}\hat{\imath}$ as it leaves the first window. It must travel across $\Delta x = w$ over time $\Delta t_{h_1\to h_2}$ to enter the other window, which means

$$\begin{aligned} \Delta x &= v_{x0}\Delta t + \frac{a_x}{2}(\Delta t)^2 \\ w &= v_{x0}\Delta t_{h_1\to h_2} + 0 \\ v_{x0} &= \frac{w}{\Delta t_{h_1\to h_2}} = w\sqrt{\frac{g}{2(h_1-h_2)}}. \end{aligned} \tag{3}$$

(4) Given the horizontal and vertical components of initial velocity, how far back from the first window do you have to stand to get the object through it?

The link between the x and y kinematics equations is Δt, which is shared between them. As such, we must find the time it takes for the object to reach $\Delta y = h_1$, then we must find how far $\Delta x = d$ it travels horizontally in that time. We begin with

$$\begin{aligned} \Delta y &= v_{y0}\Delta t + \frac{a}{2}(\Delta t)^2 \\ 0 &= \Delta y - v_{y0}\Delta t - \frac{a}{2}(\Delta t)^2 \\ &= h_1 - v_{y0}\Delta t + \frac{g}{2}(\Delta t)^2. \end{aligned} \tag{4}$$

Now we have a form to which we can apply the quadratic formula, which gives

$$\begin{aligned} \Delta t &= \frac{-(-v_{y0}) \pm \sqrt{(-v_{y0})^2 - 4(g/2)(h_1)}}{2(g/2)} \\ &= \frac{v_{y0} \pm \sqrt{v_{y0}^2 - 2gh_1}}{g}. \end{aligned} \tag{5}$$

Inserting our expression from part (2), we get

$$\begin{aligned} \Delta t &= \frac{\sqrt{2gh_1} \pm \sqrt{2gh_1 - 2gh_1}}{g} \\ &= \frac{\sqrt{2gh_1}}{g} \\ &= \sqrt{\frac{2h_1}{g}}. \end{aligned} \tag{6}$$

Without acceleration, we know that $\Delta x = v_{x0}\Delta t$. With our new expression for Δt, plus our expression for v_{x0} from part (3), we find

$$d = \Delta x = w\sqrt{\frac{g}{2(h_1 - h_2)}}\sqrt{\frac{2h_1}{g}} = w\sqrt{\frac{h_1}{h_1 - h_2}}. \tag{7}$$

(5) Given the components of initial velocity, what is the magnitude of initial velocity? At what angle with respect to the floor must the object be launched?

The magnitude of velocity is simply the result of the Pythagorean theorem:

$$v_0 = \sqrt{v_{x0}^2 + v_{y0}^2} = \sqrt{\frac{w^2 g}{2(h_1 - h_2)} + 2gh_1}. \tag{8}$$

The angle requires an application of trigonometry. If v_0 is the hypotenuse of a right triangle with legs v_{x0} and v_{y0}, then the angle θ is opposite v_{y0} and adjacent to v_{x0}. This means

$$\tan\theta = \frac{v_{y0}}{v_{x0}}. \tag{9}$$

Taking the inverse tangent of both sides will recover θ. A good sensemaking technique with recovering angles in this way is to think about the relative sizes of the triangle's legs. If $v_{y0} > v_{x0}$, then we are guaranteed that $\theta > 45°$.