Lecture 6: Projectile Motion I

Warm-Up Activity

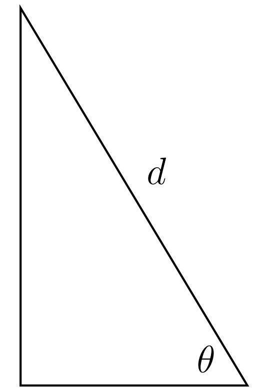
- You slide down the ramp shown, starting from rest. Consider the special case where the ramp is made horizontal.
- How long would it take you to reach the end of the ramp in this case?

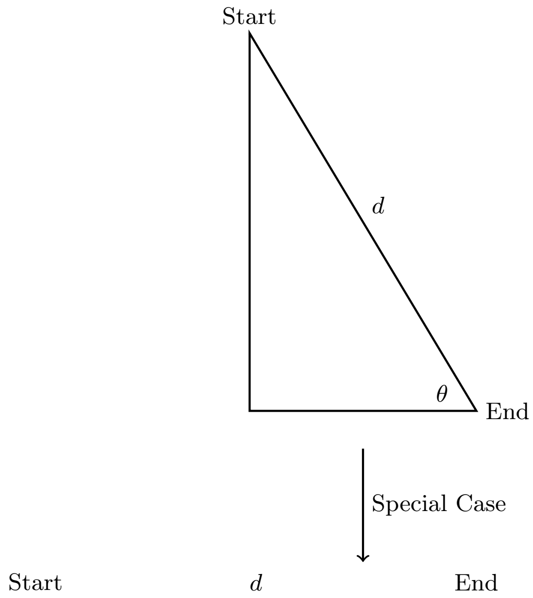
(A) No time at all.

(B) The same amount of time as when tilted.

(C) An infinite amount of time.

(D) Not enough information.





If you are sitting on flat ground, waiting to spontaneously start to slide across the floor, you will be waiting for a very long time. In this special case, the answer is (C); it will take an infinite amount of time to reach the end of a horizontal ramp.

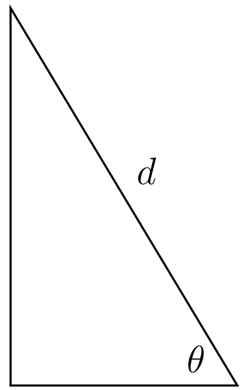
- t and θ
  - It takes an infinite amount of time to slide "down" a horizontal ramp.
  - As θ decreases toward zero, $\sin\theta$ also decreases toward zero, which means the denominator of the fraction goes to zero. This will cause the fraction (and thus the equation for t) to explode to infinity, thus agreeing with our prediction.
- t and d
  - It should take more time to slide down a longer ramp.
  - According to the equation, increasing d will increase t, which matches our prediction.
- t and g
  - If gravity is stronger, then things will be pulled down the ramp faster.
  - The equation tells us that increasing g will decrease t, which matches our prediction.

L6-1: Ramp Slide Sensemaking

- You slide down the ramp shown, starting from rest. Consider the special case where the ramp is made horizontal.
- Does the equation below agree with your prediction?

$$t = \sqrt{\frac{2d}{g\sin\theta}}$$

- Can you make sense of how t depends on d and g as well?



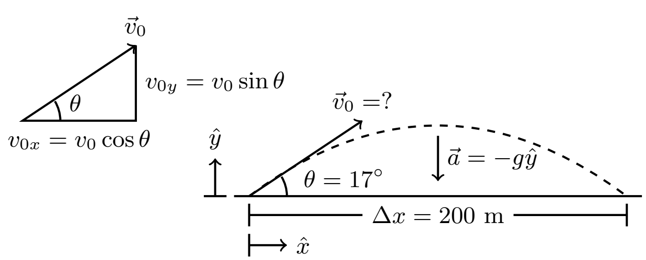
1a) Understand the Problem

- Known:
  - $\Delta x = 200$ m, $\theta = 17°$
  - By assumption: $\vec{a} = -g\hat{y}$, $g \approx 9.8$ m/s^2, $\Delta y = 0$ m
- Unknown:
  - v_0 (initial speed), t_f (time of flight), v_f final speed

1b) Identify Assumptions

- $\Delta y = 0$ m
  - Both the arrow and the target are not going to be at ground level, and archery ranges are typically in flat areas, so for simplicity, it won't be too egregious to assume that the arrow starts and ends at the same height.
- The arrow is not affected by moving through air.
  - An arrow is fast and aerodynamic, and we don't really know how to handle air resistance or the effects of wind, so we should ignore these effects to make the problem solvable.
- Near Earth's surface ($g \approx 9.8$ m/s^2)
  - So far, all known archery occurs on Earth, and the height an arrow flies is not significant enough to worry about variation in gravity.
- $\vec{a} = -g\hat{y}$
  - Since the object is in free-fall with no wind or air effects, and it isn't traveling far enough for the curvature of the Earth to matter, acceleration is straight down and purely gravitational.

1c) Represent Physically



L6-2: Long Distance Archer – Analyze & Represent

You are a long-distance archer who releases an arrow that hits a target 200 m away. Your arrow makes an initial angle with the horizontal of 17°. Following the steps for solving an A*R*C*S problem, find the initial speed of the arrow.

- **A**nalyze and **R**epresent
  - Choose appropriate symbols for the quantities you **know** and **don't know**.
  - What assumptions are you making and why are you making them?
  - Sketch a diagram showing the known and unknown quantities.

L6-2: Long Distance Archer – Calculate

You are a long-distance archer who releases an arrow that hits a target 200 m away. Your arrow makes an initial angle with the horizontal of 17°. Following the steps for solving an A*R*C*S problem, find the initial speed of the arrow.

- **C**alculate: Find a **symbolic expression** for the initial speed of the arrow.
  - *Hint 1:* **Use the symbols you defined before** to help you choose which kinematics equations to use for the x- and y-directions.
  - *Hint 2:* Think about which quantity is the same for the x- and y-directions.
  - Wait until the end to plug in numbers!

2a) Represent Principles

2-D Kinematics: Known Unknown Wanted

$$\Delta x = v_{0x} t_f + \frac{1}{2} a_x {t_f}^2 \qquad \Delta y = v_{0y} t_f + \frac{1}{2} a_y {t_f}^2$$

$$v_{fx} = v_{0x} + a_x t_f \qquad v_{fy} = v_{0y} + a_y t_f$$

$${v_{fx}}^2 = v_{0x}^2 + 2 a_x \Delta x \qquad {v_{fy}}^2 = v_{0y}^2 + 2 a_y \Delta y$$

We can avoid the bottom four equations, since we don't really want to know v_f.

Trigonometry:

$$v_{0x} = v_0 \cos\theta \qquad v_{0y} = v_0 \sin\theta$$

2b) Find Unknown(s) Symbolically

Between $\Delta x = v_{0x} t_f + \frac{1}{2} a_x {t_f}^2$ and $\Delta y = v_{0y} t_f + \frac{1}{2} a_y {t_f}^2$, we have four knowns ($\Delta x$, Δy, a_x, a_y) and three unknowns (t, v_{0x}, v_{0y}). However, v_{0x} and v_{0y} are components of $\vec{v}_0$, and we know θ, so they are sort of a single shared unknown, as is time. Two equations sharing two unknowns can be solved.

$$\begin{aligned} \Delta y &= v_{0y} t_f + \frac{1}{2} a_y {t_f}^2 \\ 0 &= v_{0y} t_f - \frac{1}{2} g t_f^2 \\ &= \left(v_{0y} - \frac{1}{2} g t_f \right) t_f \\ 0 &= v_{0y} - \frac{1}{2} g t_f \\ t_f &= \frac{2 v_{0y}}{g} \end{aligned}$$

$$\begin{aligned} \Delta x &= v_{0x} t_f + \frac{1}{2} a_x {t_f}^2 \\ \Delta x &= v_{0x} \frac{2 v_{0y}}{g} + 0 \\ &= \frac{2 v_0^2}{g} \cos\theta \sin\theta \\ v_0 &= \sqrt{\frac{\Delta x g}{2 \sin\theta \cos\theta}} \\ &= \sqrt{\frac{\Delta x g}{\sin(2\theta)}} \end{aligned}$$

2c) Plug in Numbers

$$v_0 = \sqrt{\frac{(200 \text{ m})(9.8 \text{ m/s}^2)}{\sin(2 \times 17°)}} \approx \sqrt{\frac{19600}{0.559} \frac{\text{m}^2}{\text{s}^2}} \approx 59.2 \text{ m/s}$$

If you assume $g \approx 10$ m/s^2, then $v_0 \approx 59.8$ m/s.

3a) Units

We want $[v_0] = \frac{\text{m}}{\text{s}}$, and we know $[\Delta x] = \text{m}$, $[g] = \frac{\text{m}}{\text{s}^2}$, and $[\sin(2\theta)] = 1$ (unitless), so

$$[v_0] = \sqrt{\frac{[\Delta x][g]}{[\sin(2\theta)]}} = \sqrt{\frac{\text{m} \cdot \text{m/s}^2}{1}} = \sqrt{\frac{\text{m}^2}{\text{s}^2}}$$

Note that I got a bit of a free unit check by plugging in numbers with units in 2c.

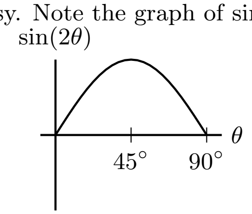
3b) Numbers

Converting into miles per hour, we find that $v_0 \approx 132$ mph. According to a quick internet search, the speed of a recurve bow shot is around 150 mph (around 204 mph for a compound bow), so we are in the right neighborhood. Side Note: In Olympic archery, the target is set 70 m from the archer, so the given $\Delta x = 200$ m is rather large in comparison to modern competition.

3c) Symbols

- Δx increases
  - If the range increases, the arrow must cover a greater distance before it hits the ground, so one would expect it to travel faster.
  - v_0 increases as Δx increases, so our equation matches this prediction.
- g increases
  - If g increases (say we do archery on Jupiter), then the arrow will be dragged to the ground more quickly, so it must go faster to cover the same distance in less time.
  - v_0 increases as g increases, so our equation matches this prediction.

  In both of the above, it is worth noting that increasing v_0 not only helps the arrow cover horizontal distance faster, but also increases its time of flight by giving it more vertical speed.
- θ increases
  - This one isn't as easy. Note the graph of $\sin(2\theta)$:



  - $\sin(2\theta)$ increases for $0° \leq \theta < 45°$, so v_0 decreases with increasing θ for shallower angles.
    - At shallow angles, the arrow gains more vertical speed (and thus time of flight) than it loses horizontal speed as θ increases, so tilting up would increase the range if v_0 did not decrease.
  - $\sin(2\theta)$ decreases for $45° < \theta \leq 90°$, so v_0 increases with increasing θ for steeper angles.
    - At steep angles, the arrow loses more horizontal speed than it gains vertical speed (and thus time of flight) as θ increases, so tilting up would shorten the range if v_0 did not increase.

L6-2: Long Distance Archer – Sensemake

You are a long-distance archer who releases an arrow that hits a target 200 m away. Your arrow makes an initial angle with the horizontal of 17°. Following the steps for solving an A*R*C*S problem, find the initial speed of the arrow.

- **S**ensemake
  - Check the units of your answer.
  - Does your number make sense?
  - Give a physical explanation for how the initial speed of the arrow would change if you increased each variable that your answer depends on.

Main Ideas

- We can use the kinematics equations to solve for any quantity of interest when the acceleration is constant.
- Motion in 2 dimensions can be broken down into independent motion in each dimension.