Studio Week 10
Using Faraday's Law

Picture credit: Wikipedia.com.

Principles for Success

- **Treat everyone with respect.**
- **Learn by doing and questioning.**
- **Everything should make sense.**

Activity 10-1

The square wire loop below is located below a long straight wire carrying current I. Each side of the circuit has length l and the loop is moving upward with constant speed v.

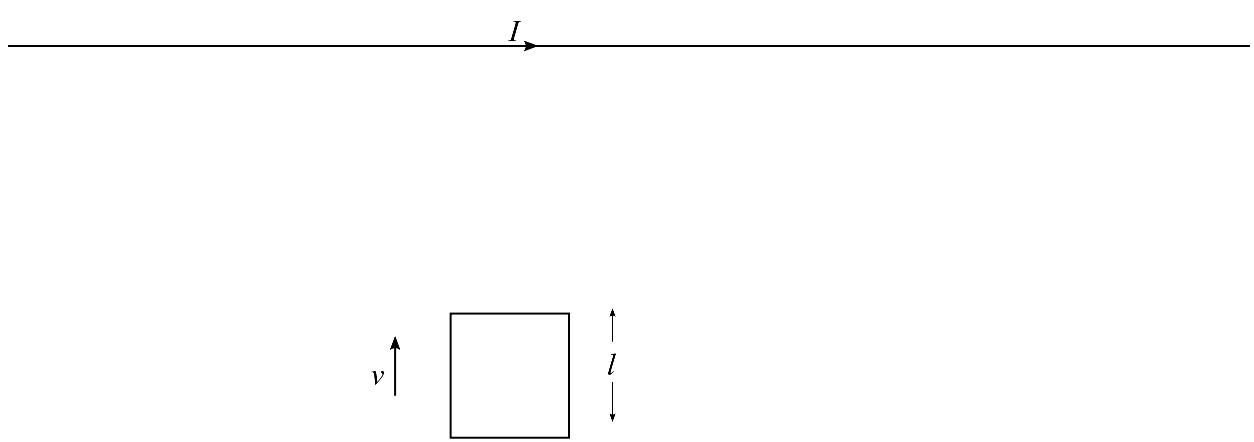
A. Make a table showing each of the following for the wire loop:
   1. The area vector of the loop
   2. The direction of the external magnetic field
   3. The sign of the magnetic flux through the loop
   4. The sign of the **change** in the magnetic flux through the loop
   5. The sign of the induced voltage in the loop
   6. The direction of the current induced in the loop (if any)
   7. The direction of the magnetic moment of the loop

B. Repeat part A (use a new column in the same table) by choosing the area vector to be in the opposite direction.

C. Repeat part A (use a new column in the same table) for the case that the loop is moving downward instead of upward.

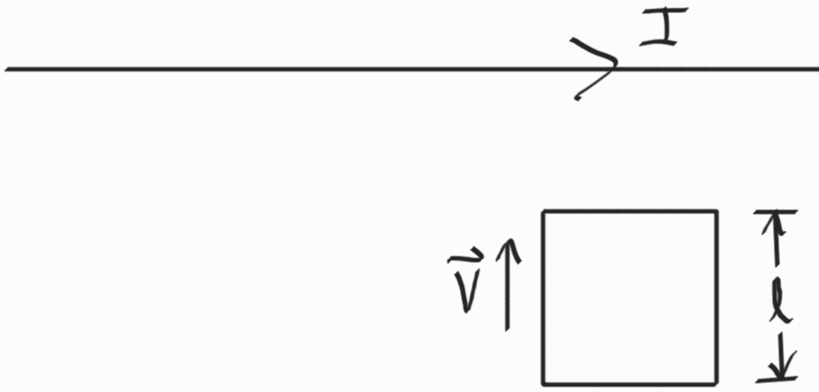
D. Repeat part A (use a new column in the same table) for the case that the loop is moving to the left instead of upward.

E. Repeat part A (use a new column in the same table) for the case that the loop is stationary but the long straight wire is moving upward with speed v.



10-1 Loop & Wire I

$$\vec{B}_{wire} = \frac{\mu_0}{2\pi}\frac{\vec{I}\times\hat{s}}{s}$$



$\vec{I}$, $\hat{s}$ $\Rightarrow \vec{B}_{wire} = \frac{\mu_0 I}{2\pi s} \otimes$

$\vec{B}$ is Into the page below the wire.

	$\boxed{A}$ Moving Up	$\boxed{B}$ Moving Up	$\boxed{C}$ Moving Down ($\vec{v}\downarrow$)	$\boxed{D}$ Moving Left ($\vec{v}\leftarrow$)	$\boxed{E}$ Wire Moving Upward
$\vec{A}$	$\odot$	$\otimes$	$\otimes$	$\otimes$	$\otimes$
$\vec{B}_{ext}$	$\otimes$	$\otimes$	$\otimes$	$\otimes$	$\otimes$
$\iint \vec{B}\cdot d\vec{A} = \Phi_B$	$-$	$+$	$+$	$+$	$+$
$\frac{d\Phi_B}{dt}$	$-$	$+$	$-$	0	$-$
$-\frac{d\Phi_B}{dt} = V_{ind}$	$+$	$-$	$+$	0	$+$
Same sign I_{ind}	$(+)$	$(-)$	$(+)$	0	$(+)$
$I\vec{A} = \vec{\mu}$	$\odot$	$\odot$	$\otimes$	$\vec{0}$	$\otimes$

A, B: $\vec{B}$ is getting Stronger as the loop approaches the wire.

C: $\vec{B}$ is getting weaker as the loop moves away from the wire.

D: $\vec{B}$ does not change as the loop moves parallel to the wire.

E: $\vec{B}$ is getting weaker as the wire moves away from the loop.

Note about $\frac{d\Phi_B}{dt}$

With $\vec{A}$ being $\odot$, opposite to $\vec{B}\otimes$, the flux Φ_B is negative $(-)$. While the wire is moving up, $\vec{B}$ is getting stronger, so Φ_B is becoming a larger negative number. Getting more negative means decreasing, so $\frac{d\Phi_B}{dt}$ is also negative $(-)$.

When we flip the area vector to $\otimes$, Φ_B is positive and getting bigger, so $\frac{d\Phi_B}{dt}$ is positive.

In summary, if the magnitude of the flux $|\Phi_B|$ is increasing, then $\frac{d\Phi_B}{dt}$ has the same sign as Φ_B. If the magnitude is decreasing, then $\frac{d\Phi_B}{dt}$ has the opposite sign.

Note about I_{ind}

$\vec{A}$ sets the direction of positive current. If you position yourself such that the area vector is pointing at you, then positive current goes counterclockwise around it: $\odot\vec{A}$ $I+$

If $\vec{A}$ points away from you, you would see positive current around it as going clockwise: $\otimes\vec{A}$ $I+$

Thus, when the loop moves up, positive current around the $\odot$ area vector is the same as negative current around the $\otimes$ area vector:

$$\underset{I+}{\odot}\vec{A} \equiv \underset{I-}{\otimes}\vec{A}$$

Physical direction of current flow does not depend on the choice of $\vec{A}$.

Note about $\vec{\mu}$

Reversing the area vector will change the sign of the induced current, so $\vec{\mu} = I\vec{A}$ will not change direction: $\vec{\mu} = I_{ind}\vec{A} = +\odot = -\otimes = \odot$,

or $\vec{\mu} = I_{ind}\vec{A} = +\otimes = -\odot = \otimes$.

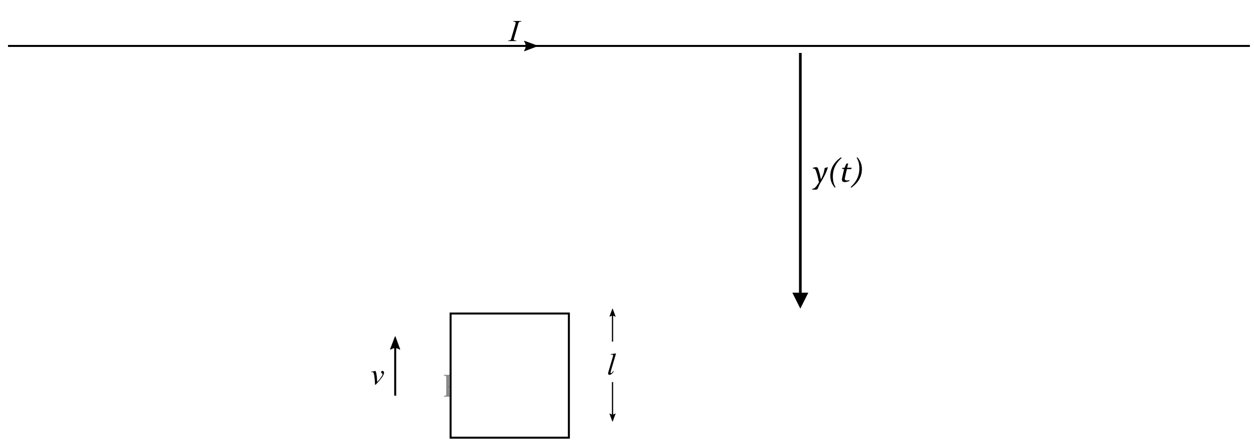
Direction of the magnetic moment does not depend on the choice of $\vec{A}$.

Activity 10-2

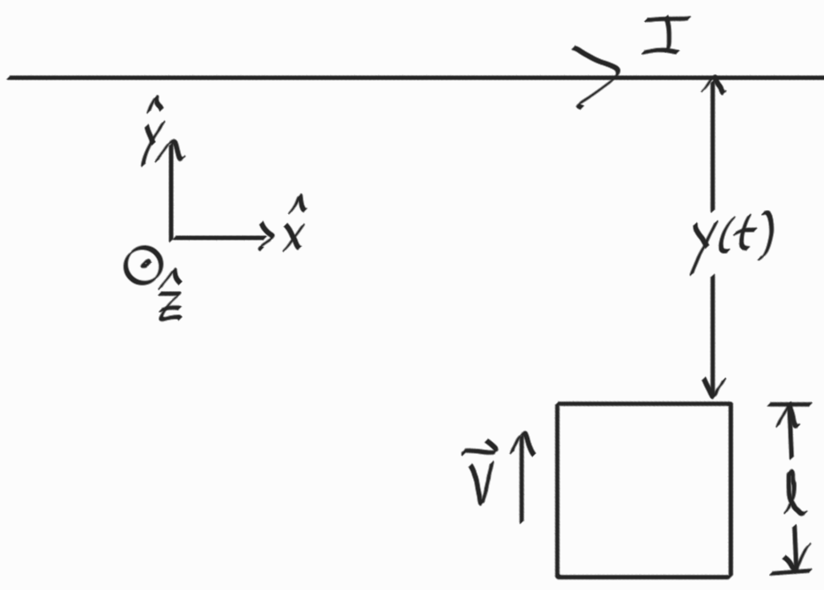
The square wire loop below is located below a long straight wire carrying current *I*. Each side of the circuit has length *l* and the loop is moving upward with constant speed *v*. The loop has resistance *R*.

A. Calculate the current induced in the loop as a function of time.

B. Make sense of your answer like you would for a homework problem.



10-2 Loop & Wire II



$$\vec{B}_{wire} = \frac{\mu_0}{2\pi}\frac{\vec{I}\times\hat{s}}{s}$$

$$\vec{I} = I\hat{x}$$

$$\hat{s} = -\hat{y} \Rightarrow \vec{B}_{wire} = \frac{\mu_0 I}{2\pi s}(-\hat{z})$$

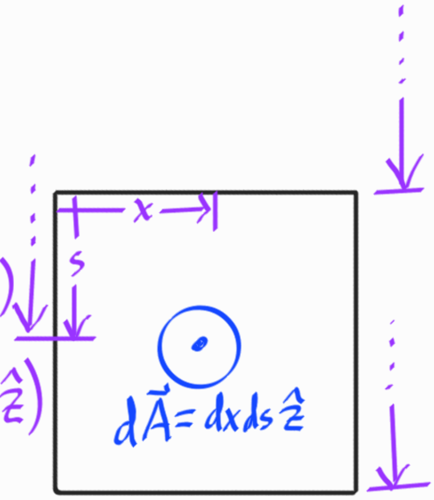
This is the distance of the top of the loop from the wire.

$$\vec{y}(t) = (y_0 - Vt)(-\hat{y})$$

$y_0 > 0$

t never gets large enough for $y_0 - Vt$ to become negative.

This gives us the desired position below the wire.



$$\vec{y}(t) = (y_0 - Vt)(-\hat{y})$$

$$\vec{B} = \frac{\mu_0 I}{2\pi(y_0 - Vt)}(-\hat{z})$$

$$(y_0 - Vt + s)(-\hat{y})$$

$$\vec{B} = \frac{\mu_0 I}{2\pi(y_0 - Vt + s)}(-\hat{z})$$

$$(y_0 - Vt + \ell)(-\hat{y})$$

$$\vec{B} = \frac{\mu_0 I}{2\pi(y_0 - Vt + \ell)}(-\hat{z})$$

$$\Phi_B = \iint \vec{B}\cdot d\vec{A} = \int_0^\ell \int_0^\ell \frac{\mu_0 I}{2\pi(y_0 - Vt + s)}(-\hat{z})\cdot dx\,ds\,\hat{z}$$

$$= -\frac{\mu_0 I}{2\pi}\ell \int_0^\ell \frac{ds}{y_0 - Vt + s} = -\frac{\mu_0 I}{2\pi}\ell\Big[\ln(y_0 - Vt + s)\Big]_{s=0}^{\ell}$$

$$= -\frac{\mu_0 I}{2\pi}\ell \ln\Bigg(\underbrace{\frac{y_0 - Vt + \ell}{y_0 - Vt}}_{\frac{m}{m} = \text{unitless}\,\checkmark}\Bigg)$$

Units: $[\mu_0] = \frac{T\cdot m}{A}$

$$[\Phi_B] = [\mu_0][I][\ell] = T\cdot m\cdot A^{-1}\cdot A\cdot m = Tm^2 \quad \checkmark$$

$$\frac{d\Phi_B}{dt} = -\frac{\mu_0 I}{2\pi} \ell \frac{d}{dt} \ln\left(1+\frac{\ell}{y_0 - vt}\right)$$

$$= -\frac{\mu_0 I}{2\pi} \ell \frac{1}{1+\frac{\ell}{y_0-vt}} \frac{d}{dt}\left[1+\ell(y_0-vt)^{-1}\right] \quad \text{Chain Rule}$$

$$= -\frac{\mu_0 I}{2\pi} \ell \frac{y_0-vt}{y_0-vt+\ell} \frac{-\ell \frac{d}{dt}(y_0-vt)}{(y_0-vt)^2} \quad \text{Chain Rule Again}$$

$$= -\frac{\mu_0 I}{2\pi} \ell \frac{\ell v}{(y_0-vt+\ell)(y_0-vt)}$$

$$= -\frac{\mu_0 I \ell^2 v}{2\pi} \left[\underbrace{(y_0-vt+\ell)(y_0-vt)}_{y_0^2+\ell y_0-(2y_0+\ell)vt+v^2t^2 \text{ if you prefer}}\right]^{-1}$$

$$I_{ind} = \frac{V_{ind}}{R} = -\frac{1}{R}\frac{d\Phi_B}{dt} = \frac{\mu_0 I \ell^2 v}{2\pi R}\left[(y_0-vt+\ell)(y_0-vt)\right]^{-1}$$

Sensemaking

Units: $\mu_0 \approx 4\pi \times 10^{-7}\ \overset{\text{henries}}{H}/m$

$\approx 1.256 \times 10^{-6}\ N \cdot A^{-2}$

$$[I_{ind}] = \frac{[\mu_0][I][\ell^2][v]}{[R][y_0-vt+\ell][y_0-vt]}$$

$$= \frac{(N\cdot A^{\cancel{-2}})\cdot \cancel{A} \cdot \cancel{m^2} \cdot \frac{m}{s}}{\Omega \cdot \cancel{m} \cdot \cancel{m}} = \frac{\cancel{N}\cdot \cancel{A^{-1}} \cdot \frac{\cancel{m}}{s}}{\cancel{N}\cdot \cancel{m}\cdot C^{-1} \cdot \cancel{A^{-1}}} = \frac{C}{s} = A \quad \checkmark$$

$$\Omega = \frac{V}{A} = \frac{J\,C^{-1}}{A} = \frac{Nm}{CA}$$

Covariation:

- I
  - A greater current will mean a greater magnetic field, which will vary more greatly over the same distance, so the flux should change more rapidly as the loop moves, making a greater current
  - $I_{ind} \propto I$ in the equation, so our solution agrees.
- R
  - Higher resistance in a loop means less current for the same voltage, so the induced current should decrease.
  - $I_{ind} \propto \frac{1}{R}$ in the equation, so our solution agrees.
- ℓ
  - Variations in magnetic field over a larger loop will make for larger changes in flux, so increasing ℓ should increase the induced current.
  - We have ℓ^2 in the numerator and ℓ in the denominator, so at the very least I_{ind} will increase as ℓ increases in the limit of large ℓ. To be sure small ℓ doesn't give us a surprise, we can check the derivative of I_{ind} with respect to ℓ.

$$\frac{dI_{ind}}{d\ell} = \frac{\mu_0 I v}{2\pi R} \frac{1}{y_0 - vt} \left(\frac{2\ell}{y_0 - vt + \ell} - \frac{\ell^2}{(y_0 - vt + \ell)^2} \right)$$

Simpler

$\frac{\ell^2}{y_0 - vt + \ell}$ Doubling ℓ will quadruple the numerator and not even double the denominator (as long as $y_0 - vt > 0$), so I_{ind} clearly increases as ℓ does.

$$= \frac{\mu_0 I v}{2\pi R} \frac{1}{y_0 - vt} \left(\frac{2\ell (y_0 - vt + \ell)}{(y_0 - vt + \ell)^2} - \frac{\ell^2}{(y_0 - vt + \ell)^2} \right)$$

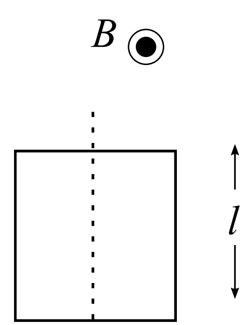
$$= \frac{\mu_0 I v}{2\pi R} \frac{2\ell (y_0 - vt) + \ell^2}{(y_0 - vt + \ell)^2 (y_0 - vt)} > 0 \quad \text{for all } \ell > 0$$

Therefore, increasing ℓ will always increase I_{ind}.

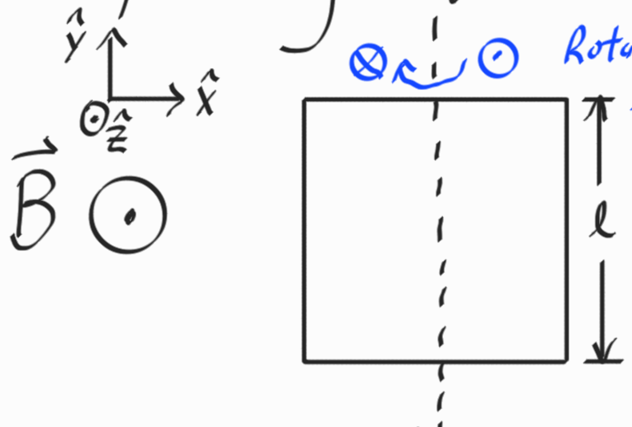
- y_0
  - A larger y_0 means starting farther from the wire, where $\vec{B}$ is weaker and not changing as rapidly with distance. As such, I_{ind} should be smaller.
  - Since $(y_0 - vt + l)(y_0 - vt)$ is in the denominator of our solution, and it gets larger as y_0 increases, our solution for I_{ind} decreases as y_0 increases.
- t
  - As t increases, the loop gets closer to the wire, where $\vec{B}$ is larger and changes more drastically with distance. As such, I_{ind} should be larger.
  - Since $(y_0 - vt + l)(y_0 - vt)$ is in the denominator of our solution, and it gets smaller as t increases, our solution for I_{ind} increases as t increases.
- v
  - Moving more quickly toward the wire means moving through the varying $\vec{B}$ field more rapidly, so increasing v should increase I_{ind}.
  - We have one instance of v in the numerator, so that will increase as v increases. We also have $(y_0 - vt + l)(y_0 - vt)$ in the denominator, which decreases as v increases, so the overall solution for I_{ind} will increase as v increases.

Activity 10-3

- The square wire loop below (resistance R) is located in a region of uniform magnetic field B. Each side of the circuit has length l and the loop is rotating clockwise about a vertical axis (when viewed from above) with constant angular speed ω. (The axis of rotation is shown as a dashed line.) The figure shows the loop at $t = 0$.
  - A. Describe the current through the loop as a function of time *qualitatively*. You may want to use a table like the one in the previous activity to help organize your thoughts.
  - B. How does the angle between the magnetic field and the area vector of the loop change with time? Give your answer in words and as a symbolic expression.
  - C. Write a symbolic expression for the magnetic flux at time t through the loop in terms of the given variables.
  - D. Determine a symbolic expression for the current induced in the loop as a function of time.
  - E. Graph Φ_B and I *vs.* t and confirm that they are related in a way that is consistent with Faraday's Law.



10-3 Spinning Square



Rotates clockwise when viewed from above with angular speed ω.

A) Top Down View ($\hat{x}$, $\hat{z}$)

	$\omega t=0$	$\omega t=\frac{\pi}{2}$	$\omega t=\pi$	$\omega t=\frac{3\pi}{2}$
$\vec{A}$	$\downarrow$	$\leftarrow$	$\uparrow$	$\rightarrow$
$\vec{B}$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
Φ_B	$+$	0	$-$	0
$\frac{d\Phi_B}{dt}$	0	$-$	0	$+$
V_{ind}	0	$+$	0	$-$
I_{ind}	0	$+$	0	$-$

When $\vec{A}$ and $\vec{B}$ are parallel (antiparallel), the flux is at a maximum (minimum), so its derivative is momentarily zero (just like a projectile at maximum height).

When $\vec{A}$ points left (right), the flux is changing from positive to negative (negative to positive), so the derivative is negative (positive).

The current begins at zero, flows counterclockwise around $\vec{A}$ for half of the rotation, becomes zero again for an instant, then flows clockwise for the second half of the rotation before going to zero one more time and starting the rotation over.

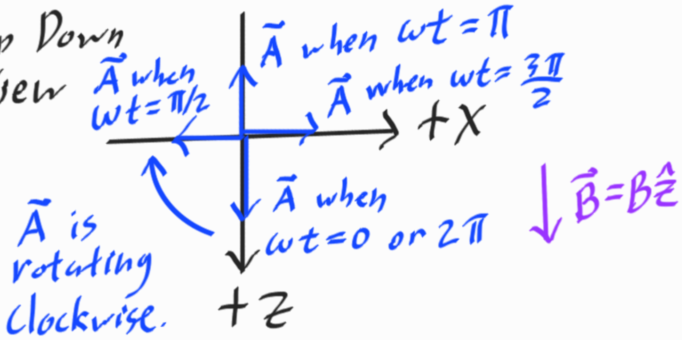
B) The angle between $\vec{A}$ and $\vec{B}$ increases from zero at a constant rate for half of the rotation, then reaches a maximum (180°) when the rotation is half complete and decreases back to zero (which can also be characterized as increasing from 180° to 360°, which gives the same result).

If $\vec{A}$ points out ($\odot = \hat{z}$) at $t=0$, then we can write

$$\vec{A}(t) = A\left(\cos(\omega t)\hat{z} - \sin(\omega t)\hat{x}\right)$$

where $\theta = \omega t$ is the angle between $\vec{A}$ and $\vec{B}$.

Top Down View



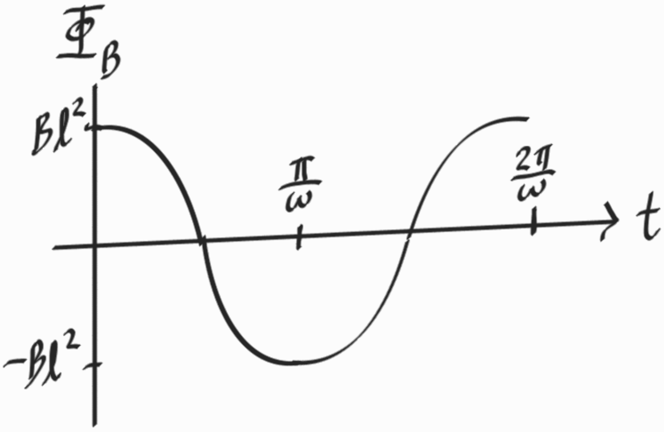
C) $$\Phi_B = \iint \vec{B}\cdot d\vec{A}(t) = \int_0^{\ell}\int_0^{\ell} B\hat{z}\cdot\left(\cos(\omega t)\hat{z} - \sin(\omega t)\hat{x}\right)dx\,dy$$

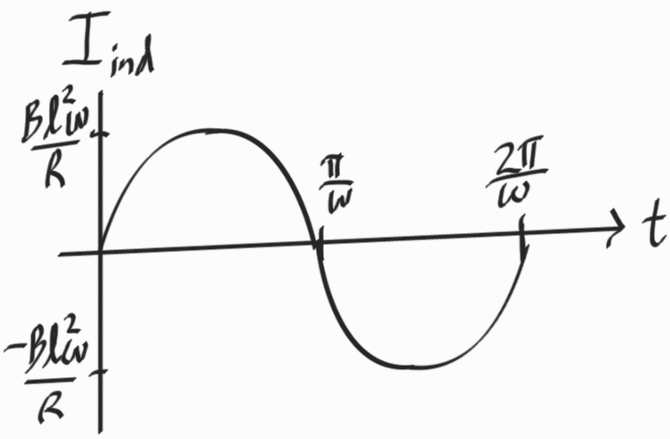
$$= B\cos(\omega t)\int_0^{\ell}\int_0^{\ell} dx\,dy$$

$$= B\ell^2\cos(\omega t)$$

D) $$I_{ind} = \frac{V_{ind}}{R} = -\frac{1}{R}\frac{d\Phi_B}{dt} = \frac{B\ell^2\omega}{R}\sin(\omega t)$$

E)





The I_{ind} graph is a sine graph, while the Φ_B graph is a cosine, and sine is the negative derivative of cosine, so the shapes of these graphs are consistent with the negative derivative relationship of Faraday's law.