Torque Analysis

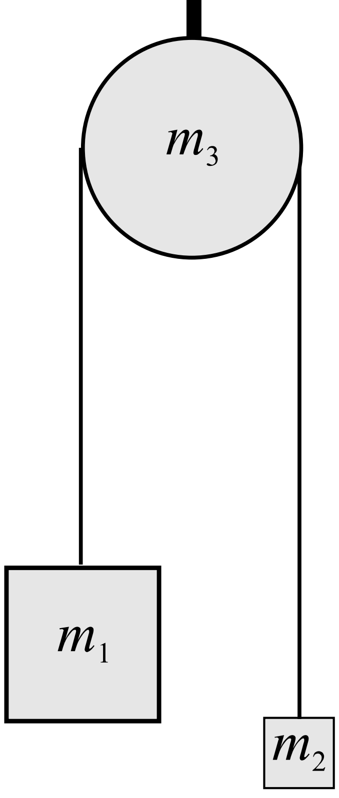
Picture credit: https://albanycarousel.com/thank-you/

Principles for Success

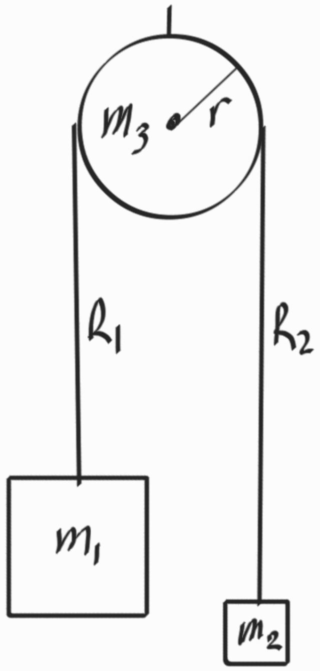
- **Treat everyone with respect.**
- **Learn by doing and questioning.**
- **Everything should make sense.**

Activity 3-1 – Massive Pulley

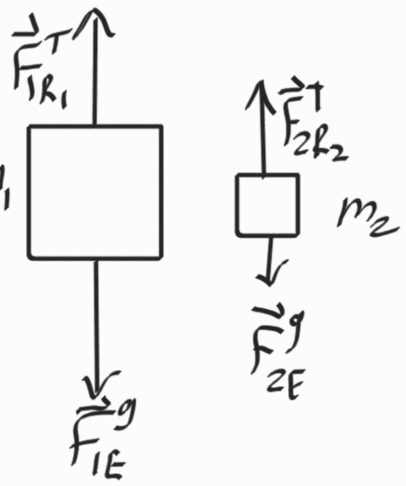
- The pulley shown has mass m_3. Two bricks (mass $m_1 > m_2$) are connected by a rope suspended over the pulley. The masses are released from rest.
  - **Understand and Plan:** Draw free-body diagrams for each object. Discuss your coordinate systems.
  - **Solve the Problem:** Find an expression for the acceleration of each block.
  - **Make Sense of Your Solution:** Use a special-case analysis and the fact that you have likely solved this problem before when the pulley is *massless*.



3-1 Massive Pulley



FBDs



Extended FBD

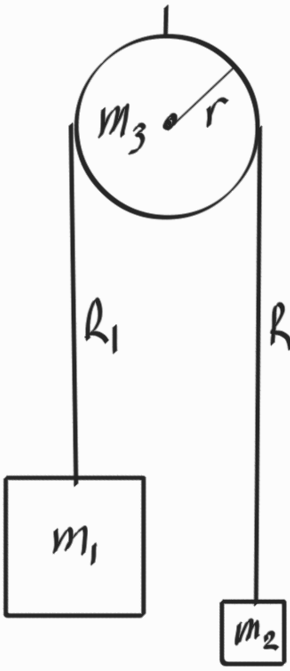
$I = \frac{1}{2} m_3 r^2$

$F_y^{net1} =$

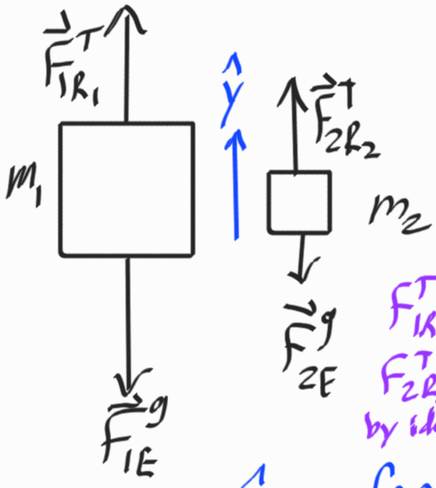
$F_y^{net2} =$

$\tau^{net} =$

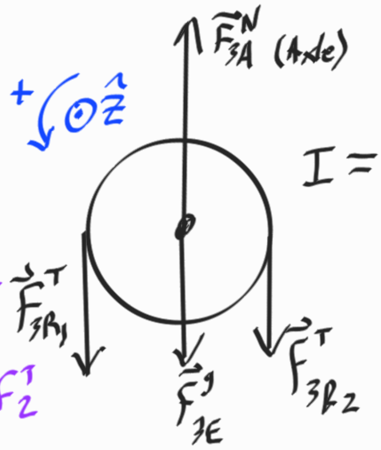
3-1 Massive Pulley



FBDs



Extended FBD



$I = \frac{1}{2} m_3 r^2$

$F^T_{1R_1} = F^T_{3R_1} = F^T_1$
$F^T_{2R_2} = F^T_{3R_2} = F^T_2$
by ideal string

By choosing $\hat{y}$ up for these linked masses, which must move in opposite directions, positive velocity or acceleration for one mass means negative velocity or acceleration for the other.

Choosing the standard coordinate system here means that counter-clockwise rotation is positive. This corresponds to m_1 lowering and m_2 rising.

$$a_1 = -a_t = -\alpha r$$
$$a_2 = a_t = \alpha r$$

$$m_1 a_1 = F_y^{net1} = F^T_{1R_1} - F^g_{1E} = F^T_1 - m_1 g$$

$$m_2 a_2 = F_y^{net2} = F^T_{2R_2} - F^g_{2E} = F^T_2 - m_2 g$$

$$I\alpha = \tau^{net} = r F^T_{3R_1} - r F^T_{3R_2} = r\left(F^T_1 - F^T_2\right)$$

$$\frac{1}{2} m_3 r^{\cancel{2}} \alpha = \cancel{r}\left[m_1(a_1+g) - m_2(a_2+g)\right]$$

$$\frac{1}{2} m_3 a_2 = m_1(-a_2+g) - m_2(a_2+g)$$

$$\left(m_1+m_2+\frac{1}{2}m_3\right) a_2 = (m_1-m_2) g$$

$$\boxed{a_2 = \frac{m_1-m_2}{m_1+m_2+\frac{1}{2}m_3}\, g}$$

$$a_1 = -a_2$$

formally done w/ l'Hôpital's rule

$$\xrightarrow{m_1 \gg m_2, m_3} a_2 \approx \frac{m_1}{m_1} g = g$$

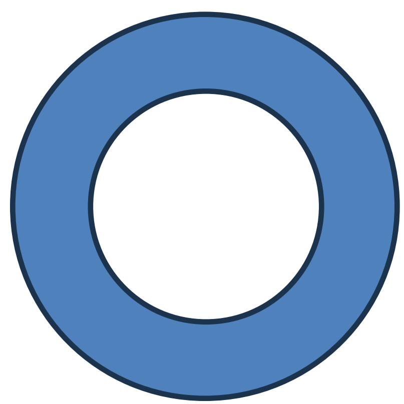
$$\xrightarrow{m_3 \gg m_1, m_2} a_2 \approx \frac{m_1-m_2}{\frac{1}{2}m_3} g \approx 0$$

$$\xrightarrow{m_3 = 0} a_2 = \frac{m_1-m_2}{m_1+m_2} g$$

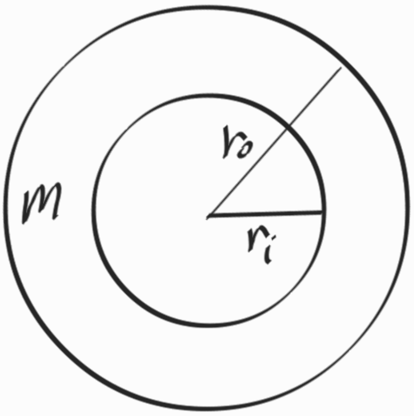
$$\xrightarrow{m_1 = m_2} a_2 = 0 \quad \text{perfect balance}$$

Activity 3-2 – The Ring

- The ring shown at right has mass m, inner radius r_i, and outer radius r_o.
  - Use an integral to find the center of mass of the ring.
  - Use an integral to find the moment of inertia of the ring about an axis passing through its center (perpendicular to the ring).
  - Use the parallel axis theorem to find the moment of inertia of the ring about an axis passing through its outer edge (perpendicular to the ring).
  - Use special-case analysis in the limit that r_i approaches r_o to make sense of your answer.



3-2 The Ring



$\sigma =$

$dm =$

$$\vec{r}_{CM} = \frac{1}{m}\int \vec{r}\, dm$$

$$I_{CM} = \int r_\perp^2\, dm$$

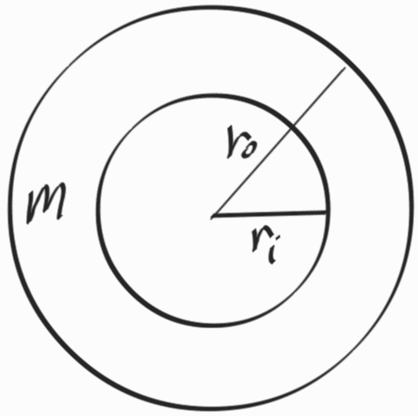
Parallel axis theorem

$$I = I_{CM} + md^2$$

Special Cases

$r_i = r_o$

3-2 The Ring



$$\sigma = \frac{m}{\pi(r_o^2 - r_i^2)}$$

$$dm = \sigma\, r\,dr\,d\theta = \frac{m r\,dr\,d\theta}{\pi(r_o^2 - r_i^2)}$$

$$\vec{r}_{CM} = \frac{1}{m}\int \vec{r}\,dm = \frac{1}{m}\int_0^{2\pi}\int_{r_i}^{r_o} (r\cos\theta\,\hat{x} + r\sin\theta\,\hat{y})\,\sigma\, r\,dr\,d\theta$$

$$= \frac{\sigma}{m}\left[\hat{x}\underbrace{\int_0^{2\pi}\cos\theta\,d\theta}_{0}\int_{r_i}^{r_o} r^2 dr + \hat{y}\underbrace{\int_0^{2\pi}\sin\theta\,d\theta}_{0}\int_{r_i}^{r_o} r^2 dr\right]$$

$$= 0\hat{x} + 0\hat{y}$$

$$I_{CM} = \int r_\perp^2\,dm = \int_0^{2\pi}\int_{r_i}^{r_o} r^2 \sigma\, r\,dr\,d\theta = 2\pi\sigma\int_{r_i}^{r_o} r^3 dr$$

$$= \frac{\pi\sigma}{2}\left[r^4\right]_{r=r_i}^{r_o}$$

$$= \frac{\pi\sigma}{2}(r_o^4 - r_i^4)$$

$$= \frac{m}{2(r_o^2 - r_i^2)}(r_o^4 - r_i^4)$$

$$= \frac{m}{2}(r_o^2 + r_i^2)$$

Parallel axis theorem

$$I = I_{CM} + md^2$$

$$I = I_{CM} + m r_o^2 = \frac{m}{2}(3r_o^2 + r_i^2)$$

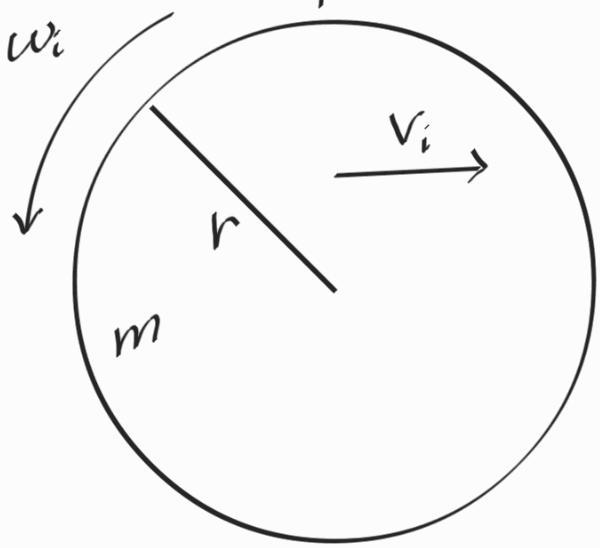
<u>Special Cases</u>

$r_i = r_o \Rightarrow I = m r_o^2$ simple hoop

Activity 3-3 – The Hoop

- A hoop with mass m and radius r is thrown to the right with initial speed v_i and initial angular speed ω_i such that the hoop is spinning backwards (opposite the direction it would need to spin to roll without slipping).
  - After observing the hoop, give a qualitative explanation, including at least one extended free-body diagram, to account for the motion of the hoop.
  - Calculate the distance the hoop slides before it is rolling without slipping.

3-3 The Hoop

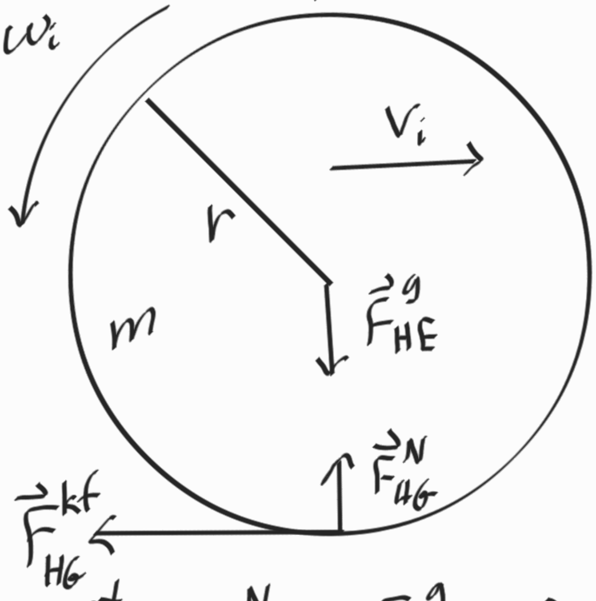


$f_y^{net} =$

$F_x^{net} =$

$\tau^{net} =$

3-3 The Hoop



The friction will slow the hoop's linear and rotational motion down until it stops, still spinning, and then the friction will continue to pull it so that it starts to go backward, eventually ceasing to slip as it rolls.

$$0 = F^{net}_y = F^N_{HG} - F^g_{HE} \Rightarrow F^N_{HG} = mg$$

$$ma = F^{net}_x = -F^{kf}_{HG} = -\mu_k F^N_{HG} = -\mu_k mg \Rightarrow a = -\mu_k g$$

$$\underset{\uparrow\, mr^2}{I}\alpha = \tau^{net} = -rF^{kf}_{HG} = -r\mu_k mg \Rightarrow \underbrace{r\alpha}_{a_t} = -\mu_k g$$

$$v_f = v_i + at = v_i - \mu_k g t$$

$$v_t = -r\omega_f = -r(\omega_i + \alpha t) = -r\omega_i + \mu_k g t$$

$$-r\omega_i + \mu_k g t = v_i - \mu_k g t$$

$$2\mu_k g t = v_i + r\omega_i$$

$$t = \frac{v_i + r\omega_i}{2\mu_k g}$$

$$x_f = v_i t + \frac{a}{2}t^2$$

$$= \frac{v_i^2 + r\omega_i v_i}{2\mu_k g} - \frac{\mu_k g}{2}\left(\frac{v_i + r\omega_i}{2\mu_k g}\right)^2$$

$$= \frac{v_i^2 + r\omega_i v_i}{2\mu_k g} - \frac{v_i^2 + 2r\omega_i v_i + r^2\omega_i^2}{8\mu_k g}$$

$$= \frac{1}{8\mu_k g}\left(3v_i^2 + 2r\omega_i v_i - r^2\omega_i^2\right)$$

$$r\omega_i = v_i \Rightarrow t = \frac{v_i}{\mu_k g} \Rightarrow v_f = 0$$

v & v_t change at same rate, so they can only reach zero simultaneously if they start with the same magnitude.

$v_f > 0$

$r\omega_i < v_i$

$t < \frac{v_i}{\mu_k g}$

$v_f = 0$

$r\omega_i = v_i$

$t = \frac{v_i}{\mu_k g}$

$v_f < 0$

$r\omega_i > v_i$

$t > \frac{v_i}{\mu_k g}$