ps1

October 11, 2020

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# importing libraries
import gurobipy as gpy
import numpy as np
import matplotlib.pyplot as plt
```

Dual problem presented in problem 1 :

minimize $12\pi_1 + 8\pi_2$

subject to :

$x_1 : 4\pi_1 + \pi_2 \geq 1$
$x_2 : \pi_1 + 2\pi_2 \geq 2$
$x_3 : 3\pi_1 - 2\pi_2 \geq -8$

0.0.1 Feasable region plot

Figure 1

0.0.2 Feasable solutions

Figure 2

As we can see in Figure 2, all feasable solutions lie in the feasable region's edge $x_2 : \pi_1 + 2\pi_2 = 2$. Such line segment has vertex $(0, 1)$, which is optimal. It has parametric equations:

$(\pi_1^*, \pi_2^*) = (0, 1) + \lambda$

Optimality condition 1: $\mathbf{x}^*$ is primal feasable as it satisfies the inequalities in the primal problem

Optimality condition 2: π^* is dual feasable as it satisfies the inequalities in the dual problem

Optimality condition 3: Complementary slackness:

$x_3 : 3\pi_1 - 2\pi_2 \geq -8$ always holds

- therefore: $x_3 = 0$

Then we have:

$4x_1 + x_2 = 12$
$x_1 + 2x_2 = 8$
$x_1 \geq 0, x_2 \geq 0$

Solving for x_1 and x_2:

$x_2 = 12 - 4x_1$
$x_1 - 8x_1 + 24 = 8$
$-7x_1 = -16$

Which results in:

$x_1 = \frac{16}{7}$
$x_2 = \frac{20}{7}$

- therefore $\mathbf{x}^* = (\frac{16}{7}, \frac{20}{7}, 0)$

The reduced costs of the primal variables are computed as following:

$\bar{c_1} = 4\pi_1^* + \pi_2^* - 1 = 4\lambda + 1 - \frac{\lambda}{2} - 1 = \frac{7\lambda}{2}$
$\bar{c_2} = \pi_1^* + 2\pi_2^* - 2 = \lambda + 2 - \lambda - 2 = 0$
$\bar{c_3} = 3\pi_1^* - 2\pi_2^* + 8 = 3\lambda - 2 - \lambda + 8 = 2\lambda + 6$

Where π^* is a dual optimal solution - therefore, the reduced costs are not unique

primal problem:

maximize $x_1 + 2x_2 - 8x_3$

subject to :

$4x_1 + x_2 + 3x_3 = 12$
$x_1 + 2x_2 - 2x_3 = 8$
$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$

Let's consider $\pi^* = (\mathbf{0}, \mathbf{1})$ we have $x_3 = 0$ and then:

$4x_1 + x_2 = 12 + \Delta b_1$
$x_1 + 2x_2 = 8 + \Delta b_2$

Solving we get:

$\hat{x}^* = (\frac{1}{7}(2\Delta b_1 - \Delta b_2 + 16), \frac{1}{7}(-\Delta b_1 + 4\Delta b_2 + 20), 0)$

Imposing primal feasability returns:

$2\Delta b_1 - \Delta b_2 \geq -16$
$-\Delta b_1 + 4\Delta b_2 \geq -20$

As a result: if $\Delta b_1 = 0$ then $\Delta b_2 \in [-5, 16]$ and if $\Delta b_2 = 0$ then $\Delta b_1 \in [-8, 20]$

The primal problem only has one feasable solution, so the optimal solution $\mathbf{x}^*$ remains optimal for the following modified problem:

maximize $(1 + \Delta r_1)x_1 + (2 + \Delta r_2)x_2 - (8 + \Delta r_3)x_3$

subject to :

$4x_1 + x_2 + 3x_3 = 12$
$x_1 + 2x_2 - 2x_3 = 8$
$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$

Following a standard analysis, we consider $\widehat{\mathbf{x}}^* = (\frac{16}{7}, \frac{20}{7}, 0)$

By complementary slackness we formulate the following 2 systems of linear equations:

system 1:

$4\pi_1 + \pi_2 = 1 + \Delta r_1$
$\pi_1 + 2\pi_2 = 2 + \Delta r_2$

resulting in:

$\hat{\pi}^* = (\frac{1}{7}(2\Delta r_1 - \Delta r_2), \frac{1}{7}(7 - \Delta r_1 + 4\Delta b_2))$

Dual feasability means that:

$3\hat{\pi}_1^* + 2\hat{\pi}_2^* + 8 - \Delta r_3 = \frac{1}{7}(42 + 8\Delta r_1 - 11\Delta r_2 - 7\Delta r_3)$

and:

$8\Delta r_1 - 11\Delta r_2 - 7\Delta r_3 \geq -42$

So, if:

$\Delta r_1 = 0, \Delta r_2 = 0$, then $\Delta r_3 \leq 6$

$\Delta r_1 = 0, \Delta r_3 = 0$, then $\Delta r_2 \leq \frac{42}{11}$

$\Delta r_2 = 0, \Delta r_3 = 0$, then $\Delta r_1 \geq \frac{-21}{4}$

system 2:

$\pi_1 + 2\pi_2 = 2 + \Delta r_2$
$3\pi_1 - 2\pi_2 = -8 + \Delta r_3$

resulting in:

$\hat{\pi}^* = (\frac{1}{4}(-14 - 3\Delta r_2 + \Delta r_3), \frac{1}{4}(-20 - 2\frac{1}{2}\Delta r_2 + 2\frac{1}{2}\Delta r_3))$

Dual feasability means that:

$4\hat{\pi}_1^* + \hat{\pi}_2^* - 1 - \Delta r_1 = \frac{1}{4}(-98 - 4\Delta r_1 - 11\Delta r_2 + 9\Delta r_3)$

and:

$-4\Delta r_1 - 11\Delta r_2 + 9\Delta r_3 \geq 98$

So, if:

$\Delta r_1 = 0, \Delta r_2 = 0$, then $\Delta r_3 \geq \frac{98}{9}$

$\Delta r_1 = 0, \Delta r_3 = 0$, then $\Delta r_2 \leq \frac{98}{11}$

$\Delta r_2 = 0, \Delta r_3 = 0$, then $\Delta r_1 \leq \frac{-49}{2}$

Combining both cases we obtain that $\mathbf{x}^*$ is optimal for any $\Delta r_j \in \mathbb{R}$

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