Stochastic Processes: Assignment 1

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Importing libraries

Problem 1

```
#> [1] "n bnq ndrsw awornoc aslnxs rzs dowfr ksfrswc ljicrwq rj nirzjwoms n ljwjcngowif gnllocs aworofz
```

Problem 2

(a)

Let $N(t)$ be the number of cars arriving at a parking lot by time t, according to the proposed scenario, we can model $N(t)$ as a non-homogenous Poisson process. Such process has almost the same process as any other Poisson process, however, its rate is a function of time.

$N(t), t \in [0, \infty)$ is the non-homogenous Poisson process with rate $\lambda(t)$ where:

- $N(0) = 0$
- $N(t)$ has independent increments

We define 8:00 as $t = 0$ with the following integrable function and each unit of t equals to 1 hour:

$$\lambda(t) = \begin{cases} 100 & 0 \leq t \leq \frac{1}{2} \\ 600t - 200 & \frac{1}{2} < t \leq \frac{3}{4} \\ 400t - 50 & \frac{3}{4} < t \leq 1 \\ -500t + 850 & 1 < t \leq 1.5 \end{cases}$$

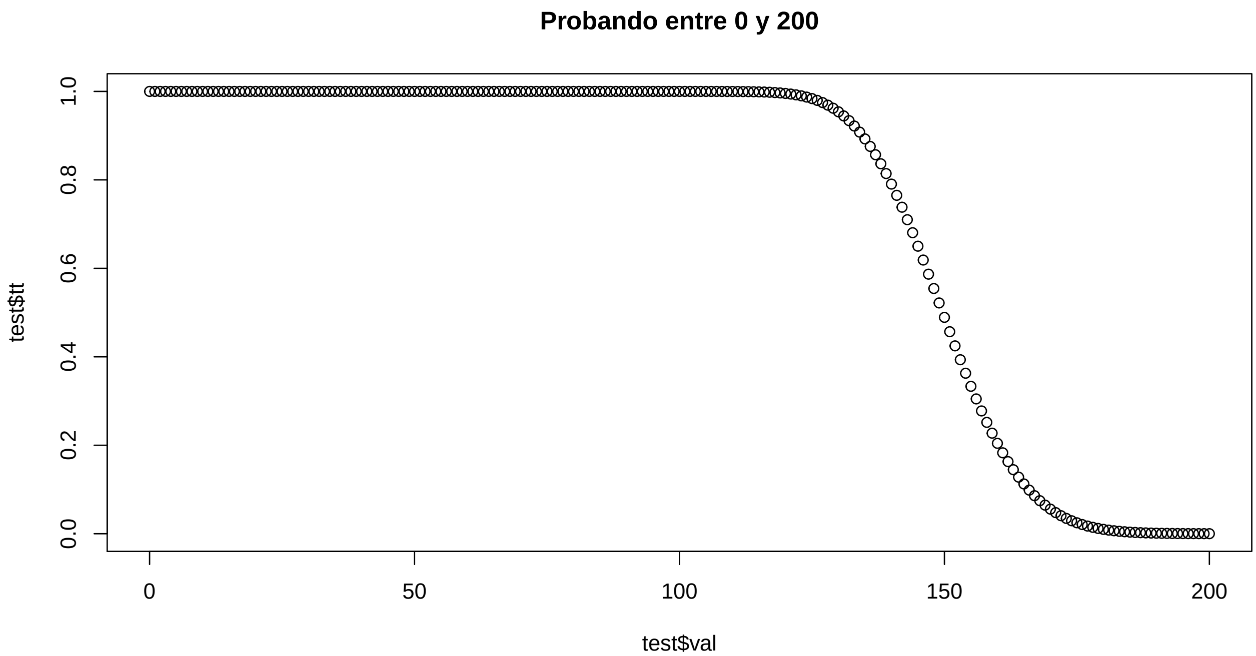
So,

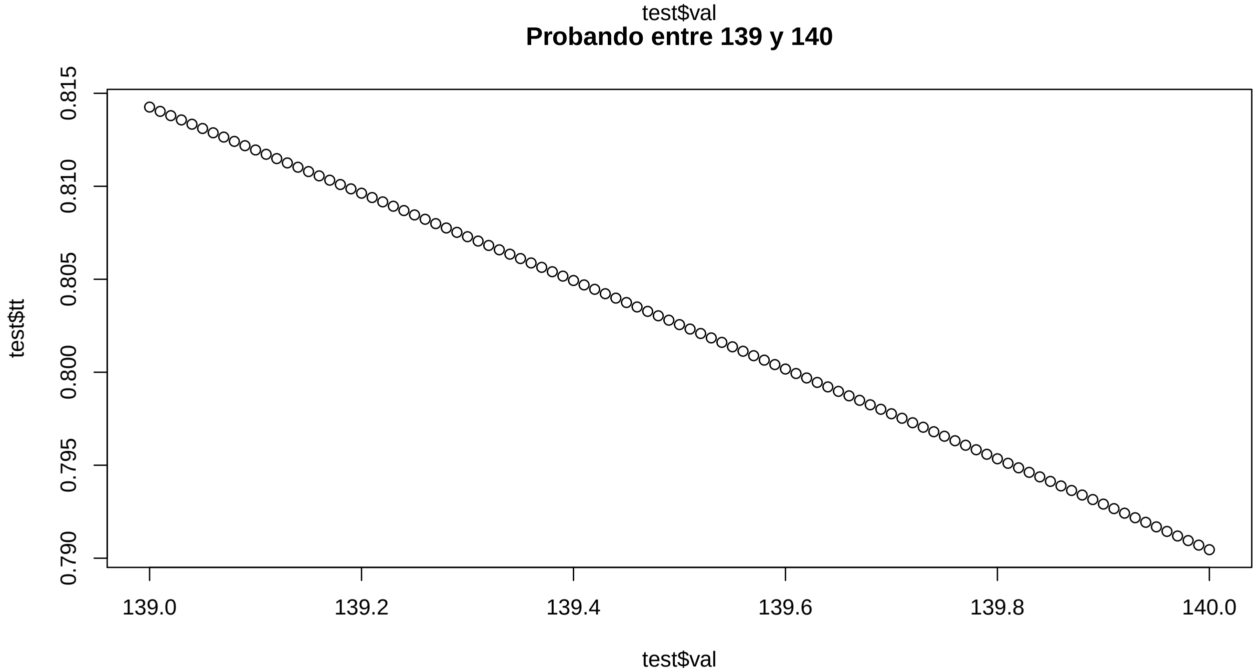
$$E[N(t)] = \begin{cases} \int_0^t 100\, dt = 100t & 0 \leq t \leq \frac{1}{2} \\ \int_{\frac{1}{2}}^t 600t - 200\, dt + 50 = 300(t^2 - \frac{1}{4}) - 200(t - \frac{1}{2}) + 50 & \frac{1}{2} < t \leq \frac{3}{4} \\ \int_{\frac{3}{4}}^t 400t - 50\, dt + 93.75 = 25(8t^2 - 2t - 3) + 93.75 & \frac{3}{4} < t \leq 1 \\ \int_1^t -500t + 850\, dt + 168.75 = -50(5t^2 - 17t + 12) + 168.75 & 1 < t \leq 1.5 \end{cases}$$

Given that there is a limit of 150 vehicles:

$$E[N(t)] = \begin{cases} 100t & 0 \leq t \leq \frac{1}{2} \\ 300(t^2 - \frac{1}{4}) - 200(t - \frac{1}{2}) + 50 & \frac{1}{2} < t \leq \frac{3}{4} \\ 25(8t^2 - 2t - 3) + 93.75 & \frac{3}{4} < t < 0.94468 \\ 150 & t \geq 0.94468 \end{cases}$$

(b)





Luego de hacer las pruebas para $\lambda(t)$ obtenemos lo siguiente:

```
lambda = 139.6
t = 0.91232
# 8:44 AM
```

Por lo que $t = 0.91232$ horas (aproximadamente a las 8:54 de la mañana).

(c)

The following function simulates a non-homogenous poisson process from a homogenous poisson process:

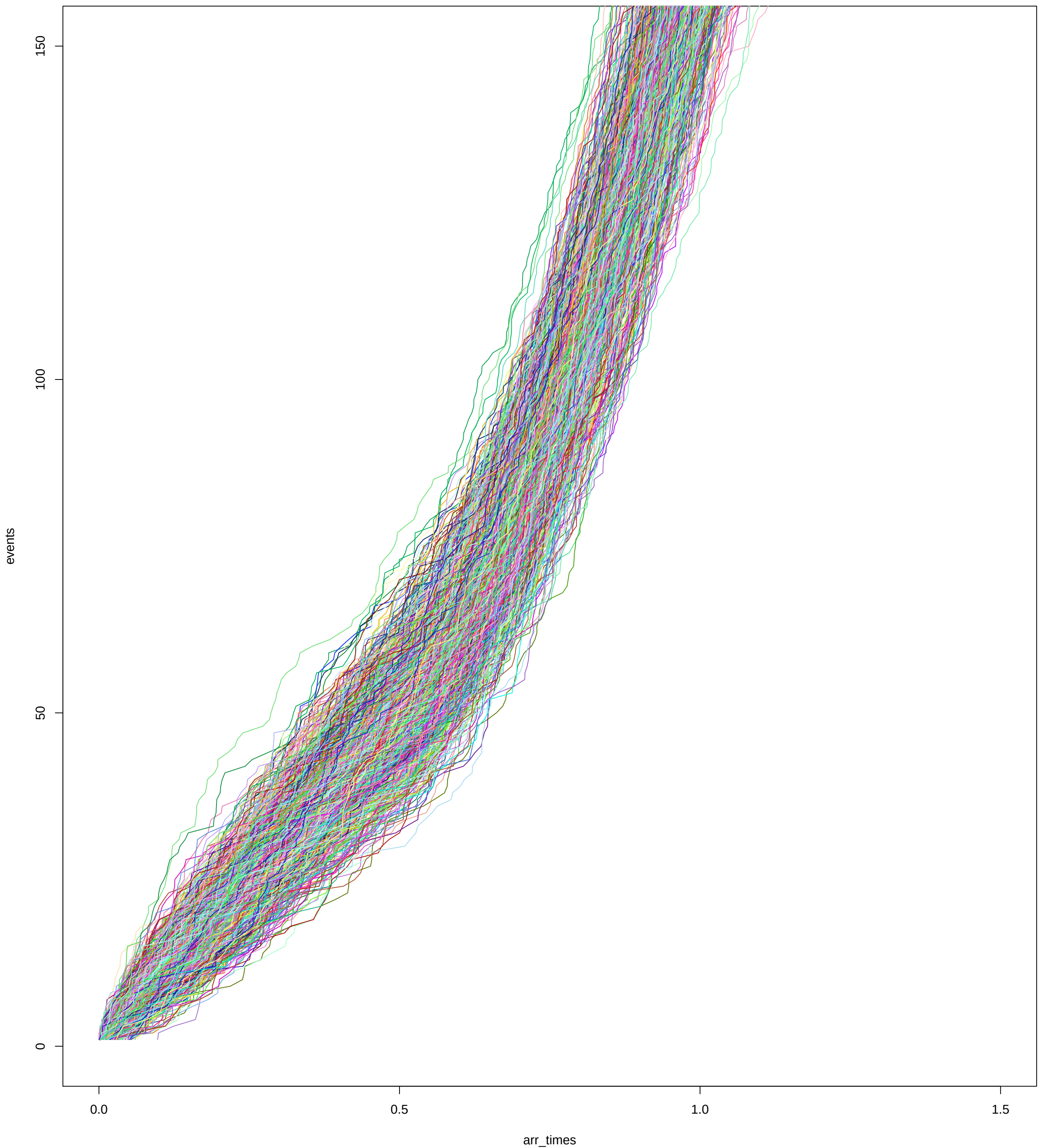
```r
non_hom_poisson <- function(fun,l,a,b,start=0) {
    # This function generates a non-homogenous poisson
    # process from a homogenous poisson process
    # PARAMS:
    # fun:   if the non-homogenous poisson process has
    #        multiple functions per time subinterval
    #        this parameters represents such function
    # l:     lambda for the homogenous poisson process
    # a:     lower bound for the time subinterval
    # b:     upper bound for the time subinterval
    # start: this parameter is used to keep track of
    #        the process count.

    # We generate the homogenous poisson process
    # arrival times
    val <- rpois(1,l*(b-a))
    intervals <- (b-a) * sort(runif(val)) + a

    # Non-homogenous poisson process
    evs <- length(intervals) # lenght of arrival times
    nh_val <- 0 + start # start of the event count
    nh_intervals <- c() # arrival times for the NHPP
    for (i in 1:evs) {
        if (runif(1) < fun(intervals[i])/l) {
            # only including intervals from the HPP which
            # match with fun(intervals[i])/l probability
            nh_intervals <- c(nh_intervals, intervals[i])
            nh_val <- nh_val+1 # adding one to the event count
        }
    }
    nh_events <- seq(1+start,nh_val,1) # events since the previous group
    return(list(arrival_times=nh_intervals, events=nh_events))
}
```

```r
simulation <- function(iters, functions, lambdas, ints) {
    # This function simulates from the NHPP
    # iters:     number of iterations to plot and add to the list of
    #            dataframes
    # functions: list of functions corresponding to the lambda function
    # lambdas:   list of lambdas for each subinterval
    # ints:      lists of vectors of 2 elements each containing the intervals
    #            that correspond to each element of lambdas and functions lists
    p <- list()
    for (i in 1:iters) {
        maximum <- 0 # start for the next NHPP simulation to continue count
        arr_times <- c() # arrival times
        events <- c() # event counts
        for (k in 1:4) {
            int <- non_hom_poisson(lambda_funs[[k]],lambdas[[k]],
                                   ints[[k]][1],ints[[k]][2],
                                   start=maximum)
            maximum <- max(int$events) # remembering last event count
            arr_times <- c(arr_times, int$arrival_times)
            events <- c(events, int$events)
        }
        p[[i]] <- data.frame(arrival_times=arr_times, events=events)
        # plots
        if (i == 1) {plot(arr_times, events, cex=0.5, pch='.',
                         col=randomColor(), xlim=c(0,1.5),
                         ylim=c(0,150))}
        lines(arr_times, events, col=randomColor())
    }
    return(p)
}
```

```r
data <- simulation(1000, lambda_funs, lambdas, ints)
```




```r
ratio <- 0
for (i in 1:length(data)) {
    df <- data.frame(data[[i]])
    cnt <- df %>% filter(arrival_times < 0.91232 & events >= 150) %>% dplyr::count()
    if (cnt[1] >= 1) {
        ratio <- ratio + 1
    }
}
```

```r
ratio/1000
#> [1] 0.2
```

Problem 3

(a)

Our infinitesimal generator is the following:

$$Q = \begin{pmatrix} -\lambda & \lambda & 0 & 0 & \dots \\ \mu & -(\lambda+\mu) & \lambda & 0 & \dots \\ 0 & 2\mu & -(2\mu+\lambda) & \lambda & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

(b)

We solve the following system:

$$\begin{cases} \sum_{i=1}^{\infty} \pi_i = 1 \\ \lambda\pi_0 = \mu\pi_1 \\ \lambda\pi_1 = 2\mu\pi_2 \\ \vdots \\ \lambda\pi_{n-1} = 2\mu\pi_n \\ \vdots \end{cases}$$

First we have:

$\pi_1 = \frac{\lambda\pi_0}{\mu}$
$\pi_2 = \frac{\lambda^2\pi_0}{2\mu^2}$
$\pi_3 = \frac{\lambda^3\pi_0}{2^2\mu^3}$
$\vdots$
$\pi_n = \frac{\lambda^n\pi_0}{2^{n-1}\mu^n}$
$\vdots$

Then:

$\sum_{i=0}^{\infty} \pi_i = \pi_0 + \frac{\lambda\pi_0}{\mu} + \frac{\lambda^2\pi_0}{2\mu^2} + \dots + \frac{\lambda^n\pi_0}{2^{n-1}\mu^n} + \dots = 1$

And so factoring π_0 we get:

$\pi_0(1 + \frac{\lambda}{\mu} + \frac{\lambda^2}{2\mu^2} + \dots) = \pi_0(1 + \sum_{i=1}^{\infty} \frac{1}{2^{i-1}}(\frac{\lambda}{\mu})^i)$

Then multiplying $\frac{2}{2}$ to the summation:

$\pi_0(1 + \frac{2}{2}\sum_{i=1}^{\infty} \frac{1}{2^{i-1}}(\frac{\lambda}{\mu})^i)$

$= \pi_0(2\sum_{i=0}^{\infty}(\frac{\lambda}{2\mu})^i - 1)$

$\pi_0(2(\frac{1}{1-\frac{\lambda}{2\mu}}) - 1) = 1$

$\pi_0 = \frac{1}{2(\frac{1}{1-\frac{\lambda}{2\mu}})-1}$

$\vdots$

$\pi_n = \frac{\lambda^n}{2^{n-1}\mu_n\pi_0}\frac{1}{2(\frac{1}{1-\frac{\lambda}{2\mu}})-1}$

finally:

$\pi_n = \frac{1}{2^{n-1}}(\frac{\lambda}{\mu})^n\pi_0$

The infinite sum converges when $|\frac{\lambda}{2\mu}| < 1$ in which case the stationary distribution P exists.

Then:

$L = \sum_{n=0}^{\infty} \pi_n * n$

Using the following sum:

$\sum_{i=0}^{n-1} ia^i = \frac{a-na^n+(n-1)a^{n+1}}{(1-a)^2}$

As n approaches infinity:

$\sum_{i=0}^{\infty} ia^i = \frac{a}{(1-a)^2}$

We get the following:

$L = \pi_0[1 + \sum_{n=0}^{infty}(\frac{\lambda}{\mu})^n * \frac{n}{2^{n-1}}]$

$L = \pi_0[1 + 2\sum_{n=0}^{infty}(\frac{\lambda}{2\mu})^n * n]$

$L = \pi_0[1 + 2\frac{\frac{\lambda}{2\mu}}{(1-\frac{\lambda}{2\mu})^2}]$

(c)

Let's consider the probabilities conditioned on the number of customers in the system that are present once our specific subject l gets into the system.

If there are no other customers when l gets into the system, there is no chance of overtaking.

$P(N^{OV} = 0|N^{PR} = 0) = 1$

With N^{OV} being the number of customers that l overtakes and N^{PR} the number of customers present in the system (queing) when l gets in the system.

If $N^{PR} = 1$, then l can overtake only 1 customer, if the time it takes to be served is shorter than the time it takes the other customers to be served. Because of the memoryless property we can assert the following:

$P(N^{OV} = 0|N^{PR} = 1) = \frac{\mu}{\mu+\mu} = \frac{1}{2}$

Actually, in general:

$P(N^{OV} = k|N^{PR} = n) = \frac{1}{n+1}$, $n \leq c-1$, $x = 0, 1$

As in this case c=2, our l subject can' t overtake more than one customer.

Now, if $n \geq c$, that is, l has to get in queue and wait to be served. When l gets served, there is also one more customer getting served. Because, again, of the memoryless property.

$P(N^{OV} = k|N^{PR} = n) = \frac{1}{c}$, $n = c$, $k = 0, 1$

In our case, it does not matter how many customers are in the system, the probability of overtaking, conditioned to the number of customers already in the system, is $\frac{1}{2}$.

Now, using Bayes' theorem and the total probability rule, we can find the probability of l overtaking another customer.

$P(A|B) = \frac{P(A \cap B)}{P(B)}$

$\frac{1}{2}\sum_{i=1}^{\infty} \pi_i = \frac{1}{2}\sum_{i=1}^{\infty}(\frac{1}{2^{i-1}})(\frac{\lambda}{\mu})^i \pi_0$

$= \sum_{i=1}^{\infty}(\frac{\lambda}{2\mu})^i \pi_0 = (\sum_{i=0}^{\infty}((\frac{\lambda}{2\mu})^i) - 1)\pi_0$

$= (\frac{1}{1-\frac{\lambda}{2\mu}} - 1)\pi_0 = \frac{1}{2(\frac{1}{1-\frac{\lambda}{2\mu}-1})}(\frac{1}{1-\frac{\lambda}{\mu}} - 1)$

$= \frac{1}{1-\frac{\lambda}{2\mu}} - 1 = \frac{1-(1-\frac{\lambda}{2\mu})}{1-\frac{\lambda}{2\mu}} = \frac{\frac{\lambda}{2\mu}}{1-\frac{\lambda}{2\mu}}$

$= 2(\frac{1}{1-\frac{\lambda}{2\mu}}) - 1 = \frac{2}{1-\frac{\lambda}{2\mu}} - 1 = \frac{2-(1-\frac{\lambda}{2\mu})}{1-\frac{\lambda}{2\mu}}$

$= \frac{1+\frac{\lambda}{2\mu}}{1-\frac{\lambda}{2\mu}}$

Then:

$\frac{\frac{\lambda}{2\mu}}{1\frac{\lambda}{2\mu}} = \frac{\frac{\lambda}{2\mu}}{\frac{2\mu+\lambda}{2\mu}} = \frac{\lambda}{2\mu+\lambda}$

So then we get:

$P(N^{OV} = k) = \frac{\lambda}{2\mu+\lambda}$, $k = c - 1 = 1$