Week 2 exercises

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Importing libraries

```r
library(ggplot2)
library(matlib)
```

We define the following function to calculate matrix powers (thanks profe!):

```r
matrixpower <- function(M,k) {
  # ARGUMENTS:
  # M: square matrix
  # k: exponent
  if(dim(M)[1]!=dim(M)[2]) return(print("Error: matrix M is not square"))
  if (k == 0) return(diag(dim(M)[1]))  # if k=0 returns the identity matrix
  if (k == 1) return(M)
  if (k > 1)  return(M %*% matrixpower(M, k-1)) # if k>1 recursively apply the function
}
```

Exercise 1

```r
P <- c(0.1, 0.4, 0.5,
       0.4, 0.6,   0,
       0.6,   0, 0.4)
P <- matrix(P,nrow=3,byrow=T)
```

First we solve $\pi P = \pi$:

$\pi(P - I) = (1, 0, \dots, 0)$

$$(\pi_1, \pi_2, \pi_3) \begin{pmatrix} 1 & 0.4 & 0.5 \\ 1 & -0.4 & 0 \\ 1 & 0 & -0.6 \end{pmatrix} = (1, 0, 0)$$

Solving the system we get:

$\pi = (\frac{6}{17}, \frac{15}{17}, \frac{10}{17})$

Which is our stationary distribution.

Exercise 2

```
stationary_dist <- function(P) {
    dim = sqrt(length(P))
    mat = matrix(P,nrow=dim, byrow=T)
    A = mat - diag(dim)
    b = c(1,rep(0,dim-1))
    A[,1] <- rep(1,dim)
    print("The system is the following:")
    showEqn(A, b)

    print("The solution is the following:")
    pi <- matlib::Solve(A, b, fractions = TRUE)
    return(pi)
}

P <- c(0.1, 0.4, 0.5,
       0.4, 0.6,   0,
       0.6,   0, 0.4)

stationary_dist(P)
```

```
## [1] "The system is the following:"
## 1*x1 + 0.4*x2 + 0.5*x3  =  1
## 1*x1 - 0.4*x2   + 0*x3  =  0
## 1*x1   + 0*x2 - 0.6*x3  =  0
## [1] "The solution is the following:"
## x1      =   6/17
##   x2    =  15/17
##     x3  =  10/17

## [1] "x1      =   6/17" "  x2    =  15/17" "    x3  =  10/17"
```

Exercise 3

```
P <- c(  0, 1/3,   0, 2/3,
       2/3,   0,   0, 1/3,
         0,   0,   1,   0,
       1/3, 2/3,   0,   0 )
P <- matrix(P,nrow=4,byrow=T)
```

a - Find the communication classes and classify the states.

We have two communication classes

b - Find the set of stationary distributions.

First we solve $\pi P = \pi$:

$\pi(P - I) = (1, 0, \ldots, 0)$

$$\begin{pmatrix}\pi_1 & \pi_2 & \pi_3 & \pi_4\end{pmatrix}\begin{pmatrix}1 & \frac{1}{3} & 0 & \frac{2}{3} \\ 1 & -1 & 0 & \frac{1}{3} \\ 1 & 0 & 0 & 0 \\ 1 & \frac{2}{3} & 0 & -1\end{pmatrix} = (1, 0, 0, 0)$$

Solving the system we get:

$\pi = (0, 0, \pi_3, 0)$

Which is our stationary distribution, where π_3 is a free variable.

c - How is the long-run behavior of the chain? Analyze $\lim_{n\to\infty} P^n$ with R

```
matrixpower(P,50)
```

```
##           [,1]      [,2] [,3]      [,4]
## [1,] 0.3333333 0.3333333    0 0.3333333
## [2,] 0.3333333 0.3333333    0 0.3333333
## [3,] 0.0000000 0.0000000    1 0.0000000
## [4,] 0.3333333 0.3333333    0 0.3333333
```

Given that there's 2 classes in this MC, our long run behavior shows the same values for columns and rows in class 1 (nodes 1,2,4) but not for class 2 (node 3).

d - Does this chain have a limiting distribution?

No, this chain does not have a limiting distribution because $\lim_{n\to\infty} \alpha P^n \neq \lambda$

Exercise 4

$X =$ number of pairs of shoes at front door

$Y =$ number of pairs of shoes at back door

$$\begin{pmatrix} \frac{3}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 \\ 0 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & 0 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 0 & 0 & 0 & \frac{1}{4} & \frac{3}{4} \end{pmatrix}$$