

Week 2 exercises

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Importing libraries

```
library(ggplot2)
library(matlib)
```

We define the following function to calculate matrix powers (thanks profe!):

```
matrixpower <- function(M,k) {
  # ARGUMENTS:
  # M: square matrix
  # k: exponent
  if(dim(M)[1]!=dim(M)[2]) return(print("Error: matrix M is not square"))
  if (k == 0) return(diag(dim(M)[1])) # if k=0 returns the identity matrix
  if (k == 1) return(M)
  if (k > 1) return(M %*% matrixpower(M, k-1)) # if k>1 recursively apply the function
}
```

Exercise 1

```
P <- c(0.1, 0.4, 0.5,
       0.4, 0.6, 0,
       0.6, 0, 0.4)
P <- matrix(P,nrow=3,byrow=T)
```

First we solve $\pi P = \pi$:

$$\pi(P - I) = (1, 0, \dots, 0)$$

$$(\pi_1, \pi_2, \pi_3) \begin{pmatrix} 1 & 0.4 & 0.5 \\ 1 & -0.4 & 0 \\ 1 & 0 & -0.6 \end{pmatrix} = (1, 0, 0)$$

Solving the system we get:

$$\pi = \left(\frac{6}{17}, \frac{15}{17}, \frac{10}{17}\right)$$

Which is our stationary distribution.

Exercise 2

```
stationary_dist <- function(P) {
  dim = sqrt(length(P))
  mat = matrix(P,nrow=dim, byrow=T)
  A = mat - diag(dim)
  b = c(1,rep(0,dim-1))
  A[,1] <- rep(1,dim)
  print("The system is the following:")
  showEqn(A, b)

  print("The solution is the following:")
  pi <- matlib::Solve(A, b, fractions = TRUE)
  return(pi)
}

P <- c(0.1, 0.4, 0.5,
      0.4, 0.6, 0,
      0.6, 0, 0.4)

stationary_dist(P)

## [1] "The system is the following:"
## 1*x1 + 0.4*x2 + 0.5*x3 = 1
## 1*x1 - 0.4*x2 + 0*x3 = 0
## 1*x1 + 0*x2 - 0.6*x3 = 0
## [1] "The solution is the following:"
## x1 = 6/17
## x2 = 15/17
## x3 = 10/17
## [1] "x1 = 6/17" " x2 = 15/17" " x3 = 10/17"
```

Exercise 3

```
P <- c( 0, 1/3, 0, 2/3,
      2/3, 0, 0, 1/3,
      0, 0, 1, 0,
      1/3, 2/3, 0, 0 )
P <- matrix(P,nrow=4,byrow=T)
```

a - Find the communication classes and classify the states.

We have two communication classes

b - Find the set of stationary distributions.

First we solve $\pi P = \pi$:

$$\pi(P - I) = (1, 0, \dots, 0)$$

$$\begin{pmatrix} \pi_1 & \pi_2 & \pi_3 & \pi_4 \end{pmatrix} \begin{pmatrix} 1 & \frac{1}{3} & 0 & \frac{2}{3} \\ 1 & -1 & 0 & \frac{1}{3} \\ 1 & 0 & 0 & 0 \\ 1 & \frac{2}{3} & 0 & -1 \end{pmatrix} = (1, 0, 0, 0)$$

Solving the system we get:

$$\pi = (0, 0, \pi_3, 0)$$

Which is our stationary distribution, where π_3 is a free variable.

c - How is the long-run behavior of the chain? Analyze $\lim_{n \rightarrow \infty} P^n$ with R

```
matrixpower(P,50)
```

```
##           [,1]      [,2] [,3]      [,4]
## [1,] 0.3333333 0.3333333    0 0.3333333
## [2,] 0.3333333 0.3333333    0 0.3333333
## [3,] 0.0000000 0.0000000    1 0.0000000
## [4,] 0.3333333 0.3333333    0 0.3333333
```

d - Does this chain have a limiting distribution?

No, this chain does not have a limiting distribution because $\lim_{n \rightarrow \infty} \alpha P^n \neq \lambda$