SOLUTIONS OF THE EXERCISES FOR "HOW TO PROVE IT" BOOK

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(may contain various errors)

2. Quantificational Logic

2.1. Quantifiers

$$\forall x(\neg L(x, j) \to L(s, x))$$

L(x, j)	L(s, x)	$\neg L(x, j) \to L(s, x)$
F	F	F
F	T	T
T	F	T
T	T	T

Exercises:

1. Analyze the logical forms of the following statements.

    (a) Anyone who has forgiven at least one person is a saint.

    (b) Nobody in the calculus class is smarter than everybody in the discrete math class.

    (c) Everyone likes Mary, except Mary herself.

    (d) Jane saw a police officer, and Roger saw one too.

    (e) Jane saw a police officer, and Roger saw him too.

(a) $\forall x$ (x has forgiven at least one person is a saint)

$\exists y F(x, y)$

F(x, y) stand for "x has forgiven y"

$\forall x(\exists y F(x, y) \to S(x))$

S(x) stand for "x is a saint"

(b) $\neg\exists x[C(x) \land \forall y(D(y) \to S(x,y))]$

S(x, y) is "x is smarter y"

C(x) is "x is in calculus class"

D(x) is "x is in discrete class"

(c) $\forall x(\neg L(m,m) \to L(x,m))$

m is "Mary"

L(x, y) is "x likes y"

(d) $\exists x(P(x) \land S(j,x)) \land \exists y(P(y) \land S(r,y))$

S(x, y) is "x saw y"

j is Jane

r is Roger

P(x) is "x is a police officer"

(e) $\exists x(P(x) \land S(j,x) \land S(r,x))$

S(x, y) is "x saw y"

j is Jane

r is Roger

P(x) is "x is a police officer"

2. Analyze the logical forms of the following statements.

(a) Anyone who has bought a Rolls Royce with cash must have a rich uncle.

(b) If anyone in the dorm has the measles, then everyone who has a friend in the dorm will have to be quarantined.

(c) If nobody failed the test, then everybody who got an A will tutor someone who got a D.

(d) If anyone can do it, Jones can.

(e) If Jones can do it, anyone can

(a) $\forall x(C(x) \rightarrow \exists y(R(y) \wedge U(y, x)))$

C(x) is "x has bought a Rolls Royce with cash"

R(x) is "x is rich"

U(x, y) is "x is uncle of y"

(b) $\exists x[(D(x) \wedge M(x)) \rightarrow \forall y(F(x, y) \wedge Q(y))]$

$\exists x[(D(x) \wedge M(x)) \rightarrow \exists z \forall y(D(z) \wedge F(z, y) \wedge Q(y))]$

M(x) is "x has the measles"

D(x) is "x is in the dorm"

F(x, y) is "x is a friend of y"

Q(x) is "x will have to be quarantined"

(c) $\neg \exists x(F(x) \rightarrow \forall y \exists z(A(y) \wedge D(z) \wedge T(y, z)))$

T(x, y) is "x will tutor y"

F(x) is "x failed the test"

A(x) is "x got an A"

D(x) is "x got a D"

(d) $\exists x(D(x) \rightarrow D(j))$

D(x) is "x can do it"

j is Jones

(e) $\forall x(D(j) \rightarrow D(x))$

D(x) is "x can do it"

j is Jones

3. Analyze the logical forms of the following statements. The universe of discourse is $\mathbb{R}$. What are the free variables in each statement?

(a) Every number that is larger than x is larger than y.

(b) For every number a, the equation $ax^2 + 4x - 2 = 0$ has at least one solution iff $a \geq 2$

(c) All solutions of the inequality $x^3 - 3x < 3$ are smaller than 10.

(d) If there is a number x such that $x^2 + 5x = w$ and there is a number y such that $4 - y^2 = w$, then w is between -10 and 10.

(a) $\forall n(n > x \to n > y)$

x and y are free variables.

(b) $\forall a \exists x(a > 2 \leftrightarrow ax^2 + 4x - 2 = 0)$

no free variables.

(c) $\forall x(x^3 - 3x < 3 \to x < 10)$

no free variables.

(d) $\forall w[(\exists x(x^2 + 5x = w) \land \exists y(4 - y^2 = w) \to (-10 < w < 10)]$

no free variables.

4. Translate the following statements into idiomatic English.

(a) $\forall x[(H(x) \land \neg \exists y M(x, y)) \to U(x)]$, where H(x) means "x is a man," M(x, y) means "x is married to y," and U(x) means "x is unhappy."

(b) $\exists z(P(z, x) \land S(z, y) \land W(y))$, where P(z, x) means "z is a parent of x," S(z, y) means "z and y are siblings," and W(y) means "y is a woman."

(a) All unmarried men are unhappy

5

(b) y is a sister of one of x's parents.

5. Translate the following statements into idiomatic mathematical English.

(a) $\forall x[(P(x) \wedge \neg(x = 2)) \rightarrow O(x)]$, where P(x) means "x is a prime number" and O(x) means "x is odd."

(b) $\exists x[P(x) \wedge \forall y(P(y) \rightarrow y \leq x)]$, where P(x) means "x is a perfect number."

(a) All x which are prime numbers and not equal to 2 should be odd.

(b) There is at least one perfect number x such that all perfect numbers are less or equal to x.

6. Are these statements true or false? The universe of discourse is the set of all people, and P(x, y) means "x is a parent of y."
  (a) $\exists x \forall y P(x, y)$.
  (b) $\forall x \exists y P(x, y)$.
  (c) $\neg \exists x \exists y P(x, y)$.
  (d) $\exists x \neg \exists y P(x, y)$.
  (e) $\exists x \exists y \neg P(x, y)$.

(a) false
(there exists person x such that x is a parent of all people)

(b) false
(all people have child)

(c) false

(nobody has a child)

(d) true

(there exists a person without kids)

(e) true

(there exist person x and person y such that x is not a parent of y)

7. Are these statements true or false? The universe of discourse is $\mathbb{N}$.
   (a) $\forall x \exists y (2x - y = 0)$.
   (b) $\exists y \forall x (2x - y = 0)$.
   (c) $\forall x \exists y (x - 2y = 0)$.
   (d) $\forall x (x < 10 \rightarrow \forall y (y < x \rightarrow y < 9))$.
   (e) $\exists y \exists z (y + z = 100)$.
   (f) $\forall x \exists y (y > x \land \exists z (y + z = 100))$.

(a) true

(b) false

(c) false

(d) true

(e) true

(f) false

8. Same as exercise 7 but with $\mathbb{R}$ as the universe of discourse.

(a) true

(b) false

(c) true

(d) false

(e) true

(f) true

9. Same as exercise 7 but with $\mathbb{Z}$ as the universe of discourse

(a) true

(b) false

(c) false

(d) true

(e) true

(f) true

2.2. Equivalences Involving Quantifiers

Quantifier Negation laws:

$\neg \exists x P(x)$ is equivalent to $\forall x \neg P(x)$

$\neg \forall x P(x)$ is equivalent to $\exists x \neg P(x)$

$\exists! x P(x) = \exists x (P(x) \wedge \neg \exists y (P(y) \wedge x \neq y))$

Abbreviations:

$\exists x \in A P(x) = \exists x (x \in A \wedge P(x))$

$\forall x \in A P(x) = \forall x (x \in A \rightarrow P(x))$

Universal quantifier distibutes over conjunction:

$\forall x (E(x) \wedge T(x)) = \forall x E(x) \wedge \forall x T(x)$

Exercises:

1. Negate these statements and then reexpress the results as equivalent positive statements. (See Example 2.2.1.)

(a) Everyone who is majoring in math has a friend who needs help with his homework.

(b) Everyone has a roommate who dislikes everyone.

(c) $A \cup B \subseteq C \setminus D$.

(d) $\exists x \forall y [y > x \rightarrow \exists z (z^2 + 5z = y)]$.

(a) $\forall x \exists y (M(x) \wedge F(x, y) \wedge H(y) \wedge x \neq y)$

M(x) is "x is majoring in math"

F(x, y) is "x has a friend y"

H(x) is "x needs help with his homework"

Negating:

$\neg\forall x\exists y(M(x) \wedge F(x,y) \wedge H(y) \wedge x \neq y)$

Quantifier negation law: $\exists x\neg\exists y(M(x) \wedge F(x,y) \wedge H(y) \wedge x \neq y)$

Quantifier negation law: $\exists x\forall y\neg(M(x) \wedge F(x,y) \wedge H(y) \wedge x \neq y)$

DeMorgan's law: $\exists x\forall y(\neg(M(x) \wedge F(x,y)) \vee \neg(H(y) \wedge x \neq y))$

Conditional law: $\exists x\forall y((M(x) \wedge F(x,y)) \rightarrow \neg(H(y) \wedge x \neq y))$

Someone who is majoring in math doesn't have friends who are needs help with their homeworks.

(b) $\forall x\exists y(R(x,y) \wedge \forall z(\neg L(y,z)))$

R(x, y) is "x a roommate y"

L(x, y) is "x likes y"

Negating:

$\neg\forall x\exists y(R(x,y) \wedge \forall z(\neg L(y,z))$

Quantifier negation law: $\exists x\neg\exists y(R(x,y) \wedge \forall z(\neg L(y,z)))$

Quantifier negation law: $\exists x\forall y\neg(R(x,y) \wedge \forall z(\neg L(y,z)))$

DeMorgan's law: $\exists x\forall y(\neg R(x,y) \vee \neg\forall z(\neg L(y,z))$

Conditional law: $\exists x\forall y(R(x,y) \rightarrow \neg\forall z(\neg L(y,z)))$

$\exists x\forall y(R(x,y) \rightarrow \exists z(L(y,z)))$

There is someone all of whose roommates like at least one person.

(c) $\forall x(x \in (A \cup B) \rightarrow x \in (C \setminus D))$

$\forall x(\neg(x \in A \vee x \in B) \vee (x \in C \wedge x \notin D))$

Nagating:

$\neg\forall x(\neg(x \in A \vee x \in B) \vee (x \in C \wedge x \notin D))$

Quantifier negation law: $\exists x\neg(\neg(x \in A \vee x \in B) \vee (x \in C \wedge x \notin D))$

DeMorgan's law: $\exists x[(x \in A \vee x \in B) \wedge (x \notin C \vee x \in D)]$

(d) $\exists x\forall y[y > x \rightarrow \exists z(z^2 + 5z = y)]$

Negating:

$\neg \exists x \forall y [y > x \rightarrow \exists z(z^2 + 5z = y)]$

$\forall x \neg \forall y [y > x \rightarrow \exists z(z^2 + 5z = y)]$ (quantifier negation law)

$\forall x \exists y \neg [y > x \rightarrow \exists z(z^2 + 5z = y)]$ (quantifier negation law)

$\forall x \exists y \neg [\neg(y > x) \vee \exists z(z^2 + 5z = y)]$ (conditional law)

$\forall x \exists y [(y > x) \wedge \neg \exists z(z^2 + 5z = y)]$ (DeMorgan's law)

$\forall x \exists y [(y > x) \wedge \forall z \neg(z^2 + 5z = y)]$ (quantifier negation law)

$\forall x \exists y [y > x \wedge \forall z(z^2 + 5z \neq y)]$

2. Negate these statements and then reexpress the results as equivalent positive statements. (See Example 2.2.1.)

(a) There is someone in the freshman class who doesn't have a roommate.

(b) Everyone likes someone, but no one likes everyone.

(c) $\forall a \in A \exists b \in B(a \in C \leftrightarrow b \in C)$.

(d) $\forall y > 0 \exists x(ax^2 + bx + c = y)$.

(a) $\exists x[F(x) \wedge \neg \exists y(R(x, y))]$

F(x) is "x is freshman"

R(x, y) is "x has a roommate y"

Negating:

$\neg \exists x[F(x) \wedge \neg \exists y(R(x, y))]$

$\forall x \neg [F(x) \wedge \neg \exists y(R(x, y))]$ (quantifier negation law)

$\forall x[\neg F(x) \vee \exists y(R(x, y))]$ (DeMorgan's law)

$\forall x[F(x) \rightarrow \exists y(R(x, y))]$ (conditional law)

All freshman have a roommate

(b) $\forall x \exists y(L(x, y)) \wedge \neg \exists z \forall w(L(z, w))$

L(x, y) is "x likes y"

Negating:

$\neg \forall x \exists y(L(x, y)) \vee \neg \neg \exists z \forall w(L(z, w))$

$\exists x \forall y \neg L(x, y) \lor \exists z \forall w L(z, w)$

There exist someone who doesn't like everybody or someone who likes everybody.

(c) $\neg \forall a \in A \exists b \in B(a \in C \leftrightarrow b \in C)$

$\exists a \in A \forall b \in B \neg (a \in C \leftrightarrow b \in C)$

$\exists a \in A \forall b \in B \neg ((a \in C \to b \in C) \land (b \in C \to a \in C))$

$\exists a \in A \forall b \in B(\neg (a \in C \to b \in C) \lor \neg (b \in C \to a \in C))$

$\exists a \in A \forall b \in B(\neg(\neg(a \in C) \lor b \in C) \lor \neg(\neg(b \in C) \lor a \in C))$

$\exists a \in A \forall b \in B((a \in C) \land \neg(b \in C)) \lor ((b \in C) \land \neg(a \in C))$

$\exists a \forall b[(a \in B \land b \in B) \land (a \in C \land b \notin C) \lor (b \in C \land a \notin C)]$

(d) $\neg \forall y > 0 \exists x(ax^2 + bx + c = y)$

$\exists y > 0 \neg \exists x(ax^2 + bx + c = y)$

$\exists y > 0 \forall x \neg(ax^2 + bx + c = y)$

$\exists y > 0 \forall x(ax^2 + bx + c \neq y)$

$\exists y \forall x(y > 0 \land ax^2 + bx + c \neq y)$

3. Are these statements true or false? The universe of discourse is $\mathbb{N}$.
   (a) $\forall x(x < 7 \to \exists a \exists b \exists c(a^2 + b^2 + c^2 = x))$.
   (b) $\exists! x((x - 4)^2 = 9)$.
   (c) $\exists! x((x - 4)^2 = 25)$.
   (d) $\exists x \exists y((x - 4)^2 = 25 \land (y - 4)^2 = 25)$.

(a) true

$0 \le x \le 6$

(b) false

x = 7

12

x = 1

(c) false

x = 9

x = -1

(d) true

4. Show that the second quantifier negation law, which says that $\neg\forall x P(x)$ is equivalent to $\exists x \neg P(x)$, can be derived from the first, which says that $\neg\exists x P(x)$ is equivalent to $\forall x \neg P(x)$. (Hint: Use the double negation law.)

Using $\neg\exists x P(x) = \forall x \neg P(x)$

prove $\neg\forall x P(x) = \exists x \neg P(x)$

$\neg\neg\forall x P(x) = \neg\exists x \neg P(x)$

$\forall x P(x) = \forall x \neg\neg P(x)$

$\forall x P(x) = \forall x P(x)$

or

$\neg\exists x \neg P(x) = \forall x \neg\neg P(x)$

$\neg\exists x \neg P(x) = \forall x P(x)$

$\exists x \neg P(x) = \neg\forall x P(x)$

5. Show that $\neg\exists x \in A P(x)$ is equivalent to $\forall x \in A \neg P(x)$.

$\neg\exists x \in A P(x)$

$\neg\exists x (x \in A \wedge P(x))$ (expanding abbreviation)

$\forall x \neg(x \in A \wedge P(x))$ (quantifier negation law)

$\forall x (\neg x \in A \vee \neg P(x))$ (DeMorgan's law)

13

$\forall x(x \in A \rightarrow \neg P(x))$ (conditional law)

$\forall x \in A \neg P(x)$ (abbreviation)

6. Show that the existential quantifier distributes over disjunction. In other words, show that $\exists x(P(x) \vee Q(x))$ is equivalent to $\exists x P(x) \vee \exists x Q(x)$. (Hint: Use the fact, discussed in this section, that the universal quantifier distributes over conjunction.)

$\exists x(P(x) \vee Q(x))$

$\neg \forall x \neg(P(x) \vee Q(x))$

$\neg \forall x(\neg P(x) \wedge \neg Q(x))$

$\neg(\forall x \neg P(x) \wedge \forall x \neg Q(x))$

$\neg \forall x \neg P(x) \vee \neg \forall x \neg Q(x)$

$\exists x \neg \neg P(x) \vee \exists x \neg \neg Q(x)$

$\exists x P(x) \vee \exists x Q(x)$

7. Show that $\exists x(P(x) \rightarrow Q(x))$ is equivalent to $\forall x P(x) \rightarrow \exists x Q(x)$.

$\exists x(P(x) \rightarrow Q(x))$

$\neg \neg \exists x(P(x) \rightarrow Q(x))$ (double negation law)

$\neg \forall x \neg(P(x) \rightarrow Q(x))$ (quantifier negation law)

$\neg \forall x \neg(\neg P(x) \vee Q(x))$ (conditional law)

$\neg \forall x(P(x) \wedge \neg Q(x))$ (DeMorgan's law)

$\neg(\forall x P(x) \wedge \forall x \neg Q(x))$ (universal quantifier distribution law)

$\neg \forall x P(x) \vee \neg \forall x \neg Q(x)$ (DeMorgan's law)

$\neg \forall x P(x) \vee \exists x \neg \neg Q(x)$ (quantifier negation law)

$\neg \forall x P(x) \vee \exists x Q(x)$ (double negation law)

$\forall x P(x) \rightarrow \exists x Q(x)$ (conditional law)

8. Show that $(\forall x \in A\, P(x)) \wedge (\forall x \in B\, P(x))$ is equivalent to $\forall x \in (A \cup B)\, P(x)$. (Hint: Start by writing out the meanings of the bounded quantifiers in terms of unbounded quantifiers.)

$(\forall x (x \in A \rightarrow P(x))) \wedge (\forall x (x \in B \rightarrow P(x)))$ (expanding abbreviation)

$\forall x ((x \in A \rightarrow P(x)) \wedge (x \in B \rightarrow P(x)))$ (universal quantifiers distribution law)

$\forall x ((\neg (x \in A) \vee P(x)) \wedge (\neg (x \in B) \vee P(x)))$ (conditional law)

$\forall x ((\neg (x \in A) \wedge \neg (x \in B)) \vee P(x))$ (distribution law)

$\forall x (\neg (x \in A \vee x \in B) \vee P(x))$ (DeMorgan's law)

$\forall x ((x \in A \vee x \in B) \rightarrow P(x))$ (conditional law)

$\forall x (x \in (A \cup B) \rightarrow P(x))$

$\forall x \in (A \cup B)\, P(x)$

9. Is $\forall x (P(x) \vee Q(x))$ equivalent to $\forall x P(x) \vee \forall x Q(x)$? Explain. (Hint: Try assigning meanings to P(x) and Q(x).)

$\forall x (P(x) \vee Q(x))$

P(x) "x is tall"

Q(x) "x is smart"

All people are either tall or smart.

$\forall x P(x) \vee \forall x Q(x)$

All people are tall or all people are smart.

$\forall x (P(x) \wedge Q(x)) \neq \forall x P(x) \vee \forall x Q(x)$

10. (a) Show that $\exists x \in A\, P(x) \vee \exists x \in B\, P(x)$ is equivalent to $\exists x \in (A \cup B)\, P(x)$.

(b) Is $\exists x \in A\, P(x) \wedge \exists x \in B\, P(x)$ equivalent to $\exists x \in (A \cap B)\, P(x)$? Explain.

(a) $\exists x \in A\, P(x) \vee \exists x \in B\, P(x)$

$\exists x[(x \in A) \wedge P(x)) \vee \exists((x \in B) \wedge P(x)]$

$\exists x[((x \in A) \wedge P(x)) \vee ((x \in B) \wedge P(x))]$

$\exists x[((x \in A) \vee (x \in B)) \wedge P(x)]$

$\exists x(x \in (A \cup B) \wedge P(x))$

$\exists x \in (A \cup B)\, P(x)$

(b) $\exists x \in A\, P(x) \wedge \exists x \in B\, P(x)$

$\exists x((x \in A) \wedge P(x)) \wedge \exists x((x \in B) \wedge P(x))$

$\exists x \in (A \cap B) P(x)$

$\exists x(x \in (A \cap B) \wedge P(x))$

$\exists x((x \in A) \wedge (x \in B) \wedge P(x))$

A is {1,2,3}

B is {4,5,7}

P(x) is "$x < 6$"

$\exists x((x \in A) \wedge P(x)) \wedge \exists x((x \in B) \wedge P(x))$

$\exists x((x \in \{1,2,3\}) \wedge x < 6) \wedge \exists x((x \in \{4,5,7\}) \wedge x < 6)$

$true \wedge true = true$

$\exists x((x \in \{1,2,3\} \wedge x \in \{4,5,7\} \wedge x < 6))$

$\exists x(x \in \varnothing \wedge x < 6)$ is false

So $\exists x \in A\, P(x) \wedge \exists x \in B\, P(x)$ is not equivalent to $\exists x \in (A \cap B)\, P(x)$.

11. Show that the statements $A \subseteq B$ and $A \setminus B = \varnothing$ are equivalent by

16

writing each in logical symbols and then showing that the resulting formulas are equivalent.

$A \subseteq B$ means $\forall x(x \in A \to x \in B)$

$A \setminus B = \varnothing$ means $\neg \exists x(x \in A \land x \notin B)$

$\forall x \neg(x \in A \land x \notin B)$

$\forall x(\neg x \in A \lor x \in B)$

$\forall x(x \in A \to x \in B)$

$A \subseteq B$

12. Let T(x, y) mean "x is a teacher of y." What do the following statements mean? Under what circumstances would each one be true? Are any of them equivalent to each other?

(a) $\exists! y T(x, y)$.

(b) $\exists x \exists! y T(x, y)$.

(c) $\exists! x \exists y T(x, y)$.

(d) $\exists y \exists! x T(x, y)$.

(e) $\exists! x \exists! y T(x, y)$.

(f) $\exists x \exists y[T(x, y) \land \neg \exists u \exists v(T(u, v) \land (u \neq x \lor v \neq y))]$.

(a) There is exactly one student who has x teacher.

Statement will be true if we choose instead of x a teacher who has exactly one student.

(b) There is at least one teacher who has exactly one student.

Statement will be true if universe of discourse has at least one teacher who has exactly one student.

(c) There is exactly one teacher who has at least one student.

Statement will be true if universe of discourse has exactly one teacher who has at least one student.

(d) There is exactly one teacher who has at least one student

$\exists y \exists! x T(x,y)$ is equivalent to $\exists! x \exists! y T(x,y)$

Statement will be true if universe of discourse has exactly one teacher who has at least one student.

(e) There is exactly one teacher who has exactly one student.

Statement will be true if universe of discourse has exactly one teacher who has exactly one student.

(f) There is exactly one teacher who has exactly one student.

$\exists x \exists y [T(x,y) \land \neg \exists u \exists v (T(u,v) \land (u \neq x \lor v \neq y))]$ is equivalent to

$\exists! x \exists! y T(x,y)$

Statement will be true if universe of discourse has exactly one teacher who has exactly one student.

(c) equivalent to (d)

(e) equivalent to (f)

2.3. More Operations on Sets