Solutions of the exercises for "How to prove it" book

by drets

(may contain various errors)

2. Quantificational Logic

2.1. Quantifiers

$$\forall x(\neg L(x, j) \rightarrow L(s, x))$$

L(x, j)	L(s, x)	$\neg L(x, j) \rightarrow L(s, x)$
F	F	F
F	T	T
T	F	T
T	T	T

Exercises:

1. Analyze the logical forms of the following statements.

(a) Anyone who has forgiven at least one person is a saint.

(b) Nobody in the calculus class is smarter than everybody in the discrete math class.

(c) Everyone likes Mary, except Mary herself.

(d) Jane saw a police officer, and Roger saw one too.

(e) Jane saw a police officer, and Roger saw him too.

(a) $\forall x$ (x has forgiven at least one person is a saint)

$\exists y F(x, y)$

F(x, y) stand for "x has forgiven y"

$\forall x(\exists y F(x, y) \rightarrow S(x))$

S(x) stand for ”x is a saint”

(b) $\neg\exists x[C(x) \wedge \forall y(D(y) \rightarrow S(x, y))]$

S(x, y) is ”x is smarter y”

C(x) is ”x is in calculus class”

D(x) is ”x is in discrete class”

(c) $\forall x(\neg L(m, m) \rightarrow L(x, m))$

m is ”Mary”

L(x, y) is ”x likes y”

(d) $\exists x(P(x) \wedge S(j, x)) \wedge \exists y(P(y) \wedge S(r, y))$

S(x, y) is ”x saw y”

j is Jane

r is Roger

P(x) is ”x is a police officer”

(e) $\exists x(P(x) \wedge S(j, x) \wedge S(r, x))$

S(x, y) is ”x saw y”

j is Jane

r is Roger

P(x) is ”x is a police officer”

2. Analyze the logical forms of the following statements.

(a) Anyone who has bought a Rolls Royce with cash must have a rich uncle.

(b) If anyone in the dorm has the measles, then everyone who has a friend in the dorm will have to be quarantined.

(c) If nobody failed the test, then everybody who got an A will tutor someone who got a D.

(d) If anyone can do it, Jones can.

(e) If Jones can do it, anyone can

(a) $\forall x(C(x) \rightarrow \exists y(R(y) \wedge U(y,x)))$

C(x) is "x has bought a Rolls Royce with cash"

R(x) is "x is rich"

U(x, y) is "x is uncle of y"

(b) $\exists x[(D(x) \wedge M(x)) \rightarrow \forall y(F(x,y) \wedge Q(y))]$

$\exists x[(D(x) \wedge M(x)) \rightarrow \exists z \forall y(D(z) \wedge F(z,y) \wedge Q(y))]$

M(x) is "x has the measles"

D(x) is "x is in the dorm"

F(x, y) is "x is a friend of y"

Q(x) is "x will have to be quarantined"

(c) $\neg \exists x(F(x) \rightarrow \forall y \exists z(A(y) \wedge D(z) \wedge T(y,z)))$

T(x, y) is "x will tutor y"

F(x) is "x failed the test"

A(x) is "x got an A"

D(x) is "x got a D"

(d) $\exists x(D(x) \rightarrow D(j))$

D(x) is "x can do it"

j is Jones

(e) $\forall x(D(j) \rightarrow D(x))$

D(x) is "x can do it"

j is Jones

3. Analyze the logical forms of the following statements. The universe of discourse is $\mathbb{R}$. What are the free variables in each statement?

(a) Every number that is larger than x is larger than y.

(b) For every number a, the equation $ax^2 + 4x - 2 = 0$ has at least one solution iff $a \geq 2$

(c) All solutions of the inequality $x^3 - 3x < 3$ are smaller than 10.

(d) If there is a number x such that $x^2 + 5x = w$ and there is a number y such that $4 - y^2 = w$, then w is between -10 and 10.

(a) $\forall n(n > x \rightarrow n > y)$

x and y are free variables.

(b) $\forall a \exists x(a > 2 \leftrightarrow ax^2 + 4x - 2 = 0)$

no free variables.

(c) $\forall x(x^3 - 3x < 3 \rightarrow x < 10)$

no free variables.

(d) $\forall w[(\exists x(x^2 + 5x = w) \wedge \exists y(4 - y^2 = w) \rightarrow (-10 < w < 10)]$

no free variables.

4. Translate the following statements into idiomatic English.

(a) $\forall x[(H(x) \wedge \neg \exists y M(x, y)) \rightarrow U(x)]$, where H(x) means "x is a man," M(x, y) means "x is married to y," and U(x) means "x is unhappy."

(b) $\exists z(P(z, x) \wedge S(z, y) \wedge W(y))$, where P(z, x) means "z is a parent of x," S(z, y) means "z and y are siblings," and W(y) means "y is a woman."

(a) All unmarried men are unhappy

(b) y is a sister of one of x's parents.

5. Translate the following statements into idiomatic mathematical English.

(a) $\forall x[(P(x) \land \neg(x = 2)) \rightarrow O(x)]$, where P(x) means "x is a prime number" and O(x) means "x is odd."

(b) $\exists x[P(x) \land \forall y(P(y) \rightarrow y \leq x)]$, where P(x) means "x is a perfect number."

(a) All x which are prime numbers and not equal to 2 should be odd.

(b) There is at least one perfect number x such that all perfect numbers are less or equal to x.

6. Are these statements true or false? The universe of discourse is the set of all people, and P(x, y) means "x is a parent of y."

(a) $\exists x \forall y P(x, y)$.

(b) $\forall x \exists y P(x, y)$.

(c) $\neg \exists x \exists y P(x, y)$.

(d) $\exists x \neg \exists y P(x, y)$.

(e) $\exists x \exists y \neg P(x, y)$.

(a) false

(there exists person x such that x is a parent of all people)

(b) false

(all people have child)

(c) false

(nobody has a child)

(d) true

(there exists a person without kids)

(e) true

(there exist person x and person y such that x is not a parent of y)

7. Are these statements true or false? The universe of discourse is $\mathbb{N}$.
   (a) $\forall x \exists y (2x - y = 0)$.
   (b) $\exists y \forall x (2x - y = 0)$.
   (c) $\forall x \exists y (x - 2y = 0)$.
   (d) $\forall x (x < 10 \to \forall y (y < x \to y < 9))$.
   (e) $\exists y \exists z (y + z = 100)$.
   (f) $\forall x \exists y (y > x \land \exists z (y + z = 100))$.

(a) true

(b) false

(c) false

(d) true

(e) true

(f) false

8. Same as exercise 7 but with $\mathbb{R}$ as the universe of discourse.

(a) true

(b) false

(c) true

(d) false

(e) true

(f) true

9. Same as exercise 7 but with $\mathbb{Z}$ as the universe of discourse

(a) true

(b) false

(c) false

(d) true

(e) true

(f) true

2.2. Equivalences Involving Quantifiers

Quantifier Negation laws:

$\neg\exists x P(x)$ is equivalent to $\forall x \neg P(x)$

$\neg\forall x P(x)$ is equivalent to $\exists x \neg P(x)$

$\exists! x P(x) = \exists x(P(x) \wedge \neg\exists y(P(y) \wedge x \neq y))$

Abbreviations:

$\exists x \in A P(x) = \exists x(x \in A \wedge P(x))$

$\forall x \in A P(x) = \forall x(x \in A \to P(x))$

Universal quantifier distibutes over conjunction:

$\forall x(E(x) \wedge T(x)) = \forall x E(x) \wedge \forall x T(x)$

Exercises:

1. Negate these statements and then reexpress the results as equivalent positive statements. (See Example 2.2.1.)

(a) Everyone who is majoring in math has a friend who needs help with his homework.

(b) Everyone has a roommate who dislikes everyone.

(c) $A \cup B \subseteq C \setminus D$.

(d) $\exists x \forall y[y > x \to \exists z(z^2 + 5z = y)]$.

(a) $\forall x \exists y(M(x) \wedge F(x, y) \wedge H(y) \wedge x \neq y)$

M(x) is "x is majoring in math"

F(x, y) is "x has a friend y"

H(x) is "x needs help with his homework"

Negating:

$\neg\forall x\exists y(M(x) \wedge F(x,y) \wedge H(y) \wedge x \neq y)$

Quantifier negation law: $\exists x\neg\exists y(M(x) \wedge F(x,y) \wedge H(y) \wedge x \neq y)$

Quantifier negation law: $\exists x\forall y\neg(M(x) \wedge F(x,y) \wedge H(y) \wedge x \neq y)$

DeMorgan's law: $\exists x\forall y(\neg(M(x) \wedge F(x,y)) \vee \neg(H(y) \wedge x \neq y))$

Conditional law: $\exists x\forall y((M(x) \wedge F(x,y)) \rightarrow \neg(H(y) \wedge x \neq y))$

Someone who is majoring in math doesn't have friends who are needs help with their homeworks.

(b) $\forall x\exists y(R(x,y) \wedge \forall z(\neg L(y,z)))$

R(x, y) is "x a roommate y"

L(x, y) is "x likes y"

Negating:

$\neg\forall x\exists y(R(x,y) \wedge \forall z(\neg L(y,z))$

Quantifier negation law: $\exists x\neg\exists y(R(x,y) \wedge \forall z(\neg L(y,z)))$

Quantifier negation law: $\exists x\forall y\neg(R(x,y) \wedge \forall z(\neg L(y,z)))$

DeMorgan's law: $\exists x\forall y(\neg R(x,y) \vee \neg\forall z(\neg L(y,z))$

Conditional law: $\exists x\forall y(R(x,y) \rightarrow \neg\forall z(\neg L(y,z)))$

$\exists x\forall y(R(x,y) \rightarrow \exists z(L(y,z)))$

There is someone all of whose roommates like at least one person.

(c) $\forall x(x \in (A \cup B) \rightarrow x \in (C \setminus D))$

$\forall x(\neg(x \in A \vee x \in B) \vee (x \in C \wedge x \notin D))$

Nagating:

$\neg\forall x(\neg(x \in A \vee x \in B) \vee (x \in C \wedge x \notin D))$

Quantifier negation law: $\exists x\neg(\neg(x \in A \vee x \in B) \vee (x \in C \wedge x \notin D))$

DeMorgan's law: $\exists x[(x \in A \vee x \in B) \wedge (x \notin C \vee x \in D)]$

(d) $\exists x\forall y[y > x \rightarrow \exists z(z^2 + 5z = y)]$

Negating:

$\neg\exists x\forall y[y > x \to \exists z(z^2 + 5z = y)]$

$\forall x\neg\forall y[y > x \to \exists z(z^2 + 5z = y)]$ (quantifier negation law)

$\forall x\exists y\neg[y > x \to \exists z(z^2 + 5z = y)]$ (quantifier negation law)

$\forall x\exists y\neg[\neg(y > x) \vee \exists z(z^2 + 5z = y)]$ (conditional law)

$\forall x\exists y[(y > x) \wedge \neg\exists z(z^2 + 5z = y)]$ (DeMorgan's law)

$\forall x\exists y[(y > x) \wedge \forall z\neg(z^2 + 5z = y)]$ (quantifier negation law)

$\forall x\exists y[y > x \wedge \forall z(z^2 + 5z \neq y)]$

2. Negate these statements and then reexpress the results as equivalent positive statements. (See Example 2.2.1.)

(a) There is someone in the freshman class who doesn't have a roommate.

(b) Everyone likes someone, but no one likes everyone.

(c) $\forall a \in A\exists b \in B(a \in C \leftrightarrow b \in C)$.

(d) $\forall y > 0\exists x(ax^2 + bx + c = y)$.

(a) $\exists x[F(x) \wedge \neg\exists y(R(x, y))]$

F(x) is "x is freshman"

R(x, y) is "x has a roommate y"

Negating:

$\neg\exists x[F(x) \wedge \neg\exists y(R(x, y))]$

$\forall x\neg[F(x) \wedge \neg\exists y(R(x, y))]$ (quantifier negation law)

$\forall x[\neg F(x) \vee \exists y(R(x, y))]$ (DeMorgan's law)

$\forall x[F(x) \to \exists y(R(x, y))]$ (conditional law)

All freshman have a roommate

(b) $\forall x\exists y(L(x, y)) \wedge \neg\exists z\forall w(L(z, w))$

L(x, y) is "x likes y"

Negating:

$\neg\forall x\exists y(L(x, y)) \vee \neg\neg\exists z\forall w(L(z, w))$

$\exists x \forall y \neg L(x, y) \vee \exists z \forall w L(z, w)$

There exist someone who doesn't like everybody or someone who likes everybody.

(c) $\neg \forall a \in A \exists b \in B(a \in C \leftrightarrow b \in C)$

$\exists a \in A \forall b \in B \neg(a \in C \leftrightarrow b \in C)$

$\exists a \in A \forall b \in B \neg((a \in C \rightarrow b \in C) \wedge (b \in C \rightarrow a \in C))$

$\exists a \in A \forall b \in B(\neg(a \in C \rightarrow b \in C) \vee \neg(b \in C \rightarrow a \in C))$

$\exists a \in A \forall b \in B(\neg(\neg(a \in C) \vee b \in C) \vee \neg(\neg(b \in C) \vee a \in C))$

$\exists a \in A \forall b \in B((a \in C) \wedge \neg(b \in C)) \vee ((b \in C) \wedge \neg(a \in C))$

$\exists a \forall b[(a \in B \wedge b \in B) \wedge (a \in C \wedge b \notin C) \vee (b \in C \wedge a \notin C)]$

(d) $\neg \forall y > 0 \exists x(ax^2 + bx + c = y)$

$\exists y > 0 \neg \exists x(ax^2 + bx + c = y)$

$\exists y > 0 \forall x \neg(ax^2 + bx + c = y)$

$\exists y > 0 \forall x(ax^2 + bx + c \neq y)$

$\exists y \forall x(y > 0 \wedge ax^2 + bx + c \neq y)$

3. Are these statements true or false? The universe of discourse is $\mathbb{N}$.
   (a) $\forall x(x < 7 \rightarrow \exists a \exists b \exists c(a^2 + b^2 + c^2 = x))$.
   (b) $\exists! x((x - 4)^2 = 9)$.
   (c) $\exists! x((x - 4)^2 = 25)$.
   (d) $\exists x \exists y((x - 4)^2 = 25 \wedge (y - 4)^2 = 25)$.

(a) true

$0 \leq x \leq 6$

(b) false

x = 7

x = 1

(c) false

x = 9

x = -1

(d) true

4. Show that the second quantifier negation law, which says that $\neg\forall x P(x)$ is equivalent to $\exists x \neg P(x)$, can be derived from the first, which says that $\neg\exists x P(x)$ is equivalent to $\forall x \neg P(x)$. (Hint: Use the double negation law.)

Using $\neg\exists x P(x) = \forall x \neg P(x)$

prove $\neg\forall x P(x) = \exists x \neg P(x)$

$\neg\neg\forall x P(x) = \neg\exists x \neg P(x)$

$\forall x P(x) = \forall x \neg\neg P(x)$

$\forall x P(x) = \forall x P(x)$

or

$\neg\exists x \neg P(x) = \forall x \neg\neg P(x)$

$\neg\exists x \neg P(x) = \forall x P(x)$

$\exists x \neg P(x) = \neg\forall x P(x)$

5. Show that $\neg\exists x \in AP(x)$ is equivalent to $\forall x \in A \neg P(x)$.

$\neg\exists x \in AP(x)$

$\neg\exists x(x \in A \wedge P(x))$ (expanding abbreviation)

$\forall x \neg(x \in A \wedge P(x))$ (quantifier negation law)

$\forall x(\neg x \in A \vee \neg P(x))$ (DeMorgan's law)

$\forall x(x \in A \to \neg P(x))$ (conditional law)

$\forall x \in A \neg P(x)$ (abbreviation)

6. Show that the existential quantifier distributes over disjunction. In other words, show that $\exists x(P(x) \vee Q(x))$ is equivalent to $\exists x P(x) \vee \exists x Q(x)$. (Hint: Use the fact, discussed in this section, that the universal quantifier distributes over conjunction.)

$\exists x(P(x) \vee Q(x))$

$\neg\forall x\neg(P(x) \vee Q(x))$

$\neg\forall x(\neg P(x) \wedge \neg Q(x))$

$\neg(\forall x\neg P(x) \wedge \forall x\neg Q(x))$

$\neg\forall x\neg P(x) \vee \neg\forall x\neg Q(x)$

$\exists x\neg\neg P(x) \vee \exists x\neg\neg Q(x)$

$\exists x P(x) \vee \exists x Q(x)$

7. Show that $\exists x(P(x) \to Q(x))$ is equivalent to $\forall x P(x) \to \exists x Q(x)$.

$\exists x(P(x) \to Q(x))$

$\neg\neg\exists x(P(x) \to Q(x))$ (double negation law)

$\neg\forall x\neg(P(x) \to Q(x))$ (quantifier negation law)

$\neg\forall x\neg(\neg P(x) \vee Q(x))$ (conditional law)

$\neg\forall x(P(x) \wedge \neg Q(x))$ (DeMorgan's law)

$\neg(\forall x P(x) \wedge \forall x\neg Q(x))$ (universal quantifier distribution law)

$\neg\forall x P(x) \vee \neg\forall x\neg Q(x)$ (DeMorgan's law)

$\neg\forall x P(x) \vee \exists x\neg\neg Q(x)$ (quantifier negation law)

$\neg\forall x P(x) \vee \exists x Q(x)$ (double negation law)

$\forall x P(x) \to \exists x Q(x)$ (conditional law)

8. Show that $(\forall x \in A\, P(x)) \wedge (\forall x \in B\, P(x))$ is equivalent to $\forall x \in (A \cup B)\, P(x)$. (Hint: Start by writing out the meanings of the bounded quantifiers in terms of unbounded quantifiers.)

$(\forall x(x \in A \rightarrow P(x))) \wedge (\forall x(x \in B \rightarrow P(x)))$ (expanding abbreviation)

$\forall x((x \in A \rightarrow P(x)) \wedge (x \in B \rightarrow P(x)))$ (universal quantifiers distribution law)

$\forall x((\neg(x \in A) \vee P(x)) \wedge (\neg(x \in B) \vee P(x)))$ (conditional law)

$\forall x((\neg(x \in A) \wedge \neg(x \in B)) \vee P(x))$ (distribution law)

$\forall x(\neg(x \in A \vee x \in B) \vee P(x))$ (DeMorgan's law)

$\forall x((x \in A \vee x \in B) \rightarrow P(x))$ (conditional law)

$\forall x(x \in (A \cup B) \rightarrow P(x))$

$\forall x \in (A \cup B)\, P(x)$

9. Is $\forall x(P(x) \vee Q(x))$ equivalent to $\forall x P(x) \vee \forall x Q(x)$? Explain. (Hint: Try assigning meanings to P(x) and Q(x).)

$\forall x(P(x) \vee Q(x))$

P(x) "x is tall"

Q(x) "x is smart"

All people are either tall or smart.

$\forall x P(x) \vee \forall x Q(x)$

All people are tall or all people are smart.

$\forall x(P(x) \wedge Q(x)) \neq \forall x P(x) \vee \forall x Q(x)$

10. (a) Show that $\exists x \in A\, P(x) \vee \exists x \in B\, P(x)$ is equivalent to $\exists x \in (A \cup B)\, P(x)$.

(b) Is $\exists x \in A\, P(x) \wedge \exists x \in B\, P(x)$ equivalent to $\exists x \in (A \cap B)\, P(x)$? Explain.

(a) $\exists x \in A\, P(x) \vee \exists x \in B\, P(x)$

$\exists x[(x \in A) \wedge P(x)) \vee \exists((x \in B) \wedge P(x)]$

$\exists x[((x \in A) \wedge P(x)) \vee ((x \in B) \wedge P(x))]$

$\exists x[((x \in A) \vee (x \in B)) \wedge P(x)]$

$\exists x(x \in (A \cup B) \wedge P(x))$

$\exists x \in (A \cup B)\, P(x)$

(b) $\exists x \in A\, P(x) \wedge \exists x \in B\, P(x)$

$\exists x((x \in A) \wedge P(x)) \wedge \exists x((x \in B) \wedge P(x))$

$\exists x \in (A \cap B) P(x)$

$\exists x(x \in (A \cap B) \wedge P(x))$

$\exists x((x \in A) \wedge (x \in B) \wedge P(x))$

A is {1,2,3}

B is {4,5,7}

P(x) is "$x < 6$"

$\exists x((x \in A) \wedge P(x)) \wedge \exists x((x \in B) \wedge P(x))$

$\exists x((x \in \{1,2,3\}) \wedge x < 6) \wedge \exists x((x \in \{4,5,7\}) \wedge x < 6)$

$true \wedge true = true$

$\exists x((x \in \{1,2,3\} \wedge x \in \{4,5,7\} \wedge x < 6))$

$\exists x(x \in \varnothing \wedge x < 6)$ is false

So $\exists x \in A\, P(x) \wedge \exists x \in B\, P(x)$ is not equivalent to $\exists x \in (A \cap B)\, P(x)$.

11. Show that the statements $A \subseteq B$ and $A \setminus B = \varnothing$ are equivalent by

writing each in logical symbols and then showing that the resulting formulas are equivalent.

$A \subseteq B$ means $\forall x(x \in A \to x \in B)$

$A \setminus B = \varnothing$ means $\neg\exists x(x \in A \wedge x \notin B)$

$\forall x \neg(x \in A \wedge x \notin B)$

$\forall x(\neg x \in A \vee x \in B)$

$\forall x(x \in A \to x \in B)$

$A \subseteq B$

12. Let T(x, y) mean "x is a teacher of y." What do the following statements mean? Under what circumstances would each one be true? Are any of them equivalent to each other?

(a) $\exists! y T(x, y)$.

(b) $\exists x \exists! y T(x, y)$.

(c) $\exists! x \exists y T(x, y)$.

(d) $\exists y \exists! x T(x, y)$.

(e) $\exists! x \exists! y T(x, y)$.

(f) $\exists x \exists y[T(x, y) \wedge \neg\exists u \exists v(T(u, v) \wedge (u \neq x \vee v \neq y))]$.

(a) There is exactly one student who has x teacher.

Statement will be true if we choose instead of x a teacher who has exactly one student.

(b) There is at least one teacher who has exactly one student.

Statement will be true if universe of discourse has at least one teacher who has exactly one student.

(c) There is exactly one teacher who has at least one student.

Statement will be true if universe of discourse has exactly one teacher who has at least one student.

(d) There is exactly one teacher who has at least one student

$\exists y \exists! x T(x, y)$ is equivalent to $\exists! x \exists! y T(x, y)$

Statement will be true if universe of discourse has exactly one teacher who has at least one student.

(e) There is exactly one teacher who has exactly one student.

Statement will be true if universe of discourse has exactly one teacher who has exactly one student.

(f) There is exactly one teacher who has exactly one student.

$\exists x \exists y [T(x, y) \wedge \neg \exists u \exists v (T(u, v) \wedge (u \neq x \vee v \neq y))]$ is equivalent to

$\exists! x \exists! y T(x, y)$

Statement will be true if universe of discourse has exactly one teacher who has exactly one student.

(c) equivalent to (d)

(e) equivalent to (f)

2.3. More Operations on Sets

$x \in \{n^2 \mid n \in \mathbb{N}\}$ means the same thing as $\exists n \in \mathbb{N}(x = n^2)$

$\mathcal{P}(A) = \{x \mid x \subseteq A\}$

$\cup$ union

$\cap$ intersection

$\cap \mathcal{F} = \{x \mid \forall A \in \mathcal{F}(x \in A)\} = \{x \mid \forall A(A \in F \rightarrow x \in A)\}$

$\cup \mathcal{F} = \{x \mid \exists A \in \mathcal{F}(x \in A)\} = \{x \mid \exists A(A \in \mathcal{F} \wedge x \in A)$

$\cap\mathcal{F} = \cap_{i\in I} A_i = \{x \mid \forall i \in I(x \in A_i)\}$

$\cup\mathcal{F} = \cup_{i\in I} A_i = \{x \mid \exists i \in I(x \in A_i)\}$

Exercises:

1. Analyze the logical forms of the following statements. You may use the symbols $\in, \notin, =, \neq, \wedge, \vee, \rightarrow, \leftrightarrow, \forall$, and $\exists$ in your answers, but not $\subseteq, \nsubseteq, \mathcal{P}, \cap, \cup, \{, \}$, or $\neg$. (Thus, you must write out the definitions of some set theory notation, and you must use equivalences to get rid of any occurrences of $\neg$.)

(a) $\mathcal{F} \subseteq \mathcal{P}(A)$.

(b) $A \subseteq \{2n+1 \mid n \in \mathbb{N}\}$.

(c) $\{n^2+n+1 \mid n \in \mathbb{N}\} \subseteq \{2n+1 \mid n \in \mathbb{N}\}$.

(d) $\mathcal{P}(\cup_{i\in I} A_i) \nsubseteq \cup_{i\in I}\mathcal{P}(A_i)$.

(a) $\forall x(x \in \mathcal{F} \rightarrow x \in \mathcal{P}(A))$

$\forall x(x \in \mathcal{F} \rightarrow x \subseteq A)$

$\forall x(x \in \mathcal{F} \rightarrow \forall y(y \in x \rightarrow y \in A))$

(b) $\forall x(x \in A \rightarrow x \in \{2n+1 \mid n \in \mathbb{N}\})$

$\forall x(x \in A \rightarrow \exists n \in \mathbb{N}(x = 2n+1))$

(c) $\forall x(x \in \{n^2+n+1 \mid n \in \mathbb{N})\} \rightarrow x \in \{2n+1 \mid n \in \mathbb{N}\})$

$\forall x(\exists n \in \mathbb{N}(x = n^2+n+1) \rightarrow \exists n \in \mathbb{N}(x = 2n+1))$

or better version:

$\forall n \in \mathbb{N}\exists m \in \mathbb{N}(n^2+n+1 = 2m+1)$

(d) $\exists x(x \in \mathcal{P}(\cup_{i\in I} A_i) \wedge x \notin \cup_{i\in I}\mathcal{P}(A_i))$

$\exists x(\forall y(y \in x \rightarrow y \in \cup_{i\in I} A_i) \wedge \neg\exists i \in I(x \in \mathcal{P}(A_i)))$

$\exists x(\forall y(y \in x \to \exists i \in I(y \in A_i)) \land \neg\exists i \in I \forall y(y \in x \to y \in A_i))$

$\exists x(\forall y(y \in x \to \exists i \in I(y \in A_i)) \land \forall i \in I \exists y(y \in x \land y \notin A_i))$

2. Analyze the logical forms of the following statements. You may use the symbols $\in, \notin, =, \neq, \land, \lor, \to, \leftrightarrow, \forall$, and $\exists$ in your answers, but not $\subseteq, \nsubseteq, \mathcal{P}, \cap, \cup, \{, \}$, or $\neg$. (Thus, you must write out the definitions of some set theory notation, and you must use equivalences to get rid of any occurrences of $\neg$.)

(a) $x \in \cup\mathcal{F} \setminus \cup\mathcal{G}$.

(b) $\{x \in B \mid x \notin C\} \in \mathcal{P}(A)$.

(c) $x \in \cap_{i\in I}(A_i \cup B_i)$.

(d) $x \in (\cap_{i\in I} A_i) \cup (\cap_{i\in I} B_i)$.

(a) $x \in \cup\mathcal{F} \land x \notin \cup\mathcal{G}$

$\exists A(A \in \mathcal{F} \land x \in A) \land \forall A \in \mathcal{G}(x \notin A)$

(b) $\forall y(y \in \{x \in B \mid x \notin C\} \to y \in A)$

$\forall y(\exists x \notin C(y = x \in B) \to y \in A)$

(c) $\forall i \in I(x \in (A_i \cup B_i))$

$\forall i \in I(x \in A_i \lor x \in B_i)$

(d) $(x \in (\cap_{i\in I} A_i)) \lor (x \in (\cap_{i\in I} B_i))$

$\forall i \in I(x \in A_i) \lor \forall i \in I(x \in B_i)$

3. We've seen that $\mathcal{P}(\varnothing) = \{\varnothing\}$, and $\{\varnothing\} \neq \varnothing$. What is $\mathcal{P}(\{\varnothing\})$?

$\mathcal{P}(\{\varnothing\})$

$\{x \mid x \subseteq \{\varnothing\}\} = \{\varnothing, \{\varnothing\}\}$

4. Suppose $\mathcal{F} = \{\{red, green, blue\}, \{orange, red, blue\}, \{purple, red, green, blue\}\}$. Find $\cap\mathcal{F}$ and $\cup\mathcal{F}$.

$\cap\mathcal{F} = \{red, blue\}$

$\cup\mathcal{F} = \{red, green, blue, orange, purple\}$

5. Suppose $\mathcal{F} = \{\{3, 7, 12\}, \{5, 7, 16\}, \{5, 12, 23\}\}$. Find $\cap\mathcal{F}$ and $\cup\mathcal{F}$.

$\cap\mathcal{F} = \varnothing$

$\cup\mathcal{F} = \{3, 5, 7, 12, 16, 23\}$

6. Let $I = \{2, 3, 4, 5\}$, and for each $i \in I$ let $A_i = \{i, i + 1, i - 1, 2i\}$.

(a) List the elements of all the sets A_i , for $i \in I$.

(b) Find $\cap_{i\in I} A_i$ and $\cup_{i\in I} A_i$.

(a) $A_2 = \{2, 3, 1, 4\}$

$A_3 = \{3, 4, 2, 6\}$

$A_4 = \{4, 5, 3, 8\}$

$A_5 = \{5, 6, 4, 10\}$

(b) $\cap_{i\in I} A_i = \{4\}$

$\cup_{i\in I} A_i = \{1, 2, 3, 4, 5, 6, 8, 10\}$

7. Let P = {Johann Sebastian Bach, Napoleon Bonaparte, Johann Wolfgang von Goethe, David Hume, Wolfgang Amadeus Mozart, Isaac Newton, George Washington} and let $Y = \{1750, 1751, 1752, \ldots, 1759\}$ For each $y \in Y$, let $A_y = \{p \in P \mid$ the person p was alive at some time during the year y$\}$. Find

$\cup_{y \in Y} A_y$ and $\cap_{y \in Y} A_y$.

B = Johann Sebastian Bach: 1685 - 1750

X = Napoleon Bonaparte: 1769 - 1821

G = Johann Wolfgang von Goethe: 1749 - 1832

D = David Hume: 1711 - 1776

M = Wolfgang Amadeus Mozart: 1756 - 1791

N = Isaac Newton: 1642 - 1726

W = George Washington: 1732 - 1799

A_{1750} = {B, G, D, W}

A_{1751} = {G, D, W}

A_{1752} = {G, D, W}

A_{1753} = {G, D, W}

A_{1754} = {G, D, W}

A_{1755} = {G, D, W}

A_{1756} = {G, D, M, W}

A_{1757} = {G, D, M, W}

A_{1758} = {G, D, M, W}

A_{1759} = {G, D, M, W}

$\cup_{y \in Y} A_y$ = {B, G, D, W, M} = {Johann Sebastian Bach, Johann Wolfgang von Goethe, David Hume, George Washington, Wolfgang Amadeus Mozart}

$\cap_{y \in Y} A_y$ = {G,D} = {Johann Wolfgang von Goethe, David Hume}

8. Let I = {2, 3}, and for each $i \in I$ let $A_i = \{i, 2i\}$ and $B_i = \{i, i+1\}$.

(a) List the elements of the sets A_i and B_i for $i \in I$.

(b) Find $\cap_{i \in I}(A_i \cup B_i)$ and $(\cap_{i \in I} A_i) \cup (\cap_{i \in I} B_i)$. Are they the same?

(c) In parts (c) and (d) of exercise 2 you analyzed the statements $x \in \cap_{i \in I}(A_i \cup B_i)$ and $x \in (\cap_{i \in I} A_i) \cup (\cap_{i \in I} B_i)$. What can you conclude from your answer to part (b) about whether or not these statements are equivalent?

(a) $A_2 = \{2,4\}$ $B_2 = \{2,3\}$

$A_3 = \{3,6\}$ $B_3 = \{3,4\}$

(b) $\cap_{i \in I}(A_i \cup B_i)$

$(A_2 \cup B_2) \cap (A_3 \cup B_3)$

$\{2,3,4\} \cap \{3,4,6\}$

$\{3,4\}$

$(\cap_{i \in I} A_i) \cup (\cap_{i \in I} B_i)$

$(A_2 \cap A_3) \cup (B_2 \cap B_3)$

$\varnothing \cup \{3\}$

$\{3\}$

$\therefore \cap_{i \in I}(A_i \cup B_i) \neq (\cap_{i \in I} A_i) \cup (\cap_{i \in I} B_i)$

(c) $x \in \cap_{i \in I}(A_i \cup B_i) \neq x \in (\cap_{i \in I} A_i) \cup (\cap_{i \in I} B_i)$

9. Give an example of an index set I and indexed families of sets $\{A_i \mid i \in I\}$ and $\{B_i \mid i \in I\}$ such that $\cup_{i \in I}(A_i \cap B_i) \neq (\cup_{i \in I} A_i) \cap (\cup_{i \in I} B_i)$.

Let I = {2, 3}, and for each $i \in I$ let $A_i = \{i, 2i\}$ and $B_i = \{i, i+1\}$.

$A_2 = \{2,4\}$ $B_2 = \{2,3\}$

$A_3 = \{3,6\}$ $B_3 = \{3,4\}$

$\cup_{i \in I}(A_i \cap B_i) = (A_2 \cap B_2) \cup (A_3 \cap B_3)$

$\{2\} \cup \{3\} = \{2,3\}$

$(\cup_{i \in I} A_i) \cap (\cup_{i \in I} B_i) = (A_2 \cup A_3) \cap (B_2 \cup B_3)$

$\{2,3,4\} \cap \{3,4,6\} = \{3,4\}$

$\therefore \cup_{i \in I}(A_i \cap B_i) \neq (\cup_{i \in I} A_i) \cap (\cup_{i \in I} B_i)$

10. Show that for any sets A and B, $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$, by showing

that the statements $x \in \mathcal{P}(A \cap B)$ and $x \in \mathcal{P}(A) \cap \mathcal{P}(B)$ are equivalent. (See Example 2.3.3.)

Logical form of $x \in \mathcal{P}(A \cap B)$ is $\forall y(y \in x \to (y \in A \land y \in B))$

Logical form of $x \in \mathcal{P}(A) \cap \mathcal{P}(B)$ is $\forall y(y \in x \to y \in A) \land \forall y(y \in x \to y \in B)$

We just need to prove that logical forms are equivalent.

$\forall y(y \in x \to y \in A) \land \forall y(y \in x \to y \in B)$

Since universal quantifier distributes over conjunction:

$\forall y((y \in x \to y \in A) \land (y \in x \to y \in B))$

Using conditional law:

$\forall y((\neg(y \in x) \lor y \in A) \land (\neg(y \in x) \lor y \in B))$

Using distribution law:

$\forall y((\neg(y \in x) \lor (y \in A \land y \in B))$

Using conditional law:

$\forall y(y \in x \to (y \in A \land y \in B))$

11. Give examples of sets A and B for which $\mathcal{P}(A \cup B) \neq \mathcal{P}(A) \cup \mathcal{P}(B)$.

A $= \{7\}$

B $= \{12\}$

$\mathcal{P}(A \cup B) = \mathcal{P}(\{7, 12\}) = \{\varnothing, \{7\}, \{12\}, \{7, 12\}\}$

$\mathcal{P}(A) = \mathcal{P}(\{7\}) = \{\varnothing, \{7\}\}$

$\mathcal{P}(B) = \mathcal{P}(\{12\}) = \{\varnothing, \{12\}\}$

$\mathcal{P}(A) \cup \mathcal{P}(B) = \{\varnothing, \{7\}\} \cup \{\varnothing, \{12\}\}$

$\mathcal{P}(A) \cup \mathcal{P}(B) = \{\varnothing, \{7\}, \{12\}\}$

$\therefore \mathcal{P}(A \cup B) \neq \mathcal{P}(A) \cup \mathcal{P}(B)$

12. Verify the following identities by writing out (using logical symbols)

what it means for an object x to be an element of each set and then using logical equivalences.

(a) $\cup_{i\in I}(A_i \cup B_i) = (\cup_{i\in I}A_i) \cup (\cup_{i\in I}B_i)$.

(b) $(\cap\mathcal{F}) \cap (\cap\mathcal{G}) = \cap(\mathcal{F} \cup \mathcal{G})$.

(c) $\cap_{i\in I}(A_i \setminus B_i) = (\cap_{i\in I}A_i) \setminus (\cup_{i\in I}B_i)$.

(a)

Logical form of $x \in \cup_{i\in I}(A_i \cup B_i)$ is $\exists i \in I(x \in (A_i \cup B_i))$

Logical form of $x \in ((\cup_{i\in I}A_i) \cup (\cup_{i\in I}B_i))$ is $\exists i \in I(x \in A_i) \vee \exists i \in I(x \in B_i)$

$\exists i \in I(x \in (A_i \cup B_i))$

Since existential quantifier distributes over disjunction:

$\exists i \in I(x \in A_i \vee x \in B_i))$

$\exists i \in I(x \in A_i) \vee \exists i \in I(x \in B_i)$

(b) $(\cap\mathcal{F}) \cap (\cap\mathcal{G})$

Logical form of $x \in \cap\mathcal{F}$ is $\forall A(A \in \mathcal{F} \to x \in A)$

Logical form of $x \in \cap\mathcal{G}$ is $\forall A(A \in \mathcal{G} \to x \in A)$

$\forall A(A \in \mathcal{F} \to x \in A) \wedge \forall A(A \in \mathcal{G} \to x \in A)$

Since universal quantifier distributes over conjunction:

$\forall A[(A \in \mathcal{F} \to x \in A) \wedge (A \in \mathcal{G} \to x \in A)]$

Conditional law:

$\forall A[(\neg(A \in \mathcal{F}) \vee x \in A)) \wedge (\neg(A \in \mathcal{G} \vee x \in A))]$

Distribution law:

$\forall A[(\neg(A \in \mathcal{F}) \wedge \neg(A \in \mathcal{G})) \vee x \in A]$

DeMorgan's law:

$\forall A[\neg((A \in \mathcal{F}) \vee (A \in \mathcal{G})) \vee x \in A]$

Conditional law:

$\forall A(((A \in \mathcal{F}) \vee (A \in \mathcal{G})) \to x \in A)$

$\cap(\mathcal{F} \cup \mathcal{G})$

Logical form of $x \in \cap(\mathcal{F} \cup \mathcal{G})$ is $\forall A(A \in (\mathcal{F} \cup \mathcal{G}) \to x \in A)$

$\forall A(((A \in \mathcal{F}) \vee (A \in \mathcal{G})) \to x \in A)$

(c) $\cap_{i \in I}(A_i \setminus B_i)$

Logical form of $x \in \cap_{i \in I}(A_i \setminus B_i)$ is $\forall i \in I(x \in (A_i \setminus B_i))$

$\forall i \in I(x \in A_i \wedge x \notin B_i)$

$\forall i \in I(x \in A_i) \wedge \forall i \in I(x \notin B_i)$

$(\cap_{i \in I} A_i) \setminus (\cup_{i \in I} B_i)$

Logical form of $x \in (\cap_{i \in I} A_i) \setminus (\cup_{i \in I} B_i)$ is $\forall i \in I(x \in A_i) \wedge \neg\exists i \in I(x \in B_i)$

$\forall i \in I(x \in A_i) \wedge \forall i \in I \neg(x \in B_i)$

$\forall i \in I(x \in A_i) \wedge \forall i \in I(x \notin B_i)$

13. Sometimes each set in an indexed family of sets has two indices. For this problem, use the following definitions: I = {1, 2}, J = {3, 4}. For each $i \in I$ and $j \in J$, let $A_{i,j} = \{i, j, i + j\}$. Thus, for example, $A_{2,3} = \{2, 3, 5\}$.

(a) For each $j \in J$ let $B_j = \cup_{i \in I} A_{i,j} = A_{1,j} \cup A_{2,j}$. Find B_3 and B_4.

(b) Find $\cap_{j \in J} B_j$. (Note that, replacing B_j with its definition, we could say that $\cap_{j \in J} B_j = \cap_{j \in J}(\cup_{i \in I} A_{i,j})$.)

(c) Find $\cup_{i \in I}(\cap_{j \in J} A_{i,j})$. (Hint: You may want to do this in two steps, corresponding to parts (a) and (b).) Are $\cap_{j \in J}(\cup_{i \in I} A_{i,j})$ and $\cup_{i \in I}(\cap_{j \in J} A_{i,j})$ equal?

(d) Analyze the logical forms of the statements $x \in \cap_{j \in J}(\cup_{i \in I} A_{i,j})$ and $x \in \cup_{i \in I}(\cap_{j \in J} A_{i,j})$. Are they equivalent?

(a) $A_{1,3} = \{1, 3, 4\}$ $A_{2,3} = \{2, 3, 5\}$

$B_3 = \{1, 2, 3, 4, 5\}$

$A_{1,4} = \{1, 4, 5\}$ $A_{2,4} = \{2, 4, 6\}$

$B_4 = \{1, 2, 4, 5, 6\}$

(b) $\{1, 2, 3, 4, 5\} \cap \{1, 2, 4, 5, 6\} = \{1, 2, 4, 5\}$

(c) $B_1 = \{1, 4\}$

$B_2 = \{2\}$

$\{1, 4\} \cup \{2\} = \{1, 2, 4\}$

$\cap_{j \in J}(\cup_{i \in I} A_{i,j}) \neq \cup_{i \in I}(\cap_{j \in J} A_{i,j})$

(d) $x \in \cap_{j \in J}(\cup_{i \in I} A_{i,j})$

$\forall j \in J(x \in \cup_{i \in I} A_{i,j})$

$\forall j \in J(x \in \exists i \in I(x \in A_{i,j}))$

$\forall j \in J \exists i \in I(x \in A_{i,j})$

$x \in \cup_{i \in I}(\cap_{j \in J} A_{i,j})$

$\exists i \in I \forall j \in J(x \in A_{i,j})$

$\exists i \in I \forall j \in J(x \in A_{i,j}) \neq \forall j \in J \exists i \in I(x \in A_{i,j})$

14. (a) Show that if $\mathcal{F} = \varnothing$, then the statement $x \in \cup\mathcal{F}$ will be false no matter what x is. It follows that $\cup\varnothing = \varnothing$.

(b) Show that if $\mathcal{F} = \varnothing$, then the statement $x \in \cap\mathcal{F}$ will be true no matter what x is. In a context in which it is clear what the universe of discourse U is, we might therefore want to say that $\cap\varnothing = U$. However, this has the unfortunate consequence that the notation $\cap\varnothing$ will mean different things in different contexts. Furthermore, when working with sets whose elements are sets, mathematicians often do not use a universe of discourse at all. (For more on this, see the next exercise.) For these reasons, some mathematicians consider the notation $\cap\varnothing$ to be meaningless. We will avoid this problem in this book by using the notation $\cap\mathcal{F}$ only in contexts in which we can be sure that $\mathcal{F} \neq \varnothing$.

(a) $x \in \cup\mathcal{F}$

$\exists A \in \mathcal{F}(x \in A)$

$\exists A(A \in \mathcal{F} \wedge x \in A)$

$\exists A(A \in \varnothing \wedge x \in A)$

$\exists A(A \in \varnothing)$ is always false.

$\therefore \cup\varnothing = \varnothing$

(b) $x \in \cap\mathcal{F}$

$\forall A \in \mathcal{F}(x \in A)$

$\forall A(A \in \mathcal{F} \rightarrow x \in A)$

$\forall A(\neg A \in \mathcal{F} \vee x \in A)$

$\forall A(A \notin \mathcal{F} \vee x \in A)$

$\forall A(A \notin \varnothing \vee x \in A)$

$\forall A(true \vee x \in A)$ is always true.

$\therefore \cap\varnothing = U$

15. In Section 2.3 we saw that a set can have other sets as elements. When discussing sets whose elements are sets, it might seem most natural to consider the universe of discourse to be the collection of all sets. However, as we will see in this problem, assuming that there is such a universe leads to contradictions. Suppose U were the collection of all sets. Note that in particular U is a set, so we would have $U \in U$. This is not yet a contradiction; although most sets are not elements of themselves, perhaps some sets are elements of themselves. But it suggests that the sets in the universe U could be split into two categories: the unusual sets that, like U itself, are elements of themselves, and the more typical sets that are not. Let R be the set of sets in the second category. In other words, $R = \{A \in U \mid A \notin A\}$. This means that for any set A in the universe U, A will be an element of R iff $A \notin A$. In other words, we have $\forall A \in U(A \in R \leftrightarrow A \notin A)$.

(a) Show that applying this last fact to the set R itself (in other words, plugging in R for A) leads to a contradiction. This contradiction was discovered by Bertrand Russell in 1901, and is known as *Russell's Paradox*.

(b) Think some more about the paradox in part (a). What do you think it tells us about sets?

(a) $\forall A \in U(A \in R \leftrightarrow A \notin A)$

$R \in R \leftrightarrow R \notin R$

(b) it's not clear how in which situation we may define a set.