Solutions for "Relations" chapter of "How to prove it" book

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(make contain various errors)

1 Ordered Pairs and Cartesian Products

1.

(a) $\{(p, c) \in P \times P \mid$ the person p is a parent of c$\} = \{$(Prince Charles, Prince William), (Prince Charles, Price Harry), $\dots\}$

(b) $\{(c, u) \in C \times U \mid$ there is someone who lives in c and attends u$\}$. If you are a university student, then let x be the city you live in, and let y be the university you attend; (x, y) will then be an element of this truth set.

2.

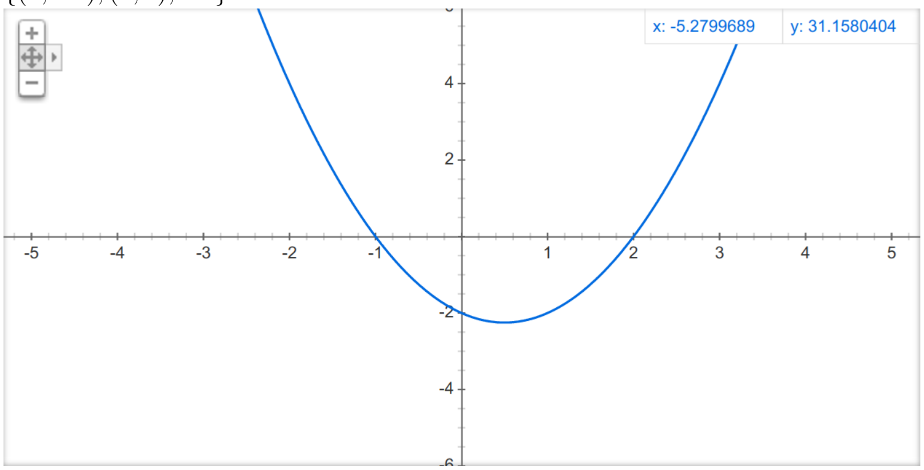
(a) $\{(p, c) \in P \times C \mid$ the person p lives in c city)$\} = \{$(drets, Poznan), (Prince William, London), $\dots\}$

(b) $\{(c, n) \in C \times \mathbb{N} \mid$ the population of c is n$\} = \{(Poznan, 600000), (Tokyo, 13600000), \dots\}$

3.

(a) $y = x^2 - x - 2$

$\{(0, -2), (2, 0), \dots\}$

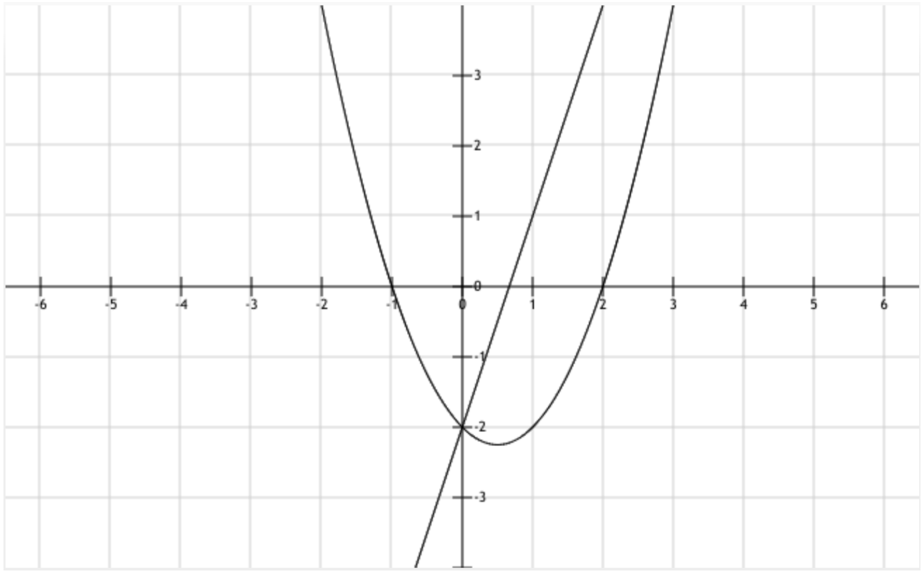


(b) $y < x$

$\{(0, 1), (0.1, 1.1), \dots\}$

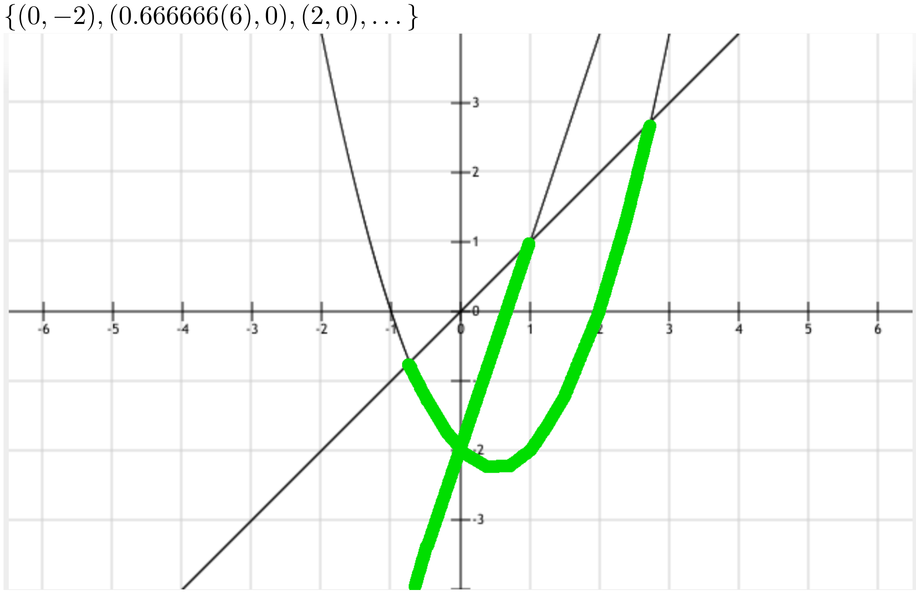
(c) Either $y = x^2 - x - 2$ or $y = 3x - 2$

$\{(-1, 0), (0, -2), (0.666666(6), 0), (2, 0), \dots\}$



(d) $y < x$, and either $y = x^2 - x - 2$ or $y = 3x - 2$

$\{(0, -2), (0.666666(6), 0), (2, 0), \dots\}$



4.

$A = \{1, 2, 3\}$

$B = \{1, 4\}$

$C = \{3, 4\}$

$D = \{5\}$

1) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

$B \cap C = \{4\}$

$A \times (B \cap C) = \{(1, 4), (2, 4), (3, 4)\}$

$A \times B = \{(1, 1), (2, 1), (3, 1), (1, 4), (2, 4), (3, 4)\}$

$A \times C = \{(1, 3), (2, 3), (3, 3), (1, 4), (2, 4), (3, 4)\}$

$(A \times B) \cap (A \times C) = \{(1, 4), (2, 4), (3, 4)\}$

2) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

$B \cup C = \{1, 3, 4\}$

$A \times (B \cup C) = \{(1, 1), (2, 1), (3, 1), (1, 3), (2, 3), (3, 3), (1, 4), (2, 4), (3, 4)\}$

$A \times B = \{(1, 1), (2, 1), (3, 1), (1, 4), (2, 4), (3, 4)\}$

$A \times C = \{(1, 3), (2, 3), (3, 3), (1, 4), (2, 4), (3, 4)\}$

$(A \times B) \cup (A \times C) = \{(1, 1), (2, 1), (3, 1), (1, 3), (2, 3), (3, 3), (1, 4), (2, 4), (3, 4)\}$

3) $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$

$A \times B = \{(1, 1), (2, 1), (3, 1), (1, 4), (2, 4), (3, 4)\}$

$C \times D = \{(3, 5), (4, 5)\}$

$(A \times B) \cap (C \times D) = \varnothing$

$A \cap C = 3$

$B \cap D = \varnothing$

$(A \cap C) \times (B \cap D) = \varnothing$

4) $(A \times B) \cup (C \times D) \subseteq (A \cup C) \times (B \cup D)$

$A \times B = \{(1, 1), (2, 1), (3, 1), (1, 4), (2, 4), (3, 4)\}$

$C \times D = \{(3, 5), (4, 5)\}$

$(A \times B) \cup (C \times D) = \{(1, 1), (2, 1), (3, 1), (1, 4), (2, 4), (3, 4), (3, 5), (4, 5)\}$

$A \cup C = \{1, 2, 3, 4\}$

$B \cup D = \{1, 4, 5\}$

$(A \cup C) \times (B \cup D) = \{(1, 1), (2, 1), (3, 1), (4, 1), (1, 4), (2, 4), (3, 4), (4, 4), (1, 5), (2, 5), (3, 5), (4, 5)\}$

5) $A \times \varnothing = \varnothing \times A = \varnothing$

$A \times \varnothing = \{1, 2, 3\} \times \varnothing = \varnothing$

$\varnothing \times A = \varnothing \times \{1, 2, 3\} = \varnothing$

5.

$A \times (B \cup C) = (A \times B) \cup (A \times C)$

1)

Givens	*Goal*
$p \in A \times (B \cup C)$	$p \in (A \times B) \cup (A \times C)$

Givens	*Goal*
$x \in A$	$p \in (A \times B) \cup (A \times C)$
$y \in B \vee y \in C$	

Givens	*Goal*
$x \in A$	$p \in (A \times B) \cup (A \times C)$
$y \in B \vee y \in C$	

Case 1.

Givens	*Goal*
$x \in A$	$p \in (A \times B) \cup (A \times C)$
$y \in B$	

Case 2.

Givens	*Goal*
$x \in A$	$p \in (A \times B) \cup (A \times C)$
$y \in C$	

2)

Givens	*Goal*
$p \in (A \times B) \cup (A \times C)$	$p \in A \times (B \cup C)$

Case 1.

Givens	*Goal*
$x \in A$	$x \in A \wedge (y \in B \vee y \in C)$
$y \in B$	

Case 2.

Givens	*Goal*
$x \in A$	$x \in A \wedge (y \in B \vee y \in C)$
$y \in C$	

Proof of 2. Let p be an arbitrary element of $A \times (B \cup C)$. Then by definition of Cartesian product, p must be an ordered pair whose first coordinate is an element of A and second coordinate is an element of $B \cup C$. In other words, $p = (x, y)$ for some $x \in A$ and $y \in B \cup C$. Since $y \in B \cup C$, $y \in B$ or $y \in C$.

Case 1. $y \in B$. Since $x \in A$ and $y \in B$, $p = (x, y) \in A \times B$. Thus, $p \in (A \times B) \cup (A \times C)$

Case 2. $y \in C$. Since $x \in A$ and $y \in C$, $p = (x, y) \in A \times C$. Thus, $p \in (A \times B) \cup (A \times C)$

Since p was an arbitrary element of $A \times (B \cup C)$, it follows that $A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$.

Now let p be an arbitrary element of $p \in (A \times B) \cup (A \times C)$. Then $p \in A \times B$ or $p \in A \times C$.

Case 1. $p \in A \times B$. Then $p = (x, y)$ for some $x \in A$ and $y \in B$. Thus, $x \in A \vee (y \in B \vee y \in C)$. Therefore, $p \in A \times (B \cup C)$

Case 2. $p \in A \times C$. Then $p = (x, y)$ for some $x \in A$ and $y \in C$. Thus, $x \in A \vee (y \in B \vee y \in C)$. Therefore, $p \in A \times (B \cup C)$.

Since p was an arbitrary element of $p \in (A \times B) \cup (A \times C)$, it follows that $(A \times B) \cup (A \times C) \subseteq A \times (B \cup C)$, so $(A \times B) \cup (A \times C) = A \times (B \cup C)$

$(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$

1)

Givens	*Goal*
$p \in (A \times B) \cap (C \times D)$	$p \in (A \cap C) \times (B \cap D)$

Givens	*Goal*
$p \in A \times B$	$p \in (A \cap C) \times (B \cap D)$
$p \in C \times D$	

Givens	*Goal*
$x \in A$	$p \in (A \cap C) \times (B \cap D)$
$y \in B$	
$x \in C$	
$y \in D$	

2)

Givens	*Goal*
$p \in (A \cap C) \times (B \cap D)$	$p \in (A \times B) \cap (C \times D)$

Givens	*Goal*
$x \in A \cap C$	$p \in (A \times B) \cap (C \times D)$
$y \in B \cap D$	

Proof of 3

Let (x, y) be an arbitrary element of $(A \times B) \cap (C \times D)$. Then $(x, y) \in A \times B$ and $(x, y) \in C \times D$. Then $x \in A$ and $x \in C$, and $y \in B$ and $y \in D$. Therefore $(x, y) \in (A \cap C) \times (B \cap D)$. Since (x, y) was an arbitrary element of $(A \times B) \cap (C \times D)$, it follows that $(A \times B) \cap (C \times D) \subseteq (A \cap C) \times (B \cap D)$

Now let (x, y) be an arbitrary element of $(A \cap C) \times (B \cap D)$. Then $x \in A$ and $x \in C$, and $y \in B$ and $y \in D$. Therefore $(x, y) \in (A \times B) \cap (C \times D)$. Since (x, y) was an arbitrary element of $(A \cap C) \times (B \cap D)$, it follows that $(A \cap C) \times (B \cap D) \subseteq (A \times B) \cap (C \times D)$, so $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$.

6. The cases are not exhaustive.

7. If A has m elements and B has n elements, $A \times B$ have $m * n$ elements.

8.

$A \times (B \setminus C) = (A \times B) \setminus (A \times C)$

1)

Givens	*Goal*
$p \in A \times (B \setminus C)$	$p \in (A \times B) \setminus (A \times C)$

Givens	*Goal*
$x \in A$	$p \in A \times B \wedge \neg(p \in A \times C)$
$y \in B$	
$y \notin C$	

Givens	*Goal*
$x \in A$	$x \in A \wedge y \in B \wedge \neg(x \in A \wedge y \in C)$
$y \in B$	
$y \notin C$	

Givens	*Goal*
$x \in A$	$x \in A \wedge y \in B \wedge (x \notin A \vee y \notin C)$
$y \in B$	
$y \notin C$	

2)

Givens	*Goal*
$p \in (A \times B) \setminus (A \times C)$	$p \in A \times (B \setminus C)$

Givens	*Goal*
$x \in A \wedge y \in B \wedge (x \notin A \vee y \notin C)$	$x \in A \wedge y \in B \wedge y \notin C$

Case 1.

Givens	*Goal*
$x \in A \wedge y \in B \wedge y \notin C$	$x \in A \wedge y \in B \wedge y \notin C$

Case 2.

Givens	*Goal*
$x \in A \wedge y \in B \wedge x \notin A$	$x \in A \wedge y \in B \wedge y \notin C$

Givens	*Goal*
$x \in A \wedge y \in B$	$x \in A$

Theorem. $A \times (B \setminus C) = (A \times B) \setminus (A \times C)$

Proof. Let (x, y) be an arbitrary element of $A \times (B \setminus C)$. Then $x \in A$, $y \in B$ and $y \notin C$. Thus $x \in A \wedge y \in B \wedge (x \notin A \vee y \notin C)$. Therefore, $(x, y) \in (A \times B) \setminus (A \times C)$. Since (x, y) was an arbitrary element of $A \times (B \setminus C)$, it follows that $A \times (B \setminus C) \subseteq (A \times B) \setminus (A \times C)$.

Now let (x, y) be arbitrary element of $(A \times B) \setminus (A \times C)$. Then $x \in A \wedge y \in B \wedge (x \notin A \vee y \notin C)$.

Case 1. $x \notin A$. $x \notin A$ contradicts to $x \in A$.

Case 2. $y \notin C$. Then $x \in A$ and $y \in B$ and $y \notin C$. Therefore, $(x, y) \in A \times (B \setminus C)$

Since (x, y) was an arbitrary element of $(A \times B) \setminus (A \times C)$, it follows that $(A \times B) \setminus (A \times C) \subseteq A \times (B \setminus C)$, so $(A \times B) \setminus (A \times C) = A \times (B \setminus C)$.

9. $(A \times B) \setminus (C \times D) = [A \times (B \setminus D)] \cup [(A \setminus C) \times B]$

1)

Givens	*Goal*
$(x, y) \in (A \times B) \setminus (C \times D)$	$(x \in A \wedge y \in B \wedge y \notin D) \vee (x \in A \wedge x \notin C \wedge y \in B)$

$(x, y) \in (A \times B) \setminus (C \times D)$

$x \in A \wedge y \in B \wedge (x \notin C \vee y \notin D$

Givens	*Goal*
$x \in A \wedge y \in B \wedge (x \notin C \vee y \notin D)$	$(x, y) \in [A \times (B \setminus D)] \cup [(A \setminus C) \times B]$

$(x, y) \in [A \times (B \setminus D)] \cup [(A \setminus C) \times B]$

$(x \in A \wedge y \in B \wedge y \notin D) \vee (x \in A \wedge x \notin C \wedge y \in B)$

Givens	*Goal*
$x \in A \wedge y \in B \wedge (x \notin C \vee y \notin D)$	$(x \in A \wedge y \in B \wedge y \notin D) \vee (x \in A \wedge x \notin C \wedge y \in B)$

Givens	*Goal*
$x \in A \wedge y \in B \wedge (x \notin C \vee y \notin D)$	$x \in A \wedge y \in B \wedge (x \notin C \vee y \notin D)$

2)

Givens	*Goal*
$(x, y) \in [A \times (B \setminus D)] \cup [(A \setminus C) \times B]$	$(x, y) \in (A \times B) \setminus (C \times D)$

Givens	*Goal*
$x \in A \wedge y \in B \wedge (x \notin C \vee y \notin D)$	$x \in A \wedge y \in B \wedge (x \notin C \vee y \notin D)$

Theorem. $(A \times B) \setminus (C \times D) = [A \times (B \setminus D)] \cup [(A \setminus C) \times B]$

Proof. Let (x, y) be an arbitrary element of $(A \times B) \setminus (C \times D)$. Then $x \in A \wedge y \in B \wedge (x \notin C \vee y \notin D)$ which is equivalent to $(x \in A \wedge y \in B \wedge y \notin D) \vee (x \in A \wedge x \notin C \wedge y \in B)$. Thus $(x, y) \in [A \times (B \setminus D)] \cup [(A \setminus C) \times B]$. Since (x, y) was an arbitrary element of $(A \times B) \setminus (C \times D)$, it follows that $(A \times B) \setminus (C \times D) \subseteq [A \times (B \setminus D)] \cup [(A \setminus C) \times B]$.

Now let (x, y) be an arbitrary element of $[A \times (B \setminus D)] \cup [(A \setminus C) \times B]$. Then $(x \in A \wedge y \in B \wedge y \notin D) \vee (x \in A \wedge x \notin C \wedge y \in B)$ which is equivalent to $x \in A \wedge y \in B \wedge (x \notin C \vee y \notin D)$. Thus $(x, y) \in (A \times B) \setminus (C \times D)$. Since (x, y) was an arbitrary element of $[A \times (B \setminus D)] \cup [(A \setminus C) \times B]$, it follows that $[A \times (B \setminus D)] \cup [(A \setminus C) \times B] \subseteq (A \times B) \setminus (C \times D)$, so $(A \times B) \setminus (C \times D) = [A \times (B \setminus D)] \cup [(A \setminus C) \times B]$.

10. If $A \times B \cap C \times D = \varnothing$ then $A \cap B = \varnothing$ or $B \cap D = \varnothing$

Givens	*Goal*
$A \times B \cap C \times D = \varnothing$	$A \cap C = \varnothing \vee B \cap D = \varnothing$

Givens	*Goal*
$p \notin A \times B \cap C \times D$	$A \cap C = \varnothing \vee B \cap D = \varnothing$

$p \notin A \times B \cap C \times D$

$\neg(x \in A \wedge y \in B \wedge x \in C \wedge y \in D)$

$x \notin A \vee x \notin C \vee y \notin B \vee y \notin D$

Givens	*Goal*
$x \notin A \vee x \notin C \vee y \notin B \vee y \notin D$	$x \notin A \cap C \vee y \notin B \cap D$

Givens	*Goal*
$x \notin A \vee x \notin C \vee y \notin B \vee y \notin D$	$x \notin A \vee x \notin C \vee y \notin B \vee y \notin D$

Theorem. If $A \times B \cap C \times D = \varnothing$ then either $A \cap B = \varnothing$ or $B \cap D = \varnothing$.

Proof. Suppose $A \times B \cap C \times D = \varnothing$. Then $(x, y) \notin A \times B \cap C \times D$. Therefore, $x \notin A \vee x \notin C \vee y \notin B \vee y \notin D$. Then

$A \cap C = \varnothing \vee B \cap D = \varnothing$

$x \notin A \cap C \vee y \notin B \cap D$

$x \notin A \vee x \notin C \vee y \notin B \vee y \notin D$

Therefore, if $A \times B \cap C \times D = \varnothing$ then either $A \cap B = \varnothing$ or $B \cap D = \varnothing$.

11.

(a)

$\cup_{i \in I}(A_i \times B_i) \subseteq (\cup_{i \in I} A_i) \times (\cup_{i \in I} B_i)$

Givens	*Goal*
$p \in \cup_{i \in I}(A_i \times B_i)$	$p \in (\cup_{i \in I} A_i) \times (\cup_{i \in I} B_i)$

Givens	*Goal*
$\exists i(i \in I \wedge p \in A_i \times B_i)$	$p \in (\cup_{i \in I} A_i) \times (\cup_{i \in I} B_i)$

Givens	*Goal*
$i \in I$	$x \in (\cup_{i \in I} A_i) \wedge y \in (\cup_{i \in I} B_i)$
$p \in A_i \times B_i$	

Givens	*Goal*
$i \in I$	$\exists(i \in I \wedge x \in A_i) \wedge \exists(i \in I \wedge y \in B_i)$
$x \in A_i$	
$y \in B_i$	

Theorem. $\cup_{i \in I}(A_i \times B_i) \subseteq (\cup_{i \in I} A_i) \times (\cup_{i \in I} B_i)$.

Proof. Let p be an arbitrary. Suppose $p \in \cup_{i \in I}(A_i \times B_i)$. Let choose some i such that $i \in I$ and $p \in A_i \times B_i$. Then by definition of Cartesian product $x \in A_i$ and $y \in B_i$. Since $i \in I$ and $x \in A_i$, $x \in (\cup_{i \in I} A_i)$. Since $i \in I$ and $y \in B_i$, $y \in (\cup_{i \in I} B_i)$. Since $x \in (\cup_{i \in I} A_i)$ and $y \in (\cup_{i \in I} B_i)$, $p \in (\cup_{i \in I} A_i) \times (\cup_{i \in I} B_i)$. Since p was an arbitrary, $\cup_{i \in I}(A_i \times B_i) \subseteq (\cup_{i \in I} A_i) \times (\cup_{i \in I} B_i)$.

(b)

$\forall (i,j) \in I \times I (C_{(i,j)} = A_i \times B_j \wedge P = I \times I)$

Givens	*Goal*
$\forall (i,j) \in I \times I (C_{(i,j)} = A_i \times B_j \wedge P = I \times I)$	$\cup_{p \in P} C_p = (\cup_{i \in I} A_i) \times (\cup_{i \in I} B_i)$

1)

Givens	*Goal*
$\forall (i,j) \in I \times I (C_{(i,j)} = A_i \times B_j \wedge P = I \times I)$	$t \in (\cup_{i \in I} A_i) \times (\cup_{i \in I} B_i)$
$t \in \cup_{p \in P} C_p$	

Givens	*Goal*
$\forall (i,j) \in I \times I (C_{(i,j)} = A_i \times B_j \wedge P = I \times I)$	$t \in (\cup_{i \in I} A_i) \times (\cup_{i \in I} B_i)$
$\exists p (p \in P \wedge t \in C_p)$	

Givens	*Goal*
$\forall (i,j) \in I \times I (C_{(i,j)} = A_i \times B_j \wedge P = I \times I)$	$t \in (\cup_{i \in I} A_i) \times (\cup_{i \in I} B_i)$
$(i,i) \in I \times I \wedge t \in C_{(i,i)}$	

Givens	*Goal*
$C_{(i,i)} = A_i \times B_i$	$t \in (\cup_{i \in I} A_i) \times (\cup_{i \in I} B_i)$
$t \in C_{(i,i)}$	

Givens	*Goal*
$t \in A_i \times B_i$	$t \in (\cup_{i \in I} A_i) \times (\cup_{i \in I} B_i)$

Givens	*Goal*
$x \in A_i$	$x \in (\cup_{i \in I} A_i) \wedge y \in (\cup_{i \in I} B_i)$
$y \in B_i$	

2)

Givens	*Goal*
$\forall (i,j) \in I \times I (C_{(i,j)} = A_i \times B_j \wedge P = I \times I)$	$t \in \cup_{p \in P} C_p$
$t \in (\cup_{i \in I} A_i) \times (\cup_{i \in I} B_i)$	

Givens	*Goal*
$\forall (i,j) \in I \times I (C_{(i,j)} = A_i \times B_j \wedge P = I \times I)$	$\exists p (p \in P \wedge t \in C_p)$
$\exists i (i \in I \wedge x \in A_i)$	
$\exists i (i \in I \wedge y \in B_i)$	

Givens	*Goal*
$\forall (i,j) \in I \times I (C_{(i,j)} = A_i \times B_j \wedge P = I \times I)$	$\exists p (p \in P \wedge t \in C_p)$
$(i,j) \in I \times I$	
$x \in A_i$	
$y \in B_j$	

Givens	*Goal*
$C_{(i,j)} = A_i \times B_j$	$\exists p (p \in P \wedge t \in C_p)$
$(i,j) \in I \times I$	
$P = I \times I$	
$t = (x,y)$	
$x \in A_i$	
$y \in B_j$	

Givens	*Goal*
$C_{(i,j)} = A_i \times B_j$	$\exists p (p \in P \wedge t \in C_p)$
$(i,j) \in P$	
$t \in A_i \times B_j$	

Givens	*Goal*
$t \in C_{(i,j)}$	$\exists p (p \in P \wedge t \in C_p)$
$(i,j) \in P$	

Theorem. Suppose $\forall(i,j) \in I \times I(C_{(i,j)} = A_i \times B_j \land P = I \times I)$. Then $\cup_{p \in P} C_p = (\cup_{i \in I} A_i) \times (\cup_{i \in I} B_i)$.

Proof. Let t be an arbitrary element of $\cup_{p \in P} C_p$. Then we can choose some p such that $p \in I \times I$ and $t \in C_p$. Since $\forall(i,j) \in I \times I(C_{(i,j)} = A_i \times B_j \land P = I \times I)$, then in particular $C_p = A_i \times B_j$ and $P = I \times I$. Since $t \in C_p$ and $C_p = A_i \times B_j$, $t \in A_i \times B_j$. Since $t \in A_i \times B_i$, $x \in A_i$ and $y \in B_i$. Since $x \in A_i$, $x \in \cup_{i \in I} A_i$. Since $y \in B_i$, $y \in \cup_{i \in I} B_i$. Since $x \in \cup_{i \in I} A_i$ and $y \in \cup_{i \in I} B_i$, $t \in (\cup_{i \in I} A_i) \times (\cup_{i \in I} B_i)$. Since t was an arbitrary element of $\cup_{p \in P} C_p$, it follows that $\cup_{p \in P} C_p \subseteq (\cup_{i \in I} A_i) \times (\cup_{i \in I} B_i)$.

Now let t be arbitrary element of $(\cup_{i \in I} A_i) \times (\cup_{i \in I} B_i)$. Then $x \in \cup_{i \in I} A_i$ and $y \in \cup_{i \in I} B_i$. Since $x \in \cup_{i \in I} A_i$ we can choose some i such that $x \in A_i$ and $i \in I$. Since $y \in \cup_{i \in I} B_i$ we can choose some j such that $y \in B_j$ and $j \in I$. Since $i \in I$ and $j \in I$, $(i,j) \in I \times I$. Since $(i,j) \in I \times I$ and $\forall(i,j) \in I \times I(C_{(i,j)} = A_i \times B_j \land P = I \times I)$, $C_{(i,j)} = A_i \times B_j$ and $P = I \times I$. Since $P = I \times I$ and $(i,j) \in I \times I$, $(i,j) \in P$. Since $C_{(i,j)} = A_i \times B_j$ and $t \in A_i \times B_j$, $t \in C_{(i,j)}$. Since $(i,j) \in P$ and $t \in C_{(i,j)}$, let $p = (i,j)$, so $t \in \cup_{p \in P} C_p$. Since t was an arbitrary element of $(\cup_{i \in I} A_i) \times (\cup_{i \in I} B_i)$, it follows that $(\cup_{i \in I} A_i) \times (\cup_{i \in I} B_i) \subseteq \cup_{p \in P} C_p$, so $\cup_{p \in P} C_p = (\cup_{i \in I} A_i) \times (\cup_{i \in I} B_i)$.

12.

Givens	*Goal*
$A \times B \subseteq C \times D$	$A \subseteq C \land B \subseteq D$

Givens	*Goal*
$A \times B \subseteq C \times D$	$A \subseteq C \land B \subseteq D$
$(a,b) \in A \times B$	

Givens	*Goal*
$A \times B \subseteq C \times D$	$A \subseteq C \wedge B \subseteq D$
$(a, b) \in A \times B$	
$(a, b) \in C \times D$	

Givens	*Goal*
$a \in C$	$A \subseteq C \wedge B \subseteq D$
$b \in D$	

Givens	*Goal*
$a \in C$	$A \subseteq C$
$a \in A$	

Givens	*Goal*
$a \in C$	$\forall x(x \in A \rightarrow x \in C)$
$a \in A$	

Givens	*Goal*
$a \in C$	$x \in C$
$a \in A$	
$x \in A$	

"Since a and b were arbitrary elements of A and B, respectively, this shows that $A \subseteq C$ and $B \subseteq D$" is wrong conclusion. Having $a \in C$ and $a \in A$, it's not possible to prove that $A \subseteq C$.

Theorem is incorrect.

Counterexample:

$A = \{1\}$

$C = \varnothing$

$B = \varnothing$

$D = \varnothing$

$A \times B = \varnothing$

$C \times D = \varnothing$

2 Relations

1.

(a) $R = \{(p, q) \in P \times P \mid \text{the person p is a parent of the person q}\}$

$Dom(R) = \{p \in P \mid \exists q \in P((p, q) \in R)\}$

$Dom(R) = \{p \in P \mid \exists q \in P(\text{the person p is a parent of the person q})\}$

$Dom(R) = \{p \in P \mid \text{the person p is a parent of some person}\}$

$Dom(R) = \{p \in P \mid \text{p has a living child}\}$

$Ran(R) = \{q \in P \mid \exists p \in P((p, q) \in R)\}$

$Ran(R) = \{q \in P \mid \exists p \in P(\text{the person p is a parent of the person q})\}$

$Ran(R) = \{q \in P \mid \text{some person is a parent of the person q}\}$

$Ran(R) = \{q \in P \mid \text{q has a living parent}\}$

(b) $L = \{(x, y) \in \mathbb{R}^2 \mid y > x^2\}$

$Dom(L) = \{x \in \mathbb{R} \mid \exists y \in \mathbb{R}((x, y) \in L)\}$

$Dom(L) = \{x \in \mathbb{R} \mid \exists y \in \mathbb{R}(y > x^2)\}$

$Dom(L) = \mathbb{R}$

$Ran(L) = \{y \in \mathbb{R} \mid \exists x \in \mathbb{R}((x, y) \in L)\}$

$Ran(L) = \{y \in \mathbb{R} \mid \exists x \in \mathbb{R}(y > x^2)\}$

$Ran(L) = \mathbb{R}^+$

2.

(a)

$P = \{(p, q) \in P \times P \mid \text{the person p is a brother of the person q}\}$

$Dom(P) = \{p \in P \mid \exists q \in P((p, q) \in P)\}$

$Dom(P) = \{p \in P \mid \exists q \in P(\text{the person p is a brother of the person q})\}$

$Dom(P) = \{p \in P \mid$ the person p is a brother of some person$\}$

$Ran(P) = \{q \in P \mid$ some person is a brother of person q$\}$

(b)

$L = \{(x, y) \in \mathbb{R}^2 \mid y^2 = 1 - 2/(x^2 + 1)\}$

$Dom(P) = \{x \in \mathbb{R} \mid \exists y \in \mathbb{R}(y^2 = 1 - 2/(x^2 + 1))\}$

$Dom(P) = \{x \in \mathbb{R} \mid |x| \geq 1\}$

$Ran(P) = \{y \in \mathbb{R} \mid \exists x \in \mathbb{R}(y^2 = 1 - 2/(x^2 + 1))\}$

$Ran(P) = \{y \in \mathbb{R} \mid |y| < 1\}$

3.

(a)

$L = \{(s, r) \in S \times R \mid$ the student s lives in the dorm room r$\}$

$L^{-1} \circ L$

Because L is a relation from S to R and L^{-1} is a relation from R to S.

$L^{-1} \circ L$ is the relation from S to S defined as follows.

$L^{-1} \circ L = \{(s, t) \in S \times S \mid \exists r \in R((s, r) \in L$ and $(r, t) \in L^{-1})\}$

$= \{(s, t) \in S{\times}S \mid \exists r \in R($the studend s lives in the dorm room r, and so is the student t$)\}$

$= \{(s, t) \in S{\times}S \mid$ there is some room that the students s and r are both live in$\}$

(b) $E \circ (L^{-1} \circ L)$

We saw in part (a) that $L^{-1} \circ L$ is a relation from S to S, and E is a relation from S to C, so $E \circ (L^{-1} \circ L)$ is the relation from S to C defined as follows.

$E \circ (L^{-1} \circ L) = \{(r, p) \in S \times C \mid \exists s \in S((r, s) \in L^{-1} \circ L$ and $(s, p) \in E)\}$

$= \{(r, p) \in S \times C \mid \exists s \in S($there is some room that the students r and s are both live in, and the student s is enrolled in the course p$)\}$

$= \{(r, p) \in S{\times}C \mid$ (some student who lives in some room with the student r is enrolled in the course p$)\}$

4.

(a) $S \circ R$ is the relation from A to B.

$S \circ R = \{(r, p) \in A \times B \mid \exists b \in B((r, b) \in R \text{ and } (b, p) \in S)\}$

$= \{(1, 5), (1, 6), (1, 4), (2, 4), (3, 6)\}$

(b) $S \circ S^{-1}$ is the relation from B to B.

$S \circ S^{-1} = \{(r, p) \in B \times B \mid \exists b \in B((r, b) \in S^{-1} \text{ and } (b, p) \in S)$

$= \{(r, p) \in B \times B \mid \exists b \in B((b, r) \in S \text{ and } (b, p) \in S)$

$= \{(5, 5), (5, 6), (6, 5), (6, 6), (4, 4)\}$

5.

(a)

S^{-1} is the relation from C to B.

R is the relation from A to C.

$S^{-1} \circ R$ is the relation from A to B.

$S^{-1} \circ R = \{(r, p) \in A \times B \mid \exists c \in C((r, c) \in R \text{ and } (c, p) \in S^{-1})\}$

$= \{(r, p) \in A \times B \mid \exists c \in C((r, c) \in R \text{ and } (p, c) \in S)\}$

$= \varnothing$

(b)

R^{-1} is the relation from C to A.

S is the relation from B to C.

$R^{-1} \circ S$ is the relation from B to A.

$R^{-1} \circ S = \{(r, p) \in B \times A \mid \exists c \in C((r, c) \in S \text{ and } (c, p) \in R^{-1})\}$

$= \{(r, p) \in B \times A \mid \exists c \in C((r, c) \in S \text{ and } (p, c) \in R)\}$

$= \varnothing$

6.

(a) $Ran(R^{-1}) = Dom(R)$

First note that $Ran(R^{-1})$ and $Dom(R)$ are both subsets of A. Now let a be an arbitrary element of A. Then

$a \in Ran(R^{-1})$ iff $\exists b \in B((b,a) \in R^{-1})$

iff $\exists b \in B((a,b) \in R)$ iff $a \in Dom(R)$.

(b)

$Dom(R^{-1}) = Ran(R)$

$(Dom(R^{-1}))^{-1} = (Ran(R))^{-1}$

$Dom((R^{-1})^{-1}) = Ran(R^{-1})$

$Dom(R) = Ran(R^{-1})$

$Ran(R^{-1}) = Dom(R)$

(c)

Now suppose $(a,d) \in (T \circ S) \circ R$. By the definition of composition, this means that we can choose some $b \in B$ such that $(a,b) \in R$ and $(b,d) \in T \circ S$. Since $(b,d) \in T \circ S$, we can again use the definition of composition and choose some $c \in C$ such that $(b,c) \in S$ and $(c,d) \in T$. Now since $(a,b) \in R$ and $(b,c) \in S$, we can conclude that $(a,c) \in S \circ R$. Similarly, since $(a,c) \in S \circ R$ and $(c,d) \in T$, it follows that $(a,d) \in T \circ (S \circ R)$

(d)

$(S \circ R)^{-1} = R^{-1} \circ S^{-1}$

Clearly $(S \circ R)^{-1}$ and $R^{-1} \circ S^{-1}$ are both relations from C to A. Let (c,a) be an arbitrary element of $C \times A$.

$(c,a) \in (S \circ R)^{-1}$ iff $(a,c) \in S \circ R$

iff $\exists B((a,b) \in R$ and $(b,c) \in S)$

iff $\exists B((b,a) \in R^{-1}$ and $(c,b) \in S^{-1})$

iff $(c,a) \in R^{-1} \circ S^{-1}$

7. $E \circ E \subseteq F$

8.

(a) $Dom(S \circ R) \subseteq Dom(R)$

$S \circ R$ is the relation from A to C.

$Dom(S \circ R)$ is subset of A.

$Dom(R)$ is subset of A.

Givens	*Goal*
	$\forall t(t \in Dom(S \circ R) \to t \in Dom(R))$

Let a be an arbitrary element from A.

Givens	*Goal*
	$a \in Dom(S \circ R) \to a \in Dom(R)$

Givens	*Goal*
$a \in Dom(S \circ R)$	$a \in Dom(R)$

$a \in Dom(S \circ R)$

$\exists c \in C((a, c) \in S \circ R)$

Let choose some $c \in C$ such that $(a, c) \in S \circ R$

Givens	*Goal*
$(a, c) \in S \circ R$	$a \in Dom(R)$
$c \in C$	

$(a, c) \in S \circ R$

$= \{(a, c) \in A \times C \mid \exists b \in B((a, b) \in R \text{ and } (b, c) \in S)\}$

$a \in Dom(R) = \exists b \in B((a, b) \in R)$

Givens	*Goal*
$\{(a,c) \in A \times C \mid \exists b \in B((a,b) \in R \text{ and } (b,c) \in S)\}$	$\exists b \in B((a,b) \in R)$
$c \in C$	

Givens	*Goal*
$(a,b) \in R$	$\exists b \in B((a,b) \in R)$
$(b,c) \in S$	
$c \in C$	
$b \in B$	

Givens	*Goal*
$(a,b) \in R$	$(a,b) \in R$
$(b,c) \in S$	
$c \in C$	
$b \in B$	

Theorem. $Dom(S \circ R) \subseteq Dom(R)$

Proof. Clearly $Dom(S \circ R)$ and $Dom(R)$ is subset of A. Let a be an arbitrary element of A. Suppose $a \in Dom(S \circ R)$. Then, let choose some $c \in C$ such that $(a,c) \in S \circ R$. Then, by definition of composition we can choose some $b \in B$ such that $(a,b) \in R$ and $(b,c) \in S$. So, since $(a,b) \in R$ and $b \in B$ we can conclude that $a \in Dom(R)$. Since a was an arbitrary element of A, it follows that $Dom(S \circ R) \subseteq Dom(R)$.

(b)

If $Ran(R) \subseteq Dom(S)$ then $Dom(S \circ R) = Dom(R)$.

R is a relation from A to B.

S is a relation from B to C.

Givens	*Goal*
$Ran(R) \subseteq Dom(S)$	$Dom(S \circ R) = Dom(R)$

$S \circ R$ is a relation from A to C.

$Dom(S \circ R)$ is a subset of A.

$Dom(R)$ is subset of A.

Let a be an arbitrary element of A.

$(\rightarrow)$

Givens	*Goal*
$Ran(R) \subseteq Dom(S)$	$a \in Dom(R)$
$a \in Dom(S \circ R)$	
$a \in A$	

$a \in Dom(S \circ R)$

iff $\exists c \in C((a, c) \in S \circ R)$

Givens	*Goal*
$Ran(R) \subseteq Dom(S)$	$a \in Dom(R)$
$c \in C$	
$(a, c) \in S \circ R$	

$(a, c) \in S \circ R$

iff $\{(a, c) \in S \times R \mid \exists b \in B((a, b) \in R \text{ and } (b, c) \in S)\}$

Givens	*Goal*
$Ran(R) \subseteq Dom(S)$	$a \in Dom(R)$
$c \in C$	
$b \in B$	
$(a, b) \in R$	
$(b, c) \in S$	

$a \in Dom(R)$

iff $\exists b \in B((a,b) \in R)$

Givens	*Goal*
$Ran(R) \subseteq Dom(S)$	$\exists b \in B((a,b) \in R)$
$c \in C$	
$b \in B$	
$(a,b) \in R$	
$(b,c) \in S$	

$(\leftarrow)$

Givens	*Goal*
$Ran(R) \subseteq Dom(S)$	$a \in Dom(S \circ R)$
$a \in A$	
$a \in Dom(R)$	

$Ran(R)$ and $Dom(S)$ are subsets of B.

$a \in Dom(R)$

iff $\exists b \in B((a,b) \in R)$

Givens	*Goal*
$\forall b \in B(b \in Ran(R) \rightarrow b \in Dom(S))$	$a \in Dom(S \circ R)$
$a \in A$	
$b \in B$	
$(a,b) \in R$	

Givens	*Goal*
$b \in Ran(R) \rightarrow b \in Dom(S)$	$a \in Dom(S \circ R)$
$a \in A$	
$(a,b) \in R$	

Givens	*Goal*
$\exists a \in A((a,b) \in R) \rightarrow b \in Dom(S)$	$a \in Dom(S \circ R)$
$a \in A$	
$(a,b) \in R$	

Givens	*Goal*
$b \in Dom(S)$	$a \in Dom(S \circ R)$
$a \in A$	
$(a,b) \in R$	

$b \in Dom(S)$
iff $\exists c \in C((b,c) \in S)$

Givens	*Goal*
$(b,c) \in S$	$a \in Dom(S \circ R)$
$a \in A$	
$c \in C$	
$(a,b) \in R$	

Givens	*Goal*
$(a,c) \in S \circ R$	$a \in Dom(S \circ R)$
$c \in C$	

Givens	*Goal*
$(a,c) \in S \circ R$	$\exists c \in C((a,c) \in S \circ R)$
$c \in C$	

Theorem. If $Ran(R) \subseteq Dom(S)$ then $Dom(S \circ R) = Dom(R)$.

Proof. Suppose $Ran(R) \subseteq Dom(S)$. Clearly $Dom(S \circ R)$ and $Dom(R)$ are subsets of A. Then, let a be an arbitrary element of A.

Suppose $a \in Dom(S \circ R)$. Let choose some $c \in C$ such that $(a, c) \in S \circ R$. By definition of composition we can choose some $b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$. Since $b \in B$ and $(a, b) \in R$, it follows that $a \in Dom(R)$.

Suppose $a \in Dom(R)$. Clearly $Ran(R)$ and $Dom(S)$ are subsets of B. Since $a \in Dom(R)$, we can choose some $b \in B$ such that $(a, b) \in R$. Since $b \in B$ and for all $b \in B$ we have $b \in Ran(R) \to b \in Dom(S)$, it follows that $b \in Ran(R) \to b \in Dom(S)$. Since $a \in A$ and $(a, b) \in R$, $b \in Ran(R)$. Since $b \in Ran(R)$ and $b \in Ran(R) \to b \in Dom(S)$, $b \in Dom(S)$. Since $b \in Dom(S)$, we can choose some $c \in C$ such that $(b, c) \in S$. Since $(a, b) \in R$ and $(b, c) \in S$, $(a, c) \in S \circ R$. Since $c \in C$ and $(a, c) \in S \circ R$, it follows that $a \in Dom(S \circ R)$.

Since a was an arbitrary element of A, $Dom(S \circ R) = Dom(R)$.

(c) $Ran(S \circ R) \subseteq Ran(S)$

$S \circ R$ is relation from A to C.

$Ran(S \circ R)$ is subset of C

$Ran(S)$ is subset of C

Givens	*Goal*
$c \in Ran(S \circ R)$	$c \in Ran(S)$

Givens	*Goal*
$\exists a \in A((a, c) \in S \circ R)$	$c \in Ran(S)$

Givens	*Goal*
$a \in A$	$c \in Ran(S)$
$(a, c) \in S \circ R$	

Givens	*Goal*
$a \in A$	$c \in Ran(S)$
$\{(a, c) \in A \times C \mid \exists b \in B((a, b) \in S \text{ and } (b, c) \in R)\}$	

Givens	*Goal*
$a \in A$	$\exists b \in B((b, c) \in S)$
$\{(a, c) \in A \times C \mid \exists b \in B((a, b) \in R \text{ and } (b, c) \in S)\}$	

Givens	*Goal*
$a \in A$	$\exists b \in B((b, c) \in S)$
$b \in B$	
$(b, c) \in S$	

Theorem. $Ran(S \circ R) \subseteq Ran(S)$

Proof. Clearly $Ran(S \circ R)$ and $Ran(S)$ are subsets of C. Let c be an arbitrary element of C. Suppose $c \in Ran(S \circ R)$. Then we can choose some $a \in A$ such that $(a, c) \in S \circ R$. By definition of composition, we can choose some $b \in B$ such that $(b, c) \in S$ and $(a, b) \in R$. Since $b \in B$ and $(b, c) \in S$, we can conclude that $c \in Ran(S)$. Since c was an arbitrary element of C, $Ran(S \circ R) \subseteq Ran(S)$.

If $Dom(S) \subseteq Ran(R)$ then $Ran(S \circ R) = Ran(S)$.

$S \circ R$ is relation from A to C.

$Ran(S \circ R)$ is subset of C.

$Ran(S)$ is subset of C.

Givens	*Goal*
$Dom(S) \subseteq Ran(R)$	$Ran(S \circ R) = Ran(S)$

Givens	*Goal*
$Dom(S) \subseteq Ran(R)$	$c \in Ran(S)$
$c \in Ran(S \circ R)$	
$c \in C$	

Givens	*Goal*
$Dom(S) \subseteq Ran(R)$	$c \in Ran(S)$
$\exists a \in A((a, c) \in S \circ R)$	

Givens	*Goal*
$Dom(S) \subseteq Ran(R)$	$c \in Ran(S)$
$a \in A$	
$\{(a, c) \in A \times C \mid \exists b \in B((a, b) \in R \text{ and } (b, c) \in S)\}$	

Givens	*Goal*
$Dom(S) \subseteq Ran(R)$	$\exists b \in B((b, c) \in S)$
$a \in A$	
$b \in B$	
$(a, b) \in R$	
$(b, c) \in S$	

$(\leftarrow)$

Givens	*Goal*
$Dom(S) \subseteq Ran(R)$	$c \in Ran(S \circ R)$
$c \in Ran(S)$	
$c \in C$	

Givens	*Goal*
$Dom(S) \subseteq Ran(R)$	$c \in Ran(S \circ R)$
$\exists b \in B((b, c) \in S)$	
$c \in C$	

Givens	*Goal*
$\forall b \in B(b \in Dom(S) \rightarrow b \in Ran(R))$	$c \in Ran(S \circ R)$
$b \in B$	
$(b, c) \in S$	
$c \in C$	

Givens	*Goal*
$b \in Dom(S) \rightarrow b \in Ran(R)$	$c \in Ran(S \circ R)$
$b \in B$	
$(b, c) \in S$	
$c \in C$	

Givens	*Goal*
$\exists c \in C((b, c) \in C) \rightarrow b \in Ran(R)$	$c \in Ran(S \circ R)$
$b \in B$	
$(b, c) \in S$	
$c \in C$	

Givens	*Goal*
$b \in Ran(R)$	$c \in Ran(S \circ R)$
$b \in B$	
$(b, c) \in S$	
$c \in C$	

Givens	*Goal*
$\exists a \in A((a, b) \in R)$	$c \in Ran(S \circ R)$
$b \in B$	
$(b, c) \in S$	
$c \in C$	

Givens	*Goal*
$(a, b) \in R$	$c \in Ran(S \circ R)$
$a \in A$	
$b \in B$	
$(b, c) \in S$	
$c \in C$	

Givens	*Goal*
$(a, c) \in S \circ R$	$c \in Ran(S \circ R)$
$c \in C$	

Givens	*Goal*
$(a, c) \in S \circ R$	$\exists a \in A((a, c) \in S \circ R)$
$a \in A$	

Theorem. If $Dom(S) \subseteq Ran(R)$ then $Ran(S \circ R) = Ran(S)$.

Proof. Suppose $Dom(S) \subseteq Ran(R)$. Clearly $Ran(S \circ R)$ and $Ran(S)$ are subsets of C. Let c be an arbitrary element from C.

Suppose $c \in Ran(S \circ R)$. Then we can choose some $ainA$ such that $(a, c) \in S \circ R$. By definition of composition we can choose some $b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$. Since $b \in B$ and $(b, c) \in S$, it follows that $c \in Ran(S)$.

Supoose $c \in Ran(S)$. Then we can choose some $b \in B$ such that $(b, c) \in S$. Since $c \in C$, $b \in B$, $(b, c) \in S$ and $Dom(S) \subseteq Ran(R)$, we can conclude that $b \in Ran(R)$. Since $b \in Ran(R)$ we can choose some $a \in A$ such that $(a, b) \in R$. Since $(b, c) \in S$ and $(a, b) \in R$, $(a, c) \in S \circ R$. Since $(a, c) \in S \circ R$ and $a \in A$, it follows that $c \in Ran(S \circ R)$.

Since c was an arbitrary element from C, $Ran(S \circ R) = Ran(S)$.

9. R is relation from A to B.

S is relation from A to B.

(a) $R \subseteq Dom(R) \times Ran(R)$

$R \subseteq A \times B$

$Dom(R)$ is subset of A

$Ran(R)$ is subset of B

Givens	*Goal*
$(a, b) \in R$	$(a, b) \in Dom(R) \times Ran(R)$
$\forall t(t \in R \rightarrow t \in A \times B)$	

Givens	*Goal*
$(a, b) \in A \times B$	$\exists b \in B((a, b) \in R) \wedge \exists a \in A((a, b) \in R)$
$a \in A$	
$b \in B$	

Theorem. $R \subseteq Dom(R) \times Ran(R)$

Proof. Clearly $Dom(R) \times Ran(R)$ is relation from A to B. Let (a, b) be an arbitrary element from A to B relation. Suppose $(a, b) \in R$. Since $(a, b) \in R$ and R is relation from A to B, it follows that $(a, b) \in A \times B$. Since $(a, b) \in A \times B$, $a \in A$ and $b \in B$. Since $b \in B$ and $(a, b) \in R$, $(a, b) \in Dom(R)$. Since $a \in A$ and $(a, b) \in R$, $(a, b) \in Ran(R)$. Since $(a, b) \in Dom(R)$ and $(a, b) \in Ran(R)$, $(a, b) \in Dom(R) \times Ran(R)$. Since $(a, b) \in R$ was an arbitrary element from A to B relation, therefore $R \subseteq Dom(R) \times Ran(R)$.

(b) If $R \subseteq S$ then $R^{-1} \subseteq S^{-1}$.

Givens	*Goal*
$R \subseteq S$	$R^{-1} \subseteq S^{-1}$

$R^{-1} \subseteq S^{-1}$

$\forall (b, a) \in B \times A((b, a) \in R^{-1} \to (b, a) \in S^{-1})$

Let (b, a) be an arbitrary element from $B \times A$.

Givens	*Goal*
$R \subseteq S$	$(a, b) \in S$
$(a, b) \in R$	

Givens	*Goal*
$\forall t(t \in R \to t \in S)$	$(a, b) \in S$
$(a, b) \in R$	

Givens	*Goal*
$(a, b) \in S$	$(a, b) \in S$

Theorem. If $R \subseteq S$ then $R^{-1} \subseteq S^{-1}$.

Proof. Suppose $R \subseteq S$. Then,

$R \subseteq S$

iff $\forall(a,b) \in A \times B((a,b) \in R \rightarrow (a,b) \in S)$

iff $\forall(b,a) \in B \times A((b,a) \in R^{-1} \rightarrow (b,a) \in S^{-1})$

iff $R^{-1} \subseteq S^{-1}$.

(c)

$(R \cup S)^{-1} = R^{-1} \cup S^{-1}$

$\rightarrow$

Givens	*Goal*
$(b,a) \in (R \cup S)^{-1}$	$(b,a) \in R^{-1} \cup S^{-1}$

Givens	*Goal*
$(b,a) \in (R \cup S)^{-1}$	$(b,a) \in R^{-1} \cup S^{-1}$

Theorem. $(R \cup S)^{-1} = R^{-1} \cup S^{-1}$

Proof. Clearly $(R \cup S)^{-1}$ and $R^{-1} \cup S^{-1}$ are relation from B to A. Let (b,a) be an arbitrary ordered pair in $B \times A$. Then

$(b,a) \in (R \cup S)^{-1}$ iff $(a,b) \in R \cup S$

iff $(a,b) \in R \vee (a,b) \in S$

iff $(b,a) \in R^{-1} \vee (b,a) \in S^{-1}$

iff $(b,a) \in (R^{-1} \cup S^{-1})$

10.

R is relation from A to B.

S is relation from B to C.

$\rightarrow$

Givens	*Goal*
$S \circ R = \varnothing$	$Ran(R) \cap Dom(S) = \varnothing$

Suppose $Ran(R) \cap Dom(S) \neq \varnothing$, then we can choose some b such that $b \in Ran(R) \cap Dom(S)$. Then $b \in Ran(R)$ and $b \in Dom(S)$.

Givens	*Goal*
$b \in Ran(R)$	$S \circ R \neq \varnothing$
$b \in Dom(S)$	

Givens	*Goal*
$b \in Ran(R)$	$\exists m(m \in S \circ R)$
$b \in Dom(S)$	

Givens	*Goal*
$\exists a \in A((a, b) \in R)$	$\exists m(m \in S \circ R)$
$\exists c \in C((b, c) \in S)$	

Givens	*Goal*
$(a, c) \in S \circ R$	$\exists m(m \in S \circ R)$
$a \in A$	
$c \in C$	

$\leftarrow$

Givens	*Goal*
$Ran(R) \cap Dom(S) = \varnothing$	$S \circ R = \varnothing$

Suppose $S \circ R \neq \varnothing$. Then $\exists m \in A \times C(m \in S \circ R)$.

Givens	*Goal*
$(a,c) \in S \circ R$	$\exists b \in B(b \in Ran(R) \cap Dom(S))$

Givens	*Goal*
$\{(a,c) \in A \times C \mid \exists b \in B((a,b) \in R \wedge (b,c) \in S)\}$	$\exists b \in B(b \in Ran(R) \cap Dom(S))$

Givens	*Goal*
$b \in B$	$\exists b \in B(\exists a \in A((a,b) \in R) \wedge \exists c \in C((b,c) \in S))$
$a \in A$	
$c \in C$	
$(a,b) \in R$	
$(b,c) \in S$	

Theorem. $S \circ R = \varnothing$ iff $Ran(R) \cap Dom(S) = \varnothing$.

Proof. Suppose $Ran(R) \cap Dom(S) \neq \varnothing$. Clearly $Ran(R) \cap Dom(S)$ is subset of B. Then we can choose some $b \in B$ such that $b \in Ran(R)$ and $b \in Dom(S)$. Since $b \in Ran(R)$, we can choose some $a \in A$ such that $(a,b) \in R$. Since $b \in Dom(S)$, we can choose some $c \in C$ such that $(b,c) \in S$. Since $(a,b) \in R$ and $(b,c) \in S$, $(a,c) \in S \circ R$. But it contradicts to $S \circ R = \varnothing$, therefore $Ran(R) \cap Dom(S) = \varnothing$.

Suppose $S \circ R \neq \varnothing$. Clearly $S \circ R$ is ordered pairs of $A \times C$. Then we can choose some $(a,c) \in A \times C$ such that $(a,c) \in S \circ R$. Since $(a,c) \in S \circ R$, it follows that $a \in A$ and $(a,b) \in R$, so $b \in Ran(R)$. Since $(a,c) \in S \circ R$, it follows that $c \in C$ and $(b,c) \in S$, so $b \in Dom(S)$. Since $b \in B$ and $b \in Ran(R)$ and $b \in Dom(S)$, it follows that $Ran(R) \cap Dom(S) \neq \varnothing$. But it contradicts to $Ran(R) \cap Dom(S) = \varnothing$, therefore $S \circ R = \varnothing$.

Therefore, $S \circ R = \varnothing$ iff $Ran(R) \cap Dom(S) = \varnothing$.

11.

R is relation from A to B.

S and T are relations from B to C.

(a)

$(S \circ R) \setminus (T \circ R) \subseteq (S \setminus T) \circ R$

$S \circ R$ is relation from A to C.

$T \circ R$ is relation from A to C.

$(S \circ R) \setminus (T \circ R)$ is relation from A to C

$S \setminus T$ is relation from B to C

$(S \setminus T) \circ R$ is relation from A to C

Givens	*Goal*
$(a, c) \in (S \circ R) \setminus (T \circ R)$	$(a, c) \in (S \setminus T) \circ R$

Givens	*Goal*
$\{(a, c) \in A \times C \mid \exists b \in B((a, b) \in R \wedge (b, c) \in S)\}$	$\{(a, c) \in A \times C \mid \exists b \in B((a, b) \in R \wedge (b, c) \in S \setminus T)\}$
$(a, c) \notin T \circ R$	

Givens	*Goal*
$b \in B$	$(a, b) \in R \wedge (b, c) \in S \setminus T$
$(a, b) \in R$	
$(b, c) \in S$	
$\neg(\{(a, c) \in A \times C \mid \exists b((a, b) \in R \wedge (b, c) \in T)\})$	

Givens	*Goal*
$b \in B$	$(a, b) \in R \wedge (b, c) \in S \setminus T$
$(a, b) \in R$	
$(b, c) \in S$	
$\forall b((a, b) \notin R \vee (b, c) \notin T)$	

Givens	*Goal*
$b \in B$	$(a,b) \in R \wedge (b,c) \in S \setminus T$
$(a,b) \in R$	
$(b,c) \in S$	
$(a,b) \notin R \vee (b,c) \notin T$	

Case 1.

$(a,b) \notin R$

Contradicts to $(a,b) \in R$.

Case 2.

$(b,c) \notin T$

Givens	*Goal*
$(b,c) \notin T$	$(a,b) \in R \wedge (b,c) \in S \setminus T$
$(a,b) \in R$	
$(b,c) \in S$	

Theorem. $(S \circ R) \setminus (T \circ R) \subseteq (S \setminus T) \circ R$

Proof. Clearly, $(S \circ R) \setminus (T \circ R)$ and $(S \setminus T) \circ R$ are relation from A to C. Then we can choose some (a,c) from ordered pairs $A \times C$. Suppose $(a,c) \in (S \circ R) \setminus (T \circ R)$. Then $(a,c) \in S \circ R$ and $(a,c) \notin T \circ R$. Since $(a,c) \in S \circ R$ we can choose some $b \in B$ such that $(a,b) \in R$ and $(b,c) \in S$. Since $(a,c) \notin T \circ R$, it follows that in particular $(a,b) \notin R \vee (b,c) \notin T$. Suppose $(a,b) \notin R$, but it contradicts to $(a,b) \in R$, so $(a,c) \in (S \setminus T) \circ R$. Now suppose $(b,c) \notin T$. Since $(a,b) \in R$, $(b,c) \in S$ and $(b,c) \notin T$, it follow that $(a,c) \in (S \setminus T) \circ R$. Therefore, $(S \circ R) \setminus (T \circ R) \subseteq (S \setminus T) \circ R$.

(b) "Similarly, since $(a,b) \in R$ and $(b,c) \notin T$, $(a,c) \notin T \circ R$" is wrong conclusion.

$(a,c) \notin T \circ R$ is $(b,c) \notin T$ and $(a,b) \notin R$.

(c) $(S \setminus T) \circ R \subseteq (S \circ R) \setminus (T \circ R)$

A = {(1,2)}

B = {(3,4)}

C = {(2,1)}

R = {(1,3),(1,4),(2,3),(2,4)}

S = {(3,2),(3,1)}

T = {(4,2),(4,1)}

$S \setminus T = \{(3, 2), (3, 1)\}$

$(S \setminus T) \circ R = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$

$S \circ R = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$

$T \circ R = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$

$\{(1, 1), (1, 2), (2, 1), (2, 2)\} \nsubseteq \varnothing$

12.

R is relation from A to B.

S and T are relations from B to C.

(a) If $S \subseteq T$ then $S \circ R \subseteq T \circ R$.

$S \circ R$ is relation from A to C.

$T \circ R$ is relation from A to C.

Givens	*Goal*
$S \subseteq T$	$(a, c) \in T \circ R$
$(a, c) \in S \circ R$	

Givens	*Goal*
$\forall (b, c) \in B \times C((b, c) \in S \to (b, c) \in T)$	$(a, c) \in T \circ R$
$(a, b) \in R$	
$(b, c) \in S$	

Givens	*Goal*
$(b, c) \in T$	$(a, c) \in T \circ R$
$(a, b) \in R$	
$(b, c) \in S$	

Givens	*Goal*
$(b, c) \in T$	$(a, c) \in T \circ R$
$(a, b) \in R$	
$(b, c) \in S$	

Theorem. If $S \subseteq T$ then $S \circ R \subseteq T \circ R$.

Proof. Suppose $S \subseteq T$. Let (a, c) be an arbitrary ordered pair from $A \times C$. Suppose $(a, c) \in S \circ R$. Then we can choose some $b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$. Since $S \subseteq T$ and $(b, c) \in S$, it follows that $(b, c) \in T$. Since $(a, b) \in R$ and $(b, c) \in T$, it follows that $(a, c) \in T \circ R$. Therefore, $S \circ R \subseteq T \circ R$.

(b) $(S \cap T) \circ R \subseteq (S \circ R) \cap (T \circ R)$.

Givens	*Goal*
$(a, c) \in (S \cap T) \circ R$	$(a, c) \in (S \circ R) \cap (T \circ R)$

Givens	*Goal*
$(a, b) \in R$	$(a, c) \in S \circ R \wedge (a, c) \in T \circ R$
$(b, c) \in S$	
$(b, c) \in T$	

Givens	*Goal*
$(a,b) \in R$	$(a,c) \in S \circ R \land (a,c) \in T \circ R$
$(b,c) \in S$	
$(b,c) \in T$	

Theorem. $(S \cap T) \circ R \subseteq (S \circ R) \cap (T \circ R)$

Proof. Let (a,c) be an arbitrary ordered pair from $A \times C$. Suppose $(a,c) \in (S \cap T) \circ R$. We can choose some $b \in B$ such that $(a,b) \in R$ and $(b,c) \in S \cap T$. Then $(b,c) \in S$ and $(b,c) \in T$. Since $(a,b) \in R$ and $(b,c) \in S$, $(a,c) \in S \circ R$. Similarly, since $(a,b) \in R$ and $(b,c) \in T$, $(a,c) \in T \circ R$. Therefore, $(a,c) \in (S \circ R) \cap (T \circ R)$. Thus, $(S \cap T) \circ R \subseteq (S \circ R) \cap (T \circ R)$.

(c) $(S \cap T) \circ R = (S \circ R) \cap (T \circ R)$

Givens	*Goal*
$(a,c) \in S \circ R$	$(a,c) \in (S \cap T) \circ R$
$(a,c) \in T \circ R$	

Givens	*Goal*
$b \in B$	$\exists b \in B((a,b) \in R \land (b,c) \in S \cap T)$
$(a,b) \in R$	
$(b,c) \in S$	
$m \in B$	
$(a,m) \in R$	
$(m,c) \in T$	

Counterexample.

A = {1,2}

B = {3,4}

C = {1,4}

R = {(1,3),(1,4),(2,3),(2,4)}

S = {(3,1),(3,4)}

T = {(4,1),(4,4)}

$(S \cap T) \circ R = (S \circ R) \cap (T \circ R)$

$S \cap T = \varnothing$

$\varnothing \circ R = \varnothing$

$S \circ R = \{(1,1),(1,4),(2,1),(2,4)\}$

$T \circ R = \{(1,1),(1,4),(2,1),(2,4)\}$

$\varnothing \neq \{(1,1),(1,4),(2,1),(2,4)\}$

(d) $(S \cup T) \circ R = (S \circ R) \cup (T \circ R)$

$(\rightarrow)$

Givens	*Goal*
$(a,c) \in (S \cup T) \circ R$	$(a,c) \in (S \circ R) \cup (T \circ R)$

$(a,c) \in (S \cup T) \circ R$

iff $(a,b) \in R \wedge (b,c) \in S \cup T$

iff $(a,b) \in R \wedge ((b,c) \in S \vee (b,c) \in T)$

Givens	*Goal*
$(a,b) \in R \wedge ((b,c) \in S \vee (b,c) \in T)$	$(a,c) \in (S \circ R) \cup (T \circ R)$

Givens	*Goal*
$(a,b) \in R \wedge ((b,c) \in S \vee (b,c) \in T)$	$(a,b) \in R \wedge (b,c) \in S$
$(a,b) \notin R \vee (b,c) \notin T$	

Case 1.

Givens	*Goal*
$(a,b) \in R \land (b,c) \in S$	$(a,b) \in R \land (b,c) \in S$
$(a,b) \notin R \lor (b,c) \notin T$	

Case 2.

Givens	*Goal*
$(a,b) \in R \land (b,c) \in T$	$(a,b) \in R \land (b,c) \in S$
$(a,b) \notin R \lor (b,c) \notin T$	

Case 2.1

Givens	*Goal*
$(a,b) \in R \land (b,c) \in T$	$(a,b) \in R \land (b,c) \in S$
$(a,b) \notin R$	

Case 2.2

Givens	*Goal*
$(a,b) \in R \land (b,c) \in T$	$(a,b) \in R \land (b,c) \in S$
$(b,c) \notin T$	

$(\leftarrow)$

Givens	*Goal*
$(a,c) \in (S \circ R) \cup (T \circ R)$	$(a,c) \in (S \cup T) \circ R$

Case 1.

Givens	*Goal*
$(a,c) \in (S \circ R)$	$(a,c) \in (S \cup T) \circ R$

Givens	*Goal*
$(a,b) \in R$	$(a,b) \in R \land ((b,c) \in S \lor (b,c) \in T)$
$(b,c) \in S$	

Case 2.

Givens	*Goal*
$(a,b) \in R$	$(a,b) \in R \land ((b,c) \in S \lor (b,c) \in T)$
$(b,c) \in T$	

Theorem. $(S \cup T) \circ R = (S \circ R) \cup (T \circ R)$

Proof. Let (a,c) be an arbitrary ordered pair from $A \times C$.

Suppose $(a,c) \in (S \cup T) \circ R$. Suppose $(a,b) \notin R \lor (b,c) \notin T$.

Case 1. $(b,c) \in S$. Since $(a,b) \in R$, $(a,b) \in R \land (b,c) \in S$.

Case 2. $(b,c) \in T$

Case 2.1 $(a,b) \notin R$. But it contradicts to $(a,b) \in R$.

Case 2.2 $(b,c) \notin T$. But it contradicts to $(b,c) \in T$.

Therefore $(a,b) \in R \land (b,c) \in S$.

Thus, $(S \cup T) \circ R \subseteq (S \circ R) \cup (T \circ R)$.

Now suppose, $(a,c) \in (S \circ R) \cup (T \circ R)$.

Case 1. $(a,c) \in (S \circ R)$. We can choose some $b \in B$ such that $(a,b) \in R$ and $(b,c) \in S$. Therefore $(a,c) \in (S \cup T) \circ R$.

Case 2. $(a,c) \in (T \circ R)$. We can choose some $b \in B$ such that $(a,b) \in R$ and $(b,c) \in T$. Therefore $(a,c) \in (S \cup T) \circ R$.

Therefore, $(S \circ R) \cup (T \circ R) \subseteq (S \cup T) \circ R$.

Thus, $(S \cup T) \circ R = (S \circ R) \cup (T \circ R)$.

3 More About Relations