SOLUTIONS OF THE EXERCISES FOR "HOW TO PROVE IT" BOOK

by drets

(may contain various errors)

2. Quantificational Logic

2.1. Quantifiers

$$\forall x(\neg L(x,j) \to L(s,x))$$

L(x, j)	L(s, x)	$\neg L(x,j) \to L(s,x)$
F	F	F
F	T	T
T	F	T
T	T	T

Exercises:

1. Analyze the logical forms of the following statements.

(a) Anyone who has forgiven at least one person is a saint.

(b) Nobody in the calculus class is smarter than everybody in the discrete math class.

(c) Everyone likes Mary, except Mary herself.

(d) Jane saw a police officer, and Roger saw one too.

(e) Jane saw a police officer, and Roger saw him too.

(a) $\forall x$ (x has forgiven at least one person is a saint)

$\exists y F(x,y)$

F(x, y) stand for ”x has forgiven y”

$\forall x(\exists y F(x,y) \to S(x))$

S(x) stand for ”x is a saint”

(b) $\neg\exists x[C(x) \wedge \forall y(D(y) \rightarrow S(x,y))]$

S(x, y) is ”x is smarter y”

C(x) is ”x is in calculus class”

D(x) is ”x is in discrete class”

(c) $\forall x(\neg L(m,m) \rightarrow L(x,m))$

m is ”Mary”

L(x, y) is ”x likes y”

(d) $\exists x(P(x) \wedge S(j,x)) \wedge \exists y(P(y) \wedge S(r,y))$

S(x, y) is ”x saw y”

j is Jane

r is Roger

P(x) is ”x is a police officer”

(e) $\exists x(P(x) \wedge S(j,x) \wedge S(r,x))$

S(x, y) is ”x saw y”

j is Jane

r is Roger

P(x) is ”x is a police officer”

2. Analyze the logical forms of the following statements.

(a) Anyone who has bought a Rolls Royce with cash must have a rich uncle.

(b) If anyone in the dorm has the measles, then everyone who has a friend in the dorm will have to be quarantined.

(c) If nobody failed the test, then everybody who got an A will tutor someone who got a D.

(d) If anyone can do it, Jones can.

(e) If Jones can do it, anyone can

(a) $\forall x(C(x) \to \exists y(R(y) \wedge U(y, x)))$

C(x) is "x has bought a Rolls Royce with cash"

R(x) is "x is rich"

U(x, y) is "x is uncle of y"

(b) $\exists x[(D(x) \wedge M(x)) \to \forall y(F(x, y) \wedge Q(y))]$

$\exists x[(D(x) \wedge M(x)) \to \exists z\forall y(D(z) \wedge F(z, y) \wedge Q(y))]$

M(x) is "x has the measles"

D(x) is "x is in the dorm"

F(x, y) is "x is a friend of y"

Q(x) is "x will have to be quarantined"

(c) $\neg\exists x(F(x) \to \forall y\exists z(A(y) \wedge D(z) \wedge T(y, z)))$

T(x, y) is "x will tutor y"

F(x) is "x failed the test"

A(x) is "x got an A"

D(x) is "x got a D"

(d) $\exists x(D(x) \to D(j))$

D(x) is "x can do it"

j is Jones

(e) $\forall x(D(j) \to D(x))$

D(x) is "x can do it"

j is Jones

3. Analyze the logical forms of the following statements. The universe of discourse is $\mathbb{R}$. What are the free variables in each statement?

(a) Every number that is larger than x is larger than y.

(b) For every number a, the equation $ax^2 + 4x - 2 = 0$ has at least one solution iff $a \geq 2$

(c) All solutions of the inequality $x^3 - 3x < 3$ are smaller than 10.

(d) If there is a number x such that $x^2 + 5x = w$ and there is a number y such that $4 - y^2 = w$, then w is between -10 and 10.

(a) $\forall n(n > x \to n > y)$

x and y are free variables.

(b) $\forall a \exists x(a > 2 \leftrightarrow ax^2 + 4x - 2 = 0)$

no free variables.

(c) $\forall x(x^3 - 3x < 3 \to x < 10)$

no free variables.

(d) $\forall w[(\exists x(x^2 + 5x = w) \wedge \exists y(4 - y^2 = w) \to (-10 < w < 10)]$

no free variables.

4. Translate the following statements into idiomatic English.

(a) $\forall x[(H(x) \wedge \neg\exists y M(x, y)) \to U(x)]$, where H(x) means "x is a man," M(x, y) means "x is married to y," and U(x) means "x is unhappy."

(b) $\exists z(P(z, x) \wedge S(z, y) \wedge W(y))$, where P(z, x) means "z is a parent of x," S(z, y) means "z and y are siblings," and W(y) means "y is a woman."

(a) All unmarried men are unhappy

(b) y is a sister of one of x's parents.

5. Translate the following statements into idiomatic mathematical English.

(a) $\forall x[(P(x) \wedge \neg(x = 2)) \rightarrow O(x)]$, where P(x) means "x is a prime number" and O(x) means "x is odd."

(b) $\exists x[P(x) \wedge \forall y(P(y) \rightarrow y \leq x)]$, where P(x) means "x is a perfect number."

(a) All x which are prime numbers and not equal to 2 should be odd.

(b) There is at least one perfect number x such that all perfect numbers are less or equal to x.

6. Are these statements true or false? The universe of discourse is the set of all people, and P(x, y) means "x is a parent of y."

(a) $\exists x \forall y P(x, y)$.

(b) $\forall x \exists y P(x, y)$.

(c) $\neg \exists x \exists y P(x, y)$.

(d) $\exists x \neg \exists y P(x, y)$.

(e) $\exists x \exists y \neg P(x, y)$.

(a) false

(there exists person x such that x is a parent of all people)

(b) false

(all people have child)

(c) false

(nobody has a child)

(d) true

(there exists a person without kids)

(e) true

(there exist person x and person y such that x is not a parent of y)

7. Are these statements true or false? The universe of discourse is $\mathbb{N}$.
   (a) $\forall x \exists y (2x - y = 0)$.
   (b) $\exists y \forall x (2x - y = 0)$.
   (c) $\forall x \exists y (x - 2y = 0)$.
   (d) $\forall x (x < 10 \rightarrow \forall y (y < x \rightarrow y < 9))$.
   (e) $\exists y \exists z (y + z = 100)$.
   (f) $\forall x \exists y (y > x \wedge \exists z (y + z = 100))$.

(a) true

(b) false

(c) false

(d) true

(e) true

(f) false

8. Same as exercise 7 but with $\mathbb{R}$ as the universe of discourse.

(a) true

(b) false

(c) true

(d) false

(e) true

(f) true

9. Same as exercise 7 but with $\mathbb{Z}$ as the universe of discourse

(a) true

(b) false

(c) false

(d) true

(e) true

(f) true

2.2. Equivalences Involving Quantifiers