332:347 – Linear Systems Lab – Fall 2016

Lab 2 – S. J. Orfanidis

In this lab, besides the theoretical and numerical questions, you will need to produce six graphs for problem 1, eight graphs for problem 2, and four graphs for problem 3.

1. This problem demonstrates the linearity and time-invariance properties of LTI systems and also studies the numerical approximation to convolution achieved by the MATLAB function, **conv**. Consider a system described by the following differential equation, with corresponding impulse response and transfer function:

$$\dot{y}(t)+ay(t)=ax(t) \quad \Rightarrow \quad h(t)=ae^{-at}u(t)\,, \quad H(s)=\frac{a}{s+a} \tag{1}$$

Let the input signal $x(t)$ be a square pulse of duration of t_d seconds, starting at $t=0$, that is,

$$x(t)=F(t,t_d)\equiv u(t)-u(t-t_d)\,, \quad t_d>0 \tag{2}$$

The corresponding exact output due to the input $F(t,t_d)$ was found in class:

$$y_{\text{exact}}(t)=G(t,t_d)\equiv e^{-at}\big[e^{a\min(t,t_d)}-1\big]u(t)=\begin{cases}1-e^{-at}\,, & 0\le t\le t_d\\ e^{-at}\big[e^{at_d}-1\big]\,, & t_d\le t\\ 0\,, & t<0\end{cases} \tag{3}$$

A numerical approximation to the convolution integral is obtained by considering the discrete time instants $t_n=nT$, where T is a small time step, and approximating:

$$y(t_n)=\int h(t_n-\tau)x(\tau)\,d\tau \approx T\sum_m h(t_n-t_m)x(t_m) \tag{4}$$

It can be implemented by the MATLAB code:

$$\texttt{y = T * conv(h,x);} \tag{5}$$

where `h,x` are the arrays $h(t_n), x(t_n)$. The functions $F(t,t_d), G(t,t_d)$ can be implemented as anonymous functions,

```
F = @(t,td) (t>=0) - (t>=td);
G = @(t,td) exp(-a*t) .* (exp(a*min(t,td)) - 1) .* (t>=0);
```

From the linearity and time-invariance of the system, we know that if one takes as input a linear combination of shifted copies of $F(t,t_d)$, then, the output would be the same linear combination of shifted copies of $G(t,t_d)$, for example,

$$\begin{aligned} x(t) &= c_1F(t-t_1,t_d)+c_2F(t-t_2,t_d)+c_3F(t-t_3,t_d)\\ y(t) &= c_1G(t-t_1,t_d)+c_2G(t-t_2,t_d)+c_3G(t-t_3,t_d)\end{aligned} \tag{6}$$

(a) Define the signals, $h(t), x(t)$, of Eqs. (1) and (6), for the following choice of parameters over a maximum time interval of $T_{\max}=25$ seconds, and pulse duration, $t_d=1$,

$$a=0.9\,, \quad T=0.05\,, \quad [c_1,c_2,c_3]=[1,\,2,\,1.5]\,, \quad [t_1,t_2,t_3]=[0,\,10,\,15]$$

with t defined to span the $[0,\,T_{\max}]$ interval in steps of T, that is, $t=0{:}T{:}T_{\max}$. Using the approximation of Eq. (5), calculate the output $y(t)$, as well as the exact output using Eq. (6), and plot both of them on the same graph, together with the input signal $x(t)$.

For plotting purposes you may wish to keep only the first $N=\text{length}(t)$ convolutional outputs. This can be accomplished by redefining the computed output vector by:

```
y = y(1:length(t));
```

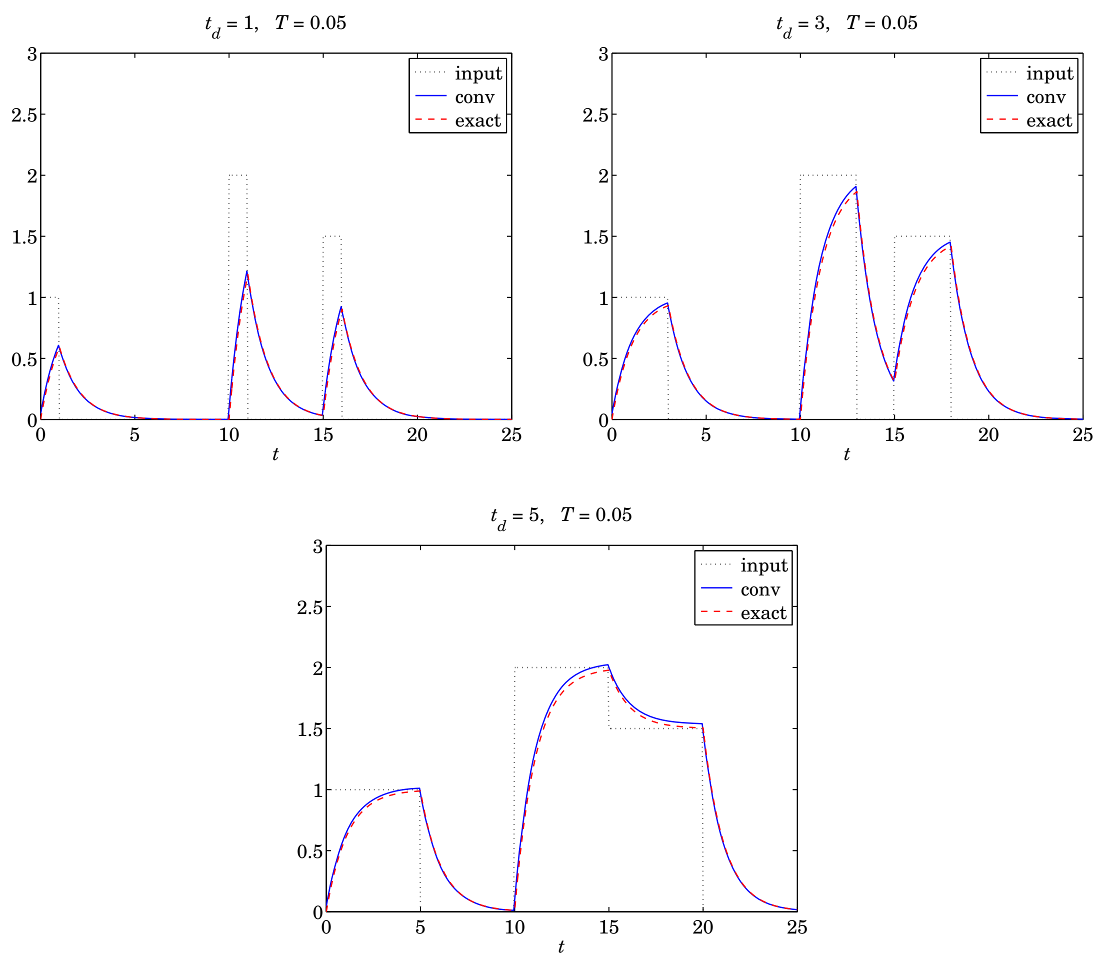
(b) Observe in part (a) that because of the very short duration of the input pulses, the outputs due to the individual pulses can be clearly discerned, each being scaled and delayed by the proper amounts. If the pulse duration is increased, these outputs will begin to overlap.

Repeat part (a) for the pulse duration value, $t_d = 3$. The input pulses do not overlap, but the output ones overlap more strongly because the transient portions are closer to each other.

Repeat part (a) for the pulse duration value, $t_d = 5$, corresponding to the case when the last two pulses are just adjacent without overlap.

(c) To assess the nature of the approximation of Eq. (5), repeat parts (a,b) for the smaller value of the time step $T = 0.01$.

Typical Outputs



2. This problem illustrates the transient and steady-state sinusoidal responses of linear systems. Consider a signal consisting of three sinusoidal bursts (shown at end):

$$x(t) = \begin{cases} \sin(\omega_1 t), & 0 \le t < 30 \\ \sin(\omega_0 t), & 30 \le t < 70 \\ \sin(\omega_1 t), & 70 \le t < 100 \end{cases}$$

where $\omega_0 = 2$ and $\omega_1 = 3$. It can be generated over the time interval, $0 \le t \le 100$, by the following MATLAB code segment:

```
w0=2; w1=3; Tmax=100; T=Tmax/2000; t = 0:T:Tmax;

x = sin(w1*t) .* F(t,30) + ...
    sin(w0*t) .* F(t-30,40) + ...
    sin(w1*t) .* F(t-70,30);
```

where the pulse function $F(t, t_d)$ was defined in the previous problem. It is desired to eliminate the middle burst by means of a notch filter:

$$H(s) = \frac{s^2 + \omega_0^2}{s^2 + \alpha s + \omega_0^2} \tag{7}$$

where $\omega_0 = 2$ is the notch frequency coinciding with the frequency of the middle burst, and $\alpha = 0.3$ is a parameter that represents the 3-dB width of the notch (see graph below), thus, the filter Q is, $Q = \omega_0/\alpha$. As discussed in class, the impulse response of this filter is:

$$h(t) = \delta(t) - g(t)\,, \quad g(t) = \alpha e^{-\alpha t/2}\left[\cos(\omega_r t) - \frac{\alpha}{2\omega_r}\sin(\omega_r t)\right]u(t)\,, \quad \omega_r = \sqrt{\omega_0^2 - \frac{\alpha^2}{4}}$$

where it is recognized as an "underdamped" case since $\omega_0 > \alpha/2$. It follows that the output signal will be:

$$y(t) = \int_{0^-}^{\infty} h(t-t')\,x(t')\,dt' = x(t) - \int_{0^-}^{\infty} g(t-t')\,x(t')\,dt'$$

which can be implemented approximately by the MATLAB code:

```
y = x - T * conv(g,x);
```
(8)

(a) Compute the output signal $y(t)$ using Eq. (8) and plot it versus t. On a separate graph, but using the same vertical and horizontal scales, plot the input signal $x(t)$. Note the removal of the middle burst after the transients have decayed.

The first and third bursts have also been attenuated by a slight amount, with a new amplitude equal approximately to $|H(j\omega_1)|$ — approximately, because steady-state is not yet reached for these.

Calculate the numerical value of $|H(j\omega_1)|$ and, on the graph for $y(t)$, add horizontal lines at that level for the first and third bursts (see the blue line segments in the example graphs below).

(b) Calculate the 40-dB time constant of this filter given in general by,

$$\tau = -\frac{\ln(10^{-40/20})}{|\,\mathrm{Re}(p)\,|} = \frac{\ln(100)}{|\,\mathrm{Re}(p)\,|}$$

where p is the stable pole closest to the $j\omega$-axis. Here, $p = -\alpha/2 + j\omega_r$. Is the value of τ consistent with the transients that you observe in the plot of $y(t)$?

(c) Calculate the output signal $y(t)$ by the alternative method of using the function, **lsim**, and make a plot of $y(t)$ using the same scales as in part (a). Compare the outputs from the **lsim** and **conv** methods by computing the percentage error as the ratio,

$$\text{Error} = 100 \cdot \frac{\|y_{\text{conv}} - y_{\text{lsim}}\|}{\|y_{\text{conv}}\|}$$

where the norm $\|y\|$ can be computed with the built-in function **norm**.

(d) The frequency and magnitude responses of the transfer function $H(s)$ of Eq. (7) are,

$$H(j\omega) = \frac{\omega_0^2 - \omega^2}{\omega_0^2 - \omega^2 + \alpha j\omega} \quad \Rightarrow \quad |H(j\omega)|^2 = \frac{(\omega^2 - \omega_0^2)^2}{(\omega^2 - \omega_0^2)^2 + \alpha^2\omega^2} \tag{9}$$

As discussed in class, the (positive) left and right 3-dB frequencies $\omega_\pm$, that is, the solutions of the 3-dB half-power condition, $|H(j\omega)|^2 = 1/2$, are given by,

$$\omega_\pm = \sqrt{\omega_0^2 + \frac{\alpha^2}{2} \pm \alpha\sqrt{\omega_0^2 + \frac{\alpha^2}{4}}} \tag{10}$$

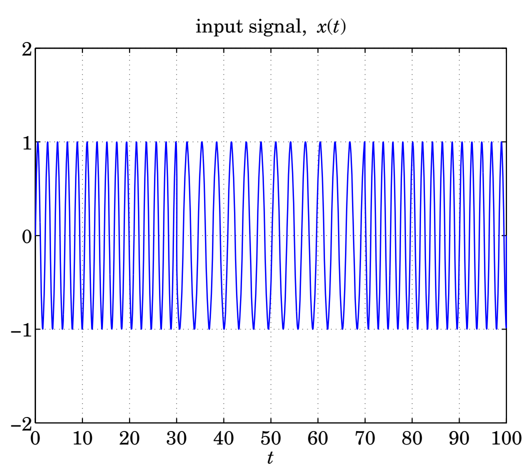
Note that these frequencies satisfy the following useful relationships,†

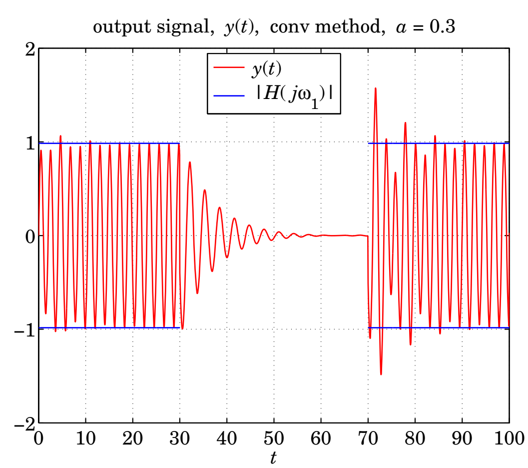
$$\boxed{\begin{aligned} \omega_+ - \omega_- &= \alpha \\ \omega_+\omega_- &= \omega_0^2 \end{aligned}} \tag{11}$$

where, $\Delta\omega = \omega_+ - \omega_- = \alpha$, represents the 3-dB width of the notch. Make a plot of the magnitude-square response $|H(j\omega)|^2$ over the interval, $0 \le \omega \le 5$, and add to it (with dots) the points at $\omega = \omega_0$ (the notch), and at $\omega = \omega_1$. Also, add the horizontal line between the two 3-dB frequencies $\omega_\pm$ in order to indicate the width of the notch.

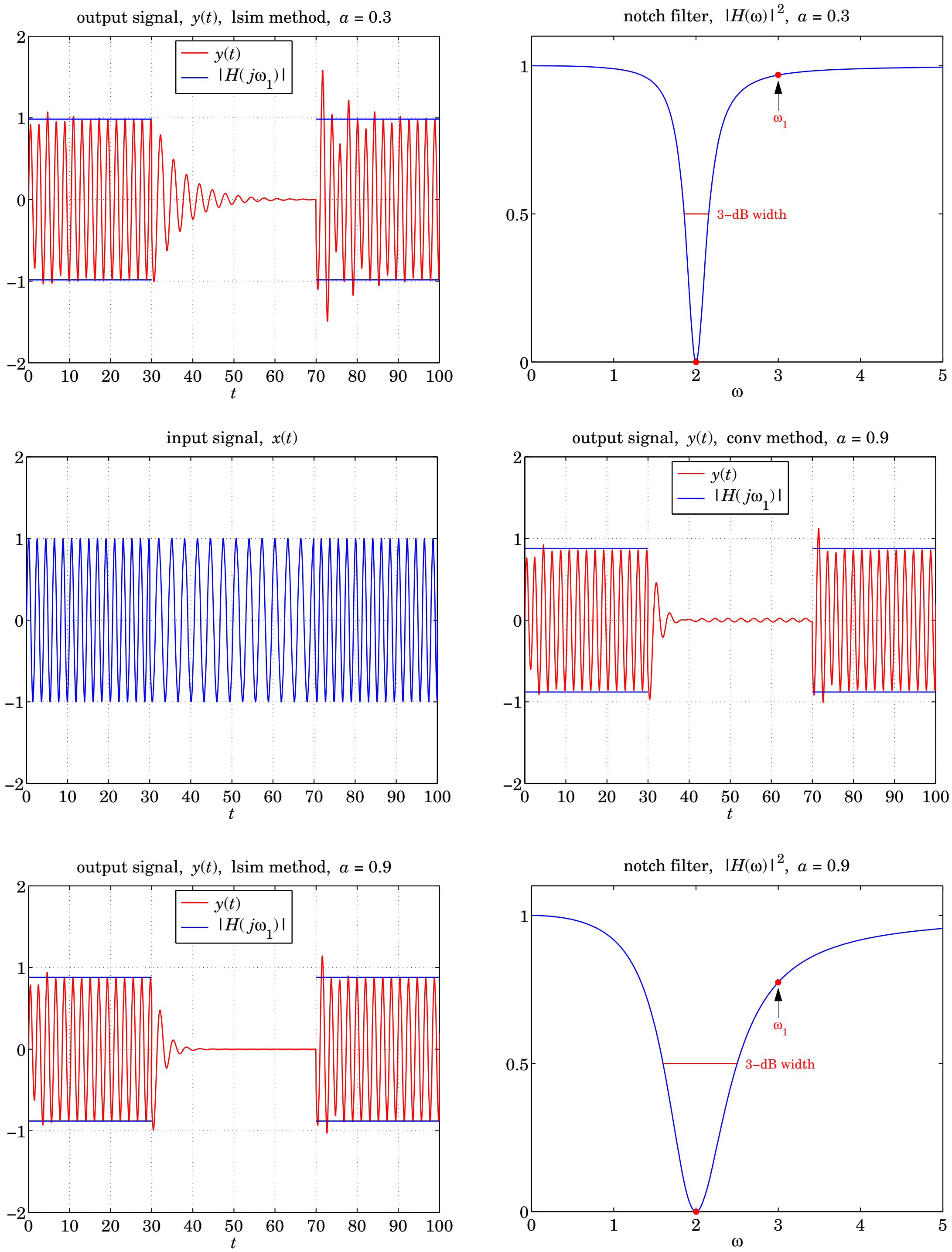
(e) Repeat parts (a–d) for the case $\alpha = 0.9$. Now the time constant and the transients will be shorter, but the notch width will be wider, and the first and last bursts at ω_1 will be more distorted in amplitude.

Typical Outputs





†First prove, $\omega_+^2\omega_-^2 = \omega_0^4$, then, use the identity, $(\omega_+ - \omega_-)^2 = \omega_+^2 + \omega_-^2 - 2\omega_+\omega_-$



output signal, $y(t)$, lsim method, $a = 0.3$
$y(t)$
$|H(j\omega_1)|$
t
notch filter, $|H(\omega)|^2$, $a = 0.3$
ω_1
3−dB width
ω
input signal, $x(t)$
t
output signal, $y(t)$, conv method, $a = 0.9$
$y(t)$
$|H(j\omega_1)|$
t
output signal, $y(t)$, lsim method, $a = 0.9$
$y(t)$
$|H(j\omega_1)|$
t
notch filter, $|H(\omega)|^2$, $a = 0.9$
ω_1
3−dB width
ω

3. *ECG processing.*† A typical problem with electrocardiogram (ECG) recordings is the 60-Hz power frequency interference picked up by the exposed electrodes that are placed on the patient's chest. The file **lab2.dat** attached to this lab, contains a 1.5 second ECG recording consisting of three half-second heartbeats, sampled at a rate of 1 kHz, that is, the sampling time interval is $T = 1/f_s = 0.001$ seconds. The data can be loaded from your MATLAB working folder by the command,

```
Y = load('lab2.dat');        % size(Y) = 1500x3
```

The first column of the matrix Y contains the sampling times t, the second column is the interference-free ECG signal $s(t)$, and the third column is the noisy signal $x(t)$ consisting of $s(t)$ plus 60 Hz interference.

(a) Design a notch filter of the type of Eq. (7) that has a notch at $f_0 = 60$ Hz and a 3-dB width of $\Delta f = 1.5$ Hz. Note, that you must convert these into radians/sec before using the formulas of the previous problem, e.g., $\omega_0 = 2\pi f_0$ and $\alpha = 2\pi\Delta f$.

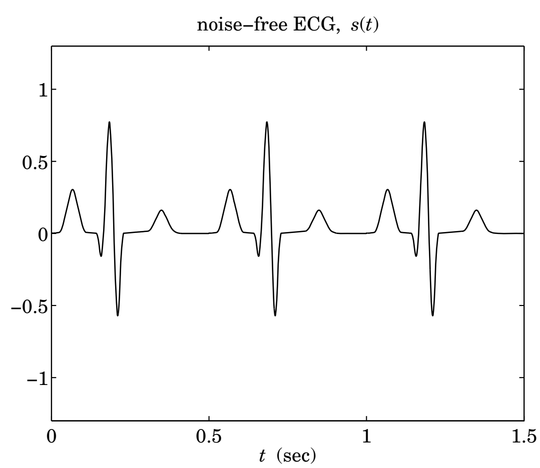
What are the values of the left and right 3-dB frequencies in Hz? What is the filter Q? What is the 40-dB time constant τ of this filter in seconds?

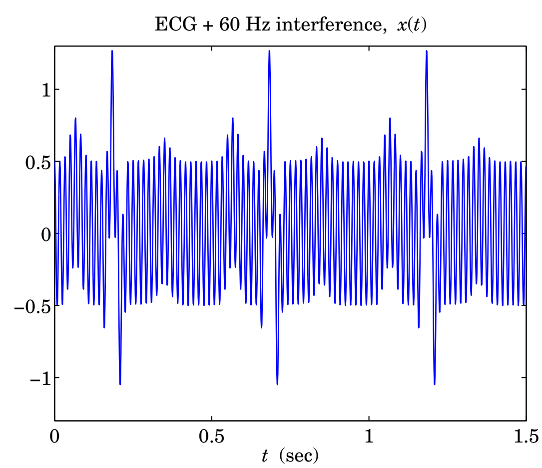
(b) Extract from Y the column vectors, $t, s(t), x(t)$. On two separate graphs, but using the same vertical scales, plot versus t the noise-free signal $s(t)$, and the noisy signal $x(t)$.

(c) Using the function **lsim**, filter the noisy ECG signal $x(t)$ through the notch filter and plot the resulting output $y(t)$ versus t, using the same vertical scales as in part (a). For reference, add the noise-free signal $s(t)$ to the graph. Is the calculated 40-dB time constant consistent with what you observe?

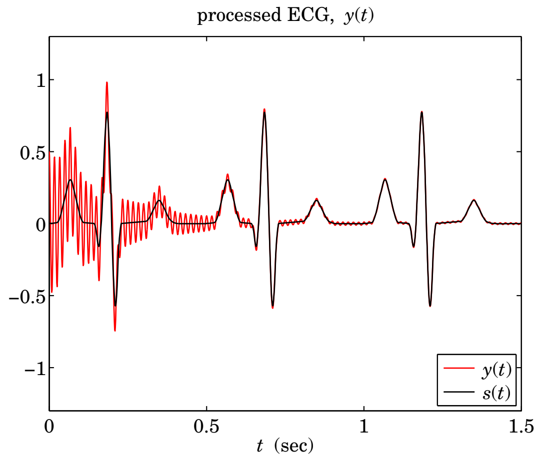
(d) Plot the magnitude response squared of the filter, $|H(j\omega)|^2$ versus frequency f in Hz over the interval, $0 \le f \le 120$ Hz, and note the narrowness of the notch at 60 Hz.

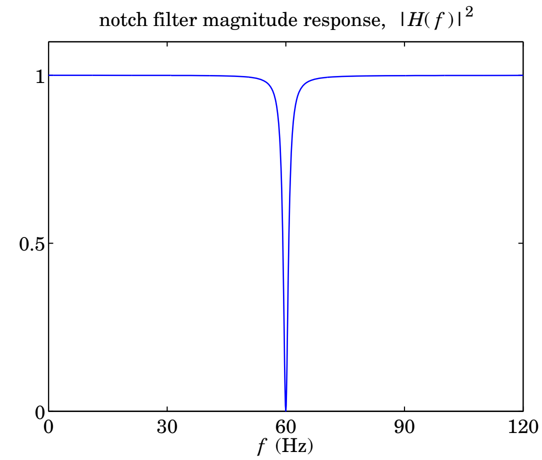
Typical Outputs





†A digital filter version of this problem was discussed in week-11 of the MATLAB course.


processed ECG, y(t)
1
0.5
0
−0.5
−1
y(t)
s(t)
0
0.5
1
1.5
t (sec)


notch filter magnitude response, |H(f)|2
1
0.5
0
0
30
60
90
120
f (Hz)