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MA3408 - ALGEBRAIC TOPOLOGY II

1 Homotopy theory

We recall the following definition.

1.1 Definition. A homotopy between $f, g\colon X \to Y$ is a continuous function $H\colon X \times I \to Y$ such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$. For a subspace $A \subseteq X$, a relative homotopy is a homotopy with $H(a,t) = f(a) = g(a)$ for all $a \in A, t \in I$.

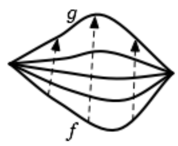


Figure 1.1: A homotopy between f and g.

1.2 Remark. Equivalently, we can specify a family of continuous maps $h_t\colon X \to Y$ such that $h_0 = f, h_1 = g$ and

$$\begin{aligned} H\colon X \times I &\to Y \\ (x,t) &\mapsto h_t(x) \end{aligned}$$

is continuous.

1.3 Notation. We will let $I_n = I^{\times n}, \partial I^n$ be the boundary of I^n, and write $[-,-]$ for homotopy classes of maps (if our spaces are based, these fix the base point).

1.4 Definition. For each $n \geq 0$ and X a topological space with $x_0 \in X$, we define

$$\pi_n(X, x_0) = [(I^n, \partial I^n), (X, x_0)].$$

1.5 Remark.
(i) When $n = 0$, we have $I^0 = \text{pt}$ and $\partial I^0 = \emptyset$, therefore $\pi_0(X)$ is the set of path components of X.

(ii) When $n = 1$, this is a group, but need not be abelian (for example, consider the wedge of two circles).

(iii) Note that $I^n/\partial I^n \simeq S^n$ and $\partial I^n/\partial I^n \simeq s_0$. By the universal property of the quotient map, we see that

$$\pi_n(X, x_0) \cong [(S_n, s_0), (X, x_0)].$$

1.6 Definition. A maps of pairs $(X, A) \to (Y, B)$ is a map $f\colon X \to Y$ with $f(A) \subseteq B$, i.e., the diagram:

$$\begin{tikzcd} A \arrow[r] \arrow[d, hook] & B \arrow[d, hook] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

commutes.

1.7 Proposition. *If $n \geq 1$, then $\pi_n(X, x_0)$ is a group with respect to the operation*

$$(f+g)(t_1, \ldots, t_n) = \begin{cases} f(2t_1, t_2, \ldots, t_n) & 0 \leq t_1 \leq 1/2 \\ g(2t_1 - 1, t_2, \ldots, t_n) & 1/2 \leq t_1 \leq 1. \end{cases}$$

Proof. The identity is given by the constant map taking all of I^n to x_0 and the inverse of f is given by

$$-f(t_1, \ldots, t_n) = f(1 - t_1, t_2, \ldots, t_n). \qquad \square$$

1.8 Remark. Call the group operation $+_1$. Note that we can also define an operation $+_i$ for $1 \leq i \leq n$ by the same formula on the i-th coordinate.

1.9 Theorem. *All of these operations agree, and for $n \geq 2$, these give $\pi_n(X, x_0)$ the structure of an abelian group.*

This is a consequence of the following exercise.

1.10 Exercise (Eckmann–Hilton argument). Let M be a set and let $*, \bullet$ be two binary operations on M, $*, \bullet \colon M \times M \to M$, both with unit elements. Suppose that

$$(a * b) \bullet (c * d) = (a \bullet c) * (b \bullet d)$$

for all $a, b, c, d \in M$. Show that the units agree, these two operations agree, and that the multiplication is commutative and associative.

1.11 Remark. Let use show that

$$(f +_1 g) +_2 (h +_1 i) \simeq (f +_2 h) +_1 (g +_2 i).$$

Indeed, both of these are the following map

$$(t_1, t_2, \ldots,) \mapsto \begin{cases} f(2t_1, 2t_2, \ldots,) & [1/2, 0] \times [1/2, 0] \\ g(2t_1 - 2, 2t_2, \ldots,) & [1/2, 1] \times [0, 1/2] \\ h(2t_1, 2t_2 - 2, \ldots) & [0, 1/2] \times [1/2, 1] \\ i(2t_1 - 1, 2t_2 - 2, \ldots) & [1/2, 1] \times [1/2, 1]. \end{cases}$$

1.12 Remark. Another approach is given by the following visualization: That is, so long as $n \geq 2$, we can shrink the domain of f and g to smaller cubes (mapping the remaining region to the base point), slide f and g past each other, and then increase the domains back again.

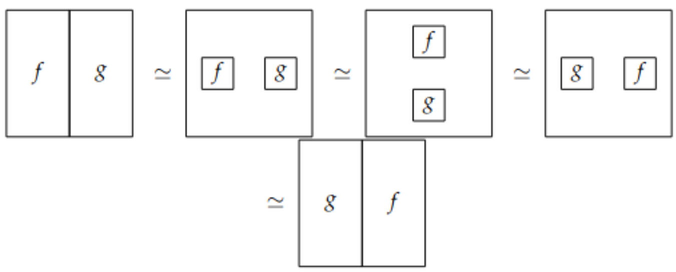


Figure 1.2: $f + g \simeq g + f$.

1.13 Exercise. Let G be a topological group with identity element e, then $\pi_1(G, e)$ is abelian.

Hint: Use Eckmann–Hilton, or note the following: A topological group is a group object in the category of topological spaces. What is a group object in the category of groups?

1.14 Proposition. *If $n \geq 1$ and X is path connected then there is an isomorphism $\beta_\gamma : \pi_n(X, x_0) \xrightarrow{\cong} \pi_n(X, x_0)$ given by $\beta_\gamma([f]) = [\gamma \circ f]$ where γ is a path in X from x_1 to x_0 and $\gamma \circ f$ is constructed by first shrinking the domain of f to a smaller cube inside of I^n, and then inserting the path γ radially from x_1 to x_0 on the boundaries of these cubes.*

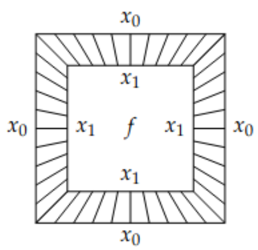
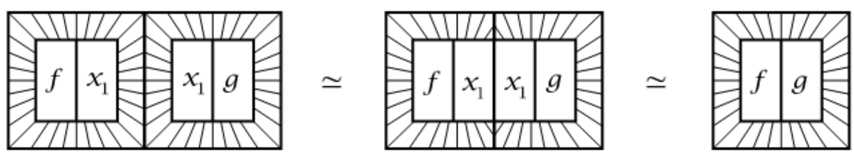


Figure 1.3: β_γ.

Proof. Observe the following:

1. $\gamma \circ (f + g) \simeq \gamma \circ f + \gamma \circ g$, i.e., β_γ is a group homomorphism.
2. $(\gamma \circ \eta) \circ f \simeq \gamma \circ (\eta \circ f)$, for η a path from x_0 to x_1.
3. $c_{x_0} \circ f \simeq f$, where c_{x_0} denotes the constant path based at x_0.
4. β_γ is well-defined with respect to homotopies of f or γ.

The only point that is perhaps not clear is (i). For this, we deform f and g to be constant on the right and left halves of I^n, respectively, producting maps we call $f + 0$ and $0 + g$. We then excise a wider symmtric middle slab of $\gamma(f + 0)$ and $\gamma(0 + g)$ until it becomes $\gamma(f + g)$: $\square$



1.15 Remark. Therefore if X is path-connected, different choices of base point x_0 yield isomorphic groups $\pi_n(X, x_0)$, which may then simply be written as $\pi_n(X)$.

1.16 Lemma. *If $\{X_\alpha\}$ is a collection of path-connected spaces, then $\pi_n(\prod_\alpha X_\alpha) \cong \prod_\alpha \pi_n(X_\alpha)$.*

Proof. Note that $\mathrm{Hom}(Y, \prod_\alpha X_\alpha) \simeq \prod_\alpha \mathrm{Hom}(Y, X_\alpha)$. In particular, a map $S^n \to \mathrm{Hom}(Y, \prod_\alpha X_\alpha)$ is determined by a collection of maps $S^n \to X_\alpha$. Likewise, a homotopy $S^n \times I \to \prod_\alpha X_\alpha$ is determined by a colletion of homotopies $S^n \times I \to X_\alpha$. This implies the result. $\square$

1.17 Proposition. *Homotopy groups are functorial: given a map $\phi\colon X \to Y$ we get group homomorphisms $\phi_*\colon \pi_n(X, x_0) \to \pi_n(X, \phi(x_0))$ given by $[f] \mapsto [\phi \circ f]$ for all $n \geq 1$.*

Proof. We have the following:

1. ϕ_* is well-defined: if $f \simeq g$ via ψ_t, then $\phi \circ \psi_t$ defines a homotopy between $\phi \circ f$ and $\phi \circ g$.
2. This is a group homomoprhism: $\phi \circ (f + g) \simeq \phi \circ g + \phi \circ g$ by the definition of the addition operation. Therefore.

$$\phi_*[f + g] = \phi_*[f] + \phi_*[g].$$

1.18 Exercise. If $\phi\colon X \to Y$ is homomotopy equivalence (not neccessarily basepoint preserving), then $\pi_*\colon \pi_n(X, x_0) \to \pi_n(Y, \phi(y_0))$ is an isomorphism.

□

1.19 Remark. We recall the following lifting property: Suppose $p\colon (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ is a covering, and there is a map $f\colon (Y, y_0) \to (X, x_0)$ with Y path-connected and locally path-connected. Then a lift $\tilde{f}$ exists if and only if $f_*\pi_1(Y, y_0) \subseteq p_*\pi_1(\tilde{X}, \tilde{x}_0)$.

$$\begin{tikzcd} & (\tilde{X}, \tilde{x}_0) \arrow[d, "p"] \\ (Y, y_0) \arrow[r, "f"] \arrow[ur, dashed, "\tilde{f}"] & (X, x_0) \end{tikzcd}$$

1.20 Proposition. *If p is a covering, then $p_*\colon \pi_n(\tilde{X}, \tilde{x}_0) \to \pi_n(X, x_0)$ is an isomorphism for all $n \geq 2$.*

Proof. Let us first show surjectivity. To that end, suppose we have a map $f\colon (S^n, s_0) \to (X, x_0)$ where $n \geq 2$. The assumption on n gives $\pi_1(S^n) = 0$, so $f_*\pi_1(S^n, s_0) \subseteq \pi_*\pi_1(\tilde{X}, \tilde{x}_0)$ holds. We therefore find a lift in the following:

$$\begin{tikzcd} & (\tilde{X}, \tilde{x}_0) \arrow[d, "p"] \\ (S^n, s_0) \arrow[r, "f"] \arrow[ur, dashed, "\tilde{f}"] & (X, x_0) \end{tikzcd}$$

Then $p_*[\tilde{f}] = [f]$, and p_* is surjective.

To see that p_* is injective, let $[\tilde{f}] \in \ker(p_*)$, i.e., $p_*[\tilde{f}] = [p \circ \tilde{f}] = 0$. Let $f = p \circ \tilde{f}$, then this is homotopic to the constant map $f \simeq c_{x_0}$ via a homotopy $\phi_t\colon (S^n, s_0) \to (X, x_0)$with $\phi_1 = f$ and $\phi_0 = c_{x_0}$. By the same argument as above, the homotopy ϕ_t can be lifted to $\tilde{\phi}_t$. This satisfies $p \circ \tilde{\phi}_1 \simeq \phi_1$ and $p \circ \tilde{\phi}_0 \simeq \phi_0$. By the uniqueness of lifts, we must have $\tilde{\phi}_1 \simeq \tilde{f}$ and $\tilde{\phi}_0 \simeq c_{x_0}$. In other words, $\tilde{\phi}_t$ gives a homotopy between $\tilde{f}$ and c_{x_0}, so that $[\tilde{f}] = 0$, and p_* is injective. □

1.21 Example. S^1 has universal cover $p\colon \mathbb{R} \to S^1$, $p(t) = e^{2\pi i t}$. Then $\pi_n(S^1) \cong \pi_n(\mathbb{R}) \cong 0$ for $n \geq 2$.

1.22 Exercise. Find two spaces X, Y with $\pi_n X \cong \pi_n Y$ but $X \not\simeq Y$.

Hint: What is the universal cover of $\mathbb{R}P^n$?

1.23 Remark (Relative homotopy groups). Suppose we have (X, x_0) and a subspace A containing x_0. We note that $i_*\colon \pi_n(A, x_0) \to \pi_n(X, x_0)$ is not injective in general (example, take S^1 into $\mathbb{R}^2$). An element in the kernel of i_* is a map $f\colon (I^n, \partial I^n) \to (A, x_0)$ such that

$$(I^n, \partial I^n) \xrightarrow{f} (A, x_0) \xrightarrow{i} (X, x_0)$$

is homotopic to c_{x_0}. This means there exists a homotopy

$$H\colon I^n \times I \to X$$

such that $H(-,1) = f, H(-,0) = c_{x_0}$ and $H|_{\partial I^n \times I} = c_{x_0}$.

If we define $J^n = I^n \times \{0\} \cup \partial I^n \times I \subseteq I^n \times I$, then this is a map of triples

$$H\colon (I^{n+1}, \partial I^{n+1}, J^n) \to (X, A, x_0).$$

1.24 Definition.

$$\pi_n(X, A, x_0) = [(I^n, \partial I^n, J^{n-1}), (X, A, x_0)]$$

1.25 Remark. Equivalently,

$$\pi_n(X, A, x_0) = [(D^n, S^{n-1}, s_0), (X, A, x_0)]$$

1.26 Proposition. *If $n \geq 2$, then $\pi_n(X, A, x_0)$ is a group, and if $n \geq 3$, then it is abelian.*

For all $n \geq 2$, a map of pairs $\phi\colon (X, A, x_0) \to (Y, B, y_0)$ induces homomorphisms $\phi_*\colon \pi_n(X, A, x_0) \to \pi_n(Y, B, y_0)$ for all $n \geq 2$.

1.27 Theorem. *The relative homotopy groups (X, A, x_0) fit into a long exact sequence*

$$\cdots \pi_n(A, x_0) \xrightarrow{i_*} \pi_n(X, x_0) \xrightarrow{j_*} \pi_n(X, A, x_0) \xrightarrow{\partial_n} \pi_{n-1}(A, x_0) \to \cdots$$

where the map ∂_n is defined by $\partial_n([f]) = [f|_{I^{n-1}}]$.

The proof relies on the following.

1.28 Lemma (Compression criterion). *A map $f\colon (D^n, S^{n-1}, x_0) \to (X, A, x_0)$ represents 0 in $\pi_n(X, A, x_0)$ if and only if $f \sim g$ rel S^{n-1}, where g is a map whose image is contained entirely in A.*

Proof. Suppose $[f] = [g]$ with g as in the statement of the lemma. Note that there is a deformation of D^n onto x_0, and so $[f] = [g] = 0$ in $\pi_n(X, A, x_0)$.

Conversely, suppose that $[f]$ represents 0 in $\pi_n(X, A, x_0)$. This means there exists a homotopy, relative to S^{n-1}, $F\colon D^n \times I \to X$ with $F|_{D^n \times \{0\}} = f$, $F|_{D^n \times 1} = c_{x_0}$ and $F|_{S^{n-1} \times I} \subseteq A$. We can restrict F to a family of n-disks in $D^n \times I$ starting with $D^n \times \{0\}$ and ending with the disk $D^n \times \{1\} \cup S^{n-1} \times \{1\}$, all the disks in the family having the same boundary, then we get a homotopy from f to a map in A, stationary on S^{n-1} (said in other words, we can deformation retract $D^n \times [0,1]$ onto $D^n \times \{1\} \cup S^{n-1} \times I$). $\square$

Proof of Theorem 1.27. **Step 1.** Let us first show exactness at $\pi_n(X, x_0)$.

We first show $\mathrm{im}(i_*) \subseteq \ker(j_*)$. Note that $j_* i_*$ is induced by the composition $j \circ i$ and that these are both inclusion maps. Therefore, for $[f] \in \pi_n(A, x_0)$ we have $j_* i_*[f] = [j \circ i \circ f]$, but this has image contained in A, and so $j_* i_*[f] = 0$. This shows $\mathrm{im}(i_*) \subseteq \ker(j_*)$.

To see the converse (namely, $\ker(j_*) \subseteq \mathrm{im}(i_*)$) let $[f] \in \ker(j_*)$, i.e. $[j \circ f] = 0$. Note that again j is an inclusion map, and by the compression criteria $f \simeq g'$ relative to S^{n-1}, where g' has image contained in A. Since $x_0 \in S^{n-1}$, the homotopy fixes the basepoint, i.e, $[f] = [g'] \in \pi_n(X, x_0)$. But because g' has image in A, $[g'] \in \pi_n(A, x_0)$ and $i_*[g'] = [i \circ g'] = [f]$, so $[f] \in \mathrm{im}(i_*)$.

Step 2. Let us now show exactness at $\pi_n(X, A, x_0)$.

Note that the composite $\partial \circ j_* = 0$ since the restriction of a map $(I^n, \partial I^n, J^{n-1}) \to (X, x_0, x_0)$ to I^{n-1} has image x_0 and so represents 0 in $\pi_{n-1}(A, x_0)$. Therefore, $\mathrm{im}(j_*) \subseteq \ker(\partial)$. For the converse, suppose $[f] \in \ker(\partial)$. This means there exists a basepoint preserving homotopy $H\colon I^{n-1} \times I \to A$ (relative to ∂I^{n-1}) from $f \mid_{I^{n-1} \times \{0\}}$ to the constant map where the image of H is contained entirely in A. We can then define another homotopy H, such that $G_0 = f$, $G_t \mid_{I^{n-1}} = H_t$ and the rest of the image of G_t is $f[I^n]$ union with the image of H_s for $0 \leq s \leq t$. This homotopy maps S^{n-1} into A at all times, so $[f] = [G_1]$. Morevoer, G_1 maps the boundary of I^n to x_0, so $[G_1] \in \pi_n(X, x_0)$. Altogether,

$$j_*[G_1] = [j \circ G_1] = [G_1] = [f]$$

so $\ker(\partial) \subseteq \mathrm{im}(j_*)$. $\square$

A
A nice category of topological spaces