
AN INVITATION TO MATHEMATICAL LOGIC

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PREFACE

01° This text may serve in a course on Mathematical Logic at the upper division undergraduate level. We suggest the subtitle:

From Zero to Gödel

For the exposition, we have adopted a relentlessly formal style, relieved, at appropriate places, by unusually detailed arguments.

02° In the central chapters of the text, we describe the syntax and semantics for the various predicate logics and we prove certain fundamental theorems: the Deduction Principle, the Interpretation Theorem, the Completeness Theorem, and the Compactness Theorem.

03° For our basic example of a predicate logic, we define the Predicate Logic for Arithmetic. We develop just enough of this important topic to support our subsequent presentation of the theorems of Tarski, Gödel, and Church. We also define the Predicate Logic for Set Theory.

04° Relative to the Standard Interpretation for Arithmetic, for which the underlying universe \mathbf{N} is composed of the familiar natural numbers, we describe the basic families of syntactically and semantically definable subsets of \mathbf{N}^r (where r is any positive integer). We apply the celebrated idea of Gödel numbering, by which one identifies the sentences of Arithmetic with corresponding natural numbers and through which one may define sentences which “refer to themselves.” We present the Diagonalization Theorem and the Deduction Theorem, which assert that, relative to Gödel numbering, the diagonalization mapping and the deduction mapping are recursive. In turn, we present the Representation Theorem, which entails that the graphs of these mappings, as subsets of \mathbf{N}^2 , are syntactically, hence semantically definable. The first of these theorems leads to the Fixed Point Theorem. We apply these developments to prove the basic theorems of Tarski, Gödel, and Church.

05° The articles which compose the text are marked by numbers, modified by superscripts: j° and k^\bullet . The latter take the form of problems. In many cases, they are critical to the text.

CHAPTER 1

LOGICS AND TREES

In this chapter, we introduce the concepts of language and logic, which underlie the description of predicate logics. We also introduce the concept of tree, which proves useful, first, for the analysis of the structure of sentences and, second, for the analysis of the structure of deductions. Often, visual inspection of an appropriate tree serves to simplify and clarify, even to displace formal argument by induction.

1.1 LOGICS

Symbol Sets

01° Let us develop the general concept of a *logic*. We begin by introducing a finite set S containing at least two members. We refer to S as a *symbol set* and to the various members of S as *symbols*. In turn, let S^* be the (countably infinite) set consisting of all finite strings σ of the form:

$$\sigma = \sigma_1 \sigma_2 \cdots \sigma_n$$

where n is any nonnegative integer and where, for each index j ($1 \leq j \leq n$), σ_j lies in S . We refer to n as the *length* of σ and denote it by $|\sigma|$. There is just one string for which $n = 0$: the *empty* string. We denote it by ϵ .

Concatenation of Strings

02° For any strings σ' and σ'' in S^* :

$$\sigma' = \sigma'_1 \sigma'_2 \cdots \sigma'_{n'}, \quad \sigma'' = \sigma''_1 \sigma''_2 \cdots \sigma''_{n''}$$

we form the *concatenation* of σ' and σ'' , as follows:

$$\sigma' \sigma'' = \sigma'_1 \sigma'_2 \cdots \sigma'_{n'} \sigma''_1 \sigma''_2 \cdots \sigma''_{n''}$$

Obviously, this operation on S^* is associative. It is not commutative. Moreover, the empty string ϵ serves as the neutral element for it.

Segments of Strings

03° In terms of the operation of concatenation on S^* , we can express various basic concepts smoothly. For instance, let σ' and σ'' be any strings in S^* . We say that σ' is a *segment* of σ'' iff there are strings ρ and τ in S^* such that:

$$\sigma'' = \rho\sigma'\tau$$

Substitution in Strings

04° In turn, let ρ' , ρ'' , σ' , and σ'' be any strings in S^* . It may happen that there are strings τ and v in S^* such that:

$$\rho'' = \tau\rho'v \quad \text{and} \quad \sigma'' = \tau\sigma'v$$

Of course, ρ' would be a segment of ρ'' and σ' would be a segment of σ'' . In such a case, we say that σ'' is defined by *substitution* of σ' for ρ' in ρ'' .

The Lexicographic Order Relation

05° Let us supply S with a linear order relation:

$$a' < a''$$

where a' and a'' are any symbols in S . In turn, let us supply S^* with the corresponding *lexicographic* order relation. To that end, let σ' and σ'' be any strings in S^* . We write:

$$\sigma' \prec \sigma''$$

iff one or the other of the following two conditions holds:

$$(O_1) \quad |\sigma'| < |\sigma''|$$

(O_2) $|\sigma'| = |\sigma''|$ and there are strings ρ , τ' , and τ'' in S^* and symbols a' and a'' in S such that:

$$\sigma' = \rho a' \tau', \quad \sigma'' = \rho a'' \tau'', \quad \text{and} \quad a' < a''$$

The latter condition expresses the conventional ordering of words in a lexicon.

06° One can easily verify that the lexicographic order relation on S^* is linear. In fact, the order structure so defined is isomorphic to that of the natural numbers \mathbf{N} . That is, there is a bijective mapping Γ carrying S^* to \mathbf{N} such that, for any strings σ' and σ'' in S^* :

$$\sigma' \prec \sigma'' \iff \Gamma(\sigma') < \Gamma(\sigma'')$$

07• Produce the mapping Γ , by the following procedure. Let b be the number of members of S . By assumption, $2 \leq b$. Let the symbols be identified with the positive integers:

$$1, 2, 3, \dots, b$$

in natural order. Let Γ be the mapping carrying S^* to \mathbf{N} , defined as follows:

$$\Gamma(\sigma) = \sum_{j=1}^n \sigma_j b^{n-j}$$

where $\sigma = \sigma_1 \sigma_2 \dots \sigma_n$ is any string in S^* . We intend that $\Gamma(\epsilon) = 0$. Show that Γ is bijective and that Γ is an order isomorphism, as just defined.

08° We refer to Γ as the Gödel Mapping. For each σ in S^* , we refer to $\Gamma(\sigma)$ as the Gödel Number for σ .

09• Describe the inverse of Γ , by the following procedure. Clearly, $\Gamma^{-1}(0) = \epsilon$. Let ℓ be any positive integer. Let q_1 and r_1 be the nonnegative integers provided by the Euclidean Algorithm:

$$\ell = q_1 b + r_1, \text{ and } 0 \leq r_1 < b$$

Let:

$$(\ell_1, s_1) = \begin{cases} (q_1 - 1, b) & \text{if } r_1 = 0 \\ (q_1, r_1) & \text{if } 0 < r_1 \end{cases}$$

In turn, let q_2 and r_2 be the nonnegative integers provided by the Euclidean Algorithm:

$$\ell_1 = q_2 b + r_2, \text{ and } 0 \leq r_2 < b$$

Let:

$$(\ell_2, s_2) = \begin{cases} (q_2 - 1, b) & \text{if } r_2 = 0 \\ (q_2, r_2) & \text{if } 0 < r_2 \end{cases}$$

Continue the computation until $\ell_n = 0$. Show that:

$$\Gamma^{-1}(\ell) = \sigma_1 \sigma_2 \dots \sigma_n \text{ where } \sigma_1 \sigma_2 \dots \sigma_n = s_n \dots s_2 s_1$$

Note that the mappings Γ and Γ^{-1} are, in any reasonable sense, computable.

10• For an illustration of problems 07• and 09•, consider the simple case in which $b = 2$. The first fifteen values of Γ stand as follows:

ϵ	1	2	11	12	21	22	111	112	121	122	211	212	221	222
\downarrow	\downarrow	\downarrow	\downarrow	\downarrow	\downarrow	\downarrow	\downarrow	\downarrow	\downarrow	\downarrow	\downarrow	\downarrow	\downarrow	\downarrow
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14

What nonnegative integer corresponds to 1222112? What string corresponds to 612? To make these computations, one might employ the Mathematica Notebook *GödelNumbering.nb* on the website:

<http://people.reed.edu/~wieting/programs.html>

Languages and Logics

11° By a *language*, we mean any nonempty subset \mathcal{L} of S^* . We refer to S as the symbol set *underlying* \mathcal{L} . We refer to the strings in \mathcal{L} as *sentences*. By a *logic*, we mean any ordered pair:

$$\Lambda = (\mathcal{L}, \mathcal{A})$$

where \mathcal{L} is a language and where \mathcal{A} is a nonempty subset of \mathcal{L} . We refer to the sentences in \mathcal{A} as *axioms*.

12° Rather cryptically, we augment the logic Λ with certain *rules of deduction*. For the sense of such rules, we appeal for now to common experience. In context of the various predicate logics, we will describe the rules of deduction carefully.

Theories

13° Let $\Lambda = (\mathcal{L}, \mathcal{A})$ be any logic and let \mathcal{H} be any subset of \mathcal{L} . We refer to the sentences in \mathcal{H} as *hypotheses*. By applying the rules of deduction, we may proceed to *derive* various sentences in \mathcal{L} from the sentences in $\mathcal{A} \cup \mathcal{H}$. Let:

$$\Theta_\Lambda(\mathcal{H})$$

be the subset of \mathcal{L} consisting of all sentences in \mathcal{L} which can be so derived. We refer to the sentences in $\Theta_\Lambda(\mathcal{H})$ as *theorems* and to the set $\Theta_\Lambda(\mathcal{H})$ itself as the *theory* of \mathcal{H} .

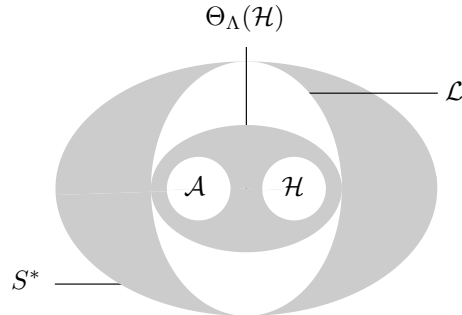


Figure 1: Logic

1.2 TREES

Trees

01° By a *graph*, we mean an ordered pair:

$$\mathcal{G} = (\mathcal{N}, \mathcal{B})$$

where \mathcal{N} is any nonempty finite set and where \mathcal{B} is any subset of $\mathcal{N} \times \mathcal{N}$. We refer to the members of \mathcal{N} and \mathcal{B} as *nodes* and *branches*, respectively. For each branch:

$$B = (N', N'')$$

in \mathcal{B} , we refer to N' as the *initial* node and to N'' as the *terminal* node of B . We require that $N' \neq N''$.

02° It may happen that a node R in \mathcal{N} is an initial node but not a terminal node. We refer to such a node as a *root*. In turn, it may happen that a node L in \mathcal{N} is a terminal node but not an initial node. We refer to such a node as a *leaf*.

03° By a *path* in \mathcal{G} , we mean a finite sequence:

$$N_0, N_1, N_2, \dots, N_k$$

of nodes in \mathcal{N} such that, for each index j ($0 \leq j < k$), $B_j = (N_j, N_{j+1})$ is a branch in \mathcal{B} . We say that the path *joins* the node N_0 to the node N_k . Of course, k is the *length* of the path.

04° We say that the graph $\mathcal{G} = (\mathcal{N}, \mathcal{B})$ is a *tree* iff it meets the following conditions:

(G_1) there is precisely one root R in \mathcal{N}

(G_2) for each node N in \mathcal{N} , if $N \neq R$ then there is precisely one path in \mathcal{G} which joins R to N

05° Let $\mathcal{G} = (\mathcal{N}, \mathcal{B})$ be a tree. Let N' and N'' be any nodes in \mathcal{N} . It may happen that there is a path in \mathcal{G} , necessarily unique, which joins N' to N'' . In such a case, we say that N' is an *ancestor* of N'' and that N'' is a *descendant* of N' . It may happen that the length of the path is 1. In such a special case, we say that N' is an *immediate* ancestor of N'' and that N'' is an *immediate* descendant of N' .

06° Let N be a node in \mathcal{N} , other than a leaf. We require that the set of immediate descendants of N be linearly ordered. In our diagrams of trees, we will display such sets of nodes in order, from left to right.

07° We define the *valence* $v(N)$ of N to be the number of its immediate descendants.

Active Trees

08° Let N be a node in \mathcal{N} . We say that N is an *active* node iff $2 \leq v(N)$ and the first of the immediate descendants of N , let it be L , is a leaf. In such a context, we refer to L as a *primary* leaf, the primary leaf for the active node N . We refer to \mathcal{G} itself as an *active* tree iff every node in \mathcal{N} other than a leaf is an active node.

09° Let L be a leaf in \mathcal{N} . We refer to L as a *secondary* leaf iff it is not a primary leaf.

Labeled Trees

10° In practice, we commonly place labels on the various nodes of a tree and we refer to the tree as a *labeled* tree. To be precise, we introduce a symbol set S and a mapping λ carrying \mathcal{N} to S^* . For each node N in \mathcal{N} , we interpret $\lambda(N)$ to be the corresponding label. Informally, we say that the labels *occupy* the nodes.

11° In the following diagram of a tree, we have placed the label r on the root of the tree, the label a on the active nodes, the labels p and s , respectively, on the primary and secondary leaves, and the label x on the rest. One should note that the root lies *below* while the leaves lie *above*. See article 42°.

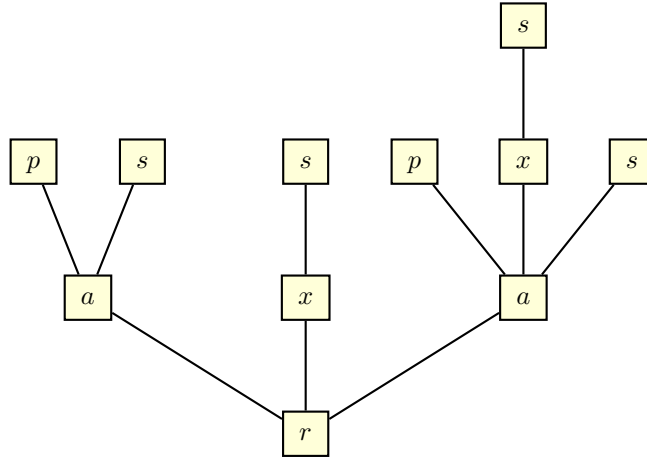


Figure 2: Labeled Tree

Relations among Trees

12° Let $\mathcal{G} = (\mathcal{N}, \mathcal{B})$ be any tree. Let N be any node in \mathcal{N} other than the root R . Let \mathcal{N}_* be the set of descendants of N . Let $\mathcal{N}_\circ = \mathcal{N}_* \cup \{N\}$, $\mathcal{N}_\bullet = \mathcal{N} \setminus \mathcal{N}_*$, $\mathcal{B}_\circ = \mathcal{B} \cap (\mathcal{N}_\circ \times \mathcal{N}_\circ)$, and $\mathcal{B}_\bullet = \mathcal{B} \cap (\mathcal{N}_\bullet \times \mathcal{N}_\bullet)$. Clearly, both $\mathcal{G}_\circ = (\mathcal{N}_\circ, \mathcal{B}_\circ)$ and $\mathcal{G}_\bullet = (\mathcal{N}_\bullet, \mathcal{B}_\bullet)$ are trees. We refer to \mathcal{G}_\circ as the *subtree* of \mathcal{G} defined by the node N and we refer to \mathcal{G}_\bullet as the subtree of \mathcal{G} *residual* to \mathcal{G}_\circ . Obviously, the common node N is the root of \mathcal{G}_\circ and it is one of the leaves of \mathcal{G}_\bullet as well. Conversely, we may say that the tree \mathcal{G} arises by *grafting* the root of \mathcal{G}_\circ to the specified leaf N of \mathcal{G}_\bullet .

13° The following figure illustrates these matters. The label n occupies the distinguished node N , while the labels \circ and \bullet mark the remaining nodes of \mathcal{G}_\circ and \mathcal{G}_\bullet , respectively.

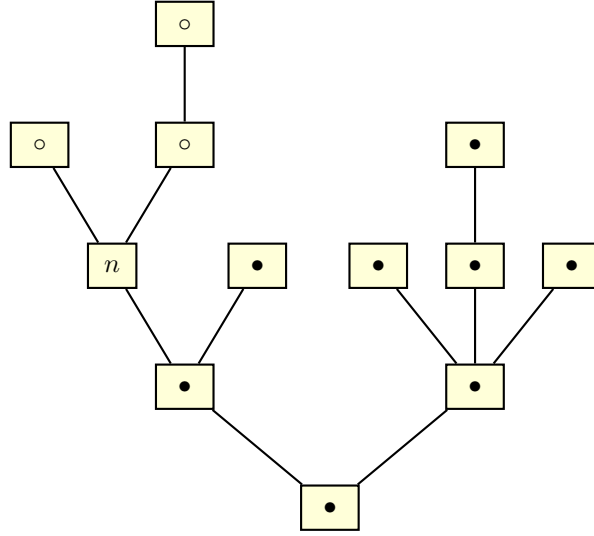


Figure 3: Graft

14° Again let $\mathcal{G} = (\mathcal{N}, \mathcal{B})$ be any tree. Let R be the root of \mathcal{G} and let N_1, N_2, \dots , and N_k be the immediate descendants of R , in order. Let $\mathcal{G}_1, \mathcal{G}_2, \dots$, and \mathcal{G}_k be the subtrees of \mathcal{G} defined by the nodes N_1, N_2, \dots , and N_k , respectively. In this special context, we say that \mathcal{G} arises by *concatenation* of the trees $\mathcal{G}_1, \mathcal{G}_2, \dots$, and \mathcal{G}_k , in order.

15° Figure 4 illustrates the concatenation of two trees. The label r occupies the root R of \mathcal{G} . The labels \circ and \bullet mark the nodes of the subtrees \mathcal{G}_1 and \mathcal{G}_2 , respectively.

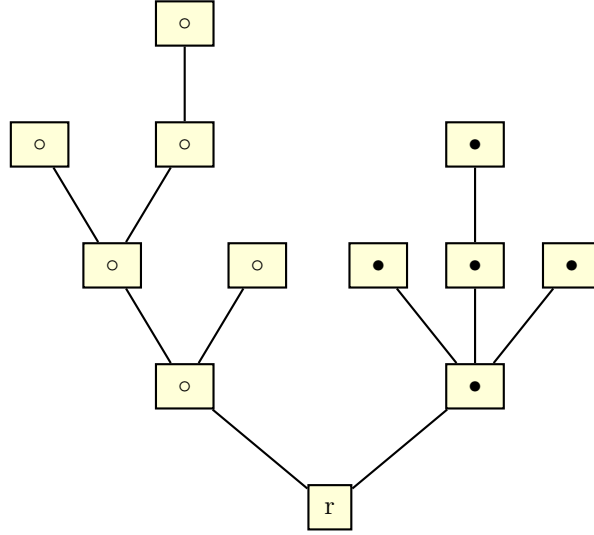


Figure 4: Concatenation

16° Of course, the foregoing trees may themselves be labeled. For the case in which \mathcal{G} arises by concatenation of the trees $\mathcal{G}_1, \mathcal{G}_2, \dots$, and \mathcal{G}_k (in order), we would presume that the root R of \mathcal{G} possesses or in some manner acquires a label of its own. For the case in which \mathcal{G} arises by grafting the root R_o of \mathcal{G}_o to a specified leaf N of \mathcal{G}_\bullet , we would require that the label on R_o displace the label on N . In context, the labels on the nodes in the path joining the root R_\bullet of \mathcal{G}_\bullet to N would, very likely, be modified as well.

Tree Forms

17° Remarkably, we can represent (labeled) trees in terms of strings of symbols, based upon appropriate symbol sets. By applying the corresponding Gödel Mapping, we can, in turn, convert (labeled) trees to nonnegative integers. This maneuver figures in the proofs of the theorems of Tarski, Gödel, and Church.

18° Let us introduce two distinguished symbols: the left angle bracket \langle and the right angle bracket \rangle . We obtain the (rather primitive) symbol set:

$$P = \{\langle, \rangle\}$$

Now let Υ be the smallest subset of P^* which meets the conditions:

$$(\Upsilon_1) \quad \langle \rangle \in \Upsilon$$

(Υ_2) for any positive integer k and for any strings v_1, v_2, \dots , and v_k in Υ , the string $\langle v_1 v_2 \dots v_k \rangle$ is in Υ

We mean to say that Υ is the intersection of all the various subsets of P^* which meet the conditions (Υ_1) and (Υ_2) .

19° Let Υ_o be the subset of P^* consisting of all strings ω of the form:

$$\omega = \langle \omega_1 \omega_2 \dots \omega_k \rangle$$

where k is any nonnegative integer and where $\omega_1, \omega_2, \dots$, and ω_k are any strings in Υ . Of course, if $k = 0$ then $\omega = \langle \rangle$. By inspection, we find that Υ_o is a subset of Υ and that Υ_o satisfies the conditions (Υ_1) and (Υ_2) . It follows that $\Upsilon_o = \Upsilon$. Hence, for each string ω in Υ , there are a nonnegative integer k and strings $\omega_1, \omega_2, \dots$, and ω_k in Υ such that:

$$(\Omega) \quad \omega = \langle \omega_1 \omega_2 \dots \omega_k \rangle$$

20° By induction, we find that the numbers of occurrences of \langle and \rangle in ω are equal.

21° Show that the form for ω displayed in relation (Ω) is unique. To that end, show that, for any strings ω' and ω'' in Υ and for any string α in P^* , if $\omega'' = \omega' \alpha$ then $\alpha = \epsilon$, so that $\omega' = \omega''$.

22° Now we contend that every tree defines a string in Υ and that every string in Υ defines a tree. The correspondence is (essentially) bijective. We are led to refer to the strings in Υ as *tree forms*.

23° Let us proceed to prove our contention. By relation (Ω) in article 19°, we see that tree forms are produced inductively from $\langle \rangle$ by concatenation. By article 14°, we know that trees are produced inductively from roots by concatenation. Of course, we may identify the tree form $\langle \rangle$ with a tree having a single node, namely, the root. These observations prove our contention.

24° For an illustration, let us retrieve the tree \mathcal{G} displayed in Figure 2 (but erase the labels) and let us introduce the tree form ω , defined as follows:

$$\omega = \langle \langle \langle \rangle \rangle \langle \rangle \rangle \langle \langle \langle \rangle \rangle \rangle \langle \rangle \rangle$$

We claim that \mathcal{G} defines ω and that ω (essentially) defines \mathcal{G} . For the first claim, we place the label $\langle \rangle$ on each of the leaves of \mathcal{G} , then proceed along the

Counters

26° The strings:

$$\omega' = \langle \langle \rangle \langle \rangle \rangle \langle \langle \rangle \langle \rangle \rangle \langle \langle \rangle \langle \rangle \rangle \langle \rangle, \quad \omega'' = \langle \langle \rangle \langle \rangle \rangle \langle \langle \rangle \langle \rangle \rangle \langle \langle \rangle \langle \rangle \rangle \langle \rangle$$

both lie in P^* . The first is a tree form but the second is not. We require an algorithm for deciding whether or not a given string ω in P^* is a tree form and, moreover, for *computing* the properties of the corresponding tree: for instance, the valences of the nodes.

27° Let m be an even positive integer. Let us introduce the symbol set:

$$Q = \{-m, \dots, -1, 0, 1, \dots, m\}$$

Let Q_m^* be the subset of Q^* consisting of all strings κ which have length m and which meet the conditions:

- (1) $|\kappa_1| = 1$
- (2) for each index j ($0 < j \leq m$), $|\kappa_j - \kappa_{j-1}| = 1$

28° In turn, let P_m^* be the subset of P^* consisting of all strings ω having length m . For each string ω in P_m^* , let κ be the string in Q_m^* defined as follows:

- (•) if $\omega_1 = \langle$ then $\kappa_1 = 1$ while if $\omega_1 = \rangle$ then $\kappa_1 = -1$
- (•) for each index j ($1 < j \leq m$), if $\omega_j = \langle$ then $\kappa_j = \kappa_{j-1} + 1$ while if $\omega_j = \rangle$ then $\kappa_j = \kappa_{j-1} - 1$

Clearly, for each index j ($1 \leq j \leq m$), κ_j is the difference between the numbers of occurrences of \langle and \rangle among the first j symbols in ω . We refer to κ as the *counter* for ω .

29° By the foregoing discussion, we may introduce the mapping C_m carrying P_m^* to Q_m^* :

$$C_m(\omega) = \kappa$$

where ω is any string in P_m^* . By elementary considerations, we find that C_m is bijective.

30° For the string:

$$\omega = \langle \rangle \langle \rangle \langle \rangle \langle \rangle \langle \rangle \langle \rangle \langle \rangle \langle \rangle \langle \rangle \langle \rangle$$

in P_{22}^* , calculate $\kappa = C_{22}(\omega)$ in Q_{22}^* .

31° Let K_m be the subset of Q_m^* consisting of all strings κ which meet not only the conditions (1) and (2) but also the conditions:

- (3) for each index j ($1 \leq j < m$), $0 < \kappa_j$
- (4) $\kappa_m = 0$

Of course, conditions (1) and (3) entail that $\kappa_1 = 1$.

32° In turn, let Υ_m be the subset of Υ consisting of all tree forms having length m . We contend that C_m carries Υ_m bijectively to K_m . Having proved the contention, we would infer that, for any string ω in P_m^* , ω is a tree form iff the corresponding counter $\kappa = C_m(\omega)$ meets the conditions (1), (2), (3), and (4).

33° Let us prove the contention. We will argue by induction on m . However, for notational clarity, we will proceed in terms of a characteristic example. Once again, let ω be the tree form:

$$\omega = \langle \langle \langle \rangle \langle \rangle \rangle \langle \rangle \rangle \langle \langle \rangle \langle \rangle \rangle \langle \rangle \rangle$$

Obviously, $|\omega| = 22$. Let \mathcal{G} stand for the tree defined by ω and let κ be the counter for ω . Referring to relation (Ω) in article 19° and also to Figure 5, we “inflate” ω to expose the three immediate descendants of the root in \mathcal{G} :

$$\omega = \langle \langle \langle \rangle \langle \rangle \rangle \langle \rangle \rangle \langle \langle \rangle \langle \rangle \rangle \langle \rangle \rangle = \langle \bar{\omega}_1 \bar{\omega}_2 \bar{\omega}_3 \rangle$$

In turn, we compute κ :

$$\kappa = 1\ 2\ 3\ 2\ 3\ 2\ 1\ 2\ 3\ 2\ 1\ 2\ 3\ 2\ 3\ 4\ 3\ 2\ 3\ 2\ 1\ 0$$

and inflate it as well:

$$\kappa = 1\ 2\ 3\ 2\ 3\ 2\ 1\ 2\ 3\ 2\ 1\ 2\ 3\ 2\ 3\ 4\ 3\ 2\ 3\ 2\ 1\ 0$$

The spaces follow the occurrences of 1. Let us display the three middle segments of κ , reduced by 1:

$$1\ 2\ 1\ 2\ 1\ 0\ 1\ 2\ 1\ 0\ 1\ 2\ 1\ 2\ 3\ 2\ 1\ 2\ 1\ 0$$

These strings $\bar{\kappa}_1$, $\bar{\kappa}_2$, and $\bar{\kappa}_3$ are the counters for $\bar{\omega}_1$, $\bar{\omega}_2$, and $\bar{\omega}_3$, respectively. Clearly, just as $\bar{\omega}_1$, $\bar{\omega}_2$, and $\bar{\omega}_3$ define ω , so also $\bar{\kappa}_1$, $\bar{\kappa}_2$, and $\bar{\kappa}_3$ define κ .

34° Let us assume that, for every even positive integer ℓ , if $\ell < m$ then C_ℓ is bijective. By the pattern explicitly developed in the foregoing example, we infer that C_m is bijective. Our contention follows by induction.

35° For further discussion, let us display the graph of κ :

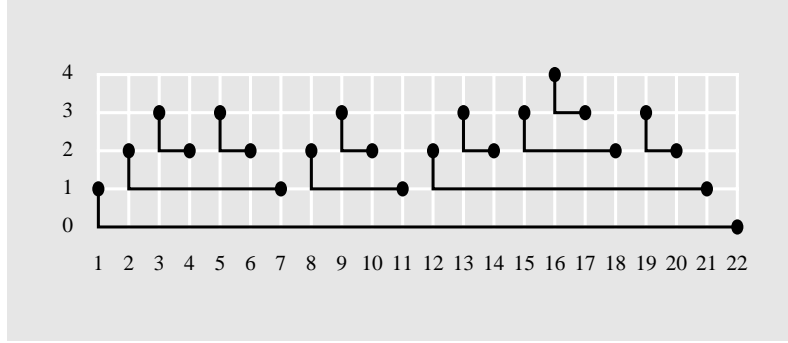


Figure 6: Graph of κ

Let j be any index ($1 \leq j < 22$). We find that $\omega_j = \langle$ iff $j = 1$ or $\kappa_j - \kappa_{j-1} = 1$. For such a value of j , let k be the smallest index such that $j < k \leq 22$ and $\kappa_j - \kappa_k = 1$. By this prescription, we obtain eleven ordered pairs (j, k) of indices. These ordered pairs are identifiable with the eleven nodes in \mathcal{G} . One may say that the particular occurrence ω_j of \langle *locates* the corresponding node as a segment of ω . The valence of the node equals the number of indices \bar{j} such that $j < \bar{j} < k$ and $\kappa_j = \kappa_{\bar{j}}$.

36° In Figure 6, we have identified these ordered pairs (j, k) of indices with “hooks.” In themselves, the hooks display the structure of \mathcal{G} .

37° By the foregoing results, we obtain the required algorithm. In fact, for any string ω in P^* , ω is a tree form iff the corresponding counter $\kappa = C_m(\omega)$ meets the conditions (1), (2), (3), and (4), where $m = |\omega|$. Moreover, by our discussion of the graph of κ , it is plain that κ yields by simple computation the basic structure of \mathcal{G} .

Labeled Tree Forms

38° In the following chapter, we will encounter two kinds of labeled trees: *syntactic* trees and *deductive* trees. We require a generalization of the relation between trees and tree forms which captures the relation between labeled trees and *labeled* tree forms.

39° We can describe the generalization most effectively by means of a simple example. Let us once again retrieve the tree \mathcal{G} displayed in Figure 2, with fresh labels:

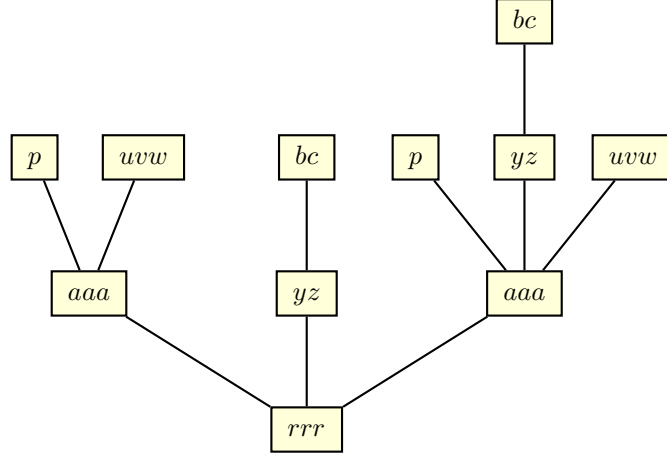


Figure 7: Labeled Tree

and let ω be the tree form corresponding to \mathcal{G} (with labels excised):

$$\omega = \langle \langle \rangle \langle \rangle \rangle \langle \langle \rangle \langle \rangle \rangle \langle \langle \rangle \langle \rangle \rangle \langle \rangle \rangle$$

We propose the string:

$$\bar{\omega} = rrr\langle aaa\langle p\langle \rangle uvw\langle \rangle \rangle yz\langle bc\langle \rangle \rangle aaa\langle p\langle \rangle yz\langle bc\langle \rangle \rangle uvw\langle \rangle \rangle$$

as the labeled tree form corresponding to the labeled tree \mathcal{G} . For each node N in \mathcal{G} , we have substituted the label on N for the empty string ϵ just to the left of the particular occurrence of \langle in ω which locates N .

40° The formal generalization proceeds without difficulty.

41° Both syntactic and deductive trees have the property that the labels on their leaves determine the labels on the other nodes, including the root. However, the details are intricate. We will discuss these matters when they inexorably arise.

Christopher Smart, c1760 (adapted)

42° “For the trees are great blessings. For the trees *are* great blessings. For the trees have their angels, even the words of God’s creation. For the tree glorifies God and the root parries the Adversary. For there is a language of trees. For the trees are, peculiarly, the poetry of Life.”

43• Memorize the Greek alphabet:

α	alpha	A
β	beta	B
γ	gamma	Γ
δ	delta	Δ
ϵ	epsilon	E
ζ	zeta	Z
η	eta	H
θ	theta	Θ
ι	iota	I
κ	kappa	K
λ	lambda	Λ
μ	mu	M
ν	nu	N
ξ	xi	Ξ
o	omicron	O
π	pi	Π
ρ	rho	P
σ	sigma	Σ
τ	tau	T
υ	upsilon	Υ
ϕ	phi	Φ
χ	chi	X
ψ	psi	Ψ
ω	omega	Ω

CHAPTER 2

PREDICATE LOGICS: SYNTAX

Let us describe the various predicate logics. In this chapter, we concentrate upon matters of *syntax*, that is, of grammar. The basic theorem is the Deduction Principle. In the following chapter, we concentrate upon matters of *semantics*, that is, of meaning. The basic theorem is the Interpretation Theorem, from which the Completeness Theorem and the Compactness Theorem follow easily.

2.1 LANGUAGES

Symbols

01° Let Σ be the symbol set consisting of the ten symbols:

$$(,), \neg, \longrightarrow, \forall, c, x, f, r, |$$

We proceed to describe the languages \mathcal{L} , as subsets of Σ^* , which figure in the predicate logics. To begin, we form the *constant strings*:

$$(c|), (c||), (c| |), (c| | |), \dots$$

the *variable strings*:

$$(x|), (x||), (x| |), (x| | |), \dots$$

the *function strings*:

$$(|f|), (|f||), (|f| |), (|f| | |), \dots ; (||f|), (||f||), (||f| |), (||f| | |), \dots ; \dots$$

and the *predicate (or relation) strings*:

$$(|r|), (|r||), (|r| |), (|r| | |), \dots ; (||r|), (||r||), (||r| |), (||r| | |), \dots ; \dots$$

Let \mathcal{V} be the subset of Σ^* consisting of all the variable strings.

02° Respecting common practice, we will refer to the foregoing strings as symbols. For any function symbol ϕ and for any predicate symbol ρ , we refer to the number of strokes $|$ to the left of f and r , respectively, as the *valence* of the symbol. We denote the valences by $v(\phi)$ and $v(\rho)$, respectively.

03° For convenience, we abbreviate the foregoing symbols in the manner suggested by the following examples:

$$c_4 \text{ for } (c|), \quad x_7 \text{ for } (x|), \quad f_5^2 \text{ for } (|f|), \quad r_4^3 \text{ for } (||r|)$$

Of course, $v(f_5^2) = 2$ and $v(r_4^3) = 3$.

04° For the predicate symbol $(|r|)$, we reserve the following special abbreviation:

$$\equiv \text{ for } r_1^2 \text{ for } (|r|)$$

Clearly, $v(\equiv) = 2$. We refer to \equiv as the *equality* symbol.

Preambles

05° Let \mathcal{C} be any subset of the set of all constant symbols, let \mathcal{F} be any subset of the set of all function symbols, and let \mathcal{P} be any subset of the set of all predicate symbols in Σ^* , constrained only by the requirement that the equality symbol \equiv be a member of \mathcal{P} . Let us refer to the ordered triple:

$$\Pi = (\mathcal{C}, \mathcal{F}, \mathcal{P})$$

as a *preamble*. In practice, we may denote the components of a preamble Π by \mathcal{C}_Π , \mathcal{F}_Π , and \mathcal{P}_Π , respectively.

06° Let us describe the terms, the atomic sentences, and the sentences which compose the predicate language \mathcal{L}_Π defined by the preamble Π .

Terms

07° Let \mathcal{T}_Π be the smallest subset of Σ^* which meets the conditions:

$$(T_1) \quad \mathcal{C} \subseteq \mathcal{T}_\Pi$$

$$(T_2) \quad \mathcal{V} \subseteq \mathcal{T}_\Pi$$

(T_3) for any positive integer k and for any function symbol ϕ in \mathcal{F} , if k is the valence of ϕ then, for any strings τ_1, τ_2, \dots , and τ_k in \mathcal{T}_Π , the string $(\phi\tau_1\tau_2 \dots \tau_k)$ is in \mathcal{T}_Π

We mean to say that \mathcal{T}_Π is the intersection of all the various subsets of Σ^* which meet conditions (T_1), (T_2), and (T_3). We refer to the strings in \mathcal{T}_Π as *terms* and, in particular, to the constant symbols in \mathcal{C} and the variable symbols in \mathcal{V} as *atomic* terms.

Atomic Sentences

08° Now we may form the various *atomic sentences*:

$$\lambda = (\rho \tau_1 \tau_2 \cdots \tau_\ell)$$

where ρ is any predicate symbol in \mathcal{P} , where τ_1, τ_2, \dots , and τ_ℓ are any terms in \mathcal{T}_Π , and where ℓ is the valence of ρ .

09° For the atomic sentence $(\equiv \tau_1 \tau_2)$, we adopt the following familiar convention:

$$(\tau_1 \equiv \tau_2) \text{ for } (\equiv \tau_1 \tau_2)$$

Sentences

10° Finally, let \mathcal{L}_Π be the smallest subset of Σ^* which meets the conditions:

- (L_1) for any atomic sentence λ in Σ^* , λ lies in \mathcal{L}_Π
- (L_2) for any string α in Σ^* , if α lies in \mathcal{L}_Π then $((\neg)\alpha)$ lies in \mathcal{L}_Π
- (L_3) for any strings β and γ in Σ^* , if β and γ lie in \mathcal{L}_Π then $((\longrightarrow)\beta\gamma)$ lies in \mathcal{L}_Π
- (L_4) for any string δ in Σ^* and for any variable symbol ζ in \mathcal{V} , if δ lies in \mathcal{L}_Π then $((\forall\zeta)\delta)$ lies in \mathcal{L}_Π

We mean to say that \mathcal{L}_Π is the intersection of all the various subsets of Σ^* which meet conditions (L_1), (L_2), (L_3), and (L_4).

11° We refer to the strings in \mathcal{L}_Π as *sentences*. For any sentences α, β, γ , and δ in \mathcal{L}_Π and for any variable symbol ζ in \mathcal{V} , we refer to $((\neg)\alpha)$ as the *negation* of α , to $((\longrightarrow)\beta\gamma)$ as the *implication* of β and γ , and to $((\forall\zeta)\delta)$ as the *generalization* of δ over ζ . We refer to the string (\neg) as the *negation* symbol, to the string (\longrightarrow) as the *implication* symbol, and to a string of the form $(\forall\zeta)$ as a *universal quantifier*.

12° The foregoing baroque deployment of the parentheses (and) will prove useful in our study of syntactic and deductive trees for the predicate logics. However, when convenient, we will use the following familiar substitutes:

$$\neg\alpha \text{ or } (\neg\alpha) \text{ for } ((\neg)\alpha), \quad (\forall\zeta)\delta \text{ for } ((\forall\zeta)\delta),$$

$$\beta \longrightarrow \gamma \text{ or } (\beta \longrightarrow \gamma) \text{ for } ((\longrightarrow)\beta\gamma)$$

Predicate Languages: $\Pi \implies \mathcal{L}_\Pi$

13° Let Π be any preamble. We refer to \mathcal{L}_Π as the *predicate language* defined by Π .

Syntactic Trees: Terms

14° Let us describe a procedure for defining all terms in \mathcal{T}_Π explicitly. (Of course, we say nothing about the atomic terms.) To that end, let $\mathcal{G} = (\mathcal{N}, \mathcal{B})$ be an active tree. We place labels on the nodes of the tree, as follows. Let L be any leaf in \mathcal{N} and let N be its immediate ancestor. If L is the primary leaf for N then we place a function symbol ϕ on L , subject to the conditions that ϕ lies in \mathcal{F} and $v(N) = v(\phi) + 1$. (If no such function symbol is available then we terminate the procedure.) If L is not the primary leaf for N then we place on L a constant symbol in \mathcal{C} or a variable symbol in \mathcal{V} , arbitrarily. In this way, we place labels on all the leaves of the tree. We place labels on the remaining nodes of the tree in accord with the following recursive instruction. Let N be any node in \mathcal{N} , other than a leaf. The label on N shall be $(\phi\tau_1\tau_2 \cdots \tau_k)$, where $\phi, \tau_1, \tau_2, \dots$, and τ_k are the labels, in order, on the immediate descendants of N . Of course, $k = v(\phi)$. By this procedure, we determine a term, namely, the term τ which occupies the root of \mathcal{G} . We refer to \mathcal{G} , now labeled, as the *syntactic tree* for the term τ .

15° Let us display an example:

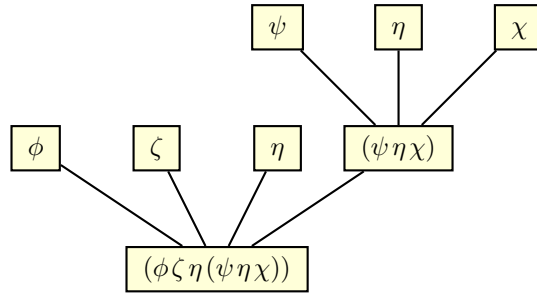


Figure 8: Term

We see a constant symbol χ , variable symbols ζ and η , and function symbols ϕ and ψ . Of course, $v(\phi) = 3$ and $v(\psi) = 2$.

16° Obviously, the syntactic trees for terms can be assembled, by grafts and concatenations, from basic labeled trees of two types. For the first type, the tree would be trivial, having one node (the root). The label on the root would be either a constant symbol or a variable symbol. For the second type, the

tree would have a root and an array of (at least two) immediate descendants. The label on the primary leaf would be a function symbol and the labels on the secondary leaves would be constant symbols and variables symbols, in any combination. One should note that the assembly proceeds in manner essentially unique.

17° Let $\bar{\mathcal{T}}_\Pi$ be the subset of \mathcal{T}_Π consisting of the atomic terms together with all the terms produced by the foregoing procedure. One can easily show that $\bar{\mathcal{T}}_\Pi$ meets the conditions (T_1) , (T_2) , and (T_3) which define \mathcal{T}_Π . Consequently, $\bar{\mathcal{T}}_\Pi = \mathcal{T}_\Pi$.

18° For a given term τ , it may happen that no variable symbols figure in its definition. In such a case, we refer to τ as a *constant* term. We denote by \mathcal{T}_Π^0 the subset of \mathcal{T}_Π consisting of all such terms.

Syntactic Trees: Atomic Sentences

19° The syntactic trees for atomic sentences require no elaboration. They derive from basic labeled trees of just one type, for instance:

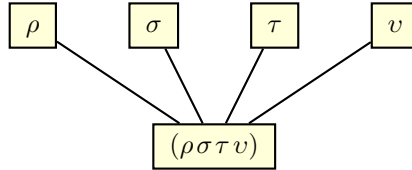


Figure 9: Atomic Sentence

Of course, ρ is a predicate symbol for which $v(\rho) = 3$, while σ , τ , and ν are terms.

Syntactic Trees: Sentences

20° Finally, let us describe a procedure for defining all sentences in \mathcal{L}_Π explicitly. To that end, let $\mathcal{G} = (\mathcal{N}, \mathcal{B})$ be an active tree, subject to the condition that, for each node N in \mathcal{N} , $v(N)$ equals 0, 2, or 3. We place labels on the nodes of the tree, as follows. Let L be any leaf in \mathcal{N} and let N be its immediate ancestor. If L is the primary leaf for N and if $v(N) = 2$ then we place either the negation symbol (\neg) or one of the universal quantifiers ($\forall\zeta$) on L . If L is the primary leaf for N and if $v(N) = 3$ then we place the implication symbol (\longrightarrow) on L . If L is not the primary leaf for N then we place an atomic sentence λ on L , arbitrarily. In this way, we place labels on all the leaves of the tree. We place labels on the remaining nodes of the tree in accord with the

following recursive instructions. Let N be any node in \mathcal{N} , other than a leaf. If $v(N) = 2$ and if the negation symbol (\neg) occupies the primary leaf for N then the label on N shall be $((\neg)\beta)$, where (\neg) and β are the labels, in order, on the two immediate descendants of N . If $v(N) = 2$ and if a universal quantifier $(\forall\zeta)$ occupies the primary leaf for N then the label on N shall be $((\forall\zeta)\beta)$, where $(\forall\zeta)$ and β are the labels, in order, on the two immediate descendants of N . If $v(N) = 3$ then the label on N shall be $((\rightarrow)\beta_1\beta_2)$, where (\rightarrow) , β_1 , and β_2 are the labels, in order, on the three immediate descendants of N . Obviously, the foregoing procedure determines a sentence in \mathcal{L}_Π , namely, the sentence α which occupies the root of \mathcal{G} . We refer to \mathcal{G} as the *syntactic tree* for α and to the various atomic sentences which occupy the secondary leaves in \mathcal{N} as the *atomic subsentences* of α .

21° In Figure 10, we display an example. Of course, η is a variable symbol while λ , μ , and ν are atomic sentences.

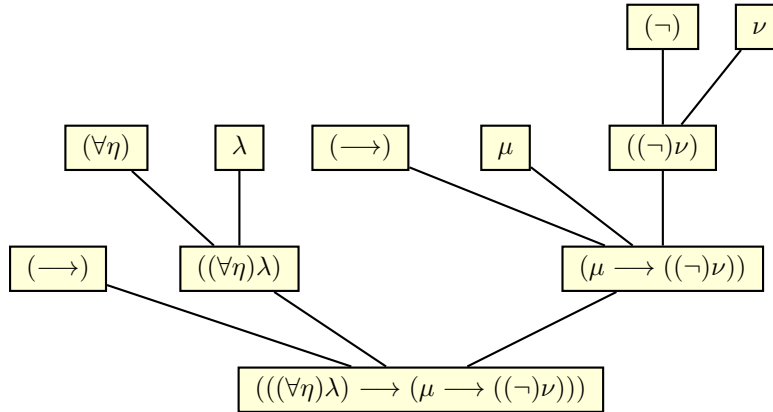


Figure 10: Sentence

The awkward disposition of nodes and branches will serve a purpose in the next following figure.

22° Obviously, the syntactic trees for sentences can be assembled, by grafts and concatenations, from basic labeled trees of three types. The types are defined by the operators (\neg) , (\rightarrow) , and $(\forall\zeta)$ of negation, implication, and generalization.

23° Let $\bar{\mathcal{L}}_\Pi$ be the subset of \mathcal{L}_Π consisting of all sentences produced by the foregoing procedure. By elementary argument, one can show that $\bar{\mathcal{L}}_\Pi$ meets the conditions (L_1) , (L_2) , (L_3) , and (L_4) which figure in the definition of \mathcal{L}_Π . Consequently, $\bar{\mathcal{L}}_\Pi = \mathcal{L}_\Pi$.

Unfoldings

24° Let α be a sentence in \mathcal{L}_Π and let $\mathcal{G} = (\mathcal{N}, \mathcal{B})$ be the syntactic tree for α . We proceed to *unfold* \mathcal{G} , in two stages, as follows.

(◦) We form the tree $\bar{\mathcal{G}} = (\bar{\mathcal{N}}, \bar{\mathcal{B}})$ by grafting to each secondary leaf L in \mathcal{N} the syntactic tree for the atomic sentence λ which occupies L .

(•) We form the tree $\hat{\mathcal{G}} = (\hat{\mathcal{N}}, \hat{\mathcal{B}})$ by grafting to each secondary leaf \bar{L} in $\bar{\mathcal{N}}$ the syntactic tree for the term v which occupies \bar{L} .

In both steps, one ought to adjust the labels on the nodes in accord with the instructions in articles 14° and 20°. However, the effect of such labor would, in general, be a tree of baffling complexity, difficult to display.

25° We refer to $\hat{\mathcal{G}}$ as the *unfolded* syntactic tree for α . To be clear, we refer to \mathcal{G} as the *basic* syntactic tree for α .

26° The following figure shows the unfolded syntactic tree for the sentence depicted in Figure 10.

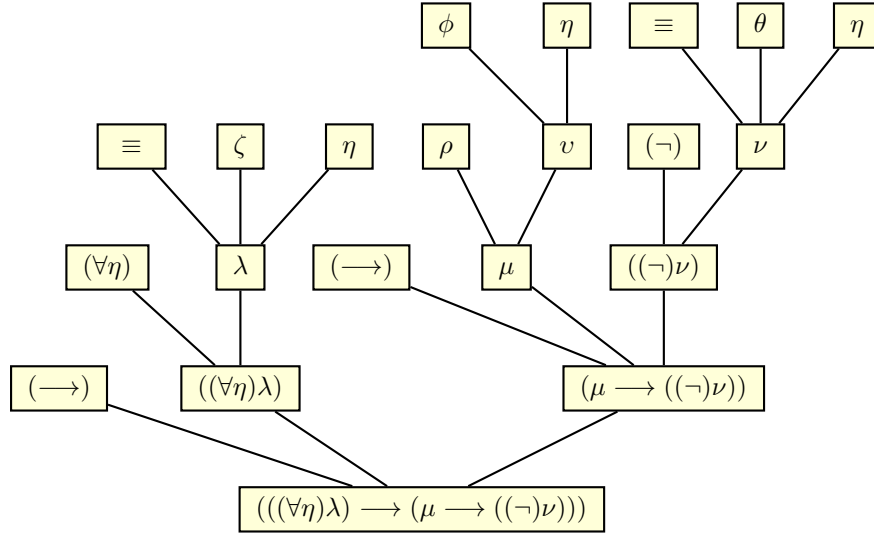


Figure 11: Unfolding

Clearly, λ , μ , and ν are the atomic sentences $(\zeta \equiv \eta)$, $(\rho\nu)$, and $(\theta \equiv \eta)$, respectively, while v is the term $(\phi\eta)$. We have not adjusted the labels on the nodes.

Subterms of Terms

27° Let τ and v be any terms in \mathcal{T}_Π . Let us emphasize that τ might be an atomic term in \mathcal{T}_Π , that is, a constant symbol χ in \mathcal{C} or a variable symbol ζ in \mathcal{V} . We say that τ is a *subterm* of v iff τ is a segment of v . We also say that τ *occurs* in v . One can easily show that τ is a subterm of v iff τ occupies one of the nodes, let it be N , of the syntactic tree for v . Of course, such a node would not be a primary leaf. Clearly, the syntactic tree for τ is the subtree of the syntactic tree for v defined by N . See Figure 8.

Subsentences of Sentences

28° Let α and β be any sentences in \mathcal{L}_Π . We say that β is a *subsentence* of α iff β is a segment of α . We also say that β *occurs* in α . One can easily show that β is a subsentence of α iff β occupies one of the nodes, let it be N , of the basic syntactic tree for α . Of course, such a node would not be a primary leaf. In any case, the basic syntactic tree for β would be the subtree of the basic syntactic tree for α defined by N . Of course, β would be an atomic subsentence of α iff N is one of the secondary leaves of the basic syntactic tree for α . See Figure 10.

Subterms of Sentences

29° Let τ be any term in \mathcal{T}_Π and let α be any sentence in \mathcal{L}_Π . Let us emphasize that τ might be an atomic term in \mathcal{T}_Π . We say that τ is a *subterm* of α iff there is an atomic subsentence λ of α such that τ is a segment of λ . We also say that τ *occurs* in α . One can easily show that τ is a subterm of α iff τ occupies one of the nodes, let it be N , of the unfolded syntactic tree for α . Of course, such a node would not be a primary leaf. In any case, the syntactic tree for τ is the subtree of the unfolded syntactic tree for α defined by N . See Figure 11.

Bound and Free Occurrences of Variable Symbols in Sentences

30° Let ζ be any variable symbol in \mathcal{V} and let α be any sentence in \mathcal{L}_Π . Let us assume that ζ occurs in α and let us focus attention upon one particular occurrence. Let λ be the atomic subsentence of α such that ζ occurs in λ . Let L be the particular secondary leaf in the basic syntactic tree for α occupied by λ and let R be the root of the tree (occupied by α itself). Of course, there is a unique path:

$$R = N_0, N_1, N_2, \dots, N_\ell = L$$

in the tree joining R to L . Let:

$$\alpha = \gamma_0, \gamma_1, \gamma_2, \dots, \gamma_\ell = \lambda$$

be the subsentences of α which occupy the corresponding nodes in the path. We say that the particular occurrence of ζ in α is *bound* iff there is at least one index q ($0 \leq q < \ell$) such that:

$$\gamma_q = ((\forall \zeta) \gamma_{q+1})$$

Clearly, each occurrence of ζ in α corresponds to the appearance of ζ as a label on one of the secondary leaves in the unfolded syntactic tree for α . Informally, we may say that a particular occurrence of ζ in α is bound iff, in course of descending through the unfolded syntactic tree for α from the root R on which α lies to the particular secondary leaf \hat{L} on which ζ lies, we encounter, at least once, an application of the universal quantifier $(\forall \zeta)$.

31° In turn, we say that the particular occurrence of ζ in α is *free* iff it is not bound.

32° In Figure 11, we find that one of the occurrences of η is bound while the other two are free. The lone occurrences of ζ and θ are free.

Closed/Open

33° Let α be any sentence in \mathcal{L}_Π . Let \mathcal{V}_α be the subset of \mathcal{V} consisting of all variable symbols ζ such that ζ occurs at least once freely in α . Of course, \mathcal{V}_α is finite. For the case in which $\mathcal{V}_\alpha = \emptyset$, we refer to α as a *closed* sentence. For the case in which $\mathcal{V}_\alpha \neq \emptyset$, we refer to α as an *open* sentence.

34° Let k be any nonnegative integer. We denote by \mathcal{L}_Π^k the subset of \mathcal{L}_Π consisting of all sentences α in \mathcal{L}_Π such that:

$$|\mathcal{V}_\alpha| = k$$

which is to say that \mathcal{V}_α contains precisely k variable symbols. Of course, \mathcal{L}_Π^0 consists of all sentences α in \mathcal{L}_Π such that α is closed.

35° Obviously:

$$\mathcal{L}_\Pi = \bigcup_{k=0}^{\infty} \mathcal{L}_\Pi^k$$

36° Let α be any sentence in \mathcal{L}_Π . Let $k = |\mathcal{V}_\alpha|$. Let us make a list, in natural order, of the variable symbols in \mathcal{V}_α :

$$x_{j_1}, x_{j_2}, \dots, x_{j_k} \quad (j_1 < j_2 < \dots < j_k)$$

In turn, let us introduce the sentence:

$$\forall\alpha = (\forall x_{j_1})(\forall x_{j_2}) \cdots (\forall x_{j_k})\alpha$$

in \mathcal{L}_Π . We refer to $\forall\alpha$ as the *closure* of α . Obviously, $\forall\alpha$ is a closed sentence. Of course, if α itself is a closed sentence then $k = 0$, the foregoing list is empty, and $\forall\alpha = \alpha$.

37° Let \mathcal{H} be any subset of \mathcal{L}_Π . Let us introduce the subset:

$$\forall\mathcal{H}$$

of \mathcal{L}_Π consisting of all sentences of the form $\forall\alpha$, where α is any sentence in \mathcal{H} . We refer to $\forall\mathcal{H}$ as the *closure* of \mathcal{H} .

38° It may happen that, for each sentence α in \mathcal{H} , α is closed, so that $\forall\mathcal{H} = \mathcal{H}$. In such a case, we say that \mathcal{H} is *closed*.

Substitution of Terms for Variable Symbols in Terms

39° Let ζ be any variable symbol in \mathcal{V} and let τ and v be any terms in \mathcal{T}_Π . Let us proceed to substitute τ for each of the occurrences of ζ in v . We denote the resulting term in \mathcal{T}_Π by $v(\tau|\zeta)$.

40° Let ℓ be the number of occurrences of ζ in v and let L_1, L_2, \dots , and L_ℓ be the corresponding leaves of the syntactic tree for v on which ζ lies. We form the syntactic tree for $v(\tau|\zeta)$ by grafting copies of the syntactic tree for τ to each of the leaves L_j ($1 \leq j \leq \ell$), then modifying the labels in accord with the inductive instruction described in article 14°.

41° Of course, there may be no occurrences of ζ in v . In such a case, we interpret $v(\tau|\zeta)$ to be v .

Substitution of Terms for Variable Symbols in Sentences

42° Let ζ be any variable symbol in \mathcal{V} , let τ be any term in \mathcal{T}_Π , and let α be any sentence in \mathcal{L}_Π . Let us proceed to substitute τ for each of the free occurrences of ζ in α . We denote the resulting sentence in \mathcal{L}_Π by $\alpha(\tau|\zeta)$.

43° Let ℓ be the number of free occurrences of ζ in α and let L_1, L_2, \dots , and L_ℓ be the corresponding leaves of the unfolded syntactic tree for α on which ζ lies. We form the unfolded syntactic tree for $\alpha(\tau|\zeta)$ by grafting copies of the syntactic tree for τ to each of the leaves L_j ($1 \leq j \leq \ell$), then modifying the labels in accord with the inductive instructions described in article 20°.

44° Of course, there may be no free occurrences of ζ in α . In such a case, we interpret $\alpha(\tau|\zeta)$ to be α .

45° For illustration, we return to the sentence depicted in Figure 11. Let us denote it by α . In turn, let us identify the term τ with $(\psi\zeta\eta)$, where ψ is a function symbol in \mathcal{F} . Obviously, the variable symbols ζ , η , and θ occur in α . Let us substitute τ for ζ in α and let us form the unfolded syntactic tree for $\alpha(\tau|\zeta)$:

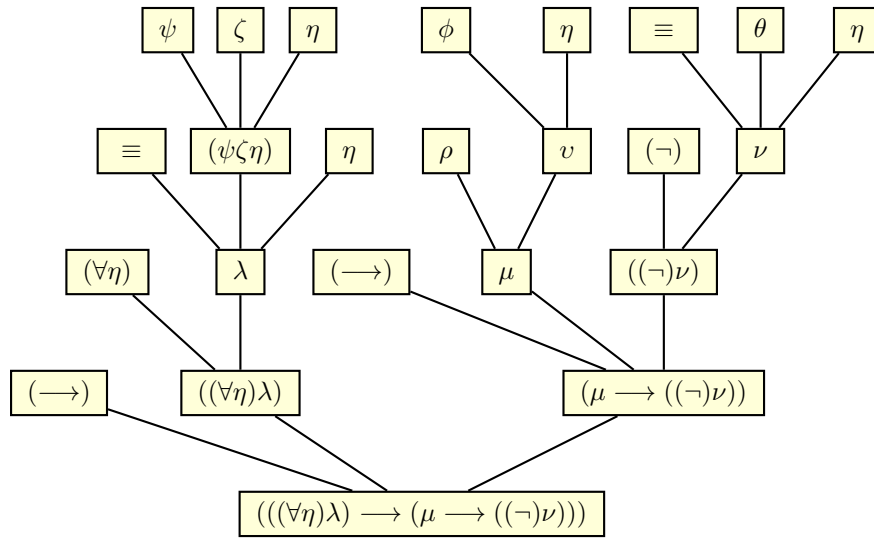


Figure 12: Substitution

Freedom

46° Let ζ be any variable symbol in \mathcal{V} , let τ be any term in \mathcal{T}_Π , and let α be any sentence in \mathcal{L}_Π . We say that τ is *free* for ζ in α iff, for any variable symbol η in \mathcal{V} , if η occurs in τ then the occurrences, if any, of η in $\alpha(\tau|\zeta)$ introduced by substitution of τ (for all the free occurrences, if any, of ζ in α) are free in $\alpha(\tau|\zeta)$.

47° In Figure 12, we find that τ is not free for ζ in α , because the occurrence of η in $\alpha(\tau|\zeta)$ (introduced by substitution of τ for the free occurrence of ζ in α) is bound. For contrast, one should note that τ is in fact free for both η and θ in α .

48• In context of an appropriate predicate language, consider the sentence:

$$\alpha = (((\forall\theta)((\forall\zeta)(\rho\eta\zeta))) \longrightarrow ((\rho\zeta\theta) \longrightarrow ((\forall\eta)(\sigma\eta))))$$

Draw the unfolded syntactic tree for α . Let τ be the term:

$$\tau = (\phi\zeta\theta)$$

Draw the unfolded syntactic tree for $\alpha(\tau|\zeta)$. Is τ free for ζ in α ? Is τ free for η in α ? Is τ free for θ in α ?

2.2 SYNTACTIC IMPLICATION

Axioms

01° Let \mathcal{A}_Π be the subset of \mathcal{L}_Π composed of all sentences of any one of the following ten forms, called Axiom Schemes:

- (A₁) $(\alpha \longrightarrow (\beta \longrightarrow \alpha))$
- (A₂) $((\alpha \longrightarrow (\beta \longrightarrow \gamma)) \longrightarrow ((\alpha \longrightarrow \beta) \longrightarrow (\alpha \longrightarrow \gamma)))$
- (A₃) $((((\neg)\beta) \longrightarrow ((\neg)\alpha)) \longrightarrow (\alpha \longrightarrow \beta))$
- (A₄) $((\forall\zeta)\alpha \longrightarrow \alpha(\tau|\zeta))$
- (A₅) $((\forall\zeta)(\alpha \longrightarrow \beta) \longrightarrow (\alpha \longrightarrow ((\forall\zeta)\beta)))$
- (E₁) $(\sigma \equiv \sigma)$
- (E₂) $((\sigma \equiv \tau) \longrightarrow (\tau \equiv \sigma))$
- (E₃) $((\sigma \equiv \tau) \longrightarrow ((\tau \equiv v) \longrightarrow (\sigma \equiv v)))$
- (E₄) $((\sigma_1 \equiv \tau_1) \longrightarrow ((\sigma_2 \equiv \tau_2) \longrightarrow \cdots$
 $((\sigma_k \equiv \tau_k) \longrightarrow ((\phi\sigma_1\sigma_2 \cdots \sigma_k) \equiv (\phi\tau_1\tau_2 \cdots \tau_k)))) \cdots))$
- (E₅) $((\sigma_1 \equiv \tau_1) \longrightarrow ((\sigma_2 \equiv \tau_2) \longrightarrow \cdots$
 $((\sigma_\ell \equiv \tau_\ell) \longrightarrow ((\rho\sigma_1\sigma_2 \cdots \sigma_\ell) \longrightarrow (\rho\tau_1\tau_2 \cdots \tau_\ell)))) \cdots))$

where α , β and γ are any sentences in \mathcal{L}_Π , where ζ is any variable symbol in \mathcal{V} , where ϕ is any function symbol in \mathcal{F} , where ρ is any predicate symbol in \mathcal{P} , and where the various σ 's, τ 's, and v 's are any terms in \mathcal{T} . Of course, $k = v(\phi)$ and $\ell = v(\rho)$. In case (A₄), we require that τ be free for ζ in α . In case (A₅), we require that each occurrence (if any) of ζ in α be bound.

02° Supplemented by appropriate rules of deduction, the foregoing set \mathcal{A}_Π of axioms yields cogent proofs of the Deduction Principle and the Interpretation Theorem.

Predicate Logics: $\Pi \implies \Lambda_\Pi = (\mathcal{L}_\Pi, \mathcal{A}_\Pi)$

03° Let Π be any preamble. We refer to the ordered pair:

$$\Lambda_\Pi = (\mathcal{L}_\Pi, \mathcal{A}_\Pi)$$

as the *predicate logic* defined by Π . However, to complete our description of it, we must specify the *rules of deduction* and we must describe the concept of *syntactic implication*.

Deductive Trees

04° Let \mathcal{H} be any subset of \mathcal{L}_Π . We refer to the sentences in \mathcal{H} as *hypotheses*. Let $\mathcal{G} = (\mathcal{N}, \mathcal{B})$ be any tree. We say that \mathcal{G} is a *deductive tree* from \mathcal{H} iff it meets the following conditions:

- (D₁) \mathcal{G} is active and labeled
- (D₂) for each node N in \mathcal{N} , $v(N)$ equals 0, 2, or 3
- (D₃) if N is a secondary leaf then the label α which occupies N is an axiom in \mathcal{A}_Π or an hypothesis in \mathcal{H}
- (D₄) if $v(N) = 2$ then the labels which occupy the immediate descendants of N are, in order:

$$(\forall\zeta), \beta$$

and the label which occupies N is the formula $((\forall\zeta)\beta)$ in \mathcal{L}_Π , where ζ is a variable symbol in \mathcal{V} and β is a sentence in \mathcal{L}_Π

- (D₅) if $v(N) = 3$ then the labels which occupy the immediate descendants of N are, in order:

$$(\longrightarrow), \gamma_1, (\gamma_1 \longrightarrow \gamma_2)$$

and the label which occupies N is γ_2 , where γ_1 and γ_2 are sentences in \mathcal{L}_Π

Let R be the root in \mathcal{N} and let δ be the sentence in \mathcal{L}_Π which occupies R . We refer to δ as the *consequence* of \mathcal{G} .

05° Let $\tilde{\mathcal{H}}$ be the subset of \mathcal{H} consisting of all sentences α in $\mathcal{H} \setminus \mathcal{A}_\Pi$ which occupy at least one of the secondary leaves in \mathcal{G} . We refer to the sentences in $\tilde{\mathcal{H}}$ as the *material hypotheses* for \mathcal{G} .

06° Under condition (D_4) , we say that the sentence $((\forall\zeta)\beta)$ which occupies the node N follows by Generalization from the sentence β which occupies the second of the immediate descendants of N :

$$\beta \implies ((\forall\zeta)\beta)$$

in particular, by generalization over the variable symbol ζ . Briefly, we say that \mathcal{G} *justifies* $((\forall\zeta)\beta)$ by Generalization (over ζ).

07° Under condition (D_5) , we say that the sentence γ_2 which occupies the node N follows by Modus Ponens from the sentences γ_1 and $(\gamma_1 \longrightarrow \gamma_2)$ which occupy the second and third of the immediate descendants of N :

$$\gamma_1, (\gamma_1 \longrightarrow \gamma_2) \implies \gamma_2$$

Briefly, we say that \mathcal{G} *justifies* γ_2 by Modus Ponens.

08° For the various predicate logics Λ_Π , we refer to **Modus Ponens** and to **Generalization** as the **Rules of Deduction**.

09° Let ζ be a variable symbol in \mathcal{V} and let α and β be sentences in \mathcal{L}_Π . Let \mathcal{H} be the set of hypotheses, defined as follows:

$$\mathcal{H} = \{\alpha, ((\forall\zeta)(\alpha \longrightarrow \beta))\}$$

We offer the following simple example of a deductive tree from \mathcal{H} :

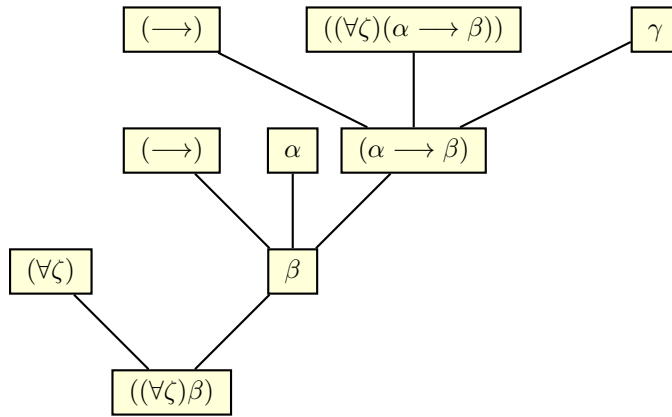


Figure 13: Deductive Tree

where:

$$\gamma = (((\forall\zeta)(\alpha \longrightarrow \beta)) \longrightarrow (\alpha \longrightarrow \beta))$$

Clearly, the sentence γ is an instance of Axiom Scheme (A_4) . Of course, the consequence of the tree is $((\forall\zeta)\beta)$.

10° Let $\mathcal{G} = (\mathcal{N}, \mathcal{B})$ be a deductive tree from \mathcal{H} . Let N be any node in \mathcal{N} , other than a primary leaf and let $\mathcal{G}_o = (\mathcal{N}_o, \mathcal{B}_o)$ be the subtree of \mathcal{G} defined by N , with labels intact. Obviously, \mathcal{G}_o is a deductive tree from \mathcal{H} and the consequence of \mathcal{G}_o is the sentence in \mathcal{L}_Π which occupies N .

Proper Deductive Trees

11° The practice of justification by Generalization requires restraint. To preserve the Deduction Principle and, in due course, the Interpretation Theorem, we must forbid generalization over a variable which occurs freely in one of the material hypotheses. Let us be more precise.

12° Let $\mathcal{G} = (\mathcal{N}, \mathcal{B})$ be any deductive tree from \mathcal{H} . Let N be any node in \mathcal{N} for which $v(N) = 2$. Of course, there must be a variable symbol ζ in \mathcal{V} and a sentence β in \mathcal{L}_Π such that $(\forall\zeta)$ and β are the labels, in order, on the immediate descendants of N and such that $((\forall\zeta)\beta)$ is the label on N . Let \mathcal{G}_o be the subtree of \mathcal{G} defined by N . As noted earlier, \mathcal{G}_o is a deductive tree from \mathcal{H} and $((\forall\zeta)\beta)$ is its consequence. We say that N is an *improper* node for \mathcal{G} iff ζ occurs at least once freely in at least one of the material hypotheses for \mathcal{G}_o . Such a node marks an improper application of Generalization.

13° We say that \mathcal{G} is *improper* iff there is at least one improper node for \mathcal{G} . Naturally, we say that \mathcal{G} is *proper* iff it is not improper.

14• Review the deductive tree (from \mathcal{H}) displayed in Figure 13. Let each occurrence (if any) of ζ in α be bound. Verify that the tree is proper. Using Axiom Scheme (A_5) , design a simpler proper deductive tree from \mathcal{H} having consequence $((\forall\zeta)\beta)$.

Syntactic Implication

15° Let \mathcal{H} be any subset of \mathcal{L}_Π . We refer to the sentences in \mathcal{H} as *hypotheses*. Let δ be any sentence in \mathcal{L}_Π . We say that \mathcal{H} *syntactically implies* δ iff there is a proper deductive tree \mathcal{G} from \mathcal{H} for which δ is the consequence. To express this relation, we write:

$$\mathcal{H} \Vdash \delta$$

We say that \mathcal{G} is a proper deductive tree for δ from \mathcal{H} .

16° Let \mathcal{H}_1 and \mathcal{H}_2 be any subsets of \mathcal{L}_Π . We say that \mathcal{H}_1 *syntactically implies* \mathcal{H}_2 iff, for each sentence δ in \mathcal{H}_2 , $\mathcal{H}_1 \Vdash \delta$. To express this relation, we write:

$$\mathcal{H}_1 \Vdash \mathcal{H}_2$$

In practice, it may happen that one or both of the sets \mathcal{H}_1 and \mathcal{H}_2 are finite:

$$\mathcal{H}_1 = \{\beta_1, \beta_2, \dots, \beta_k\} \text{ or/and } \mathcal{H}_2 = \{\gamma_1, \gamma_2, \dots, \gamma_\ell\}$$

In such cases, we write not $\mathcal{H}_1 \Vdash \mathcal{H}_2$ but:

$$\begin{aligned} \beta_1, \beta_2, \dots, \beta_k &\Vdash \mathcal{H}_2 \\ \mathcal{H}_1 &\Vdash \gamma_1, \gamma_2, \dots, \gamma_k \\ \beta_1, \beta_2, \dots, \beta_k &\Vdash \gamma_1, \gamma_2, \dots, \gamma_k \end{aligned}$$

17° Obviously, for any subsets \mathcal{H}_1 and \mathcal{H}_2 of \mathcal{L}_Π and for any sentence δ in \mathcal{L}_Π :

$$\mathcal{H}_1 \subseteq \mathcal{H}_2, \mathcal{H}_1 \Vdash \delta \implies \mathcal{H}_2 \Vdash \delta$$

Modus Ponens

18° Obviously, for any subset \mathcal{H} of \mathcal{L}_Π and for any sentences γ and δ in \mathcal{L}_Π :

$$(\mathbf{MP}) \quad \mathcal{H} \Vdash \gamma, \mathcal{H} \Vdash (\gamma \longrightarrow \delta) \implies \mathcal{H} \Vdash \delta$$

Generalization

19° Obviously, for any subset \mathcal{H} of \mathcal{L}_Π , for any variable symbol ζ in \mathcal{V} , and for any sentence β in \mathcal{L}_Π , if ζ does not occur freely in any one of the sentences in \mathcal{H} then:

$$(\mathbf{GN}) \quad \mathcal{H} \Vdash \beta \implies \mathcal{H} \Vdash ((\forall \zeta)\beta)$$

In practice, we will introduce applications of Modus Ponens and Generalization simply by mentioning the abbreviations **(MP)** and **(GN)**.

Syntactic Theories

20° Let \mathcal{H} be any subset of \mathcal{L}_Π . We refer to the sentences in \mathcal{H} as *hypotheses*. Let:

$$\Theta_\Pi(\mathcal{H})$$

be the subset of \mathcal{L}_Π consisting of all sentences δ such that $\mathcal{H} \Vdash \delta$. We refer to the sentences in $\Theta_\Pi(\mathcal{H})$ as *syntactic theorems* and to $\Theta_\Pi(\mathcal{H})$ itself as the *syntactic theory* of \mathcal{H} . Of course:

$$\mathcal{H} \Vdash \Theta_\Pi(\mathcal{H})$$

A Trivial Syntactic Implication

21° Let us illustrate syntactic implication by showing that, for any sentence α in \mathcal{L}_Π :

$$(SI_1) \quad \emptyset \Vdash (\alpha \longrightarrow \alpha)$$

To prove (SI_1) , we introduce the following proper deductive tree $\hat{\mathcal{G}}$ from \emptyset :

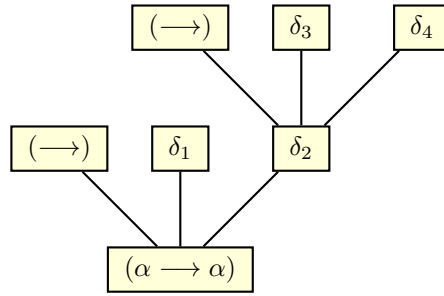


Figure 14: (SI_1)

where:

$$\begin{aligned} \delta_1 &= (\alpha \longrightarrow (\alpha \longrightarrow \alpha)) \\ \delta_2 &= ((\alpha \longrightarrow (\alpha \longrightarrow \alpha)) \longrightarrow (\alpha \longrightarrow \alpha)) \\ \delta_3 &= (\alpha \longrightarrow ((\alpha \longrightarrow \alpha) \longrightarrow \alpha)) \\ \delta_4 &= ((\alpha \longrightarrow ((\alpha \longrightarrow \alpha) \longrightarrow \alpha)) \longrightarrow ((\alpha \longrightarrow (\alpha \longrightarrow \alpha)) \longrightarrow (\alpha \longrightarrow \alpha))) \end{aligned}$$

Clearly, δ_1 and δ_3 are instances of the Axiom Scheme (A_1) and δ_4 is an instance of the Axiom Scheme (A_2) .

22° The foregoing peculiar argument is a cost of the design of the set \mathcal{A}_Π of axioms. In general, the design yields seamless proofs, notably, of the Deduction Principle and the Interpretation Theorem. However, for (SI_1) , the proof is baroque.

2.3 THE DEDUCTION PRINCIPLE

The Deduction Principle

01° Let \mathcal{H} be any subset of \mathcal{L}_Π and let α and β be any sentences in \mathcal{L}_Π . We contend that:

$$(\mathbf{DP}) \quad \mathcal{H} \Vdash \beta \implies \mathcal{H} \setminus \{\alpha\} \Vdash (\alpha \longrightarrow \beta)$$

We refer to this fundamental fact as the Deduction Principle. It plays a critical role in the design of deductions.

02° To prove **(DP)**, we argue as follows. Let $\mathcal{G} = (\mathcal{N}, \mathcal{B})$ be a proper deductive tree for β from \mathcal{H} . Let $\tilde{\mathcal{H}}$ be the subset of \mathcal{H} consisting of all material hypotheses for \mathcal{G} . We claim that there is a proper deductive tree $\bar{\mathcal{G}} = (\bar{\mathcal{N}}, \bar{\mathcal{B}})$ for $(\alpha \longrightarrow \beta)$ from $\tilde{\mathcal{H}} \setminus \{\alpha\}$. Having proved the claim, we will have proved **(DP)**.

03° We begin by introducing the following labeled tree, as a utility:

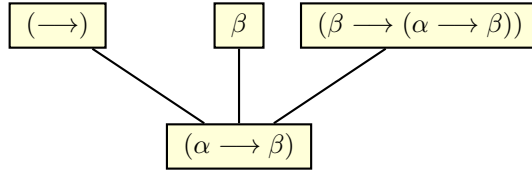


Figure 15: Utility

Of course, the sentence:

$$(\beta \longrightarrow (\alpha \longrightarrow \beta))$$

is an instance of the Axiom Scheme (A_1) . Let us assume that $\alpha \notin \tilde{\mathcal{H}}$. Under this assumption, we define $\bar{\mathcal{G}}$ to be the tree obtained by grafting the tree \mathcal{G} to the leaf in the displayed tree (Figure 15) which carries the label β . Clearly, $\bar{\mathcal{G}}$ is a proper deductive tree for $(\alpha \longrightarrow \beta)$ from $\tilde{\mathcal{H}} \setminus \{\alpha\} = \tilde{\mathcal{H}}$.

04° Hereafter, we assume that $\alpha \in \tilde{\mathcal{H}}$. We proceed to argue by induction on the number of nodes in \mathcal{G} . Let R be the root of \mathcal{G} .

05° Let us consider the case in which $v(R) = 0$. By definition, there is just one node in \mathcal{G} (namely, R) and $\beta \in \mathcal{A}_\Pi \cup \tilde{\mathcal{H}}$. If $\beta = \alpha$ then we define $\bar{\mathcal{G}}$ to be the tree displayed in Figure 14. Clearly, $\bar{\mathcal{G}}$ is a proper deductive tree for $(\alpha \longrightarrow \beta)$ from \emptyset . If $\beta \neq \alpha$ then we define $\bar{\mathcal{G}}$ to be the tree displayed in Figure 15. Clearly, $\bar{\mathcal{G}}$ is a proper deductive tree for $(\alpha \longrightarrow \beta)$ from $\tilde{\mathcal{H}} \setminus \{\alpha\}$.

06° Now let us consider the case in which $v(R) = 2$. Let P and N be the immediate descendants of R , in order. Let $(\forall\zeta)$ and γ be the labels which occupy P and N , respectively, where ζ is a variable symbol in \mathcal{V} and where γ is a sentence in \mathcal{L}_{Π} . Of course, $\beta = ((\forall\zeta)\gamma)$. Let us emphasize that ζ does not occur freely in any one of the sentences in $\tilde{\mathcal{H}}$. In particular, ζ does not occur freely in α . Let \mathcal{G}_o be the subtree of \mathcal{G} defined by the node N . Obviously, \mathcal{G}_o is a proper deductive tree for γ from $\tilde{\mathcal{H}}$ and the number of nodes in \mathcal{G}_o is smaller than the number of nodes in \mathcal{G} . By induction, we may introduce a proper deductive tree $\tilde{\mathcal{G}}_o$ for $(\alpha \rightarrow \gamma)$ from $\tilde{\mathcal{H}} \setminus \{\alpha\}$. Let us introduce the following (labeled) tree, as a utility:

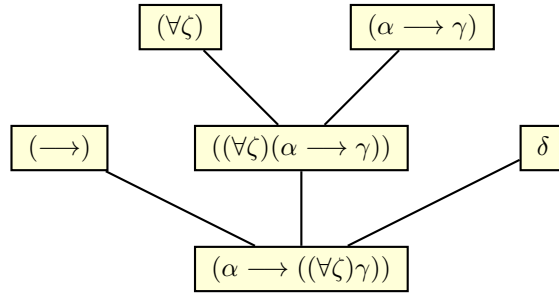


Figure 16: Utility

where:

$$\delta \equiv (((\forall\zeta)(\alpha \rightarrow \gamma)) \rightarrow (\alpha \rightarrow ((\forall\zeta)\gamma)))$$

Clearly, δ is an instance of the Axiom Scheme (A_5) . At this point, we define $\bar{\mathcal{G}}$ to be the tree obtained by grafting the tree $\tilde{\mathcal{G}}_o$ to the leaf in the displayed tree (Figure 16) which carries the label $(\alpha \rightarrow \gamma)$. Clearly, $\bar{\mathcal{G}}$ is a proper deductive tree for $(\alpha \rightarrow \beta)$ from $\tilde{\mathcal{H}} \setminus \{\alpha\}$.

07° Finally, let us consider the case in which $v(R) = 3$. Let P , N_1 , and N_2 be the immediate descendants of R , in order. Let (\rightarrow) , γ_1 , and γ_2 be the labels which occupy P , N_1 , and N_2 , respectively, where γ_1 and γ_2 are sentences in \mathcal{L}_{Π} . Of course, $\gamma_2 = (\gamma_1 \rightarrow \beta)$. Let \mathcal{G}_1 and \mathcal{G}_2 be the subtrees of \mathcal{G} defined by the nodes N_1 and N_2 , respectively. Obviously, \mathcal{G}_1 and \mathcal{G}_2 are proper deductive trees for γ_1 and γ_2 , respectively, from $\tilde{\mathcal{H}}$ and the numbers of nodes in \mathcal{G}_1 and \mathcal{G}_2 are smaller than the number of nodes in \mathcal{G} . By induction, we may introduce proper deductive trees $\tilde{\mathcal{G}}_1$ and $\tilde{\mathcal{G}}_2$ for $(\alpha \rightarrow \gamma_1)$ and $(\alpha \rightarrow \gamma_2)$, respectively, from $\tilde{\mathcal{H}} \setminus \{\alpha\}$. Let us introduce the following (labeled) tree, as a utility:

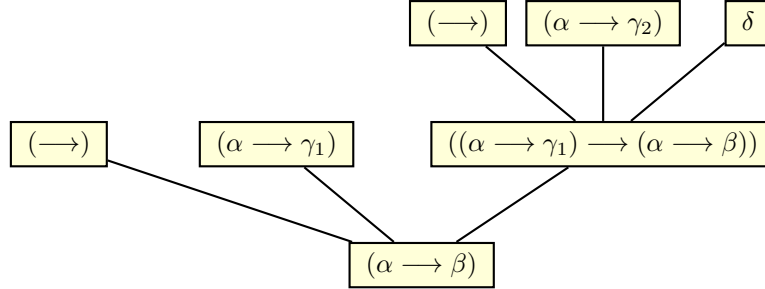


Figure 17: Utility

where:

$$\delta \equiv ((\alpha \rightarrow \gamma_2) \rightarrow ((\alpha \rightarrow \gamma_1) \rightarrow (\alpha \rightarrow \beta)))$$

Since $\gamma_2 \equiv (\gamma_1 \rightarrow \beta)$, δ is an instance of the Axiom Scheme (A_2) . At this point, we define $\bar{\mathcal{G}}$ to be the tree obtained by grafting the trees $\bar{\mathcal{G}}_1$ and $\bar{\mathcal{G}}_2$ to the leaves in the displayed tree (Figure 17) which carry the labels $(\alpha \rightarrow \gamma_1)$ and $(\alpha \rightarrow \gamma_2)$, respectively. Clearly, $\bar{\mathcal{G}}$ is a proper deductive tree for $(\alpha \rightarrow \beta)$ from $\tilde{\mathcal{H}} \setminus \{\alpha\}$.[‡]

The Transitivity Principle

08° For any subsets \mathcal{H}_1 , \mathcal{H}_2 , and \mathcal{H}_3 of \mathcal{L}_{Π} , we contend that if $\mathcal{H}_1 \Vdash \mathcal{H}_2$ and if $\mathcal{H}_2 \Vdash \mathcal{H}_3$ then $\mathcal{H}_1 \Vdash \mathcal{H}_3$:

$$(\mathbf{TP}) \quad \mathcal{H}_1 \Vdash \mathcal{H}_2, \mathcal{H}_2 \Vdash \mathcal{H}_3 \implies \mathcal{H}_1 \Vdash \mathcal{H}_3$$

We refer to this fundamental fact as the Transitivity Principle.

09° For the proof of **(TP)**, we may assume that \mathcal{H}_3 is a singleton. Let γ be the lone sentence in \mathcal{H}_3 , so that $\mathcal{H}_3 = \{\gamma\}$. In turn, we may assume that \mathcal{H}_2 is finite. We argue by induction on the number of sentences in \mathcal{H}_2 . The arguments for the initial step and for the inductive step take the same form. Let β be any sentence in \mathcal{H}_2 . We apply **(DP)** to obtain $\mathcal{H}_2 \setminus \{\beta\} \Vdash (\beta \rightarrow \gamma)$. By induction, we infer that $\mathcal{H}_1 \Vdash (\beta \rightarrow \gamma)$. Of course, $\mathcal{H}_1 \Vdash \beta$. By **(MP)**, we infer that $\mathcal{H}_1 \Vdash \gamma$.[‡]

(DP) and **(TP)** in Action

10• Let ζ be a variable symbol in \mathcal{V} and let α and β be sentences in \mathcal{L}_{Π} . Prove that:

$$\emptyset \Vdash (((\forall \zeta)(\alpha \rightarrow \beta)) \rightarrow (((\forall \zeta)\alpha) \rightarrow ((\forall \zeta)\beta)))$$

To that end, explain the following syntactic implications:

$$\begin{aligned}
& ((\forall\zeta)\alpha), ((\forall\zeta)(\alpha \longrightarrow \beta)) \Vdash \alpha, (\alpha \longrightarrow \beta) \\
& \quad \alpha, (\alpha \longrightarrow \beta) \Vdash \beta \\
& ((\forall\zeta)\alpha), ((\forall\zeta)(\alpha \longrightarrow \beta)) \Vdash \beta \\
& ((\forall\zeta)\alpha), ((\forall\zeta)(\alpha \longrightarrow \beta)) \Vdash ((\forall\zeta)\beta) \\
& ((\forall\zeta)(\alpha \longrightarrow \beta)) \Vdash (((\forall\zeta)\alpha) \longrightarrow ((\forall\zeta)\beta)) \\
& \quad \emptyset \Vdash (((\forall\zeta)(\alpha \longrightarrow \beta)) \longrightarrow (((\forall\zeta)\alpha) \longrightarrow ((\forall\zeta)\beta)))
\end{aligned}$$

Compare the sentences of the form just described with the various instances of Axiom Scheme (A_5). By symmetry of structure, one might prefer the former to the latter in the design of the scheme. However, the latter fit seamlessly into the proof of the Deduction Principle. See article 06°.

Several Syntactic Implications

11° Granted **(DP)** and **(TP)**, let us prove several syntactic implications, which serve as lemmas for subsequent arguments. For the statements and proofs of these implications, we adopt some of the notational substitutions described in article 1.12°:

$$\begin{aligned}
(SI_2) \quad & (\neg\alpha), \alpha \Vdash \beta \\
(SI_3) \quad & (\neg(\neg\alpha)) \Vdash \alpha \\
& \quad \alpha \Vdash (\neg(\neg\alpha)) \\
(SI_4) \quad & (\alpha \longrightarrow \beta) \Vdash ((\neg\beta) \longrightarrow (\neg\alpha)) \\
(SI_5) \quad & \beta \Vdash (\alpha \longrightarrow \beta) \\
& \quad (\neg\alpha) \Vdash (\alpha \longrightarrow \beta) \\
& \quad (\neg(\alpha \longrightarrow \beta)) \Vdash \alpha, (\neg\beta) \\
& \quad \alpha, (\neg\beta) \Vdash (\neg(\alpha \longrightarrow \beta)) \\
(SI_6) \quad & ((\neg\alpha) \longrightarrow \beta), (\alpha \longrightarrow \beta) \Vdash \beta
\end{aligned}$$

where α and β are any sentences in \mathcal{L}_Π . We will present the proofs as series' of implications, with pointers to justifications entered on the margin. As a utility, let $\delta = (\gamma \longrightarrow \gamma)$, where γ is any sentence in \mathcal{L}_Π .

For (SI_2) :

$$\begin{array}{ll}
 (\neg\alpha) \Vdash ((\neg\beta) \longrightarrow (\neg\alpha)) & (A_1), (\mathbf{MP}) \\
 ((\neg\beta) \longrightarrow (\neg\alpha)) \Vdash (\alpha \longrightarrow \beta) & (A_3), (\mathbf{MP}) \\
 (\neg\alpha) \Vdash (\alpha \longrightarrow \beta) & (\mathbf{TP}) \\
 (\neg\alpha), \alpha \Vdash \beta & (\mathbf{MP})
 \end{array}$$

For (SI_3) :

$$\begin{array}{ll}
 (\neg(\neg\alpha)), (\neg\alpha) \Vdash (\neg\delta) & (SI_2) \\
 (\neg(\neg\alpha)) \Vdash (\delta \longrightarrow \alpha) & (\mathbf{DP}), (A_3), (\mathbf{MP}) \\
 (\neg(\neg\alpha)) \Vdash \alpha & (SI_1), (\mathbf{MP}) \\
 (\neg(\neg(\neg\alpha))) \Vdash (\neg\alpha) & \\
 \alpha \Vdash (\neg(\neg\alpha)) & (\mathbf{DP}), (A_3), (\mathbf{MP})
 \end{array}$$

For (SI_4) :

$$\begin{array}{ll}
 (\neg(\neg\alpha)), (\alpha \longrightarrow \beta), (\neg\beta) \Vdash (\neg\delta) & (SI_3), (\mathbf{MP}), (SI_2) \\
 (\alpha \longrightarrow \beta), (\neg\beta) \Vdash (\delta \longrightarrow (\neg\alpha)) & (\mathbf{DP}), (A_3), (\mathbf{MP}) \\
 (\alpha \longrightarrow \beta), (\neg\beta) \Vdash (\neg\alpha) & (SI_1), (\mathbf{MP}) \\
 (\alpha \longrightarrow \beta) \Vdash ((\neg\beta) \longrightarrow (\neg\alpha)) & (\mathbf{DP})
 \end{array}$$

For (SI_5) :

$$\begin{array}{ll}
 \beta \Vdash (\alpha \longrightarrow \beta) & (A_1), (\mathbf{MP}) \\
 (\neg\alpha) \Vdash ((\neg\beta) \longrightarrow (\neg\alpha)) & (A_1), (\mathbf{MP}) \\
 (\neg\alpha) \Vdash (\alpha \longrightarrow \beta) & (A_3), (\mathbf{MP}) \\
 \emptyset \Vdash (\beta \longrightarrow (\alpha \longrightarrow \beta)) & (\mathbf{DP}) \\
 \emptyset \Vdash ((\neg\alpha) \longrightarrow (\alpha \longrightarrow \beta)) & (\mathbf{DP}) \\
 (\neg(\alpha \longrightarrow \beta)) \Vdash \alpha, (\neg\beta) & (SI_4), (SI_3), (\mathbf{MP}) \\
 \alpha, (\neg\beta), (\alpha \longrightarrow \beta) \Vdash (\neg\delta) & (\mathbf{MP}), (SI_2) \\
 \alpha, (\neg\beta) \Vdash ((\alpha \longrightarrow \beta) \longrightarrow (\neg\delta)) & (\mathbf{DP}) \\
 \alpha, (\neg\beta) \Vdash ((\neg(\neg\delta)) \longrightarrow (\neg(\alpha \longrightarrow \beta))) & (SI_4), (\mathbf{MP}) \\
 \alpha, (\neg\beta) \Vdash (\neg(\alpha \longrightarrow \beta)) & (SI_3), (SI_1), (\mathbf{MP})
 \end{array}$$

For (SI_6) :

$$\begin{array}{ll}
 (\alpha \longrightarrow \beta), ((\neg\alpha) \longrightarrow \beta), (\neg\beta) \Vdash (\neg(\neg\alpha)), (\neg\alpha) & (SI_4), (\mathbf{MP}) \\
 (\alpha \longrightarrow \beta), ((\neg\alpha) \longrightarrow \beta), (\neg\beta) \Vdash (\neg\delta) & (SI_2), (\mathbf{TP}) \\
 (\alpha \longrightarrow \beta), ((\neg\alpha) \longrightarrow \beta) \Vdash (\delta \longrightarrow \beta) & (\mathbf{DP}), (A_3), (\mathbf{MP}) \\
 (\alpha \longrightarrow \beta), ((\neg\alpha) \longrightarrow \beta) \Vdash \beta & (SI_1), (\mathbf{MP})
 \end{array}$$

12• Following the pointers, write detailed and formally precise explanations of the syntactic implications (SI_2) , (SI_3) , (SI_4) , (SI_5) , and (SI_6) .

13• Prove that:

$$(SI_7) \quad \alpha \Vdash \beta \implies (\forall\zeta)\alpha \Vdash (\forall\zeta)\beta$$

where α and β are any sentences in \mathcal{L}_Π and where ζ is any variable symbol in \mathcal{V} .

Abbreviations

14° Let Π be any preamble. Let α , β , and γ be any sentences in \mathcal{L}_Π and let ζ be any variable symbol in \mathcal{V} . Let us introduce the following abbreviations, which provide links to conventional expressions:

$$\begin{array}{lll}
 (\alpha \vee \beta) & \text{for} & (((\neg)\alpha) \longrightarrow \beta) \\
 (\alpha \wedge \beta) & \text{for} & ((\neg)(\alpha \longrightarrow ((\neg)\beta))) \\
 (\alpha \longleftrightarrow \beta) & \text{for} & ((\alpha \longrightarrow \beta) \wedge (\beta \longrightarrow \alpha)) \\
 ((\exists\zeta)\gamma) & \text{for} & ((\neg)((\forall\zeta)((\neg)\gamma)))
 \end{array}$$

15• Let α and β be any sentences in \mathcal{L}_Π . To animate the foregoing abbreviations, prove the following implications:

$$\begin{array}{l}
 \alpha \Vdash (\alpha \vee \beta) \\
 \beta \Vdash (\alpha \vee \beta) \\
 (\alpha \wedge \beta) \Vdash \alpha \\
 (\alpha \wedge \beta) \Vdash \beta \\
 \alpha, \beta \Vdash (\alpha \wedge \beta)
 \end{array}$$

Should one expect the (seemingly plausible) contention:

$$(\alpha \vee \beta) \Vdash \alpha \quad \text{or} \quad (\alpha \vee \beta) \Vdash \beta$$

to be true?

16• Let us say that α and β are *syntactically equivalent* iff::

$$\alpha \Vdash \beta \quad \text{and} \quad \beta \Vdash \alpha$$

We express this relation by writing $\alpha \approx \beta$. Show that:

$$\alpha \approx \beta \iff \emptyset \Vdash (\alpha \longleftrightarrow \beta)$$

Substitution and Syntactic Implication

17• Let α , β , and $\bar{\beta}$ be sentences in \mathcal{L}_Π . Let β be a subsentence of α and let $\bar{\alpha}$ be the sentence defined by substituting $\bar{\beta}$ for β in α . Prove that:

$$\beta \approx \bar{\beta} \implies \alpha \approx \bar{\alpha}$$

To that end, one should argue by induction on the length of the path in the basic syntactic tree for α which joins the root R , carrying the label α , to the particular node N , carrying the label β . The argument can be fabricated from the following implications:

$$\beta \approx \bar{\beta} \implies \begin{cases} \neg\beta \approx \neg\bar{\beta} \\ (\forall\zeta)\beta \approx (\forall\zeta)\bar{\beta} \\ (\beta \longrightarrow \delta) \approx (\bar{\beta} \longrightarrow \delta) \\ (\delta \longrightarrow \beta) \approx (\delta \longrightarrow \bar{\beta}) \end{cases}$$

where ζ is any variable symbol in \mathcal{V} and where δ is any sentence in \mathcal{L}_Π .

18• Let β and γ be sentences in \mathcal{L}_Π and let η and θ be variable symbols in \mathcal{V} . Prove that if:

$$\theta \notin \mathcal{V}_\beta, \theta \text{ free for } \eta \text{ in } \beta, \gamma = \beta(\theta|\eta)$$

then:

$$\eta \notin \mathcal{V}_\gamma, \eta \text{ free for } \theta \text{ in } \gamma, \beta = \gamma(\eta|\theta)$$

and:

$$(\forall\eta)\beta \approx (\forall\theta)\gamma$$

The Freedom Maneuver

19° Let α be a sentence in \mathcal{L}_Π , let τ be a term in \mathcal{T}_Π , and let ζ be a variable symbol in \mathcal{V} . It may happen that τ is not free for ζ in α . Nevertheless, by a procedure to be described in a moment, we can design a sentence $\bar{\alpha}$ in \mathcal{L}_Π such that:

- (F₁) τ is free for ζ in $\bar{\alpha}$
- (F₂) $\alpha \approx \bar{\alpha}$
- (F₃) the basic syntactic trees for α and $\bar{\alpha}$ have the same number of nodes

We refer to this procedure as the Freedom Maneuver (**FM**).

20° To design $\bar{\alpha}$, we proceed by induction on the number of nodes in the basic syntactic tree for α . Three of the cases are simple:

$$\begin{aligned} \alpha = \lambda &\implies \bar{\alpha} = \alpha \\ \alpha = (\neg\beta) &\implies \bar{\alpha} = (\neg\bar{\beta}) \\ \alpha = (\gamma \longrightarrow \delta) &\implies \bar{\alpha} = (\bar{\gamma} \longrightarrow \bar{\delta}) \end{aligned}$$

where λ is any atomic sentence and where β , γ , and δ are any sentences in \mathcal{L}_Π . Let us consider the fourth, more complex case:

$$\alpha = (\forall\eta)\beta$$

where η is any variable symbol in \mathcal{V} and where β is any sentence in \mathcal{L}_Π .

21° It may happen that $\eta = \zeta$. In such a case, τ is free for ζ in α by default, so we can take $\bar{\alpha}$ to be α .

22° Let us assume that $\eta \neq \zeta$. In this case, we might leap to define $\bar{\alpha}$ as follows:

$$\bar{\alpha} = (\forall\eta)\bar{\beta}$$

Obviously, $\bar{\alpha}$ (so defined) would meet (F₃) and, by (SI₇), it would meet (F₂) as well. However, it might not meet (F₁), because η might occur in τ . We provide a remedy by introducing a new variable symbol θ in \mathcal{V} such that θ does not occur in τ , θ does not occur freely in $\bar{\beta}$, and θ is free for η in $\bar{\beta}$. Now we define $\bar{\alpha}$ as follows:

$$\bar{\alpha} = (\forall\theta)\gamma \quad \text{where} \quad \gamma = \bar{\beta}(\theta|\eta)$$

Clearly, $\bar{\alpha}$ meets (F₁) and (F₃). By article 18°, $\bar{\alpha}$ meets (F₂) as well. \natural

2.4 CONSISTENCY

Inconsistency

01° Let \mathcal{H} be any subset of \mathcal{L}_Π . We say that \mathcal{H} is *inconsistent* iff there is a sentence α in \mathcal{L}_Π such that:

$$\mathcal{H} \Vdash (\neg\alpha), \alpha$$

One may say that \mathcal{H} syntactically implies a *contradiction*. We express this property of \mathcal{H} by writing:

$$Inc(\mathcal{H})$$

By (SI_2) and **(TP)**, if \mathcal{H} is inconsistent then, for any (!) sentence β in \mathcal{L}_Π :

$$\mathcal{H} \Vdash \beta$$

Reductio ad Absurdum

02° For any subset \mathcal{H} of \mathcal{L}_Π and for any sentence γ in \mathcal{L}_Π , we contend that:

$$\begin{aligned} \text{(RA)} \quad Inc(\mathcal{H} \cup \{(\neg\gamma)\}) &\implies \mathcal{H} \Vdash \gamma \\ Inc(\mathcal{H} \cup \{\gamma\}) &\implies \mathcal{H} \Vdash (\neg\gamma) \end{aligned}$$

These simple but subtle facts set the base for arguments by contradiction in Mathematics.

03° To prove these contentions, we argue as follows. First, let $\mathcal{H} \cup \{(\neg\gamma)\}$ be inconsistent. By article 01°, $\mathcal{H} \cup \{(\neg\gamma)\} \Vdash \gamma$. By **(DP)**, $\mathcal{H} \Vdash ((\neg\gamma) \longrightarrow \gamma)$. By (SI_1) , $\emptyset \Vdash (\gamma \longrightarrow \gamma)$. By (SI_6) and **(TP)**, $\mathcal{H} \Vdash \gamma$. Second, let $\mathcal{H} \cup \{\gamma\}$ be inconsistent. By (SI_3) , $\mathcal{H} \cup \{(\neg(\neg\gamma))\}$ is inconsistent, so that, by the first contention, $\mathcal{H} \Vdash (\neg(\neg(\neg\gamma)))$. By (SI_3) and **(TP)**, $\mathcal{H} \Vdash (\neg\gamma)$. \spadesuit

Consistency

04° Let \mathcal{H} be any subset of \mathcal{L}_Π . We say that \mathcal{H} is *consistent* iff \mathcal{H} is not inconsistent. We express this property of \mathcal{H} by writing:

$$Con(\mathcal{H})$$

In the following chapter, we will prove that \mathcal{H} is consistent iff it admits an *interpretation* in the context of informal Set Theory under which every sentence in \mathcal{H} is *true*. This foundation stone of Mathematics is the substance of the Interpretation Theorem.

Sublogics

05° Let \mathcal{C}' and \mathcal{C}'' be any subsets of the set of all constant symbols for which $\mathcal{C}' \subseteq \mathcal{C}''$. Let \mathcal{F} and \mathcal{P} be any subsets of the sets of all function symbols and predicate symbols, respectively, where \equiv is a member of \mathcal{P} . Let:

$$\Pi' = (\mathcal{C}', \mathcal{F}, \mathcal{P}) \quad \text{and} \quad \Pi'' = (\mathcal{C}'', \mathcal{F}, \mathcal{P})$$

be the corresponding preambles and let:

$$\Lambda' = (\mathcal{L}', \mathcal{A}') \quad \text{and} \quad \Lambda'' = (\mathcal{L}'', \mathcal{A}'')$$

be the predicate logics which they define. Obviously, $\mathcal{L}' \subseteq \mathcal{L}''$ and $\mathcal{A}' \subseteq \mathcal{A}''$. Now Λ' is a *sublogic* of Λ'' , in a sense which requires no explanation.

06° For precise expression, let us introduce the symbols:

$$\Vdash', \quad \Vdash''$$

to distinguish the relations of syntactic implication in Λ' and Λ'' .

The Consistency Principle

07° Let \mathcal{H} be any subset of \mathcal{L}' . Of course, \mathcal{H} is a subset of \mathcal{L}'' as well. We contend that if \mathcal{H} is consistent with respect to the logic Λ' then \mathcal{H} is also consistent with respect to the logic Λ'' . We refer to this fact as the Consistency Principle (**CP**).

08° We prepare for the proof of (**CP**) by describing an operation on trees. Let \mathcal{G}'' be a labeled tree for which the labels on the nodes of \mathcal{G}'' are sentences in \mathcal{L}'' . Let:

$$\chi_1, \chi_2, \dots, \chi_k$$

be a list (without repetitions) of the constant symbols in $\mathcal{C}'' \setminus \mathcal{C}'$ which occur in some one of the labels on the nodes of \mathcal{G}'' . Let:

$$\theta_1, \theta_2, \dots, \theta_k$$

be a list (without repetitions) of variable symbols in \mathcal{V} which do not occur in any one of the labels on the nodes of \mathcal{G}'' . Let N be any node in \mathcal{G}'' . Let α'' be the sentence in \mathcal{L}'' which serves as the label on N . For each index j ($1 \leq j \leq k$), let us substitute θ_j for each occurrence of χ_j in α'' , to obtain a new sentence α' . Of course, α' is a sentence in \mathcal{L}' . Let us replace the label α'' for N by the new label α' . In this way, we obtain a new labeled tree. Let us denote it by \mathcal{G}' .

09° By systematic review of the relevant definitions, one can easily show that if \mathcal{G}'' is a proper deductive tree from \mathcal{H} relative to Λ'' then \mathcal{G}' is a proper deductive tree from \mathcal{H} relative to Λ' . Obviously:

$$\mathcal{H} \Vdash'' \delta'' \implies \mathcal{H} \Vdash' \delta'$$

where δ' and δ'' are the consequences for \mathcal{G}' and \mathcal{G}'' , respectively.

10° Now the proof of **(CP)** is routine. We argue by contradiction. Let us assume that \mathcal{H} is consistent with respect to Λ' but let us suppose that \mathcal{H} is inconsistent with respect to Λ'' . By our supposition, we may introduce a sentence δ'' in \mathcal{L}'' such that:

$$\mathcal{H} \Vdash'' (\neg\delta''), \delta''$$

By the foregoing discussion:

$$\mathcal{H} \Vdash' (\neg\delta'), \delta'$$

contradicting our assumption. \natural

The Remote Constant Principle

11° In the context just described, let α be a sentence in \mathcal{L}' , let ζ be a variable symbol in \mathcal{V} , and let χ be a constant symbol in $\mathcal{C}'' \setminus \mathcal{C}'$. We contend that:

$$\mathcal{H} \Vdash'' \alpha(\chi|\zeta) \implies \mathcal{H} \Vdash' (\forall\zeta)\alpha$$

We refer to this implication as the Remote Constant Principle **(RC)**.

12° To prove **(RC)**, we follow the procedure in articles 08° and 09°. Let $\alpha'' = \alpha(\chi|\zeta)$. Let us assume that:

$$\mathcal{H} \Vdash'' \alpha''$$

Let \mathcal{G}'' be a proper deductive tree from \mathcal{H} relative to Λ'' , for which the consequence is α'' . Let \mathcal{G}' be the corresponding proper deductive tree from \mathcal{H} relative to Λ' , and let α' be its consequence. By design, $\alpha' = \alpha(\theta|\zeta)$, where θ is a variable symbol which does not occur in any one of the labels on the nodes of \mathcal{G}'' . Of course:

$$\mathcal{H} \Vdash' \alpha'$$

By article 18• in the foregoing section:

$$\mathcal{H} \Vdash' (\forall\zeta)\alpha$$

13° Let us pause to observe that **(FM)**, **(CP)**, and **(RC)** all play critical roles in the proof of the Interpretation Theorem.

2.5 EXAMPLES

Examples

01° In later chapters, we will develop the Predicate Logic for Arithmetic and the Predicate Logic for Set Theory. These logics set the context for the fundamental theorems of Tarski, Gödel, and Church. They require careful preparation. For now, we provide four simple instances of predicate logics, which will serve as illustrations for the Interpretation Theorem, soon to follow: Ordered Fields, Groups, Abstract Lines, and Boolean Rings.

Ordered Fields

02° Let Π_F be the preamble defined by the following specifications:

$$\mathcal{C}_F = \{0, 1\}, \quad \mathcal{F}_F = \{-, \iota, +, \times\}, \quad \mathcal{P}_F = \{\equiv, <\}$$

where $0 = (c|)$ and $1 = (c|)$, where $- = (|f|)$ and $\iota = (|f|)$, where $+$ and \times are the *addition* and *multiplication* symbols, respectively; to $-$ and ι as the *additive inversion* symbol and the *multiplicative inversion* symbol, respectively; and to $<$ as the *order* symbol. As usual, we write:

$$(\tau_1 + \tau_2)$$

$$(\tau_1 \times \tau_2)$$

$$(-\tau_1)$$

$$\tau_2^{-1}$$

$$(\tau_1 \equiv \tau_2)$$

$$(\tau_1 \not\equiv \tau_2)$$

$$(\tau_1 < \tau_2)$$

$$(\tau_1 \not< \tau_2)$$

instead of:

$$\begin{aligned}
& (+\tau_1\tau_2) \\
& (\times\tau_1\tau_2) \\
& (-\tau_1) \\
& (\iota\tau_2) \\
& (\equiv\tau_1\tau_2) \\
& ((\neg)(\equiv\tau_1\tau_2)) \\
& (<\tau_1\tau_2) \\
& ((\neg)(<\tau_1\tau_2))
\end{aligned}$$

respectively, where τ_1 and τ_2 are any terms in \mathcal{T}_F . Let \mathcal{L}_F be the predicate language defined by Π_F and let \mathcal{A}_F be the corresponding set of axioms. Let Λ_F be the predicate logic defined by Π_F :

$$\Lambda_F = (\mathcal{L}_F, \mathcal{A}_F)$$

We refer to Λ_F as the Predicate Logic for Ordered Field Theory. Let \mathcal{H}_F be the subset of \mathcal{L}_F consisting of all sentences in any one of the following schemes:

$$\begin{aligned}
& \forall (\zeta + \eta \equiv \eta + \zeta) \\
& \forall (\zeta \times \eta \equiv \eta \times \zeta) \\
& \forall ((\zeta + \eta) + \theta \equiv \zeta + (\eta + \theta)) \\
& \forall ((\zeta \times \eta) \times \theta \equiv \zeta \times (\eta \times \theta)) \\
& \forall (\zeta \times (\eta + \theta) \equiv (\zeta \times \eta) + (\zeta \times \theta)) \\
& \forall (\zeta + 0 \equiv \zeta) \\
& \forall (\zeta \times 1 \equiv \zeta) \\
& \forall (\zeta + (-\zeta) \equiv 0) \\
(\mathcal{H}_F) \quad & \forall (\zeta \neq 0 \longrightarrow (\zeta \times \zeta^{-1} \equiv 1)) \\
& (0 \neq 1) \\
& \forall (\zeta \not< \zeta) \\
& \forall ((\zeta < \eta) \wedge (\eta < \theta) \longrightarrow \zeta < \theta) \\
& \forall (\zeta \neq \eta \longrightarrow (\zeta < \eta) \vee (\eta < \zeta)) \\
& \forall ((\zeta < \eta) \longrightarrow (\zeta + \theta) < (\eta + \theta)) \\
& \forall ((\zeta < \eta) \wedge (0 < \theta) \longrightarrow (\zeta \times \theta) < (\eta \times \theta))
\end{aligned}$$

We refer to the sentences in \mathcal{H}_F as the hypotheses underlying Ordered Field Theory. Let:

$$\Theta_F(\mathcal{H}_F)$$

be the subset of \mathcal{L}_F consisting of all sentences δ for which $\mathcal{H}_F \Vdash \delta$. We refer to $\Theta_F(\mathcal{H}_F)$ as the syntactic Theory of Ordered Fields.

03• Should one expect that:

$$\mathcal{H}_F \Vdash \forall ((0 < \zeta) \longrightarrow (\exists \eta)(\zeta \equiv (\eta \times \eta))) \quad ?$$

Groups

04° Let Π_G be the preamble defined by the following specifications:

$$\mathcal{C}_G = \{e\}, \quad \mathcal{F}_G = \{\iota, \mu\}, \quad \mathcal{P}_G = \{\equiv\}$$

where $e = (c|)$, where $\iota = (|f|)$, and where $\mu = (||f|)$. We write:

$$\tau_1^{-1} \quad \text{for} \quad (\iota\tau_1) \quad \text{and} \quad (\tau_2 \cdot \tau_3) \quad \text{or} \quad \tau_2 \cdot \tau_3 \quad \text{for} \quad (\mu\tau_2\tau_3)$$

where τ_1, τ_2 and τ_3 are any terms in \mathcal{T}_G . Let Λ_G be the predicate logic defined by Π_G :

$$\Lambda_G = (\mathcal{L}_G, \mathcal{A}_G)$$

We refer to Λ_G as the Predicate Logic for Group Theory. Let \mathcal{H}_G be the subset of \mathcal{L}_G consisting of all sentences in any one of the following schemes:

$$\begin{aligned} (\mathcal{H}_G) \quad & \forall ((e \cdot \zeta \equiv \zeta) \wedge (\zeta \cdot e \equiv \zeta)) \\ & \forall ((\eta^{-1} \cdot \eta \equiv e) \wedge (\eta \cdot \eta^{-1} \equiv e)) \\ & \forall (\zeta \cdot (\eta \cdot \theta) \equiv (\zeta \cdot \eta) \cdot \theta) \end{aligned}$$

We refer to the sentences in \mathcal{H}_G as the hypotheses underlying Group Theory. Let:

$$\Theta_G(\mathcal{H}_G)$$

be the subset of \mathcal{L}_G consisting of all sentences δ for which $\mathcal{H}_G \Vdash \delta$. We refer to $\Theta_G(\mathcal{H}_G)$ as the syntactic Theory of Groups.

05• Should one expect that:

$$\mathcal{H}_G \Vdash \forall (\zeta \cdot \eta \equiv \eta \cdot \zeta) \quad ?$$

Abstract Lines

06° Let Π_L be the preamble defined by the following specifications:

$$\mathcal{C}_L = \emptyset, \mathcal{F}_L = \emptyset, \mathcal{P}_L = \{\equiv, <\}$$

where $< = (||f||)$. We refer to $<$ as the *order* symbol. Note that $v(<) = 2$. As usual, we write:

$$(\tau_1 \equiv \tau_2)$$

$$(\tau_1 \not\equiv \tau_2)$$

$$(\tau_1 < \tau_2)$$

$$(\tau_1 \not< \tau_2)$$

instead of:

$$(\equiv \tau_1 \tau_2),$$

$$(\neg)(\equiv \tau_1 \tau_2))$$

$$(< \tau_1 \tau_2)$$

$$((\neg)(< \tau_1 \tau_2))$$

respectively, where τ_1 and τ_2 are any terms in \mathcal{T}_L . Let Λ_L be the predicate logic defined by Π_L :

$$\Lambda_L = (\mathcal{L}_L, \mathcal{A}_L)$$

We refer to Λ_L as the Predicate Logic for Abstract Lines. Let \mathcal{H}_L be the subset of \mathcal{L}_L consisting of all sentences in any one of the following schemes:

$$\begin{aligned}
 (\mathcal{H}_L) \quad & (\forall \zeta) \neg(\zeta < \zeta) \\
 & (\forall \zeta)(\forall \eta)(\forall \theta)((\zeta < \eta) \wedge (\eta < \theta)) \longrightarrow (\zeta < \theta) \\
 & (\forall \zeta)(\forall \eta)((\zeta < \eta) \vee (\zeta \equiv \eta) \vee (\eta < \zeta)) \\
 & (\forall \zeta)(\exists \theta)(\theta < \zeta) \\
 & (\forall \zeta)(\forall \eta)((\zeta < \eta) \longrightarrow (\exists \theta)((\zeta < \theta) \wedge (\theta < \eta))) \\
 & (\forall \eta)(\exists \theta)(\eta < \theta)
 \end{aligned}$$

We refer to the sentences in \mathcal{H}_L as the hypotheses underlying the Theory of Abstract Lines. Let:

$$\Theta_L(\mathcal{H}_L)$$

be the subset of \mathcal{L}_L consisting of all sentences δ for which $\mathcal{H}_L \Vdash \delta$. We refer to $\Theta_L(\mathcal{H}_L)$ as the syntactic Theory of Abstract Lines.

07• Should one expect that \mathcal{H}_L be consistent?

Boolean Rings

08° Let Π_B be the preamble defined by the following specifications:

$$\mathcal{C}_B = \{0, 1\}, \quad \mathcal{F}_B = \{+, \times\}, \quad \mathcal{P}_B = \{\equiv\}$$

where $0 = (c|)$ and $1 = (c|)$, and where $+$ $= (||f|)$ and $\times = (||f|)$. We refer to 0 and 1 as the *zero* symbol and the *one* symbol, respectively, and to $+$ and \times as the *addition* symbol and the *multiplication* symbol, respectively. As usual, we write:

$$(\tau_1 + \tau_2)$$

$$(\tau_1 \times \tau_2)$$

$$(\tau_1 \equiv \tau_2)$$

$$(\tau_1 \not\equiv \tau_2)$$

instead of:

$$(+\tau_1\tau_2)$$

$$(\times\tau_1\tau_2)$$

$$(\equiv\tau_1\tau_2)$$

$$((\neg)(\equiv\tau_1\tau_2))$$

respectively, where τ_1 and τ_2 are any terms in \mathcal{T}_F . Let Λ_B be the predicate logic defined by Π_B :

$$\Lambda_B = (\mathcal{L}_B, \mathcal{A}_B)$$

We refer to Λ_B as the Predicate Logic for Boolean Ring Theory. Let \mathcal{H}_B be the subset of \mathcal{L}_B consisting of all sentences in any one of the following schemes:

$$\begin{aligned}
 (\mathcal{H}_B) \quad & \forall ((\zeta + \eta) \equiv (\eta + \zeta)) \\
 & \forall (((\zeta + \eta) + \theta) \equiv (\zeta + (\eta + \theta))) \\
 & \forall (((\zeta \times \eta) \times \theta) \equiv (\zeta \times (\eta \times \theta))) \\
 & \forall ((\zeta \times (\eta + \theta)) \equiv ((\zeta \times \eta) + (\zeta \times \theta))) \\
 & \forall (((\eta + \theta) \times \zeta) \equiv ((\eta \times \zeta) + (\theta \times \zeta))) \\
 & \forall ((\zeta + 0) \equiv \zeta) \\
 & \forall ((\zeta \times 1) \equiv \zeta) \\
 & (0 \not\equiv 1) \\
 & \forall ((\zeta \times \zeta) \equiv \zeta)
 \end{aligned}$$

We refer to the sentences in \mathcal{H}_B as the hypotheses underlying Boolean Ring Theory. Let:

$$\Theta_B(\mathcal{H}_B)$$

be the subset of \mathcal{L}_B consisting of all sentences δ for which $\mathcal{H}_B \Vdash \delta$. We refer to $\Theta_B(\mathcal{H}_B)$ as the syntactic Theory of Boolean Rings.

09• Show that:

$$\mathcal{H}_B \Vdash \forall (\zeta + \zeta \equiv 0) \quad \text{and} \quad \mathcal{H}_B \Vdash \forall (\eta \times \theta \equiv \theta \times \eta)$$

CHAPTER 3

PREDICATE LOGICS: SEMANTICS

To assign *meaning* to the sentences in a predicate language \mathcal{L}_Π , we must introduce structure sufficient to interpret the preamble Π which defines the language. For that purpose, we turn, without apology, to the intuitive concepts of Set Theory.

3.1 INTERPRETATIONS

Interpretations

01° Let Π be any preamble:

$$\Pi = (\mathcal{C}, \mathcal{F}, \mathcal{P})$$

By an *interpretation* I of Π , we mean an ordered pair:

$$I = (\Omega, S)$$

where Ω is any nonempty set and where S is an ordered triple:

$$S = (C, F, P)$$

for which C , F , and P are any mappings having domains \mathcal{C} , \mathcal{F} , and \mathcal{P} , respectively, and assigning values as follows:

- (IN₁) for each χ in \mathcal{C} , $C(\chi)$ is a member of Ω
- (IN₂) for each ϕ in \mathcal{F} , $F(\phi)$ is a mapping carrying Ω^k to Ω
- (IN₃) for each ρ in \mathcal{P} , $P(\rho)$ is a subset of Ω^ℓ

In this context, $k = v(\phi)$ and $\ell = v(\rho)$. We require that:

$$P(\equiv) = \{(\omega', \omega'') \in \Omega^2 : \omega' = \omega''\}$$

We refer to Ω as the *universe* and to S as the *structure* underlying I .

Assignments

02° By an *assignment* for \mathcal{V} in Ω , we mean any mapping A having domain \mathcal{V} and assigning values as follows:

$$(IN_4) \quad \text{for each } \zeta \text{ in } \mathcal{V}, A(\zeta) \text{ is a member of } \Omega$$

Valuations

03° Let I be an interpretation of Π and let A be an assignment for \mathcal{V} in the universe Ω underlying I . Let \mathcal{L}_Π be the predicate language defined by Π and let δ be a sentence in \mathcal{L}_Π . Relative to I and A , we plan to inquire whether δ is *false* or *true*.

04° To that end, let:

$$\mathcal{Z} = \{0, 1\}$$

be the set composed of the *false* sign 0 and the *true* sign 1. In terms of I and A , we shall define a mapping I_A having domain \mathcal{L}_Π and assigning values in \mathcal{Z} in accord with a certain recursive pattern, soon to be described. We will refer to I_A as the *valuation* defined by I and A . Then we will say that δ is *false* if $I_A(\delta) = 0$ and that δ is *true* if $I_A(\delta) = 1$.

05° For the definition of I_A , we must prepare the way by extending the definitions of C and A . Let S_A be the mapping having domain \mathcal{T}_Π and codomain Ω , assigning values by the following recursive pattern:

$$(S_1) \quad \text{for each } \chi \text{ in } \mathcal{C}, S_A(\chi) = C(\chi)$$

$$(S_2) \quad \text{for each } \zeta \text{ in } \mathcal{V}, S_A(\zeta) = A(\zeta)$$

$$(S_3) \quad \text{for each } \phi \text{ in } \mathcal{F} \text{ and for any } \tau_1, \tau_2, \dots, \text{ and } \tau_k \text{ in } \mathcal{T}_\Pi:$$

$$S_A((\phi \tau_1 \tau_2 \dots \tau_k)) = F(\phi)(S_A(\tau_1), S_A(\tau_2), \dots, S_A(\tau_k))$$

Of course, $k = v(\phi)$. Let us emphasize that S_A depends upon Ω , C , F , and A but not upon P .

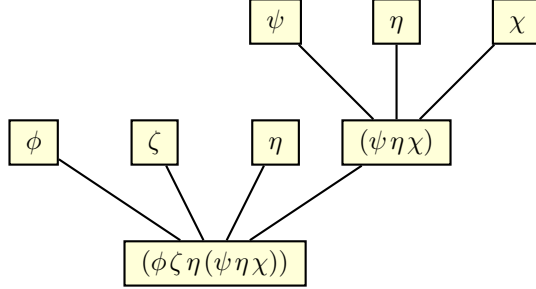
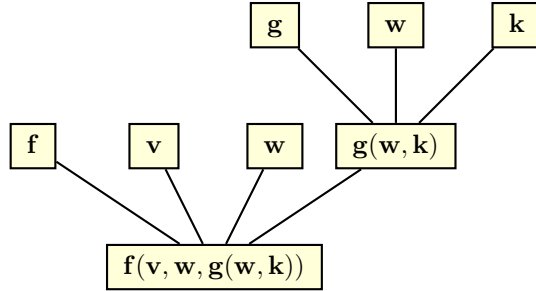
06° For illustration, let us recover the syntactic tree \mathcal{G} for the term τ from article 1.15° in Chapter 2:

$$\tau = ((\phi \zeta \eta(\psi \eta \chi))$$

See Figure 8. To calculate $S_A(\tau)$, we apply the foregoing prescription. In Figures 19 and 20, soon to follow, we display two copies of \mathcal{G} , one with labels appropriate to τ and one with labels appropriate to the calculation of $S_A(\tau)$.

We denote $C(\chi)$ by \mathbf{k} ; $A(\zeta)$ and $A(\eta)$ by \mathbf{v} and \mathbf{w} , respectively; and $F(\phi)$ and $F(\psi)$ by \mathbf{f} and \mathbf{g} , respectively. We find that:

$$S_A((\psi\eta\chi)) = \mathbf{g}(\mathbf{w}, \mathbf{k}), \quad S_A(\tau) = S_A((\phi\zeta\eta(\psi\eta\chi))) = \mathbf{f}(\mathbf{v}, \mathbf{w}, \mathbf{g}(\mathbf{w}, \mathbf{k}))$$

Figure 19: τ Figure 20: $S_A(\tau)$

The Joint Valuation Mapping J

07° Let I be an interpretation of Π :

$$I = (\mathbf{\Omega}, S)$$

where $\mathbf{\Omega}$ is the universe and S is the structure underlying I . Let \mathbf{V} be the set of all assignments for \mathcal{V} in $\mathbf{\Omega}$. We plan to define, by recursion, the *joint valuation* mapping J carrying the set:

$$\mathbf{V} \times \mathcal{L}_\Pi$$

to the set:

$$\mathcal{Z} = \{0, 1\}$$

Specifically, for each sentence α in \mathcal{L}_Π , we will define the values of J on the set:

$$\mathbf{V} \times \{\alpha\}$$

in terms of its values, defined prior, on the sets:

$$\mathbf{V} \times \{\beta\}$$

where β is any *proper* subsentence of α , that is, where β is any subsentence of α other than α itself. We will organize the definition in terms of four rules:

$$(J_1), (J_2), (J_3), (J_4)$$

which reflect the various characteristics of the sentence α .

08° Let us proceed. Let A be any assignment in \mathbf{V} and let α be any atomic sentence in \mathcal{L}_Π :

$$\alpha = (\rho\tau_1\tau_2 \cdots \tau_\ell)$$

where ρ is any predicate symbol in \mathcal{P} and where τ_1, τ_2, \dots , and τ_ℓ are any terms in \mathcal{T}_Π . Of course, $\ell = v(\rho)$. We define:

$$(J_1) \quad J(A, \alpha) = 1 \iff (S_A(\tau_1), S_A(\tau_2), \dots, S_A(\tau_\ell)) \in P(\rho)$$

In particular, if:

$$\alpha = (\tau_1 \equiv \tau_2)$$

where τ_1 and τ_2 are any terms in \mathcal{T}_Π , then:

$$J(A, \alpha) = 1 \iff S_A(\tau_1) = S_A(\tau_2)$$

09° Let A be any assignment in \mathbf{V} , let β be any sentence in \mathcal{L}_Π , and let:

$$\alpha = ((\neg)\beta)$$

We define:

$$(J_2) \quad J(A, \alpha) = 1 \iff J(A, \beta) = 0$$

10° Let A be any assignment in \mathbf{V} , let γ_1 and γ_2 be any sentences in \mathcal{L}_Π , and let:

$$\alpha = (\gamma_1 \longrightarrow \gamma_2)$$

We define:

$$(J_3) \quad J(A, \alpha) = 1 \iff J(A, \gamma_1) = 0 \text{ or } J(A, \gamma_2) = 1$$

11° Let A be any assignment in \mathbf{V} , let β be any sentence in \mathcal{L}_Π , let ζ be any variable symbol in \mathcal{V} , and let:

$$\alpha = ((\forall\zeta)\beta)$$

We define:

$$(J_4) \quad J(A, \alpha) = 1 \iff \text{for any } \omega \text{ in } \Omega, J(B, \beta) = 1$$

where B is the assignment in \mathbf{V} such that A and B coincide on $\mathcal{V} \setminus \{\zeta\}$ while $B(\zeta) = \omega$.

12° Let A be an assignment for \mathcal{V} in Ω and let ζ be a variable symbol in \mathcal{V} . For each member ω of Ω , let $A(\omega|\zeta)$ denote the assignment for \mathcal{V} in Ω defined as follows:

$$A(\omega|\zeta)(\eta) = \begin{cases} A(\eta) & \text{if } \eta \neq \zeta \\ \omega & \text{if } \eta = \zeta \end{cases}$$

The rule (J_4) now takes the form:

$$(J_4) \quad J(A, \alpha) = 1 \iff \text{for any } \omega \text{ in } \Omega, J(A(\omega|\zeta), \beta) = 1$$

The Valuation I_A

13° Finally, we can produce the valuation I_A defined by I and A :

$$I_A(\delta) = J(A, \delta)$$

where δ is any sentence in \mathcal{L}_Π . As noted, we say that δ is false if $I_A(\delta) = 0$ and that δ is true if $I_A(\delta) = 1$.

An Example

14° With reference to article 5.04° in Chapter 2, let us introduce the preamble Π_G , the Predicate Logic:

$$\Lambda_G = (\mathcal{L}_G, \mathcal{A}_G)$$

and the set \mathcal{H}_G of hypotheses underlying Group Theory. Let us adopt the notation introduced in the articles just mentioned. In turn, let I be the interpretation:

$$I = (\Omega, S), \quad S = (C, F, P)$$

of Π_G , where Ω is the set consisting of all invertible matrices having two rows and two columns and having rational entries and where:

$$\begin{aligned} C(e) &= \mathbf{e} \\ F(\iota)(\mathbf{m}) &= \mathbf{m}^{-1} \\ F(\mu)(\mathbf{m}', \mathbf{m}'') &= \mathbf{m}'\mathbf{m}'' \end{aligned}$$

In the foregoing relations, \mathbf{e} is the identity matrix in Ω , while \mathbf{m} , \mathbf{m}' , and \mathbf{m}'' are any matrices in Ω , \mathbf{m}^{-1} is the inverse of \mathbf{m} in Ω , and $\mathbf{m}'\mathbf{m}''$ is the product of \mathbf{m}' and \mathbf{m}'' in Ω . Let ζ and η be distinct variable symbols in \mathcal{V} and let α and β be the sentences:

$$\alpha = (\exists\zeta)\beta = (\neg)((\forall\zeta)((\neg)\beta)), \quad \beta = (\eta \equiv \zeta \cdot \zeta)$$

in \mathcal{L}_G . Let A' and A'' be assignments for \mathcal{V} in Ω such that, in particular:

$$A'(\eta) = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \quad A''(\eta) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

We find that:

$$I_{A'}(\alpha) = J(A', \alpha) = 1$$

because $J(A'(\mathbf{m}|\zeta), \beta) = 1$, where:

$$\mathbf{m} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

However:

$$I_{A''}(\alpha) = J(A'', \alpha) = 0$$

because, for any matrix \mathbf{m} in Ω , $J(A''(\mathbf{m}|\zeta), \beta) = 1$ would imply that $\det(\mathbf{m})$ equals $\sqrt{2}$, an irrational number.

The Coincidence Principle

15° Let α be any sentence in \mathcal{L}_Π . Let \mathcal{V}_α be, as usual, the subset of \mathcal{V} consisting of all variable symbols η such that η occurs at least once freely in α . Let I be any interpretation of Π and let Ω be the universe underlying I . Let A' and A'' be any assignments for \mathcal{V} in Ω . We contend that:

$$(\mathbf{KP}) \quad \text{for any } \eta \text{ in } \mathcal{V}_\alpha, A'(\eta) = A''(\eta) \implies J(A', \alpha) = J(A'', \alpha)$$

We refer to this fact as the Coincidence Principle.

16° To prove the contention, we prepare the unfolded syntactic tree for α , then argue by induction on the number of nodes in the basic syntactic tree for α .

17° For rules (J_1) , (J_2) , and (J_3) , the arguments are obvious. For rule (J_4) , in context of which $\alpha = ((\forall\zeta)\beta)$, we note that, $\mathcal{V}_\beta \subseteq \mathcal{V}_\alpha \cup \{\zeta\}$, while $\zeta \notin \mathcal{V}_\alpha$. By hypothesis, for any η in \mathcal{V}_α , $A'(\eta) = A''(\eta)$. It follows that, for any ω in Ω and for any η in \mathcal{V}_β , $A'(\omega|\zeta)(\eta) = A''(\omega|\zeta)(\eta)$. By induction, we infer that $J(A'(\omega|\zeta), \beta) = J(A''(\omega|\zeta), \beta)$. Therefore, $J(A', \alpha) = J(A'', \alpha)$. \dagger

Closed Sentences

18° Let α be any sentence in \mathcal{L}_Π^0 . That is, let α be a closed sentence in \mathcal{L}_Π . For such a sentence, **(KP)** implies that, for any assignments A' and A'' for \mathcal{V} in Ω , $I_{A'}(\alpha) = I_{A''}(\alpha)$. Naturally, we simplify notation by writing:

$$I(\alpha) \quad \text{for} \quad I_A(\alpha)$$

where A is any assignment for \mathcal{V} in Ω .

The Substitution Principle

19° Let α be any sentence in \mathcal{L}_Π , let ζ be any variable symbol in \mathcal{V} , and let τ be any term in \mathcal{T}_Π . Let I be any interpretation of Π and let A be any assignment for \mathcal{V} in the universe Ω underlying I . Let $\omega = S_A(\tau)$. We contend that:

$$\text{(SP)} \quad \tau \text{ free for } \zeta \text{ in } \alpha \implies J(A, \alpha(\tau|\zeta)) = J(A(\omega|\zeta), \alpha)$$

We refer to this fact as the Substitution Principle.

20° To prove the contention, we prepare the unfolded syntactic trees for α and $\alpha(\tau|\zeta)$, then argue by induction on the number of nodes in the basic syntactic tree for α .

21° For rule (J_1) , in context of which $\alpha = (\rho \tau_1 \tau_2 \dots \tau_\ell)$ and $\ell = v(\rho)$, we note that $\alpha(\tau|\zeta) = (\rho \tau_1(\tau|\zeta) \tau_2(\tau|\zeta) \dots \tau_\ell(\tau|\zeta))$ and that, for each index j ($1 \leq j \leq \ell$), $S_A(\tau_j(\tau|\zeta)) = S_B(\tau_j)$, where $B = A(\omega|\zeta)$. We find that:

$$\begin{aligned} J(A, \alpha(\tau|\zeta)) = 1 &\iff (S_A(\tau_1(\tau|\zeta)), S_A(\tau_2(\tau|\zeta)), \dots, S_A(\tau_\ell(\tau|\zeta))) \in P(\rho) \\ &\iff (S_B(\tau_1), S_B(\tau_2), \dots, S_B(\tau_\ell)) \in P(\rho) \\ &\iff J(A(\omega|\zeta), \alpha) = 1 \end{aligned}$$

22° For rules (J_2) and (J_3) , the arguments are obvious. For rule (J_4) , in context of which $\alpha = ((\forall\eta)\beta)$ and $\eta \neq \zeta$, we note that $\alpha(\tau|\zeta) = (\forall\eta)\beta(\tau|\zeta)$.

Since τ is free for ζ in α , η does not occur in τ . Hence, for any \mathbf{v} in Ω , $S_B(\tau) = S_A(\tau)$, where $B = A(\mathbf{v}|\eta)$. We find that:

$$\begin{aligned} J(A, \alpha(\tau|\zeta)) = 1 &\iff \text{for all } \mathbf{v} \text{ in } \Omega, J(A(\mathbf{v}|\eta), \beta(\tau|\zeta)) = 1 \\ &\iff \text{for all } \mathbf{v} \text{ in } \Omega, J(A(\mathbf{v}|\eta)(\omega|\zeta), \beta) = 1 \\ &\iff \text{for all } \mathbf{v} \text{ in } \Omega, J(A(\omega|\zeta)(\mathbf{v}|\eta), \beta) = 1 \\ &\iff J(A(\omega|\zeta), \alpha) = 1 \end{aligned}$$

The second of the foregoing equivalences follows by induction.

23° To be thorough, let us note that if $\eta = \zeta$ then $\alpha(\tau|\zeta) = \alpha$ and, by **(KP)**, $J(A, \alpha) = J(A(\omega|\zeta), \alpha)$. \ddagger

Valuations of Axioms

24° Let I be any interpretation of Π and let A be any assignment for \mathcal{V} in the universe Ω underlying I . Let α be any axiom in \mathcal{A}_Π . We contend that:

$$I_A(\alpha) = J(A, \alpha) = 1$$

This unsurprising result is fundamental to our subject.

25° To prove the contention, one need only apply the rules. The cases (A_1) , (A_2) , and (A_3) follow by inspection. The cases (E_1) , (E_2) , (E_3) , (E_4) , and (E_5) follow from the meaning of S_A . Let us sketch the arguments for the more complicated cases (A_4) and (A_5) .

26° For the case (A_4) , we introduce the sentence:

$$\alpha = ((\forall \eta)\beta \longrightarrow \beta(\tau|\eta))$$

where β is any sentence in \mathcal{L}_Π , η is any variable symbol in \mathcal{V} , and τ is any term in \mathcal{T}_Π . Of course, τ is free for η in β . Let us suppose that $J(A, \alpha) = 0$, so that:

$$J(A, (\forall \eta)\beta) = 1 \quad \text{and} \quad J(A, \beta(\tau|\eta)) = 0$$

By the first relation, it would follow that, for any \mathbf{v} in Ω , $J(A(\mathbf{v}|\eta), \beta) = 1$. By the second relation and by **(SP)**, it would follow that $J(A(\omega|\eta), \beta) = 0$, where $\omega = S_A(\tau)$. By this contradiction, we conclude that $J(A, \alpha) = 1$.

27° For the case (A_5) , we introduce the sentence:

$$\alpha = ((\forall \eta)(\beta \longrightarrow \gamma) \longrightarrow (\beta \longrightarrow (\forall \eta)\gamma))$$

where β and γ are any sentences in \mathcal{L}_Π and where η is any variable symbol in \mathcal{V} . Of course, each occurrence (if any) of η in β is bound. Hence, for any ω in Ω , the restrictions of A and $A(\omega|\eta)$ to \mathcal{V}_β coincide. By **(KP)**, we infer that, for any ω in Ω , $J(A, \beta) = J(A(\omega|\eta), \beta)$. Let us suppose that $J(A, \alpha) = 0$, so that:

$$J(A, (\forall\eta)(\beta \longrightarrow \gamma)) = 1 \quad \text{and} \quad J(A, (\beta \longrightarrow (\forall\eta)\gamma)) = 0$$

By the first relation, it would follow that, for any \mathbf{v} in Ω , $J(A(\mathbf{v}|\eta), \beta) = 0$ or $J(A(\mathbf{v}|\eta), \gamma) = 1$. By the second relation, it would follow that $J(A, \beta) = 1$ and $J(A, (\forall\eta)\gamma) = 0$. Hence, for some ω in Ω , $J(A(\omega|\eta), \gamma) = 0$ and, by our prior inference, $J(A(\omega|\eta), \beta) = J(A, \beta) = 1$. By this contradiction, we conclude that $J(A, \alpha) = 1$. \dagger

3.2 SEMANTIC IMPLICATION

Semantic Implication

01° Let \mathcal{H} be any subset of \mathcal{L}_Π . We refer to the sentences in \mathcal{H} as *hypotheses*. Let δ be any sentence in \mathcal{L}_Π . We say that \mathcal{H} *semantically implies* δ iff, for any interpretation I of Π and for any assignment A for \mathcal{V} in the universe Ω underlying I , if, for any sentence α in \mathcal{H} , $I_A(\alpha) = 1$ then $I_A(\delta) = 1$. To express this relation, we write:

$$\mathcal{H} \models \delta$$

02° In parallel with our description of syntactic implication in Chapter 2, we are led to the expressions:

$$\begin{aligned} \mathcal{H}_1 &\models \mathcal{H}_2 \\ \beta_1, \beta_2, \dots \beta_k &\models \mathcal{H}_2 \\ \mathcal{H}_1 &\models \gamma_1, \gamma_2, \dots \gamma_k \\ \beta_1, \beta_2, \dots \beta_k &\models \gamma_1, \gamma_2, \dots \gamma_k \end{aligned}$$

and to the implication:

$$\mathcal{H}_1 \subseteq \mathcal{H}_2, \mathcal{H}_1 \models \delta \implies \mathcal{H}_2 \models \delta$$

Semantic Theories

03° Let \mathcal{H} be any subset of \mathcal{L}_Π . We refer to the sentences in \mathcal{H} as *hypotheses*. Let:

$$T_\Pi(\mathcal{H})$$

be the subset of \mathcal{L}_Π consisting of all sentences δ such that $\mathcal{H} \models \delta$. We refer to the sentences in $T_\Pi(\mathcal{H})$ as *semantic theorems* and to $T_\Pi(\mathcal{H})$ itself as the *semantic theory* of \mathcal{H} . Of course:

$$\mathcal{H} \models T_\Pi(\mathcal{H})$$

The Soundness Theorem

04° Let \mathcal{H} be any subset of \mathcal{L}_Π and let δ be any sentence in \mathcal{L}_Π . We contend that if \mathcal{H} syntactically implies γ then \mathcal{H} semantically implies γ :

$$(\mathbf{ST}) \quad \mathcal{H} \Vdash \gamma \implies \mathcal{H} \models \gamma$$

This fact is the *simple half* of the Completeness Theorem, often called the Soundness Theorem. We prove it now because we require it in the proof of the Interpretation Theorem.

05° To prove the contention, we argue as follows. Let \mathcal{G} be any proper deductive tree from \mathcal{H} and let δ be its consequence. Let I be any interpretation of Π and let A be any assignment for \mathcal{V} in the universe Ω underlying I such that, for each sentence α in \mathcal{H} , $J(A, \alpha) = 1$. We claim that $J(A, \delta) = 1$. Having proved the claim, we will have proved **(ST)**. To prove the claim, we will argue by induction on the number of nodes in \mathcal{G} .

06° Let R be the root of \mathcal{G} . Let $\tilde{\mathcal{H}}$ be the subset of \mathcal{H} consisting of all material hypotheses for \mathcal{G} .

07° Let us consider the case in which $v(R) = 0$. By definition, the number of nodes in \mathcal{G} equals 1, so that R is a leaf. Hence, $\delta \in \mathcal{A}_\Pi \cup \mathcal{H}$. Obviously, $J(A, \delta) = 1$.

08° Let us consider the case in which $v(R) = 2$. Let P and N be the immediate descendants of R , in order. Let $(\forall\zeta)$ and γ be the labels which occupy P and N , respectively, where ζ is a variable symbol in \mathcal{V} and where γ is a sentence in \mathcal{L}_Π . Of course, $\delta = ((\forall\zeta)\gamma)$. Let \mathcal{G}_o be the subtree of \mathcal{G} defined by the node N . Obviously, \mathcal{G}_o is a proper deductive tree for γ from $\tilde{\mathcal{H}}$ and the number of nodes in \mathcal{G}_o is smaller than the number of nodes in \mathcal{G} . Let ω be any member of Ω . Let α be any sentence in $\tilde{\mathcal{H}}$. Since ζ does not occur freely in α , we find that the restrictions of A and $A(\omega|\zeta)$ to \mathcal{V}_α coincide. By **(KP)**, $J(A(\omega|\zeta), \alpha) = J(A, \alpha) = 1$. By induction, $J(A(\omega|\zeta), \gamma) = 1$. We conclude that $J(A, \delta) = 1$.

09° Finally, let us consider the case in which $v(R) = 3$. Let P , N_1 , and N_2 be the immediate descendants of R , in order. Let (\longrightarrow) , γ_1 , and γ_2 be the labels

which occupy P , N_1 , and N_2 , respectively, where γ_1 and γ_2 are sentences in \mathcal{L}_Π . Of course, $\gamma_2 = (\gamma_1 \longrightarrow \delta)$. Let \mathcal{G}_1 and \mathcal{G}_2 be the subtrees of \mathcal{G} defined by the nodes N_1 and N_2 , respectively. Obviously, \mathcal{G}_1 and \mathcal{G}_2 are proper deductive trees for γ_1 and γ_2 , respectively, from $\tilde{\mathcal{H}}$ and the numbers of nodes in \mathcal{G}_1 and \mathcal{G}_2 are smaller than the number of nodes in \mathcal{G} . By induction, $J(A, \gamma_1) = 1$ and $J(A, \gamma_2) = 1$. We conclude that $J(A, \delta) = 1$. \spadesuit

Satisfaction

10° Let \mathcal{H} be any subset of \mathcal{L}_Π . We say that \mathcal{H} is *satisfiable* iff there are an interpretation I of Π :

$$I = (\mathbf{\Omega}, S)$$

and an assignment A for \mathcal{V} in the universe $\mathbf{\Omega}$ underlying I such that, for any sentence δ in \mathcal{H} , $I_A(\delta) = 1$. We express this property of \mathcal{H} by writing:

$$Sat(\mathcal{H})$$

11° When \mathcal{H} is closed, we suppress reference to A .

3.3 THE INTERPRETATION THEOREM

The Interpretation Theorem

01° Let \mathcal{H} be any subset of \mathcal{L}_Π . We contend that \mathcal{H} is consistent iff \mathcal{H} is satisfiable:

$$(\mathbf{IT}) \quad Con(\mathcal{H}) \iff Sat(\mathcal{H})$$

We refer to this fundamental fact as the Interpretation Theorem, a cornerstone of Mathematics.

02° To prove **(IT)**, we must prove:

$$(SC) \quad Sat(\mathcal{H}) \implies Con(\mathcal{H})$$

$$(CS) \quad Con(\mathcal{H}) \implies Sat(\mathcal{H})$$

To prove **(SC)**, we assume that \mathcal{H} is satisfiable but we suppose that \mathcal{H} is inconsistent. Accordingly, we introduce an interpretation I of Π and an assignment A for \mathcal{V} in the universe $\mathbf{\Omega}$ underlying I such that, for any sentence α in \mathcal{H} , $I_A(\alpha) = 1$ and we introduce a sentence γ in \mathcal{L}_Π such that $\mathcal{H} \Vdash (\neg\gamma), \gamma$. By the Soundness Theorem, $\mathcal{H} \models (\neg\gamma), \gamma$. Hence, $I_A((\neg\gamma)) = 1$ and $I_A(\gamma) = 1$, a contradiction. We infer that our supposition is untenable, hence, that \mathcal{H} is consistent.

03° To prove (CS), we assume that \mathcal{H} is consistent. Following a method designed by L. Henkin, we will define an interpretation I of Π and an assignment A for \mathcal{V} in the universe Ω underlying I such that, for any sentence α in \mathcal{H} , $I_A(\alpha) = 1$. Having done so, we may infer that \mathcal{H} is satisfiable.

04° We begin by assuming, provisionally, that \mathcal{H} is *maximally consistent*, which is to say that the following conditions hold:

- (H₁) \mathcal{H} is consistent
- (H₂) for any subset $\bar{\mathcal{H}}$ of \mathcal{L}_Π , if $\bar{\mathcal{H}}$ is consistent and if $\mathcal{H} \subseteq \bar{\mathcal{H}}$ then $\bar{\mathcal{H}} = \mathcal{H}$

and that \mathcal{H} is *universal*, which is to say that the following condition holds:

- (H₃) for any variable symbol ζ in \mathcal{V} and for any sentence β in \mathcal{L}_Π , if $(\forall\zeta)\beta \notin \mathcal{H}$ then there is a constant symbol χ in \mathcal{C} such that $\beta(\chi|\zeta) \notin \mathcal{H}$

One interprets χ as a *witness* to the relation $(\forall\zeta)\beta \notin \mathcal{H}$.

05° Under the foregoing provisional assumptions, we will proceed to define I and A . Later, we will show that conditions (H₂) and (H₃) can, in a sense, be justified.

06° One can easily show that conditions (H₁) and (H₂) imply the following conditions:

- (H₄) for any sentence γ in \mathcal{L}_Π , $(\neg\gamma) \in \mathcal{H}$ or $\gamma \in \mathcal{H}$
- (H₅) for any sentence γ in \mathcal{L}_Π , $\gamma \in \mathcal{H}$ iff $\mathcal{H} \Vdash \gamma$
- (H₆) $\mathcal{A}_\Pi \subseteq \mathcal{H}$
- (H₇) for any sentences γ and δ in \mathcal{L}_Π :

$$(\gamma \longrightarrow \delta) \in \mathcal{H} \quad \text{iff} \quad (\neg\gamma) \in \mathcal{H} \quad \text{or} \quad \delta \in \mathcal{H}$$

For pointers, let us note that **(RA)** figures in the proof of (H₄) and (SI₅) figures in the proof of (H₇).

07° Let us design I and A . To that end, we introduce a relation \sim on the set \mathcal{T}_Π of terms:

$$\tau_1 \sim \tau_2 \iff \mathcal{H} \Vdash (\tau_1 \equiv \tau_2)$$

where τ_1 and τ_2 are any terms in \mathcal{T}_Π . By (E₁), (E₂), and (E₃) and by (H₅), (H₆), and (H₇), one can easily verify that \sim is *reflexive*, *symmetric*, and *transitive*, that is, that \sim is an equivalence relation. Let Ω be the set of all

equivalence classes in \mathcal{T}_Π following \sim . For each term τ in \mathcal{T}_Π , let us denote by $[\tau]$ the equivalence class defined by τ :

$$\tau \in [\tau]$$

08° Let C be the mapping defined as follows:

$$C(\chi) = [\chi]$$

where χ is any constant symbol in \mathcal{C} . Let F be the mapping defined as follows:

$$F(\phi)([\tau_1], [\tau_2], \dots, [\tau_k]) = [(\phi\tau_1\tau_2 \dots \tau_k)]$$

where ϕ is any function symbol in \mathcal{F} and where τ_1, τ_2, \dots , and τ_k are any terms in \mathcal{T}_Π . Of course, $k = v(\phi)$. By (E_4) and by (H_5) , (H_6) , and (H_7) , one can easily verify that $F(\phi)$ is a properly defined mapping carrying Ω^k to Ω . Let P be the mapping defined as follows:

$$([\tau_1], [\tau_2], \dots, [\tau_\ell]) \in P(\rho) \iff \mathcal{H} \Vdash (\rho\tau_1\tau_2 \dots \tau_\ell)$$

where ρ is any predicate symbol in \mathcal{P} and where τ_1, τ_2, \dots , and τ_ℓ are any terms in \mathcal{T}_Π . Of course, $\ell = v(\rho)$. By (E_5) and by (H_5) , (H_6) , and (H_7) , one can easily verify that $P(\rho)$ is a properly defined subset of Ω^ℓ . Moreover:

$$P(\equiv) = \{([\tau_1], [\tau_2]) \in \Omega^2 : [\tau_1] = [\tau_2]\}$$

09° Let I be the interpretation of Π defined as follows:

$$I = (\Omega, S), \quad S = (C, F, P)$$

In turn, let A be the assignment for \mathcal{V} in Ω defined as follows:

$$A(\zeta) = [\zeta]$$

where ζ is any variable symbol in \mathcal{V} . One can easily prove that, for any term τ in \mathcal{T}_Π :

$$S_A(\tau) = [\tau]$$

To that end, one should argue by induction on the number of nodes in the syntactic tree for τ .

10° Now we claim that, for every sentence α in \mathcal{L}_Π :

$$\alpha \in \mathcal{H} \iff I_A(\alpha) = 1$$

Having proved the claim, we will have proved that \mathcal{H} is satisfiable. To prove the claim, we argue by induction on the number of nodes in the basic syntactic tree for α . We apply, tacitly, rules (J_1) through (J_4) and conditions (H_3) through (H_7) , as needed.

11° Let α be any atomic sentence in \mathcal{L}_Π :

$$\alpha = (\rho\tau_1\tau_2 \cdots \tau_\ell)$$

where ρ is any predicate symbol in \mathcal{P} and where τ_1, τ_2, \dots , and τ_ℓ are any terms in \mathcal{T}_Π . Of course, $\ell = v(\rho)$. We find that:

$$\begin{aligned} I_A(\alpha) = 1 &\iff (S_A(\tau_1), S_A(\tau_2), \dots, S_A(\tau_\ell)) \in P(\rho) \\ &\iff ([\tau_1], [\tau_2], \dots, [\tau_\ell]) \in P(\rho) \\ &\iff \mathcal{H} \Vdash (\rho\tau_1\tau_2 \cdots \tau_\ell) \\ &\iff \alpha \in \mathcal{H} \end{aligned}$$

12° Let β be any sentence in \mathcal{L}_Π and let:

$$\alpha = (\neg\beta)$$

We find that:

$$\begin{aligned} I_A(\alpha) = 1 &\iff I_A(\beta) = 0 \\ &\iff \beta \notin \mathcal{H} \\ &\iff \alpha \in \mathcal{H} \end{aligned}$$

13° In turn, let β and γ be any sentences in \mathcal{L}_Π and let:

$$\alpha = (\beta \longrightarrow \gamma)$$

We find that:

$$\begin{aligned} I_A(\alpha) = 1 &\iff I_A(\beta) = 0 \text{ or } I_A(\gamma) = 1 \\ &\iff \beta \notin \mathcal{H} \text{ or } \gamma \in \mathcal{H} \\ &\iff \alpha \in \mathcal{H} \end{aligned}$$

14° Finally, let β be any sentence in \mathcal{L}_Π , let ζ be any variable symbol in \mathcal{V} , and let:

$$\alpha = (\forall \zeta)\beta$$

Let us assume that $\alpha \notin \mathcal{H}$. Since \mathcal{H} is universal, we may introduce a constant symbol χ in \mathcal{C} such that $\beta(\chi|\zeta) \notin \mathcal{H}$. By induction, $J(A, \beta(\chi|\zeta)) = 0$. By **(SP)**, $J(A([\chi]|\zeta), \beta) = 0$. By rule (J_4) , we infer that $J(A, \alpha) = 0$.

15° Let us assume that $J(A, \alpha) = 0$. By rule (J_4) , there is a term τ in \mathcal{T}_Π such that $J(A([\tau]|\zeta), \beta) = 0$. Presuming that τ is free for ζ in β , we may apply **(SP)** to obtain $J(A, \beta(\tau|\zeta)) = 0$. By induction, $\beta(\tau|\zeta) \notin \mathcal{H}$. By (H_6) and (A_4) , $((\forall \zeta)\beta \rightarrow \beta(\tau|\zeta)) \in \mathcal{H}$. By (H_7) , $\neg((\forall \zeta)\beta) \in \mathcal{H}$. Under our presumption, we infer that $\alpha \notin \mathcal{H}$.

16° We hasten to admit that we have no basis for presuming that τ is free for ζ in β . However, by **(FM)**, that is, by the Freedom Maneuver, we can design a sentence $\bar{\beta}$ in \mathcal{L}_Π such that:

(F₁) τ is free for ζ in $\bar{\beta}$

(F₂) $\beta \Vdash \bar{\beta}$ and $\bar{\beta} \Vdash \beta$

(F₃) the basic syntactic trees for β and $\bar{\beta}$ have the same number of nodes

By (F_2) and **(ST)**, we obtain $\bar{\beta} \models \beta$. Since $J(A([\tau]|\zeta), \beta) = 0$, we find that $J(A([\tau]|\zeta), \bar{\beta}) = 0$. By (F_1) and (F_3) and by the presumptuous argument in the preceding article, we find that $\neg((\forall \zeta)\bar{\beta}) \in \mathcal{H}$. By (F_2) and (SI_7) , $(\forall \zeta)\beta \Vdash (\forall \zeta)\bar{\beta}$. Without presumption, we infer that $\alpha \notin \mathcal{H}$.

17° Now let us return to our original assumption that \mathcal{H} is consistent but let us rescind our provisional assumptions that \mathcal{H} is maximally consistent and that \mathcal{H} is universal.

18° To prove the Interpretation Theorem, we propose a naive plan. We will design a subset \mathcal{H}^* of \mathcal{L}_Π such that $\mathcal{H} \subseteq \mathcal{H}^*$ and such that \mathcal{H}^* is maximally consistent and universal. That done, we will apply the foregoing argument to prove that \mathcal{H}^* is satisfiable. Of course, it would follow that \mathcal{H} is satisfiable as well.

19° However, there is an obstacle to the plan. In certain troublesome cases, the conditions that \mathcal{H}^* be (maximally) consistent and universal are incompatible, because there is a shortage of “independent” witnesses. The following simple example serves to illustrate the matter.

20• Let ζ and η be distinct variable symbols in \mathcal{V} . Let \mathcal{H} be the subset of \mathcal{L}_Π consisting of the sentence:

$$\neg(\forall\zeta)(\forall\eta)(\zeta \equiv \eta)$$

together with all sentences of the form:

$$(\chi' \equiv \chi'')$$

where χ' and χ'' are any constant symbols in \mathcal{C} . Show that \mathcal{H} is satisfiable, hence consistent. Let \mathcal{H}^* be any subset of \mathcal{L}_Π for which $\mathcal{H} \subseteq \mathcal{H}^*$. Suppose that \mathcal{H}^* is both consistent and universal. Let β be the sentence in \mathcal{L}_Π defined as follows:

$$\beta = (\forall\eta)(\zeta \equiv \eta)$$

Note that $(\forall\zeta)\beta \notin \mathcal{H}^*$. Hence, there is a constant symbol χ_1 in \mathcal{C} such that $\beta(\chi_1|\zeta) \notin \mathcal{H}^*$. Let γ be the sentence in \mathcal{L}_Π defined as follows:

$$\gamma = (\chi_1 \equiv \eta)$$

Note that $\beta(\chi_1|\zeta) = (\forall\eta)\gamma$, so that $(\forall\eta)\gamma \notin \mathcal{H}^*$. Hence, there is a constant symbol χ_2 in \mathcal{C} such that $\gamma(\chi_2|\eta) \notin \mathcal{H}^*$. However:

$$\gamma(\chi_2|\eta) = (\chi_1 \equiv \chi_2)$$

which, by design, is contained in \mathcal{H} . By this contradiction, it follows that \mathcal{H}^* is not both consistent and universal.

21° Let us propose a more sophisticated plan. We will define a new preamble:

$$\Pi^* = (\mathcal{C}^*, \mathcal{F}, \mathcal{P}) \quad (\mathcal{C} \subseteq \mathcal{C}^*)$$

by introducing a rich supply of new constant symbols to serve as witnesses. Of course, $\mathcal{L}_\Pi \subseteq \mathcal{L}_{\Pi^*}$. Then we will produce a subset \mathcal{H}^* of \mathcal{L}_{Π^*} such that $\mathcal{H} \subseteq \mathcal{H}^*$ and such that, relative to the new logic Λ_{Π^*} , \mathcal{H}^* is maximally consistent and universal. Finally, we will apply the foregoing argument to prove that \mathcal{H}^* is satisfiable. Of course, it would follow that \mathcal{H} is satisfiable as well.

22° The idea of the plan is simple but the implementation of it requires careful organization of the new constant symbols.

23° Let us implement the plan. Let \mathcal{C}^* be the set consisting of all (!) constant symbols:

$$c_1, c_2, \dots, c_n, \dots$$

Let Π^* be the preamble:

$$\Pi^* = (\mathcal{C}^*, \mathcal{F}, \mathcal{P})$$

and let Λ^* be the predicate logic defined by Π^* :

$$\Lambda^* = (\mathcal{L}^*, \mathcal{A}^*)$$

Of course, we can present \mathcal{C}^* as an indexed array:

$$c_{k\ell m}$$

without repetitions, where k , ℓ , and m run through all positive integers. For each positive integer j , let \mathcal{D}_j be the set of all constant symbols of the form:

$$c_{j\ell m}$$

where ℓ and m run through all positive integers. For each positive integer k , let:

$$\mathcal{C}_k = \bigcup_{j=1}^k \mathcal{D}_j$$

Let Π_k be the preamble:

$$\Pi_k = (\mathcal{C}_k, \mathcal{F}, \mathcal{P})$$

and let Λ_k be the predicate logic defined by Π_k :

$$\Lambda_k = (\mathcal{L}_k, \mathcal{A}_k)$$

Of course, the sequences:

$$\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_k, \dots; \quad \mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_k, \dots$$

are increasing, and:

$$\mathcal{C}^* = \bigcup_{k=1}^{\infty} \mathcal{C}_k; \quad \mathcal{L}^* = \bigcup_{k=1}^{\infty} \mathcal{L}_k$$

Without loss of generality, we can identify \mathcal{C} with a subset of \mathcal{D}_1 , so that:

$$\mathcal{L}_{\Pi} \subseteq \mathcal{L}_1$$

24° Let us denote \mathcal{H} by \mathcal{H}_1 . By **(CP)**, \mathcal{H}_1 is consistent with respect to Λ_1 . We intend to generate an increasing sequence:

$$\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3, \mathcal{H}_4, \dots$$

of subsets of \mathcal{L}^* such that, for each positive integer k , \mathcal{H}_k is a subset of \mathcal{L}_k and such that the union:

$$\mathcal{H}^* = \bigcup_{k=1}^{\infty} \mathcal{H}_k$$

is maximally consistent and universal with respect to Λ^* . We proceed by induction.

25° Let k be a positive integer. Let \mathcal{H}_k be a subset of \mathcal{L}_k such that \mathcal{H}_k is consistent with respect to Λ_k . Let us make a list, without repetitions, of the sentences in \mathcal{L}_k :

$$\alpha_1, \alpha_2, \alpha_3, \dots$$

Let $\mathcal{K}_0 = \mathcal{H}_k$ and let:

$$\mathcal{K}_0, \mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3, \dots$$

be the increasing sequence of subsets of \mathcal{L}_k defined, by induction, as follows:

$$\mathcal{K}_j = \begin{cases} \mathcal{K}_{j-1} & \text{if } Inc(\mathcal{K}_{j-1} \cup \{\alpha_j\}) \\ \mathcal{K}_{j-1} \cup \{\alpha_j\} & \text{if } Con(\mathcal{K}_{j-1} \cup \{\alpha_j\}) \end{cases}$$

Let:

$$\mathcal{H}_k^\circ = \bigcup_{j=0}^{\infty} \mathcal{K}_j$$

Obviously, $\mathcal{H}_k \subseteq \mathcal{H}_k^\circ$. One can easily check that \mathcal{H}_k° is maximally consistent with respect to Λ_k . In turn, let us form the list, in natural order, of the variable symbols in \mathcal{V} :

$$x_1, x_2, \dots, x_\ell, \dots$$

and let us make a list, without repetitions, of the sentences in \mathcal{H}_k° :

$$\beta_1, \beta_2, \dots, \beta_m, \dots$$

Let \mathcal{H}_k^\bullet be the subset of \mathcal{L}_{k+1} consisting of all sentences of the form:

$$\beta_{\ell m} = (\beta_m(c_{(k+1)\ell m}(x_\ell) \longrightarrow (\forall x_\ell)\beta_m))$$

where ℓ and m run through all positive integers. Finally, let us introduce \mathcal{H}_{k+1} :

$$\mathcal{H}_{k+1} = \mathcal{H}_k^\circ \cup \mathcal{H}_k^\bullet$$

26° Of course, \mathcal{H}_{k+1} is a subset of \mathcal{L}_{k+1} . To complete our inductive design, we will prove that \mathcal{H}_{k+1} is consistent with respect to Λ_{k+1} . We argue by contradiction. Let us suppose that \mathcal{H}_{k+1} is inconsistent with respect to Λ_{k+1} . Under this supposition, we may introduce a finite subset \mathcal{E} of \mathcal{H}_k^\bullet and a sentence $\bar{\beta}$ in $\mathcal{H}_k^\circ \setminus \mathcal{E}$ such that $\mathcal{H}_k^\circ \cup \mathcal{E}$ is consistent with respect to Λ_{k+1} while

$\mathcal{H}_k^\circ \cup \mathcal{E} \cup \{\bar{\beta}\}$ is inconsistent with respect to Λ_{k+1} . Of course, $\bar{\beta}$ must stand in the form:

$$\bar{\beta} = (\beta(\bar{\chi}|\zeta) \longrightarrow (\forall\zeta)\beta)$$

where β is a formula in \mathcal{H}_k° , where ζ is a variable symbol in \mathcal{V} , and where $\bar{\chi}$ is the appropriate constant symbol in \mathcal{D}_{k+1} . By **(RA)**:

$$\mathcal{H}_k^\circ \cup \mathcal{E} \Vdash \neg(\beta(\bar{\chi}|\zeta) \longrightarrow (\forall\zeta)\beta)$$

with respect to Λ_{k+1} . By **(SI₅)**:

$$\mathcal{H}_k^\circ \cup \mathcal{E} \Vdash \beta(\bar{\chi}|\zeta), \neg(\forall\zeta)\beta$$

with respect to Λ_{k+1} . Now let us invoke **(RC)**, the Remote Constant Principle. See article 4.10° in Chapter 2. To be precise, we interpret the set \mathcal{C}' of constant symbols to be the set \mathcal{C}_k augmented by the constant symbols in \mathcal{D}_{k+1} which occur in the sentences composing \mathcal{E} . In turn, we interpret the set \mathcal{C}'' of constant symbols to be $\mathcal{C}' \cup \{\bar{\chi}\}$. Now, by **(RC)**, we obtain:

$$\mathcal{H}_k^\circ \cup \mathcal{E} \Vdash (\forall\zeta)\beta, \neg(\forall\zeta)\beta$$

with respect to Λ_{k+1} . We infer that $\mathcal{H}_k^\circ \cup \mathcal{E}$ is inconsistent with respect to Λ_{k+1} , contrary to our initial supposition. We conclude that \mathcal{H}_{k+1} is consistent with respect to Λ_{k+1} .

27° By induction, we form the increasing sequence:

$$\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3, \mathcal{H}_4, \dots$$

of subsets of \mathcal{L}^* . Of course, for each positive integer k , $\mathcal{H}_k \subseteq \mathcal{H}_k^\circ \subseteq \mathcal{H}_{k+1}$. Let:

$$\mathcal{H}^* = \bigcup_{k=1}^{\infty} \mathcal{H}_k = \bigcup_{k=1}^{\infty} \mathcal{H}_k^\circ$$

We contend that, with respect to the logic Λ^* , \mathcal{H}^* is maximally consistent and universal. To prove the contention, we require conditions (H_4) and (H_7) . See articles 04° and 06°.

28° For each positive integer k , \mathcal{H}_k is consistent with respect to Λ_k . By **(CP)**, \mathcal{H}_k is consistent with respect to Λ^* . By observations now familiar, we infer that \mathcal{H}^* is consistent with respect to Λ^* . Let δ be any formula in \mathcal{L}^* . Of course, there is a positive integer k such that δ is a member of \mathcal{H}_k° . By design, \mathcal{H}_k° is maximally consistent with respect to Λ_k . By (H_4) , $\neg\delta \in \mathcal{H}_k^\circ$ or $\delta \in \mathcal{H}_k^\circ$. Therefore, $\neg\delta \in \mathcal{H}^*$ or $\delta \in \mathcal{H}^*$. We conclude that, with respect to Λ^* , \mathcal{H}^* is maximally consistent.

29° Let δ be any formula in \mathcal{H}^* and let ζ be any variable symbol in \mathcal{V} . Let us assume that $(\forall\zeta)\delta \notin \mathcal{H}^*$. Of course, there is a positive integer k such that $\delta \in \mathcal{H}_k^\circ$. By design, there is a constant symbol χ in \mathcal{D}_{k+1} such that the formula:

$$(\delta(\chi|\zeta) \longrightarrow (\forall\zeta)\delta)$$

is contained in \mathcal{H}_k^\bullet . Of course, it must be contained in \mathcal{H}_{k+1}° as well. By design, \mathcal{H}_{k+1}° is maximally consistent with respect to Λ_{k+1} . Obviously, $(\forall\zeta)\delta \notin \mathcal{H}_{k+1}^\circ$. By (H_7) , $\neg\delta(\chi|\zeta) \in \mathcal{H}_{k+1}^\circ$, so that $\neg\delta(\chi|\zeta) \in \mathcal{H}^*$. Consequently, $\delta(\chi|\zeta) \notin \mathcal{H}^*$. We conclude that \mathcal{H}^* is universal. \ddagger

Skolem

30° In the context just described, the universe Ω underlying the interpretation I is countable (that is, finite or countably infinite). In fact, it is a quotient of the countably infinite set \mathcal{T}_Π . This simple observation presents a first impression of the Theorem of Skolem:

$$(\mathbf{SK}) \quad \text{Con}(\mathcal{H}) \implies \text{Sat}_\circ(\mathcal{H})$$

Informally, it asserts that if \mathcal{H} is consistent then \mathcal{H} admits a “countable interpretation.”

3.4 COMPLETENESS/COMPACTNESS

The Completeness Theorem

01° Let \mathcal{H} be any subset of \mathcal{L}_Π and let δ any sentence in \mathcal{L}_Π . We contend that \mathcal{H} syntactically implies δ iff \mathcal{H} semantically implies δ :

$$(\mathbf{CT}) \quad \mathcal{H} \Vdash \delta \iff \mathcal{H} \models \delta$$

Consequently:

$$\Theta_\Pi(\mathcal{H}) = T_\Pi(\mathcal{H})$$

We refer to this central result as the Completeness Theorem.

02° To prove **(CT)**, we argue as follows. Directly, let us assume that $\mathcal{H} \Vdash \delta$. By **(ST)**, $\mathcal{H} \models \delta$. Conversely, let us assume that $\mathcal{H} \not\models \delta$. By **(RA)**, $\mathcal{H} \cup \{(\neg\delta)\}$ is consistent. By **(IT)**, $\mathcal{H} \cup \{(\neg\delta)\}$ is satisfiable, so there are an interpretation I of Π and an assignment A for \mathcal{V} in the universe Ω underlying I such that, for any sentence α in \mathcal{H} , $I_A(\alpha) = 1$ and such that $I_A(\neg\delta) = 1$ as well. Hence, $\mathcal{H} \not\models \delta$. \ddagger

Tautologies

03° It may happen that the set \mathcal{H} of hypotheses is empty: $\mathcal{H} = \emptyset$. In that case, **(CT)** states that, for any sentence δ in \mathcal{L}_Π :

$$\emptyset \Vdash \delta \iff \emptyset \models \delta$$

By definition, $\emptyset \models \delta$ iff:

(T_1) for any interpretation I of Π and for any assignment A for \mathcal{V} in the universe Ω underlying I , $I_A(\delta) = 1$

Just as well, $\emptyset \models \delta$ iff:

(T_2) for any interpretation I of Π , $I(\forall\delta) = 1$

Under condition (T_1) and/or (T_2), we say that δ is a *tautology*. We also say that δ is *valid* or even that δ is true in “all possible worlds.”

The Compactness Theorem

04° Let \mathcal{H} be any subset of \mathcal{L}_Π . We contend that $Sat(\mathcal{H})$ iff, for every finite subset \mathcal{F} of \mathcal{H} , $Sat(\mathcal{F})$:

$$\mathbf{(KT)} \quad Sat(\mathcal{H}) \iff (\forall \mathcal{F}) [(\mathcal{F} \subseteq \mathcal{H} \wedge Fin(\mathcal{F})) \implies Sat(\mathcal{F})]$$

This remarkable result is called the Compactness Theorem.

05° To prove **(KT)**, we note that, for any deductive tree \mathcal{G} from \mathcal{H} , the set $\tilde{\mathcal{H}}$ consisting of all material hypotheses for \mathcal{G} is finite. Hence, \mathcal{H} is consistent iff, for every finite subset \mathcal{F} of \mathcal{H} , \mathcal{F} is consistent. Now **(IT)** implies **(KT)**. \natural

NonStandard Interpretations

06° We have taken care to design a proof of the Interpretation Theorem which applies to arbitrary (consistent) sets \mathcal{H} of hypotheses, even those that contain open sentences. Of course, the Completeness Theorem and the Compactness Theorem inherit the same level of generality. We will find that, in context of the Predicate Logic for Arithmetic, the Compactness Theorem yields interpretations of Arithmetic for which the underlying universe contains “infinite numbers.” One refers to such interpretations as NonStandard.

3.5 EQUIVALENCE/ISOMORPHISM

Models

01° Let I be an interpretation of Π and let A be an assignment for \mathcal{V} with values in the universe Ω underlying I . Let:

$$\mathcal{T}(I, A)$$

be the subset of \mathcal{L}_Π consisting of all formulae δ such that $I_A(\delta) = 1$, which is to say that, relative to I and A , δ is true.

02• Review articles 3.04° and 3.06°. Then prove that $\mathcal{T}(I, A)$ is a maximally consistent subset of \mathcal{L}_Π .

03° In turn, let \mathcal{H} be any subset of \mathcal{L}_Π . We say that (I, A) is a *model* for \mathcal{H} iff:

$$\mathcal{H} \subseteq \mathcal{T}(I, A)$$

04• Let $\bar{\mathcal{H}}$ be a maximally consistent subset of \mathcal{L}_Π . Show that there is a model (I, A) for $\bar{\mathcal{H}}$ such that $\mathcal{T}(I, A) = \bar{\mathcal{H}}$.

Equivalence/Isomorphism

05° Let I' and I'' be any interpretations of Π and let A' and A'' be any assignments for \mathcal{V} with values in the universes Ω' and Ω'' for I' and I'' , respectively. We say that the models (I', A') and (I'', A'') are *equivalent* iff:

$$\mathcal{T}(I', A') = \mathcal{T}(I'', A'')$$

Let us display the component parts of I' and I'' , respectively, as follows:

$$I' = (\Omega', (C', F', P')), \quad I'' = (\Omega'', (C'', F'', P''))$$

We say that (I', A') and (I'', A'') are *isomorphic* iff there exists a bijective mapping H carrying Ω' to Ω'' such that:

$$(IS_1) \quad C'' = H.C'$$

$$(IS_2) \quad A'' = H.A'$$

$$(IS_3) \quad F'' = H.F'$$

$$(IS_4) \quad P'' = H.P'$$

Condition (IS_1) means that, for each χ in \mathcal{C} :

$$H(C'(\chi)) = C''(\chi)$$

Condition (IS_2) means that, for each ζ in \mathcal{V} :

$$H(A'(\zeta)) = A''(\zeta)$$

Condition (IS_3) means that, for each ϕ in \mathcal{F} and for any $\omega_1, \omega_2, \dots$, and ω_k in Ω' :

$$H(F'(\phi)(\omega_1, \omega_2, \dots, \omega_k)) = F''(\phi)(H(\omega_1), H(\omega_2), \dots, H(\omega_k))$$

Condition (IS_4) means that, for each ρ in \mathcal{P} and for any $\omega_1, \omega_2, \dots$, and ω_ℓ in Ω' :

$$(\omega_1, \omega_2, \dots, \omega_\ell) \in P'(\rho) \text{ iff } (H(\omega_1), H(\omega_2), \dots, H(\omega_\ell)) \in P''(\rho)$$

Of course, $k = v(\phi)$ and $\ell = v(\rho)$.

06• Show that if (I', A') and (I'', A'') are isomorphic then they are equivalent. Should one expect that the converse be true?

Closed Sentences

07° Very often, one chooses to restrict the foregoing developments to closed sentences. One replaces \mathcal{L}_Π by \mathcal{L}_Π^0 and one considers just those subsets \mathcal{H} of \mathcal{L}_Π which are subsets of \mathcal{L}_Π^0 . One suppresses all references to assignments A . One is led to replace (I, A) by I and $\mathcal{T}(I, A)$ by $\mathcal{T}(I)$. The adjustments to the definitions of equivalence and isomorphism are obvious.

An Example

08• With reference to article 5.06° in Chapter 2, let us recover the Predicate Logic for Abstract Lines:

$$\Lambda_L = (\mathcal{L}_L, \mathcal{A}_L)$$

defined by the preamble Π_L . Let \mathcal{H}_L be the set of hypotheses underlying the syntactic Theory of Abstract Lines. Of course, \mathcal{H}_L is a closed set of hypotheses. Let I' and I'' be interpretations of \mathcal{H}_L and let Ω' and Ω'' be the underlying universes. Show that if I' and I'' are models of \mathcal{H}_L :

$$\mathcal{H}_L \subseteq \mathcal{T}(I') \cap \mathcal{T}(I'')$$

and if Ω_1 and Ω_2 are countable then I' and I'' are isomorphic. To do so, you might proceed as follows:

- (1) Identify Ω' with the set of dyadic rationals.
- (2) Interpret $<$ in the usual manner.
- (3) Display the members of Ω'' as a sequence of distinct elements.
- (4) Define the required isomorphism carrying Ω'' to Ω' by induction.

CHAPTER 4

ARITHMETIC

In this chapter, we introduce the predicate logic:

$$\Lambda_a = (\mathcal{L}_a, \mathcal{A}_a)$$

for Arithmetic. We will develop just enough of the theory of Arithmetic to support the celebrated Representation Theorem, which connects the relation of syntactic implication with the property of decidability for subsets of the Natural Numbers. In turn, this connection figures in the proofs of the basic theorems of Tarski, Gödel, and Church.

4.1 THE PREDICATE LOGIC FOR ARITHMETIC

The Predicate Logic for Arithmetic

01° Let Π_a be the preamble:

$$\Pi_a = (\mathcal{C}_a, \mathcal{F}_a, \mathcal{P}_a)$$

defined as follows:

$$\mathcal{C}_a = \{\bar{0}, \bar{1}\}, \quad \mathcal{F}_a = \{+, \times\}, \quad \mathcal{P}_a = \{\equiv\}$$

where $\bar{0}$ and $\bar{1}$ stand for selected constant symbols:

$$\bar{0} = (c|), \quad \bar{1} = (c||)$$

and where $+$ and \times stand for selected function symbols:

$$+ = (||f|), \quad \times = (||f||)$$

Of course, the latter have valence 2. We refer to $\bar{0}$ and $\bar{1}$ as the *zero* symbol and *one* symbol, respectively, and to $+$ and \times as the *addition* symbol and *multiplication* symbol, respectively. As usual, we write:

$$\begin{aligned}
&(\tau_1 + \tau_2) \\
&(\tau_1 \times \tau_2) \\
&(\tau_1 \equiv \tau_2) \\
&(\tau_1 \not\equiv \tau_2)
\end{aligned}$$

instead of:

$$\begin{aligned}
&(+ \tau_1 \tau_2) \\
&(\times \tau_1 \tau_2) \\
&(\equiv \tau_1 \tau_2) \\
&(\neg(\equiv \tau_1 \tau_2))
\end{aligned}$$

respectively, where τ_1 and τ_2 are any terms in \mathcal{T}_a .

02° Let \mathcal{L}_a be the predicate language defined by Π_a and let \mathcal{A}_a be the corresponding set of axioms. Let Λ_a be the predicate logic defined by Π_a :

$$\Lambda_a = (\mathcal{L}_a, \mathcal{A}_a)$$

We refer to Λ_a as the Predicate Logic for Arithmetic.

The Set \mathcal{H}_a of Hypotheses

03° Let \mathcal{H}_a be the subset of \mathcal{L}_a composed of all sentences of any one of the following forms, called Hypothesis Schemes:

$$\begin{aligned}
&\forall ((\zeta + \eta) \equiv (\eta + \zeta)) \\
&\forall ((\zeta \times \eta) \equiv (\eta \times \zeta)) \\
&\forall (((\zeta + \eta) + \theta) \equiv (\zeta + (\eta + \theta))) \\
&\forall (((\zeta \times \eta) \times \theta) \equiv (\zeta \times (\eta \times \theta))) \\
&\forall ((\zeta \times (\eta + \theta)) \equiv ((\zeta \times \eta) + (\zeta \times \theta))) \\
(\mathcal{H}_a) \quad &\forall ((\zeta + \bar{0}) \equiv \zeta) \\
&\forall ((\zeta \times \bar{1}) \equiv \zeta) \\
&\forall (((\zeta + \theta) \equiv (\eta + \theta)) \longrightarrow (\zeta = \eta)) \\
&\forall (((\zeta \times \eta) \equiv \bar{0}) \longrightarrow ((\zeta \equiv \bar{0}) \vee (\eta \equiv \bar{0}))) \\
&\forall ((\alpha(\bar{0}|\zeta) \wedge ((\forall \zeta)(\alpha \longrightarrow \alpha((\zeta + \bar{1})|\zeta)))) \longrightarrow ((\forall \zeta)\alpha))
\end{aligned}$$

where ζ , η , and θ are any variable symbols in \mathcal{V} and where α is any sentence in \mathcal{L}_a . By design, \mathcal{H}_a is closed. The last of the foregoing schemes is called the Mathematical Induction Scheme.

04° For the closed terms:

$$(\bar{1} + \bar{1}), ((\bar{1} + \bar{1}) + \bar{1}), (((\bar{1} + \bar{1}) + \bar{1}) + \bar{1}), (((((\bar{1} + \bar{1}) + \bar{1}) + \bar{1}) + \bar{1}) + \bar{1}), \dots$$

in \mathcal{L}_a , we introduce the familiar abbreviations:

$$\bar{2}, \bar{3}, \bar{4}, \bar{5}, \dots$$

That is, for any nonnegative integer n :

$$\overline{n+1} = (\bar{n} + \bar{1})$$

05° We denote by:

$$\Theta_a(\mathcal{H}_a)$$

the subset of \mathcal{L}_a consisting of all sentences δ such that \mathcal{H}_a syntactically implies δ :

$$\mathcal{H}_a \Vdash \delta$$

We refer to $\Theta_a(\mathcal{H}_a)$ as the *syntactic theory* of Arithmetic. It contains a legion of sentences expressing, under suitable interpretation, the familiar facts of Arithmetic.

Mathematical Induction

06° The Mathematical Induction Scheme yields the following implication, itself called the Mathematical Induction Principle:

$$(\mathbf{MI}) \quad \mathcal{H}_a \Vdash \alpha(\bar{0}|\zeta), (\forall\zeta) \left(\alpha \longrightarrow \alpha((\zeta + \bar{1})|\zeta) \right) \implies \mathcal{H}_a \Vdash (\forall\zeta) \alpha$$

where α is any sentence in \mathcal{L}_a and where ζ is any variable symbol in \mathcal{V} . In practice, we will introduce applications of the Mathematical Induction Principle simply by mentioning the abbreviation **(MI)**.

Polynomial Equations

07° Terms in \mathcal{T}_a are polynomials with nonnegative integer coefficients. For instance:

$$\bar{6} \times \zeta \times \eta \times \eta + \bar{11} \times \theta \times \theta \times \theta$$

Atomic sentences in \mathcal{L}_a are polynomial equations with arbitrary integer coefficients. For instance:

$$\bar{2} \times \zeta \times \zeta \times \eta \equiv \bar{7} \times \theta \times \theta \times \theta \times \theta$$

One might write:

$$\bar{2} \times \zeta \times \zeta \times \eta + (-\bar{7}) \times \theta \times \theta \times \theta \times \theta \equiv \bar{0}$$

Order

08° For convenience, we introduce two new relation symbols by the following abbreviations:

$$\begin{aligned} (\zeta < \eta) & \quad \text{for} \quad (\exists \theta)((\zeta + \theta + \bar{1}) \equiv \eta) \\ (\zeta \leq \eta) & \quad \text{for} \quad (\exists \theta)((\zeta + \theta) \equiv \eta) \end{aligned}$$

The following syntactic implications are evident:

$$\begin{aligned} \mathcal{H}_a & \Vdash \forall((\zeta \leq \eta) \longleftrightarrow ((\zeta < \eta) \vee (\zeta \equiv \eta))) \\ \mathcal{H}_a & \Vdash \forall(\neg(\zeta < \zeta)) \\ \mathcal{H}_a & \Vdash \forall(((\zeta < \eta) \wedge (\eta < \theta)) \longrightarrow (\zeta < \theta)) \\ \mathcal{H}_a & \Vdash \forall((\zeta < \eta) \vee (\zeta \equiv \eta) \vee (\eta < \zeta)) \\ \mathcal{H}_a & \Vdash (\bar{0} < \bar{1}) \\ \mathcal{H}_a & \Vdash \forall((\zeta < \eta) \longrightarrow ((\zeta + \theta) < (\eta + \theta))) \\ \mathcal{H}_a & \Vdash \forall(((\zeta < \eta) \wedge (\bar{0} < \theta)) \longrightarrow ((\zeta \times \theta) < (\eta \times \theta))) \end{aligned}$$

Complete Induction/The Least Integer Principle

09° From **(MI)**, we obtain the following implication, called Complete Induction:

$$\textbf{(CI)} \quad \mathcal{H}_a \Vdash (\forall \zeta) \left((\forall \eta) \left((\eta < \zeta) \longrightarrow \alpha(\eta|\zeta) \right) \longrightarrow \alpha \right) \implies \mathcal{H}_a \Vdash (\forall \zeta) \alpha$$

We also obtain the Least Integer Principle:

$$\begin{aligned} \textbf{(LI)} \quad & \mathcal{H}_a \Vdash (\exists \zeta) \alpha \\ \implies & \mathcal{H}_a \Vdash (\exists \eta) \left(\alpha(\eta|\zeta) \wedge \left((\forall \theta) ((\theta < \eta) \longrightarrow \neg(\alpha(\theta|\zeta))) \right) \right) \end{aligned}$$

Of course, we intend that the variable symbols η and θ be free for ζ in α .

10° Now one may prove the following medley of syntactic implications:

$$\begin{aligned}
\mathcal{H}_a &\Vdash (\forall \zeta)(\forall \eta)((\bar{0} < \eta) \longrightarrow (\zeta < (\zeta + \eta))) \\
\mathcal{H}_a &\Vdash (\forall \zeta)((\zeta \equiv \bar{0}) \vee (\bar{0} < \zeta)) \\
\mathcal{H}_a &\Vdash (\forall \zeta)((\zeta \equiv \bar{0}) \vee (\zeta \equiv \bar{1}) \vee (\bar{1} < \zeta)) \\
\mathcal{H}_a &\Vdash (\forall \zeta)((\zeta \equiv \bar{0}) \vee (\zeta \equiv \bar{1}) \vee (\zeta \equiv \bar{2}) \vee (\bar{2} < \zeta)) \\
&\vdots \\
\mathcal{H}_a &\Vdash (\forall \zeta)(\forall \eta)((\zeta < \eta) \longrightarrow (\exists \theta)((\bar{0} < \theta) \wedge (\eta \equiv (\zeta + \theta)))) \\
\mathcal{H}_a &\Vdash (\forall \zeta)(\forall \eta)((\exists \theta)((\bar{0} < \theta) \wedge (\eta \equiv (\zeta + \theta))) \longrightarrow (\zeta < \eta)) \\
\mathcal{H}_a &\Vdash (\forall \zeta)((\bar{0} < \zeta) \longrightarrow (\exists \eta)((\eta + \bar{1}) \equiv \zeta)) \\
\mathcal{H}_a &\Vdash (\forall \zeta)(\forall \eta)((\zeta < \eta) \longrightarrow \neg(\eta < (\zeta + \bar{1})))
\end{aligned}$$

In turn, one may show that, for any nonnegative integers m and n :

$$\begin{aligned}
\mathcal{H}_a &\Vdash (\overline{m + n} \equiv (\bar{m} + \bar{n})) \\
\mathcal{H}_a &\Vdash (\overline{m \times n} \equiv (\bar{m} \times \bar{n}))
\end{aligned}$$

Moreover:

$$m < n \iff \mathcal{H}_A \Vdash (\bar{m} < \bar{n})$$

We will call upon these results as needed.

4.2 THE STANDARD INTERPRETATION

The Standard Interpretation

01° By the Standard Interpretation of Π_a , we mean the ordered pair:

$$\mathbf{I} = (\mathbf{N}, S)$$

where \mathbf{N} is the set of natural numbers and where S is the ordered triple:

$$S = (C, F, P)$$

for which C , F , and P are the mappings having domains \mathcal{C} , \mathcal{F} , and \mathcal{P} , respectively, and assigning values as follows:

- (I_1) $\mathbf{C}(\bar{0}) = 0$ and $\mathbf{C}(\bar{1}) = 1$ are the neutral elements in \mathbf{N} for the operations of addition and multiplication on \mathbf{N} , respectively
- (I_2) $\mathbf{F}(+)$ and $\mathbf{F}(\times)$ are the operations $+$ and \times themselves
- (I_3) and, as usual, $\mathbf{P}(\equiv)$ is the equality relation $=$ on \mathbf{N}

Let \mathbf{V} be the set of all assignments for \mathcal{V} with values in \mathbf{N} . Following Section 3.1, we may introduce the joint valuation mapping \mathbf{J} carrying the set:

$$\mathbf{V} \times \mathcal{L}_a$$

to the set:

$$\mathcal{Z} = \{0, 1\}$$

02•

03° One can easily check that, for any nonnegative integer n :

$$S(\bar{n}) = n$$

04° For each sentence δ in \mathcal{H}_a , $\mathbf{I}_a(\delta) = 1$. Hence, \mathbf{I}_a is a model for \mathcal{H}_a . We refer to \mathbf{I}_a as the Standard Model for the Natural Numbers. We denote by:

$$T(\mathbf{I}_a)$$

the subset of \mathcal{L}_a consisting of all sentences δ in \mathcal{L}_a for which $\mathbf{I}_a(\forall\delta) = 1$. By the Soundness Theorem:

$$\Theta_a(\mathcal{H}_a) \subseteq T(\mathbf{I}_a)$$

NonStandard Interpretations

05°

CHAPTER 5

RECURSIVE MAPPINGS

CHAPTER 6

TARSKI, GÖDEL, AND CHURCH

For now, let us append a rough sketch of the theorems of Tarski, Gödel, and Church.

6.1 TARSKI, GÖDEL, AND CHURCH

Introduction

06° Let Π_A be the preamble for Arithmetic and let $\Lambda_A = (\mathcal{L}_A, \mathcal{A}_A)$ be the corresponding predicate logic. Let \mathcal{H}_A be a set of hypotheses adequate for the syntactic theory of Arithmetic. In turn, let \mathbf{N} be the set of natural numbers and let \mathbf{I} be the standard interpretation of Arithmetic. By definition, \mathbf{N} is the universe underlying \mathbf{I} . We plan to describe the Theorems of Tarski, Gödel, and Church, bearing upon the Incompleteness of Arithmetic.

Gödel Numbers

07° Let Σ be the symbol set for the various predicate logics:

$$(\ , \), \neg, \longrightarrow, \forall, \exists, c, x, f, r, |, \langle, \rangle$$

and let Σ^* be the set of finite strings of symbols drawn from Σ . Relative to the displayed linear ordering of the twelve symbols in Σ , let us introduce the corresponding degree-lexicographic ordering \prec on Σ^* . For instance:

$$\langle \rangle fr \prec (\forall ccc \text{ and } \neg \neg fxc) \prec \neg \neg f \rangle (($$

Let Γ be the order isomorphism carrying Σ^* to \mathbf{N} , defined by the base twelve presentation of natural numbers in \mathbf{N} , subject to the condition that the digits shall run not from 0 to 11 but from 1 to 12. For instance:

$$\Gamma((x|)|) = 1 \cdot 12^4 + 7 \cdot 12^3 + 10 \cdot 12^2 + 12 \cdot 12^1 + 10 \cdot 12^0 = 34426$$

For each string α in Σ^* , we refer to $\Gamma(\alpha)$ as the *Gödel number* of α .

Notation

08° Let k be a positive integer. Let α be a sentence in \mathcal{L}_A^k . Let \mathcal{V}_α be the set consisting of the variable symbols which occur at least once freely in α . By definition, \mathcal{V}_α contains k members. When useful, we will emphasize the relation between α and the variable symbols in \mathcal{V}_α by writing α in *functional form*:

$$\alpha(\zeta_1, \zeta_2, \dots, \zeta_k) \quad \text{for} \quad \alpha$$

The variable symbols shall appear in natural order. Moreover, for any terms τ_1, τ_2, \dots , and τ_k , we will write:

$$\alpha(\tau_1, \tau_2, \dots, \tau_k) \quad \text{for} \quad \alpha(\tau_1|\zeta_1)(\tau_2|\zeta_2) \cdots (\tau_k|\zeta_k)$$

Truth

09° Let \mathcal{L}_A^t be the subset of \mathcal{L}_A consisting of all sentences β for which:

$$\mathbf{I}(\forall\beta) = 1$$

It is the same to say that, relative to \mathbf{I} , $\forall\beta$ is *true*. Let:

$$\mathbf{T} = \Gamma(\mathcal{L}_A^t)$$

We refer to \mathbf{T} as the *truth set* for the standard interpretation \mathbf{I} of Arithmetic.

Proof

10° Let \mathcal{L}_A^p be the subset of \mathcal{L}_A consisting of all sentences β for which:

$$\mathcal{H}_A \Vdash \forall\beta$$

Let:

$$\mathbf{P} = \Gamma(\mathcal{L}_A^p)$$

We refer to \mathbf{P} as the *proof set* for the theory of Arithmetic.

Semantically Definable Sets

11° Let T be any subset of \mathbf{N} . We say that T is *semantically definable* iff there is a sentence $\alpha(\zeta)$ in \mathcal{L}_A^1 such that, for each natural number j :

$$j \in T \iff \mathbf{I}(\alpha(\bar{j})) = 1$$

where \bar{j} is the constant term corresponding to j .

12° Let W be any subset of $\mathbf{N} \times \mathbf{N}$. We say that W is *semantically definable* iff there is a sentence $\delta(\eta, \theta)$ in \mathcal{L}_A^2 such that, for all ordered pairs (k, ℓ) of natural numbers:

$$(k, \ell) \in W \iff \mathbf{I}(\delta(\bar{k}, \bar{\ell})) = 1$$

Syntactically Definable Sets

13° Let T be any subset of \mathbf{N} . We say that T is *syntactically definable* iff there is a sentence $\alpha(\zeta)$ in \mathcal{L}_A^1 such that, for each natural number j :

- (1) $j \in T \implies \mathcal{H}_A \Vdash \alpha(\bar{j})$
- (2) $j \notin T \implies \mathcal{H}_A \Vdash \neg\alpha(\bar{j})$

14° Let W be any subset of $\mathbf{N} \times \mathbf{N}$. We say that W is *syntactically definable* iff there is a sentence $\delta(\eta, \theta)$ in \mathcal{L}_A^2 such that, for any ordered pair (k, ℓ) of natural numbers:

- (1) $(k, \ell) \in W \implies \mathcal{H}_A \Vdash \delta(\bar{k}, \bar{\ell})$
- (2) $(k, \ell) \notin W \implies \mathcal{H}_A \Vdash \neg\delta(\bar{k}, \bar{\ell})$

It may happen that W is the graph of a mapping D carrying \mathbf{N} to \mathbf{N} . In such a case, we claim that W is syntactically definable iff there is a sentence $\bar{\delta}(\eta, \theta)$ in \mathcal{L}_A^2 such that, for any natural number k :

$$(3) \quad \mathcal{H}_A \Vdash (\forall\theta)(\bar{\delta}(\bar{k}, \theta) \longleftrightarrow (\overline{D(\bar{k})} \equiv \theta))$$

For the proof of the claim, see article 30°.

A Basic Implication

15° By the Soundness Theorem, it is plain that Syntactically Definable sets are Semantically Definable.

The Diagonalization Theorem

16° For each sentence $\alpha(\zeta)$ in \mathcal{L}_A^1 , let $a = \Gamma(\alpha)$ and let $\bar{\alpha} = \alpha(\bar{a})$. Let Δ° be the mapping carrying Σ^* to itself, defined as follows:

$$\Delta^\circ(\alpha) = \begin{cases} \epsilon & \text{if } \alpha \notin \mathcal{L}_A^1 \\ \bar{\alpha} & \text{if } \alpha \in \mathcal{L}_A^1 \end{cases}$$

We refer to Δ° as the Diagonalization Mapping. Let D° be the corresponding mapping carrying \mathbf{N} to \mathbf{N} , defined by conjugation of Δ° by Γ as follows:

$$D^\circ = \Gamma \cdot \Delta^\circ \cdot \Gamma^{-1}$$

Let W° be the graph of D° , a subset of $\mathbf{N} \times \mathbf{N}$. We contend that D° is recursive. We refer to this basic fact as the Diagonalization Theorem. It follows, in turn, that W° is decidable.

17° To prove our contention, we introduce the recursive mappings λ carrying \mathbf{N} to \mathbf{N} and ρ carrying $\mathbf{N} \times \mathbf{N}$ to \mathbf{N} such that, for each m in \mathbf{N} , if $m \neq 0$ then:

$$m = \sum_{k=0}^{\ell} \rho(m, k) 12^k \quad (\ell = \lambda(m), 1 \leq \rho(m, k) \leq 12)$$

In this way, we set the base for computing Γ^{-1} . Then, by relentless analysis, one may proceed to prove that D° is recursive. See article 31°.

18° To show that W° is decidable, we display the characteristic mapping for W° :

$$1_{W^\circ}(k, \ell) = 1 \ominus |\ell - D^\circ(k)|$$

where k and ℓ are any natural numbers. Clearly, 1_{W° is recursive.

The Representation Theorem

19° Let T be any subset of \mathbf{N} . We contend that if T is decidable then T is syntactically definable. In turn, let W be any subset of $\mathbf{N} \times \mathbf{N}$. We contend that if W is decidable then W is syntactically definable. We refer to these fundamental facts as the Representation Theorem. For the proofs of these contentions, see article 32°.

The Fixed Point Theorem

20° Let $\alpha(\zeta)$ be any sentence in \mathcal{L}_A^1 . We contend that there is a sentence β in \mathcal{L}_A^0 such that:

$$\mathcal{H}_A \Vdash (\beta \longleftrightarrow \alpha(\bar{b}))$$

where $b = \Gamma(\beta)$. We refer to this basic fact as the strong (syntactic) form of the Fixed Point Theorem.

21° Let us prove the contention. By conjoining the Diagonalization Theorem and the Representation Theorem, we may introduce a sentence $\delta^\circ(\eta, \theta)$ in \mathcal{L}_A^2 such that $\delta^\circ(\eta, \theta)$ syntactically defines W° . We mean to say that condition (3) in article 09° is valid for the mapping D° carrying \mathbf{N} to \mathbf{N} . Without loss

of generality, we may assume that $\zeta \neq \eta$ and $\zeta \neq \theta$. Let $\gamma(\eta)$ be the sentence in \mathcal{L}_A^1 defined as follows:

$$\gamma(\eta) = (\forall \theta)(\delta^\circ(\eta, \theta) \longrightarrow \alpha(\theta))$$

Let $c = \Gamma(\gamma)$. Let β be the sentence in \mathcal{L}_A^0 defined as follows:

$$\beta = \Delta^\circ(\gamma) = \bar{\gamma} = \gamma(\bar{c}) = (\forall \theta)(\delta^\circ(\bar{c}, \theta) \longrightarrow \alpha(\theta))$$

Let $b = \Gamma(\beta)$. By definition, $D^\circ(c) = b$. By condition (3):

$$\mathcal{H}_A \Vdash (\forall \theta)(\delta^\circ(\bar{c}, \theta) \longleftrightarrow (\bar{b} \equiv \theta))$$

By elementary steps, we complete the proof:

$$\begin{aligned} \mathcal{H}_A &\Vdash ((\forall \theta)((\bar{b} \equiv \theta) \longrightarrow \alpha(\theta)) \longleftrightarrow (\forall \theta)(\delta^\circ(\bar{c}, \theta) \longrightarrow \alpha(\theta))) \\ \mathcal{H}_A &\Vdash ((\forall \theta)((\bar{b} \equiv \theta) \longrightarrow \alpha(\theta)) \longleftrightarrow \alpha(\bar{b})) \\ \mathcal{H}_A &\Vdash (\beta \longleftrightarrow \alpha(\bar{b})) \end{aligned}$$

22° Let $\alpha(\zeta)$ be any sentence in \mathcal{L}_A^1 . We contend that there is a sentence β in \mathcal{L}_A^0 such that:

$$\mathbf{I}(\beta) = 1 \iff \mathbf{I}(\alpha(\bar{b})) = 1$$

where $b = \Gamma(\beta)$. We refer to this basic fact as the weak (semantic) form of the Fixed Point Theorem.

23° To prove the contention, we need only review the foregoing argument. Of course, $\delta^\circ(\eta, \theta)$ semantically defines W° . By straightforward inspection, we find that, relative to \mathbf{I} , β is true iff $\alpha(\bar{b})$ is true.

Tarski

24° We contend that the truth set \mathbf{T} for the standard interpretation of Arithmetic is not semantically definable. This assertion is the substance of the Theorem of Tarski. To prove the contention, we argue by contradiction. Let us suppose that there is a sentence $\alpha(\zeta)$ in \mathcal{L}_A^1 which semantically defines \mathbf{T} . By the weak (semantic) form of the Fixed Point Theorem, we may introduce a sentence β in \mathcal{L}_A^0 such that β is true iff $(\neg\alpha)(\bar{b})$ is true iff $\alpha(\bar{b})$ is false, where $b = \Gamma(\beta)$. Hence:

$$b \in \mathbf{T} \iff b \notin \mathbf{T}$$

By this contradiction, we infer that \mathbf{T} is not semantically definable.

The Deduction Theorem

25° Let \mathcal{D}_A be the subset of Σ^* consisting of all strings λ which are identifiable with proper deductions from \mathcal{H}_A . For each proper deduction λ in \mathcal{D}_A , let δ_λ be the consequence of λ , a sentence in \mathcal{L}_A . Let Δ^\bullet be the mapping carrying Σ^* to itself, defined as follows:

$$\Delta^\bullet(\lambda) = \begin{cases} \epsilon & \text{if } \lambda \notin \mathcal{D}_A \\ \delta_\lambda & \text{if } \lambda \in \mathcal{D}_A \end{cases}$$

We refer to Δ^\bullet as the *Deduction Mapping*. Let D^\bullet be the corresponding mapping carrying \mathbf{N} to itself, defined by conjugation of Δ^\bullet by Γ as follows:

$$D^\bullet = \Gamma \cdot \Delta^\bullet \cdot \Gamma^{-1}$$

Let W^\bullet be the graph of D^\bullet , a subset of $\mathbf{N} \times \mathbf{N}$. We contend that D^\bullet is recursive. We refer to this basic fact as the Deduction Theorem. It follows, in turn, that W^\bullet is decidable.

26° For the proof of this contention, see article 33°. To show that W^\bullet is decidable, one need only review article 13°.

Gödel

27° Let \mathbf{P} be the proof set for the theory of Arithmetic and let \mathbf{T} be the truth set for the standard interpretation of Arithmetic. See articles 04° and 05°. We contend that:

$$\mathbf{T} \setminus \mathbf{P} \neq \emptyset$$

We may say that there exist sentences which are true, relative to the standard interpretation of Arithmetic, but not provable. This fundamental fact is the substance of the Incompleteness Theorem of Gödel.

28° By conjoining the Representation Theorem and the Deduction Theorem, we may introduce a sentence $\delta^\bullet(\eta, \theta)$ in \mathcal{L}_A^2 which semantically defines the graph W^\bullet of D^\bullet . Let γ be the sentence in \mathcal{L}_A^1 defined as follows:

$$\gamma(\theta) = (\exists \eta) \delta^\bullet(\eta, \theta)$$

We claim that γ semantically defines $\mathbf{P} \cup \{0\}$.

29° Obviously, $\mathbf{P} \cup \{0\} = \text{ran}(D^\bullet)$. To prove the claim, we argue as follows. Let ℓ be any natural number in \mathbf{N} . Of course, $\gamma(\ell) = (\exists \eta) \delta^\bullet(\eta, \ell)$. Clearly, $\gamma(\ell)$ is true iff there is some natural number k in \mathbf{N} such that $\delta^\bullet(k, \ell)$ is true. Moreover, $\delta^\bullet(k, \ell)$ is true iff $D^\bullet(k) = \ell$. Hence, $\gamma(\ell)$ is true iff $\ell \in \text{ran}(D^\bullet)$.

30° Now let us prove our contention. By the weak (semantic) form of the Fixed Point Theorem, we may introduce a sentence β in \mathcal{L}_A^0 such that β is true iff $(\neg\gamma)(\bar{b})$ is true iff $\gamma(\bar{b})$ is false, where $b = \Gamma(\beta)$. Hence:

$$b \in \mathbf{T} \iff b \notin \mathbf{P} \cup \{0\}$$

Of course, $b \neq 0$. If b were not a member of \mathbf{T} then, by the Soundness Theorem, b would not be a member of \mathbf{P} . By the foregoing equivalence, we infer that $b \in \mathbf{T} \setminus \mathbf{P}$.

Syntax versus Semantics

31° It seems interesting that the proofs of the Theorems of Tarski and Gödel depend not upon the strong (syntactic) form of the Fixed Point Theorem but upon the weak (semantic) form. However, the Theorem of Church, soon to follow, requires the full strength of the theorem.

Church

32° Let \mathbf{P} be the proof set for the theory of Arithmetic. See article 05°. We contend that \mathbf{P} is enumerable but that \mathbf{P} is not decidable. One refers to these fundamental facts as the Theorem of Church.

33° For the first contention, we argue as follows. By the Deduction Theorem, the mapping D^\bullet is recursive. It follows that the range of D^\bullet is enumerable. Of course, the range of D^\bullet is $\mathbf{P} \cup \{0\}$. Consequently:

$$\mathbf{P} = (\mathbf{P} \cup \{0\}) \setminus \{0\}$$

is enumerable.

34° For the second contention, we argue by contradiction. Let us suppose that \mathbf{P} is decidable. Let $\mathbf{Q} = \mathbf{N} \setminus \mathbf{P}$. Of course, \mathbf{Q} would be decidable. By the Representation Theorem, there would be a sentence $\alpha(\zeta)$ in \mathcal{L}_A^1 such that α syntactically defines \mathbf{Q} . That is, for each natural number k :

- (1) $k \in \mathbf{Q} \implies \mathcal{H}_A \Vdash \alpha(\bar{k})$
- (2) $k \notin \mathbf{Q} \implies \mathcal{H}_A \Vdash \neg\alpha(\bar{k})$

By the strong (syntactic) form of the Fixed Point Theorem, we may introduce a sentence β in \mathcal{L}_A^0 such that:

$$\mathcal{H}_A \Vdash (\beta \longleftrightarrow \alpha(\bar{b}))$$

where $b = \Gamma(\beta)$. Now we would find that:

$$\begin{aligned}
 b \in \mathbf{Q} &\implies \mathcal{H}_A \Vdash \alpha(\bar{b}) \\
 &\implies \mathcal{H}_A \Vdash \beta \\
 &\implies b \in \mathbf{P} \\
 &\implies b \notin \mathbf{Q} \\
 &\implies \mathcal{H}_A \Vdash \neg\alpha(\bar{b}) \\
 &\implies \mathcal{H}_A \Vdash \neg\beta \\
 &\implies \mathcal{H}_A \nVdash \beta \\
 &\implies b \notin \mathbf{P} \\
 &\implies b \in \mathbf{Q}
 \end{aligned}$$

a bald contradiction. Consequently, \mathbf{P} is not decidable.

Basic Support

35• In context of article 09°, we claim that there is a sentence $\delta(\eta, \theta)$ in \mathcal{L}_A^2 such that, for any ordered pair (k, ℓ) of natural numbers:

$$\begin{aligned}
 (1) \quad (k, \ell) \in W &\implies \mathcal{H}_A \Vdash \delta(\bar{k}, \bar{\ell}) \\
 (2) \quad (k, \ell) \notin W &\implies \mathcal{H}_A \Vdash \neg\delta(\bar{k}, \bar{\ell})
 \end{aligned}$$

iff there is a sentence $\bar{\delta}(\eta, \theta)$ in \mathcal{L}_A^2 such that, for any natural number k :

$$(3) \quad \mathcal{H}_A \Vdash (\forall\theta)(\bar{\delta}(\bar{k}, \theta) \longleftrightarrow (\overline{D(k)} \equiv \theta))$$

To prove the claim, we note that if condition (3) holds for $\bar{\delta}$ then conditions (1) and (2) hold for $\bar{\delta}$ as well. In turn, we contend that if conditions (1) and (2) hold for δ then condition (3) holds for $\bar{\delta}$, defined as follows:

$$(*) \quad \bar{\delta}(\eta, \theta) = \delta(\eta, \theta) \wedge \left((\forall\zeta) \left((\zeta \leq \theta) \longrightarrow (\delta(\eta, \zeta) \longrightarrow (\zeta \equiv \theta)) \right) \right)$$

Let us prove the contention.

36• Let us prove the Diagonalization Theorem.

37• Let us prove the Representation Theorem.

38• Let us prove the Deduction Theorem.

Addendum

These are the saddest of possible words:
Tinker to Evers to Chance.
Trio of bear cubs, and fleeter than birds,
Tinker and Evers and Chance.
Ruthlessly pricking our gonfalon bubble,
Making a Giant hit into a double
Words that are heavy with nothing but trouble:
Tinker to Evers to Chance.

Franklin Pierce Adams (1910)

CHAPTER 7

SET THEORY

CHAPTER 8

THE CATEGORY OF PREDICATE LOGICS

BIBLIOGRAPHY

01° For further development of our subject, one might turn to any one of the following excellent books:

Computability and Logic

D. E. Cohen, Ellis Horwood Limited, 1987

Mathematical Logic

E. B. Ebbinghaus, J. Flum, and W. Thomas, Springer-Verlag, 1984

Principles of Mathematical Logic

D. Hilbert and W. Ackermann, Chelsea, 1950

Notes on Logic and Set Theory

S. C. Kleene, Van Nostrand, 1952

A Course in Mathematical Logic

Yu. I. Manin, Springer-Verlag, 1977

Introduction to Mathematical Logic

Elliott Mendelson, Chapman and Hall/CRC, 1997

Mathematical Logic

Joel Robbin, W. A. Benjamin, 1969

Gödel's Incompleteness Theorems

Raymond M. Smullyan, Oxford University Press, 1992

The fifth and seventh books played important roles in the design of our own exposition.

02° For a first impression of the theorems of Gödel, one should study the classic booklet:

Gödel's Proof

Ernest Nagel and James R. Newman, New York University Press, 2001

and the recently published gem:

Gödel's Theorem: An Incomplete Guide to Its Use and Abuse

Torkel Franzén, A. K. Peters, 2005, New York University Press, 2001

For a study of the philosophical context and significance of the theorems, one might consult the book:

Incompleteness: The Proof and Paradox of Kurt Gödel

Rebecca Goldstein, W. W. Norton and Company, 2005

For a general survey, one should read:

A Tour Through Mathematical Logic

Richard S. Wolf, Mathematical Association of America, 2007

03° As acts of respect, one should study the classical papers by K. Gödel:

“Die Vollständigkeit der Axiome des logischen Funktionenkalküls”

Monatshefte für Mathematik und Physik **37** (1930), 349-360

“Über formal unentscheidbare Sätze der Principia Mathematica ... I”

Monatshefte für Mathematik und Physik **38** (1931), 173-198

and L. Henkin:

“The completeness of the first-order functional calculus”

Journal of Symbolic Logic **14** (1949) 159-166

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