FE-620 Final Project

Pricing and Hedging American Options on SPY

Drew Carranti

Nicholas Suppiah

Aaron Kamal

Harsh Singh

Pricing and Hedging

Professor Pirjol

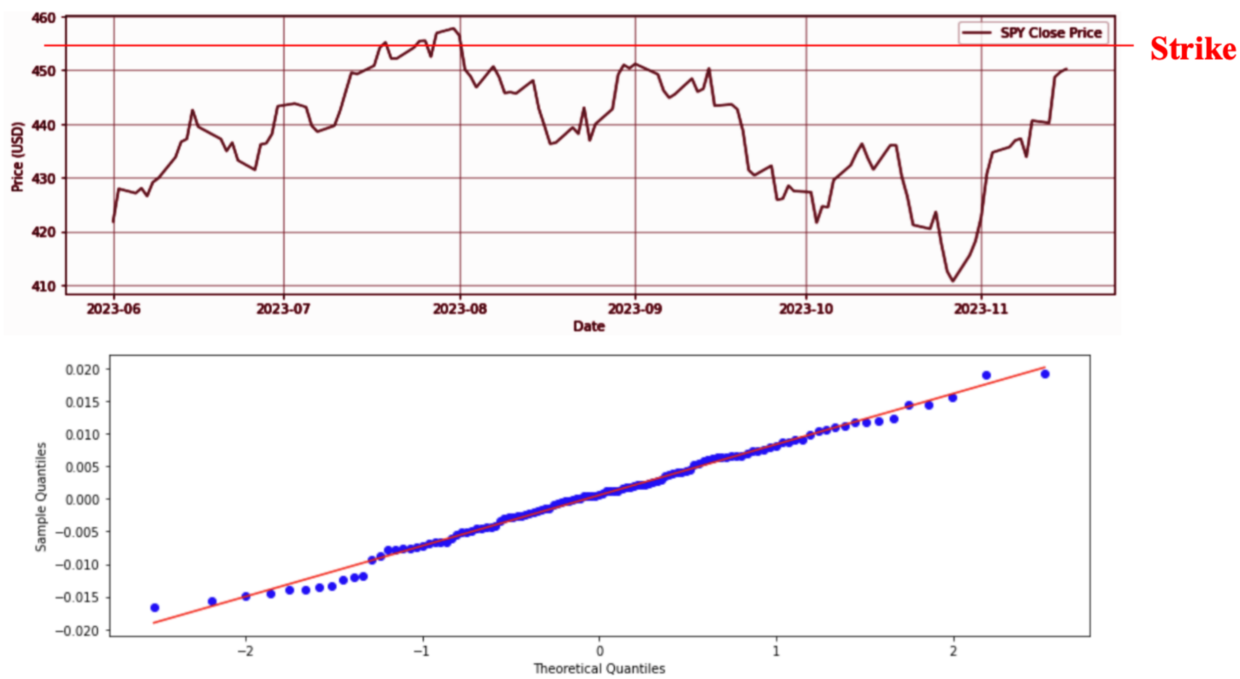
15 December 2023

Asset Description:

Problem Setting and Choice of Underlying Stock

In this study, we focus on American option pricing within the context of the SPDR S&P 500 ETF Trust (SPY) for the period spanning June 1, 2023, to November 17, 2023. Our primary objective is the development and implementation of a robust valuation algorithm tailored specifically for pricing American options on SPY. Notably, this algorithm incorporates the consideration of early exercise opportunities, providing real-time pricing estimates through the utilization of binomial trees.

SPY Price and Log-Returns Q-Q Plot (June 1, 2023, to November 17, 2023)



Our investigation delves into the market dynamics of the SPY options, particularly emphasizing the liquidity of this financial instrument. As of December 13, 2023, the SPY options exhibit substantial liquidity, with an open interest of 250,847 contracts. The bid/ask spreads on SPY options, often as narrow as a penny, contribute to reduced transaction costs for individuals seeking to hedge or speculate on the S&P 500. Notably, SPY stands as the largest exchange-traded fund globally, boasting $446,561.78 million in assets under management (AUM) as of December 12, 2023.

Underlying Asset: SPDR S&P 500 (SPY) Option Characteristics

The SPY option derives its value from the renowned Standard & Poor's 500 (S&P 500) index, making it a compelling choice for market participants seeking exposure to this influential US benchmark. Each SPY option contract encompasses 100 underlying SPY ETF shares, contributing to its significance and tradability.

Liquidity and Market Dynamics

SPY options enjoy a preeminent status in terms of liquidity, underlined by a total open interest of 250,847 contracts. The put-call ratio for options expiring on December 13, 2023, is recorded at 1.46, further substantiating the widespread interest and activity in the SPY options market. This liquidity makes SPY options an attractive choice, fostering an environment where bid/ask spreads remain minimal and enhancing accessibility for market participants.

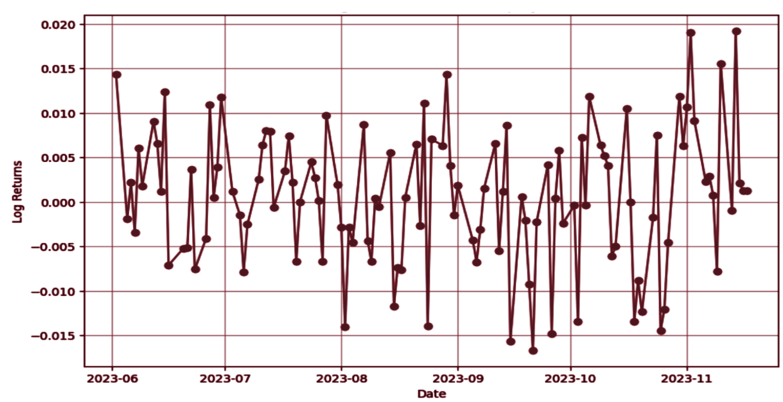
Economic Use and Trading Purposes for SPY American Options

Engaging in the trading of American options on SPY offers a range of strategic benefits for both traders and investors. Speculation plays a crucial role as market participants leverage SPY options to implement bullish or bearish strategies aligned with their market outlook. This speculative approach allows them to capitalize on directional views and market trends.

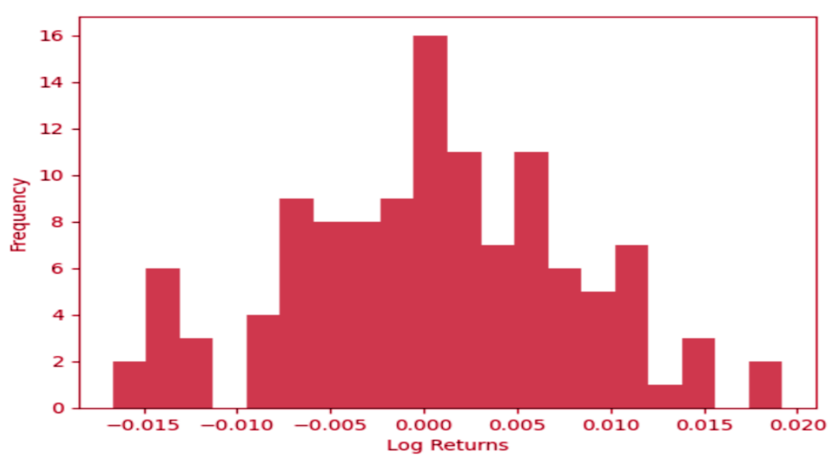
Additionally, the realm of American options on SPY presents arbitrage opportunities where traders exploit price differentials between SPY options and the underlying ETF (S&P 500) through strategic buying and selling. This dynamic activity is geared towards profiting from market inefficiencies, ultimately contributing to the enhancement of overall market liquidity. In conclusion, our exploration of American options on SPY integrates practical valuation methods with a keen understanding of the unique characteristics and dynamics of this widely traded financial instrument.

Market Data Analysis:

Log Return Of SPY



Histogram of Log Returns



For the analysis conducted, the primary data source utilized was Bloomberg, encompassing various key metrics such as stock price, 455 call option price expiring on 11/17/23, 455 put option price expiring on 11/17/23, implied volatility, 1-month SOFR, 3-month SOFR, 6-month SOFR, and 1-year SOFR. The annualized volatility (σ) was calculated using the formula $\sigma = \sqrt{252} * SD(\text{log returns})$, resulting in a volatility value of 12.15%. This signifies the market's expectation of the SPY's price fluctuation within a range of ±12.15% around its current price over the next year. Subsequent to the volatility calculation, two normality tests were performed. The Jarque-Bera Test yielded a test statistic of 0.39 and a p-value of 0.82, suggesting that the distribution of log returns does not significantly deviate from normality. Similarly, the Normal Test produced a test statistic of 0.23 and a p-value of 0.89, reinforcing the conclusion that log returns are approximately normally distributed.

Calculating Greeks:

Delta is a numerical measure ranging from 0 to 1 for call options and -1 to 0 for put options, helping assess the likelihood of an option expiring in-the-money (ITM). For call options, a higher delta, approaching 1, indicates a greater probability of expiring ITM, as the underlying security's price rise positively influences the option's value. Conversely, for put options, a lower delta, approaching -1, suggests a higher likelihood of expiring ITM, as the option benefits from a decrease in the underlying security's price. Delta essentially quantifies the sensitivity of an option's price to changes in the underlying asset's market price, aiding traders in evaluating risk and potential profitability. As options approach expiration, delta tends to move towards 1 or -1, emphasizing the increasing certainty of the option expiring ITM.

Gamma is a measure that helps estimate the rate of change in Delta based on fluctuations in the stock price. It quantifies the sensitivity of Delta, indicating how much Delta might change with a one-point movement in the underlying security's price. A higher Gamma suggests that Delta is more responsive to price changes, emphasizing the dynamic nature of options pricing. Traders use Gamma to assess and manage the risk associated with Delta, especially in rapidly changing market conditions.

Delta & Gamma Comparisons (Calls and Puts):

	Calls	***Puts***
Delta	- + Delta - ATM delta near 0.50 - As options near expiration - ITM options -> 1 - OTM options -> 0	- - Delta - ATM delta near -0.50 - As options near expiration - ITM options -> -1 - OTM options -> 0
Gamma	- Measures rate of change of delta over time - Same for puts	- Measures rate of change of delta over time - Same for calls

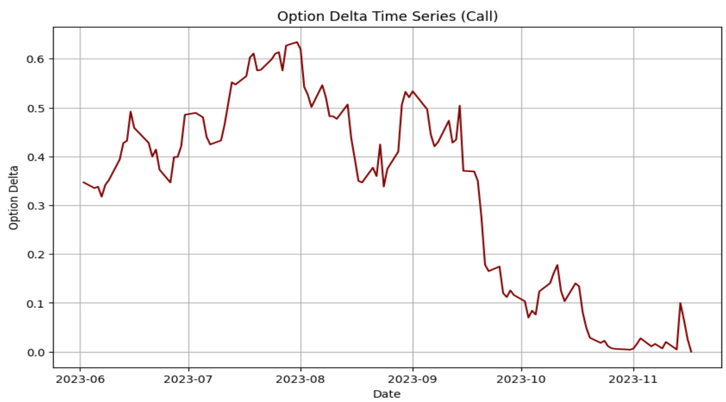
Sample data frame for calculating delta on $455 call:

Date	SPY US 11/17/23 C455 Equity	SPY US Equity	6M SOFR	Implied Vol	Option Delta
2023-06-01	5.46	421.82	5.28528	12.576	
2023-06-02	7.45	427.92	5.24547	12.57	0.3468714130956470
2023-06-05	7.67	427.1	5.28773	12.515	0.33509707804259600
2023-06-06	6.64	428.03	5.27695	12.16	0.3379688402703240
2023-06-07	6.64	426.55	5.27788	11.923	0.3174948756498900
2023-06-08	6.63	429.13	5.29207	11.756	0.34150546829962400
2023-06-09	6.87	429.9	5.26715	11.99	0.35159530217406300
2023-06-12	8.43	433.8	5.28611	12.204	0.39392289509456300
2023-06-13	9.8	436.66	5.29464	12.508	0.4274397823135240
2023-06-14	9.56	437.18	5.26695	12.56	0.4319714985100260
2023-06-15	12.51	442.6	5.28853	12.776	0.4918205995490630

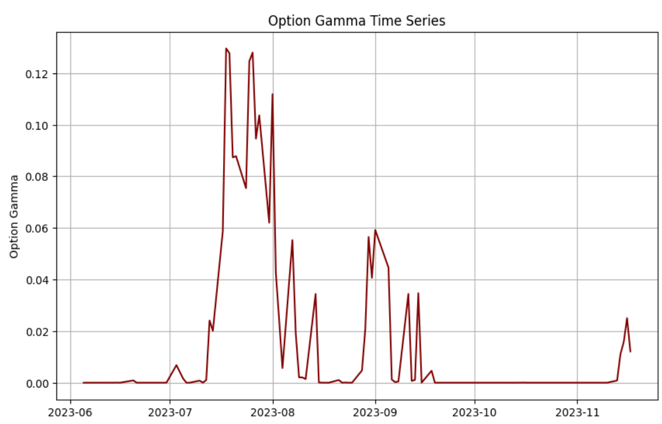
Sample data frame for calculating gamma on $455 call:

Date	SPY US 11/17/23 C455 Equity	SPY US Equity	6M SOFR	Implied Vol	Option Gamma
2023-06-01	5.46	421.82	5.28528	12.576	
2023-06-02	7.45	427.92	5.24547	12.57	
2023-06-05	7.67	427.1	5.28773	12.515	1.85027661503519E-08
2023-06-06	6.64	428.03	5.27695	12.16	1.81995869434149E-21
2023-06-07	6.64	426.55	5.27788	11.923	1.12153293680236E-24
2023-06-08	6.63	429.13	5.29207	11.756	4.34082571036349E-21
2023-06-09	6.87	429.9	5.26715	11.99	3.30963077389889E-19
2023-06-12	8.43	433.8	5.28611	12.204	9.22083019934262E-06
2023-06-13	9.8	436.66	5.29464	12.508	4.35106242780498E-10
2023-06-14	9.56	437.18	5.26695	12.56	1.5461556166831E-09
2023-06-15	12.51	442.6	5.28853	12.776	2.93195766518678E-05

Time series of $455 strike call option delta:



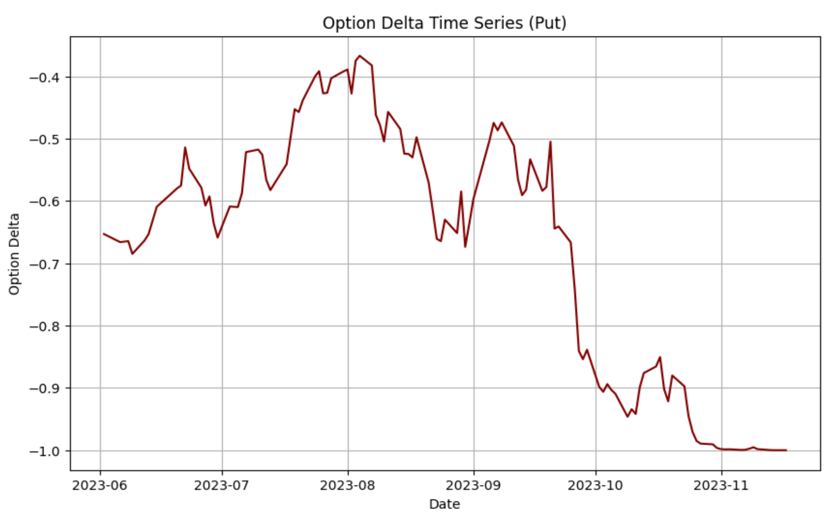
Time series of $455 strike call option gamma:



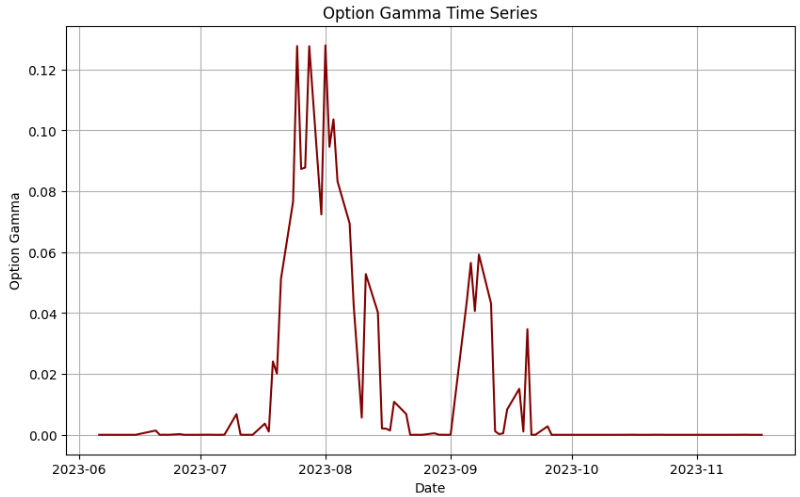
The behavior of delta and gamma for call options is closely monitored in relation to the underlying asset's price movement. Specifically, delta exhibits an upward trajectory as the underlying asset's price increases, reflecting heightened sensitivity to market fluctuations.

Conversely, for put options, delta experiences a decrease as the underlying asset's price rises. Throughout the observed period, gamma maintains a generally positive trend for both call and put options. The Delta & Gamma Time Series Graphs for Call Options provide a visual representation of these dynamics. The Delta Graph vividly illustrates the changing delta values with shifts in the underlying asset's price, offering insights into the option's sensitivity. Simultaneously, the Gamma Graph visually depicts the correlation between alterations in the underlying asset's value and the corresponding changes in the delta of a call option. Key observations from these graphs indicate that the call option's delta approaches 0 as the option's strike is $455, but the stock's expiration value is $450.79, signifying that the option contract expires deep out of the money. Additionally, the call gamma chart exhibits substantial changes within the delta time steps and closely resembles the put gamma chart.

Time series of $455 strike call option delta:

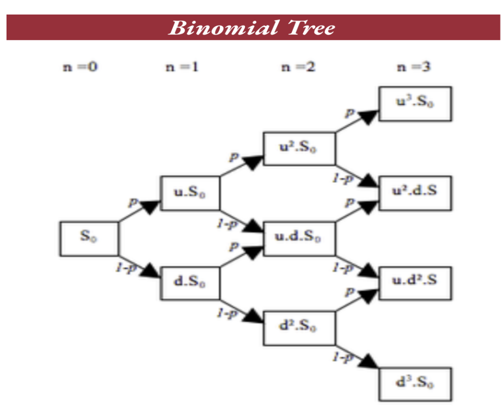


Time series of $455 strike call option gamma:



The Delta Graph reveals that the delta of the put option gradually approaches -1. This trend is explained by the fact that the strike price of the option is $455, and at expiration, the stock's value is $450.79, indicating that the option contract expires deep in the money. The negative delta reflects the inverse relationship between the put option's value and the underlying asset's price. Furthermore, the Put Gamma Chart illustrates significant changes within the delta time steps, sharing similarities with the call gamma chart. These observations highlight the dynamic nature of put options, particularly in terms of delta's response to price movements and the consistency of gamma's behavior across both call and put options.

Pricing Algorithm:



Our strategy explores the application of the risk-neutral binomial model in the pricing of American options. The focus is on the binom_rn_pricer function, which orchestrates a series of mathematical algorithms encapsulated within the functions _get_binom_rn_params, _calc_maturity_value, and _back_prop. These algorithms leverage the binomial tree structure to provide a comprehensive and efficient method for valuing options with early exercise features. The pricing of American options poses a unique challenge due to the embedded flexibility of early exercise rights. Traditional closed-form solutions, such as the Black-Scholes model, may not capture the intricacies of American-style options. The risk-neutral binomial model, rooted in discrete-time pricing, offers an alternative approach that accommodates early exercise. This paper delves into the binom_rn_pricer function, highlighting its role in utilizing the binomial tree to price American options

The binomial tree is a discrete-time model representing the possible future paths of an underlying asset's price. The up and down factors, u and d, capture the potential magnitude of price movements. The risk-neutral probabilities, p and q, ensure that the current price is equal to the expected future payoff discounted at the risk-free rate (r). This risk-neutral pricing approach allows for the creation of a replicating portfolio that hedges against the option's risk with no risk premium required. When programming the binomial tree model we import necessary libraries for financial data retrieval (yfinance), data manipulation (pandas), numerical operations (numpy), and operating system-related functions (os). The _get_binom_rn_params function plays an important role in determining the parameters required for constructing the risk-neutral binomial tree. These parameters are fundamental to option pricing. The risk-neutral probability p is particularly significant, as it allows us to create a portfolio that replicates the option's payoff. By calculating p and related factors, the function sets the stage for modeling future stock price movements under a risk-neutral framework. This modeling of future stock price will enable pricing of the options contract.

The parameters include:

- dt = t/n: Time step size.
- u = exp(hvol * sqrt(dt)): Up factor.
- d = 1/u: Down factor.
- p = (exp(r * dt) - d) / (u - d): Probability of an up move.
- q = 1 - p: Probability of a down move.

The _print_model function is a helper function used when printing the binomial tree results, specifying the type of contract (put or call). Additionally, the _get_div_idx helper function adjusts the format of the dividend dictionary, converting date keys to step indices, for the incorporation of discrete dividends into the pricing algorithm. The _calc_maturity_value function focuses on calculating option prices at maturity for the option contract. Finding the value of the contract when held to maturity is an important first step when calculating the price of an American option. This gives us a starting point before we account for early exercise.
In the context of American options, this is critical as it allows for the possibility of early exercise. The calculations involve considering potential dividend payments, which are significant in real-world scenarios. Dividends affect both the stock price and the option value. For example, a call option holder may exercise the option early if the stock's ex-dividend date falls within the option's time to maturity. If there is a dividend at maturity, it will be accounted for in our calculations.

The backward propagation performed by the _back_prop function is central to the risk-neutral pricing approach. By iterating backward through the binomial tree, the algorithm computes the option values at each node by discounting future values to present value terms. This mirrors the concept of risk-neutral valuation, where the expected future payoff is discounted at the risk-free rate. The function ensures that the entire tree is consistent with the risk-neutral framework, making it suitable for option pricing. The _print_tree() and _orint_stock() functions are primarily focused on visualization, they contribute to the understanding of how stock prices and option values evolve over time. Visualization is vital in comprehending the dynamics of option pricing. Observing how stock prices change at each time step and how option values propagate through the tree enhances the intuitive grasp of the risk-neutral binomial model. The binomial model, implemented in this context, leverages a discrete-time framework represented by a binomial tree structure, providing an efficient method for valuing options with early exercise features. This function uses a series of algorithms within the functions _get_binom_rn_params, _calc_maturity_value, and _back_prop.

American options, with their embedded early exercise rights, pose a challenge for traditional closed-form solutions like the Black-Scholes model. The risk-neutral binomial model offers an alternative approach by discretizing time and accommodating the flexibility of early exercise. The binom_rn_pricer function orchestrates the construction of the binomial tree, employing algorithms that ensure a consistent and unbiased valuation of American options. The up and down factors (u and d) capture potential price movements, while the risk-neutral probabilities (p and q) ensure that the expected future payoff is discounted at the risk-free rate (r). The _get_hvol function serves a crucial role in the binomial pricing model, specifically in the context of pricing options. This function is responsible for obtaining historical volatility, an essential parameter in the option pricing model, which reflects the magnitude of past price fluctuations of the underlying asset. The function calculates the percentage change in the adjusted closing prices, which represents the daily returns of the stock. The standard deviation of these returns over the specified historical window is calculated, and this value is then annualized by multiplying it by the square root of the number of trading days in a year (252).

Below is a sample stock price and option price tree constructed by the model:

```
--------------------------------------------------
time 0:                 [421.82]
time 1:              [427.44, 416.28]
time 2:           [421.82, 433.13, 410.81]
time 3:        [416.28, 427.44, 438.9, 405.41]
time 4:  [410.81, 421.82, 433.13, 444.74, 400.08]
time 5: [405.41, 416.28, 427.44, 438.9, 450.66, 394.82]
time 6: [389.63, 400.08, 410.81, 421.82, 433.13, 444.74, 456.67]
--------------------------------------------------
```

```
--------------------------------------------------
     time 0:                  [9.52]
     time 1:              [11.07, 8.04]
     time 2:           [0.0, 21.65, 16.73]
     time 3:           [0, 0, 9.75, 34.82]
--------------------------------------------------
```

Output using Real Data

Hedging Analysis

	Date	SPY US 11/17/23 P455 Equity	SPY US Equity	P&L Unhedged	P&L Hedged
99	2023-10-27	45.58	420.46	-6.39	-5.670751
100	2023-10-30	38.63	423.63	6.95	3.813303
101	2023-10-31	37.14	417.55	1.49	7.512936
102	2023-11-01	32.47	412.55	4.67	9.652034
103	2023-11-02	24.42	410.68	8.05	9.916674
104	2023-11-03	20.49	415.59	3.93	-0.973744
105	2023-11-06	19.60	418.20	0.89	-1.716137
106	2023-11-07	17.53	422.66	2.07	-2.388192
107	2023-11-08	18.07	430.76	-0.54	-8.634478
108	2023-11-09	21.45	434.69	-3.38	-7.300507
109	2023-11-10	14.49	435.69	6.96	5.964698
110	2023-11-13	14.66	436.93	-0.17	-1.407841
111	2023-11-14	6.15	437.25	8.51	8.190082
112	2023-11-15	6.00	433.84	0.15	3.559911
113	2023-11-16	4.84	440.61	1.16	-5.609999
114	2023-11-17	4.47	440.19	0.37	0.790000

Delta hedging aims to achieve directional neutrality, mitigating the exposure to market movements. This strategy operates by offsetting directional risk, thereby isolating changes in volatility. The primary objective of delta hedging is to safeguard profits derived from an option or stock position without unwinding the long-term holding. Throughout the period spanning October 27, 2023, to November 17, 2023, with a comprehensive dataset commencing from June 1, 2023, the closing prices for SPY are considered. Notably, for a put option with a strike price (K) of 455, concluding in-the-money on November 17th, a specific hedging strategy is deployed. Leveraging the Delta values from the provided table, a dynamically hedged portfolio is constructed using the relationship $D(t) - D(t-1) = P(t) - P(t-1) + \Delta(t-1)(S(t) - S(t-1))$. It is crucial to acknowledge that the hedge is not flawless, encountering imperfections due to factors such as time decay, gamma contributions, and temporal errors.

Future Objectives

In our pursuit to deepen our comprehension of American option pricing on SPY, we have outlined key objectives aimed at refining trading strategies, improving hedge effectiveness, and enhancing overall accuracy. One focal point involves investigating the correlation between Vega (volatility sensitivity) and Rho (interest rate sensitivity) with the days remaining until maturity, recognizing the importance of understanding these sensitivities as options approach expiration for optimizing trading strategies and hedging techniques. Another objective is to employ European options as a control variate to enhance the accuracy of tree pricing, leveraging their closed-form solutions as a benchmark to control errors in the more complex American option pricing tree model. Additionally, we aim to introduce advanced models like the Trinomial model, which provides a more granular pricing approach with three possible price movements per period, requiring a careful evaluation of trade-offs between accuracy and computational cost. We will also compare the performance of option pricing models with continuous vs discrete dividend adjustments, evaluating their impact on model performance and informing best practices for incorporating dividends into valuations. Furthermore, we will explore the efficacy of risk-neutral assumptions against historical calibration methods and other numerical methods to gain insights into the robustness of different modeling approaches. Lastly, we will investigate the potential of using implied volatility as a forward-looking replacement for historical volatility, aiming to create more dynamic and forward-looking option pricing models.

Appendix:

Delta Hedge PnL:

```
df['P&L Unhedged'] = 0.0
df['P&L Hedged'] = 0.0

for i in range(1, len(df)):
    delta_s = df.loc[i, 'SPY US Equity'] - df.loc[i - 1, 'SPY US Equity']
    delta_c = df.loc[i, 'SPY US 11/17/23 P455 Equity'] - df.loc[i - 1, 'SPY US 11/17/23

    pnl_unhedged = -delta_c

    pnl_hedged = -delta_c + df.loc[i - 1, 'Option Delta'] * delta_s

    df.at[i, 'P&L Unhedged'] = pnl_unhedged
    df.at[i, 'P&L Hedged'] = pnl_hedged
```

Sources:

Muroi, Y. and Suda, S. (2017) Computation of Greeks Using Binomial Tree. *Journal of Mathematical Finance*, **7**, 597-623. doi: 10.4236/jmf.2017.73031.