STAT 4640/7640 **Homework 2**

Due: February 8, 2022

- **Instructions**: Make your answers are clearly written.

1. Let's return to the example of the Mizzou kickers from Homework 1. Recall we assumed that each attempt was an independent Bernoulli trial with probability of success, θ. In 2019, Mizzou kickers were successful on 16 out of 23 in point after attempts.

   (a) Compute the maximum likelihood estimator of θ. How does this compare to the posterior mean estimate under the Beta(1,1) prior?

   (b) For the Beta(1,1) prior, compute the shrinkage weight, w. Does this suggest more weight from the data or the prior?

   (c) Recall from your introductory statistics course that the Frequentist 95% condifence interval for θ is computed

   $$\hat{\theta} \pm 1.96 \times \sqrt{\frac{\hat{\theta}(1-\hat{\theta})}{n}}$$

   where $\hat{\theta}$ is the sample proportion and n is the number of trials. Compute the 95% confidence interval for θ.

   (d) Using the Beta(1,1) prior, compute the Bayesian 95% credible interval for θ. The following R command computes the 10th and 90th percentile of a Beta(2,5) distribution.

   ```
   qbeta(c(.1,.9),2,5)
   ```

   (e) Using Monte Carlo sampling with samples of size $S = 100$ and $S = 10000$, compute the posterior mean, variance, and 95% credible interval for θ assuming the Beta(4,4) prior. Repeat the sampling multiple times and comment on the sensitivity of the posterior estimates to S. Compare these results to their analytical counterparts. The following R command generates 50 independent samples from a Beta(2,5) distribution.

   ```
   rbeta(50,2,5)
   ```

   (f) Using the Beta(4,4) prior, compute the 95% highest posterior density (HPD) interval for θ and compare it to the 95% central interval. The highest posterior density interval can be obtained using Monte Carlo sampling. Make sure to choose a value for S such that you feel confident in your results. The following R package and command will return the 80% HPD interval for a Beta(2,5) using a sample of size $S = 50$.

   ```
   library(HDInterval)
   hdi(rbeta(50,2,5),credMass=0.80)
   ```

   (g) Let's assume the Mizzou kicker is going to attempt $n^* = 5$ more extra points and let y^* denote the number of successful attemps. What are the possible values of y^*?

(h) Assuming a Beta(1,1) prior, compute the probabilities of the posterior predictive distribution for y^* and plot them. Note that these can be easily be computed by hand (using a calculator). However, the `LearnBayes` package in R has a lot of useful functions for Bayesian inference. The `pbetap` function computes the probabilities of the posterior predictive distribution of the Beta-Binomial model. In the following example code, the posterior distribution is $[\theta|y] \sim$Beta(7,53), $n^* = 10$, and the function will return probabilities for $y^* = 2, 4, 6, 8$.

```
library(LearnBayes)
pbetap(c(7,53),10,c(2,4,6,8))
```

(i) Recompute the probabilities of the posterior predictive distribution for y^* using the Monte Carlo sampling approach. The following code is an exerpt from Page 33 in the text:

```
S=10000
#Sample theta.star from it's posterior distribution,
#which I will assume is Beta(2,5)
theta.star=rbeta(S,2,5)
#With n.star =10, sample Y.star from it's posterior predictive
#distribution using theta.star
Y.star=rbinom(S,10,theta.star)
#Summarize Y.star
table(Y.star)/S
```

2. Refer to the data in problem 11 of Chapter 1. The number of species detected each year, Y_t are i.i.d. samples from a Poisson distribution with parameter λ. That is, $Y_t \sim$ Poisson(λ) and

$$P(Y_t = y|\lambda) = \frac{\lambda^y e^{-\lambda}}{y!}$$

Assume a $\lambda \sim$ Gamma(10,0.3) prior distribution.

(a) Using all the data, derive the posterior distribution $[\lambda|Y_{2010}, \ldots, Y_{2015}]$.
Note: Recall that the likelihood is the product of six univariate Poisson densities. e.g., $[Y_{2010}, \ldots, Y_{2015}|\lambda] = [Y_{2010}|\lambda] \times \cdots \times [Y_{2015}|\lambda]$.

(b) Make a plot of the posterior distribution and overlay the prior distribution. Is there evidence of Bayesian learning?

(c) Compute the posterior mean, variance, 90% central credible interval, and 90% HPD interval for λ.

3. Problem 5 of Chapter 2: Over the past 50 years, California has experience an average of $\lambda_0 = 75$ large wildefires per year. For the next 10 years, you will record the number of large fires in California and then fit a Poisson-Gamma model to these data. Let λ be the rate of large fires in this future period, and have prior $\lambda \sim$ Gamma(a, b). Select a and b such that the prior is uninformative with prior variance around 100 and has prior probability approximately P($\lambda > \lambda_0$) =0.5 so that the prior places equal probability on both hypotheses in the test for a change in rate. Present a plot of the Gamma prior distribution you choose.

4. Problem 6 of Chapter 2 (slightly modified): An assemly line relies on accurate measurements from an image-recognition algorithm at the first stage of the process. It is known that the algorithm is unbiased, so assume that measurements follow a normal distribution with mean zero, $Y_i|\sigma^2 \stackrel{iid}{\sim} \text{Normal}(0, \sigma^2)$. Some errors are permissible, but if σ^2 exceeds the threshold c, then the algorithm must be replaced. You make $n = 20$ measurements and observe $\sum_{i=1}^n Y_i = -2$ and $\sum_{i=1}^n Y_i^2 = 15$. Conduct a Bayesian analysis with a $\sigma^2 \sim \text{InverseGamma}(a, b)$ prior. Compute the posterior probability that $\sigma^2 > c$ for:

   (a) $c = 1$ and $a = b = 0.2$

   (b) $c = 1$ and $a = b = 2.0$

   (c) $c = 2$ and $a = b = 0.2$

   (d) $c = 2$ and $a = b = 2.0$

   For each value of c, compute the ratio of probabilities for the two priors (i.e., $a = b = 0.2$ and $a = b = 2.0$). Which, if any, of these results are sensitive to the prior? Note: The function `rinvgamma()` in the R package `MCMCpack` can be used to simulated from the Inverse-Gamma distribution.