Homework 3

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1.a)

$$\mu = 0.95,\ \sigma = 0.05 \implies 0.95 = \frac{a}{a+b},\ 0.05^2 = \frac{ab}{(a+b)^2(a+b+1)}$$

$$\Downarrow$$

$$19b = a,\quad 0.05^2 = \frac{19b^2}{(20b)^2(20b+1)}$$

$$\Downarrow$$

$$\boxed{\text{Expert 1} \sim \text{Beta}(17.1, 0.9) \text{ prior}} \qquad b = 0.9,\ a = 17.1$$

$$\mu = 0.5,\ \sigma = 1/12 \implies 0.5 = \frac{a}{a+b},\ (1/12)^2 = \frac{ab}{(a+b)^2(a+b+1)}$$

$$\Downarrow \qquad \Downarrow$$

$$b = a,\quad (1/12)^2 = \frac{b^2}{(2b)^2(2b+1)}$$

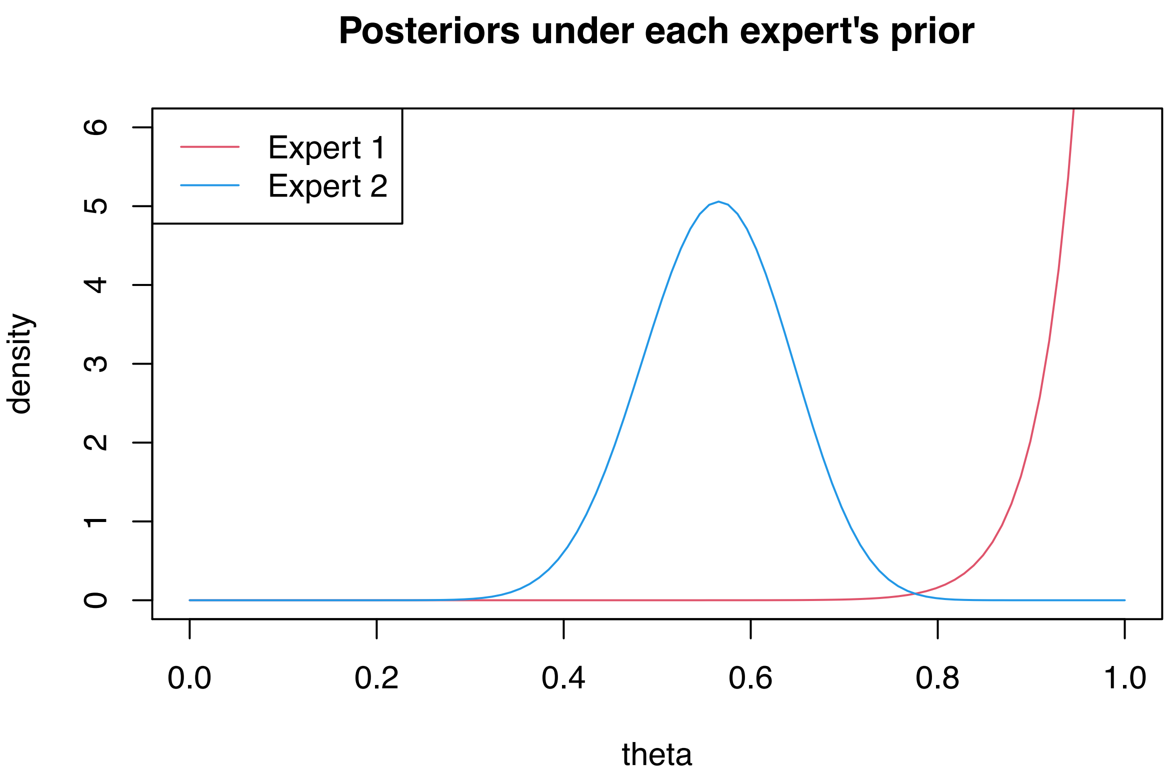
$$\Downarrow$$

$$\boxed{\text{Expert 2} \sim \text{Beta}(17.5, 17.5) \text{ prior}} \qquad b = 35/2,\ a = 35/2$$

Expert 2's prior is uninformative since solving for the desired mean and standard deviation yiels a = b = 17.5.

1.b)

The posterior for Expert 1 is relatively unchanged given all 5 mice test positive, since it expected 0.95 probability for each mouse, whereas the posterior for Expert 2 has a greater mean and lower variance.



2.a)

The Negative Binomial distribution is used to count the number of failures from independent trails with probability of success θ before recording the mth success.

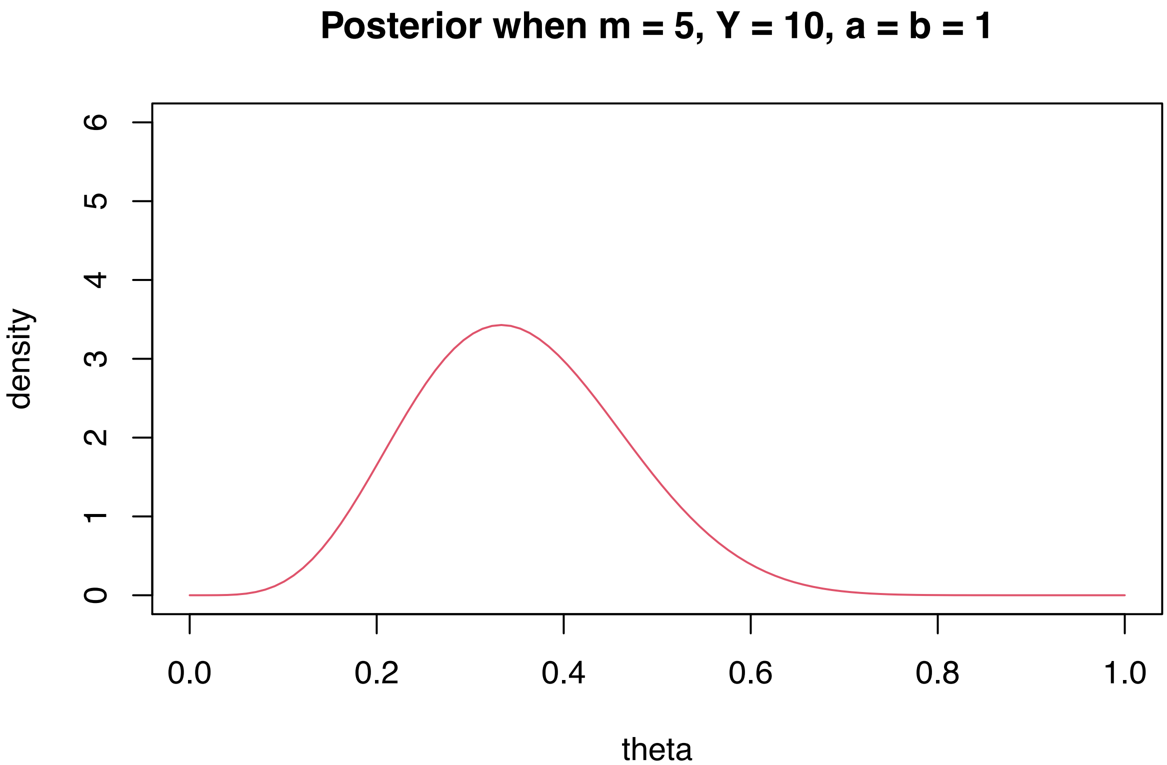
2.b) $Y|\theta \sim \text{NegBinom}(\theta, m)$, $\theta \sim \text{Beta}(a,b)$

$$p(\theta|Y) \propto \theta^{a-1}(1-\theta)^{b-1}\cancel{\binom{Y+m-1}{Y}}\theta^{m}(1-\theta)^{Y}$$

$$\propto \theta^{a+m-1}(1-\theta)^{b+Y-1}$$

$$= \text{Beta}(a+m, b+Y)$$

2.c)

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95% credible interval = (0.1519837, 0.5866206).
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3)

$$p(\lambda \mid Y) = \frac{p(Y \mid \lambda)\, p(\lambda)}{\int p(Y \mid \lambda)\, p(\lambda)\, d\lambda}$$

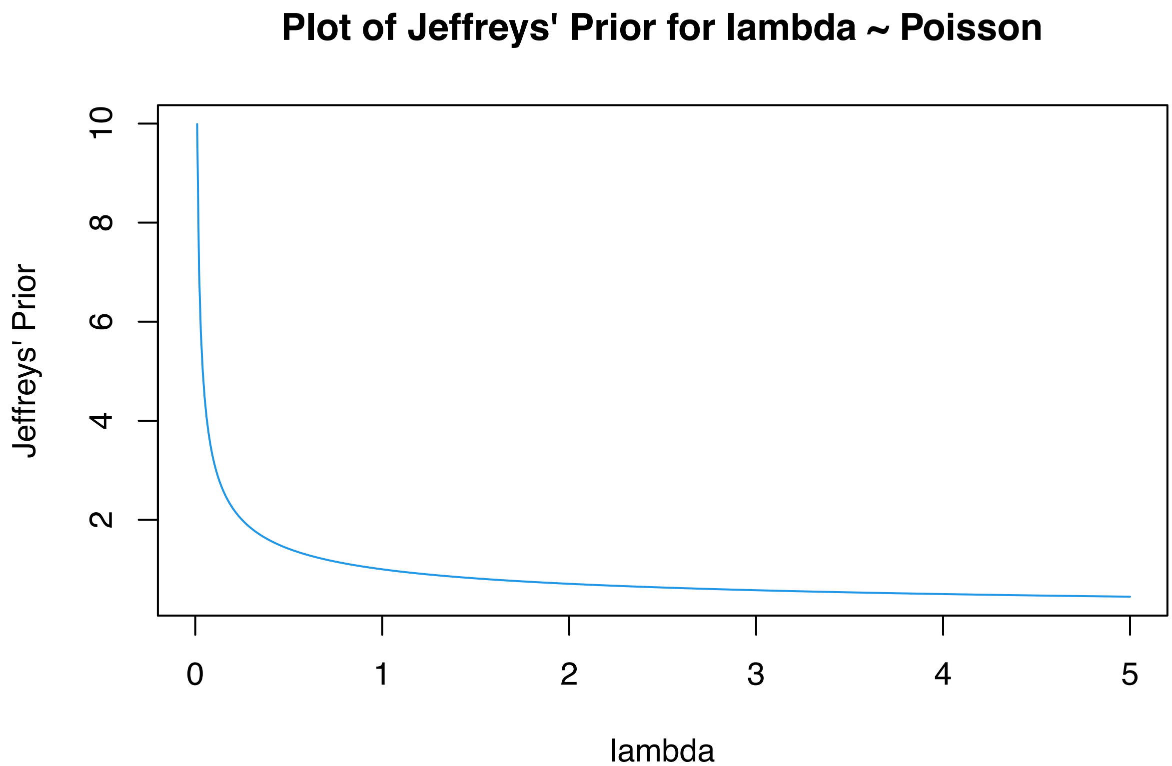
Top: $\frac{\lambda^{Y} e^{-\lambda}}{Y!} \cdot \frac{b^a}{\Gamma(a)} \lambda^{a-1} e^{-b\lambda} = \frac{b^a}{Y!\,\Gamma(a)} \lambda^{a+Y-1} e^{-(b+1)\lambda}$

Bottom: $\int \frac{\lambda^{Y} e^{-\lambda}}{Y!} \cdot \frac{b^a}{\Gamma(a)} \lambda^{a-1} e^{-b\lambda}\, d\lambda = \frac{b^a}{Y!\,\Gamma(a)} \frac{\Gamma(a+Y)}{(b+1)^{a+Y}} \underbrace{\int \frac{(b+1)^{a+Y}}{\Gamma(a+Y)} \lambda^{a+Y-1} e^{-(b+1)\lambda}\, d\lambda}_{\text{Gamma} \implies =1.}$

Thus, $p(\lambda \mid Y) = \frac{\cancel{\frac{b^a}{Y!\,\Gamma(a)}} \lambda^{a+Y-1} e^{-(b+1)\lambda}}{\cancel{\frac{b^a}{Y!\,\Gamma(a)}} \frac{\Gamma(a+Y)}{(b+1)^{a+Y}}} = \frac{(b+1)^{a+Y} \lambda^{a+Y-1} e^{-(b+1)\lambda}}{\Gamma(a+Y)}$

4.a) $\log(L(\lambda|Y)) \propto -\lambda - \ln(Y!) + \ln(\lambda)Y$

$$\frac{\partial^2 \log(L(\lambda|Y))}{\partial\lambda^2} = \frac{-Y}{\lambda^2}. \text{ Taking } -E(Y) \text{ gives } 1/\lambda. \implies \boxed{p(\lambda) \propto \lambda^{-1/2}}$$



4.b)

No, this is not a proper prior since the integral does not converge. $\int_0^\infty \lambda^{-1/2}\, d\lambda$ doesn't converge, i.e. it's improper.

4.c)

$$p(\lambda|Y) = \frac{\lambda^Y e^{-\lambda}}{Y!}\lambda^{-1/2} = \frac{\lambda^{Y-1/2} e^{-\lambda}}{Y!} = \text{Gamma}(Y+1/2, 1)$$

$Y \geq 0$ ensures this is a proper posterior.