Math-M413 Homework 10

Chapter 3, Section 4: 1, 2, 6, 9

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3.4.1

If P is a perfect set and K is compact, is the intersection $P \cap K$ always compact? Always perfect?

Proof. By definition, a perfect set P is closed and has no isolated points. So, the intersection of a closed set P and a closed and bounded set K will give us a closed and bounded set $P \cap K$. So since $P \cap K$ is closed and bounded it is always compact. However $P \cap K$ is not always perfect. For example, let $P = \mathbb{R}$, then $P \cap K = K$ and we already know that K is not necessarily perfect by our questions definition of K. $\square$

3.4.2

Does there exist a perfect set consisting of only rational numbers?

No. By Theorem 1.4.11 We know that $\mathbb{Q}$ is countable, and since $\mathbb{Q}$ is countable, any subset is also countable. We also know that a non-empty perfect set is uncountable. Since a set can't be countable and uncountable we have a contradiction, and We know that there is no perfect set that consists of only rational numbers.

3.4.6

Prove Theorem 3.4.6

Theorem 3.4.6

A set $E \subseteq R$ is connected if and only if, for all nonempty disjoint sets A and B satisfying $E = A \cup B$, there always exists a convergent sequence $(x_n) \to x$ with (x_n) contained in one of A or B, and x an element of the other.

Proof. Since this is an IFF proof we have to show that,

1. $E \subseteq \mathbb{R} \implies [(x_n) \to x \text{ where } (x_n), x \in A \text{ or } (x_n), x \in B]$
2. $[(x_n) \to x \text{ where } (x_n), x \in A \text{ or } (x_n), x \in B] \implies E \subseteq \mathbb{R}$

First let it be true that a set $E \subseteq \mathbb{R}$ is connected. Since it is connected, we know that E is connected if and only if $E = A \cup B$ where A and B are disjoint and non empty subsets of E. Then it is either the case that $\bar{A} \cap B = \emptyset$ or $A \cap \bar{B} = \emptyset$. Since A and B are both disjoint sets, WOLOG, we'll look at the case of $\bar{A} \cap B = \emptyset$, since we would get the same logical answer if we chose $A \cap \bar{B} = \emptyset$. So consider an element $x \in \bar{A} \cap B$. Then,

$$
\begin{aligned}
&x \in \bar{A} \cap B \\
\implies &x \in \bar{A} \text{ and } x \in B
\end{aligned} \tag{1}
$$

So, since $x \in \bar{A}$. There exists a sequence $(x_n) \in A$, such that $(x_n) \to x$, since $\bar{A}$ is closed.

Now, to show the other way. Suppose that $E = A \cup B$, where A and B are nonempty disjoint subsets of E and there is a sequence $(x_n) \in A$ or B such that $(x_n) \to x$ where and whatever set (x_n) is in, x is in the other.

So, we WTS that E is connected. To do so, we need to show that either $\bar{A} \cap B \neq \emptyset$ or that $A \cap \bar{B} \neq \emptyset$. So, consider the sequence (x_n). Let $(x_n) \in A$. WOLOG, we will show that $\bar{A} \cap B \neq \emptyset$. So, since $(x_n) \in A$, By the definition of the question we know that $x \in B$. But, since $(x_n) \to x$, this implies that $x \in \bar{A}$. Now, since $x \in \bar{A}$ and $x \in B$, $x \in \bar{A} \cap B \implies \bar{A} \cap B \neq \emptyset$. Tus, E is connected.

So, a set $E \subseteq R$ is connected if and only if, for all nonempty disjoint sets A and B satisfying $E = A \cup B$, there always exists a convergent sequence $(x_n) \to x$ with (x_n) contained in one of A or B, and x an element of the other.

$\square$

3.4.9

Follow these steps to show that the Cantor set $C_n = \bigcap_{n=0}^{\infty} C_n$ described in Section 3.1 is totally disconnected in the sense described in Exercise 3.4.8.

(a.) Given $x, y \in C$, with $x < y$, set $\epsilon = y - x$. For each $n = 0, 1, 2, \ldots$, the set C_n consists of a finite number of closed intervals. Explain why there must exist an N large enough so that it is impossible for x and y both to belong to the same closed interval of C_N .

(b.) Argue that there exists a point$z \notin C$ such that $x < z < y$. Explain how this proves that there can be no interval of the form (a, b) with $a < b$ contained in C.

(c.) Show that C is totally disconnected.

Notes

Definitions:

Perfect

A set $P \subseteq \mathbb{R}$ is **perfect** if it is closed and contains no isolated points.

Compact

A set $K \subseteq \mathbb{R}$ is **compact** if every sequence in K has a subsequence that converges to a limit that is also in K.

Theorems:

Theorem 1.4.11

(i) The set $\mathbb{Q}$ is countable. (ii) The set $\mathbb{R}$ is uncountable.