Math-M413 Homework 10

Chapter 3, Section 4: 1, 2, 6, 9

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3.4.1

If P is a perfect set and K is compact, is the intersection $P \cap K$ always compact? Always perfect?

Proof. By definition, a perfect set P is closed and has no isolated points. So, the intersection of a closed set P and a closed and bounded set K will give us a closed and bounded set $P \cap K$. So since $P \cap K$ is closed and bounded it is always compact. However $P \cap K$ is not always perfect. For example, let $P = \mathbb{R}$, then $P \cap K = K$ and we already know that K is not necessarily perfect by our questions definition of K. $\square$

3.4.2

Does there exist a perfect set consisting of only rational numbers?

No. By Theorem 1.4.11 We know that $\mathbb{Q}$ is countable, and since $\mathbb{Q}$ is countable, any subset is also countable. We also know that a non-empty perfect set is uncountable. Since a set can't be countable and uncountable we have a contradiction, and We know that there is no perfect set that consists of only rational numbers.

3.4.6

Prove Theorem 3.4.6

Theorem 3.4.6

A set $E \subseteq R$ is connected if and only if, for all nonempty disjoint sets A and B satisfying $E = A \cup B$, there always exists a convergent sequence $(x_n) \to x$ with (x_n) contained in one of A or B, and x an element of the other.

Proof. Since this is an IFF proof we have to show that,

1. $E \subseteq \mathbb{R} \implies [(x_n) \to x \text{ where } (x_n), x \in A \text{ or } (x_n), x \in B]$

   Proof. Begin by assuming that $E \subseteq \mathbb{R}$ is connected. That is for any nonempty disjoint sets A and B such that $E = A \bigcup B$, there exists a convergent sequence $(x_n) \to x$ with (x_n) in A or B and x in the set that (x_n) is not in.

   Since E is connected, it can't be separated into two disjoint nonempty open sets. If $A \cap \bar{B} \neq \emptyset$ or $B \cap \bar{A} \neq \emptyset$, then we would have a sequence

(x_n), WOLOG, contained in A converging to a point $x \in \bar{B} \implies x \in B$, which is expected.

WOLOG, in the case of $\bar{A} \cap B \neq \emptyset$. Since $x \in \bar{A} \cap B$, $\exists (x_n) \in A$ such tath $(x_n) \to x$. Thus, $x \in B$. □

2. $[(x_n) \to x \text{ where } (x_n), x \in A \text{ or } (x_n), x \in B] \implies E \subseteq \mathbb{R}$

Proof. Now, assume that for every pair of non empty disjoint sets A and B such that $E = A \cup B$ there exists a sequence $(x_n) \subset A \text{or} (x_n) \subset B$ where $(x_n) \to x$ and $x \in A$ or $x \in B$, with x being in set that (x_n) is not in.

We WTS that E is connected. Suppose BWOC that E is disconnected. Thus, we can write E as the union of two nonempty disjoint open sets A and B, thus $E = A \cup B$ where $A \cap B = \emptyset$ and $A \neq \emptyset \neq B$.

WOLOG, assume that $\exists (x_n) \subset A$ such that $(x_n) \to x$ where $x \in B$. Since A and B are disjoint this contradicts are assumption that A and B are disjoint and open, because a sequence in A that converges to an $x \in B$ forces a x to belong to the closure of A, which implies that $A \cap \bar{B} \neq \emptyset$. This is a contradiction to what we assumed.

Thus, E must be connected.

□

So, a set $E \subseteq R$ is connected if and only if, for all nonempty disjoint sets A and B satisfying $E = A \cup B$, there always exists a convergent sequence $(x_n) \to x$ with (x_n) contained in one of A or B, and x an element of the other.

□

3.4.9

Let $r_1, r_2, r_3, ...$ be an enumeration of the rational numbers, and for each $n \in \mathbb{N}$ set $\epsilon_n = (1/2)^n$. Define $O = \bigcup_{n=1}^{\infty} (V_{\epsilon_n})(r_n)$, and let $F = O^c$

(a.) Argue that F is closed, nonempty set consisting only of irrational numbers

Let $\{r_1, r_2, r_3, ...\}$ be an enumeration of the rational numbers. Also let $(V_{\epsilon_n})(r_n) = (r_n - \frac{1}{2^n}, r_n + \frac{1}{2^n})$ be epsilon neighborhoods around each rational number. If we were take the union of all of the neighborhoods we just defined, call it $O = \bigcup_{n=1}^{\infty} (V_{\epsilon_n})(r_n)$, the complement of this union by definition would be all the elements not in any of the epsilon neighborhoods defined.

Thus, O is an open set by definition since it is the countable union of open intervals (example 3.2.2). Also, since O is open, by Theorem 3.2.13, F is closed.

Since the irrational numbers are dense in $\mathbb{R}$, between any two numbers in $\mathbb{R}$, there exists an irrational number. So, since O covers intervals surrounding every rational number, the irrational numbers outside of those intervals will be contained in F. Thus F contains irrational numbers, and is thus nonempty.

Thus, F is a closed, nonempty set consisting of only irrational numbers.

(b.)Does F contain any nonempty open intervals? is F totally disconnected?

An open interval $(a, b) \in F$ if every point in (a, b) is not covered by any epsilon neighborhood $V_{\epsilon_n}(r_n)$. But, since the rational numbers are dense in $\mathbb{R}$, for any open interval around any irrational number, there are rational numbers that will fall within the interval (a, b). Thus, F does not contain any nonempty open intervals.

F is totally disconnected if the only connected subsets are singletons. Since between any two irrational numbers, there exists rational numbers that are in O, any segment of irratiional numbers does not form a continuous connection. Thus, F is totally disconnected.

(c.) Is it possible to know whether F is perfect? If not, can we modify this construction to produce a nonempty perfect set of irrational numbers?

F is perfect if it is closed and contains no isolated points. Although F is closed, it contains isolated points. Consider an irrational number x. There exists a neighborhood around x such that some rational numbers are included. Thus making the irrational points not limit points of F. So, since F contains isolated points, it is not a perfect set.

If we were to modify it, we would need to create a nonempty perfect set of irrational numbers. We could do this by removing a countable dense set of rationals from a closed interval like $[0, 1]$, in a specific way making sure that the limit points of irrational numbers are retained.

Notes

Definitions:

Perfect

A set $P \subseteq \mathbb{R}$ is **perfect** if it is closed and contains no isolated points.

Compact

A set $K \subseteq \mathbb{R}$ is **compact** if every sequence in K has a subsequence that converges to a limit that is also in K.

Theorems:

Theorem 1.4.11

(i) The set $\mathbb{Q}$ is countable. (ii) The set $\mathbb{R}$ is uncountable.