Math-M413 Exam 2

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Take Home Exam 2

1. Supposed that $K \in \mathbb{R}$ is compact and $F \subseteq K$ is closed.

(a) Prove that F is compact.

Proof. Given that K is in $\mathbb{R}$ and is compact tells us that K is closed and bounded by Theorem 3.3.4.

The question tells us that $F \in K$ is closed. Since F is a subset of K and K is bounded, F is also bounded by K's bounds. So, since F is closed and bounded, by Theorem 3.3.4 it is compact. □

(b) Use the previous answer to prove that $F \bigcap K$ is compact.

Proof. Given that $F \subseteq K$ is closed, that K is compact and F is closed, WTS that $F \bigcap K$ is compact. To do so, we will again use Theorem 3.3.4, and show that $K \bigcap F$ is closed and bounded.

We are given that F is closed and that K is compact which tells us that K is closed. Since these are both closed, we know that the intersection of two closed sets must also be closed by Theorem 3.2.14. Thus, $F \bigcap K$ is closed.

Given that $F \subseteq K$ where K is compact, we know that K bust be bounded by Theorem 3.3.4. So, all of the elements of $F \bigcap K$ must be contained within the bounds of the set K, telling us that $F \bigcap K$ is bounded.

Thus, since $F \bigcap K$ has been found to be closed and bounded, by Theorem 3.3.4, $F \bigcap K$ must be compact. □

2. Prove the partial sums of an alternating series form a Cauchy sequence.

Using the definition provided in Exc 2.7.1. So let (a_n) be a decreasing nonnegative sequence which convergest to zero. WTS that $(s_n) = (\sum_{k=1}^{n}(-1)^{k+1}a_k)$ is cauchy where (s_n) is the sequence of partial sums;

Proof. Let $\epsilon > 0$ be arbitary. By the question definition we are given that (a_n) is nonnegative and decreasing, thus, $a_n \to 0$. So, there must exist an $N \in \mathbb{N}$ such that $a_n < \epsilon, \forall n \geq N$.

Now, let $n, m \in \mathbb{N}$ such that $n > m$. Then, combining all of these statements together, we have that, $0 \leq \sum_{k=m+1}^{n}(-1)^k a_k \leq a_{m+1}$.

Then, for each $n, m \in \mathbb{N}$ where $n > n \geq N$, we can get the difference of partial sums.

$$
\begin{aligned}
|s_n - s_m| &= |\sum_{k=1}^{n}(-1)^{k+1}a_k - \sum_{k=1}^{m}(-1)^{k+1}a_k| \\
&= |\sum_{k=m+1}^{n}(-1)^{k+1}a_k| \leq a_{m+1} < \epsilon
\end{aligned}
$$

Thus the elements in s_n get close enough together to conclude that they converge by the Cauchy Criterion.

So, the sequence of partial sums s_n is a cauchy sequence.

$\square$

3. Prove that if $\sum |a_n|$ converges absolutely, then $\sum |(a_n)^2|$ also converges absolutely.

Given that the $\sum |a_n|$ converges, we WTS that $\sum |(a_n)^2|$ converges absolutely.

The $\sum |a_n|$ converging tells us that as $n \to \infty$, $|a_n| \to 0$. Now since $|a_n| \to 0$ this implieas that $(a_n)^2$ will approach 0 faster as it goes towards infinity.

So, since it appears that $|a_n| \geq |(a_n)^2 \forall n \in \mathbb{N}$, we will use the comparison test to prove that $\sum (a_n)^2$ converges absolutely.

Given that the $\sum |a_n|$ converges

4. Define sets A, B as $A = \{(-1)^{n+1} + \frac{5}{n} : n \in \mathbb{N}\}$, and $B = \{x \in \mathbb{Q} : 0 < x < 1\}$ answer the following:

(a) What are the limit points?

The limit points of A are -1 and 1

The limit points of B are 0 and 1.

(b) Is the set open, closed, or neither?

The set A is neither.

The set B is neither.

(c) Does the set contain isolated points? Identify or describe them if yes.

All of the points in set A are isolated.

The set B does not have any isolated points.

(d) Find the closure of each set.

The closure of $A = \bar{A} \bigcup \{-1, 1\}$.

Since as $n \to \infty$, $\frac{5}{n} \to 0$ and $(-1)^{n+1}$ bounces between -1 and 1

The closure of $B = [0, 1]$

5. Provide a counterexample for the following claim: $(\bar{E})^c = (\bar{E^c})$

Consider the set $E = (-1, 0) \cup (0, 1)$, then $\bar{E} = [-1, 1]$ and $(\bar{E})^c = (-\infty, 1) \cup (1, \infty)$, and $(\bar{E^c}) = (-\infty, -1] \cup 0 \cup [1, \infty)$. Thus, $(\bar{E})^c \neq (\bar{E^c})$.

6. Find an explicit cover for the set $\{ 1, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots \}$ that does not have a finite subcover. Conclude that this set is not compact.

As the set goes towards infinity the elements are also members of a set that can be defined by $S = \{\frac{n}{n+1} : n \in \mathbb{N}\}$ (I know that 1 is not in this set but all others are.). As this set S goes to infinity, the elements get arbitrarly close to 1 but never reach 1.

Now consider the collection of intervals for each $n \in \mathbb{N}$.

7. Let A, B be nonempty subsets of $\mathbb{R}$. Show that if there exist disjoint open sets U, V with $A \subseteq U$ and $B \subseteq V$, then A, B are separated.

Proof. Given that $A \subseteq U$ and $B \subseteq V$ where $U \cap V = \emptyset$. We WTS that A and B are separated.

To show that A and B are separated we need to show that $A \cap \bar{B} = \emptyset$ and $\bar{A} \cap B = \emptyset$. To start we will show that $A \cap \bar{B} = \emptyset$.

Let b be any point in B. Since B is a subset of V, $b \in V$. Consider the case where b is a point in B, that is also within $\bar{A}$. Since A is closed, every neighborhood of b will intersect A. Thus either b is in A or is a limit point of A.

But, since $A \subseteq U$ and U and V are disjoint open sets. Since $b \in V$, it cannot be in U, and cannot be a limit point of A. Thus, $b \notin \bar{A}$ because a point in B cannot be in $\bar{A}$ while being in the disjoint open set V.

So, it must be true that $A \cap \bar{B} = \emptyset$.

WLOG, the argument is the same to show the other direction of $\bar{A} \cap B = \emptyset$.

So, since $A \cap \bar{B} = \emptyset$ and $\bar{A} \cap B = \emptyset$ we know that by definition A and B are separated. So, if there exist disjoint open sets U, V with $A \subseteq U$ and $B \subseteq V$, then A, B are separated

□

8. Prove each limit statement

(a) $\lim_{x \to -1} \left(3x^2 + x - 4\right) = -2$

The function $f(x) = 3x^2 + x - 4$ is continuous so we should be able to just plug in -1 for x and get our expected limit as $x \to -1$.

$$\begin{aligned} f(-1) &= 3(-1)^2 + (-1) - 4 \\ f(-1) &= 3(1) - 1 - 4 \\ f(-1) &= 3 - 1 - 4 \\ f(-1) &= 2 \end{aligned}$$

Thus the $\lim_{x \to -1} \left(3x^2 + x - 4\right) = -2$.

(b) $\lim_{x\to 2}\left(\frac{1}{x^2}\right) = 1/4$

Again the function $g(x) = \frac{1}{x^2}$ is continuous. So,

$$g(2) = \frac{1}{(2)^2}$$
$$g(2) = \frac{1}{4}$$

So the $\lim_{x\to 2}\left(\frac{1}{x^2}\right) = 1/4$.

9. Consider the Cantor like set defined by removing the middle α from $[0,1]$ where $\alpha = \frac{1}{k}$ for $k \geq 3$.

(a) For $\alpha = \frac{1}{4}$, that is $k = 4$, compute the length of the Cantor like set.

(b) Is this Cantor like set compact?

(c) Consider the formal equation $mC = [0,2]$ where C is a Cantor like set and $m \in \mathbb{N}$. For the Cantor like set with $k = 4$, what is the value of m? Use this value of m to compute the dimension of this Cantor like set.

10. Give an example of each or state why the request is impossible/reference appropriate Theorems.

(a) Two functions f, g neither one continuous at 0 but buth $f(x)g(x)$ and $f(x)+g(x)$ are continuous at 0. Note: you may only use two functions for this part.

Let,

$$f(x) = \begin{cases} 1 & \text{if } x \geq 0, \\ 0 & \text{if } x > 0. \end{cases} \tag{1}$$

and

$$g(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x \leq 0. \end{cases} \tag{2}$$

Then,

Then, $f(x)g(x) = 0, \forall n \in \mathbb{N}$. Also, $f(x) + g(x) = 1, \forall n \in \mathbb{N}$. Both of these are continuous at 0.

(b) Two functions f, g such that f is continuous at 0, g is not continuous at 0, but $f(x)g(x)$ is continuous at 0.

(c) A function $f(x)$ that is not continuous at 0 such that $f(x)+\frac{1}{f(x)}$ is continuous at 0.