#4 PAGE 9

Ω - domain
S set, $\neq \emptyset$ for Ω
- S open
- $\Omega \setminus S$ open

$\Big\}$ $S = \Omega$ or $\Omega \setminus S = \emptyset$

Using Ω here instead of R_2 in the book.

Proof.

By definition, $\Omega \setminus S$ is closed since S is open. Since $\Omega \setminus S$ is given to be open, $\Omega \setminus S$ is both open and closed. Hence either $\Omega \setminus S = \emptyset$ or $\Omega \setminus S = \Omega$. If $\Omega \setminus S = \Omega$, $S = \emptyset$ but S is nonempty, thus $\Omega \setminus S = \emptyset$.

Picking up from tuesday...

Ex. 1 $u(x,y) = x - x^2 + 3y^2$, $z_0 = (1,2)$, $\vec{v} = \langle 1/\sqrt{2}, 1/\sqrt{2} \rangle$

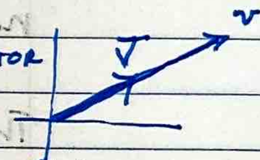
"The Long Way" $u'(z; \vec{v}) = u_x(z)k_1 + u_y(z)k_2$

NORMALIZE VECTOR

$v = \langle 3, 5 \rangle$

$\|v\| = \sqrt{9+25}$

$\hat{v} = \frac{v}{\sqrt{34}} = \langle \frac{3}{\sqrt{34}}, \frac{5}{\sqrt{34}} \rangle$



$u_x(z) = 1 - 2x \quad u_y(z) = 6y$

$u_x(z) = -1, \quad u_y(z) = 12$

$u'(z; \vec{v}) = -1\left(\frac{1}{\sqrt{2}}\right) + 12\left(\frac{1}{\sqrt{2}}\right)$

$= \boxed{\frac{11}{\sqrt{2}}}$ Slope of the tangent line at the point z_0 but not necessarily the tangent line.

Definition |

Let Ω be a domain ~~$\mathcal{E}$~~ and $u \in C^1(\Omega)$. Then the gradient vector of u at the point z is

$$\nabla u(z) = \langle u_x(z), u_y(z) \rangle$$

Fact: $u'(z_0; \vec{v}) = \nabla u(z_0) \cdot \vec{v}$

Notation: $\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right\rangle$

Ex. 1 $u(x,y) = x - x^2 + 3y^2$; $z_0 = (1,2)$

$\nabla u(z) = \langle 1-2x, 6y \rangle$

$\nabla u(z_0) = \langle 1-2, 6\cdot 2 \rangle$

$= \langle -1, 12 \rangle$

$\nabla u(z_0) \cdot v = \langle -1, 12 \rangle \cdot \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$

$= -\frac{1}{\sqrt{2}} + \frac{12}{\sqrt{2}} = \frac{11}{\sqrt{2}}$

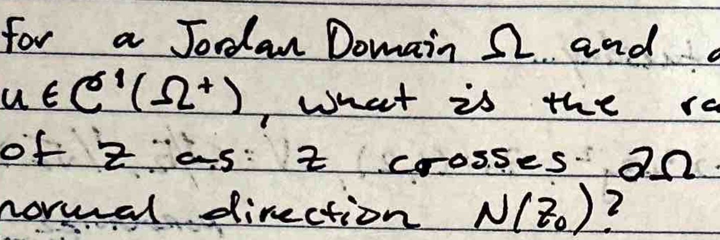
∇ = "nabla"

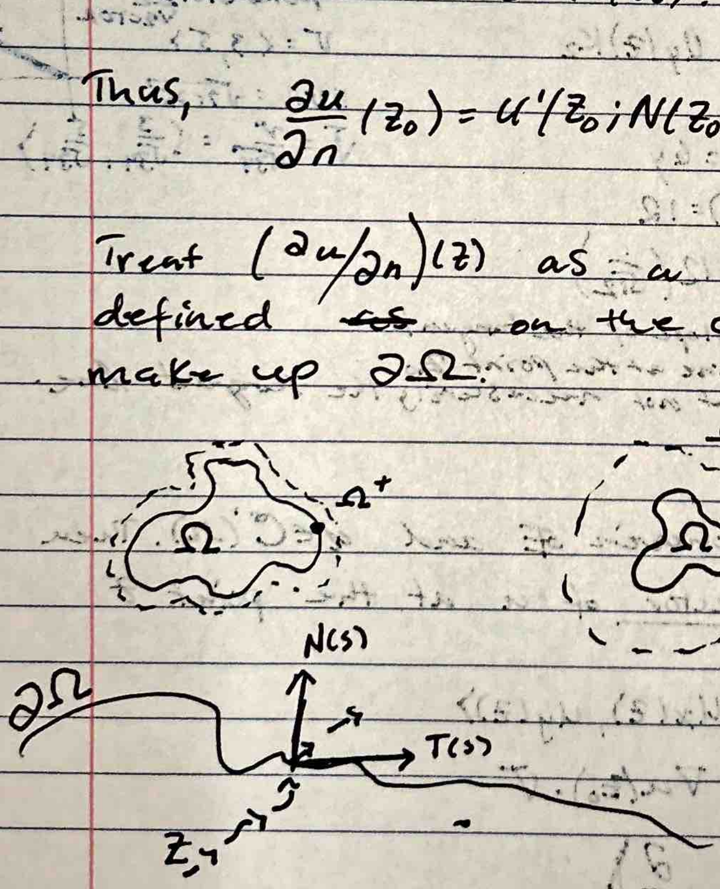
Outward Normal Derivative

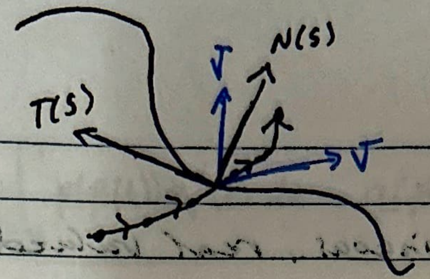
For a Jordan Domain Ω and a function $u \in C^1(\Omega^+)$, what is the rate of change of z as z crosses $\partial\Omega$ in the normal direction $N(z_0)$?

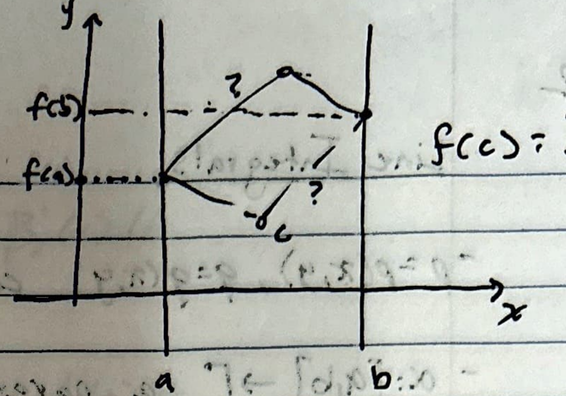
Thus, $\frac{\partial u}{\partial n}(z_0) = u'(z_0; N(z_0)) = \nabla u(z_0) \cdot N(z_0)$

Treat $(\partial u / \partial n)(z)$ as a function of z defined ~~as~~ on the curves that make up $\partial\Omega$.



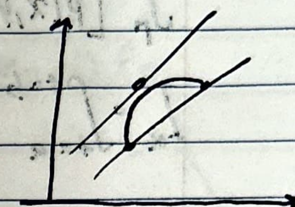




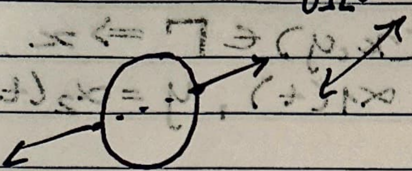


$\partial[a,b] = \{a,b\}$

If f is diff. on $[a,b)$ there exists $c \in [a,b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$



Ex. $\Omega = C(0;r)$, $\int_{\partial\Omega} \frac{\partial u}{\partial n}(z)\,dz = 0$



Laplacian I

If u_{xy} and u_{yx} exist and are continuous, then they are equal.

Note:

$\mathcal{C}(\Omega) \supset \mathcal{C}^1 \supset \dots \supset \mathcal{C}^\infty(\Omega)$

where $\mathcal{C}^\infty(\Omega)$ is the set of functions with continuos partial derivatives of all order.

Laplacian II

Def. Let $u \in \mathcal{C}^2(\Omega)$. Then the <u>laplacian</u> of u is the second derivative, that is $u_{xx} + u_{yy}$ and denoted

$$\Delta u(z) = \Delta u(x,y) = u_{xx}(x,y) + u_{yy}(x,y)$$

Def. A function $u \in C^2(\Omega)$ is <u>harmonic</u> if and only if $\Delta u = 0$.

$\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right\rangle$, $\nabla\nabla = \left\langle \frac{\partial^2}{\partial x^2}, \frac{\partial^2}{\partial y^2} \right\rangle$ ~~$\nabla \cdot \nabla$~~ $\Delta \approx \nabla^2$

<u>Line Integrals</u>

- $p = p(x,y)$, $q = q(x,y)$ continuous, real valued

- $\alpha: [a,b] \to \Gamma$ a parameterization;

$\alpha(t) = (\alpha_1(t), \alpha_2(t))$

$\int_\Gamma [p(x,y)dx + q(x,y)dy]$

$$\boxed{\begin{array}{c} p(x,y) = p(\alpha(t))\alpha_1'(t) \\ \alpha(t) = \langle \alpha_1(t), \alpha_2(t) \rangle \\ = \langle x, y \rangle \\ x = \alpha_1(t) \\ dx = \alpha_1'(t)dt, \end{array}}$$

$I_\alpha = \int_{t=a}^{t=b} [p(\alpha(t))\alpha_1'(t)dt + q(\alpha(t))\alpha_2'(t)dt]$

- By calculus and $(x,y) \in \Gamma \Rightarrow x = \alpha_1(t)$, $y = \alpha_2(t)$

<u>Theorem</u>

Let α, β be equivalent Parameterizations of Γ. Then $I_\alpha = I_\beta$

Find 2 inequivalent parametrizations for Γ. Compute I_α and I_β and see that you get diff. #'s.

<u>Ex.</u> Let ~~$p(x,y), q$~~ $p(x,y) = x^2y$, $q(x,y) = 2$, $\Gamma = \{z : |z| = 1, y \geq 0\}$ compute $\int_\Gamma [pdx + qdy]$

$\alpha(t) = (\cos(2\theta), \sin(2\theta))$, $\theta \in [0, 2\pi]$

$\beta(t) = (-t, \sqrt{1-t^2})$, $t \in [-1, 1]$

$\int_0^{\pi/2} p(\alpha(t))\alpha_1'(t)dt + q(\alpha(t))\alpha_2'(t)dt$

$$\int_{-1}^{1} p(\beta(t))\beta_1'(t)\,dt + q(\beta(t))\beta_2'(t)\,dt$$

$$\int_0^{\pi/2} p(\alpha(t))\alpha_1'(t)\,dt + q(\alpha(t))\alpha_2'(t)\,dt$$

$$= \int_0^{\pi/2} -\cos^2(2\theta)\sin(2\theta)\sin(2\theta)\cdot 2\,d\theta \; + \; 2\cos(2\theta)\cdot 2\,d\theta$$

$$= -2\int_0^{\pi/2} \cos^2(2\theta)\sin^2(2\theta)\,d\theta + 4\int_0^{\pi/2} \cos^2(2\theta)\,d\theta$$

These are supposed to end up being equal.

E has nothing to do w/ p and q

$$\int_{-1}^{1} \left[t^2\sqrt{1-t^2}(-1) + 2\cdot\tfrac{1}{2}(1-t^2)^{-1/2}(-2t)\right]dt$$

$$= -1\int_{-1}^{1} t^2\sqrt{1-t^2}\,dt + -2\int (1-t^2)^{-1/2}(t)\,dt$$