Complex Analysis

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Outline I

Outline II

Outline III

- Argument Principle
- Real Integrals

Introduction

Objectives Overview I

1. Complex numbers

Exercises

- Mostly basic; but when it isn't…

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- PLTS: proof left to student

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- Polar coordinates frequently ‘used’; not explicitly (in this book anyway)

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- however, usually considered as complete objects (this represents the area of a circle, so I know the answer)

Domains

Objectives

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- Real valued functions of two variables

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2. The set of all ordered pairs $\{(x, y) : x, y \in \mathbb{R}\}$
3. Addition and subtraction are defined component-wise
4. Scalar multiplication is identical to vectors

Definition

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Definition

Let $z = (x, y)$ *and* $z_0 = (x_0, y_0)$. *Then the* **distance** *between* z, z_0 *is*

$$|z - z_0| = \sqrt{(x - x_0)^2 + (y - y_0)^2}$$

Definition

Let $r > 0$ and z_0 be a fixed point in $\mathbb{R}^2$. The **disc of radius** r **with center** z_0 *is*

$$D(z_0; r) = \{z \in \mathbb{R}^2 : |z - z_0| < r\}$$

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Definition

A point z is an **interior point** *of a set $\Omega \subset \mathbb{R}^2$ if and only if there is a disc with center z contained entirely within Ω.*

Examples

Domain Examples:

- Let $\Omega = \mathbb{R}^2$

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- Let $\Omega = \{(x, y) : 0 \leq x \leq 1, 0 < y \leq 1\}$

Closed Discs

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- Note that closed discs have points which are interior points and points which are not interior points.
- Furthermore, a disc which is "partially closed" (includes some points on the circle but not all) is NOT a closed disc (or a closed set).

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Connected and Disconnected Subsets

Definition

An open subset $\Omega \subset \mathbb{R}^2$ *is* **disconnected** *if and only if it can be written as* $\Omega = \Omega_1 \cup \Omega_2$ *where* Ω_1, Ω_2 *are non-empty and disjoint.*

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Definition

An open set $\Omega \subset \mathbb{R}^2$ is **connected** *if and only if it is not disconnected.*

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- $D(z_0; r) \backslash \{z_0\}$
- $A = \{z : a < |z| < b\}$ for $a < b \in \mathbb{R}$

A Flawed Proof the Unit Disc Is Disconnected

Find the flaw in the following attempt to prove that $D(z_0; r)$ is disconnected:

- Let $U = \{z \in D(z_0; r) : x < 0\}$

Notes:

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Notes:

- $U = D(z_0; r) \cap \{z : x < 0\}$
- $V = D(z_0; r) \cap \{z : x > 0\}$
- Hence both U, V are each the finite intersection of open sets, hence are themselves open.

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Loosely speaking, open sets possibly containing holes but only 1 component. Ask: "Would the paint bucket tool in MS Paint work?"

The Boundary and Boundary Points

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A set $S \subset \mathbb{R}^2$ is **bounded** *if and only if there exists $r > 0$ such that $S \subset D(0; r)$.*

Note that a boundary point of S is also a boundary point of $\mathbb{R}^2 \backslash S$.

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Definition

The **boundary** *of a set $S \subset \mathbb{R}^2$ is the set of all boundary points of S; denoted by ∂S.*

Examples

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- $H = \{(x,y) \in \mathbb{R}^2 : y > 0\}$; $\partial H = \{(x,y) \in \mathbb{R}^2 : y = 0\}$
- $\mathbb{R}^2$; $\partial \mathbb{R}^2 = \emptyset$

Plane Curves

Conventions and Definitions I

- $0 = (0, 0)$

Conventions and Definitions II

Circle

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- Also $\gamma : [0, 1] \to C(0; r)$ with $\gamma(t) = (r\cos(2\pi t), r\sin(2\pi t))$.

Parameterization of Curves

Definition

Let $\Gamma \subset \mathbb{R}^2$ *and* $[a, b] \subset \mathbb{R}$ *with* $a < b$. *Any function* $\alpha = (\alpha_1, \alpha_2) : [a, b] \to \Gamma$ *is a* **parameterization** *of* Γ *if and only if*

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3. *if* $\alpha(a) = \alpha(b)$, *then* $\alpha'(a) = \alpha'(b)$

Definition

A parameterization α is **simple** *if and only if $\alpha|_{(a,b)}$ is injective.*

Definitions

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A parameterization α *is* **closed** *if* $\alpha(a) = \alpha(b)$.

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- The circle $C(0;1)$ is often parameterized by $\alpha(t) = (r\cos(t), r\sin(t))$ where $t \in [0, 2\pi]$ or $\beta(t) = (r\cos(2\pi t), r\sin(2\pi t))$ for $t \in [0,1]$

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- The parameterization $\eta(t) = (t, t^3)$ for $t \in [-1,1]$ is a simple, not closed parameterization between $(-1,-1)$ and $(1,1)$

Piecewise-smooth Parameterizations I

Definition

A parameterization $\alpha : [a, b] \to \Gamma$ *is a* **piecewise-smooth parameterization** *if and only if there exists a finite set of values* $a = a_0 < a_1 < a_2 < \cdots < a_{n-1} < a_n = b$ *such that the function* α *restricted to the intervals* $[a_{i-1}, a_i]$ *gives a smooth parameterization of* $\Gamma_i = \{\alpha(t) : t \in [a_{i-1}, a_i]\}$.

Piecewise-smooth Parameterizations II

Examples:

Smooth Curves

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Smooth Curves

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- If α is a smooth parameterization of Γ, then the pair (Γ, α) is called a **piecewise-smooth curve**.
- It is customary to omit mention of α and speak of "the (smooth) curve Γ"
- The situation is delicate however; see the Peano curve

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- Then $\alpha'(t) = z_1 - z_0$ (independent of t)
- So
$$\int_0^1 \alpha'(t)\mathrm{d}t = (z_1 - z_0)(1 - 0)$$

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If Γ *is a smooth curve given by* $\alpha : [a, b] \to \Gamma$, *then the* **length of** α *is*

$$\textit{length } (\alpha) = \int_a^b |\alpha'(t)| \mathrm{d}t$$

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Since α is smooth, the integrand is piecewise continuous and bounded, hence the integral exists.

Examples

- $C(0; r)$ with $\alpha(t) = (r\cos(t), r\sin(t))$ for $t \in [0, 2\pi]$

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- $C(0; r)$ with $\beta(t) = (r\cos(t), r\sin(t))$ for $t \in [0, 6\pi]$

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- Goal: adapt length integral to be new parameter

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- $\sigma(s)$ will have the property that $\sigma(s_1) - \sigma(s_0) = s_1 - s_0$
- $\sigma(s)$ will be "equivalent" to α

Equivalence of Intervals

Definition

Let $[a_1, b_1]$ *and* $[a_2, b_2]$ *be intervals on the* s, t *axes respectively. A function* $E : [a_1, b_1] \to [a_2, b_2]$ *is an* **equivalence** *if and only if it is:*

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This is a formalization of the idea that $E(a_1) = a_2$, $E(b_1) = b_2$, and as s increases from a_1 to b_1, it is also the case that $t = E(s)$ increases from a_2 to b_2.

Theorem

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Let $\alpha : [a, b] \to \Gamma$ be a piecewise-smooth parameterization with length $L > 0$. Then there exists a unique piecewise-smooth parameterization $\sigma : [0, L] \to \Gamma$ equivalent to α which satisfies:

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1. *the tangent vector* $\sigma'(s)$ *has unit length;* $|\sigma'(s)| = 1$

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1. *the tangent vector* $\sigma'(s)$ *has unit length;* $|\sigma'(s)| = 1$
2. *the distance traveled along* Γ *from* $\sigma(0)$ *to* $\sigma(s)$ *is* s

Proof

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$\square$

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- Note that if σ satisfies property (1), then the distance from $\sigma(0)$ to $\sigma(s)$ is $\int_0^s |\sigma'(t)| \mathrm{d}t = s$. It is sufficient to prove the result for smooth parameterizations; hence assume α is smooth. Define $F(\tau) = \int_a^\tau |\alpha'(t)| \mathrm{d}t$ for $\tau \in [a, b]$. By Calculus, $F'(\tau) = |\alpha'(\tau)| > 0$ and continuous.

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- Define E as the inverse of F. By the Chain Rule, $E'(s) > 0$ and E is an equivalence from $[0, L]$ to $[a, b]$.

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- Define E as the inverse of F. By the Chain Rule, $E'(s) > 0$ and E is an equivalence from $[0, L]$ to $[a, b]$.
- Define $\sigma(s) = \alpha(E(s))$ and verify $|\sigma'(s)| = 1$ by the Chain Rule.

$\square$

Definitions

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A **Jordan curve** *is a simple, closed, piecewise smooth curve.*

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Definition

A **Jordan domain** *is a bounded domain* Ω *whose boundary* $\partial\Omega$ *is a Jordan curve.*

Jordan Curve Theorem

Theorem (Jordan Curve Theorem)

If Γ is a Jordan curve, then its complement $\mathbb{R}^2 \backslash \Gamma$ consists of two disjoint domains, one bounded (the "inside") and one unbounded (the "outside"); both have Γ as boundary. If a point inside Γ is joined by a path to a point outside Γ, then the path must cross Γ.

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Omitted. And difficult for generic Jordan curves. □

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A Jordan domain is *positively oriented* if the given parameterization has the interior to the left.

Holes in Domains

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Definition

A Jordan domain Ω *is* **k-connected** *if its boundary consists of* k *Jordan curves.*

Note that the complement of a k-connected Jordan domain is therefore k disjoint connected components.

Normal Vectors I

Definition

Let Ω be a Jordan domain with boundary $\partial\Omega$ and Γ one of the Jordan curves comprising $\partial\Omega$ with parameterization $\sigma : [0, L] \to \Gamma$

is called the **outward normal vector**.

Normal Vectors II

Example:

Calculus in the Plane

Overview

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- In particular, how these functions change in the vicinity of a point $z \in \mathbb{R}^2$ - differentiation
- The Chain Rule will be used offensively often

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This will be "accomplished" using Green III

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- Since $z = (x, y)$, $u(z) = u(x, y)$
- $u : \Omega \to \mathbb{R}$
- For $S \subset \mathbb{R}$, $u^{-1}(S) = \{z \in \mathbb{R}^2 : u(z) \in S\}$

Continuity

Definition

Let $U \subset \mathbb{R}^2$. A function $u : \Omega \to \mathbb{R}$ is **continuous** *if and only if for every open interval $S \subset \mathbb{R}$, the set $u^{-1}(S)$ is open in $\mathbb{R}^2$.*

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- Notation: $\mathfrak{C}^k(\Omega)$
- In particular, $u \in \mathfrak{C}^1(\Omega)$ implies $u_x, u_y \in \mathfrak{C}(\Omega)$
- That is, $\mathfrak{C}^1(\Omega)$ is the set of all continuously differentiable functions on Ω

Theorem I

Theorem (Linear Approximation)

Let $u \in \mathfrak{C}^1(\Omega)$ and $z_0 = (x_0, y_0) \in \Omega$. For each point $z = (x, y) \in \Omega$, we may write

$$u(z) = u(z_0) + u_x(z_0)(x - x_0) + u_y(z_0)(y - y_0) + \epsilon_1(x - x_0) + \epsilon_2(y - y_0)$$

where $\epsilon_i \to 0$ as $z \to z_0 \in \Omega$.

Notation:

$$u(z) = L(z; z_0) + E(z; z_0)$$

Note: $E(z; z_0)$ tends to zero rapidly; specifically

$$\lim_{z \to z_0} \frac{|E(z; z_0)|}{|z - z_0|} = 0$$

Theorem II

Theorem

Let $u \in \mathfrak{C}^1(\Omega), z_0 \in \Omega$. *Then* u *is continuous at* z_0. *That is*

$$\mathfrak{C}^1(\Omega) \subset \mathfrak{C}(\Omega)$$

Theorem (Chain Rule)

Let $u \in \mathfrak{C}^1(\Omega)$ *and* $\alpha(t) = (\alpha_1(t), \alpha_2(t))$ *parameterize* $\Gamma \subset \Omega$. *Then*

$$\frac{\mathrm{d}}{\mathrm{d}t} u(\alpha(t)) = u_x(\alpha(t))\alpha_1'(t) + u_y(\alpha(t))\alpha_2'(t)$$

Directional Derivatives I

Definition

Let $z = (x, y)$ be a point in the domain Ω, V a unit vector with tail at z, and u a function defined on Ω. Then the **derivative of u at z in the direction V** *is denoted $u'(z; V)$ and given by*

$$u'(z; V) = \frac{\mathrm{d}}{\mathrm{d}s} u(z + sV)\Big|_{s=0}$$

Alternatively

$$u'(z; V) = \lim_{s \to 0} \frac{u(z + sV) - u(z)}{s} \tag{1}$$

Directional Derivatives II

Theorem

Let $u \in \mathfrak{C}^1(\Omega)$ and $z \in \Omega$. If $V = \langle k_1, k_2 \rangle$ is a unit vector, then

$$u'(z; V) = u_x(z)k_1 + u_y(z)k_2$$

Gradient Vector

Definition

Let Ω be a domain and $u \in \mathfrak{C}^1(\Omega)$. Then the **gradient vector** *of u at the point z is*

$$\nabla u(z) = \langle u_x(z), u_y(z) \rangle$$

Fact:

$$u'(z_0; V) = \nabla u(z_0) \cdot V$$

Notation: $\nabla = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \rangle$

Outward Normal Derivative

Motivation: For a Jordan domain Ω and function $u \in \mathfrak{C}^1(\Omega^+)$, what is the rate of change of z as z crosses $\partial\Omega$ in the normal direction $N(z_0)$?

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Treat $(\partial u/\partial n)(z)$ as a function of z defined on the curves that make up $\partial\Omega$.

The Laplacian I

- If u_{xy} and u_{yx} exist and are continuous, then they are equal.

Note:

$$\mathfrak{C}(\Omega) \supset \mathfrak{C}^1(\Omega) \supset \mathfrak{C}^2(\Omega) \cdots \supset \mathfrak{C}^\infty(\Omega)$$

where $\mathfrak{C}^\infty(\Omega)$ is the set of functions with continuous partial derivatives of all order.

The Laplacian II

Definition

Let $u \in \mathfrak{C}^2(\Omega)$. Then the **Laplacian** *of u is the second derivative, that is $u_{xx} + u_{yy}$ and denoted*

$$\Delta u(z) = \Delta u(x, y) = u_{xx}(x, y) + u_{yy}(x, y)$$

Definition

A function $u \in \mathfrak{C}^2(\Omega)$ is **harmonic** *if and only if $\Delta u = 0$.*

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- by calculus and $(x, y) \in \Gamma \Rightarrow x = \alpha_1(t), y = \alpha_2(t)$

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Let α, β be equivalent parameterizations of Γ. Then

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Proof.

Apply the definition of equivalence and the Chain Rule. $\square$

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- Find the equivalence between α, β

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- Given $p\,\mathrm{d}x + q\,\mathrm{d}y$, does there exist u such that $u_x = p$ and $u_y = q$?
- Integrals of such differentials are "nice"

Exact Differential Diagram

$$\begin{tikzcd}
& u(x,y) = C \arrow[dl, "\frac{\partial}{\partial x}"'] \arrow[dr, "\frac{\partial}{\partial y}"] & \\
p(x,y)\,\mathrm{d}x \arrow[d, "\frac{\partial}{\partial y}"'] & + & q(x,y)\,\mathrm{d}y = 0 \arrow[d, "\frac{\partial}{\partial x}"] \\
p_y & = & q_x
\end{tikzcd}$$

Figure: Exact Equation Model

Path Independence Theorem

Theorem

Let Γ *be a piecewise-smooth curve from* z_0 *to* z_1 *lying inside a domain* Ω*. Let* $u \in \mathfrak{C}^1(\Omega)$*, then*

$$\int_\Gamma \mathrm{d}u = u(z_1) - u(z_0)$$

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Proof.

Denote the parameterization by $\alpha : [a, b] \to \Gamma$. Apply the definition of $\mathrm{d}u$ and the Chain Rule. $\square$

Double Integrals

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- If $u(x, y) \geq 0$ for all $(x, y) \in \bar{\Omega}$, then

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represents the volume of the right cylinder whose base is $\bar{\Omega}$ and whose height above (x, y) is $u(x, y)$.

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  represents the volume of the right cylinder whose base is $\bar{\Omega}$ and whose height above (x, y) is $u(x, y)$.
- Alternate notation:

$$\iint_{\Omega} u \,\mathrm{d}x\mathrm{d}y$$

Computation of Double Integrals

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- Method: iterated integrals
- Not always successful - Ω or u may not be "nice enough" for this to work

Green's Theorem

Theorem (Green's Theorem)

Let Ω be a k-connected Jordan domain and suppose $p(x, y), q(x, y)$ are functions in $\mathfrak{C}^1(\Omega^+)$ where Ω^+ is a domain containing both Ω and $\partial\Omega$. Then

$$\int_{\partial\Omega} [p \,\mathrm{d}x + q \,\mathrm{d}y] = \iint_{\Omega} (q_x - p_y) \,\mathrm{d}x\mathrm{d}y \tag{2}$$

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Proof of Green's Theorem

Proof.

Let $\Omega = \{(x, y) : x_0 < x < x_1, y_0 < y < y_1\}$ and denote $\partial\Omega = \Gamma_i$ for $i = 1, 2, 3, 4$ with positive orientation. Then compute $-\iint_\Omega p_y \mathrm{d}x\mathrm{d}y$ iteratively to get $\int_{\partial\Omega} p(x, y)\mathrm{d}x$ and observing that $\mathrm{d}x = 0$ on vertical edges. Now compute $\iint_\Omega q_x \mathrm{d}x\mathrm{d}y$ iteratively to get $\int_{\partial\Omega} q\mathrm{d}y$ using a similar observation about $\mathrm{d}y$ on $\partial\Omega$. Integral properties give the desired result from here. $\square$

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- Recall that Ω will be parameterized by arc-length; then $N = \langle \sigma_2'(s), -\sigma_1'(s) \rangle$ so that $\frac{\partial q}{\partial n}\mathrm{d}s = q_x \mathrm{d}y - q_y \mathrm{d}x$

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- Thus $\frac{\partial q}{\partial n} \mathrm{d}s$ is a "hybrid notation" for a line integral

Green's Identities I

Theorem (Green I)

Let Ω be a k connected Jordan domain and $p, q \in \mathfrak{C}^2(\Omega^+)$. Then

$$\iint_\Omega \nabla p \cdot \nabla q \, \mathrm{d}x\mathrm{d}y = \int_{\partial\Omega} p \frac{\partial q}{\partial n} \, \mathrm{d}s - \iint_\Omega p \Delta q \, \mathrm{d}x\mathrm{d}y \tag{3}$$

Interpretation: two dimensional integration by parts

Green's Identities II

Theorem (Green II)

Let Ω be a k connected Jordan domain and $p, q \in \mathfrak{C}^2(\Omega^+)$. Then

$$\int_{\partial\Omega} \left(p\frac{\partial q}{\partial n} - q\frac{\partial p}{\partial n} \right) \mathrm{d}s = \iint_{\Omega} (p\Delta q - q\Delta p)\, \mathrm{d}x\mathrm{d}y \tag{4}$$

Proof.

Note that the left side of 4 is symmetric in p, q, therefore exchanging roles produces the same value. Performing the swap on the right side and subtracting gives the identity. □

Green's Identities III

Theorem (Green III)

Let Ω be a k connected Jordan domain with $\zeta \in \Omega$ and $q \in \mathfrak{C}^2(\Omega^+)$. Let $z = (x, y), r = |z - \zeta|$, then

$$q(\zeta) = \frac{1}{2\pi} \iint_\Omega \ln r \Delta q(z) \, \mathrm{d}x\mathrm{d}y - \frac{1}{2\pi} \int_{\partial\Omega} \left(\ln r \frac{\partial q}{\partial n}(z) - q(z) \frac{\partial \ln r}{\partial n} \right) \mathrm{d}s \tag{5}$$

Application

Theorem (Inside-Outside Theorem)

Let Ω be a k connected Jordan domain and $q \in \mathfrak{C}^2(\Omega^+)$. Then

$$\int_{\partial\Omega} \frac{\partial q}{\partial n}\,\mathrm{d}s = \iint_{\Omega} \Delta q \,\mathrm{d}x\mathrm{d}y \tag{6}$$

Interpretation: $\int_{\partial\Omega} \frac{\partial q}{\partial n}\,\mathrm{d}s$ is "net change" of q across $\partial\Omega$ (outside) is equal to "total change" of q over Ω $\iint_{\Omega} \Delta q \,\mathrm{d}x\mathrm{d}y$ (inside).

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Proof.

Take $p = 1$ in Green I. $\square$

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Fix $\zeta \in \mathbb{R}^2$ *and define* $r = |z - \zeta|$ *for any* $z \in \mathbb{R}^2$. *Then*

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Proof.

1. Observe that r is constant on $C(\zeta; r)$
2. Let $z = (x, y), \zeta = (\xi, \eta)$ and compute r using Euclidean distance. Then use properties of natural log and partial derivatives.

□

Recall:

Theorem (Green III)

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$$q(\zeta) = \frac{1}{2\pi} \iint_{\Omega} \ln r \Delta q(z) \, \mathrm{d}x\mathrm{d}y - \frac{1}{2\pi} \int_{\partial\Omega} \left(\ln r \frac{\partial q}{\partial n}(z) - q(z) \frac{\partial \ln r}{\partial n} \right) \mathrm{d}s \tag{7}$$

Proof strategy: Apply Green II with $p = \ln r$ on $\Omega \backslash \bar{D}(\zeta; \epsilon)$ and let $\epsilon \to 0$.

Green III Proof II

Proof Continued

Proof.

$\square$

Proof Continued

Proof.

1. Consider $\int_0^{2\pi} \left(\ln \epsilon \frac{\partial q}{\partial n} + q \frac{1}{\epsilon} \right) \epsilon \, \mathrm{d}\theta$

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Proof Continued

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4. Consider $\epsilon \ln \epsilon \int_0^{2\pi} \frac{\partial q}{\partial n} \, \mathrm{d}\theta$.
5. Claim: $\epsilon \ln \epsilon \to 0$ as $\epsilon \to 0$ by L'Hôpital's Rule

$\square$

Proof Continued Again

Proof.

□

Interpretation: theoretical use only!

Proof Continued Again

Proof.

1. Now 8 reduces to

$$2\pi q(\zeta) = \lim_{\epsilon \to 0} \iint_{\Omega \setminus \bar{D}(\zeta;\epsilon)} \ln r \Delta q \, \mathrm{d}x\mathrm{d}y - \int_{\partial\Omega} \left(\ln r \frac{\partial q}{\partial n} - q \frac{\partial \ln r}{\partial n} \right) \mathrm{d}s \tag{9}$$

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2. The limit must exist and equal

$$\iint_{\Omega} \ln r \Delta q \,\mathrm{d}x\mathrm{d}y \tag{10}$$

□

Interpretation: theoretical use only!

Future Useage

Note that $\frac{\partial \ln r}{\partial n} = -\frac{\partial \ln r}{\partial r}$ on the boundary of a disc (that is, on circles). This fact will be treated as "common knowledge" from this point onwards.

Harmonic Functions

Objectives

- Properties

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- Principles

Examples

- Constant functions

Verify $\Delta u = 0$ as exercise. Also construct/guess/determine two harmonic functions such that the product is not harmonic.

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- $u(x, y) = e^x \sin(y), v(x, y) = e^x \cos(y)$
- $u(x, y) = v(x, y) + w(x, y)$ where v, w are harmonic functions

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Fundamental Theorem

Theorem

Let Ω *be a* k *connected Jordan domain and* u *a harmonic function in* $\mathfrak{C}^2(\Omega^+)$*. Then*

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1. $\int_{\partial\Omega} \frac{\partial u}{\partial n}(z)\,\mathrm{d}s = 0$
2. *for* $\zeta \in \Omega$, $z \in \partial\Omega$,

$$u(\zeta) = -\frac{1}{2\pi}\int_{\partial\Omega}\left(\ln r\frac{\partial u}{\partial n}(z) - u(z)\frac{\partial \ln r}{\partial n}\right)\,\mathrm{d}s \tag{11}$$

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where $r = |z-\zeta|$.

Proof.

Note that $\Delta u = 0$ and apply Theorem 13 (part 1) then use Green III (part 2). □

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- Note that heat flows throughout the domain (not along isotherms) from hot spots to cold spots
- Assume that the temperature does not vary with time (only location)
- Goal: if u is the temperature, then $\Delta u = 0$

- For any small portion of the domain, the heat flow in must equal the heat flow out (why?)

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- Suppose Γ is a small, closed loop inside the domain.
- The net heat flow across Γ must be zero: $\int_\Gamma \frac{\partial u}{\partial n} \, \mathrm{d}s = 0$
- This is based on the assumption $\Delta u = 0$

Characterization of Harmonic Functions

Theorem

Let Ω be a plane domain and $u \in \mathfrak{C}^2(\Omega)$. Then u is harmonic in Ω if and only if for every Jordan curve Γ inside Ω whose interior lies inside Ω,

$$\int_\Gamma \frac{\partial u}{\partial n} \, \mathrm{d}s = 0 \tag{12}$$

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Hence a steady state temperature inside a Jordan domain is harmonic. And conversely!

Bump Principle

Lemma (Bump Principle)

Let Ω be a domain and $q \in \mathfrak{C}(\Omega)$ with $q(z) \geq 0$ for all $z \in \Omega$. Then

$$\iint_{\Omega} q \, \mathrm{d}x\mathrm{d}y = 0 \Leftrightarrow q \equiv 0 \tag{13}$$

Boundary Mean-Value Theorem

Theorem (Boundary Mean-Value Theorem)

Let u be harmonic in a domain Ω, $\zeta \in \Omega$ and suppose $\bar{D}(\zeta; R) \subset \Omega$. Then $u(\zeta)$ is determined by the values $u(z)$ for $z \in \partial D$:

$$u(\zeta) = \frac{1}{2\pi R} \int_{\partial D} u(z)\,\mathrm{d}z = \frac{1}{2\pi} \int_0^{2\pi} u(R, \theta)\,\mathrm{d}\theta \tag{14}$$

Note: $u(R, \theta)$ is defined using polar coordinates centered at ζ.

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Theorem (Solid Mean-Value Theorem)

Let u be harmonic in a domain Ω, $\zeta \in \Omega$ and suppose $\bar{D}(\zeta; R) \subset \Omega$. Then

$$u(\zeta) = \frac{1}{2\pi R^2} \iint_D u \,\mathrm{d}x\mathrm{d}y \tag{15}$$

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Proof.

Apply Theorem 17 to a disc with $0 < r \leq R$; multiply both sides by $r\,\mathrm{d}r$ and integrate from $r = 0$ to $r = R$. Change coordinates to rectangular and evaluate. □

Harmonic functions are characterized by their boundary values.

Definition

Let Ω be a domain. Then $u \in \mathfrak{C}^2(\Omega)$ has the **circumferential mean value property** *in Ω if and only if for each $D = D(\zeta; R)$ with $\bar{D} \subset \Omega$*

$$u(\zeta) = \frac{1}{2\pi R} \int_{\partial D} u \, \mathrm{d}s \tag{16}$$

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Let Ω be a domain and $u \in \mathfrak{C}^2(\Omega)$. Then u is harmonic if and only if it has the circumferential mean value property in Ω.

Proof.

Differentiate defining equation (in polar coordinates centered at ζ) and multiply by r; simplify. $\square$

Strong Maximum Principle

It is not possible for a non-constant harmonic function to attain its maximum value at an interior point of Ω; hence the maximum value is only attainable on $\partial\Omega$.

Lemma (Strong Maximum Princple)

A non-constant harmonic function on a domain Ω does not assume its maximum or minimum on Ω.

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Lemma (Strong Maximum Princple)

A non-constant harmonic function on a domain Ω does not assume its maximum or minimum on Ω.

For example, $u(x, y) = 1 - x^2 - y^2$ has maximum value on $\mathbb{R}^2$ at $(0, 0)$ by definition.

Strong Maximum Proof

Proof.

Assume u attains its maximum, c, at $\zeta \in \Omega$. If u is not constant, then the set $A = \{w : u(w) < c\}$ is non-empty and open by continuity. Furthermore $B = \{t : u(t) = c\}$ is not open or else $\Omega = A \cup B$ and Ω is disconnected. Hence there exists some $\zeta_1 \in \Omega$ and $r > 0$ such that $\bar{D}(\zeta_1; r) \subset \Omega$ with $u(\zeta_1) = c$ and at least one $z \in \partial D$ with $u(z) < c$. Using 2 on the arc near z gives a set of values z such that $u(z) < c$ for all z on this arc. But then

$$u(\zeta_1) = \frac{1}{2\pi} \int_0^{2\pi} u(r, \theta)\, \mathrm{d}\theta < \frac{1}{2\pi} \int_0^{2\pi} u(\zeta_1)\, \mathrm{d}\theta = u(\zeta_1)$$

which contradicts the expected value for $u(\zeta_1)$ from Theorem 17 as u is harmonic. □

Applying this argument to the function $w = -u$ proves the minimum value of a harmonic function must also occur on the boundary.

Weak Maximum Principle

Lemma (Weak Maximum Principle)

Let Ω be a bounded domain with $u \in \mathfrak{C}(\bar{\Omega})$ and u harmonic on Ω. Then either u is constant on $\bar{\Omega}$ or u assumes its maximum and minimum values on $\partial\Omega$ only.

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Let Ω be a bounded domain with $u \in \mathfrak{C}(\bar{\Omega})$ and u harmonic on Ω. Then either u is constant on $\bar{\Omega}$ or u assumes its maximum and minimum values on $\partial\Omega$ only.

Proof.

Observe that $\bar{\Omega}$ is compact, hence $\{u(z) : z \in \bar{\Omega}\}$ is a closed interval of finite length. Note that for any $z \in \Omega$, $u(z)$ is not the maximum value by the strong maximum principle applied to a disc $D(z; r) \subset \Omega$. $\square$

Application of Maximum Principles

Theorem

Suppose Ω is a bounded domain and u, v are continuous functions on $\bar{\Omega}$ with $u(z) = v(z)$ for all $z \in \partial\Omega$. Then $u(\zeta) = v(\zeta)$ for all $\zeta \in \bar{\Omega}$.

Application of Maximum Principles

Theorem

Suppose Ω is a bounded domain and u, v are continuous functions on $\bar{\Omega}$ with $u(z) = v(z)$ for all $z \in \partial\Omega$. Then $u(\zeta) = v(\zeta)$ for all $\zeta \in \bar{\Omega}$.

Proof.

Note that $u - v$ is harmonic in Ω and vanishes on $\partial\Omega$, hence has maximum value 0. $\square$

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A harmonic function with domain $\mathbb{R}^2$ is called **entire**.

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A harmonic function with domain $\mathbb{R}^2$ *is called* **entire**.

Note that no consideration is given to unbounded domains which are not $\mathbb{R}^2$ aside from the Strong Maximum Principle.

Lemma (Harnack's Inequality)

Let $D = D(z_0; R)$ be an open disc and u be harmonic on D such that $u(z) \geq 0$ for all $z \in D$. Then for all $z \in D$

$$0 \leq u(z) \leq \left(\frac{R}{R - |z - z_0|}\right)^2 u(z_0) \tag{17}$$

Interpretation: the growth rate of harmonic functions is limited by the distance to the boundary of a disc.

Proof of Harnack's Inequality

Proof.

Apply Theorem 18 to a disc $D'(z; R - |z - z_0|)$

$$0 \leq u(z) = \frac{1}{\pi(R - |z - z_0|)^2} \iint_{D'} u \, \mathrm{d}x\mathrm{d}y$$

but since $D' \subset D$

$$\iint_{D'} u \, \mathrm{d}x\mathrm{d}y \leq \iint_{D} u \, \mathrm{d}x\mathrm{d}y = \pi R^2 u(z_0)$$

$\square$

Note that the growth rate bound is independent of derivatives of u - an unusual situation wholly dependent on the harmonic nature of u.

Liouville's Theorem

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If u is entire, harmonic and bounded either above or below, then u is constant.

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If u is entire, harmonic and bounded either above or below, then u is constant.

- Alternative phrasing: non-constant harmonic functions have unbounded images.
- Using heat: a non-constant steady state temperature achieves every temperature somewhere.

Liouville's Theorem Proof

Proof.

Suppose u is bounded below: $u(z) \geq \alpha$ for some α. Then $v = u - \alpha \geq 0$ is harmonic and constant if and only if u is constant. Therefore WLOG $\alpha = 0$. Choose any two points $z, \zeta \in \mathbb{R}^2$. For any $R > |z - \zeta|$,

$$0 \leq u(z) \leq \left(\frac{R}{R - |z - \zeta|} \right)^2 u(\zeta)$$

As $R \to \infty$, $0 \leq u(z) \leq u(\zeta)$. Interchanging z, ζ gives the reverse inequality hence $u(z) = u(\zeta)$. If u is bounded above, apply the argument to the harmonic function $w = -u$. $\square$

Complex Numbers and Functions

Objectives

- Construct C

Objectives

- Construct C
- Complex functions

Objectives

- Construct $\mathbb{C}$
- Complex functions
  - Derivatives

Objectives

- Construct C
- Complex functions
  - Derivatives
  - Analytic

Objectives

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    - Transcendental

Construction

- Endow $\mathbb{R}^2$ with a sensible multiplication

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- Define $i^2 = -1$; $1i = i$, $a(bi) = (ab)i$, $ai = ia$
- $(x + yi) \cdot (\xi + \eta i) = (x\xi - y\eta) + (x\eta + y\xi)i$

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Definition

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- Example(s)

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- Example(s)

Compute the following:

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If $z = x + yi$, then $\bar{z} = x - yi$ is the **conjugate** *of z.*

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- Hence point multiplication (not a dot product) is sensible

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- but w varies with z; hence $u = u(z)$ and $v = v(z)$ also
- therefore $f(z) = u(z) + iv(z)$ is the sum of the real and imaginary functions

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Given $z_0 \in \mathbb{C}$, *the symbols*

$$\lim_{z \to z_0} f(z) = w$$

mean that for all $\epsilon > 0$, *there exists* $\delta > 0$ *such that* $z \in D(z_0; \delta)$ *implies* $f(z) \in D(w; \epsilon)$.

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- Complex limits are compatible with arithmetic operations

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Note that this complex limit is path independent.
Additionally, the value $f'(z_0)$ is not to be interpreted as "slope".

Derivative Rules

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3. $\left[\frac{f(z)}{g(z)}\right]' = \frac{g(z)f'(z) - f(z)g'(z)}{(g(z))^2}$

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Simple case ($g(\zeta) = a\zeta + \beta$): multiply the difference quotient of the composite function by $1 = \frac{z-z_0}{z-z_0}$. $\square$

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The exponent formula works: $f(z) = z^n$ has $f'(z) = nz^{n-1}$ for n a natural number. PLTS.

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Extend to "analytic on Ω" in obvious way.

Theorem (Goursat's Theorem)

Examples

- constant functions

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Examples

- constant functions
- polynomial functions
- rational functions
- counter-example: $f(z) = \bar{z}$ is nowhere analytic. PLTS.

Motivation

Goal: determine analyticity of $f(z) = u(z) + iv(z)$ based on partial derivatives of u, v.

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Lemma (Cauchy-Riemann Equations)

If f is analytic at $z \in \Omega$, then

$$f'(z) = u_x(z) + iv_x(z) = v_y(z) - iu_y(z) \tag{20}$$

so that

$$u_x = v_y \qquad u_y = -v_x$$

Conversely, if $u, v \in \mathfrak{C}^1(\Omega)$ and satisfy these equations for all $z \in \Omega$, then $f = u + iv$ is analytic in Ω.

Cauchy-Riemann Equations Proof Outline

Proof.

Let z approach z_0 vertically, horizontally. Observe $f'(z_0)$ is unique. Apply Theorem 3 and the Cauchy-Riemann equations. Using the triangle inequality, $\epsilon_i \to 0$ as $z \to z_0$ hence f' exists; moreover is continuous. $\square$

CR Equations in Polar Coordinates

Lemma

For $f(z) = u(r, \theta) + iv(r, \theta)$,

$$u_r = \frac{1}{r} v_\theta \qquad u_\theta = -r v_r \tag{21}$$

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Proof.

Apply the Chain Rule. ∎

Analytic and Harmonic Functions

Lemma

If $f(z) = u(z) + iv(z)$ is analytic with $u_x = v_y$ and $u_y = -v_x$, then u, v are harmonic functions.

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That is, the real and imaginary parts of analytic functions are themselves harmonic functions. What???!!!

The Exponential Function

Definition

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Compute $\exp(z + 2\pi i)$.

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- Polar multiplication:

$$z_1 z_2 = r_1 r_2 e^{\theta_1 + \theta_2}$$

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$$e^{i(\theta+2k\pi)} = e^{i\theta}e^{i2k\pi} = e^{i\theta}1$$

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- Claim: $\{\zeta_k\}$ are all distinct

Uniqueness of Roots

Proof.

Suppose $\zeta_k = \zeta_m$ for $0 \leq k < m \leq n-1$. Then

$$\zeta_k \zeta_m^{-1} = 1 = (r^{1/n} e^{i(\theta + 2\pi k)/n})(r^{-1/n} e^{-i(\theta+2\pi m)/n}) = e^{(2i(k-m)\pi)/n}$$

But $e^{2\pi i t} = 1$ if and only if t is an integer. Since $(k-m)/n$ is not an integer, contradiction. Furthermore, if $w^n = z$, then $w \in \{\zeta_k\}$: consider the factorization of $w^n - z = 0$ using $\{\zeta_k\}$. $\square$

Roots of Unity

Definition

Define $\omega_k = e^{2\pi ik/n}$ *so that* ω_k *is a solution to* $z^n = 1$ *for* $k = 0, 1, \ldots, n-1$. *Then* ω_k *is an* **nth root of unity***.*

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Lemma

If $\zeta^n = z$, *then all solutions are given by*

$$\zeta\omega_0, \zeta\omega_1, \dots, \zeta\omega_{n-1}$$

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- Therefore a complete inverse function does not exist:
- It is reasonable to assign $\log 1 = 0$ and $\log 1 = 2\pi i$

Resolution

Theorem

Let $S = \{z : -\pi < Imz \leq \pi\}$. *Then* $\exp : S \to \mathbb{C} \backslash \{0\}$ *in a bijective fashion. In particular, the line* $y = \pi$ *is mapped onto the negative real axis.*

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This is the same idea used to force $\sqrt{x}$ to be a function: restrict the domain.

Proof

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Suppose $w \neq 0$. Then $w = re^{i\theta}$ for some r, θ with $\theta \in (-\pi, \pi]$. Define $x = \ln r, y = \theta$ and let $z = x + iy$. Then $z \in S$ by construction and $e^z = w$ hence exp is surjective.
Suppose $e^{z_1} = e^{z_2}$. Then $1 = e^{z_1 - z_2}$ but then $z_1 - z_2 = 2\pi ik$ for some integer k. Since $z_1, z_2 \in S$, $|z_1 - z_2| < 2\pi$ thus $k = 0$. □

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In general the range of log can be any strip of width 2π. The choice of branch is up to the user but must be declared.

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- $\log z_1 z_2 = \log z_1 + \log z_2 + 2\pi i k$

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- Analytic automatically
- Derivatives follow giving expected results
- Remaining functions defined in usual way

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- Fact: $[z^w]' = wz^{w-1}$: PLTS

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Definition

Suppose that u is a real valued harmonic function defined on Ω. Then v is a **harmonic conjugate** *if*

$$f(z) = u(z) + iv(z)$$

is analytic on Ω.

Construction Process

Lemma

If Ω is simply connected and $z, z_0 \in \Omega$ then

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is path independent.

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If Ω is simply connected and $z, z_0 \in \Omega$ then

$$\int_{z_0}^{z} -u_y \,\mathrm{d}x + u_x \,\mathrm{d}y$$

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- Applying the previous lemma,

$$v(z) - v(z_0) = \int_{z_0}^{z} -u_y \,\mathrm{d}x + u_x \,\mathrm{d}y$$

where $v(z_0)$ is arbitrary (real).

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- Therefore, $v(x, y) = \psi(x, y)$ for $c \in \mathbb{R}$ is a harmonic conjugate of $u(x, y)$
- Hence $f(z) = u(x, y) + iv(x, y) = z^2 + ic$ is analytic

Summary Theorem

Theorem

Given a real harmonic function $u(x, y)$ on a simply connected domain Ω, there exist infinitely many harmonic conjugates $v(x, y)$ each differing by an additive constant. Hence the given u is the real part of infinitely many analytic functions

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Note that this implies $\ln z$ does not have a harmonic conjugate on $\mathbb{C}\backslash\{0\}$ yet is harmonic on this domain.

Integration

Objectives

- Complex line integrals

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- Theorems about integration

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- Theorems about integration
- Complex versions of previous theorems about harmonic functions

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mean and how to compute it?
- Strategy: Separate into real and imaginary parts; use previous knowledge

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- Compute using parameterization (as usual)

Example

(get from another book)

Fundamental Theorem of Calculus

Theorem

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2. *$G'(x) = g(x)$*
3. *if also $F'(x) = g(x)$, then $F(x) - G(x)$ is constant and*

$$\int_a^b g(t)\,\mathrm{d}t = G(b) - G(a)$$

Complex FTC Version 1

Theorem

Let $f(z)$ be continuous in a domain Ω and suppose that $F(z)$ is analytic with $F'(z) = f(z)$ for all $z \in \Omega$. If Γ is a curve from z_0 to z_1 in Ω, then

$$\int_\Gamma f(z)\,\mathrm{d}z = \int_\Gamma F'(z)\,\mathrm{d}z = F(z_1) - F(z_0) \tag{22}$$

Notes

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- Need to know $f(z) = F'(z)$ for some analytic $F(z)$ beforehand
- The Cauchy Integral Theorem 30 will void the previous point

Crucial Example

- Compute $\int_\Gamma \frac{\mathrm{d}z}{z}$ where Γ is the unit circle parameterized in the counter clockwise direction.

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- Parameterize: $z = e^{i\theta}$ for $\theta \in [0, 2\pi]$ so that $\mathrm{d}z = ie^{i\theta}\mathrm{d}\theta$
- Manually compute (no theorems):

$$\int_\Gamma \frac{\mathrm{d}z}{z} = \int_0^{2\pi} e^{-i\theta}(ie^{i\theta})\,\mathrm{d}\theta = 2\pi i$$

Integral Estimates

Theorem (ML Inequality)

Let $f(z)$ be continuous and defined on Γ. Suppose that $|f(z)| \leq M$ for all $z \in \Gamma$ and $L =$ length Γ. Then

$$\left| \int_\Gamma f(z)\, \mathrm{d}z \right| \leq \int_\Gamma |f(z)||\mathrm{d}z| = \int_\Gamma |f(z)|\, \mathrm{d}s \leq ML \tag{23}$$

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Common application: show some integral is zero.

Proof

Proof.

Denote $I = \int_\Gamma f(z)\,\mathrm{d}z$. Then there exists ω such that $I = |I|e^{i\omega}$. Thus

$$|I| = \int_\Gamma e^{-i\omega} f(z)\,\mathrm{d}z$$

can be parameterized by $z(t)$. The integrand can be written as $U(t) + iV(t)$ and all functions, $|I|$, U, V, are real. $\square$

Cauchy Integral Theorem

Theorem

Let $f(z)$ be analytic in a domain Ω and let Γ be a closed Jordan curve inside Ω whose interior is contained in Ω so that $f(z)$ is analytic on and inside Γ. Then

$$\int_{\Gamma} f(z)\,\mathrm{d}z = 0 \tag{24}$$

Proof

Proof.

Apply Green's Theorem 2 and the Cauchy-Riemann equations. □

Examples

- $\int_\Gamma \cos(\sin z)\,\mathrm{d}z$

For $\Gamma = C(0;1)$.

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- $\int_\Gamma \cos(\sin z)\,\mathrm{d}z$
- $\int_\Gamma e^{z^2}\,\mathrm{d}z$

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Strong Cauchy Integral Theorem

Theorem

Let $f(z)$ be analytic in a domain Ω and let Γ be a closed curve in Ω which can be shrunk to a point within Ω. Then

$$\int_{\Gamma} f(z)\,\mathrm{d}z = 0 \tag{25}$$

Corollary

Let $f(z)$ be analytic in Ω and Γ_1, Γ_2 be piecewise smooth curves from z_0 to z_1 all lying inside Ω such that all points between Γ_1, Γ_2 also lie in Ω. Then

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Proof.

Note that $\Gamma = \Gamma_1 + (-\Gamma_2)$ is a closed loop. □

Corollary

If $f(z)$ is analytic in the simply connected domain Ω and z, z_0 are points of Ω, then

$$F(z) = \int_{z_0}^{z} f(\zeta)\,\mathrm{d}\zeta \tag{27}$$

is a value independent of the path from z_0 to z in Ω hence defines a continuous function $F = F(z)$ in Ω.

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- It is true that $F(z)$ is analytic and $F'(z) = f(z)$.
- A different choice of z_0 results in a different F; how much?

Complex Series

Objectives

- Singularities

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- Laurent series

Questions

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- Is this even possible??

Assumptions, Singularity, Notation

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Definition

If f is analytic on Ω' and z_0 is an interior point of Ω, then z_0 is an **isolated singularity**.

Types of Singularities

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Definition

If f is analytic on Ω' and z_0 is a singular point which is not removable and not a pole, then z_0 is an **essential singularity**.

Examples

- If f is analytic on Ω, then $\lim_{z \to z_0} f(z) = w_0$ for all z_0; puncture and replace appropriately

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- $1/z, 1/z^2, z - 1$ all have poles
- $h(z) = e^{1/z}$ on $\mathbb{C} \setminus \{0\}$
- Compute $\lim_{z \to 0^+} h(z)$ and $\lim_{z \to 0^-} h(z)$

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Let $f(z)$ be analytic on Ω' where z_0 is an interior point of Ω which is a removable singularity. Then there exists w_0 such that defining $f(z_0) = w_0$ guarantees f is analytic at z_0, hence on all of Ω.

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Note that there is no mention of integrals. Warning: the proof is long. Also z, z_0, ρ are constants.

Proof

Let $\hat{f}$ be the extension of f. Clearly $\hat{f}$ is continuous on Ω. Choose $C(z_0; R) \subset \Omega'$ and define

$$F(z) = \frac{1}{2\pi i} \int_{C(z_0;R)} \frac{f(\zeta)}{\zeta - z} \mathrm{d}\zeta$$

for $|z - z_0| < R$.
Claim: F is analytic at z_0. This is an integral of Cauchy type, hence differentiation under the integral sign is permitted thus $F'(z)$ exists.
Claim: $F = f$ on $0 < |z - z_0| < R$. Suppose z_1 satisfies $0 < |z_1 - z_0| < R$ and consider

$$F(z_1) - f(z_1) = \frac{1}{2\pi i} \int_{C(z_0;R)} \frac{f(\zeta)}{\zeta - z_1} \mathrm{d}\zeta - \frac{1}{2\pi i} \int_{C(z_1;r_1)} \frac{f(\zeta)}{\zeta - z_1} \mathrm{d}\zeta$$

Proof Continued

Now compute

$$\frac{1}{2\pi i}\int_{C(z_0;\rho)} \frac{f(\zeta)}{\zeta - z_1}\mathrm{d}\zeta$$

$\frac{f(\zeta)}{\zeta - z_1}$ is analytic on $D(z_0;R)\backslash(\bar{D}(z_1;r_1)\cup\bar{D}(z_0;\rho))$ and using the oriented boundary circles:

$$\int_{C(z_0;R)} - \int_{C(z_1;r_1)} - \int_{C(z_0;\rho)} = 0$$

Therefore $\frac{1}{2\pi i}\int_{C(z_0;\rho} \frac{f(\zeta)}{\zeta - z_1}\mathrm{d}\zeta = F(z_1) - f(z_1)$
Claim: $\left|\frac{f(\zeta)}{\zeta - z_1}\right| < M$ for some M. For $|\zeta - z_0| = \rho$, $|f(\zeta)|$ is bounded since $\lim\limits_{\rho\to 0} f(\zeta) = w_0$ is finite. Furthermore, $|\zeta - z_1| \geq |z_1 - z_0| - \rho$ because ζ is 'far away' from z_1 (See graphic). Applying Theorem 29 to $|F(z_1) - f(z_1)|$ gives $|F(z_1) - f(z_1)| < M\rho$ for all sufficiently small ρ.

Proof Illustrations

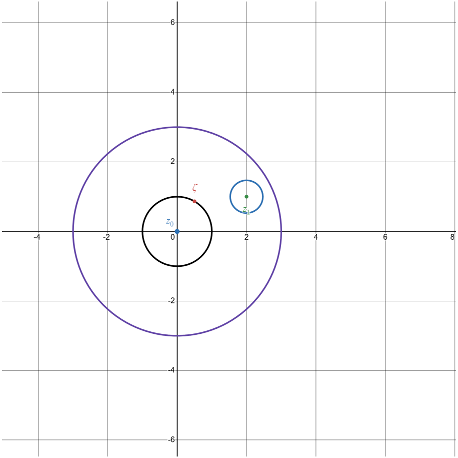

Figure: Desmos

Key Proof Ideas

- Extending a function

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- Extending a function
- Use of true circles (not arbitrary curves)

- Note that f is **not** analytic at z_0 by assumption and definition, hence does not have a Taylor series at z_0.

Singularities by Series

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- Thus any series must have terms $b_k(z - z_0)^{-k}$
- What is the radius of convergence (if this concept is even reasonable anymore)??

Convergence

- Consider $\sum b_k(z - z_0)^{-k} = \sum b_k \zeta^k$ for $\zeta = (z - z_0)^{-1}$

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- This is convergence *outside* some disc!

Notation

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- $$\sum_{n=-\infty}^{\infty} c_n(z-z_0)^n$$
- $$\sum_{n=0}^{\infty} a_n(z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$$

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Theorem

Suppose f is analytic in the annulus $R_1 < |z - z_0| < R_2$. Then $f(z)$ is represented by a Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n$$

which converges to f throughout the annulus. Moreover

$$c_n = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \, \mathrm{d}\zeta$$

where C is any circle centered at z_0 contained in the annulus.

Comments

- If f is analytic for $|z - z_0| < R_2$, then $c_{-n} = 0$ for $n = 1, 2, \ldots$ and f has a Taylor series. Why?

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- c_n not usually computed using this formula.

Proof Notes

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- Multiply by $-\frac{f(\zeta_1)}{2\pi i}$ and integrate

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- $1/(z^2+1)$ at $z = i$

Poles

Lemma

Suppose f is analytic and has a pole at z_0. Then $g(z) = 1/f(z)$ is analytic at z_0 which an isolated zero. Conversely, if z_0 is an isolated zero of an analytic function f, then $g(z) = 1/f(z)$ has a pole at z_0.

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Proof.

Consider $|f|$ and definitions. □

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- Use the same hypotheses as Lemma; $g(z)$ has isolated zero

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- $g(z) = (z - z_0)^m g_1(z)$ where $g_1(z_0) \neq 0$ is analytic in $V_\delta(z_0)$

Order

- Use the same hypotheses as Lemma; $g(z)$ has isolated zero
- $g(z) = (z - z_0)^m g_1(z)$ where $g_1(z_0) \neq 0$ is analytic in $V_\delta(z_0)$
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- If $m = 1$, z_0 is a *simple pole*

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- The **polar part** is

$$\sum_{n=-1}^{\infty} c_n (z - z_0)^n = \sum_{n=1}^{\infty} c_{-n} (z - z_0)^{-n}$$

Classification of Singularities

Theorem

Suppose z_0 *is an isolated singularity of the analytic function* f *which has polar part* $\sum_{n=1}^{\infty} c_{-n}(z-z_0)^{-n} = \sum_{n=1}^{\infty} \frac{c_{-n}}{(z-z_0)^n}$. *Then*

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3. z_0 *is an essential singularity if and only if the polar part has infinitely many nonzero terms.*

Examples Revisited

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- Consider the set $\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$; the Riemann sphere

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- Expand the domain of f to $\bar{\mathbb{C}}$ and define $f(z_0) = \infty$
- Then f is continuous at z_0!

Essential Singularities

Recall: z_0 an isolated singularity of analytic f is removable if and only if $|f(z)|$ is bounded in $V_\delta(z_0)\backslash\{z_0\}$.

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Theorem (Cassorati-Weierstrass)

In every open neighborhood of an isolated essential singularity z_0, the function f assumes values $f(z)$ arbitrarily close to every complex number w. That is, if D' is any punctured disc centered at z_0 and $D(w;\rho)$ is any disc about any point w, then there exists $z \in D'$ such that $f(z) \in D(w;\rho)$.

Application

Recall the surjective nature of exp on horizontal strips of height 2π: Theorem 25.
Consider $f(z) = e^{1/z}$. Let $w \in \mathbb{C}\backslash\{0\}$. Claim: for all $\delta > 0$, there exists $z_0 \in V = V_\delta(0)\backslash\{0\}$ such that $f(z_0) = w$. Note that $z \mapsto 1/z$ maps V to the set $Y = V_{1/\delta}(\bar{0})\backslash\{0\}^C$. Thus there exists $X \subset Y$ with "height" 2π. Since $\exp : X \to \mathbb{C}\backslash\{0\}$ is surjective, there exists $\zeta \in X$ with $e^\zeta = w$ so let $\zeta = 1/z$.

Calculus of Residues

Objectives

- Residue Theorem

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- Argument Principle

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Now recall $c_n = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\,\mathrm{d}\zeta$
Choosing $n = -1$ and C to be a circle around z_0 containing no other singularities of f:

$$c_{-1} = \mathrm{Res}(f; z_0) = \int_C f(\zeta)\,\mathrm{d}\zeta$$

Theorem

The natural extension to a finite number of singularities is:

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Theorem (Residue Theorem)

Suppose $f(z)$ is analytic in a domain Ω except for isolated singularities at $z_1, \ldots, z_m$ in Ω. Let Γ be a positively oriented smiple closed curve in Ω which has all z_i in its interior. Then

$$\int_\Gamma f(z)\,\mathrm{d}z = 2\pi i \sum_{i=1}^{m} \operatorname{Res}(f; z_i)$$

Examples

- $\int_{\Gamma} \mathrm{d}z/z = 2\pi i(1)$

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Observation: Cauchy's Integral Formula is a special case of the Residue Theorem.

Example 1

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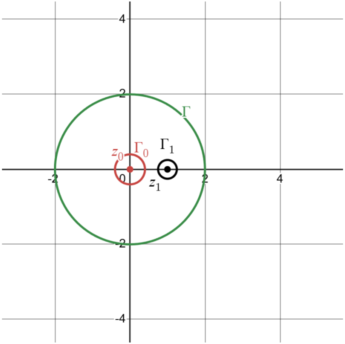

Figure: Desmos

- Recall $\frac{\mathrm{d}}{\mathrm{d}z}[f(z)]\Big|_{z=\zeta} = \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{(z-\zeta)^2}\,\mathrm{d}z$

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- Hence the final value: $2\pi i(e-2)$

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- $\int_{|z|=1} \frac{\cos z}{\sin z}\,\mathrm{d}z = \int_{|z|=1} \frac{\cos z/g(z)}{z}\,\mathrm{d}z = 2\pi i\,\frac{\cos z}{z}\Big|_{z=0} = 2\pi i$

Summary

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- If not, use residues

Techniques

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Examples B

Consider

$$f(z) = \frac{(z-i)^3(z+1)}{z(z-2)^4}$$

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- $\mathrm{Res}(f; 0) = \lim_{z \to 0} zf(z) = zf(z)|_{z=0} = i/16$
- Therefore $\int_\Gamma f(z)\,\mathrm{d}z = -\pi/8$ provided $2 \notin \Gamma^\circ$

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Can you find the pattern for a pole of order m?

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Example

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- $-1/z^2\big|_{z=i} = 1 = \mathrm{Res}(f; i)$
- $zf(z) = (z+1)/(z-i)^2$
- $\lim\limits_{z\to 0}(z+1)/(z-i)^2 = -1 = \mathrm{Res}(f; 0)$

Example Summary

Hence there are four possibilities for the value $\int_\Gamma f(z)\,\mathrm{d}z$:

$$\int_\Gamma f(z)\,\mathrm{d}z = \begin{cases} 0 & \text{no pole is in } \Gamma^\circ \\ 2\pi i & i \in \Gamma^\circ \text{ and } 0 \notin \Gamma^\circ \\ -2\pi i & 0 \in \Gamma^\circ \text{ and } i \notin \Gamma^\circ \\ 0 & 0, i \in \Gamma^\circ \end{cases}$$

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Three-ish Numbers

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Note that the second must be an integer and the order matters.

Notation and Note

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Recall that counting is hard! Using the order (of poles and zeros) is the proper way to count these items.

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- For $\Gamma = |z| = 1$, $N_0(\Gamma) = 2, N_\infty(\Gamma) = 5$
- For $\Gamma = |z| = 4$, $N_0(\Gamma) = 2, N_\infty(\Gamma) = 9$

Lemma

Let Γ *be a piecewise smooth positively oriented simple closed curve and* f *an analytic function on and inside* Γ *except for a finite number of poles with no poles or zeros on* Γ. *Then*

$$N_0(\Gamma) - N_\infty(\Gamma) = \frac{1}{2\pi i} \int_\Gamma \frac{f'(z)}{f(z)} \, \mathrm{d}z$$

Lemma

Let Γ be a piecewise smooth positively oriented simple closed curve and f an analytic function on and inside Γ except for a finite number of poles with no poles or zeros on Γ. Then

$$N_0(\Gamma) - N_\infty(\Gamma) = \frac{1}{2\pi i}\int_\Gamma \frac{f'(z)}{f(z)}\,\mathrm{d}z$$

Proof.

□

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- $f'(z)/f(z) = 3/z$
- $\frac{1}{2\pi i}\int_\Gamma 3/z\,\mathrm{d}z = 3$
- f has a zero of order 3 and no poles

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- What if $t \in [0, 5\pi]$?

Winding Numbers

- Let $f(t) = (\cos t, \sin t), t \in [0, 6\pi]$
- Suppose a camera at the origin is facing $(1, 0)$ and a rabbit runs the path traced by f
- Hence the location of the rabbit at time t is $(t, f(t))$
- If the camera takes a picture when the rabbit passes $(1, 0)$, how many photos are taken?
- What if $t \in [0, 5\pi]$?
- What if $t \in [0, 7\pi]$?

Formalization

Definition

Given a closed curve Γ *and* $z_0 \notin \Gamma$, *define* $n(\Gamma; z_0)$ *to be the* **winding number of** Γ *with respect to* z_0

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Intuitively this quantity answers the question "How many times does Γ wind/wrap around z_0?"

Lemma

Let Γ be a piecewise smooth closed curve that does not pass through z_0. then

$$\frac{1}{2\pi i}\int_\Gamma \frac{f'(z)}{f(z)}\,\mathrm{d}z$$

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Note that the proof uses several “accessory functions”.

Proof I

Define

$$\phi(\tau \int_a^\tau \frac{\gamma'(t)}{\gamma(t) - z_0}\,\mathrm{d}t$$

and compute $\phi(b)$. Note that ϕ is continuous and

$$\phi'(\tau) = \frac{\gamma'(\tau)}{\gamma(\tau) - z_0}$$

wherever $\gamma'(t)$ is continuous. Define

$$\Phi(t) = (\gamma(t) - z_0)e^{-\phi(t)}$$

Observe that $\Phi(t)$ is continuous on $[a, b]$ except possibly for finitely many points. Also $\Phi'(t) = 0$ where Φ' exists. Therefore Φ is constant so that

$$\Phi(t) = \Phi(a) = (\gamma(a) - z_0)e^{-\phi(a)} = \gamma(a) - z_0$$

Proof II

Therefore

$$e^{\phi(t)} = \frac{\gamma(t) - z_0}{\gamma(a) - z_0}$$

Since Γ is closed it follows that $e^{\phi(b)} = 1$ and the result follows.

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Lemma

Let Γ be a closed curve and z_0, z_1 points in the same component of $\mathbb{C}\backslash\Gamma$. Then $n(\Gamma; z_0) = n(\Gamma; z_1)$; moreover if z_0 is outside Γ, then $n(\Gamma; z_0) = 0$.

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- $f(\Gamma) = \{f(z) : z \in \Gamma\}$
- If Γ is parameterized by $\gamma : [a, b] \to \Gamma$, then $f(\Gamma) = f(\gamma(t))$
- Therefore $f(\Gamma)$ is a curve

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Note that $w = f(z)$ so that $\mathrm{d}w = f'(z)\mathrm{d}z$ by Change of Variables. Hence the answer is the same as the difference between the zeros and poles of $f(z) - w_0$ inside Γ.

Complete Statement

Theorem

Let Γ be a piecewise smooth positively oriented simple closed curve in the z-place and f be analytic and non-constant on a domain containing Γ and its interior, except perhaps fora finite number of poles strictly inside Γ. Let w_0 be a point of the w-plane not on the curve $f(\Gamma)$. Then

$$N_{w_0}(\Gamma) - N_\infty(\Gamma) = \frac{1}{2\pi i}\int_\Gamma \frac{f'(z)}{f(z) - w_0}\,\mathrm{d}z = n(f(\Gamma); w_0)$$

Examples

- z^k

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First define $\int_{\mathbb{R}} f(x)\,\mathrm{d}x = \int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = \lim_{R\to\infty} \int_{-R}^{R} f(x)\,\mathrm{d}x$; the **principal value**

Example: Type A

Lemma

Let $f(z)$ be analytic in a domain containing the closed half-plane $y \geq 0$ except for a finite number of poles in the open half-plane $y > 0$. Suppose further that $\lim_{|z|\to\infty} zf(z) = 0$ uniformly for $z \in \{w \in \mathbb{C} : \Im(w) \geq 0\}$. Then $I = \int_{-\infty}^{\infty} f(x)\,\mathrm{d}x$ is $2\pi i$ times the sum of residues of $f(z)$ at poles in the upper half-plane.

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- Hence the integral has value $2\pi i(-i/2) = \pi$

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Let $g(z)$ be analytic in a domain containing the closed upper half-plane except perhaps a finite number of poles in the open upper half-plane. Suppose further that

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Corollary

$\int_{\mathbb{R}} g(x)e^{ikx}\,\mathrm{d}x$ *is* $2\pi i$ *times the sum of residues of* $g(z)e^{ikz}$ *in the upper half-plane.*

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- Which has real part
$$\frac{\pi}{\sqrt{3}}e^{-\sqrt{3}}\cos(1)$$

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- Observe that ϕ must be a rational function
- If ϕ has a pole between $0, 2\pi$, this transformed integral is not possible.

Example

- Show $\int_0^{2\pi} \frac{\cos(3\theta)}{5-4\cos(\theta)}\,\mathrm{d}z = \frac{\pi}{12}$

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- Note that substituting for $\cos(3\theta)$ results in a significantly more tedious integral