1-21-2025

Equivalence of Intervals

Definition:

Let $[a_1, b_1]$ and $[a_2, b_2]$ be intervals on the s, t axes respectively. A function $E : [a_1, b_1] \to [a_2, b_2]$ is an *equivalence* if and only if it is:

① bijective

② continuous ③ increasing

④ piecewise diff. with everywhere positive first deriv.

This is a formalization of the idea that $E(a_1) = a_2$, $E(b_1) = b_2$ and as s increases from $a_1 \to b_1$, $t = E(s) \uparrow$ from $a_2 \to b_2$

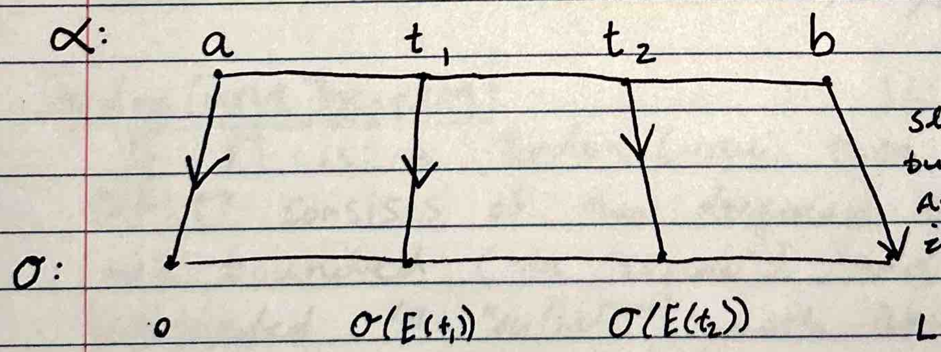
Theorem:

Let $\alpha : [a, b] \to \Gamma$ be a piecewise smooth parameterization with length $L > 0$. Then there exists a unique piecewise-smooth parameterization $\sigma : [0, L] \to \Gamma$ equivalent to α which satisfies:

① The tangent vector $\sigma'(s)$ has unit length $|\sigma'(s)| = 1$

② The distance traveled along Γ from $\sigma(0)$ to $\sigma(s)$ is s.

Ex. 1



Slope may be dif. but that's okay. As long as all is nice.

Proof. 1.

Note that if σ satisfies property (1), then the distance from $\sigma(0)$ to $\sigma(s)$ is $\int_0^s |\sigma'(t)|\,dt = s$.

It is sufficient to prove the result for smooth parameterizations; hence assume α is smooth. Define $F(\tau) = \int_a^\tau |\alpha'(r)|\,dt$ for $\tau \in [a,b]$. By Calculus, $F'(\tau) = |\alpha'(\tau)| > 0$ and continuous.

Define E as the inverse of F. By the Chain Rule, $E'(s) > 0$ and E is an equivalence from $[0,L] \to [a,b]$.

define $\sigma(s) = \alpha(E(s))$ and verify $|\sigma'(s)| = 1$ by the chain Rule. ■

Definitions

A Jordan Curve is a simple, closed, piecewise smooth curve

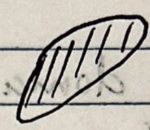
A Jordan Domain is a bounded domain Ω whose boundary $\partial\Omega$ is a Jordan Curve.

Ex.1

$\alpha : [a, b] \to \Gamma$

$\alpha(a) = \alpha(L)$

Domain closed



2D...

closed

2D

Real Analysis:

Let $A \subseteq \mathbb{R}$. A is bounded if $\forall a \in A$, $\exists M \in \mathbb{R}^+$ such that $\boxed{|a| \leq M}$.

complex:

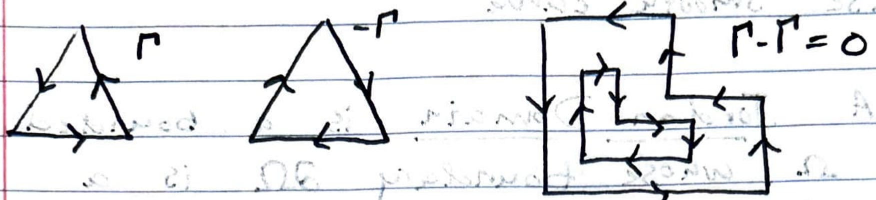
Let $\Omega \subset \mathbb{R}^2$. Ω is bounded if $\exists R > 0$ such that $\Omega \subseteq D(0;R)$. $\forall z \in D(0;R)$, $\boxed{|z| < R}$

Jordan Curve Theorem

If Γ is a Jordan Curve, then it's complement $\mathbb{R}^2 \setminus \Gamma$ consists of two disjoint domains, one bounded (the "inside") and one unbounded (the "outside"); both have a Γ as a boundary. If a point inside Γ is joined by a path to a point outside of Γ, then the path must cross Γ.

A Jordan domain is positively oriented if the given parameterization has the interior to the left.

$\alpha'(E(s))\cdot E'(s)$ $\quad E=f'$ $\quad \frac{1}{f(?)}$



Jordan Domains can have holes.

Def. 1

A Jordan domain Ω is K-connected if it's boundary consists of K jordan curves.

The complement of a K-connected Jordan ~~curve~~ Domain is thus K-disjoint connected components.

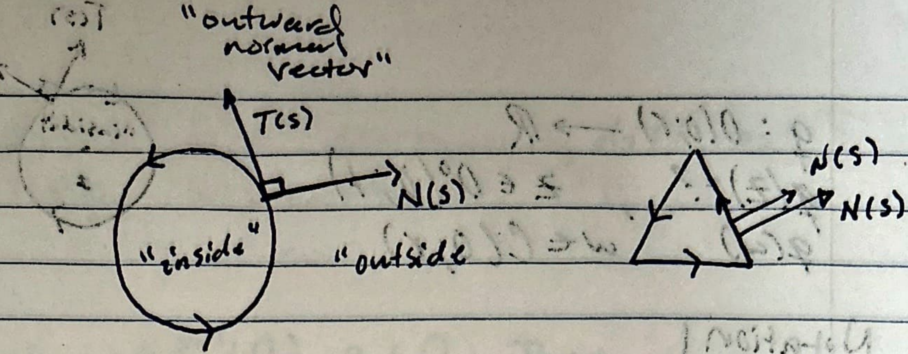
Definition

Let Ω be a Jordan Domain with boundary $\partial\Omega$ and Γ one of the Jordan Curves comprising $\partial\Omega$ with parameterization $\sigma:[0,L]\to\Gamma$

1. $|N(s)|=1$
2. $N(s)\perp T(s)$ (Perp to tangent vector)
3. $N(s)$ points outward from Ω

is called the outward normal vector

Ex:



- Consider $C(0;r)$ with $\sigma(s) = (r\cos(s/r), r\sin(s/r))$
- Then $T(s) = \langle -\sin(s/r), \cos(s/r)\rangle$
- thence $N(s) = \langle \sigma_2'(s), \sigma_1'(s)\rangle = \langle \cos(s/r), \sin(s/r)\rangle$
- Verification of outward left as exercise (graph).

$T(s) = \langle \sigma_1'(s), \sigma_2'(s)\rangle$

$C(0;r) = \langle \sigma_1(s), \sigma_2(s)\rangle$

$N(s) = \langle \beta(s), \gamma(s)\rangle$

$T(s)\cdot N(s) = \sigma_1'(s)\beta(s) + \sigma_2'(s)\gamma(s)$

$\beta(s) = \sigma_2'(s), \quad \gamma(s) = \sigma_1'(s)$

The study of plane curves is study of $f: \mathbb{R} \to \mathbb{R}^2$
The study of plane Calculus is the study of $f: \mathbb{R}^2 \to \mathbb{R}$
In particular how these functions change in the vicinity of a point $z \in \mathbb{R}^2$ - differentiation

* The chain Rule will be used often.

<u>Objective</u>:

- determine the values of $g(z)$ for $z \in \Omega$ given:
  - $g(z)$ exists and is twice differentiable on $\bar{\Omega}$
  - $\Delta g(z)$ is known $\forall z \in \Omega$
  - $g(z)$ is known $\forall z \in \partial\Omega$
  - $\nabla g(z)$ is known $\forall z \in \partial\Omega$

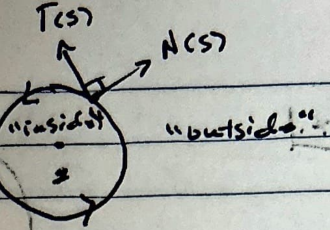
This will be accomplished using Green Th

16,

$g : D(0;r) \to \mathbb{R}$

$g(z) = ?, \quad z \in D^{\circ}(0;r)$

$g(w), \quad w \in C(0;r)$



Notation|

Ω - domain (open, connected)

function u assigns val $u(z)$ to point $z \in \Omega$

Since $z=(x,y)$: $u(z) = u(x,y)$.

$u : \Omega \to \mathbb{R}$

For $S \subset \mathbb{R}$, $u^{-1}(S) = \{ z \in \mathbb{R}^2 : u(z) \in S \}$

Definition|

Let $U \subset \mathbb{R}^2$. A function $u : \Omega \to \mathbb{R}$ is <u>continuous</u> if and only if for every open interval $S \subset \mathbb{R}$, the set $u^{-1}(S)$ is open in $\mathbb{R}^2$.

Ex. 1|

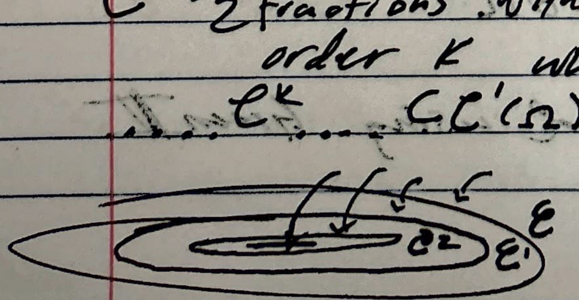
$\bigcup_S \underline{u^{-1}(S)}$ is open if and only if $\bigcup u^{-1}(S)$ is always open.

<u>Differentiability</u>

- If $u(z) = u(x,y)$, then $u_x(x,y)$ and $u_y(x,y)$
- Notation: $\mathcal{C}^k(\Omega)$
- In particular, $u \in \mathcal{C}^1(\Omega)$ implies $u_x, u_y \in \mathcal{C}(\Omega)$
- That is, $\mathcal{C}^1(\Omega)$ is the set of all continuously differentiable functions on Ω

$\mathcal{C}^k = \{$functions with derivatives up to order k which are continuous$\}$

$\mathcal{C}^k \ldots \subset \mathcal{C}^1(\Omega) \subset \underbrace{\mathcal{C}(\Omega)}_{\text{largest}},$



$\mathcal{C}^\infty(\Omega)$ example is e^x. continuously diff. but doesn't have a limit.

Theorem |

Let $u \in \mathcal{C}^1(\Omega)$, $z_0 \in \Omega$. Then u is continuous at z_0. That is,

$$\mathcal{C}^1(\Omega) \subset \mathcal{C}(\Omega)$$

Theorem (Chain Rule) |

Let $u \in \mathcal{C}^1(\Omega)$ and $\alpha(t) = (\alpha_1(t), \alpha_2(t))$ parameterizes $\Gamma \in \Omega$. Then

$$\frac{d}{dt} u(\alpha(t)) = u_x(\alpha(t))\alpha_1'(t) + u_y(\alpha(t))\alpha_2'(t)$$

Directional Derivative

Definition |

Let $z = (x, y)$ be a point in the domain Ω, V a unit vector with tail at z, and u a function defined on Ω. Then the derivative of u at z in the direction V is denoted $u'(z; V)$ and given by

$$u'(z;V) = \frac{d}{ds} u(z + sV)\Big|_{s=0}$$

Alternatively,

$$u'(z;V) = \lim_{s \to 0} \frac{u(z+sV) - u(z)}{s}$$

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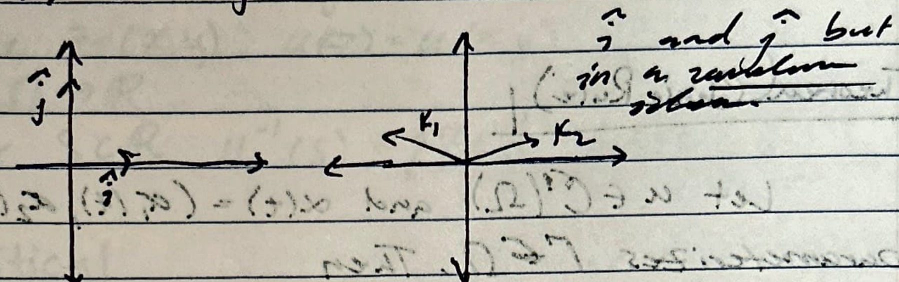
Theorem

Let $u \in C^1(\Omega)$ and $z \in \Omega$. If $V = \langle k_1, k_2 \rangle$ is a unit vector, then

$$u'(z; V) = u_x(z)\, k_1 + u_y(z)\, k_2$$

$$\langle 3,4 \rangle = 3\hat{i} + 4\hat{j}$$



Theorem (Linear Approx.)

Let $u \in C^1(\Omega)$ and $z_0 = (x_0, y_0) \in \Omega$. For each point $z = (x, y) \in \Omega$ we may write

$$u(z) = u(z_0) + u_x(z_0)(x - x_0) + u_y(z_0)(y - y_0) + \varepsilon_1 (x - x_0) + \varepsilon_2 (y - y_0)$$

where $\varepsilon_i \to 0$ as $z \to z_0 \in \Omega$

Notation:

$$u(z) = L(z; z_0) + E(z; z_0)$$

Note: $E(z; z_0) \to 0$ rapidly; specifically

$$\lim_{z \to z_0} \frac{|E(z; z_0)|}{|z - z_0|} = 0$$