$\underline{1.1.1 \quad \#1\text{-}2}$ ✓

1.) Let $z = (2,-2)$, $\zeta = (-1,5)$. Compute

(a) $z + \zeta = (2,-2) + (-1,5)$
$\quad = (2+(-1), -2+5) = (1,3)$

(b) $\zeta - z = (-1,5) - (2,-2)$
$\quad = (-1-2, 5-(-2)) = (-3,7)$

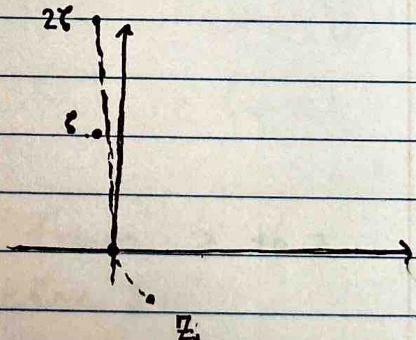
(c) $2z - \zeta = 2(2,-2) - (-1,5)$
$\quad = (2\cdot 2, -2\cdot 2) - (-1,5)$
$\quad = (4,-4) - (-1,5)$
$\quad = (4+1, -4-5) = (5,-9)$

(d) $z + 4\zeta = (2,-2) + 4(-1,5)$
$\quad = (2,-2) + (-4,20)$
$\quad = (2+(-4), -2+20) = (-2,18)$

2.) Given z, ζ as in (1) solve the following for $w = (u,v)$:

(a) $z + 2\zeta + 3w = 0$

$\cancel{(2,-2) + 2(-1,5) + 3(u,v) = 0}$
$\cancel{(2,-2) + (-2,10) + (3u,3v) = 0}$
$\cancel{(0,8) + (3u+3v) = 0}$



$3w = (-z) + (-2\zeta) + 0$
$\quad = (-2,2) + (2,-10)$
$\quad = (0,-8)$

$\begin{cases} 3u = 0 \Rightarrow u = 0 \\ 3v = 8 \Rightarrow v = 2 \end{cases}$ Thus, $w = (0,2)$.

(b) $2z + w = -\xi$

$w = -\xi + (-2z)$

$= -(-1,5) + (-2(2,-2))$

$= (1,-5) + (-4,4)$

$= (1-4, -5+4)$

$w = (-3,-1)$

1.1.2 # 1-4

1.) Let $z = (-1, 4)$, $z_0 = (2, 2)$ compute

(a) $|z| = \sqrt{(-1)^2 + 4^2}$
$= \sqrt{1+16}$
$= \sqrt{17}$

(b) $|z_0| = \sqrt{2^2 + 2^2} = \sqrt{8}$

(c) $|z - z_0| = \sqrt{(-1-2)^2 + (4-2)^2}$
$= \sqrt{(-3)^2 + (2)^2}$
$= \sqrt{9+4}$
$= \sqrt{13}$

(d) $|z_0 - z| = \sqrt{(2-(-1))^2 + (2-4)^2}$
$= \sqrt{(3)^2 + (-2)^2}$
$= \sqrt{9+4}$
$= \sqrt{13}$

2.) Compute the distance from z_0 to z, with z, z_0 as in exercise 1.

$|z_0 - z| = \sqrt{13}$ as in 1d.

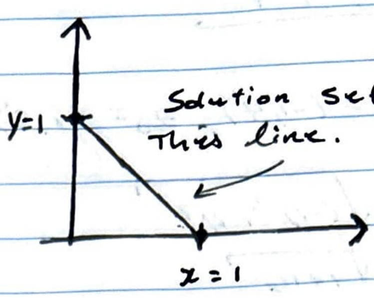
3.) Sketch in the plane the set of points z determined by the following conditions. Here, $z_0 = (1,1)$.

(a) $|z| = 1$

$\sqrt{x^2+y^2} = 1$

$x^2+y^2 = 1$

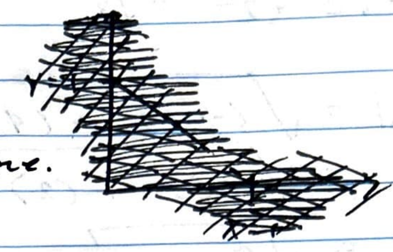
$x = \pm\sqrt{1-y^2}$

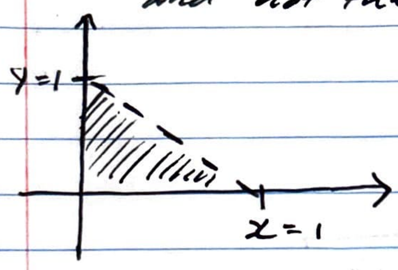


if $y = 0$, $x = +1$ WOLOG same for y.

(b) $|z| < 1$

similar to (a),
just including <1
and not the line.

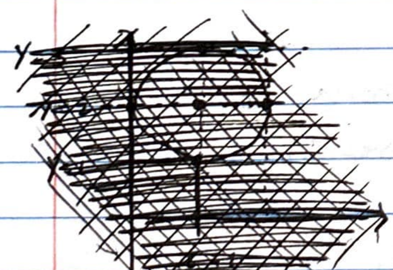


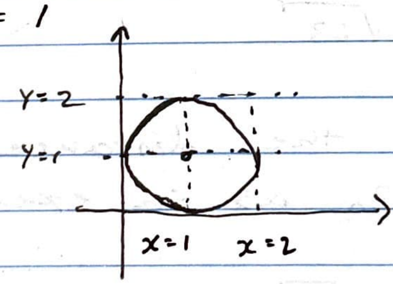


(c) $|z - z_0| = 1$ $\qquad z_0 = (1,1)$

$\sqrt{(x-1)^2+(y-1)^2} = 1$

$(x-1)^2+(y-1)^2 = 1$



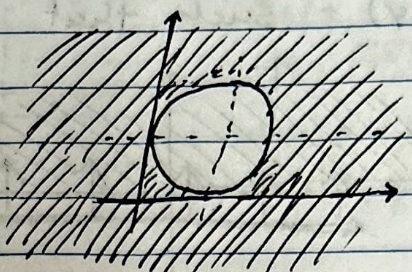


So the elements that satisfy z such that $|z - z_0| = 1$ are any along the line of the circle centered at $(1,1)$ with $r = 1$.

(d) $|z - z_0| \geq 1$

$\sqrt{(x-1)^2+(y-1)^2} \geq 1$

$(x-1)^2+(y-1)^2 \geq 1$

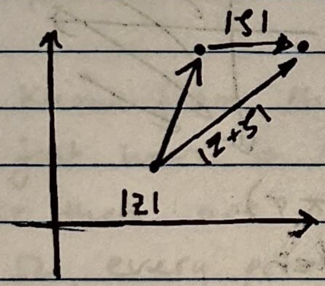


So the points that satisfy z are any on the line of the circle or in the outter shaded region.

4. Establish the following useful inequalities.

(a) $|z+\zeta| \leq |z| + |\zeta|$

$\sqrt{(x-x_0)^2+(y-y_0)^2} \leq \sqrt{x^2+y^2} + \sqrt{x_0^2+y_0^2}$



Basically this will always hold since on the right we are not subtracting at all.

(b) $|x| \leq |z|, |y| \leq |z|$

$\sqrt{x^2} \leq \sqrt{x^2+y^2}$

y will always be at least 0 and of course $\sqrt{x^2} \leq \sqrt{x^2}$ at least. So the addition of y confirms this

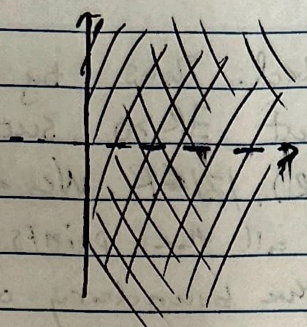
WOLOG, this is the same for $|y| \leq |z|$.

1.1.3

1.1.3 #1-4, 6

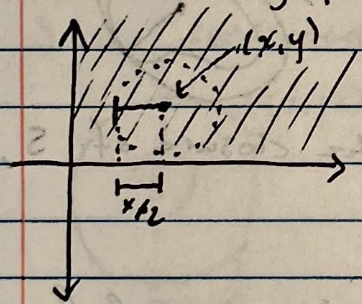
(1) ~~Sket~~

(a) Sketch the set S of points $z=(x,y)$ satisfying $x \geq 0$



(b) Verify that the subset of interior points of S is determined by the condition $x > 0$.

Consider a point $z \in S^\circ$. Let $z = (x, y)$. by def. $x > 0$, So consider the circle $C(z_0; x/2)$ this circle is guaranteed to contain only points within S°.



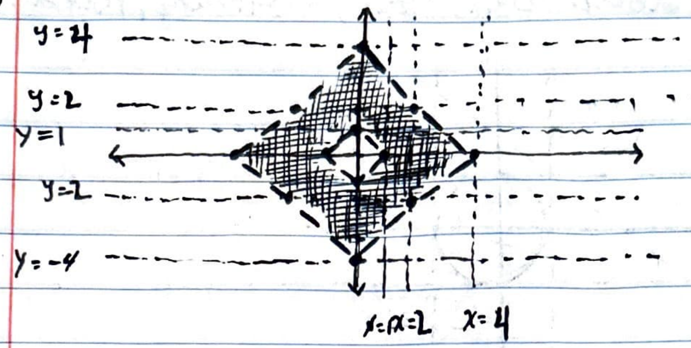
Then, $\forall w \in C(z; x/2)$, $w \in S^\circ$. So $w = (u, v)$ where $u > 0$.

(c) Is the subset in (b) a domain?

A domain is an open, connected subset of $\mathbb{R}^2$. this subset is open since every point of it is well within S. It is also connected since it cannot be created using two disjoint sets. So yes, the subset in (b) is a domain.

(2.) Sketch ~~an~~ the "annulus" $\Omega = \{ z \mid 1 < |z| < 2 \}$.

(a)



$z = (x, y)$

$|z| = \sqrt{x^2 + y^2}$

$1 < \sqrt{x^2 + y^2} < 2$

$1 < x^2 + y^2 < 4$

(b) There is a hole in Ω, is Ω connected?

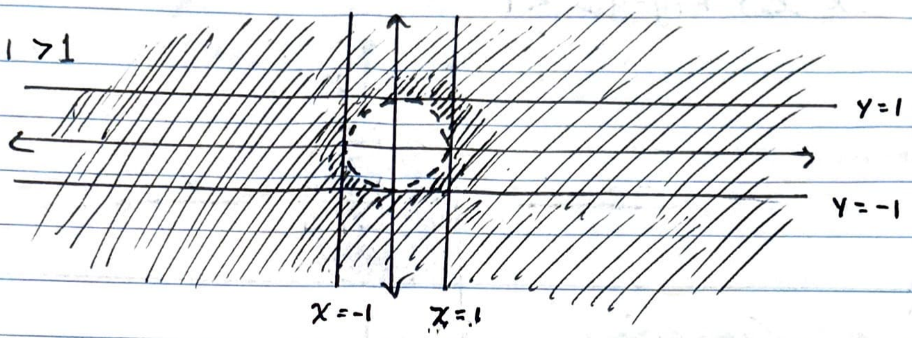
Yes Ω is connected. There aren't 2 disjoint subsets Ω_1, Ω_2 s.t. $\Omega_1 \cup \Omega_2 = \Omega$, so it is connected.

(c) Verify that Ω is a domain

We know from (b) that Ω is connected so we just have to show that Ω ~~it~~ open. Since there are no points on the boundary of Ω, every point is well inside the set Ω.

(3.) Sketch the sets determined by the following and decide whether it is a domain. z_0 is arbitrary, but fixed.

(a) $|z| > 1$

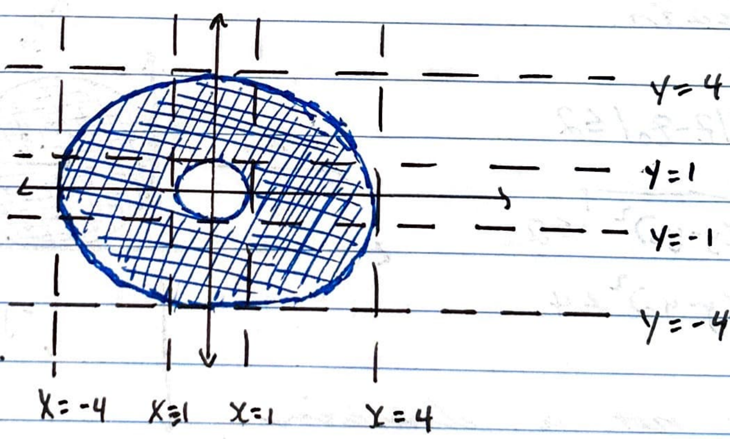


This is a domain. There are no points on the boundary. Likewise ~~it is~~ there are no disjoint subsets that's union is Ω. Thus Ω is a domain.

(b.) $1 \le |z| \le 2$

$1 \le \sqrt{x^2+y^2} \le 2$

$1 \le x^2+y^2 \le 4$.



Not a domain.

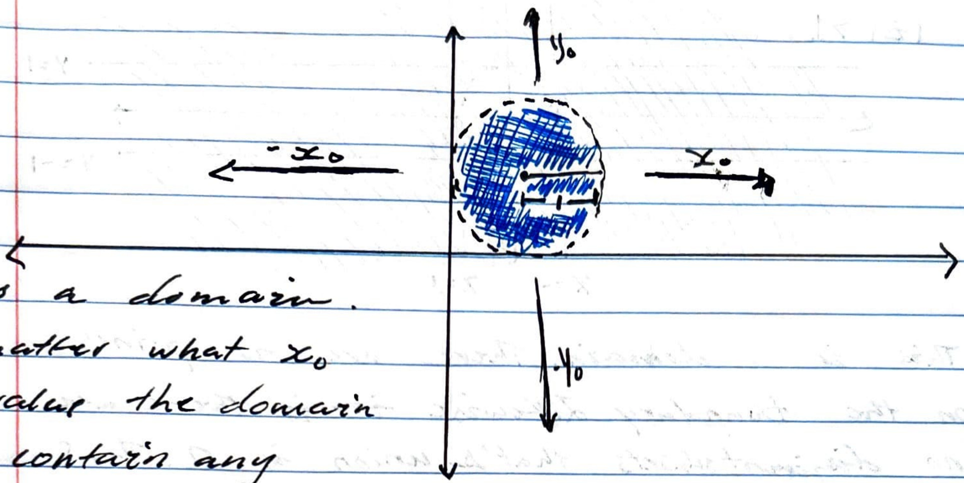
This contains points on the boundary, so we know that it is closed, and not a domain.

1.1.3

(c) $|z-z_0|<1$

$\sqrt{(x-x_0)^2+(y-y_0)^2}<1$

$(x-x_0)^2+(y-y_0)^2<1$

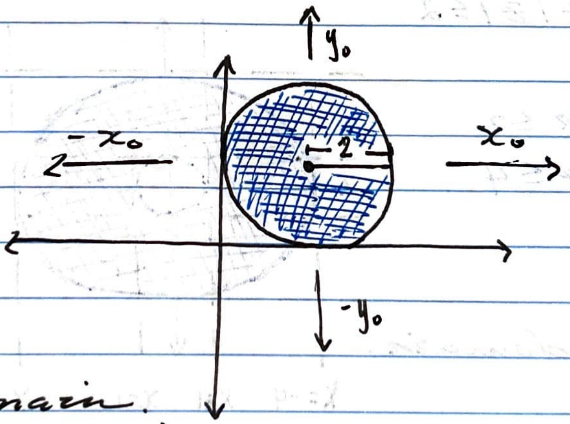


This is a domain.
No matter what x_0 or y_0 value the domain doesn't contain any bondary points and can't be created by the union of two disjoint subsets of $|z-z_0|<1$. So, this is a domain.

(d) $|z-z_0|\leq 2$

$\sqrt{(x-x_0)^2+(y-y_0)^2}\leq 2$

$(x-x_0)^2+(y-y_0)^2\leq 4$



This is not a domain.
This contains boundary points, thus it is closed, making it not a domain.

(4.) Let Ω be a domain and let S be a nonempty subset of Ω satisfying
(i.) S is open.
(ii) Its complement $\Omega \setminus S$ is open.
Prove that $S = \Omega$.

Proof.

Given that S is open, we know that it's complement $\Omega \setminus S$ must be closed (Page 8). ~~Given that $\Omega \setminus S$ is open, but the question by the question, but this fact tells us it is closed.~~
Now from the question and this fact, we have that $\Omega \setminus S$ is both open and closed. So, either $\Omega \setminus S = \emptyset$ or $\Omega \setminus S = \Omega$. But, if $\Omega \setminus S = \Omega$, then we know that $S = \emptyset$. However $S \neq \emptyset$ by the definition of S in the question. So, it must be the case that $\Omega \setminus S = \emptyset$ and thus $S = \Omega$.

$\blacksquare$

$\boxed{1.1.3}$

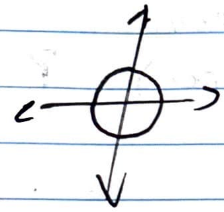
1.1.3

(6) Is a Disc a Circle?

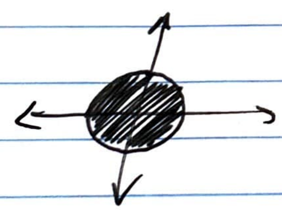
No.

A circle is the outter edge of a disc. A disc is more like a puck or a plate and a circle is more like a ring or a tire with no rim.

circle defined as $x^2+y^2=1$



disc defined as $x^2+y^2 \geq 1$



1.1.4 #1-3

(1) which of the following sets are bounded.

bounded if contained by some disc $D(z_0; r)$ where r is finite.

(a) $|z| \geq 1$

$z = (x, y)$

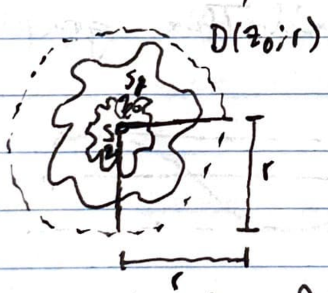
$|z| = \sqrt{x^2+y^2} \geq 1$

$\Rightarrow x^2+y^2 \geq 1$

This is not bounded.

(b) A subset of a bounded set

Given a set that is bounded, any subset must also be bounded by the same bounds as the larger set. So let $S_2 \subseteq S_1$, visually this looks like



which is contained by $D(z_0; r)$, thus S_2 must also be bounded.

$\boxed{1.1.4}$

(c) $0 < |z - z_0| < 1$

:

$0 < |z - z_0| < 1$

$\Rightarrow 0 < \sqrt{(x^2+y^2) - (x_0^2+y_0^2)} < 1$

$0 < (x^2 - x_0^2) + (y^2 - y_0^2) < 1$

This is bounded. It's a circle with an inner circle not included. It depends on x_0, y_0, but in general looks like,

d) The graph of $y = \sin(x)$

Assuming that this is along the y, it is bounded. The graph,

is bounded by $[-1, 1]$. So, it is bounded.

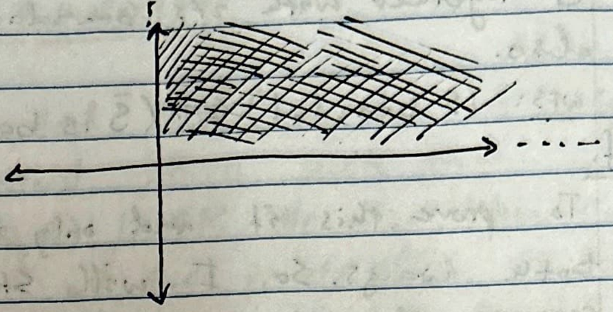
$\boxed{1.1.4}$

(2) Determine the boundaries of the following sets. As usual, $z=(x,y)$

(a) $x>0, y>0.$

Let this set be S, thus,

$\partial S = \{x=0, y=0\}$



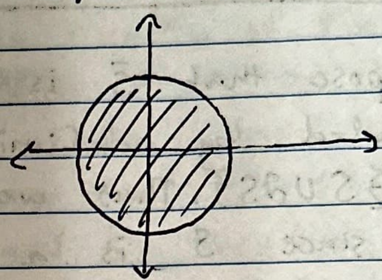
(b) $|z-z_0| \leq 2$ Let this set be S, then,

$(x^2-x_0^2)+(y^2-y_0^2) \leq 4$

we only want the points on the circle edge so

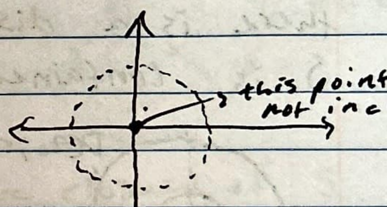
~~$\partial S = \{|z-z_0|=2\}$~~

$\partial S = \{|z-z_0|=2\}$



(c) $0<|z-z_0|<2$

Let this set be called S. thus, the graph looks like,



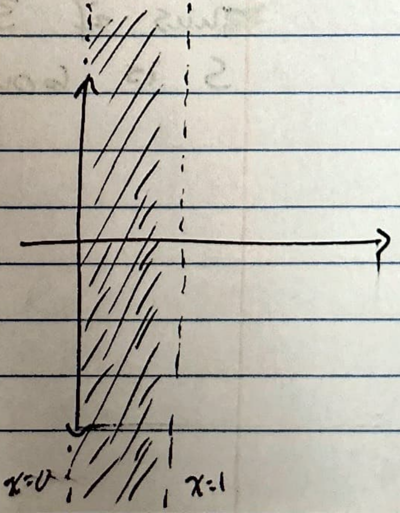
So our boundary points are the center point, and the outside rim of the circle.

So it would look like $\partial S = \{z_0, |z-z_0|=2\}$

(d) $0<x<1$, y arbitrary

Let this set be called S.

since y is any value the boundary is the set $\partial S = \{x=0, x=1\}$. It is just any points on the lines $x=0$ ~~and~~ or $x=1$ vertical lines.



$\boxed{1.1.4}$

(3) Prove that a plane set S is bounded if and only if its closure $\bar{S}$ (that is, S together with its boundary ∂S) is bounded also.

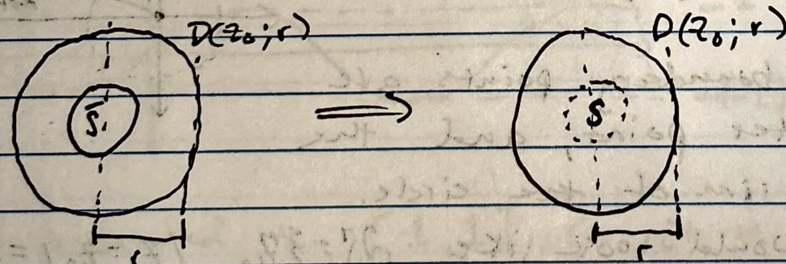
<u>WTS</u>: (S bounded) $\Leftrightarrow$ ($\bar{S}$ is bounded)

<u>Proof.</u>

To prove this if and only if we have to prove both ways. So, I will start by showing the reverse of above.

$\boxed{(\bar{S} \text{ bounded}) \rightarrow (S \text{ bounded})}$

So Suppose that $\bar{S}$ is bounded. Given that it is bounded by definition we know that $\bar{S} = \{S \cup \partial S\}$. Thus we know that $S \subseteq \bar{S}$. So since S is a subset or equivalent to $\bar{S}$ we know that it has the same boundary, $D(z_0;r)$, that makes $\bar{S}$ bounded. Thus, there is a disc with finite r such that S is contained in $D(z_0;r)$.



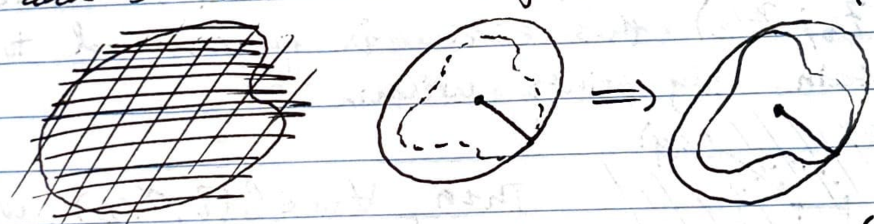
Thus if $\bar{S}$ is bounded we know that S is bounded as well.

$\boxed{1.1.4}$

Now to show the second way

(S bounded) $\Rightarrow$ ($\bar{S}$ bounded)

So let S be bounded. Thus by def. of being bounded we know that $\exists R > 0$ such that $S \subseteq D(0;R)$, $\forall z \in D(0;R)$, $|z| < R$. We also know that $\bar{S} = S \cup \partial S$. Thus $\bar{S}$ is all the points of S, as well as the points on the boundary of S. So since S is bounded, the points in S are all within R distance from the center of the circle. So if we set R to be the distance from the center of the circle to the $\max\{z_0 - s : s \in \partial S\}$ then the $D(0;r)$ is guaranteed to bound S, and $\bar{S}$. That is S and $\bar{S}$ are bounded by the same disc. Thus,



These are supposed to be circles...

So since S is bounded, the closure of S, $\bar{S}$ is also bounded.

So S is bounded iff $\bar{S}$ is bounded. $\blacksquare$