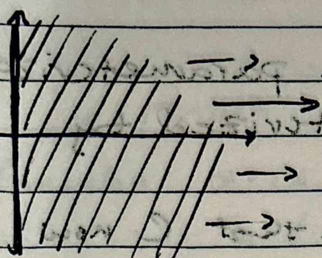


connected set not bounded

Definition | A set is bounded if and only if $\exists r > 0$ such that $S \subset D(0;r)$.

Def. | A point $z \in S \subset \mathbb{R}^2$ is a boundary point if and only if every disc $D(z;r)$ contains points in S and points not in S.

Note: A Boundary Point of S is also a boundary point of $\mathbb{R}^2/S$

Def. | The boundary of a set $S \subset \mathbb{R}^2$ is the set of all boundary points of S, denoted by ∂S

Exs. |

$D(0;r)$; $\partial D(0;r) = C(0;r) = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 = 1\}$

$H = \{(x,y) \in \mathbb{R}^2 : y > 0\}$; $\partial H = \{(x,y) \in \mathbb{R}^2 : y = 0\}$

$\mathbb{R}^2$; $\partial \mathbb{R}^2 = \emptyset$

Conventions and Definitions

$0 = (0,0)$

$C = C(0;r)$ for $r > 0$

$C(0;r)$ equiv. to $|z| = r$

$\alpha = \alpha(t)$

- $C(0;r)$ is "static"
- We can add movement with parameterization.
- $C(0;r)$ can be parameterized by $\alpha : [0, 2\pi] \to C$ where $\alpha(t) = (r \cdot \cos(t), r \cdot \sin(t))$. Note that C now has an orientation, in this case counterclockwise around the origin.
- Moreso, $\alpha(t)$ can be written as $\alpha(t) = (\alpha_1(t), \alpha_2(t))$; that is α has component functions α_1, α_2.
- Additionally, $\alpha'(t)$ is defined as $(\alpha_1'(t), \alpha_2'(t))$.

Uniqueness of Parameterization

- not unique

Ex.1

$\alpha : [0, 2\pi] \to C(0;r)$ with $\alpha(t) = (r \times \cos(t), r \times \sin(t))$

$\beta : [0, 2\pi] \to C(0;r)$ w/ $\beta(t) = (r \times \cos(3t), r \times \sin(3t))$

and $\gamma : [0,1] \to C(0;r)$ w/ $\gamma(t) = (r \times \cos(2\pi t), r \times \sin(2\pi t))$

Parameterization of Curves

Def.

Let $\Gamma \subset \mathbb{R}^2$ and $[a,b] \subset \mathbb{R}$ with $a<b$. Any function $\alpha = (\alpha_1, \alpha_2) : [a,b] \to \Gamma$ is a parameterization, of Γ if and only if

① α is continuous; that is α_i are all cont.

② α is surjective; that is for every $z \in \Gamma$ there exists $t \in [a,b]$ such that $\alpha(t) = z$

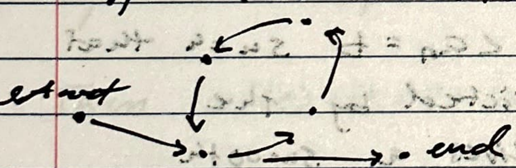
Def.

α is called Smooth or continuosly diff. if in addition,

① α_i' all exist and are continuosly for every $t \in [a,b]$

② for each $t \in [a,b]$, the velocity vector $\alpha'(t) = \langle a_1'(t), a_2'(t) \rangle \neq \langle 0, 0 \rangle$

③ if $\alpha(a) = \alpha(b)$, then $\alpha'(a) = \alpha'(b)$



Definition

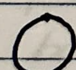
A parameterization α is simple if and only if $\alpha|_{(a,b)}$ is injective

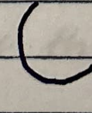
$\alpha : [a,b] \to \Gamma$, $\alpha|_{(a,b)} \leftrightarrow \alpha(t)$ for $t \in (a,b)$

Def.

A parameterization α is closed if $\alpha(a) = \alpha(b)$.

 closed, not simple

 closed, simple

 not closed, simple.

Exs.]

- The line from z_0 to z_1 is parameterized by $\gamma(t) := (1-t)z_0 + t.z_1$ where $t \in [0,1]$

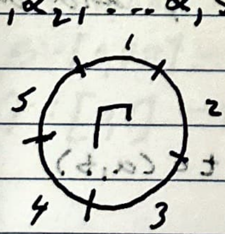
- The circle $C(0;1)$ is often parameterized by $\alpha(t) = (r\cos(t), r\sin(t))$ where $t \in [0, 2\pi]$ or $\beta(t) = (r\cos(2\pi t), r\sin(2\pi t))$ for $t \in [0,1]$

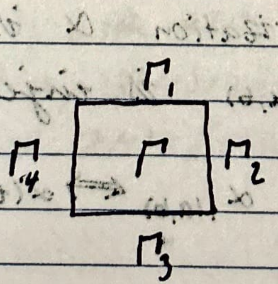
- The parameterization $\eta(t) = (t, t^3)$ for $t \in [-1,1]$ is a simple, not closed param. between $(-1,-1)$ and $(1,1)$.

PieceWise - Smooth

Def.] A param. $\alpha : [a,b] \to \Gamma$ is a piecewise-smooth parameterization if and only if $\exists$ a finite set of values $a = a_0 < a_1 < a_2 < \dots < a_{n-1} < a_n = b$ such that the function α restricted by the intervals $[a_{i-1}, a_i]$ gives a smooth parameterization of $\Gamma_i = \{\alpha(t) : t \in [a_{i-1}, a_i]\}$.

$\alpha_1, \alpha_2, \dots \alpha_5$





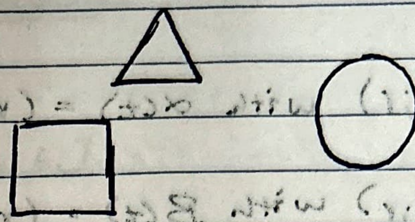
P.W. Smooth

not smooth

$$\int_\Gamma f(z)dz = \int_{\Gamma_1} f(z)dz + \int_{\Gamma_2} f(z)dz$$

Examples|

- The Unit Circle
- $(t,t^2) \in [-1,3]$?
- Triangle



Smooth Curves

- If α is a smooth param of Γ, the pair (Γ,α) is called a piecewise-smooth curve.
- It is customary to omit "mention of α and speak of "the smooth curve of Γ"

$$f:[a,b] \longrightarrow [0,1]\times[0,1]$$

Motivation|

Consider the line from z_0 to z_1, parameterized by $\alpha(t) = tz_1 + (1-t)z_0 = z_0 + t(z_1 - z_0)$

Then $\alpha'(t) = z_1 - z_0$ (independent of t)

So, $\displaystyle\int_0^1 \alpha'(t)\,dt = (z_1 - z_0)(1-0)$

Definition|

If Γ is a smooth curve given by $\alpha:[a,b]\to\Gamma$, then the length of α is

$$\text{length}(\alpha) = \int_a^b |\alpha'(t)|\,dt$$

Since α is smooth, the integrand is piecewise continuous and bounded, hence the integral exists.

P10

Exs.

$C(0;r)$ with $\alpha(t) = (r\cos(t), r\sin(t))$ for $t \in [0, 2\pi]$

$C(0;r)$ with $\beta(t) = (r\cos(t), r\sin(t))$ for $t \in [0, 6\pi]$

$\alpha''(t) = \langle -r\sin(t), r\cos(t) \rangle$

$|\alpha'(t)| = \sqrt{r^2\sin^2(t) + r^2\cos^2(t)}$

$\int_0^{2\pi} r\,dt = \boxed{r \cdot 2\pi}$ $\int_0^{6\pi} r\,dt = r \cdot 6\pi$ $\boxed{= 3 \cdot 2\pi r}$

Parameter Selection for parameterization:

- Using an arbitrary parameter for curves is nonsensical
- Prefer to use an intrinsic property of the curve itself
- The desired property is arc-length
- Goal: Adapt length integral to be new parameter.

Adaptation of length:

- Denote the new parameterization of a given curve (Γ, α) by $\sigma(s)$.
- $\sigma(s)$ will have the property that $\sigma(s_1) - \sigma(s_0) = (s_1 - s_0)$
- $\sigma(s)$ will be equivalent to α.