DREW HEARD

MA3408 - ALGEBRAIC TOPOLOGY II

Contents

1
Homotopy theory

1.1 Review of basics on homotopy theory

We begin with a recollection of some facts that have been covered in Algebraic Topology I and Introduction to Topology.

1.1.1 Notation. We let $I = [0,1]$ denote the unit interval. For a pointed topological space X we will denote the basepoint by x_0 or $*$.

We recall the following definition.

1.1.2 Definition. A homotopy between $f, g\colon X \to Y$ is a continuous function $H\colon X \times I \to Y$ such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$ and $H(x_0, t) = y_0$ for all $t \in I$. We will write $f \simeq g$, or $f \simeq_H g$, if we need to make the choice of homotopy clear.

For a subspace $A \subseteq X$, a relative homotopy is a homotopy with $H(a,t) = f(a) = g(a)$ for all $a \in A, t \in I$.

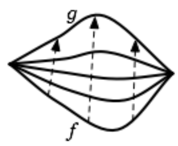


Figure 1.1: A homotopy between f and g.

1.1.3 Remark. Equivalently, we can specify a family of continuous maps $h_t\colon X \to Y$ such that $h_0 = f, h_1 = g$ and

$$\begin{aligned} H\colon X \times I &\to Y \\ (x,t) &\mapsto h_t(x) \end{aligned}$$

is continuous. We will switch between the two equivalent definitions without comment, using whatever is more convenient.

1.1.4 Proposition. *For all spaces X and Y, homotopy is an equivalence relation on the set of maps from X to Y. Furthermore, if we are given $k\colon A \to X, \ell\colon Y \to B$ and homotopic maps $f \simeq g\colon X \to Y$, then $fk \simeq gk\colon A \to Y$ and $\ell f \simeq \ell g\colon X \to B$.*

Proof. Let $f, g\colon X \to Y$, then

1. $f \simeq_F f$ via $F(x,t) = f(x)$ for all $x \in X, t \in I$.
2. If $f \simeq_F g$, then $g \simeq_G f$ where $G(x,t) = F(x, 1-t)$.
3. If $f \simeq_F g$ and $g \simeq_G h$, then $f \simeq_H h$ via

$$H(x,t) = \begin{cases} F(x,2t) & \text{if } 0 \leq t \leq 1/2 \\ G(x,2t-1) & \text{if } 1/2 \leq t \leq 1. \end{cases}$$

For the last part of the proposition let f_t be a homotopy between f and g, then $f_t k$ and ℓf_t give the required homotopy. $\square$

1.1.5 Definition. For a map $f\colon X \to Y$, we let $[f]$ denote the equivalence class containing f. The collection of all homotopy classes of maps from X to Y is denoted $[X, Y]$.[1]

[1] If our spaces are based, then these should be homotopy classes of *based* maps.

1.1.6 Remark. Note that if $\alpha = [f] \in [Y, Z]$ and $\beta = [g] \in [X, Y]$, then $\alpha\beta = [f \circ g] \in [X, Z]$, i.e., we can form the category $hTop_*$ whose objects are topological spaces, and whose morphisms are homotopy classes of maps.

1.1.7 Remark. We now very quickly review a number of standard topological constructions.

- Let X be a space and $A \subseteq X$. A map $r\colon X \to A$ such that $ri(a) = a$ for all $a \in A$ is called a retraction of X onto A, and A is called a retract of X.
- Let $i\colon A \hookrightarrow X$ be the inclusion, so that $ri = \mathrm{id}_A$. If $ir \simeq \mathrm{id}_X$, we call this a deformation retraction, and say that A is a deformation retract of X.
- If $f\colon X \to Y$, then a section of f is a map $s\colon Y \to X$ such that $f \circ s = \mathrm{id}_Y$. We can also ask for a *homotopy* section by requiring only that $f \circ s \simeq \mathrm{id}_Y$.

1.1.8 Definition. A map $f\colon X \to Y$ is called null-homotopic if $f\colon c_y\colon X \to Y$ where $c_y X \to Y$ is the constant map sending all of X to the point $y \in Y$. A homotopy between f and c_y is called a null-homotopy. A space X is contractible if id_X is null-homotopic.

1.1.9 Definition. Let (X, x_0) be a based topological space and $X \times I$ the cylinder on X. The quotient

$$CX = (X \times I)/(X \times \{1\} \cup \{x_0\} \times I)$$

with the base-point the equivalence class of $(x_0, 1)$ is called the (reduced) cone on X. Note that we have a natural inclusion $X \to CX$ of based maps given by $x \mapsto [x, 0]$.

1.1.10 Lemma. *The cone CX is contractible.*

Proof. Define $F\colon CX \times I \to CX$ by

$$F([x, t], s) = [x, s + (1 - s)t].$$

Note then that we have

$$F([x, t], 0) = [x, t] \quad \text{and} \quad F([x, t], 1) = [x, 1]. \qquad \square$$

1.1.11 Lemma. *The following are equivalent:*

(i) $f\colon X \to Y$ *is null-homotopic.*

(ii) f *can be extended to* CX*:*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, hook, "i"'] & Y \\ CX \arrow[ur, dashed, "\exists \tilde{f}"'] & \end{tikzcd}$$

Proof. $(i) \implies (ii)$: Suppose f is null-homotopic, so $f \simeq_F *$. Then $F(X \times \{1\} \cup \{*\} \times I) = *$, so by the universal property of the quotient, we can find $\tilde{F}: CX \to Y$ such that $\tilde{f} \circ i = f$.

$(ii) \implies (i)$: Suppose $\tilde{f} \circ i = f$, then because CX is contractible (Lemma 1.1.10), we have $f = \tilde{f} \circ \mathrm{id}_{CX} \circ i \simeq \tilde{f} \circ (*_{CX}) \circ i \simeq *$, so that f is null-homotopic. □

1.1.12 *Definition.* A map $f: X \to Y$ is a homotopy equivalence if there exists $g: Y \to X$ such that $fg \simeq \mathrm{id}_Y$ and $gf \simeq \mathrm{id}_X$. We write $X \simeq Y$.

1.1.13 *Example.* (i) X is contractible if and only if $X \simeq *$.

(ii) If $i: A \hookrightarrow X$, and $r: X \to A$ is a deformation retract, then i and r are homotopy equivalences, and $A \simeq X$.

1.2 Higher homotopy groups

1.2.1 *Notation.* We will let $I_n = I^{\times n}, \partial I^n$ be the boundary of I^n, and write $[-,-]$ for homotopy classes of maps (if our spaces are based, these fix the base point).

1.2.2 *Definition.* For each $n \geq 0$ and X a topological space with $x_0 \in X$, we define

$$\pi_n(X, x_0) = [(I^n, \partial I^n), (X, x_0)].$$

1.2.3 *Remark.* (i) When $n = 0$, we have $I^0 = \mathrm{pt}$ and $\partial I^0 = \emptyset$, therefore $\pi_0(X)$ is the set of path components of X.

(ii) When $n = 1$, this is a group, but need not be abelian (for example, consider the wedge of two circles).

(iii) Note that $I^n/\partial I^n \simeq S^n$ and $\partial I^n/\partial I^n \simeq s_0$. By the universal property of the quotient map, we see that

$$\pi_n(X, x_0) \cong [(S_n, s_0), (X, x_0)].$$

1.2.4 *Definition.* A maps of pairs $(X, A) \to (Y, B)$ is a map $f: X \to Y$ with $f(A) \subseteq B$, i.e., the diagram:

$$\begin{tikzcd} A \arrow[r] \arrow[d, hook] & B \arrow[d, hook] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

commutes.

1.2.5 Proposition. *If $n \geq 1$, then $\pi_n(X, x_0)$ is a group with respect to the operation*

$$(f+g)(t_1, \ldots, t_n) = \begin{cases} f(2t_1, t_2, \ldots, t_n) & 0 \leq t_1 \leq 1/2 \\ g(2t_1 - 1, t_2, \ldots, t_n) & 1/2 \leq t_1 \leq 1. \end{cases}$$

Proof. The identity is given by the constant map taking all of I^n to x_0 and the inverse of f is given by

$$-f(t_1,\ldots,t_n) = f(1-t_1,t_2,\ldots,t_n). \qquad \square$$

1.2.6 Remark. Call the group operation $+_1$. Note that we can also define an operation $+_i$ for $1 \leq i \leq n$ by the same formula on the i-th coordinate.

1.2.7 Theorem. *All of these operations agree, and for $n \geq 2$, these give $\pi_n(X,x_0)$ the structure of an abelian group.*

This is a consequence of the following exercise, known as the Eckmann–Hilton lemma.

Exercise 1: Eckmann–Hilton lemma

Let M be a set and let $*, \bullet$ be two binary operations on M, $*, \bullet \colon M \times M \to M$, both with unit elements. Suppose that

$$(a * b) \bullet (c * d) = (a \bullet c) * (b \bullet d)$$

for all $a,b,c,d \in M$. Show that the units agree, these two operations agree, and that the multiplication is commutative and associative.

1.2.8 Remark. Let use show that

$$(f +_1 g) +_2 (h +_1 i) \simeq (f +_2 h) +_1 (g +_2 i).$$

Indeed, both of these are the following map

$$(t_1,t_2,\ldots,) \mapsto \begin{cases} f(2t_1,2t_2,\ldots,) & [1/2,0] \times [1/2,0] \\ g(2t_1-2,2t_2,\ldots,) & [1/2,1] \times [0,1/2] \\ h(2t_1,2t_2-2,\ldots) & [0,1/2] \times [1/2,1] \\ i(2t_1-1,2t_2-2,\ldots) & [1/2,1] \times [1/2,1]. \end{cases}$$

1.2.9 Remark. Another approach is given by the following visualization: That is, so long as $n \geq 2$, we can shrink the domain of f and g

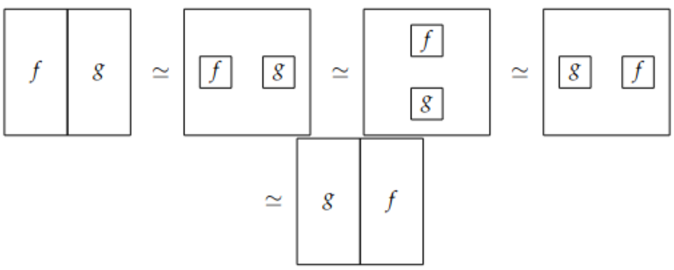


Figure 1.2: $f+g \simeq g+f$.

to smaller cubes (mapping the remaining region to the base point), slide f and g past each other, and then increase the domains back again.

Exercise 2

Let G be a topological group with identity element e, then $\pi_1(G, e)$ is abelian.
Hint: Use Eckmann–Hilton, or note the following: A topological group is a group object in the category of topological spaces. What is a group object in the category of groups?

1.2.10 Proposition. *If $n \geq 1$ and X is path connected, then there is an isomorphism $\beta_\gamma : \pi_n(X, x_0) \xrightarrow{\cong} \pi_n(X, x_0)$ given by $\beta_\gamma([f]) = [\gamma \circ f]$ where γ is a path in X from x_1 to x_0 and $\gamma \circ f$ is constructed by first shrinking the domain of f to a smaller cube inside of I^n, and then inserting the path γ radially from x_1 to x_0 on the boundaries of these cubes.*

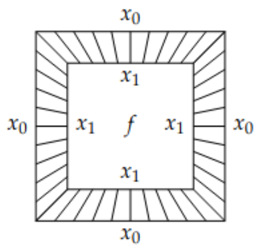
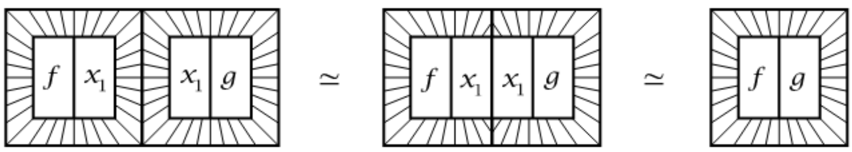


Figure 1.3: β_γ.

Proof. Observe the following:

1. $\gamma \circ (f + g) \simeq \gamma \circ f + \gamma \circ g$, i.e., β_γ is a group homomorphism.
2. $(\gamma \circ \eta) \circ f \simeq \gamma \circ (\eta \circ f)$, for η a path from x_0 to x_1.
3. $c_{x_0} \circ f \simeq f$, where c_{x_0} denotes the constant path based at x_0.
4. β_γ is well-defined with respect to homotopies of f or γ.

The only point that is perhaps not clear is (i). For this, we deform f and g to be constant on the right and left halves of I^n, respectively, producing maps we call $f + 0$ and $0 + g$. We then excise a wider symmetric middle slab of $\gamma(f + 0)$ and $\gamma(0 + g)$ until it becomes $\gamma(f + g)$: □



1.2.11 Remark. Therefore if X is path-connected, different choices of base point x_0 yield isomorphic groups $\pi_n(X, x_0)$, which may then simply be written as $\pi_n(X)$.

1.2.12 Lemma. *If $\{X_\alpha\}$ is a collection of path-connected spaces, then $\pi_n(\prod_\alpha X_\alpha) \cong \prod_\alpha \pi_n(X_\alpha)$.*

Proof. Note that $\mathrm{Hom}(Y, \prod_\alpha X_\alpha) \simeq \prod_\alpha \mathrm{Hom}(Y, X_\alpha)$. In particular, a map $S^n \to \mathrm{Hom}(Y, \prod_\alpha X_\alpha)$ is determined by a collection of maps $S^n \to X_\alpha$. Likewise, a homotopy $S^n \times I \to \prod_\alpha X_\alpha$ is determined by a colletion of homotopies $S^n \times I \to X_\alpha$. This implies the result. □

1.2.13 Proposition. *Homotopy groups are functorial: given a map $\phi\colon X \to Y$ we get group homomorphisms $\phi_*\colon \pi_n(X, x_0) \to \pi_n(X, \phi(x_0))$ given by $[f] \mapsto [\phi \circ f]$ for all $n \geq 1$.*

Proof. We have the following:

1. ϕ_* is well-defined: if $f \simeq g$ via ψ_t, then $\phi \circ \psi_t$ defines a homotopy between $\phi \circ f$ and $\phi \circ g$.
2. This is a group homomorphism: $\phi \circ (f + g) \simeq \phi \circ g + \phi \circ g$ by the definition of the addition operation. Therefore.

$$\phi_*[f + g] = \phi_*[f] + \phi_*[g].$$

□

Exercise 3

If $\phi\colon X \to Y$ is homotopy equivalence (not necessarily base-point preserving), then $\pi_*\colon \pi_n(X, x_0) \to \pi_n(Y, \phi(y_0))$ is an isomorphism.

1.2.14 Remark. We recall the following lifting property: Suppose $p\colon (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ is a covering, and there is a map $f\colon (Y, y_0) \to (X, x_0)$ with Y path-connected and locally path-connected. Then a lift $\tilde{f}$ exists if and only if $f_*\pi_1(Y, y_0) \subseteq p_*\pi_1(\tilde{X}, \tilde{x}_0)$.

$$\begin{tikzcd} & (\tilde{X}, \tilde{x}_0) \arrow[d, "p"] \\ (Y, y_0) \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"] & (X, x_0) \end{tikzcd}$$

1.2.15 Proposition. *If p is a covering, then $p_*\colon \pi_n(\tilde{X}, \tilde{x}_0) \to \pi_n(X, x_0)$ is an isomorphism for all $n \geq 2$.*

Proof. Let us first show surjectivity. To that end, suppose we have a map $f\colon (S^n, s_0) \to (X, x_0)$ where $n \geq 2$. The assumption on n gives $\pi_1(S^n) = 0$, so $f_*\pi_1(S^n, s_0) \subseteq p_*\pi_1(\tilde{X}, \tilde{x}_0)$ holds. We therefore find a lift in the following:

$$\begin{tikzcd} & (\tilde{X}, \tilde{x}_0) \arrow[d, "p"] \\ (S^n, s_0) \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"] & (X, x_0) \end{tikzcd}$$

Then $p_*[\tilde{f}] = [f]$, and p_* is surjective.

To see that p_* is injective, let $[\tilde{f}] \in \ker(p_*)$, i.e., $p_*[\tilde{f}] = [p \circ \tilde{f}] = 0$. Let $f = p \circ \tilde{f}$, then this is homotopic to the constant map $f \simeq c_{x_0}$

via a homotopy $\phi_t\colon (S^n, s_0) \to (X, x_0)$ with $\phi_1 = f$ and $\phi_0 = c_{x_0}$. By the same argument as above, the homotopy ϕ_t can be lifted to $\tilde{\phi}_t$. This satisfies $p \circ \tilde{\phi}_1 \simeq \phi_1$ and $p \circ \tilde{\phi}_0 \simeq \phi_0$. By the uniqueness of lifts, we must have $\tilde{\phi}_1 \simeq \tilde{f}$ and $\tilde{\phi}_0 \simeq c_{x_0}$. In other words, $\tilde{\phi}_t$ gives a homotopy between $\tilde{f}$ and c_{x_0}, so that $[\tilde{f}] = 0$, and p_* is injective. □

1.2.16 Example. S^1 has universal cover $p\colon \mathbb{R} \to S^1$, $p(t) = e^{2\pi i t}$. Then $\pi_n(S^1) \cong \pi_n(\mathbb{R}) \cong 0$ for $n \geq 2$.

Exercise 4

Find two spaces X, Y with $\pi_n X \cong \pi_n Y$ but $X \not\simeq Y$.
Hint: What is the universal cover of $\mathbb{R}P^n$?

1.2.17 Remark (Relative homotopy groups). Suppose we have (X, x_0) and a subspace A containing x_0. We note that $i_*\colon \pi_n(A, x_0) \to \pi_n(X, x_0)$ is not injective in general (example, take S^1 into $\mathbb{R}^2$). An element in the kernel of i_* is a map $f\colon (I^n, \partial I^n) \to (A, x_0)$ such that

$$(I^n, \partial I^n) \xrightarrow{f} (A, x_0) \xrightarrow{i} (X, x_0)$$

is homotopic to c_{x_0}. This means there exists a homotopy

$$H\colon I^n \times I \to X$$

such that $H(-,1) = f, H(-,0) = c_{x_0}$ and $H|_{\partial I^n \times I} = c_{x_0}$.

If we define $J^n = I^n \times \{0\} \cup \partial I^n \times I \subseteq I^n \times I$, then this is a map of triples

$$H\colon (I^{n+1}, \partial I^{n+1}, J^n) \to (X, A, x_0).$$

1.2.18 Definition.

$$\pi_n(X, A, x_0) = [(I^n, \partial I^n, J^{n-1}), (X, A, x_0)]$$

1.2.19 Remark. Equivalently,

$$\pi_n(X, A, x_0) = [(D^n, S^{n-1}, s_0), (X, A, x_0)]$$

1.2.20 Proposition. *If $n \geq 2$, then $\pi_n(X, A, x_0)$ is a group, and if $n \geq 3$, then it is abelian.*

For all $n \geq 2$, a map of pairs $\phi\colon (X, A, x_0) \to (Y, B, y_0)$ induces homomorphisms $\phi_*\colon \pi_n(X, A, x_0) \to \pi_n(Y, B, y_0)$ for all $n \geq 2$.

Proof. This is similar to the case of $\pi_n(X)$ itself, and the details are left to the reader. □

1.2.21 Theorem. *The relative homotopy groups (X, A, x_0) fit into a long exact sequence*

$$\cdots \to \pi_n(A, x_0) \xrightarrow{i_*} \pi_n(X, x_0) \xrightarrow{j_*} \pi_n(X, A, x_0) \xrightarrow{\partial_n} \pi_{n-1}(A, x_0) \to \cdots$$

where the map ∂_n is defined by $\partial_n([f]) = [f|_{I^{n-1}}]$.

The proof relies on the following.

1.2.22 Lemma (Compression criterion). *A map $f\colon (D^n, S^{n-1}, x_0) \to (X, A, x_0)$ represents 0 in $\pi_n(X, A, x_0)$ if and only if $f \sim g$ rel S^{n-1}, where g is a map whose image is contained entirely in A.*

Proof. Suppose $[f] = [g]$ with g as in the statement of the lemma. Note that there is a deformation of D^n onto x_0, and so $[f] = [g] = 0$ in $\pi_n(X, A, x_0)$.

Conversely, suppose that $[f]$ represents 0 in $\pi_n(X, A, x_0)$. This means there exists a homotopy, relative to S^{n-1}, $F\colon D^n \times I \to X$ with $F\mid_{D^n\times\{0\}} = f$, $F\mid_{D^n\times 1} = c_{x_0}$ and $F\mid_{S^{n-1}\times I} \subseteq A$. We can restrict F to a family of n-disks in $D^n \times I$ starting with $D^n \times \{0\}$ and ending with the disk $D^n \times \{1\} \cup S^{n-1} \times \{1\}$, all the disks in the family having the same boundary, then we get a homotopy from f to a map in A, stationary on S^{n-1} (said in other words, we can deformation retract $D^n \times [0, 1]$ onto $D^n \times \{1\} \cup S^{n-1} \times I$). □

We now prove the existence of the long exact sequence.[2]

[2] This is the type of proof that is best done by the reader themselves.

Proof of Theorem 1.2.21. **Step 1.** Let us first show exactness at $\pi_n(X, x_0)$.

We first show $\mathrm{im}(i_*) \subseteq \ker(j_*)$. Note that $j_* i_*$ is induced by the composition $j \circ i$ and that these are both inclusion maps. Therefore, for $[f] \in \pi_n(A, x_0)$ we have $j_* i_*[f] = [j \circ i \circ f]$, but this has image contained in A, and so $j_* i_*[f] = 0$. This shows $\mathrm{im}(i_*) \subseteq \ker(j_*)$.

To see the converse (namely, $\ker(j_*) \subseteq \mathrm{im}(i_*)$) let $[f] \in \ker(j_*)$, i.e. $[j \circ f] = 0$. Note that again j is an inclusion map, and by the compression criteria $f \simeq g'$ relative to S^{n-1}, where g' has image contained in A. Since $x_0 \in S^{n-1}$, the homotopy fixes the basepoint, i.e, $[f] = [g'] \in \pi_n(X, x_0)$. But because g' has image in A, $[g'] \in \pi_n(A, x_0)$ and $i_*[g'] = [i \circ g'] = [f]$, so $[f] \in \mathrm{im}(i_*)$.

Step 2. Let us now show exactness at $\pi_n(X, A, x_0)$.

Note that the composite $\partial \circ j_* = 0$ since the restriction of a map $(I^n, \partial I^n, J^{n-1}) \to (X, x_0, x_0)$ to I^{n-1} has image x_0 and so represents 0 in $\pi_{n-1}(A, x_0)$. Therefore, $\mathrm{im}(j_*) \subseteq \ker(\partial)$. For the converse, suppose $[f] \in \ker(\partial)$. This means there exists a basepoint preserving homotopy $H\colon I^{n-1} \times I \to A$ (relative to ∂I^{n-1}) from $f\mid_{I^{n-1}\times\{0\}}$ to the constant map where the image of H is contained entirely in A. We can then define another homotopy H, such that $G_0 = f$, $G_t\mid_{I^{n-1}} = H_t$ and the rest of the image of G_t is $f[I^n]$ union with the image of H_s for $0 \le s \le t$. This homotopy maps S^{n-1} into A at all times, so $[f] = [G_1]$. Moreover, G_1 maps the boundary of I^n to x_0, so $[G_1] \in \pi_n(X, x_0)$. Altogether,

$$j_*[G_1] = [j \circ G_1] = [G_1] = [f]$$

so $\ker(\partial) \subseteq \mathrm{im}(j_*)$.

Step 3: Exactness at $\pi_n(A, x_0)$.

Let $[f] \in \pi_n(X, A, x_0)$ then $i_*\partial \in \pi_{n-1}(X, x_0)$ is the class represented by $f\mid_{I^{n-1}}$ and this is homotopic relative J^{n-2} to the constant map to x_0, via f viewed as a homotopy. So this implies $\mathrm{im}(\partial_*) \subseteq \ker(i_*)$. Conversely, let $[f] \in \ker(i_*)$ i.e., $i_*[f] = [i \circ f] = 0$.

Therefore, there exists a homotopy H between f and a constant map through a homotopy that has image in X and preserves x_0. Since $H_0 = f$ has image in A and H_1 has image $\{x_0\}$, and H_0 takes the boundary to $\{x_0\}$, we see that $[H] \in \pi_n(X, A, x_0)$, and moreover $\partial([H]) \simeq f$. Therefore, $[f] \in \mathrm{im}(\partial)$, and $\mathrm{im}(\partial) = \ker(i_*)$. □

1.2.23 Definition. A pair (X, A) with basepoint x_0 is said to be n-connected if $\pi_i(X, A) = 0$ for all $i \leq n$.[3]

[3] A 0-connected space is exactly a path-connected space.

1.2.24 Lemma. *A pair (X, A) is n-connected if and only if $\pi_i(A) \xrightarrow{i_*} \pi_i(X)$ is an isomorphism for $i < n$ and a surjection for $i = n$.*

Proof. Use the long exact sequence in homotopy. □

Exercise 5

Let X be a path-connected space, and CX the cone on X. Show that

$$\pi_n(CX, X, X_0) \cong \pi_{n-1}(X, x_0)$$

for $n \geq 1$.

1.3 Cofibrations and the homotopy extension property

1.3.1 Definition. Let $\mathcal{C}$ be a class of topological spaces. A map $i\colon A \to X$ has the homotopy extension property (HEP) if, for every $Y \in \mathcal{C}$, the following extension property has a solution[4]

[4] Here $i_0(x) = (x, 0)$.

$$\begin{tikzcd}
A \arrow[r, hook, "i_0"] \arrow[d, "i"'] & A \times I \arrow[d, "i\times \mathrm{id}"] \arrow[ddr, bend left, "H"] & \\
X \arrow[r, hook, "i_0"'] \arrow[drr, bend right, "f"'] & X \times I \arrow[dr, dashed, "\exists \tilde{H}"] & \\
& & Y
\end{tikzcd}$$

A map $f\colon A \to X$ is a cofibration if it has the HEP with respect to all spaces Y.[5]

[5] We will see later that cofibrations are always inclusions, and, if X is Hausdorff, are always closed maps.

1.3.2 Remark. Note that we do not ask that $\tilde{H}$ is unique.

1.3.3 Remark. If we are in a 'nice' category of topological spaces (see CREF), which we always assume, then we have an adjunction

$$\mathrm{Hom}(X, \mathrm{Hom}(Y, Z)) \cong \mathrm{Hom}(X \otimes Y, Z)$$

of topological spaces, where $\mathrm{Hom}(Y, Z)$ is given the compact open topology. Writing, $Z^Y := \mathrm{Hom}(Y, Z)$, the homotopy extension property admits a reformulation in the following diagram

$$\begin{tikzcd}
A \arrow[r, "h"] \arrow[d, "i"'] & Y^I \arrow[d, "p"] \\
X \arrow[r, "f"'] \arrow[ur, dashed, "\exists \tilde{h}"] & Y
\end{tikzcd}$$

where $p\colon Y^I \to Y$ is the evaluation at 0 map. It is often easier to work with this equivalent diagram.

> **Exercise 6**
>
> Let (X, A) have the HEP, and assume moreover that $i\colon A \to X$ is a retract up to homotopy. Show that A is a retract of X.

1.3.4 Lemma. *Let $J = [0,1]$.*

(i) The inclusion $i_0\colon X \to X \times J$ has the homotopy extension property for all Y.

(ii) The inclusion $i_0\colon X \to CX$ has the homotopy extension property for all Y.

Proof. The proof in both cases is very similar; we do the first case in some detail. We are claiming there exists a lift $\tilde{H}$ in the following diagram:

$$\begin{tikzcd}
X \arrow[r, hook, "i_0"] \arrow[d, "i"'] & X \times I \arrow[d, "i\times\mathrm{id}"] \arrow[ddr, bend left, "H"] & \\
X \times J \arrow[r, hook, "i_0"'] \arrow[drr, bend right, "f"'] & X \times J \times I \arrow[dr, dashed, "\exists\tilde{H}"] & \\
 & & Y
\end{tikzcd}$$

Geometrically, we will do this in two parts: we will define a map that "stacks" the two intervals on top of each other, i.e., we construct a map $G\colon X \times J \times I \to X \times [0,2]$. We will then do H on one part of the cylinder, and f on the remaining part.

For the first part, let $G\colon X \times J \times I \to X \times [0,2]$ be defined as[6]

$$G(x,t,s) = (x, t(1+s)).$$

[6] To see what is going on it is worth testing some cases and drawing pictures. For example, when $t = 0$ we have $G(x,0,s) = (x,0)$. When $t = 1$ we have $G(x,1,s) = (x,1+s)$. When $s = 0$ we have $G(x,t,0) = (x,t)$ and when $s = 1$ we have $G(x,t,1) = (x,2t)$.

We then define $F\colon X \times [0,2] \to Y$ by

$$F(x,k) = \begin{cases} f(x,k) & 0 \le k \le 1 \\ H(x,k/2) & 1 \le k \le 2. \end{cases}$$

Putting these together and defining $\tilde{H} := F \circ G$, we see that[7]

$$\tilde{H}((x,t),s) = \begin{cases} f(x, 1-(1-t)(1+s)), & (1-t)(1+s) \le 1 \\ H(x, (1-t)(1+s)-1), & (1-t)(1+s) \ge 1. \end{cases}$$

[7] Again, it is worthwhile to consider some cases. For example, if $t = 0$, then $(1-t)(1+s) = (1+s) \ge 1$ for all s, so $\tilde{H}((x,0),s) = H(x,s)$. At the other extreme, if $t = 1$, then $(1-t)(1+s) = 0 \le 1$ for all s, so $\tilde{H}((x,1),s) = f(x,1)$.

One verifies directly that this gives the required extension. □

1.3.5 Remark. We recall that given a map $f\colon X \to Y$, the mapping cylinder (see Figure 1.4) is the pushout

$$\begin{tikzcd}
X \arrow[r, "i_0"] \arrow[d, "f"'] & X \times I \arrow[d] \\
Y \arrow[r] & M_f \arrow[ul, phantom, "\lrcorner", very near start]
\end{tikzcd}$$

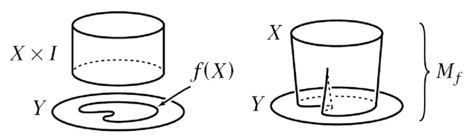


Figure 1.4: The mapping cylinder.

In formulas,

$$M_f = ((X \times I) \coprod Y)/((0,x) \sim f(x),\ \forall x \in X)$$

Note that M_f deformation retracts on Y by sliding each point $(x,t) \in M_f$ to the end-point. Note that we have a natural map $j\colon X \to M_f$ sending x to $(x,1)$.

1.3.6 Lemma. *The map $j\colon X \to M_f$ has the HEP for all spaces Y.*

Proof. The proof is similar to the previous lemma; one just has to modify the end point by defining

$$\tilde{H}|_{Y\times I}(y,s) = f(y,0).$$

□

1.3.7 Corollary. *The inclusion $S^{n-1} \to D^n$ is a cofibration.*

Proof. Simply note that $D^n \simeq CS^{n-1}$. □

There is a universal test space for cofibrations.

1.3.8 Proposition. *Let $i\colon A \to X$, and let M_i be the mapping cylinder. Then $i\colon A \to X$ is a cofibration if and only if there exists a map $r\colon X \times I \to M_i$ making the diagram*

$$\begin{tikzcd}
A \arrow[rr, "i_0"] \arrow[dd, "i"'] & & A \times I \arrow[dl, "i\times id"'] \arrow[dd] \\
& X \times I \arrow[dr, dashed, "\exists r"] & \\
X \arrow[ur, "i_0"'] \arrow[rr] & & M_i
\end{tikzcd}$$

commute.

Proof. If i is a cofibration, then the map r exists as a consequence of the HEP.

For the other direction, if r exists, then for any maps $f\colon X \to Y$ and $H\colon A \times I \to Y$ making the obvious diagram commute, the universal property of the pushout gives us a map $H'\colon M_i \to Y$. Then let $\tilde{H} = H' \circ r$, and we are done. □

1.3.9 Corollary. *If $A \subseteq X$, then $I\colon A \to X$ is a cofibration if and only if $X \times I$ is a retract of $M_i = X \times \{0\} \cup A \times I$.*

1.3.10 Corollary. *A cofibration $i\colon A \to X$ is an injection. If X is Hausdorff, then $i(A)$ is closed in X.*

Proof. Let $J\colon A \times I \to M_i$ be the canonical map (arising from the definition of M_i as a pushout). Then, $J(a,1) = r(i(a),1)$, and observe that $J\mid_{A\times\{1\}}$ is the identity, as it is the top of the mapping cylinder. So, $i(a) \neq i(a')$ if $a \neq a'$, i.e., i is injective.

Because $i\colon A \to X$ is a cofibration, so is $i(A) \to X$. Hence $X \times I$ retracts onto $X \times \{0\} \cup i(A) \times I$ (Corollary 1.3.9). For a Hausdorff space, the image of a retract is closed, and so $X \times \{0\} \cup i(A) \times I$ is a closed subspace of $X \times I$. Intersecting with $X \times \{1\}$, we see that $i(A) \times \{1\}$ is closed in $X \times \{1\}$, i.e, $i(A)$ is closed in X. □

The following (rather pathological) example shows that i is not always a closed map if X is not Hausdorff.

Exercise 7

Let $A = \{a\}$ and $X = \{a, b\}$ with the trivial topology. Show that the inclusion $A \to X$ is a cofibration whose image is not closed.

1.3.11 Remark. The next goal is to show that CW-complexes (X, A) are always cofibrations. The key is the following exercise.

Exercise 8

(a) Suppose $\{(X_i, A_i)\}$ are a collection of spaces satisfying the HEP, then so does $\{(\coprod X_i, \coprod A_i)\}$.

(b) Suppose (X, A) satisfies the HEP, and $f\colon A \to B$ is a continuous map. Let $Y = X \cup_f B$ be the pushout, then (Y, B) satisfies the HEP.

(c) Suppose $A = X_0 \subseteq X_1 \subseteq \cdots \subseteq X_n \subseteq X_{n+1} \subseteq \cdots$.

Let $X = \operatorname{colim} X_i$. If each (X_i, X_{i-1}) satisfies the HEP, then so does (X, A).

1.3.12 Theorem. *A relative CW-complex (X, A) satisfies the HEP.*

Proof. Using Corollary 1.3.7 and the previous exercise we see that (S^{n-1}, D^n) satisfies the HEP $\implies$ $(\coprod S^{n-1}, \coprod D^n)$ satisfies the HEP. Inductively, (X_{n-1}, A) satisfies the HEP and by the exercise (X, A) satisfies the HEP. □

1.3.13 Remark. One can also prove this directly by constructing a deformation retract $r\colon X \times I \to X \times \{0\} \cup A \times I$.

1.3.14 Remark. One can consider the following question: Suppose that $A \subset X$ with A contractible, then is $X \simeq X/A$? Surprisingly, this is not true in general. Indeed, let $A = S^1 \setminus \{(1, 0)\}$ and consider $A \to S^1$. Then $S^1/A \cong T$, the $T = \{a, b\}$ the two point space with open sets $\emptyset, \{a\}, \{a, b\}$ (this is the Sierpiński space). One can check that this space is contractible.[8] The exact condition we need is that $A \to X$ is a cofibration.

[8] See `https://math.stackexchange.com/a/264789/64273`.

1.3.15 Definition. A contracting homotopy is a map $H\colon X \times I \to X$ such that $H(x, 0) = \mathrm{id}_X$ and $H(x, 1) = c_{x_0}$, the constant map at x_0.

1.3.16 Proposition. *Suppose $A \subseteq X$ and $x_0 \in A$. Suppose there exists a map $H\colon X \times I \to X$ such that $H|_{X \times \{0\}} = \mathrm{id}_X$ and $H|_{A \times I}$ has image in A and is a contacting homotopy for A. Then $q\colon X \to X/A$ is a homotopy equivalence.*

Proof. We need to find $p\colon X/A \to X$ such that $q \circ p \simeq \mathrm{id}_{X/A}$ and $p \circ q \simeq \mathrm{id}_X$. The quotient map has a set-theoretic section given by

$$s(\overline{x}) = \begin{cases} x & x \notin A \\ x_0 & x \in A \end{cases}$$

Define $p\colon X/A \to X$ by the following diagram

$$\begin{tikzcd} X \arrow[r, "q"] & X/A \arrow[r, "s"] \arrow[dr, "p"'] & X \arrow[d, "H|_{X\times\{1\}}"] \\ & & X \end{tikzcd}$$

Assume for a moment that p is continuous. Then $p \circ q = H\,|_{X\times\{1\}}$, and so H gives a homotopy between id_X and $p \circ q = H\,|_{X\times\{1\}}$. Likewise, if we define G by

$$\begin{tikzcd} X/A \times I \arrow[r, "s\times\mathrm{id}"] \arrow[drr, "G"'] & X \times I \arrow[r, "H"] & X \arrow[d, "q"] \\ & & X/A \end{tikzcd}$$

and assume that G is continuous, then

$$G(\bar{x}, 1) = q \circ (H\,|_{X\times\{1\}} \circ s) = q \circ p,$$

so that G is a homotopy between $\mathrm{id}_{X/A}$ and $q \circ p$. To see that p is continuous, let $U \subset X$ be open, then

$$q^{-1}p^{-1}(U) = (p \circ q)^{-1}(U) = (H\,|_{X\times\{1\}})^{-1}(U)$$

is open in X by the continuity of $H\,|_{X\times\{1\}}$, hence $p^{-1}(U)$ is open in X/A by the definition of the quotient topology, and so p is continuous. We leave the proof of continuity of G to the reader. $\square$

1.3.17 Theorem. *Let $A \subseteq X$ be a subspace with A contractible. Suppose that the inclusion $i\colon A \to X$ is a cofibration, then $X \to X/A$ is a homotopy equivalence.*

Proof. Let $h\colon A \to I \to A$ be a contracting homotopy. Let $H\colon A \times I \to X$ be the composition of h with the inclusion map of A into X, i.e., the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, hook, "i_0"] \arrow[d, "i"'] & A \times I \arrow[d, "i\times\mathrm{id}"] \arrow[ddr, bend left, "H"] & \\ X \arrow[r, hook, "i_0"'] \arrow[drr, bend right, "\mathrm{id}_X"'] & X \times I \arrow[dr, dashed, "\exists\tilde{H}"] & \\ & & X \end{tikzcd}$$

By the HEP, the dotted map $\tilde{H}$ exists as in the diagram. This map satisfies the conditions of Proposition 1.3.16:

(i) $\tilde{H}\colon X \times \{0\} \to X$ is the identity.

(ii) $\tilde{H}(A \times I) = H(A \times I) = h(A \times I) \subseteq A$.

(iii) $\tilde{H}(A \times \{1\}) = x_0$.

Therefore, $q\colon X \to X/A$ is a homotopy equivalence, as claimed. $\square$

Exercise 9: Cofibrations are pushout closed.

Let $i\colon A \to X$ be a cofibration, and $g\colon A \to B$ any map, then the induced map $B \to B \cup_g X$ is a cofibration.

1.4 Fibrations and the homotopy lifting property

The dual notion of a cofibration is a fibration, where the homotopy extension property is replaced by the homotopy lifting property.

1.4.1 Definition. Let $\mathcal{E}$ be a class of topological spaces. Assume that $p\colon E \to B$ is a continuous map, then we say that p has the homotopy lifting property (with respect to $\mathcal{E}$) if for every $X \in \mathcal{E}$, and map $f\colon X \to E$ and every homotopy $H\colon X \times I \to B$ that begins with $p \circ f$, we can lift it to a homotopy $\tilde{H}\colon X \times I \to E$ that begins with f, i.e., $p \circ \tilde{H} = H$ and $\tilde{H}(x, 0) = f(x)$. In a diagram, we require the lift $\tilde{H}$ in the following:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "i_0"'] & E \arrow[d, "p"] \\ X \times I \arrow[r, "H"'] \arrow[ur, dashed, "\exists \tilde{H}"] & B \end{tikzcd}$$

If $\mathcal{E}$ is the class of all topological spaces, then p is called a (Hurewicz) fibration, while if $\mathcal{E} = \{I^n\}$ (or equivalently, the class of CW-complexes), then p is called a Serre fibration.

1.4.2 Remark. As in Remark 1.3.3, there is an equivalent way to present the homotopy lifting property: we ask for the lift $\tilde{h}$ as shown in the following

$$\begin{tikzcd} X \arrow[drr, bend left, "f"] \arrow[dr, dashed, "\exists \tilde{h}"] \arrow[ddr, bend right, "h"'] & & \\ & E^I \arrow[r, "ev_0"] \arrow[d, "p_*"'] & E \arrow[d, "p"] \\ & B^I \arrow[r, "ev_0"'] & B \end{tikzcd}$$

This makes it clear how the homotopy lifting property is dual to the homotopy extension property.

1.4.3 Remark. We can also talk about the homotopy lifting property with respect to a pair (X, A): namely, a map $p\colon E \to B$ has the homotopy lifting property with respect to a pair (X, A) if each homotopy $H\colon X \times I \to B$ lifts to a homotopy $\tilde{H}\colon X \times I \to E$ which agrees with a given homotopy H_A on $A \times I$. In a diagram, we ask for the lift $\tilde{H}$ in the following:

$$\begin{tikzcd} X \cup (A \times I) \arrow[r, "f \cup H_A"] \arrow[d, "i_0"'] & E \arrow[d, "p"] \\ X \times I \arrow[r, "H"'] \arrow[ur, dashed, "\exists \tilde{H}"] & B \end{tikzcd}$$

1.4.4 Theorem. *The following are equivalent:*

(i) *p is a Serre fibration.*

(ii) *p has the homotopy lifting property with respect to all n-discs D^n.*

(iii) *p has relative homotopy property with respect to all pairs (D^n, S^{m-1})*

(iv) p has the relative homotopy property with respect to all CW-pairs (X, A).

Proof sketch. $(i) \implies (ii)$ is immediate from the definitions.

$(ii) \implies (iii)$ follows because the pairs $(D^n \times I, D^n \times \{0\})$ and $(D^n \times I, D^n \times \{0\} \cup S^{n-1} \times I)$ are homeomorphic.

$(iii) \implies (iv)$ by induction over the skeleton of X; one reduces to the case (iii).

$(iv) \implies (i)$ by taking $A = \emptyset$. □

Exercise 10

Show that the composition of fibrations is a fibration.

1.4.5 Definition. We recall the construction of pullbacks in topological spaces: given maps $p\colon E \to B$ and $f\colon B' \to B$, we let

$$E' = \{(b', e) \in B' \times E \mid p(e) = f(b')\}.$$

This comes with natural projection maps $f'\colon E' \to E$ and $p'\colon E' \to B'$. Then E' is the pull-back in topological spaces, and so we often also denote it by f^*E.

The following is dual to Exercise 9.

1.4.6 Lemma. *If $p\colon E \to B$ satisfies the HLP with respect to the class $\mathcal{E}$, then so does $p'\colon E' \to B'$.*

Proof. Consider the following diagram:

$$\begin{tikzcd} X \arrow[r] \arrow[d, "i_0"'] & E' \arrow[r, "f'"] \arrow[d, "p'"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "p"] \\ X \times I \arrow[r] & B' \arrow[r, "f"'] & B \end{tikzcd}$$

Because $p\colon E \to B$ satisfies the HLP, there is a lift $\tilde{H}'\colon X \times I \to E$ of $X \times I \to B$. Then, by the universal property of the pullback, we get a map $\tilde{H}\colon X \times I \to E'$ satisfying the desired properties. □

1.4.7 Definition. If $p\colon E \to B$ is a fibration, then $F := p^{-1}(*)$ is called the fiber, E is called the total space, and B is the base space. We write this as

$$F \to E \to B.$$

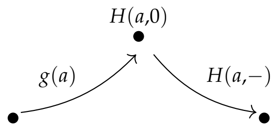
1.4.8 Example. Given a based space X, let

$$PX = \mathrm{Hom}_*(I, X) = \{f\colon I \to X \mid f(0) = *\}$$

be the space of paths starting at the base-point. Then $PX \xrightarrow{p_1} X$ is a fibration with fiber ΩX, the loop space in X (i.e., $f(0) = f(1) = *$). To see this, consider our test diagram, where we must show that $\tilde{H}$ exists:

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "i_0"'] & PX \arrow[d, "p_1"] \\ A \times I \arrow[r, "H"'] \arrow[ur, dashed, "\exists \tilde{H}"] & X \end{tikzcd}$$

Note that for each $a \in A$, $g(a)$ is a path in X which ends at $p_1 g(a) = H(a,0)$. This point is the start of the path $H(a,-)$.



We will define $\tilde{H}(a,s)(t)$ to be a path running along $g(a)$ and then part-way along $H(a,-)$ ending at $H(a,s)$. In symbols,

$$\tilde{H}(a,s)(t) = \begin{cases} g(a)((1+s)t) & 0 \le t \le 1/(1+s) \\ H(a,(1+s)t-1) & 1/(1+s) \le t \le 1. \end{cases}$$

Then $\tilde{H}(a,0) = g(a)$ and $p_1\tilde{H}(a,s) = \tilde{H}(a,s)(1) = H(a,s)$, as required.

The same argument shows that there is a fibration

$$p_* Y \to Y^I \xrightarrow{p_1} Y$$

where $p_* Y$ is the space of paths with end-point $*$.

1.4.9 Remark (The path-space fibration). The fibration $\Omega X \to PX \to B$ is known as the path-space fibration. Note that the space PX is contractible.

1.4.10 Definition. Given $f\colon X \to Y$ the mapping path space P_f (or mapping cocylinder), is the pullback of f along $Y^I \xrightarrow{p_1} Y$, i.e.,

$$\begin{tikzcd} P_f \arrow[r] \arrow[d, "p'"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y^I \arrow[d, "p_1"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

Note that $P_f \simeq X$.

1.4.11 Proposition. *The map $p\colon P_f \to Y$ given by $p(x,\alpha) = \alpha(1)$ is a fibration.*

Proof. This is very similar to Example 1.4.8. Our test diagram is the following:

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "i_0"'] & P_f \arrow[d, "p"] \\ A \times I \arrow[r, "H"'] \arrow[ur, dashed, "\exists \tilde{H}"] & Y \end{tikzcd}$$

Note that $g(a) \in P_f \subset X \times Y^I$, so we can write $g(a) = (g_1(a), g_2(a))$. Here $g_1(a)$ maps via f to the starting point of the path $g_2(a)$ and the commutativity of the diagram implies that the endpoint of the path $g_2(a)$ is the starting point of $H(a,-)$. The lift $\tilde{H}$ will have two components. The x component will be constant in s, i.e., $\tilde{H}_1(a,s) = g_1(a)$. Overall, we define

$$\tilde{H}(a,s) = (g_1(a), \tilde{H}_2(a,s)(-)) \in P_f$$

where[9]

$$\tilde{H}_2(a,s)(t) = \begin{cases} g_2(a)((1+s)t) & 0 \leq t \leq 1/(1+s) \\ H(a,(1+s)t-1) & 1/(1+s) \leq t \leq 1. \end{cases}$$

One check directly that $\tilde{H}(a,s)$ has the required properties. □

[9] Compare this to the formula in Example 1.4.8.

As with the homotopy extension property, we have a universal test space. The details (which are dual to Proposition 1.3.8) are left to the reader.

1.4.12 Proposition. *Let $f\colon E \to B$ be a continuous map, then f is a fibration if and only if there exists $s\colon P_f \to E^I$ making the following diagram commute:*

$$\begin{tikzcd}
P_f \arrow[dr, dashed, "\exists s"] \arrow[drr, bend left, "\pi_E"] \arrow[ddr, bend right, "\pi_{B_I}"'] & & \\
& E^I \arrow[r, "ev_0"] \arrow[d, "f_*"'] & E \arrow[d, "f"] \\
& B^I \arrow[r, "ev_0"'] & B
\end{tikzcd}$$

where π_{B_I} and π_E are the projection maps coming from the construction of P_f as a pullback.

1.4.13 Remark. One property of cofibrations that does not dualize to fibrations is that cofibrations are inclusions, but fibrations need not be surjective. Indeed, given $p\colon E \to B$ a fibration, then the composite

$$E \xrightarrow{p} B \hookrightarrow B \coprod *$$

is also a fibration, but is not surjective.

1.4.14 Remark. We will want to talk about exact sequences where the terms appearing may not have a group structure, but are rather only sets with base-points. Therefore, given a sequence of functions

$$A \xrightarrow{f} B \xrightarrow{g} C$$

of sets with base-points, we say that this is exact at B if $f(A) = g^{-1}(c_0)$ where c_0 is the base-point of C. Note that if A, B, C are groups with base-points the identity elements of the group, then exactness of sets corresponds to exactness of groups.

1.4.15 Theorem. *Let $p\colon E \to B$ be a fibration with fiber F and B path-connected. Let Y be any space, then*

$$[Y,F] \xrightarrow{i_*} [Y,E] \xrightarrow{p_*} [Y,B]$$

is exact.

Proof. For one direction, it is clear that $p_*(i_*[g])) = 0$.

Suppose $f \in [Y,E]$ is such that $p_*[f] = [\text{const}]$, i.e., $p \circ f$ is null-homotopic. Let $G\colon Y \times I \to B$ be a null-homotopy, and let

$H\colon Y \times I \to E$ be a solution to the lifting problem indicated in the following diagram, using that p is a fibration:

$$\begin{tikzcd} Y \times \{0\} \arrow[r, "f"] \arrow[d, hook, "i_0"'] & E \arrow[d, "p"] \\ Y \times I \arrow[r, "G"'] \arrow[ur, dashed, "H"] & B \end{tikzcd}$$

Note now that $p \circ H(y,1) = G(y,1) = b_0$, so that $H(y,1) \in F := p^{-1}(b_0)$. It follows that $[f] = i_*[H(-,1)]$. $\square$

We have an analogous result for cofibration.

1.4.16 Theorem. *Let $i\colon A \to X$ be a cofibration, and $q\colon X \to X/A$ the quotient map. Let Y be any path-connected space, then the sequence of pointed sets*

$$[X/A, Y] \xrightarrow{q^*} [X, Y] \xrightarrow{i^*} [A, Y]$$

is exact.

Proof. Again, one inclusion is clear: we have $i^*(g^*([g])) = [g \circ q \circ i] = [\text{const}]$.

Now suppose that $f\colon X \to Y$ is a map with $f\ |_A\colon A \to Y$ null-homotopic. Let $h\colon A \times I \to Y$ be a hull-homotopy, and let $F\colon X \times I \to Y$ be the extension as shown in the following diagram:

$$\begin{tikzcd} A \arrow[r, hook, "i_0"] \arrow[d, "i"'] & A \times I \arrow[d, "i \times \mathrm{id}"] \arrow[ddr, bend left, "H"] & \\ X \arrow[r, hook, "i_0"'] \arrow[drr, bend right, "f"'] & X \times I \arrow[dr, dashed, "F"] & \\ & & Y \end{tikzcd}$$

Let $f' := F(-,1)$. Then, $f \sim f'$ and $f'(A) = F(A,1) = y_0$. By the universal property of the quotient, we can find $g\colon X/A \to Y$ making the following diagram commute:

$$\begin{tikzcd} X \arrow[r, "f'"] \arrow[d, "q"'] & Y \\ X/A \arrow[ur, dashed, "g'"'] & \end{tikzcd}$$

Therefore $[f] = [f'] = q^*[g']$. $\square$

As an extension of Theorem 1.4.15 we have the following.

1.4.17 Theorem. *Given a (Serre) fibration $p\colon E \to B$, and base points $b \in B$ and $e \in F := f^{-1}(b)$, then there is an isomorphism $p_*\colon \pi_n(E, F, e) \xrightarrow{\simeq} \pi_n(B, b)$ for all $n \geq 1$. Hence, if B is path-connected, there is a long exact sequence of homotopy groups*

$$\cdots \pi_n(F, e) \to \pi_n(E, e) \xrightarrow{p_*} \pi_n(B, b) \to \pi_{n-1}(F, e) \to \cdots$$
$$\cdots \to \pi_0(E, e) \to 0.$$

Proof. We first show that p_* is surjective. Let $[f] \in \pi_n(B,b)$, represented by a map $f\colon (I^n, \partial I^n) \to (B,b)$. Note that $I^{n-1} \times \{0\} \subseteq \partial I^n$, so we can form the diagram

$$\begin{tikzcd} I^{n-1} \times \{0\} \arrow[r, "*"] \arrow[d] & E \arrow[d, "p"] \\ I^n \arrow[r, "f"'] \arrow[ur, dashed, "\tilde{f}"] & B \end{tikzcd}$$

where the lift $\tilde{f}$ exists because p is a Serre fibration. Because $f(\partial I^n) = b$, we have $\tilde{f}(\partial I^n) \subseteq F$. So $\tilde{f}$ represents an element of $\pi_n(E,F,e)$ with $p_*([\tilde{f}]) = [p \circ \tilde{f}] = [f]$.

To show injectivity, let $\tilde{f}_0, \tilde{f}_1\colon (I^n, \partial I^n, J^{n-1}) \to (E,F,e)$ be such that $p_*(\tilde{f}_0) = p_*(\tilde{f}_1)$. Let $H\colon (I^n \times I, \partial I^n \times I) \to (B,b)$ be a homotopy from $p\tilde{f}_0$ to $p\tilde{f}_1$. We can find a lift in the following diagram:

$$\begin{tikzcd} W \arrow[r, "f"] \arrow[d] & E \arrow[d, "p"] \\ I^n \times I \arrow[r, "H"'] \arrow[ur, dashed, "\tilde{H}"] & B \end{tikzcd}$$

where $W = I^n \times \{0\} \cup I^n \times \{1\} \cup \partial I^n \times I$, and f is $\tilde{f}_0$ on $I^n \times \{0\}$, $\tilde{f}_1$ on $I^n \times \{1\}$ and f is constant on $\partial I^n \times I$. The homotopy lifting property gives $\tilde{H}$ defining a homotopy between $\tilde{f}_0$ and $\tilde{f}_1$.

The result then follows (modulo some noise in the low homotopy groups, which can be checked by hand) from Theorem 1.2.21. $\square$

1.4.18 Example (Hopf fibrations). Let $\mathbb{F} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ and fix an integer $d = 1, 2$ or 4, respectively.

Let

$$\mathbb{F}^{n+1} = \begin{cases} \mathbb{R}^{n+1} & \mathbb{F} = \mathbb{R} \\ \mathbb{C}^{n+1} \cong \mathbb{R}^{2(n+1)} & \mathbb{F} = \mathbb{C} \\ \mathbb{H}^{n+1} \cong \mathbb{R}^{4(n+1)} & \mathbb{F} = \mathbb{H}. \end{cases}$$

In other words, $\mathbb{F}^{n+1} \cong \mathbb{R}^{d(n+1)}$. We define the $d(n+1)-1$ dimensional sphere inside $\mathbb{F}^{n+1}$:

$$S^{d(n+1)-1} = \{(u_0, \ldots, u_n) \mid u_i \in \mathbb{F}, \sum_{k=0}^{n} |u_k|^2 = 1\}.$$

We define the $\mathbb{F}$-projective space by

$$\mathbb{F}P^n := \mathbb{F}^{n+1} \setminus \{0\} / \sim$$

where $(u_0, \ldots, u_n) \simeq (v_0, \ldots, v_n)$ if and only if there exists $\lambda \in \mathbb{F} \setminus \{0\}$ such that $v_i = \lambda u_i$ for $i = 0, \ldots, n$.

Now we have a map $\phi\colon S^{d(n+1)-1} \to \mathbb{F}P^n$ that sends $(u_0, \ldots, u_n)$ to its equivalence class $[u_0, \ldots, u_n]$. Let $F = \phi^{-1}[1, \ldots, 0] = \{(\lambda, 0, \ldots, 0) \mid \lambda \in \mathbb{F}, |\lambda| = 1\} \cong S^{d-1}$.

We will see later in the course that $S^{d-1} \to S^{d(n+1)-1} \to \mathbb{R}P^n$ is a fibration. Explicitly, the fibrations are

$$\begin{aligned} &S^0 \to S^n \to \mathbb{R}P^n \\ &S^1 \to S^{2n+1} \to \mathbb{C}P^n \\ &S^3 \to S^{4n+3} \to \mathbb{H}P^n. \end{aligned}$$

The case $n = 1$ is of interest, as then projective spaces are just spheres, and we obtain the following Hopf fibrations

$$\begin{aligned} S^0 &\to S^1 \to S^1 \\ S^1 &\to S^3 \to S^2 \\ S^3 &\to S^7 \to S^4. \end{aligned}$$

There is also a fibration $S^7 \to S^{15} \to S^8$. It is a difficult theorem of Adams that these are the only fibrations between spheres.

1.5 The homotopy extension and lifting property

We recall that given $f\colon X \to Y$ we defined the mapping path space P_f in Definition 1.4.10, and that $P_f \to Y$ is a fibration.

1.5.1 Definition. The homotopy fiber F_f of $f\colon X \to Y$ is the fiber of the fibration $P_f \to Y$. This is well-defined up to homotopy.

The following is an extremely useful definition in homotopy theory; as we will see later, any weak equivalence between CW-complexes is in fact a homotopy equivalence.

1.5.2 Definition. A map $f\colon (X, x_0) \to (Y, y_0)$ is a weak equivalence if $f_0\colon \pi_0(X, x_0) \to \pi_0(Y, y_0)$ is a bijection and $f_*\colon \pi_k(X, x_0) \to \pi_k(Y, y_i)$ is an isomorphism for all $k \geq 1$.

1.5.3 Lemma. *If $f\colon X \to Y$ is a weak-equivalence, then $\pi_k(F_f) = 0$ for all $k \geq 0$.*

Proof. This follows from the long exact sequence of a fibration (Theorem 1.4.17). □

1.5.4 Remark. We now make a series of remarks about a map $f\colon X \to Y$ with homotopy fiber F_f.

(i) A map $\phi\colon S^{n-1} \to F_f$ corresponds to a diagram

$$\begin{tikzcd} S^{n-1} \arrow[r, "g"] \arrow[d, hook] & X \arrow[d, "f"] \\ C(S^{n-1}) \cong D^n \arrow[r, "h"'] & Y \end{tikzcd} \tag{1.5.5}$$

where g is the composite $S^{n-1} \xrightarrow{\phi} F_f \to X$ (use Lemma 1.1.11).

(ii) The boundary map $\pi_n(Y) \to \pi_{n-1}(F_f)$ in the long exact sequence corresponds to the map sending the class of $\bar{h}\colon S^n \to Y$ to the class of $\pi_{n-1}(F_f)$ represented by the diagram

$$\begin{tikzcd} S^{n-1} \arrow[r, "c"] \arrow[d, hook] & X \arrow[d, "f"] \\ D^n \arrow[r, "h"'] & Y \end{tikzcd}$$

where $c = c_{x_0}$ is the constant map, and h is the composite $D^n \to D^n/S^{n-1} \cong S^n \xrightarrow{\bar{h}} Y$.

(iii) Similarly, the map $\pi_{n-1}(F_f) \to \pi_{n-1}(X)$ corresponds to sending the diagram (1.5.5) to the class $[g]$.

(iv) In particular, $\pi_{n-1}(F_f) = 0$ is equivalent to completing the diagram (1.5.5) in the following way:

$$\begin{tikzcd}
S^{n-1} \arrow[rr, "g"] \arrow[dddd, hook] \arrow[dr] & & X \arrow[dddd, "f"] \\
& C(S^{n-1}) \arrow[ur, dashed, "G"] \arrow[d, hook] & \\
& C(D^n) \arrow[ddr, dashed, "H"] & \\
& & \\
D^n \arrow[rr, "h"'] \arrow[uur] & & Y
\end{tikzcd}$$

We can restate the last remark in the following lemma.

1.5.6 Lemma. *Suppose $f\colon X \to Y$ is a map with homotopy fiber F_f. Then $\pi_{n-1}(F_f) = 0$ if and only if each diagram*

$$\begin{tikzcd}
S^{n-1} \times \{1\} \arrow[r, "g"] \arrow[d, hook] & X \arrow[d, "f"] \\
S^{n-1} \times I \cup D^n \times \{0\} \arrow[r, "h"'] & Y
\end{tikzcd}$$

can be completed to a diagram

$$\begin{tikzcd}
S^{n-1} \times \{1\} \arrow[rr, "g"] \arrow[dddd, hook] \arrow[dr, hook] & & X \arrow[dddd, "f"] \\
& D^n \times \{1\} \arrow[ur, dashed, "G"] \arrow[d, hook] & \\
& D^n \times I \arrow[ddr, dashed, "H"] & \\
& & \\
S^{n-1} \times I \cup D^n \cup \{0\} \arrow[rr, "h"'] \arrow[uur, hook] & & Y
\end{tikzcd}$$

Proof. For any disk we have a homeomorphism $CD^n \cong D^n \times I$ which sends the cone point to the center of $D^n \times \{1\}, D^n$ to $S^{n-1} \times I \cup D^n\{0\}, S^{n-1}$ to $S^{n-1} \times \{1\}$ and CS^{n-1} to $D^n \times \{1\}$. Thus the statement follows from the last part of the remark. $\square$

This extends to relative CW-complexes.[10]

[10] The following result is perhaps difficult to remember, but very useful!

1.5.7 Theorem (Homotopy extension and lifting property (HELP))**.** *Let (X, A) be a relative CW-pair and $f\colon Y \to Z$ a weak equivalence. Then every diagram*

$$\begin{tikzcd}
A \times \{1\} \arrow[r, "g"] \arrow[d, hook] & Y \arrow[d, "f"] \\
A \times I \cup X \times \{0\} \arrow[r, "h"'] & Z
\end{tikzcd}$$

can be completed to a diagram

$$
\begin{tikzcd}
A \times \{1\} \arrow[rr, "g"] \arrow[dr, hook] \arrow[ddd, hook] & & Y \arrow[ddd, "f"] \\
& X \times \{1\} \arrow[ur, dashed, "G"] \arrow[d, hook] & \\
& X \times I \arrow[dr, dashed, "H"] & \\
A \times I \cup X \times \{0\} \arrow[ur, hook] \arrow[rr, "h"'] & & Z
\end{tikzcd}
$$

Proof. The proof is by induction over the n-skeleton, with the base case being straightforward. For the inductive step, one reduces to attaching a single cell using the diagram

$$
\begin{tikzcd}
X_{n-1} \times \{1\} \arrow[rr, "G_{n-1}"] \arrow[dr, hook] \arrow[ddd, hook] & & Y \arrow[ddd, "f"] \\
& X_n \times \{1\} \arrow[ur, dashed, "G_n"] \arrow[d, hook] & \\
& X_n \times I \arrow[dr, dashed, "H_n"] & \\
X_{n-1} \times I \cup X_n \times \{0\} \arrow[ur, hook] \arrow[rr, "H_{n-1} \cup h"'] & & Z
\end{tikzcd}
$$

□

1.5.8 Remark. If (X, A) is a relative CW-complex of dimension n, and $f\colon Y \to Z$ is an n-equivalence[11], the same argument goes through to show that the conclusion of HELP also holds in this case.

[11] That is, the homotopy fiber of f is $(n-1)$-conneceted

Exercise 11

Show that if $f = \mathrm{id}$ in HELP, then we recover the homotopy extension property.

Our first application of this will be Whitehead's theorem. We start with the folliwing lemma.

1.5.9 Lemma. *For any weak equivalence $f\colon Y \to Z$ and any CW-complex X, the induced map $f_*\colon [X, Y] \to [X, Z]$ is a bijection.*

Proof. We first show surjectivity. The pair $X = (X, \emptyset)$ is a relative CW-complex, and so we can apply HELP. Then, for any $h\colon X \to Z$

we have a diagram

$$
\begin{tikzcd}
* \arrow[rr] \arrow[dr, hook] \arrow[ddd, hook] & & Y \arrow[ddd, "f"] \\
& X \times \{1\} \arrow[ur, dashed, "G"] \arrow[d, hook] & \\
& X \times I \arrow[dr, dashed, "H"] & \\
X \times \{0\} \arrow[ur, hook] \arrow[rr, "h"'] & & Z
\end{tikzcd}
$$

The homotopy $H\colon X \times I \to Z$ satisfies $H_0 = h$ and $H_1 = f \circ G$. Therefore, $[h] = f_*[G]$.

Now assume that $g_0, g_1 \in [X, Y]$ with $f_*[g_0] = f_*[g_1]$. Let $F\colon X \times I \to Z$ be a homotopy between $f \circ g_0$ and $f \circ g_1$. Consider the pair $(X \times I, X \times \partial I)$. This is a relative CW-pair, and HELP gives a diagram

$$
\begin{tikzcd}
X \times \partial I \times \{1\} \arrow[rr, "g"] \arrow[dr, hook] \arrow[ddd, hook] & & Y \arrow[ddd, "f"] \\
& X \times I \times \{1\} \arrow[ur, dashed, "G"] \arrow[d, hook] & \\
& X \times I \times J \arrow[dr, dashed, "H"] & \\
X \times \partial I \times J \cup X \times I \times \{0\} \arrow[ur, hook] \arrow[rr, "h"'] & & Z
\end{tikzcd}
$$

Here $g\colon X \times \partial I \to Y$ sends (X, ν) to $g_\nu(x)$ for $\nu = 1, 2$ and $h\colon X \times \partial I \times J \to Z$ sends (x, ν, s) to $f \circ g_\nu(x)$. The lift $G\colon X \times I \to Y$ gives a homotopy between g_0 and g_1, i.e., $[g_0] = [g_1]$, and so f_* is injective. $\square$

1.5.10 Remark. Using Remark 1.5.8 we have the following variant of Lemma 1.5.9: If $f\colon Y \to Z$ is n-connected, then for any CW-complex X, the induced map $f_*\colon [X, Y] \to [X, Z]$ is an isomorphism if $n < \dim(X)$ and is surjective if $n = \dim(X)$.

1.5.11 Theorem (Whitehead theorem)**.** *If $f\colon X \to Y$ is weak-equivalence between CW-complexes, then it is a homotopy equivalence.*

Proof. Suppose $f\colon X \to Y$ is a weak equivalence, so $f_*\colon [Y, X] \xrightarrow{\simeq} [Y, Y]$. In other words, there exists a $g\colon Y \to X$ such that $f_*[g] = [f \circ g] = [\mathrm{id}_Y]$, i.e., $f \circ g \simeq \mathrm{id}_Y$. Then, $f \circ g \circ f \simeq f$ as well. But, we also have $f_*\colon [X, X] \xrightarrow{\simeq} [X, Y]$, which sends id_X to f and $g \circ f$ to $f \circ g \circ f \simeq f$. Therefore, $\mathrm{id}_X \simeq g \circ f$, and so $X \simeq Y$. $\square$

1.5.12 Corollary. *If X is a CW-complex with $\pi_i(X) = 0$ for all i, then X is contractible.*

Proof. Apply Whitehead's theorem to the unique map $X \to *$. $\square$

1.5.13 Remark. We cannot drop any assumptions from this theorem, as the following examples show:

(i) We must have a map inducing the weak equivalence; the homotopy groups cannot be abstractly isomorphism, e.g., consider $\mathbb{R}P^2 \times S^3$ and $\mathbb{R}P^3 \times S^2$.

(ii) The Warsaw circle[12] is an example of a space with $\pi_n X = 0$ for all n, but for which X is not contractible.

[12] See, for example, https://wildtopology.com/bestiary/warsaw-circle/

Exercise 12

Use Whitehead's theorem to show that a CW complex is contractible if it is the union of an increasing sequence of sub-complexes $X_1 \subseteq X_2 \subseteq \cdots$ such that each inclusion $X_i \to X_{i+1}$ is null-homotopic.

Exercise 13

Let $f\colon X \to Y$ be a weak homotopy equivalence. Assuming X is a CW-complex, and Y has the homotopy type of a CW-complex, show that f is a homotopy equivalence.

1.6 *The cellular approximation theorem*

The next important theorem is the cellular approximation theorem.

1.6.1 Definition. If X and Y are CW-complexes, and $g\colon X \to Y$ a map, then g is cellular if g carries the n-skeleton of X into the n-skeleton of Y, i.e., $f(X^n) \subseteq Y^n$ for all $n \geq 0$. Similarly, for relative CW-complexes (X, A) and (Y, B) a map $g\colon (X, A) \to (Y, B)$ is cellular if $g((X, A)^n) \subseteq (Y, B)^n$ for all $n \geq 0$.

The main result of this section is the following.

1.6.2 Theorem (Cellular approximation theorem)**.** *Suppose $f\colon (X, A) \to (Y, B)$ is a map of relative CW-complexes, then f is homotopic rel A to a cellular map of pairs.*

We will use the following lemma.

1.6.3 Lemma. *If Z is obtained from Y by attaching cells of dimension $> n$, then $\pi_k(Z, Y) = 0$ for all $k \leq n$, i.e., (Z, Y) is n-connected.*

Proof. We can reduce to the case where $Z = Y \cup_\alpha D^r$ for $\alpha\colon S^{r-1} \to Y, r \geq n+1$. Then $\pi_k(Z, Y)$ corresponds to a map of pairs $g\colon (D^k, S^{k-1}) \to (Z, Y)$. By smooth or simplicial approximation,[13] we can find a map $g'\colon D^k \to Z$ such that $g' = g$ on S^{k-1} and g' misses a point p in the interior of D^k. We can deform $X \setminus \{p\}$ onto A and so deform g' to a map in Y. □

[13] See https://ncatlab.org/nlab/show/simplicial+approximation+theorem

Proof of Theorem 1.6.2. The proof is by induction, with the base case left to the reader. So, by induction, we have $g_{n-1}\colon X^{n-1} \to Y^{n-1}$

and a homotopy $H_{n-1}\colon X^{n-1} \times I \to Y$ such that $H_0 = f$ and $H_1 = g_{n-1}$. Now consider the diagram

$$\begin{tikzcd}
X^{n-1} \times \{1\} \arrow[rr, "\iota_n \circ g_{n-1}"] \arrow[dr, hook] \arrow[ddd] & & Y^n \arrow[ddd] \\
& X^n \times \{1\} \arrow[ur, dashed, "g^n"] \arrow[d, hook] & \\
& X^n \times I \arrow[dr, dashed, "H_n"] & \\
X^{n-1} \times I \cup X^n \times \{0\} \arrow[ur, hook] \arrow[rr, "H_{n-1} \cup f"] & & Y
\end{tikzcd}$$

Here $\iota_n\colon Y^{n-1} \to Y^n$ is the inclusion map. We can apply the version of HELP given in Remark 1.5.8 since $Y^n \hookrightarrow Y$ is an n-equivalence by Lemma 1.6.3, which gives the required extensions g_n and H_n. $\square$

There is also a relative version, whose proof we omit.

1.6.4 Theorem. *Suppose $f\colon (X, A) \to (Y, B)$ is a map of relative CW-complexes which is cellular on a subspace (X', A') of (X, A), then there is a cellular map $g\colon (X, A) \to (Y, B)$ homotopic to f relative to Y such that $g|_{X'} = f$.*

1.6.5 Example. Suppose $i < n$. Taking the standard CW structure on the k-sphere with one 0-cell and one k-cell, we see that any map $S^i \to S^n$ can be made cellular. Because the i-skeleton of S^n is a point, we see that such a map is null-homotopic, and deduce that $\pi_i(S^n) = 0$ for $i < n$.

1.6.6 Corollary. *Let $A \subseteq X$ be CW-complexes, and suppose that all cells of $X \setminus A$ have dimension $> n$. Then $\pi_i(X, A) = 0$ for $i \leq n$.*

Proof. Let $[f] \in \pi_i(X, A)$, i.e., $f\colon (D^i, S^{i-1}) \to (X, A)$. We can use cellular approximation to replace f with a cellular map g with $g(D^i) \subseteq X^i$. But for $i \leq n$ we have $X^i \subseteq A$, so the image of g is contained in A. By the compression criterion (Lemma 1.2.22) we have $[f] = [g] = 0$. $\square$

1.6.7 Corollary. *If X is a CW-complex, then $\pi_i(X, X_n) = 0$ for all $i \leq n$.*

Exercise 14

Use cellular approximation to show that the n-skeletons of homotopy equivalent CW-complexes without cells of dimension $n + 1$ are also homotopy equivalent.

1.7 *Excision and the Freudenthal suspension theorem*

On of the most powerful results in (co)homology is excision. As we will see in this section, things are more complicated for homotopy groups. This is one of the reasons why homotopy groups are (generally) more complicated to compute than homology groups.

1.7.1 Definition. An excisive triad $(X; A, B)$ consists of a space X along with two subspaces $A, B \subseteq X$ such that $X = A^\circ \cup B^\circ$.

1.7.2 Remark. In homology $(A, A \cap B) \to (X, B)$ induces an isomorphism in homology (by excision). This fails in homotopy, as the following example shows.

1.7.3 Example. Let $X = S^2 \vee S^2$ and let $A = C_+$ and $B = C_-$, the southern and northern hemispheres, with a small overlap between the two hemispheres (see Figure 1.5).

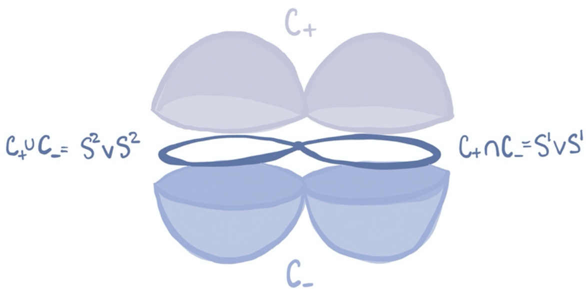


Figure 1.5: The decomposition of $X = S^2 \vee S^2$ into the upper/lower hemispheres $C_\pm$, which intersect along the equator $S^1 \vee S^1$.

Then $C_+ \cap C_- \simeq S^1 \vee S^1$ and $C_+ \cup C_- = S^2 \vee S^2$. Note that both C_+ and C_- are contractible (they have the homotopy type of a wedge of two discs). By the long exact sequence in homotopy we have

$$\pi_i(S^2 \vee S^2, C_-) \cong \pi_i(S^2 \vee S^2) \quad \text{and} \quad \pi_i(C_+, S^1 \vee S^2) \cong \pi_{i-1}(S^1 \vee S^2).$$

In particular, when $i = 2$ we have $\pi_2(S^2 \vee S^2, C_-) \cong \pi_2(S^2 \vee S^2)$ is the free abelian group on two generators while $\pi_2(C_+, S^1 \vee S^1) \cong \pi_1(S^1 \vee S^1)$ is the free group on two generators. Therefore $(C_+, S^1 \vee S^1) \to (S^2 \vee S^2, C_-)$ does not induce an isomorphism on homotopy.

1.7.4 Remark. The following is the homotopy theoretic version of excision. We will not give a full proof as it is quite involved. The full details can be found in May's book, for example.

1.7.5 Theorem (Homotopy excision/Blakers–Massey theorem)**.** *Let $(X; A, B)$ be an excisive triad such that $C = A \cap B$ is non-empty and (A, C) and (B, C) are relative CW-complexes. Suppose $(A, C, *)$ is n-connected and $(B, C, *)$ is m-connected for every choice of base-point $* \in C$. Then the map*

$$\pi_i(A, C) \to \pi_i(X, B)$$

induced by the inclusion is an isomorphism for $i < n + m$ and a surjection for $i = n + m$, i.e., it is an $(n + m)$-equivalence.

Sketch of proof. The proof proceeds by a number of reductions.

Reduction 1: It suffices to prove this when A is built from C by attaching cells of dimension greater than n and B is built by attaching cells of dimension greater than m. Indeed, we claim we can replace the pair (A, C) with an n-connected pair (A', C) such that the following diagram commutes:

$$\begin{tikzcd} C \arrow[r, hook] \arrow[d] & A' \arrow[dl, "\sim"] \\ A & \end{tikzcd}$$

and A' is built from C by attaching cells of dimension greater than n only. To show this, we build up a CW complex from C by adding cells which represent elements of $\pi_i(A)$ or gets rid of elements which should not be there.[14] Since $\pi_i(C) \cong \pi_i(A)$ for all $i < n$, we only need to add cells of dimension greater than n to make this work. This procedure can be carried out for (B, C) as well.

[14] For example, the first step is to kill the kernel of the surjection $\pi_n(C) \to \pi_n(A)$ by attaching cells to A. We will discuss this procedure in more detail when we discuss Eilenberg–Maclane spaces.

Reduction 2: It suffices to prove excision when each of A and B is built from C by attaching one cell apiece. To see this, let us say that a pair of extensions $C \to A$ and $C \to B$ is of size (p, q) if A is obtained by attaching p-cells (of dimension greater than n) and B is obtained by attaching q-cells (of dimension greater than m). The claim is that excision holds for size $(1, 1)$ if it holds for size (p, q). The proof is inductive, via a long each sequence of 'triad homotopy groups' and the 5-lemma.

The following lemma, whose proof is omitted, then completes the proof of homotopy excision. $\square$

1.7.6 Lemma. *Suppose that $X = A \cup_C B$ where $A = C \cup e$ and $B = C \cup e'$ are built from C by attaching cells of dimension $> n$ and $> m$, respectively. Then, $\pi_i(A, C) \to \pi_i(X, B)$ is an isomorphism for $i < n + m$ and a surjection for $i = n + m$.*

Our main application will be the Freudenthal suspension theorem. We first make a definition.

1.7.7 Definition. Let (X, x_0) be a based space. The suspension homomorphism is the map $\Sigma_* \colon \pi_i(X) \to \pi_{i+1}(\Sigma X)$ which sends $[f]$ to $[\Sigma f]$, where $\Sigma f \colon S^{i+1} \to \Sigma X$ sends $[s, t]$ to $[f(s), t]$.

1.7.8 Theorem. *Let X be an $(n-1)$-connected CW-complex, then the suspension homomorphism*

$$\Sigma_* \colon \pi_i(X) \to \pi_{i+1}(\Sigma X)$$

is an isomorphism for $i < 2n - 1$ and a surjection for $i = 2n - 1$.

Proof. Write $\Sigma X = C_+ X \cup C_- X$ for the decomposition of ΣX into its upper and lower cone. Now consider the diagram

$$\begin{tikzcd} \pi_{i+1}(C_+X, X) \arrow[r] \arrow[d, "\cong"', "\partial"] & \pi_{i+1}(\Sigma X, C_-X) \arrow[d, "\partial"', "\cong"] \\ \pi_i(X) \arrow[r, "\Sigma_*"'] & \pi_{i+1}(\Sigma X) \end{tikzcd}$$

which can be shown to commute. Then it suffices to show that the upper diagram is an isomorphism/surjection in the appropriate range. To see this, note that if X is $(n-1)$-connected, then $(C_\pm X, X)$

are n-connected (use the long exact sequence and contractibility of $C_{\pm}X$). By excision,

$$\pi_{i+1}(C_+, X) \to \pi_{i+1}(\Sigma X, C_- X)$$

is an isomorphism for $i+1 < 2n$ and a surjection for $i+1 = 2n$, and the result follows. $\square$

1.7.9 Example. The n-sphere has $\pi_i(S^n) = 0$ for $i < n$ (Example 1.6.5). So by the Freudenthal suspension theorem

$$\Sigma_*\colon \pi_i(S^n) \to \pi_{i+1}(S^{n+1})$$

is an isomorphism for $i < 2n-1$. In particular, $\pi_n(S^n) \to \pi_{n+1}(S^{n+1})$ is an isomorphism for $n < 2n-1$, i.e., for $n \geq 2$. In particular, there is a surjection $\mathbb{Z} \cong \pi_1(S^1) \to \pi_2(S^2)$ and isomorphisms $\pi_2(S^2) \cong \pi_3(S^3) \cong \cdots \pi_n(S^n)$. In fact, the Hopf fibration $S^1 \to S^2 \to S^3$ shows that $\pi_2(S^2) \cong \mathbb{Z}$, and so we have $\pi_n(S^n) \cong \mathbb{Z}$ for all $n \geq 1$.

1.7.10 Remark. Let X be a CW-complex. By the suspension theorem $\Sigma^n X$ is always $(n+1)$-connected. Thus,

$$\Sigma_*\colon \pi_i(\Sigma n X) \to \pi_{i+1}(\Sigma^{n+1} X)$$

is an isomorphism for $i < 2n-1$. This means that for a fixed value of k, the maps in the sequence

$$\pi_k(X) \to \pi_{k+1}(\Sigma X) \to \pi_{k+2}(\Sigma^2 X) \to \cdots \Sigma_{k+i}(\Sigma^i X)$$

eventually become isomorphisms. This is known as the k-th stable homotopy group of X.

1.7.11 Remark. There is an equivalent way to state homotopy excision. Suppose $f\colon A \to X$ is an m-equivalence and $g\colon A \to Y$ is an n-equivalence. We can form the following diagram

$$\begin{tikzcd}
F_f \arrow[r] \arrow[d, dashed, "\tilde{g}"'] & A \arrow[r, "f"] \arrow[d, "g"'] & X \arrow[d] \\
F_z \arrow[r] & Y \arrow[r, "z"'] \arrow[ur, phantom, "\lrcorner"] & Z
\end{tikzcd}$$

Then $\tilde{f}\colon F_f \to F_z$ is an $(n+m-1)$-equivalence. This follows because $\pi_n(X, A) \cong \pi_{n-1}F_f$ (use Theorem 1.4.17 and the 5-lemma, for example).

Exercise 15

Show that if $f\colon X \to Y$ is an n-connected map between spaces with X an $(m-1)$-connected CW-complex, then the comparison map $F(f) \to \Omega C(f)$ is $(m+n-1)$-connected.

1.8 The CW-approximation theorem

In this next section we show that, up to weak homotopy equivalence, every space is a CW-complex. We begin with the following.

1.8.1 Lemma. *Let X be any space. Then there exists a space Y and a map $i\colon X \to Y$ such that i induces isomorphisms $\pi_q\colon \pi_q X \xrightarrow{\simeq} \pi_q Y$ for $0 \leq q \leq n$ and $\pi_{n+1}Y = 0$.*

Proof. The idea of the proof is to attach cells to X to kill the classes we don't want to exist. To that end, let J be a set of representatives for each $[j] \in \pi_{n+1}X$. Form the pushout diagram

$$\begin{tikzcd} \coprod_{j\in J} S^{n+1} \arrow[r] \arrow[d] & X \arrow[d] \\ \coprod_{j\in J} D^{n+2} \arrow[r] & Y \end{tikzcd}$$

Note that (Y, X) is a relative CW-complex with $Y^{n+1} = X$ so $X \to Y$ is an $(n+1)$-equivalence. We claim that $\pi_{n+1}Y = 0$. Indeed, let $f\colon (S^{n+1}, *) \to (Y, *)$ be a representative of $\pi_{n+1}(Y)$. By cellular approximation we can assume that f factors through the $(n+1)$-skeleton of Y, which is just X. In other words, $f \simeq j$ for some $j \in J$. But $j\colon S^{n+1} \to X \to Y$ is null-homotopic by assumption, and so $\pi_{n+1}Y = 0$. □

1.8.2 Proposition (CW-approximation)**.** *Given any topological space X there exists a CW complex ΓX and a weak equivalence $\Gamma X \xrightarrow{\gamma} X$.*[15]

[15] In slightly fancy language, this is cofibrant replacement in a certain model structure on the category of topological spaces.

Proof. We can assume that X is path-connected, or we can work one path component at a time. Choose a set of representatives $J = \{j_q \mid [j] \in \pi_q X, q \geq 1\}$. Let $X_1 = \bigvee_{j_q \in J} S^q$ (with its standard CW structure) and $\gamma_1 := \vee j_q\colon X_1 \to X$. By construction, γ_1 is a 1-equivalence. Suppose by induction that we have constructed a CW complex X_n and an n-equivalence $X_n \to X$. Once again, let $J = \{j \mid [j] \in \pi_n X_n, [\gamma_n \circ j] = 0 \in \pi_n X\}$. By construction, for each $j \in J$ there exists an extension h_j of $S^n \xrightarrow{\gamma_n \circ j} X$ to D^{n+1}. We then construct X_{n+1} by the pushout

$$\begin{tikzcd} \coprod_{j\in J} S^n \arrow[r, "\coprod j"] \arrow[d] & X_n \arrow[d] \arrow[ddr, bend left, "\gamma_n"] & \\ \coprod_{j\in J} D^{n+1} \arrow[r] \arrow[drr, bend right, "\coprod h_j"'] & X_{n+1} \arrow[dr, dashed, "\gamma_{n+1}"] & \\ & & X \end{tikzcd}$$

Any map $S^q \to X_{n+1}$ for $q \leq n$ factors through X_n, so $\pi_q\gamma_{n+1}$ factors through $\pi_q\gamma_n$ for $q \leq n$. Since $X^n \to X^{n+1}$ is an n-equivalence, we have $\pi_q\gamma_{n+1} = \pi_q\gamma_n$ is an isomorphism for $0 \leq q < n$. For $q = n$

we have a commutative diagram

$$\begin{tikzcd} \pi_n X^n \arrow[r, two heads] \arrow[dr, "\pi_n\gamma_n"'] & \pi_n X^{n+1} \arrow[d, "\pi_n\gamma_{n+1}"] \\ & \pi_n X \end{tikzcd}$$

so that $\pi_n\gamma_{n+1}$ is a surjection. Moreover, any map $j \in \pi_n X^n$ such that $[\gamma_n \circ j] = 0 \in \pi_{n+1}$ extends to D^{n+1} in X_{n+1}, hence maps to zero in $\pi_{n+1}X_{n_1}$. Therefore, $\pi_n\gamma_{n+1}$ is also injective, and thus an isomorphism.

Finally, setting $\Gamma X = \operatorname{colim}_n X_n$ and $\gamma = \operatorname{colim}_n \gamma_n$ gives the required CW-approximation. □

1.8.3 Remark. There is also a version of CW-approximation for pairs. Namely, if (X, A) is pair then one can product a CW-pair $(\Gamma X, \Gamma A)$ weakly-homotopic to (X, A) such that ΓA is a sub-complex of ΓX.

Exercise 16

Let $f\colon X \to Y$ be a map of topological spaces. Show that there is an induced map, $\Gamma f\colon \Gamma X \to \Gamma Y$, unique up to homotopy, between the CW-approximations to X and Y, such that the following diagram commutes:

$$\begin{tikzcd} \Gamma X \arrow[r, "\Gamma f"] \arrow[d, "\gamma'"'] & \Gamma Y \arrow[d, "\gamma"] \\ X \arrow[r, "f"'] & Y \end{tikzcd} .$$

Deduce that CW-approximations are unique up to homotopy.

Exercise 17

Assume given maps $f\colon X \to Y$ and $g\colon Y \to X$ such that $g \circ f$ is homotopic to the identity. (We say that Y "dominates" X.) Suppose that Y is a CW complex. Prove that X has the homotopy type of a CW complex.

1.9 *Eilenberg–Maclane spaces*

1.9.1 Definition. A space X having just one non-trivial homotopy group $\pi_n(X) = G$ is called an Eilenberg–MacLane space $K(G, n)$.

1.9.2 Example. We have seen that S^1 is a $K(\mathbb{Z}, 1)$ (Example 1.2.16). We will see later that $\mathbb{C}P^\infty$ is a $K(\mathbb{Z}, 2)$.

1.9.3 Remark. We have not yet shown that Eilenberg–MacLane spaces exist in any generality. In fact, since π_n is abelian for $n \geq 2$ we see that for $n \geq 2$ Eilenberg–Maclane spaces can only exist for G abelian. The goal of this section is to show that this is the only obstruction: for any group G, abelian if $n \geq 2$, the Eilenberg–MacLane space $K(G, n)$ exists. We begin with a lemma.

1.9.4 Lemma. *For $n \geq 2$ we have $\pi_n(\bigvee_\alpha S^n_\alpha)$ is free abelian, generated by the inclusion of the factors.*

Proof. Suppose first that we have only finitely many factors. Then we can regard $\bigvee_\alpha S^n_\alpha$ as the n-skeleton of $\prod_\alpha S^n_\alpha$. Taking the usual cell structure on S^n we see that $\prod_\alpha S^n_\alpha$ has a cell structure with one zero cell and the n-cells

$$\bigcup_\alpha (\prod_{\beta \neq \alpha} D^0_\beta) \times D^n_\alpha$$

and together these form the n-skeleton of $\prod_\alpha S^n_\alpha$. Hence $\prod_\alpha S^n_\alpha \setminus \bigvee_\alpha S^n_\alpha$ has only cells of dimension at least $2n$, so that the pair $(\prod_\alpha S^n_\alpha, \bigvee_\alpha S^n_\alpha)$ is $(2n-1)$-connected by Corollary 1.6.7. Therefore, we have (recall we fix $n \geq 2$)

$$\pi_n(\bigvee_\alpha S^n_\alpha) \cong \pi_n(\prod_\alpha S^n_\alpha) \cong \prod_\alpha \pi_n(S^n_\alpha) \cong \bigoplus_\alpha \mathbb{Z}.$$

This handles the case of finitely many summands. The infinite case can be reduced to this case in the following way: Let $\Phi\colon \bigoplus_\alpha \pi_n(S^n_\alpha) \to \pi_n(\bigvee_\alpha S^n_\alpha)$ be induced by the inclusions. Then, any map $f\colon S^n \to \bigvee_\alpha S^n_\alpha$ has compact image contained in the wedge of finitely many summands, so such that $[f]$ is in the image of Φ by the finite case, and Φ is surjective. Similarly, a null-homotopy of f has compact image and again by the finite case Φ must be injective. □

1.9.5 Remark. If $n = 1$ then the Seifert–Van Kampen theorem shows that $\pi_1(\bigvee_\alpha S^1_\alpha)$ is the free group on the components; as soon as we have more than one sphere, this group is not abelian.

1.9.6 Lemma. *If a CW-pair (X, A) is r-connected $(r \geq 1)$ and A is s-connected $(s \geq 0)$, then the map $\pi_i(X, A) \to \pi_i(X/A)$ induced by the quotient map $X \to X/A$ is an isomorphism if $i \leq r + s$ and onto if $i = r + s - 1$.*

Proof. Let $i\colon A \to X$ be the inclusion and $C(i)$ the mapping cone, $C(i) = X \cup_A CA$. Since CA is contractible and $(CA, C(i))$ is a cofibration the quotient map

$$q\colon C(i) \to C(i)/CA \simeq X/A$$

is a homotopy equivalence (Theorem 1.3.17). So we have a sequence of homomorphisms

$$\pi_i(X, A) \to \pi_i(C(i), CA) \xleftarrow{\cong} \pi_i(C(i)) \xrightarrow{\cong} \pi_i(X/A),$$

where the first and second maps are induced by the inclusion of pairs and the third map is the isomorphism q_*. The second map is an isomorphism by the long exact sequence of the pair $(C(i), CA)$.

Now we know that (X, A) is r-connected and $CA, A)$ is $(s+1)$-connected which follows from the assumption on A on the long exact sequence in homotopy. The result now follows from excision, Theorem 1.7.5. □

1.9.7 Remark. Suppose $n \geq 2$, and we are given maps $\phi_\beta \colon S^n_\beta \to \bigvee_\alpha S^n_\alpha$. Then we construct a space X as the pushout

$$\begin{tikzcd} \bigvee_\beta S^n_\beta \arrow[r, "(\phi_\beta)"] \arrow[d, hook] & \bigvee_\alpha S^n_\alpha \arrow[d] \\ \bigvee_\beta D^{n+1}_\beta \arrow[r, "(\Phi_\beta)"'] & X \end{tikzcd}$$

1.9.8 Lemma. *We have*

$$\pi_n(X) \cong \pi_n(\bigvee_\alpha S^n_\alpha)/\langle \phi_\beta \rangle \cong (\bigoplus_\alpha \mathbb{Z})/\langle \phi_\beta \rangle.$$

Proof. The pair $(X, \bigvee_\alpha S^n_\alpha)$ is n-connected and fits in a long exact sequence

$$\pi_{n+1}(X, \bigvee_\alpha S^n_\alpha) \xrightarrow{\partial} \pi_n(\bigvee_\alpha S^n_\alpha) \to \pi_n(X) \to \pi_n(X, \bigvee_\alpha S^n_\alpha) = 0,$$

where the final equality follows as a consequence of cellular approximation, see Corollary 1.6.7. It follows that $\pi_n(X) \cong \pi_n(\bigvee_\alpha S^n_\alpha)/\operatorname{im}(\partial)$.

We have $X/\bigvee_\alpha S^n_\alpha \simeq \bigvee S^{n+1}_\beta$, and so by Lemma 1.9.4 and Lemma 1.9.6 we have $\pi_{n+1}(X, \bigvee_\alpha S^n_\alpha) \cong \pi_{n+1}(\bigvee_\beta S^{n+1}_\beta)$ is free with a basis consisting of the (restriction of the) characteristic maps Φ_β of the attaching cells. Since $\partial([\Phi_\beta]) = [\phi_\beta]$, the claim follows. □

1.9.9 Example. Using the previous results, any abelian group G can be realized as $\pi_n(X)$ for $n \geq 2$ of some space X. Indeed, choosing a presentation $G = \langle g_\alpha \mid r_\beta \rangle$ we can take

$$X = \left(\bigvee_\alpha S^n_\alpha\right) \cup \bigcup_\beta D^{n+1}_\beta,$$

with the S^n_α's corresponding to the generators, and the discs are attached by maps $f \colon S^n_\beta \to \bigvee_\alpha S^n_\alpha$ satisfying $[f] = r_\beta$. Then, the previous lemma says that $\pi_n(X) \cong G$. In fact, $\pi_i(X) = 0$ for $i < n$ by cellular approximation, but we have no control over the higher homotopy groups. In order to construct Eilenberg–Maclane spaces, we have to kill higher homotopy groups.

1.9.10 Theorem. *For any $n \geq 1$ and any group G (abelian if $n \geq 2$) there exists an Eilenberg–Maclane space $K(G, n)$.*

Proof. Let $X_{n+1} = (\bigvee_\alpha S^n_\alpha) \cup \bigcup_\beta D^{n+1}_\beta$ be as in Example 1.9.9. We want to build a space X_{n+2} which agrees with X_{n+1} in homotopy up to π_n but has $\pi_{n+1} X_{n+2} = 0$. We have seen exactly how to do this in Lemma 1.8.1. Repeating this procedure inductively and taking colimits, we product a space X with the correct homotopy groups. □

Exercise 18

Exhibit a fibration $F \to E \to B$ where, up to weak homotopy equivalence, F is a $K(G, n-1)$, B is a $K(G, n)$ and E is contractible.

1.9.11 Remark. Eilenberg–Maclane spaces represent cohomology in the following sense.[16]

[16] We will talk about representability more when we discuss Brown representability

1.9.12 Theorem. *There are natural bijections $T\colon [X, K(G, n)] \xrightarrow{\cong} H^n(X; G)$ for all CW-complexes and all $n > 0$ with G any abelian group. Such a T has the form $T([f]) = f^*(\alpha)$ for a distinguished class $\alpha \in H^n(K(G, n); G)$.*

Proof. Recall that if h^* is an unreduced cohomology theory on the category of CW-pairs, and $h^n(*) = 0$ for $n \neq 0$, then there exists a natural isomorphism $h^n(X, A) \cong H^n(X, A; h^0(*))$ for all CW-pairs (X, A) and all n.

Now define $h^n(X) = [X, K(G, n)]$. This defines a reduced cohomology theory and the coefficient groups $h^n(S^i) = \pi_i(K(G, n)$ are the same as $\tilde{H}^i(S^i; G)$. Therefore, the previous paragraph (translated into reduced cohomology) gives the representability result. It remains to be seen that T has the claimed form. This is formal: let $\alpha = T(\mathrm{id})$ for $\mathrm{id}\colon K(G, n) \to K(G, n)$ the identity map. Then,

$$T([f]) = T(f^*(\mathrm{id})) \cong f^*(T(\mathrm{id})) = f^*(\alpha). \qquad \square$$

We can use this to prove a uniqueness theorem for Eilenberg–MacLane spaces. We begin with a lemma.

1.9.13 Lemma. *If X is $(n-1)$-connected, then $H^n(X; H) \cong \mathrm{Hom}(H_n(X); G)$.*

Proof. This follows from a form of the Universal Coefficient theorem: there is an exact sequence

$$0 \to \mathrm{Ext}(H_{n-1}(X), G) \to H^n(X; G) \to \mathrm{Hom}(H_n(X), G) \to 0,$$

but the first term is zero by assumption. $\square$

1.9.14 Remark. Let $G = \pi_n(X)$, then for $(n-1)$-connected X the lemma gives $H^n(X; \pi_n(X)) \cong \mathrm{Hom}(H_n(X), \pi_n(X))$. The fundamental class ι is the class in $H^n(X; \pi_n(X))$ which corresponds to the inverse of a certain isomorphism $h_n\colon \pi_n(X) \cong H_n(X)$ we will construct in Section 1.10. In particular, to $K(G, n)$ we can associate a fundamental class $\iota_n \in H^n(K(G, n); G)$.

The following is a consequence of the representability of cohomology (Theorem 1.9.12).

1.9.15 Corollary. *There is a bijection*

$$[K(G, n), K(G', n)] \xrightarrow{\cong} \mathrm{Hom}(G, G').$$

Proof. We have bijections

$$\begin{aligned} [K(G, n), K(G', n)] &\cong H^n(K(G, n); G') \\ &\cong \mathrm{Hom}(H_n(K(G, n)), G') \\ &\cong \mathrm{Hom}(G, G'). \end{aligned}$$

□

1.9.16 Corollary. *The homotopy type of a CW-complex $K(G,n)$ is uniquely determined by G and n.*

Proof. If $G \cong G'$ then this can be realized by a map $K(G,n) \to K(G',n)$ by the previous corollary, and since all other homotopy groups are trivial, it follows from Whitehead's theorem that this map is a homotopy equivalence. □

1.9.17 Remark (Moore spaces). There is a homology version of an Eilenberg–MacLane space, known as a Moore space: there exists a space $M(G,n)$ for an abelian group G such that

$$\widetilde{H}_k(M(G,n)) \cong \begin{cases} G & k = n \\ 0 & \text{else.} \end{cases}$$

These can in fact be used to construct Eilenberg–MacLane spaces:[17] Let $\mathrm{Sp}^n(X) := X^{\times n}/\Sigma_n$, and let the infinite symmetric product of a space X, denoted $\mathrm{Sp}^\infty(X)$, be the colimit of the $\mathrm{Sp}^n(X)$. Then, for any connected space X, we have $\widetilde{H}_n(X) \cong \pi_n(\mathrm{Sp}^\infty(X)$. In particular, $\mathrm{Sp}^\infty(M(G,n))$ is the Eilenberg–MacLane space $K(G,n)$.

[17] This is the Dold–Thom theorem.

1.10 The Hurewicz theorem

Define a homomorphism $h\colon \pi_i(X) \to H_i(X)$ by choosing a generator u of $H_i(S^i) \cong \mathbb{Z}$ and defining $h([f]) = f_*(u)$. This is the Hurewicz homomorphism, and extends the Hurewicz homomorphism studied previously for π_1.

1.10.1 Theorem. *If a space X is $(n-1)$-connected and $n \geq 2$, then $\tilde{H}_i(X) = 0$ for $i < n$ and $\pi_n(X) \cong H_n(X)$. Moreover, if a pair (X,A) is $(n-1)$-connected with $n \geq 2$ and $\pi_i(A) = 0$, then $H_i(X,A) = 0$ for all $i \leq n$ and $\pi_n(X,A) \cong H_n(X,A)$.*

1.10.2 Remark. The isomorphism in the theorem is induced by h_n, but we do not show this.

Proof. For note that the statements only involve homology and homotopy groups, so it suffices to use a CW approximation for X, or for (X,A).[18] Secondly, the relative case can be reduced to the absolute case: (X,A) is $(n-1)$-connected and A is 1-connected, so Lemma 1.5.9 applies to show that $\pi_i(X,A) \cong \pi_i(X/A)$ for $i \leq n$, while $H_i(X,A) \cong \tilde{H}_i(X/A)$ always holds for CW-pairs.

[18] Note that a weak equivalence in homotopy also induces a weak equivalence in (co)homology.

By a version of CW-approximation, we can assume X has $(n-1)$-skeleton a point. In particular, $\tilde{H}_i(X) = 0$ for $i < n$. For showing $\pi_n(X) \cong H_n(X)$ we may ignore cells of dimension $> n+1$ since these do not change π_n or H_n. So we can assume $X = (\bigvee_\alpha S^n_\alpha) \cup \bigcup_\beta D^{n+1}_\beta$. As we have seen, we then have $\pi_n(X) \cong \oplus_\alpha \mathbb{Z}/\langle \phi_\beta \rangle$, and the same is true for $H_n(X)$ by cellular homology. □

1.10.3 Corollary (The homology Whitehead theorem)**.** *If X and Y are simply connected and $f\colon X \to Y$ a map inducing an isomorphism on homology, then f is a homotopy equivalence.*

Proof. By using the mapping cylinder M_f we can assume f is an inclusion. Then by the long exact sequence in homology we have $H_n(Y, X) = 0$ for all n. Since X and Y are simply-connected $\pi_1(Y, X) = 0$. So by the relative Hurewicz theorem, the first non-zero $\pi_n(Y, X)$ is equal to the first non-zero $H_n(YmX)$. Therefore, $\pi_n(Y, X) = 0$ for all n. Therefore, $f_*\colon \pi_n(X) \to \pi_n(Y)$ is an isomorphism for all n. By Whitehead's theorem, f is a homotopy equivalence. □

Exercise 19

Show that a map between simply-connected CW complexes is a homotopy equivalence if its mapping cone is contractible.

Exercise 20

Show that a simply-connected closed 3-manifold is homotopy equivalent to S^3. (You may assume that every closed manifold is homotopy equivalent to a CW-complex.)

1.10.4 Remark. This exercise is a little bit tricky, but you should try and compute the homology groups of such a manifold, and then use the homology Whitehead theorem. You should begin by noting that simply-connected manifolds are orientable.

1.11 *Brown representability*

In this section we explain Brown's representability theorem, which one can use to give an abstract proof of Theorem 1.9.12.[19]

[19] There are no proofs given in this section.

1.11.1 Definition. Let $F\colon \mathcal{C} \to \mathrm{Set}$ be a functor. Then F is representable if it is naturally isomorphic to $\mathrm{Hom}_{\mathcal{C}}(A, -)$ for some $A \in \mathcal{C}$.

1.11.2 Remark. If such an A exists, then it is unique up to unique isomorphism.

1.11.3 Example.
- The forgetful functor $\mathrm{Ab} \to \mathrm{Set}$ is representable. Indeed, $\mathrm{Hom}_{\mathrm{Ab}}(\mathbb{Z}, M) \to M$ sending f to $f(1)$ is an isomorphism.
- The forgetful functor $\mathrm{Ring} \to \mathrm{Set}$ is represented by $\mathbb{Z}[x]$: the map $\mathrm{Hom}_{\mathrm{Ring}}(\mathbb{Z}[x], M) \to M$ sending f to $f(x)$ is a bijection.
- The functor $\mathrm{Top}^{\mathrm{op}} \to \mathrm{Set}$ that sends a topological space (X, τ_X) to its topology (the family of open sets) is representable. Indeed, let $Y = \{0, 1\}$ with topology $\{\emptyset, \{1\}, \{0, 1\}\}$. Then, consider $\mathrm{Hom}_{\mathrm{Top}}(X, Y) \to \tau_X$ sending f to $f^{-1}(\{1\})$. This map is a bijection: the inverse is given by sending an open set U to the

characteristic function $\mathbb{1}_U$, i.e., the functor $\mathbb{1}_U \colon X \to \{0,1\}$ defined by

$$\mathbb{1}_U(x) = \begin{cases} 1, & x \in U \\ 0, & x \notin U. \end{cases}$$

1.11.4 Remark. Note that representable functors preserve limits, because $\mathrm{Hom}_{\mathcal{C}}(A, -)$ preserves limits. This, if we want a functor to be representable it should at least preserve limits. Brown's representability theorem gives sufficient additional conditions for a functor from CW-complexes to Sets to be representable.

Let CW denote the homotopy category of based connected CW complexes.

1.11.5 Theorem. *Let $F \colon CW^{\mathrm{op}} \to \mathrm{Set}$ be a functor satisfying:*

1. $F(\bigvee_\alpha X_\alpha) = \prod_\alpha F(X_\alpha)$
2. *Let X be an object of CW. Consider a cover $X = Y \cup Z$ by sub-complexes such that $Y, Z, Y \cap Z \in CW$. Then, for all $y \in F(Y)$ and $z \in F(Z)$ that restrict to the same elect of $F(Y \cap Z)$, there exists some $x \in F(x)$ that restricts to $z \in F(Z)$ and $y \in F(Y)$.*

Then F is representable: there exists some $C \in CW$ and $c \in F(c)$ such that for all $X \in CW$, the map

$$[X, C] \to F(x), f \mapsto f^*(c)$$

is a bijection.

1.11.6 Example. Consider the functor $\tilde{H}^n(-; G) \colon CW^{\mathrm{op}} \to \mathrm{Ab} \to \mathrm{Set}$. This satisfies the above conditions (the second condition is essentially Mayer–Vietoris), and so there exists some $C \in CW$ and $c \in \widetilde{H}^n(C; G)$ such that

$$[X, C] \to \widetilde{H}^n(X; G), f \mapsto f^*(c).$$

is a bijection. In particular, if we take $X = S^i$, then

$$[S^i, C] = \pi_i(C) \cong \widetilde{H}^n(S^i; G) \cong \begin{cases} G & i = n \\ 0 & \text{otherwise.} \end{cases}$$

In other words, C is a $K(G, n)$, and we recover Theorem 1.9.12.

1.11.7 Example. Let X be *any* space, and consider the functor $F_X \colon CW^{\mathrm{op}} \to \mathrm{Set}$ given by $Z \mapsto [Z, X]$. This is representable, and so there exists $Y \in CW$ and a map $f \colon Y \to X$ such that $[Z, Y] \xrightarrow{\cong} [Z, X]$. Taking Z to be the spheres, we see that $Y \to X$ is a CW-approximation.

2
Spectral sequences

Spectral sequences are a powerful computation tool in topology. Computing with spectral sequences is a bit like computing integral in calculus; it is helpful to have ingenuity and a big bag of tricks - and even that may not be enough!

2.1 Filtered complexes

We begin our discussion on spectral sequences by discussing filtered complexes.

2.1.1 Remark. Let $C_\bullet$ be a chain complex and $F_0C_\bullet$ a sub-complex. Then we have a short exact sequence

$$0 \to F_0C_\bullet \to C_\bullet \to C_\bullet/F_0C_\bullet \to 0$$

which gives rise to a long exact sequence in homology

$$\cdots \to H_i(F_0C_\bullet) \to H_i(C_\bullet) \to H_i(C_\bullet/F_0C_\bullet) \xrightarrow{\partial} H_{i-1}(F_0C_\bullet) \to \cdots$$

Suppose we know $H_*(F_0C_\bullet)$ and $H_*(C_\bullet/F_0C_\bullet)$. Can we compute $H_*(C_\bullet)$? We can split the long exact sequence into short exact sequences

$$0 \to \operatorname{coker}(\partial) \to H_*(C_\bullet) \to \ker(\partial) \to 0$$

which gives the following procedure for computing $H_*(C_\bullet)$:

1. Compute $H_*(F_0C_\bullet)$ and $H_*(C_\bullet/F_0C_\bullet)$
2. Consider the two-term chain complex

   $$H_*(C_\bullet/F_0C_\bullet) \xrightarrow{\partial} H_*(F_0C_\bullet).$$

   Denote its homology groups by G_1H_* and G_0H_*.
3. There is a short exact sequence

   $$0 \to G_0H_* \to H_*(C_\bullet) \to G_1H_* \to 0.$$

   This determines $H_*(C_\bullet)$ up to extension.[1]

How would we handle the situation if we have a longer filtration: $\cdots F_pC_\bullet \subseteq F_{p+1}C_\bullet \subseteq \cdots$?

[1] This is a common phenomenon for a spectral sequence. For example, if we have a short exact sequence $0 \to \mathbb{Z}/2 \to M \to \mathbb{Z}/2 \to 0$, can you say what the middle group is? Not without further information!

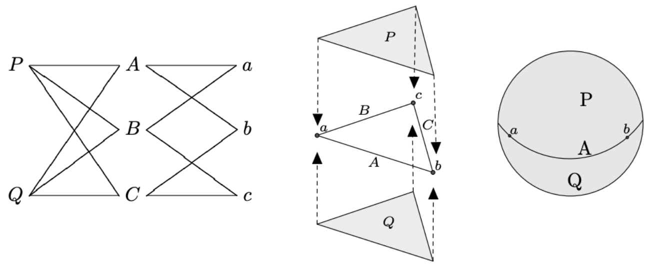


Figure 2.1: Simplicial model of S^2

2.1.2 Example. Consider a (semi-simplicial) model of the 2-sphere S^2 with vertices $\{a, b, c\}$, edges $\{A, B, C\}$ and solid triangles $\{P, Q\}$ and with inclusions as soon in Figure 2.1.[2] The associated chain complex is $C_\bullet$

$$0 \to \mathbb{Z}\{P, Q\} \xrightarrow{d} \mathbb{Z}\{A, B, C\} \xrightarrow{d} \mathbb{Z}\{a, b, c\} \to 0$$

with

$$d(P) = C - B + A \quad d(Q) = C - B + A$$

and

$$d(A) = b - a \quad d(B) = c - a \quad d(C) = c - b.$$

One can check directly that $H_i(C_\bullet; \mathbb{Z}) \cong \mathbb{Z}$ for $i = 0, 2$ and is zero otherwise. Alternatively, we use the following filtration:

$$\begin{array}{ccccccccc}
0 & \to & \mathbb{Z}\{P, Q\} & \to & \mathbb{Z}\{A, B, C\} & \to & \mathbb{Z}\{a, b, c\} & \to & 0 \\
0 & \to & 0 & \to & \mathbb{Z}\{A, B\} & \to & \mathbb{Z}\{a, b, c\} & \to & 0 \\
0 & \to & 0 & \to & \mathbb{Z}\{A\} & \to & \mathbb{Z}\{a, b\} & \to & 0.
\end{array}$$

The differentials are induced from d_1 and d_2 and a direct check shows that they are still chain complexes. Passing to the quotient, we get a chain complex we call E_0:

$$\begin{array}{ccccccccc}
0 & \to & \mathbb{Z}\{P, Q\} & \to & \mathbb{Z}\{C\} & \to & 0 & \to & 0 \\
0 & \to & 0 & \to & \mathbb{Z}\{B\} & \to & \mathbb{Z}\{c\} & \to & 0 \\
0 & \to & 0 & \to & \mathbb{Z}\{A\} & \to & \mathbb{Z}\{a, b\} & \to & 0
\end{array}
\qquad
\begin{array}{c}
d_0(P) = C, d_0(Q) = C \\
d_0(B) = c \\
d_0(A) = b - a.
\end{array}$$

Taking homology with respect to d_0 we obtain E^1:

$$\begin{array}{ccccccccc}
0 & \to & \mathbb{Z}\{P - Q\} & \to & 0 & \to & 0 & \to & 0 \\
0 & \to & 0 & \to & 0 & \to & 0 & \to & 0 \\
0 & \to & 0 & \to & 0 & \to & \mathbb{Z}\{\bar{a}\} & \to & 0.
\end{array}$$

The general theory of spectral sequences will tell us that we have computed the homology of $H_*(C_\bullet)$; there is a $\mathbb{Z}$ in degree 2, generated by $P - Q$ and a $\mathbb{Z}$ in degree 0, generated by $\bar{a}$.

This leads us to the theory of filtered modules.

2.1.3 Definition. A filtered R-module is an R-module A together with an increasing sequence of submodules $F_p A \subseteq F_{p+1} A$ indexed by $p \in \mathbb{Z}$ such that $\cup_p F_p A = A$ and $\cap_p F_p A = \{0\}$. The filtration is

[2] This example comes from Example 2.1 of https://arxiv.org/pdf/1702.00666.pdf.

bounded if $F_pA = \{0\}$ for p sufficiently small, and $F_pA = A$ for p sufficiently large. The associated graded module is defined by

$$G_pA = F_pA/F_{p-1}A.$$

2.1.4 Definition. A filtered chain complex is a chain complex $(C_\bullet, \partial)$ together with a filtration $\{F_pC_i\}$ of each C_i such that the differential preserves the filtration: $\partial(F_pC_i) \subseteq F_pC_{i-1}$. Then, ∂ induces $\partial\colon G_pC_i \to G_pC_{i-1}$ on the associated graded modules.

2.1.5 Remark. The filtration on $C_\bullet$ induces a filtration on the homology of $C_\bullet$ by

$$F_pH_i(C_\bullet) = \{\alpha \in H_i(C_\bullet) \mid \exists x \in F_pC_i, \alpha = [x]\}.$$

This has associated graded pieces $G_pH_i(C_\bullet)$.

2.1.6 Remark. Suppose we want to compute $H_*(C_\bullet)$ and that we can compute the homology of the associated graded pieces $H_*(G_pC_\bullet)$. Does this determine $G_pH_*(C_\bullet)$? This leads to the idea of the spectral sequence of a filtered complex.

2.2 *The spectral sequence of a filtered complex*

2.2.1 Definition. Let $(F_pC_\bullet, \partial)$ be a filtered chain complex. Let us write

$$E^0_{p,q} := G_pC_{p+q} = F_pC_{p+q}/F_{p-1}C_{p+q}.$$

The differential ∂ induces a differential on E^0,

$$\partial_0\colon E^0_{p,q} \to E^0_{p,q-1}.$$

We denote the homology of the associated graded by

$$E^1_{p,q} := H_{p+q}(G_pC_\bullet, \partial_0).$$

2.2.2 Remark. We can think of $E^1_{p,q}$ as a "first order approximation" to $H_*(C_\bullet)$. We can also define a differential

$$\partial_1\colon E^1_{p,q} \to E^1_{p-1,q}$$

as follows: a homology class $\alpha \in E^1_{p,q}$ can be represented by a chain $x \in F_pC_{p+1}$ such that $\partial x \in F_{p-1}C_{p+q-1}$. We define $\partial_1(\alpha) = [\partial x]$. Because $\partial^2 = 0$, we can check that $\partial_1^2 = 0$ and that ∂_1 is well defined.

2.2.3 Definition. With notation as above, we define

$$E_2^{p,q} = \ker(\partial_1\colon E^1_{p,q} \to E^1_{p-1,q}) / \operatorname{im}(\partial_1\colon E^1_{p+1,q} \to E^1_{p,q}).$$

2.2.4 Remark. We can continue this procedure, and define an "r"-th order approximation to $G_pH_{p+q}(C_\bullet)$ by

$$E^r_{p,q} = \frac{x \in F_pC_{p+q} \mid \partial x \in F_{p-r}C_{p+q-1}}{F_{p-1}C_{p+q} + \partial(F_{p+r-1}C_{p+q+1}}).$$

The notation denotes the quotient of the numerator by the intersection with the denominator.

So instead of considering cycles, we consider chains in F_p whose differentials vanishes "to order r", and instead of modding out by the entire image, we only mod out by $\partial(F_{p+r-1})$.

The main result regarding these groups is the following.

2.2.5 Lemma. *Let $(F_pC_\bullet, \partial)$ denote a filtered chain complex, and define $E^r_{p,q}$ as above. Then,*

1. *∂ induces a map*
$$\partial_r \colon E^r_{p,q} \to E^r_{p-r,q+r-1}$$
satisfying $\partial_r^2 = 0$.
2. *E^{r+1} is the homology of the chain complex (E^r, ∂_r), i.e.,*
$$E^{r+1}_{p,q} = \ker(\partial_r \colon E^r_{p,q} \to E^r_{p-r,q+r-1}) / \operatorname{im}(\partial_r \colon E^r_{p+r,q+r-1} \to E^r_{p,q}).$$
3. *$E^1_{p,q} = H_{p+q}(G_pC_\bullet)$.*
4. *If the filtration of C_i is bounded for each i, then for every p, q if r is sufficiently large, then*
$$E^r_{p,q} = G_pH_{p+q}(C_\bullet).$$

Proof. This is a rather tedious diagram chase,[3] which generalizes the argument that a short exact sequence of chain complexes induces a long exact sequence on homology. □

[3] For example, see http://www.math.uchicago.edu/~may/MISC/SpecSeqPrimer.pdf

2.2.6 Example. In this example[4] we show that the singular and cellular homology groups of a CW-complex X agree. To that end, let $C_*(X)$ denote the singular chain complex of X. We filter this by

$$F_pC_*(X) := C_*(X^p)$$

where X^p denotes the p-skeleton of X. The associated graded is

$$E^0_{p,q} = C_{p+q}(X^p)/C_{p+q}(X^{p-1}).$$

By definition, the homology is

$$E^1_{p,q} = H_{p+q}(X^p, X^{p-1}),$$

the relative homology of the pair (X^p, X^{p-1}). We have

$$H_{p+q}(X^p, X^{p-1}) \cong \begin{cases} C_p^{cell}(X) & q = 0 \\ 0, & q \neq 0 \end{cases}$$

where $C_p^{cell}(X)$ is the cellular chains on X, the free $\mathbb{Z}$-module with one generator for each p-cell. The cellular differential $\partial \colon C_p^{cell}(X) \to C_{p-1}^{cell}(X)$ is exactly the boundary map $E^1_{p,0} \to E^1_{p-1,0}$. Therefore, we have

$$E^2_{p,q} = \begin{cases} H_p^{cell}(X), & q = 0 \\ 0, & q \neq 0. \end{cases}$$

We must have $\partial_r = 0$ for $r \geq 2$ as either the domain or the range is zero. So, $E_r^{p,q} = E^2_{p,q}$ for all $r \geq 2$. If X is finite-dimensional, then the filtration is bounded and so $H_p(X) = H_p^{cell}(X)$ by Lemma 2.2.5.[5]

[4] See page 67 of Mosher–Tangor, *Cohomology Operations and Applications in Homotopy Theory*

[5] One can allow arbitrary X by, for example, using colimits.

2.3 *Homological spectral sequences*

We have managed to so far avoid defining exactly what a spectral sequence is. Let us change that now.

2.3.1 Definition. A (homological) spectral sequence is a sequence

$$\{E^r_{*,*}, d^r_{*,*}\}_{r\geq 0}$$

of chain complexes of abelian groups, such that

$$E^{r+1}_{*,*} = H_*(E^r_{*,*})$$

where the homology is taken with respect to maps (called differentials)

$$d^r_{p,q} \colon E^r_{p,q} \to E^r_{p-r,q+r-1}$$

such that $(d^r)^2 = 0$.

2.3.2 Remark. We say that a spectral sequence is first quadrant if $E^r_{p,q} = 0$ whenever $p < 0$ or $q < 0$. Note that this implies that $d^r_{p,q} = 0$ for $r \gg 0$ (as either the source or the target is zero). In particular,

$$E^r_{p,q} = E^{r+1}_{p,q} = \cdots = E^\infty_{p,q}.$$

We say that the spectral sequence collapses or degenerates at E^r.

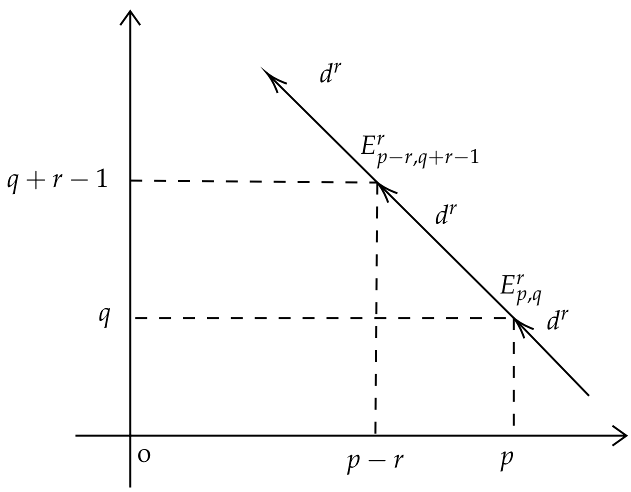


Figure 2.2: The E^r-page of a homological spectral sequence

2.3.3 Definition. If $\{H_n\}_n$ are groups, then we say that the spectral sequence converges, or abuts, to H_*, denoted $E^2_{*,*} \implies H_*$, if for each n there is a filtration

$$H_n = D_{n,0} \subseteq D_{n-1,1} \subseteq \cdots \subseteq D_{1,n-1} \subseteq D_{0,n} \subseteq 0$$

such that, for all p, q,

$$E^\infty_{p,q} = D_{p,q}/D_{p-1,q+1}.$$

2.3.4 Remark. In more straightforward terms: the if we look along the n-th diagonal of the spectral sequence, then the E_∞-page computes the associated graded of the filtration on H_n. For example, if $E^\infty_{p,q} = 0$ for all $p+q=n$, then $H_n = 0$. If there is only a single non-zero term, say $E^\infty_{p,n-p}$, then the filtration is trivial, and $H_n = E^\infty_{p,n-p}$. If we have two non-zero terms, then H_n fits into a short exact sequence, and so on.

2.3.5 Example. We have previously discussed the spectral sequence of a filtered complex without explicitly mentioning it. Indeed, if $C_\bullet$ is a filtered chain complex, then there is a spectral sequence with $E^1_{p,q} = H_{p+q}(G_pC_\bullet)$, such that if the filtration of C_i is bounded for each i the spectral sequence converges to $H_{p+q}(C_\bullet)$.[6]

[6] Recall what this means: we have $E^\infty_{p,q} = G_pH_{p+q}(C_\bullet)$.

2.4 *The spectral sequence of a double complex*

An important example where a filtered complex arises is from a double complex.

2.4.1 Definition. A double complex is a bi-indexed family $\{C_{p,q}\}$ of abelian groups, with two differentials

$$d' \colon C_{p,q} \to C_{p-1,q}, \quad d'' \colon C_{p,q} \to C_{p,q-1}$$

such that $d'd' = 0, d''d'' = 0$, and $d'd'' + d''d' = 0$. For simplicity, we also assume that $C_{p,q} = 0$ for $p < 0$ or $q < 0$.

2.4.2 Example. Suppose that (A, d_A) and (B, d_B) are chain complexes. If we define $C_{p,q} = A_p \otimes B_q$ and define $d' = d_A \otimes 1$ and $d'' = (-1)^p 1 \otimes d_B$, then $C_{p,q}$ is a double complex.[7]

[7] Try and verify this to make sure you understand the definitions.

2.4.3 Construction . A double complex gives rise to a chain complex (the total complex), defined by $C_n = \sum_{p+q=n} C_{p,q}$ and $d = d' + d''$. This has two obvious filtrations, by row and by column:

1. ${}'C^p_n = \sum_{j+q=n, j \le p} C_{j,q}$.
2. ${}''C^p_n = \sum_{p+q=n, k \le q} C_{p,k}$.

The spectral sequence of a filtered complex (Example 2.3.5) gives us two spectral sequences:

1. ${}'E^1_{p,q} = H_{p+q}({}'C^p/{}'C^{p-1}) = C_{p,n-p}$.
2. ${}''E^1_{p,q} = H_{p+q}({}''C^q/{}''C^{q-1}) = C_{q,n-q}$.

One checks that ${}'E^1$ is computed via means of d'' and that d^1 is induced by d', while in ${}''E^1$ the role of the two indices are exchanged. We can therefore write:

1. ${}'E^2_{p,q} = H'_p H''_q(C)$.
2. ${}''E^2_{p,q} = H''_q H'_p(C)$.

Moreover, both spectral sequences converge to $H_*(C)$, and the idea is to compare the two spectral sequences.

It is constructive to do an example.

2.4.4 *Example.* Let ${}'\operatorname{Tor}(A,B)$ be defined as follows: take a free resolution of A, $0 \to R' \to F' \to A \to 0$, then ${}'\operatorname{Tor}(A,B)$ is defined by

$$0 \to {}' \operatorname{Tor}(A,B) \to R' \otimes B \to F' \otimes B \to A \otimes B \to 0.$$

Similarly, let ${}''\operatorname{Tor}(A,B)$ be defined as follows: take a free resolution of B, $0 \to R'' \to F'' \to B \to 0$, then ${}''\operatorname{Tor}(A,B)$ is defined by

$$0 \to {}''\operatorname{Tor}(A,B) \to A \otimes R'' \to A \otimes F'' \to A \otimes B \to 0.$$

It is a classical theorem of homological algebra that $\operatorname{Tor}(A,B) = {}''\operatorname{Tor}(A,B)$. Let us prove this via a spectral sequence argument.

Let X be the chain complex $0 \to R' \xrightarrow{d'} F' \to 0$ and let Y be the chain complex $0 \to R'' \xrightarrow{d''} F'' \to 0$. We can build a double complex $C_{*,*}$ as in Example 2.4.2, which we write as a matrix:

$$\begin{bmatrix} C_{p,q} \end{bmatrix} = \begin{bmatrix} F' \otimes R'' & R' \otimes R'' \\ F' \otimes F'' & R' \otimes F'' \end{bmatrix}$$

We have two spectral sequences: the first is take vertical and then horizontal homology:

$$H_q''(C_{p,q}) = \begin{bmatrix} {}''\operatorname{Tor}(F',B) & {}'\operatorname{Tor}(R',B) \\ F' \otimes B & R' \otimes B \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ F' \otimes B & R' \otimes B \end{bmatrix}$$

and

$$H_p H_q''(C_{p,q}) = \begin{bmatrix} 0 & 0 \\ A \otimes B & {}'\operatorname{Tor}(A,B) \end{bmatrix}$$

In other words, the total complex has $H_0(C) = A \otimes B$ and $H_1(C) = {}'\operatorname{Tor}(A,B)$.

However, we can use the second spectral sequence, which first takes horizontal and then vertical homology:

$$H_p(C_{p,q}) \begin{bmatrix} A \otimes R'' & {}'\operatorname{Tor}(A,R'') \\ A \otimes F'' & {}'\operatorname{Tor}(A,F'') \end{bmatrix} = \begin{bmatrix} A \otimes R'' & 0 \\ A \otimes F'' & 0 \end{bmatrix}$$

and then

$$H''_q H_p(C_{p,q}) = \begin{bmatrix} {}''\operatorname{Tor}(A,B) & 0 \\ A \otimes B & 0 \end{bmatrix}$$

In this case we see that $H_0(C) = A \otimes B$ and $H_1(C) = {}''\operatorname{Tor}(A,B)$. Therefore, ${}'\operatorname{Tor}(A,B) = {}''\operatorname{Tor}(A,B)$.

Exercise 21: The snake lemma

Show, using spectral sequences, the following result in homological algebra (the snake lemma):
Given a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, "f"] & B \arrow[r] \arrow[d, "g"] & C \arrow[r] \arrow[d, "h"] & 0 \\ 0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0 \end{tikzcd}$$

in an abelian category with exact rows, there is a long exact sequence

$$0 \to \ker(f) \to \ker(g) \to \ker(h) \\ \to \operatorname{coker}(f) \to \operatorname{coker}(g) \to \operatorname{coker}(h) \to 0.$$

Exercise 22

(1) Suppose we have a commutative triangle

$$\begin{tikzcd} A \arrow[rr, "i"] \arrow[dr, "f"'] & & C \arrow[dl, "q"] \\ & B & \end{tikzcd}$$

Show using the snake lemma that

$$\ker(\operatorname{coker} f \to \operatorname{coker} q) \cong \operatorname{im}(q) / \operatorname{im}(f)$$

and

$$\operatorname{coker}(\operatorname{coker} f \to \operatorname{coker} q) = 0.$$

(2) Using Part (1), prove the following 'butterfly lemma': given a commutative diagram

$$\begin{tikzcd} A \arrow[dr, "i"] \arrow[dd, "f"'] & & D \arrow[dl, "j"'] \arrow[dd, "g"] \\ & C \arrow[dl, "q"] \arrow[dr, "p"'] & \\ B & & E \end{tikzcd}$$

of abelian groups, in which the diagonals pi and qj are exact at C, there is an isomorphism

$$\frac{\operatorname{im} q}{\operatorname{im} f} \cong \frac{\operatorname{im} p}{\operatorname{im} g}.$$

2.5 *The Serre spectral sequence*

For us the most important example of a spectral sequence will be the Serre spectral sequence. We will state the theorem now and then return to the proof after some examples and applications.

2.5.1 Theorem (The Serre spectral sequence)**.** *Let $\pi\colon E \to F$ be a fibration with fiber F and assume that $\pi_1(B) = 0$ and $\pi_0(F) = 0$, then there is a first quadrant spectral sequence*

$$E^2_{p,q} = H_p(B; H_q(F)) \implies H_{p+q}(E).$$

In particular, this means there is a filtration

$$H_n(E) = D_{n,0} \supseteq D_{n-1,1} \supseteq \ldots \supseteq D_{0,n} \supseteq D_{-1,n+1} = 0$$

such that $E^\infty_{p,q} = D_{p,q}/D_{p-1,q+1}$.

2.5.2 Remark. There is a version of this spectral sequence where $\pi_1(B) \neq 0$; the E_2-page is then given by the cohomology of B with local coefficients $\mathcal{H}_q(F)$. This will not play a role in this course.

2.5.3 Example. Consider the Hopf fibration $S^1 \to S^3 \to S^2$. We have

$$E^2_{p,q} = H_p(S^2; H_q(S^1)) \cong \begin{cases} \mathbb{Z} & p = 0,2 \text{ and } q = 0,1 \\ 0 & \text{otherwise.} \end{cases}$$

The E^2-term is as follows (and we have $E^3 = E^\infty$ for degree reasons):

$$H_*(S^1)\quad \begin{array}{c|cccc} 2 & & & & \\ 1 & \mathbb{Z} & & \mathbb{Z} & \\ 0 & \mathbb{Z} & & \mathbb{Z} & \\ \hline & 0 & 1 & 2 & 3 \end{array}$$

$$H_*(S^2)$$

$\cdot n$

There are three possibilities for the d_2-differential (which is multiplication by $n \in \mathbb{Z}$ as indicated): either $n = 0, n = \pm 1$ or $n \neq 0, \pm 1$, which lead to the following $E^3 = E^\infty$-page:

$$H_*(S^1)\ \begin{array}{c|ccc} 1 & \mathbb{Z} & & \mathbb{Z} \\ 0 & \mathbb{Z} & & \mathbb{Z} \\ \hline & 0 & 1 & 2 \end{array} \qquad H_*(S^1)\ \begin{array}{c|ccc} 1 & & & \mathbb{Z} \\ 0 & \mathbb{Z} & & \\ \hline & 0 & 1 & 2 \end{array} \qquad H_*(S^1)\ \begin{array}{c|ccc} 1 & \mathbb{Z}/n & & \mathbb{Z} \\ 0 & \mathbb{Z} & & \\ \hline & 0 & 1 & 2 \end{array}$$

$$\begin{array}{ccc} H_*(S^2) & H_*(S^2) & H_*(S^2) \\ n = 0 & n = \pm 1 & n \neq 0, \pm 1 \end{array}$$

We see that taking $n = \pm 1$ computes the correct answer for $H_*(S^3)$; we have a copy of $\mathbb{Z}$ in the $p + q = 0$ and $p + q = 3$ columns, as required.

2.5.4 Example. There is a fibration $SU(n-1) \to SU(n) \to S^{2n-1}$. Taking $n = 3$ and using $SU(2) \cong S^3$, we obtain a fibration $S^3 \to SU(3) \to S^5$. We have

$$E^2_{p,q} \cong \begin{cases} \mathbb{Z} & p = 0,5 \text{ and } q = 0,3 \\ 0 & \text{otherwise.} \end{cases}$$

The E^2-term is as follows:

$$
\begin{array}{c|cccccc}
 & & & & & & \\
3 & \mathbb{Z} & & & & & \mathbb{Z} \\
2 & & & & & & \\
1 & & & & & d_2 & \\
0 & \mathbb{Z} & & & & & \mathbb{Z} \\
\hline
 & 0 & 1 & 2 & 3 & 4 & 5
\end{array}
$$

Vertical axis: $H_*(S^3)$; horizontal axis: $H_*(S^5)$

Note that there are no differentials for degree reasons, as shown for d_2. Therefore, the spectral sequence collapses and we see that

$$H_i(SU(3)) \cong \begin{cases} \mathbb{Z}, & i = 0,3,5,8 \\ 0, & \text{otherwise.} \end{cases}$$

2.5.5 Example. We can continue the previous example and take $n = 4$ to get a fibration $SU(3) \to SU(4) \to S^7$. We can compute the E^2-term using the previous example

$$E^2_{p,q} = H_p(S^7; H_q(SU(3))) \cong \begin{cases} \mathbb{Z}, & p = 0,7, q = 0,3,5,8 \\ 0, & \text{otherwise.} \end{cases}$$

The E^2-term is as follows:

$$
\begin{array}{c|cccccccc}
8 & \mathbb{Z} & & & & & & & \mathbb{Z} \\
7 & & & & & & & & \\
6 & & & & & & & & \\
5 & \mathbb{Z} & & & & & & & \mathbb{Z} \\
4 & & & & & & & & \\
3 & \mathbb{Z} & & & & & & & \mathbb{Z} \\
2 & & & & & & & & \\
1 & & & & & & & & \\
0 & \mathbb{Z} & & & & & & & \mathbb{Z} \\
\hline
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7
\end{array}
$$

Vertical axis: $H_*(SU(3))$; horizontal axis: $H_*(S^7)$

Note that there are no differentials for degree reasons, and we compute

$$H_i(SU(4)) \cong \begin{cases} \mathbb{Z}, & i = 0,3,5,7,8,10,12,15 \\ 0, & \text{otherwise.} \end{cases}$$

2.5.6 Remark. If one tries the same argument for $SU(5)$ there are possible differentials. We will see later that it is easier to use cohomology, where one can use multiplicative structures to rule out differentials.

2.5.7 Remark (Naturality of the Serre spectral sequence). The Serre spectral sequence is natural in the following sense. Suppose we are given two fibrations satisfying the hypothesis of the Serre spectral sequence, and a map between them:

$$\begin{tikzcd} F \arrow[r, hook] \arrow[d] & E \arrow[r] \arrow[d, "\tilde{f}"] & B \arrow[d, "f"] \\ F' \arrow[r, hook] & E' \arrow[r] & B' \end{tikzcd}$$

Then the following hold:

1. There are induced maps $f_*^r\colon E^r_{p,q} \to' E^r_{p,q}$ commuting with differentials, i,e, the diagram

$$\begin{tikzcd} E^r_{p,q} \arrow[r, "d_r"] \arrow[d, "f_*^r"'] & E^r_{p-r,q+r-1} \arrow[d, "f_*^r"] \\ {}'E^r_{p,q} \arrow[r, "'d_r"'] & {}'E^r_{p-r,q+r-1} \end{tikzcd}$$

   commutes, and moreover f_*^{r+1} is the map induced on homology by f_*^r.

2. The map $\tilde{f}_*\colon H_*(E) \to H_*(E')$ preserves filtrations, inducing a map on associated graded which is exactly f_*^∞.

3. Under the isomorphisms $E^2_{p,q} \cong H_p(B; H_q(F))$ and ${}'E^2_{p,q} \cong H_p(B'; H_q(F'))$ the map f_*^2 corresponds to the map induced by the maps $B \to B'$ and $F \to F'$.

   Once again, we can demonstrate this with an example.

2.5.8 Example. We recall the Hopf fibration $S^1 \to S^3 \to S^2$. This factors through $\mathbb{R}P^3 = S^3/\{\pm 1\}$ as in the following diagram:

$$\begin{tikzcd} S^1 \arrow[r, hook] \arrow[d, two heads, "q"'] & S^3 \arrow[r] \arrow[d, two heads, "q"'] & S^2 \arrow[d, equal] \\ S^1/\{\pm 1\} \arrow[r, hook] & S^3/\{\pm 1\} \arrow[r] & S^2 \end{tikzcd}$$

We see that we have a fibration $S^1 \to \mathbb{R}P^3 \to S^2$. The ${}'E^2$-term of this spectral sequence is as for the Hopf fibration:

$$E^2_{p,q} = H_p(S^2; H_q(S^1)) \cong \begin{cases} \mathbb{Z} & p = 0,2 \text{ and } q = 0,1 \\ 0 & \text{otherwise.} \end{cases}$$

As in Example 2.5.3 there is only one possible differential, which is ${}'d_2\colon {}'E^2_{2,0} \to' E^2_{0,1}$, and this is given by multiplication by an integer n. We use naturality to determine what this is. We note that we have a commutative diagram

$$\begin{tikzcd} H_2(S^2; H_0(S^1)) \arrow[r, "d_2", "\cong"'] \arrow[d, "q_*"', "\cong"] & H_0(S^2; H_1(S^1)) \arrow[d, "\cdot 2"] \\ H_2(S^2; H_0(S^1/\{\pm 1\})) \arrow[r, "'d_2"'] & H_0(S^2; H_1(S^1/\{\pm 1\})) \end{tikzcd}$$

The right hand arrow is multiplication by 2 because the map induced on homology by $S^1 \to S^1/\{\pm 1\}$ has degree 2 (it is the attaching map for the top cell of $\mathbb{R}P^2$). Commutativity of the diagram implies that $'d_2$ is multiplication by 2. Therefore, the E^2 and $E^3 = E^\infty$-terms are as follows:

$$
H_*(S^1)\ \begin{array}{c|cccc}
2 & & & & \\
1 & \mathbb{Z} & \cdot 2 & \mathbb{Z} & \\
0 & \mathbb{Z} & & \mathbb{Z} & \\
\hline
 & 0 & 1 & 2 & 3
\end{array}
\qquad
H_*(S^1)\ \begin{array}{c|cccc}
2 & & & & \\
1 & \mathbb{Z}/2 & & \mathbb{Z} & \\
0 & \mathbb{Z} & & & \\
\hline
 & 0 & 1 & 2 & 3
\end{array}
$$

$$H_*(S^2) \qquad\qquad\qquad H_*(S^2)$$

We deduce that

$$
H_i(\mathbb{R}P^3) \cong \begin{cases} \mathbb{Z}, & i = 0, 3 \\ \mathbb{Z}/2, & i = 1 \\ 0, & \text{otherwise.} \end{cases}
$$

2.5.9 *Remark.* It is also possible to deduce some information about $H^*(F)$ or $H^*(B)$ in certain cases, as the following example demonstrates.

2.5.10 *Example.* There is a fibration $S^1 \to S^\infty \to \mathbb{C}P^\infty$. Note that $\pi_1(\mathbb{C}P^\infty) = 0$, so we can run the Serre spectral sequence. We have

$$
E^2_{p,q} = \begin{cases} H_p(\mathbb{C}P^\infty), & q = 0, 1 \\ 0, & \text{otherwise.} \end{cases}
$$

We also know that the spectral sequence converges to $H_{p+q}(S^\infty, \mathbb{Z})$, which is only non-zero when $p + q = 0$. In particular, the E^∞ page should be zero except for $E^\infty_{0,0}$. Now consider the E_2-page of the spectral sequence:

$$
H_*(S^1)\ \begin{array}{c|cccc}
2 & & & & \\
1 & \mathbb{Z} & H_1(\mathbb{C}P^\infty) & H_2(\mathbb{C}P^\infty) & H_3(\mathbb{C}P^\infty) \\
0 & \mathbb{Z} & H_1(\mathbb{C}P^\infty) & H_2(\mathbb{C}P^\infty) & H_3(\mathbb{C}P^\infty) \\
\hline
 & 0 & 1 & 2 & 3
\end{array}
$$

$$H_*(\mathbb{C}P^\infty)$$

Note that for degree reasons $E^2_{1,0} \cong H_1(\mathbb{C}P^\infty)$ survives the spectral sequences, and so must be 0. So the E^2-page is as follows:

$$
H_*(S^1)\qquad
\begin{tikzcd}
2 & & & & \\
1 & \mathbb{Z} & 0 & H_2(\mathbb{C}P^\infty) & H_3(\mathbb{C}P^\infty) \\
0 & \mathbb{Z} & 0 & H_2(\mathbb{C}P^\infty) \arrow[llu] & H_3(\mathbb{C}P^\infty) \arrow[llu] \\
 & 0 & 1 & 2 & 3
\end{tikzcd}
$$

$$H_*(\mathbb{C}P^\infty)$$

By the same argument $E^2_{3,0} \cong H_3(\mathbb{C}P^\infty)$ survives the spectral sequences, and so must be 0. Inductively, we deduce that $H_n(\mathbb{C}P^\infty) = 0$ for all n odd. Since $E^2_{0,1} \cong \mathbb{Z}$ must also die in the spectral sequence, we see that we must have $H_2(\mathbb{C}P^\infty) \cong \mathbb{Z}$, and that d^2 must be an isomorphism. Continuing inductively, we get

$$H_n(\mathbb{C}P^\infty) \cong \begin{cases} \mathbb{Z}, & n \text{ even} \\ \mathbb{Z}, & 0 \text{ odd.} \end{cases}$$

2.5.11 Example. In our next example, we compute $H_*(\Omega S^n)$ for $n > 1$. We use the path-space fibration of Remark 1.4.9. In this case, this takes the form

$$\Omega S^n \to PS^n \to S^n$$

where we recall that PS^n is contractible, i.e. $H_0(PS^n) = \mathbb{Z}$ and is zero otherwise. In particular, the only non-zero term on the E^∞-page of the spectral sequence is a copy of $\mathbb{Z}$ when $p + q = 0$. Now consider a small portion of the E_2-term:

$$
\begin{array}{c|cc}
3 & H_3(\Omega S^n) & H_3(\Omega S^n) \\
2 & H_2(\Omega S^n) & H_2(\Omega S^n) \\
1 & H_1(\Omega S^n) & H_1(\Omega S^n) \\
0 & \mathbb{Z} & \mathbb{Z} \\
\hline
 & 0 & n
\end{array}
$$

Note that the only possible differential is a d_n, and so goes $n-1$-terms upwards. We immediately see that $H_i(\Omega S^n) = 0$ for $0 < i < n-1$. Moreover, the only way to get rid of the $\mathbb{Z}$ in $E^2_{n,0} = E^n_{n,0}$ is that $H_{n-1}(\Omega S^n) \cong \mathbb{Z}$, and that d_n is an isomorphism. We can inductively repeat this argument, getting the following, where all the differentials shown are isomorphisms:

$$
\begin{tikzcd}[row sep=small]
3n-3 & H_{3n-3}(\Omega S^n) & H_{3n-3}(\Omega S^n) \\
2n-2 & H_{2n-2}(\Omega S^n) & H_{2n-2}(\Omega S^n) \arrow[ul] \\
n-1 & H_{n-1}(\Omega S^n) & H_{n-1}(\Omega S^n) \arrow[ul] \\
0 & \mathbb{Z} & \mathbb{Z} \arrow[ul] \\
 & 0 & n
\end{tikzcd}
$$

We conclude that

$$
H_i(\Omega S^n) \cong \begin{cases} \mathbb{Z}, & i = k(n-1) \\ 0, & \text{otherwise.} \end{cases}
$$

2.5.12 Remark. So far we have only considered examples where the extension problem is trivial; we have had at most one non-zero term in each diagonal on the E^∞-page. The following gives an example where this is not the case.

2.5.13 Example. Consider the Serre spectral sequence of the fibration

$$
S^1 \to U(2) \to \mathbb{R}P^3
$$

where we identify $S^1 \cong U(1)$ and the first map is given by

$$
\lambda \mapsto \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}
$$

The E^2-page is given by

$$
H_p(\mathbb{R}P^3; H_q(S^1)) \cong \begin{cases} \mathbb{Z}, & p = 0,3, q = 0,1 \\ \mathbb{Z}/2 & p = 1, q = 0,1 \\ 0 & \text{else.} \end{cases}
$$

The E^2-page looks as follows:

$$
\begin{tikzcd}[row sep=small]
 & 1 & \mathbb{Z} & \mathbb{Z}/2 & & \mathbb{Z} \\
H_*(S^1) & 0 & \mathbb{Z} & \mathbb{Z}/2 & & \mathbb{Z} \arrow[ull] \\
 & & 0 & 1 & 2 & 3 \\
 & & & H_*(\mathbb{R}P^3) & &
\end{tikzcd}
$$

We know (or take it as fact) that $H_2(U(2)) = 0$; the only way that this is compatible with the spectral sequence is if the differential shown is a surjection, and we get the following $E^3 = E^\infty$-page:

$$
\begin{array}{c c|c c c c}
\multirow{2}{*}{$H_*(S^1)$} & 1 & \mathbb{Z} & & & \mathbb{Z} \\
 & 0 & \mathbb{Z} & \mathbb{Z}/2 & & \\
\hline
 & & 0 & 1 & 2 & 3
\end{array}
$$

$H_*(\mathbb{R}P^3)$

Now, in fact we have that[8]

$$H_i(U(2)) \begin{cases} \mathbb{Z}, & i = 0, 1, 3, 4 \\ 0, & \text{else.} \end{cases}$$

Note that in the E^∞-page shown we have two non-zero terms in the $p + q = 1$ column, a $\mathbb{Z}$ in $(0, 1)$ and $\mathbb{Z}/2$ in $(1, 0)$. This means there is an extension[9]

$$0 \to \mathbb{Z} \to H_1(U(2)) \to \mathbb{Z}/2 \to 0.$$

From the calculations above we know that this extension must be non-trivial. Yet, if we didn't know another way to compute $H_1(U(2))$ we could not determine (without more information) if $H_1(U(2))$ was $\mathbb{Z}$ or $\mathbb{Z} \oplus \mathbb{Z}/2$.

2.5.14 *Remark.* We now return to the Hurweicz theorem, giving a second proof of Theorem 1.10.1.

2.5.15 Theorem. *If X is $(n-1)$-connected, $n \geq 2$, then $\widetilde{H}_i(X) = 0$ for $i \leq n - 1$ and $\pi_n(X) \cong H_n(X)$.*

Proof. We use the path-space fibration

$$\Omega X \to PX \to X,$$

and the fact that PX is contractible. The E^2-page of the Serre spectral sequence is

$$E^2_{p,q} = H_p(X; H_q(\Omega X)) \implies H_{p+q}(PX).$$

We prove the theorem by induction on n. When $n = 2$, we have $H_1(X) = 0$ because X is simply connected by assumption. Moreover, we have

$$\pi_2(X) \cong \pi_1(\Omega X) \cong H_1(\Omega X)$$

where the first isomorphism follows by the long exact sequence of the fibration, and the second follows from the fact that $\pi_1(\Omega X)$ is abelian, so that $H_1(\Omega X) \cong \pi_1(\Omega X)^{ab} \cong \pi_1(\Omega X)$. It remains to show that $H_1(\Omega X) \cong H_2(X)$. We will use the Serre spectral sequence to show this. Note that $E^2_{2,0} = H_2(X)$ and $E^2_{0,1} = H_1(\Omega X)$, so it suffices to show that

$$d^2 \colon E^2_{2,0} = H_2(X) \to E^2_{0,1} = H_1(\Omega X)$$

is an isomorphism. We consider then then a portion of the E^2-page:

[8] For example, note that $U(2) \cong SU(2) \times U(1)$

[9] In fact, the spectral sequence shows that we have filtered $H_1(U(2))$ as follows $0 \subseteq 2\mathbb{Z} \subseteq \mathbb{Z} \subseteq \mathbb{Z} = H_1(U(2))$.

$H_*(\Omega X)$

$$\begin{tikzcd} 1 & H_1(\Omega X) & & \\ 0 & \mathbb{Z} & H_1(X) & H_2(X) \arrow[ull] \\ & 0 & 1 & 2 \end{tikzcd}$$

$H_*(X)$

Note that if d_2 is not an isomorphism, then both of these groups will persist to the E^∞-page, giving a contradiction to the fact that PX is contractible. So, d_2 must be an isomorphism, as required. This gives the base case of the induction.

We now assume the statement of the theorem holds for $n-1$ and deduce it for n. Since X is $(n-1)$-connected, ΩX is $(n-2)$-connected, and so by the inductive hypothesis, we have that $\widetilde{H}_i(\Omega X) = 0$ for $i < n-1$ and $\pi_{n-1}(\Omega X) \cong H_{n-1}(\Omega X)$. In particular, we get isomorphisms

$$\pi_n(X) \cong \pi_{n-1}(\Omega X) \cong H_{n-1}(\Omega X),$$

and so it suffices to show that $H_{n-1}(\Omega X) \cong H_n(X)$. We do this via the Serre spectral sequence. We have

$$\begin{aligned} E^2_{p,q} &= H_p(X; H_q(\Omega)) \\ &\cong H_p(X) \otimes H_q(X) \oplus \operatorname{Tor}(H_{p-1}(X), H_q(\Omega X)) \\ 0 & \end{aligned}$$

for $0 < q < n-1$ by the inductive hypothesis. Now consider the Serre spectral sequence:

$H_*(\Omega X)$

$$\begin{tikzcd} H_{n-1}(\Omega X) & & & & & \\ 0 & & & & & \\ \vdots & & & & & \\ 0 & & & & & \\ \mathbb{Z} & H_1(X) & H_2(X) & \cdots & H_{n-1}(X) & H_n(X) \arrow[uuuulllll] \end{tikzcd}$$

The only differentials that interact with $H_n(X)$ and $H_{n-1}(\Omega X)$ is the d_n differential shown, and so this must be an isomorphism in order for these terms to die in the spectral sequence. Moreover, the terms $H_i(X)$ for $1 \leq i \leq n-1$ have no differentials at all in the spectral sequence; in particular, we must have $H_i(X) = 0$ for $1 \leq i \leq n-1$ and $d_n \colon H_n(X) \to H_{n-1}(\Omega X)$ is an isomorphism. $\square$

Exercise 23

Show, using the Serre spectral sequence, that if $S^k \to S^m \to S^n$ is a fibration with $n \geq 2$, then $k = n-1$ and $m = 2n-1$.

2.6 *The Serre spectral sequence in cohomology*

The Serre spectral sequence in cohomology looks much like the homology version:

2.6.1 Theorem (The Serre spectral sequence in cohomology)**.** *Let $\pi\colon E \to F$ be a fibration with fiber F and assume that $\pi_1(B) = 0$ and $\pi_0(F) = 0$, then there is a first quadrant spectral sequence*

$$E_2^{p,q} = H^p(B; H^q(F)) \implies H^{p+q}(E).$$

In particular, this means there is a filtration

$$H^n(E) = D^{0,n} \supseteq D^{1,n-1} \supseteq \ldots \supseteq D^{n,0} \supseteq D^{n+1,-1} = 0$$

such that $E_\infty^{p,q} = D^{p,q}/D^{p+1,q-1}$.

The differentials run $d_r^{p,q}\colon E_r^{p,q} \to E_r^{p+r,q+r-1}$.

2.6.2 Remark. Apart from the direction of differentials, this looks much like the Serre spectral sequence in homology. However, there is one major difference: each E_r page has a bilinear product, i.e., a map

$$\bullet\colon E_r^{p,q} \times E_r^{p',q'} \to E_r^{p+p',q+q'}$$

or equivalently,

$$E_r^{p,q} \otimes E_r^{p',q'} \to E_r^{p+p',q+q'}$$

satisfying the Leibniz rule

$$d_r(x \bullet y) = d_r(x) \bullet y + (-1)^{\deg(x)} x \bullet d_r(y).$$

where $\deg(x) = p+q$. Moreover, on the E_2-page, this product is induced by the cup product.

Once again, it is instructive to do an example.

2.6.3 Example. Consider the fibration

$$S^1 \to S^\infty \simeq * \to \mathbb{C}P^\infty$$

The E_2-page looks as follows

$$\begin{tikzcd}
 & 2 & & & & \\
H^*(S^1) & 1 & \mathbb{Z} \arrow[rrd] & H^1(\mathbb{C}P^\infty) \arrow[rrd] & H^2(\mathbb{C}P^\infty) & H^3(\mathbb{C}P^\infty) \\
 & 0 & \mathbb{Z} & H^1(\mathbb{C}P^\infty) & H^2(\mathbb{C}P^\infty) & H^3(\mathbb{C}P^\infty) \\
 & & 0 & 1 & 2 & 3 \\
 & & & H^*(\mathbb{C}P^\infty) & &
\end{tikzcd}$$

Running an argument similar to Example 2.5.10 it is not too hard to compute the additive structure: we must have $E_2^{2k+1,0} = 0$, and $d_2 \colon E_2^{p,1} \to E_2^{p+2,0}$ is an isomorphism $\mathbb{Z} \to \mathbb{Z}$. In particular, we have

$$H^i(\mathbb{C}P^\infty) \begin{cases} \mathbb{Z}, & i = \text{ even} \\ 0, & i = \text{ odd.} \end{cases}$$

Now we wish to compute the multiplicative structure. Let use note that by the universal coefficient theorem in cohomology[10] we have

$$E_2^{p,q} = H^p(\mathbb{C}P^\infty) \otimes H^q(S^1).$$

Let $\mathbb{Z} = \langle x \rangle = H^1(S^1)$ and let $\mathbb{Z} = \langle y \rangle = H^2(\mathbb{C}P^\infty)$, chosen so that $d_2(x) = y$. Then we have

$$E_2^{2,1} = H^2(\mathbb{C}P^\infty) \otimes H^1(S^1) = \mathbb{Z} \otimes \mathbb{Z} \cong \mathbb{Z}.$$

The pairing

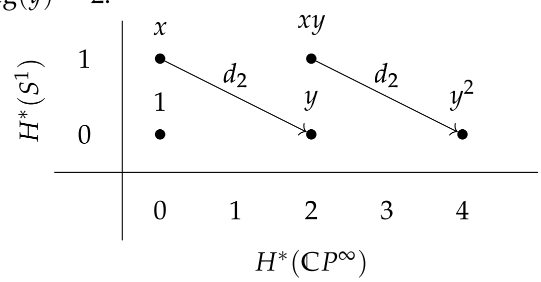
$$\bullet \colon E_2^{2,0} \times E_2^{0,1} \to E_2^{2,1}$$

is induced by the cup product, and unwinding the definitions, sends (x, y) to xy, i.e., xy generates $E_2^{2,1}$.

Let z be a generator of $H^4(\mathbb{C}P^\infty)$. We want to show that $z = y^2$. By the Leibniz rule,

$$d_2(xy) = d_2(x)y + (-1)^{\deg(x)} x d_2(y) = y^2.$$

Noting that d_2 is an isomorphism, we see that $d_2(xy) = y^2 = z$, as needed. Arguing inductively, we see that $d_2(xy^{n-1}) = y^n$ is a generator of $H^{2n}(\mathbb{C}P^\infty)$ and we deduce that $H^*(\mathbb{C}P^\infty) \cong \mathbb{Z}[y]$ with $\deg(y) = 2$.

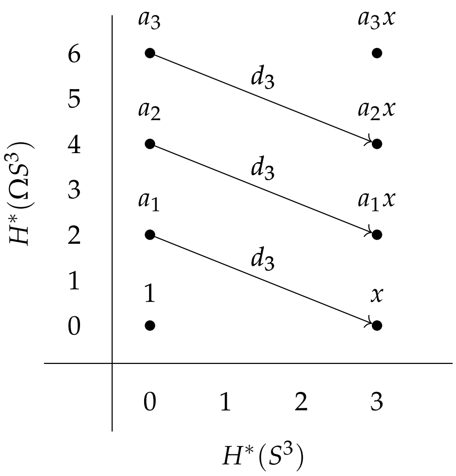


2.6.4 Example. We now consider the cohomology ring $H^*(\Omega S^3)$, leaving the general case of $H^*(\Omega S^n)$ as an exercise. To do this, we use the Serre spectral sequence of the fibration $\Omega S^3 \to PS^3 \simeq * \to S^3$. The additive structure can be determined much as in Example 2.5.11,[11] and the spectral sequence looks as follows:

[10] In case this was not covered or you need a reminder, this states there is a natural short exact sequence

$$0 \to H^n(X;\mathbb{Z}) \otimes M \to H^n(X;M) \to \mathrm{Tor}(H^{n+1}(X;\mathbb{Z}), M) \to 0.$$

[11] Convince yourself of this!



That is, additively, we have

$$H^i(\Omega S^3) \cong \begin{cases} \mathbb{Z}, & i = 2k \\ 0, \text{else.} \end{cases}$$

In order to work out the multiplicative structure, we need to work out how the classes a_i relate to each other. For example, is $a_1^2 = a_2$? We have chosen the generators such that $d_3(a_i) = a_{i-1}x$ where $a_0 = 1$. Now we use the Leibniz rule to see that[12]

[12] Here it is important that all our classes are in even total degrees!

$$d_3(a_1^2) = d_3(a_1)a_1 + a_1 d_3(a_1) = 2a_1x = d_3(2a_2).$$

Because d_3 is an isomorphism, we deduce that $a_1^2 = 2a_2$. What about a_3? Note that

$$\begin{aligned} d_3(a_1a_2) &= d_3(a_1)a_2 + a_1d_3(a_2) = xa_2 + a_1^2x \\ &= xa_2 + 2xa_2 = 3xa_2 \\ &= d_3(3a_3). \end{aligned}$$

Because d_3 is an isomorphism, we deduce that $a_1a_2 = 3a_3$. Said another way, $a_1^3 = a_1a_1^2 = 2a_1a_2 = 3 \cdot 2 \cdot a_3$. By an inductive argument, we deduce that $a_1^n = n!a_n$, where a_n generates $E_2^{0,2n}$. We see that $H^*(\Omega S^3) \cong \Gamma_{\mathbb{Z}}[a_1]$, the divided polynomial algebra on a class a_1 in degree 2.[13]

[13] In general, the divided polynomial algebra on a ring R, denoted $\Gamma_R[\alpha]$ where α has (even) degree n is the algebra with additive generators α_i in degree ni and multiplication $\alpha_1^k = k!\alpha_k$ (and hence $\alpha_i\alpha_j = \binom{i+j}{i}\alpha_{i_j}$). Note that if $R = \mathbb{Q}$, then $\Gamma_{\mathbb{Q}}[\alpha] \cong \mathbb{Q}[\alpha]$, but in general it is more complex. For example, if $R = \mathbb{F}_p$, then $\Gamma_{\mathbb{F}_p}[\alpha] \cong \otimes_{i\geq 0}\mathbb{F}_p[\alpha_{p_i}]/(\alpha_{p^i}^p)$, a tensor product of truncated polynomial rings.

You should now attempt the following exercise.[14]

[14] Here $\Lambda_{\mathbb{Z}}[x] \cong \mathbb{Z}[x]/(x^2)$ is the exterior algebra

Exercise 24

Use the cohomological Serre spectral sequence associated to the path fibration

$$\Omega S^n \to PS^n \to S^n$$

to show the following: If n is odd, then

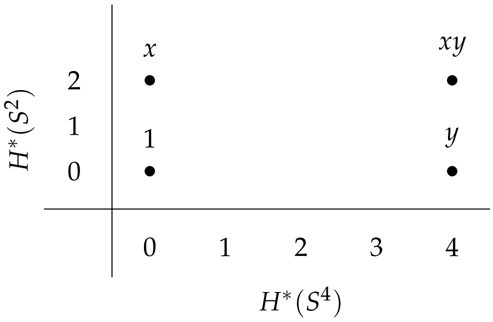
$$H^*(\Omega S^n) \cong \Gamma_{\mathbb{Z}}[x]$$

where $|x| = n-1$. If n is even, then

$$H^*(\Omega S^n) \cong \Lambda_{\mathbb{Z}}[x] \otimes \Gamma_{\mathbb{Z}}[y]$$

where $|x| = n-1$ and $|y| = 2n-2$.

2.6.5 Example. Much like in the additive case, we can sometimes have multiplicative extensions that we cannot solve without additional information. For example, there is a fibration $S^2 \to \mathbb{C}P^3 \to S^4$, and the associated spectral sequence looks as follows:



There is no room for differentials, and so

$$H^i(\mathbb{C}P^\infty) \cong \begin{cases} \mathbb{Z}, & i = 0,2,4,6 \\ 0, & \text{else.} \end{cases}$$

Yet from the spectral sequence, we cannot deduce (without further information) that $y = x_2^2$, which we know holds.[15]

2.6.6 Remark. A useful way to compute multiplicative extensions is the following theorem:[16] If there is a spectral sequence converging to H_* as an algebra and the E_∞-term is a free, graded-commutative, bigraded algebra, then H_* is a free, graded commutative algebra isomorphic to the total complex $E_\infty^{*,*}$, i.e.,

$$H_i \cong \bigoplus_{p+q=i} E_\infty^{p,q}.$$

2.6.7 Example. Recall that we have a fiber sequence

$$SU(n-1) \to SU(n) \to S^{2n-1}$$

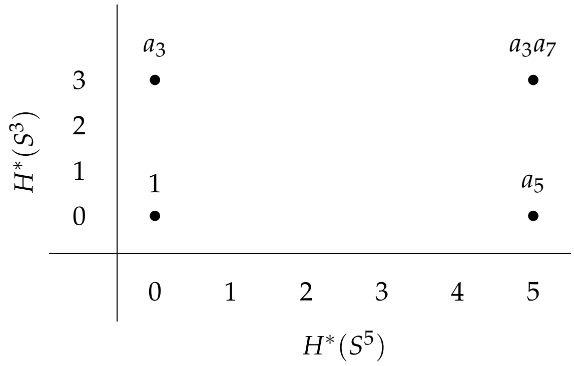
Taking $n = 3$, this has the form $S^3 \to SU(3) \to S^5$. The E^2-term is

$$E_2^{p,q} = H^p(S^5; H^q(S^3)) \cong H^p(S^5) \otimes H^q(S^3).$$

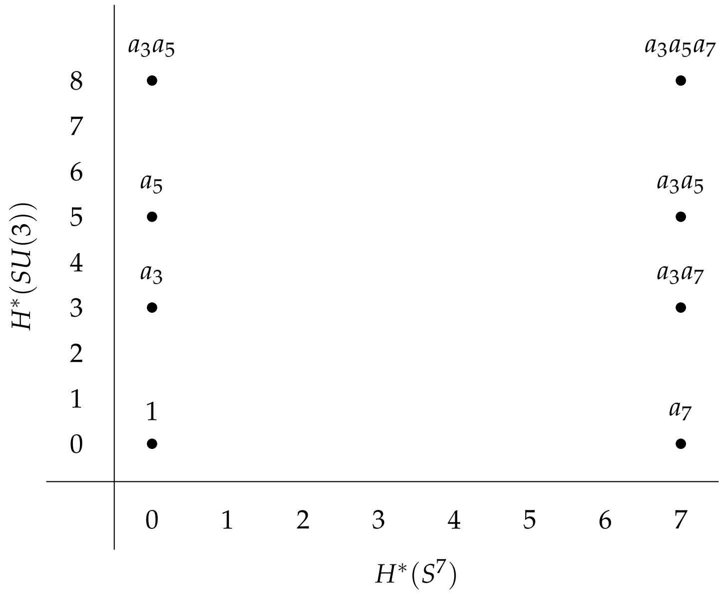
[15] Recall that $H^*(\mathbb{C}P^3) \cong \mathbb{Z}[x]/(x^4)$ for $|x| = 2$.

[16] See Example 1.K of McCleary's "A user's guide to spectral sequences"

The $E^2 = E^\infty$-page looks as follows:



Unlike in the previous example, there are no possible multiplicative extension problems, and we deduce that $H^*(SU(3)) \cong \Gamma_{\mathbb{Z}}(a_3, a_5)$, the exterior algebra on two generators a_3 and a_5. Now we consider the case $n = 4$. We leave it for the reader to deduce the following $E^2 = E^\infty$-page



The extension problem can be solved by Remark 2.6.6, and we get $H^*(SU(4)) \cong \Gamma_{\mathbb{Z}}(a_3, a_5, a_7)$. These two examples suggest a general pattern: is $H^*(SU(n)) \cong \Gamma_{\mathbb{Z}}(a_3, a_5, \ldots, a_{2n-1})$? The next exercise is to show that this is true.[17]

[17] However, unlike the previous two examples, there will be differentials to deal with. The hint is to use the multiplicative structure, in particular the Leibniz rule.

Exercise 25

Show, using induction, that

$$H^*(SU(n)) \cong \Gamma_{\mathbb{Z}}(a_3, a_5, \ldots, a_{2n-1}).$$

2.7 *The Gysin and Wang sequences*

The Gysin and Wang sequences are long exact sequences derived from special cases of the Serre spectral sequence. We will prove the following.

2.7.1 Theorem (The Gysin sequence)**.** *Let $S^n \to E \to B$ be a fibration with $\pi_1 B = 0$ and $n \geq 1$. Then, there exists an exact sequence*

$$\cdots H_r(E) \to H_r(B) \to H_{r-n-1}(B) \to H_{r-1}(E) \to \cdots$$

We begin with two algebraic lemmas, whose proof we leave as exercises for the reader.

2.7.2 Lemma. *Let $A \to B \xrightarrow{f} C$ and $D \to E \xrightarrow{g}$ be exact sequences of abelian groups. Suppose there exists an isomorphism $\phi\colon \operatorname{coker}(f) \cong \ker(g)$, then there is an exact sequence*

$$A \to B \xrightarrow{f} C \xrightarrow{\phi} D \to E \xrightarrow{g} F,$$

where $c \mapsto \phi(\bar{c})$, for $\bar{c}$ the class of c in $\operatorname{coker}(f)$.

2.7.3 Lemma. *Given the following diagram of abelian groups:*

$$\begin{tikzcd}
 & A \arrow[d, "f"] & & \\
 & B \arrow[d, "g"] \arrow[dr, "hg"] & & \\
0 \arrow[r] & C \arrow[r, "h"'] & D \arrow[r, "k"'] & E
\end{tikzcd}$$

with rows and columns exact, then the sequence $A \xrightarrow{f} B \xrightarrow{hg} D \xrightarrow{k} E$ is exact.

We now return to the Gysin sequence.

Proof of Theorem 2.7.1. We consider the Serre spectral sequence of the fibration. This has E_2-term

$$E^2_{p,q} \cong H_p(B; H_q(S^n)) \cong \begin{cases} H_p(B) & q = 0, n \\ 0 & \text{else.} \end{cases}$$

and so is as follows:

$$\begin{array}{c|cccc}
n & H_0(B) & H_1(B) & H_2(B) & H_3(B) \\
 & & & & \\
0 & H_0(B) & H_1(B) & H_2(B) & H_3(B) \\
\hline
 & & & &
\end{array}$$

$H_*(S^n)$

$H_*(B)$

We observe that there is only one possible differential, namely $d_{n+1}\colon E^{n+1}_{p,0} \to E^{n+1}_{p-n-1,n}$, and so $E^2 = E^{n+1}$ and $E^{n+2} = E^\infty$. Therefore, using Lemma 2.7.2 we get a short exact sequence

$$0 \to E^\infty_{p,0} \to E^{n+1}_{p,0} \xrightarrow{d_{n+1}} E^{n+1}_{p-n-1,n} \to E^\infty_{p-n-1,n} \to 0. \qquad (2.7.4)$$

The filtration on $H_i(E)$ is $0 \subseteq E^\infty_{i-n,n} = D_{i-n,n} \subseteq D_{i,0} = H_i(E)$, i.e., we have a short exact sequence:

$$0 \to E^\infty_{i-n,n} \to H_i(E) \to E^\infty_{i,0} \to 0. \tag{2.7.5}$$

Pasting (2.7.4) and (2.7.5) together we get a diagram of the form:

$$\begin{tikzcd}
 & H_r(E) \arrow[d] \arrow[dr, dashed] & & & 0 \arrow[d] & \\
0 \arrow[r] & E^\infty_{r,0} \arrow[r] \arrow[d] & \underbrace{E^2_{r,0}}_{=H_r(B)} \arrow[r, "d_{n+1}"] & \underbrace{E^2_{r-n-1,n}}_{H_{r-n-1}(B)} \arrow[r] \arrow[dr, dashed] & E^\infty_{r-n-1,n} \arrow[r] \arrow[d] & 0 \\
 & 0 & & & H_{r-1}(E) \arrow[d] \arrow[dr, dashed] & \\
 & & & 0 \arrow[r] & E^\infty_{r-1,0} \arrow[r] \arrow[d] & \cdots \\
 & & & & 0 &
\end{tikzcd}$$

and Lemma 2.7.3 implies that the sequence in red is exact. $\square$

2.7.6 Example. Consider the fiber sequence $S^1 \to S^{2n-1} \to \mathbb{C}P^n$ for $n \geq 1$. One recalls that $H_p(\mathbb{C}P^n) = 0$ for $p > 2n$ using, for example, cellular homology. We will show that

$$H_p(\mathbb{C}P^n) \cong \begin{cases} \mathbb{Z} & p \text{ even} \\ 0 & p \text{ odd} \end{cases}$$

using the Gysin sequence. The sequence tell us that

$$0 = H_{2n+2}(\mathbb{C}P^n) \to H_n(\mathbb{C}P^n) \to H_{2n+1}(S^{2n+1}) \cong \mathbb{Z} \to H_{2n+1}(\mathbb{C}P^n) = 0$$

is exact, and so $H_n(\mathbb{C}P^n) \cong \mathbb{Z}$. Next, observe that we have an exact sequence

$$0 = H_{2n}(S^{2n+1}) \to H_{2n}(\mathbb{C}P^n) \cong \mathbb{Z} \to H_{2n-2}(\mathbb{C}P^n) \to H_{2n-1}(S^{2n-1})$$

so that $H_{2n-2}(\mathbb{C}P^n) \cong \mathbb{Z}$. Moreover, the exact sequence

$$0 = H_{2n+1}(\mathbb{C}P^n) \to H_{2n-1}(\mathbb{C}P^n) \to H_{2n}(S^{2n+1}) = 0$$

shows that $H_{2n-1}(\mathbb{C}P^n) = 0$. Inductively continuing, we get the claimed result.

A

A nice category of topological spaces

A.1 The compact open topology

In this appendix we briefly discuss how to give the set of continuous maps between topological spaces X and Y a topology, such that the product is left adjoint to the Hom functor. To begin, we fix some notation.

A.1.1 Remark. Let X and Y be topological spaces. Let $M(X, Y)$ denote the *set* of continuous homomorphisms from X to Y. There is an evaluation map

$$e' \colon \mathrm{Hom}_{Sets}(X, Y) \times X \to Y$$

given by $e'(f, x) = f(x)$. This restricts to a function

$$e \colon M(X, Y) \times X \to Y.$$

A.1.2 Definition. A topology on $M(X, Y)$ is called admissible if e is continuous with respect to this topology.

A.1.3 Remark. It is possible that $M(X, Y)$ has no admissible topologies.

A.1.4 Definition. The compact-open topology on $M(X, Y)$ has as a sub-base the family of sets

$$U^K = \{f \in M(X, Y) \mid f(K) \subseteq U\}$$

where $K \subseteq U$ is compact and U is open in Y.

A.1.5 Proposition. *If X is a locally compact*[1] *Hausdorff space, the the compact-open topology on $M(X, Y)$ is admissible.*

[1] i.e., every point in X has a compact neighborhood

A.1.6 Remark. The compact-open topology is the coarsest admissible topology: for any admissible topology τ we have $\tau_{\mathrm{co}} \subseteq \tau$.

A.1.7 Remark. Suppose we have sets X, Y, X. Then there is an adjoint equivalence

$$\mathrm{Hom}_{Sets}(X \times Y, Z) \underset{\psi}{\overset{\phi}{\underset{\sim}{\rightleftarrows}}} \mathrm{Hom}_{Sets}(X, \mathrm{Hom}_{Sets})(Y, Z)$$

given by

$$\phi(f)(x)(y) = f(x, y) \quad \text{and} \quad \psi(g)(x, y) = g(x)(y).$$

A.1.8 Proposition. *If X, Y, X are topological spaces with Y Hausdorff, locally compact, then*

$$\phi\colon M(X \times Y, Z) \xrightarrow{\cong} M(X, M(Y, Z))$$

is an isomorphism of sets. If X is Hausdorff, then it is a homeomorphism (using the compact-open topology).