1
Homotopy theory

1.1 Review of basics on homotopy theory

We begin with a recollection of some facts that have been covered in Algebraic Topology I and Introduction to Topology.

1.1.1 Notation. We let $I = [0,1]$ denote the unit interval. For a pointed topological space X we will denote the basepoint by x_0 or $*$.

We recall the following definition.

1.1.2 Definition. A homotopy between $f, g\colon X \to Y$ is a continuous function $H\colon X \times I \to Y$ such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$ and $H(x_0,t) = y_0$ for all $t \in I$. We will write $f \simeq g$, or $f \simeq_H g$, if we need to make the choice of homotopy clear.

For a subspace $A \subseteq X$, a relative homotopy is a homotopy with $H(a,t) = f(a) = g(a)$ for all $a \in A, t \in I$.

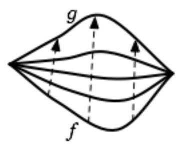


Figure 1.1: A homotopy between f and g.

1.1.3 Remark. Equivalently, we can specify a family of continuous maps $h_t\colon X \to Y$ such that $h_0 = f, h_1 = g$ and

$$\begin{aligned} H\colon X \times I &\to Y \\ (x,t) &\mapsto h_t(x) \end{aligned}$$

is continuous. We will switch between the two equivalent definitions without comment, using whatever is more convenient.

1.1.4 Proposition. *For all spaces X and Y, homotopy is an equivalence relation on the set of maps from X to Y. Furthermore, if we are given $k\colon A \to X, \ell\colon Y \to B$ and homotopic maps $f \simeq g\colon X \to Y$, then $fk \simeq gk\colon A \to Y$ and $\ell f \simeq \ell g\colon X \to B$.*

Proof. Let $f, g\colon X \to Y$, then

1. $f \simeq_F f$ via $F(x,t) = f(x)$ for all $x \in X, t \in I$.
2. If $f \simeq_F g$, then $g \simeq_G f$ where $G(x,t) = F(x, 1-t)$.
3. If $f \simeq_F g$ and $g \simeq_G h$, then $f \simeq_H h$ via

$$H(x,t) = \begin{cases} F(x,2t) & \text{if } 0 \le t \le 1/2 \\ G(x,2t-1) & \text{if } 1/2 \le t \le 1. \end{cases}$$

For the last part of the proposition let f_t be a homotopy between f and g, then $f_t k$ and ℓf_t give the required homotopy. $\square$

1.1.5 Definition. For a map $f\colon X \to Y$, we let $[f]$ denote the equivalence class containing f. The collection of all homotopy classes of maps from X to Y is denoted $[X, Y]$.[1]

[1] If our spaces are based, then these should be homotopy classes of *based* maps.

1.1.6 Remark. Note that if $\alpha = [f] \in [Y, Z]$ and $\beta = [g] \in [X, Y]$, then $\alpha\beta = [f \circ g] \in [X, Z]$, i.e., we can form the category $hTop_*$ whose objects are topological spaces, and whose morphisms are homotopy classes of maps.

1.1.7 Remark. We now very quickly review a number of standard topological constructions.

- Let X be a space and $A \subseteq X$. A map $r\colon X \to A$ such that $ri(a) = a$ for all $a \in A$ is called a retraction of X onto A, and A is called a retract of X.
- Let $i\colon A \hookrightarrow X$ be the inclusion, so that $ri = \mathrm{id}_A$. If $ir \simeq \mathrm{id}_X$, we call this a deformation retraction, and say that A is a deformation retract of X.
- If $f\colon X \to Y$, then a section of f is a map $s\colon Y \to X$ such that $f \circ s = \mathrm{id}_Y$. We can also ask for a *homotopy* section by requiring only that $f \circ s \simeq \mathrm{id}_Y$.

1.1.8 Definition. A map $f\colon X \to Y$ is called null-homotopic if $f\colon c_y\colon X \to Y$ where $c_y X \to Y$ is the constant map sending all of X to the point $y \in Y$. A homotopy between f and c_y is called a null-homotopy. A space X is contractible if id_X is null-homotopic.

1.1.9 Definition. Let (X, x_0) be a based topological space and $X \times I$ the cylinder on X. The quotient

$$CX = (X \times I)/(X \times \{1\} \cup \{x_0\} \times I)$$

with the base-point the equivalence class of $(x_0, 1)$ is called the (reduced) cone on X. Note that we have a natural inclusion $X \to CX$ of based maps given by $x \mapsto [x, 0]$.

1.1.10 Lemma. *The cone CX is contractible.*

Proof. Define $F\colon CX \times I \to CX$ by

$$F([x,t], s) = [x, s + (1-s)t].$$

Note then that we have

$$F([x,t], 0) = [x,t] \quad \text{and} \quad F([x,t],1) = [x,1]. \qquad \square$$

1.1.11 Lemma. *The following are equivalent:*

(i) $f\colon X \to Y$ *is null-homotopic.*

(ii) f *can be extended to* CX*:*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, hook, "i"'] & Y \\ CX \arrow[ur, dashed, "\exists \tilde{f}"'] & \end{tikzcd}$$

Proof. $(i) \implies (ii)$: Suppose f is null-homotopic, so $f \simeq_F *$. Then $F(X \times \{1\} \cup \{*\} \times I) = *$, so by the universal property of the quotient, we can find $\tilde{F}\colon CX \to Y$ such that $\tilde{f} \circ i = f$.

$(ii) \implies (i)$: Suppose $\tilde{f} \circ i = f$, then because CX is contractible (Lemma 1.1.10), we have $f = \tilde{f} \circ \mathrm{id}_{CX} \circ i \simeq \tilde{f} \circ (*_{CX}) \circ i \simeq *$, so that f is null-homotopic. □

1.1.12 Definition. A map $f\colon X \to Y$ is a homotopy equivalence if there exists $g\colon Y \to X$ such that $fg \simeq \mathrm{id}_Y$ and $gf \simeq \mathrm{id}_X$. We write $X \simeq Y$.

1.1.13 Example. (i) X is contractible if and only if $X \simeq *$.

(ii) If $i\colon A \hookrightarrow X$, and $r\colon X \to A$ is a deformation retract, then i and r are homotopy equivalences, and $A \simeq X$.

1.2 *Higher homotopy groups*

1.2.1 Notation. We will let $I_n = I^{\times n}, \partial I^n$ be the boundary of I^n, and write $[-,-]$ for homotopy classes of maps (if our spaces are based, these fix the base point).

1.2.2 Definition. For each $n \geq 0$ and X a topological space with $x_0 \in X$, we define

$$\pi_n(X, x_0) = [(I^n, \partial I^n), (X, x_0)].$$

1.2.3 Remark. (i) When $n = 0$, we have $I^0 = \mathrm{pt}$ and $\partial I^0 = \emptyset$, therefore $\pi_0(X)$ is the set of path components of X.

(ii) When $n = 1$, this is a group, but need not be abelian (for example, consider the wedge of two circles).

(iii) Note that $I^n/\partial I^n \simeq S^n$ and $\partial I^n/\partial I^n \simeq s_0$. By the universal property of the quotient map, we see that

$$\pi_n(X, x_0) \cong [(S_n, s_0), (X, x_0)].$$

1.2.4 Definition. A maps of pairs $(X, A) \to (Y, B)$ is a map $f\colon X \to Y$ with $f(A) \subseteq B$, i.e., the diagram:

$$\begin{tikzcd} A \arrow[r] \arrow[d, hook] & B \arrow[d, hook] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

commutes.

1.2.5 Proposition. *If $n \geq 1$, then $\pi_n(X, x_0)$ is a group with respect to the operation*

$$(f+g)(t_1, \dots, t_n) = \begin{cases} f(2t_1, t_2, \dots, t_n) & 0 \leq t_1 \leq 1/2 \\ g(2t_1 - 1, t_2, \dots, t_n) & 1/2 \leq t_1 \leq 1. \end{cases}$$

Proof. The identity is given by the constant map taking all of I^n to x_0 and the inverse of f is given by

$$-f(t_1, \dots, t_n) = f(1 - t_1, t_2, \dots, t_n). \qquad \square$$

1.2.6 Remark. Call the group operation $+_1$. Note that we can also define an operation $+_i$ for $1 \leq i \leq n$ by the same formula on the i-th coordinate.

1.2.7 Theorem. *All of these operations agree, and for $n \geq 2$, these give $\pi_n(X, x_0)$ the structure of an abelian group.*

This is a consequence of the following exercise, known as the Eckmann–Hilton lemma.

Exercise 1: Eckmann–Hilton lemma

Let M be a set and let $*, \bullet$ be two binary operations on M, $*, \bullet \colon M \times M \to M$, both with unit elements. Suppose that

$$(a * b) \bullet (c * d) = (a \bullet c) * (b \bullet d)$$

for all $a, b, c, d \in M$. Show that the units agree, these two operations agree, and that the multiplication is commutative and associative.

1.2.8 Remark. Let use show that

$$(f +_1 g) +_2 (h +_1 i) \simeq (f +_2 h) +_1 (g +_2 i).$$

Indeed, both of these are the following map

$$(t_1, t_2, \dots,) \mapsto \begin{cases} f(2t_1, 2t_2, \dots,) & [1/2, 0] \times [1/2, 0] \\ g(2t_1 - 2, 2t_2, \dots,) & [1/2, 1] \times [0, 1/2] \\ h(2t_1, 2t_2 - 2, \dots) & [0, 1/2] \times [1/2, 1] \\ i(2t_1 - 1, 2t_2 - 2, \dots) & [1/2, 1] \times [1/2, 1]. \end{cases}$$

1.2.9 Remark. Another approach is given by the following visualization: That is, so long as $n \geq 2$, we can shrink the domain of f and g

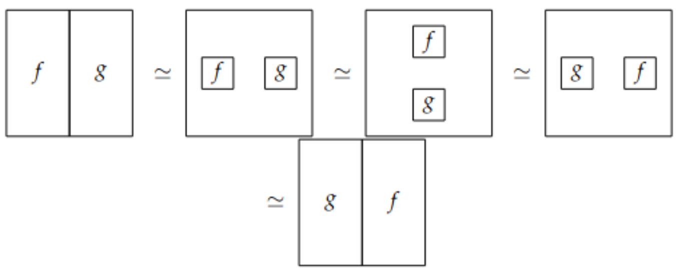


Figure 1.2: $f + g \simeq g + f$.

to smaller cubes (mapping the remaining region to the base point), slide f and g past each other, and then increase the domains back again.

Exercise 2

Let G be a topological group with identity element e, then $\pi_1(G, e)$ is abelian.
Hint: Use Eckmann–Hilton, or note the following: A topological group is a group object in the category of topological spaces. What is a group object in the category of groups?

1.2.10 Proposition. *If $n \geq 1$ and X is path connected, then there is an isomorphism $\beta_\gamma : \pi_n(X, x_0) \xrightarrow{\simeq} \pi_n(X, x_0)$ given by $\beta_\gamma([f]) = [\gamma \circ f]$ where γ is a path in X from x_1 to x_0 and $\gamma \circ f$ is constructed by first shrinking the domain of f to a smaller cube inside of I^n, and then inserting the path γ radially from x_1 to x_0 on the boundaries of these cubes.*

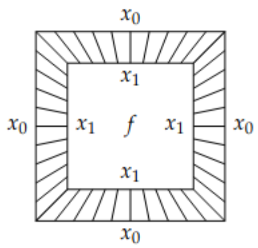
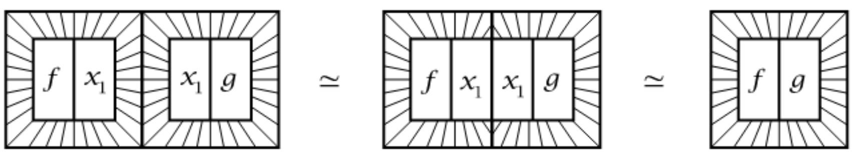


Figure 1.3: β_γ.

Proof. Observe the following:

1. $\gamma \circ (f + g) \simeq \gamma \circ f + \gamma \circ g$, i.e., β_γ is a group homomorphism.
2. $(\gamma \circ \eta) \circ f \simeq \gamma \circ (\eta \circ f)$, for η a path from x_0 to x_1.
3. $c_{x_0} \circ f \simeq f$, where c_{x_0} denotes the constant path based at x_0.
4. β_γ is well-defined with respect to homotopies of f or γ.

The only point that is perhaps not clear is (i). For this, we deform f and g to be constant on the right and left halves of I^n, respectively, producing maps we call $f + 0$ and $0 + g$. We then excise a wider symmetric middle slab of $\gamma(f + 0)$ and $\gamma(0 + g)$ until it becomes $\gamma(f + g)$: □



1.2.11 Remark. Therefore if X is path-connected, different choices of base point x_0 yield isomorphic groups $\pi_n(X, x_0)$, which may then simply be written as $\pi_n(X)$.

1.2.12 Lemma. *If $\{X_\alpha\}$ is a collection of path-connected spaces, then $\pi_n(\prod_\alpha X_\alpha) \cong \prod_\alpha \pi_n(X_\alpha)$.*

Proof. Note that $\mathrm{Hom}(Y, \prod_\alpha X_\alpha) \simeq \prod_\alpha \mathrm{Hom}(Y, X_\alpha)$. In particular, a map $S^n \to \mathrm{Hom}(Y, \prod_\alpha X_\alpha)$ is determined by a collection of maps $S^n \to X_\alpha$. Likewise, a homotopy $S^n \times I \to \prod_\alpha X_\alpha$ is determined by a colletion of homotopies $S^n \times I \to X_\alpha$. This implies the result. $\square$

1.2.13 Proposition. *Homotopy groups are functorial: given a map $\phi\colon X \to Y$ we get group homomorphisms $\phi_*\colon \pi_n(X, x_0) \to \pi_n(X, \phi(x_0))$ given by $[f] \mapsto [\phi \circ f]$ for all $n \geq 1$.*

Proof. We have the following:

1. ϕ_* is well-defined: if $f \simeq g$ via ψ_t, then $\phi \circ \psi_t$ defines a homotopy between $\phi \circ f$ and $\phi \circ g$.
2. This is a group homomorphism: $\phi \circ (f + g) \simeq \phi \circ g + \phi \circ g$ by the definition of the addition operation. Therefore.

$$\phi_*[f + g] = \phi_*[f] + \phi_*[g].$$

$\square$

Exercise 3

If $\phi\colon X \to Y$ is homotopy equivalence (not necessarily base-point preserving), then $\pi_*\colon \pi_n(X, x_0) \to \pi_n(Y, \phi(y_0))$ is an isomorphism.

1.2.14 Remark. We recall the following lifting property: Suppose $p\colon (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ is a covering, and there is a map $f\colon (Y, y_0) \to (X, x_0)$ with Y path-connected and locally path-connected. Then a lift $\tilde{f}$ exists if and only if $f_*\pi_1(Y, y_0) \subseteq p_*\pi_1(\tilde{X}, \tilde{x}_0)$.

$$\begin{tikzcd} & (\tilde{X}, \tilde{x}_0) \arrow[d, "p"] \\ (Y, y_0) \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"] & (X, x_0) \end{tikzcd}$$

1.2.15 Proposition. *If p is a covering, then $p_*\colon \pi_n(\tilde{X}, \tilde{x}_0) \to \pi_n(X, x_0)$ is an isomorphism for all $n \geq 2$.*

Proof. Let us first show surjectivity. To that end, suppose we have a map $f\colon (S^n, s_0) \to (X, x_0)$ where $n \geq 2$. The assumption on n gives $\pi_1(S^n) = 0$, so $f_*\pi_1(S^n, s_0) \subseteq p_*\pi_1(\tilde{X}, \tilde{x}_0)$ holds. We therefore find a lift in the following:

$$\begin{tikzcd} & (\tilde{X}, \tilde{x}_0) \arrow[d, "p"] \\ (S^n, s_0) \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"] & (X, x_0) \end{tikzcd}$$

Then $p_*[\tilde{f}] = [f]$, and p_* is surjective.

To see that p_* is injective, let $[\tilde{f}] \in \ker(p_*)$, i.e., $p_*[\tilde{f}] = [p \circ \tilde{f}] = 0$. Let $f = p \circ \tilde{f}$, then this is homotopic to the constant map $f \simeq c_{x_0}$

via a homotopy $\phi_t\colon (S^n, s_0) \to (X, x_0)$ with $\phi_1 = f$ and $\phi_0 = c_{x_0}$. By the same argument as above, the homotopy ϕ_t can be lifted to $\tilde{\phi}_t$. This satisfies $p \circ \tilde{\phi}_1 \simeq \phi_1$ and $p \circ \tilde{\phi}_0 \simeq \phi_0$. By the uniqueness of lifts, we must have $\tilde{\phi}_1 \simeq \tilde{f}$ and $\tilde{\phi}_0 \simeq c_{x_0}$. In other words, $\tilde{\phi}_t$ gives a homotopy between $\tilde{f}$ and c_{x_0}, so that $[\tilde{f}] = 0$, and p_* is injective. □

1.2.16 Example. S^1 has universal cover $p\colon \mathbb{R} \to S^1$, $p(t) = e^{2\pi i t}$. Then $\pi_n(S^1) \cong \pi_n(\mathbb{R}) \cong 0$ for $n \geq 2$.

Exercise 4

Find two spaces X, Y with $\pi_n X \cong \pi_n Y$ but $X \not\simeq Y$.
Hint: What is the universal cover of $\mathbb{R}P^n$?

1.2.17 Remark (Relative homotopy groups). Suppose we have (X, x_0) and a subspace A containing x_0. We note that $i_*\colon \pi_n(A, x_0) \to \pi_n(X, x_0)$ is not injective in general (example, take S^1 into $\mathbb{R}^2$). An element in the kernel of i_* is a map $f\colon (I^n, \partial I^n) \to (A, x_0)$ such that

$$(I^n, \partial I^n) \xrightarrow{f} (A, x_0) \xrightarrow{i} (X, x_0)$$

is homotopic to c_{x_0}. This means there exists a homotopy

$$H\colon I^n \times I \to X$$

such that $H(-,1) = f, H(-,0) = c_{x_0}$ and $H|_{\partial I^n \times I} = c_{x_0}$.

If we define $J^n = I^n \times \{0\} \cup \partial I^n \times I \subseteq I^n \times I$, then this is a map of triples

$$H\colon (I^{n+1}, \partial I^{n+1}, J^n) \to (X, A, x_0).$$

1.2.18 Definition.

$$\pi_n(X, A, x_0) = [(I^n, \partial I^n, J^{n-1}), (X, A, x_0)]$$

1.2.19 Remark. Equivalently,

$$\pi_n(X, A, x_0) = [(D^n, S^{n-1}, s_0), (X, A, x_0)]$$

1.2.20 Proposition. *If $n \geq 2$, then $\pi_n(X, A, x_0)$ is a group, and if $n \geq 3$, then it is abelian.*

For all $n \geq 2$, a map of pairs $\phi\colon (X, A, x_0) \to (Y, B, y_0)$ induces homomorphisms $\phi_*\colon \pi_n(X, A, x_0) \to \pi_n(Y, B, y_0)$ for all $n \geq 2$.

Proof. This is similar to the case of $\pi_n(X)$ itself, and the details are left to the reader. □

1.2.21 Theorem. *The relative homotopy groups (X, A, x_0) fit into a long exact sequence*

$$\cdots \to \pi_n(A, x_0) \xrightarrow{i_*} \pi_n(X, x_0) \xrightarrow{j_*} \pi_n(X, A, x_0) \xrightarrow{\partial_n} \pi_{n-1}(A, x_0) \to \cdots$$

where the map ∂_n is defined by $\partial_n([f]) = [f|_{I^{n-1}}]$.

The proof relies on the following.

1.2.22 Lemma (Compression criterion). *A map $f\colon (D^n, S^{n-1}, x_0) \to (X, A, x_0)$ represents 0 in $\pi_n(X, A, x_0)$ if and only if $f \sim g$ rel S^{n-1}, where g is a map whose image is contained entirely in A.*

Proof. Suppose $[f] = [g]$ with g as in the statement of the lemma. Note that there is a deformation of D^n onto x_0, and so $[f] = [g] = 0$ in $\pi_n(X, A, x_0)$.

Conversely, suppose that $[f]$ represents 0 in $\pi_n(X, A, x_0)$. This means there exists a homotopy, relative to S^{n-1}, $F\colon D^n \times I \to X$ with $F|_{D^n \times \{0\}} = f$, $F|_{D^n \times 1} = c_{x_0}$ and $F|_{S^{n-1} \times I} \subseteq A$. We can restrict F to a family of n-disks in $D^n \times I$ starting with $D^n \times \{0\}$ and ending with the disk $D^n \times \{1\} \cup S^{n-1} \times \{1\}$, all the disks in the family having the same boundary, then we get a homotopy from f to a map in A, stationary on S^{n-1} (said in other words, we can deformation retract $D^n \times [0,1]$ onto $D^n \times \{1\} \cup S^{n-1} \times I$). □

We now prove the existence of the long exact sequence.[2]

[2] This is the type of proof that is best done by the reader themselves.

Proof of Theorem 1.2.21. **Step 1.** Let us first show exactness at $\pi_n(X, x_0)$.

We first show $\mathrm{im}(i_*) \subseteq \ker(j_*)$. Note that $j_* i_*$ is induced by the composition $j \circ i$ and that these are both inclusion maps. Therefore, for $[f] \in \pi_n(A, x_0)$ we have $j_* i_* [f] = [j \circ i \circ f]$, but this has image contained in A, and so $j_* i_* [f] = 0$. This shows $\mathrm{im}(i_*) \subseteq \ker(j_*)$.

To see the converse (namely, $\ker(j_*) \subseteq \mathrm{im}(i_*)$) let $[f] \in \ker(j_*)$, i.e. $[j \circ f] = 0$. Note that again j is an inclusion map, and by the compression criteria $f \simeq g'$ relative to S^{n-1}, where g' has image contained in A. Since $x_0 \in S^{n-1}$, the homotopy fixes the basepoint, i.e, $[f] = [g'] \in \pi_n(X, x_0)$. But because g' has image in A, $[g'] \in \pi_n(A, x_0)$ and $i_*[g'] = [i \circ g'] = [f]$, so $[f] \in \mathrm{im}(i_*)$.

Step 2. Let us now show exactness at $\pi_n(X, A, x_0)$.

Note that the composite $\partial \circ j_* = 0$ since the restriction of a map $(I^n, \partial I^n, J^{n-1}) \to (X, x_0, x_0)$ to I^{n-1} has image x_0 and so represents 0 in $\pi_{n-1}(A, x_0)$. Therefore, $\mathrm{im}(j_*) \subseteq \ker(\partial)$. For the converse, suppose $[f] \in \ker(\partial)$. This means there exists a basepoint preserving homotopy $H\colon I^{n-1} \times I \to A$ (relative to ∂I^{n-1}) from $f|_{I^{n-1} \times \{0\}}$ to the constant map where the image of H is contained entirely in A. We can then define another homotopy H, such that $G_0 = f$, $G_t|_{I^{n-1}} = H_t$ and the rest of the image of G_t is $f[I^n]$ union with the image of H_s for $0 \leq s \leq t$. This homotopy maps S^{n-1} into A at all times, so $[f] = [G_1]$. Moreover, G_1 maps the boundary of I^n to x_0, so $[G_1] \in \pi_n(X, x_0)$. Altogether,

$$j_*[G_1] = [j \circ G_1] = [G_1] = [f]$$

so $\ker(\partial) \subseteq \mathrm{im}(j_*)$.

Step 3: Exactness at $\pi_n(A, x_0)$.

Let $[f] \in \pi_n(X, A, x_0)$ then $i_* \partial \in \pi_{n-1}(X, x_0)$ is the class represented by $f|_{I^{n-1}}$ and this is homotopic relative J^{n-2} to the constant map to x_0, via f viewed as a homotopy. So this implies $\mathrm{im}(\partial_*) \subseteq \ker(i_*)$. Conversely, let $[f] \in \ker(i_*)$ i.e., $i_*[f] = [i \circ f] = 0$.

Therefore, there exists a homotopy H between f and a constant map through a homotopy that has image in X and preserves x_0. Since $H_0 = f$ has image in A and H_1 has image $\{x_0\}$, and H_0 takes the boundary to $\{x_0\}$, we see that $[H] \in \pi_n(X, A, x_0)$, and moreover $\partial([H]) \simeq f$. Therefore, $[f] \in \operatorname{im}(\partial)$, and $\operatorname{im}(\partial) = \ker(i_*)$. □

1.2.23 Definition. A pair (X, A) with basepoint x_0 is said to be n-connected if $\pi_i(X, A) = 0$ for all $i \leq n$.

1.2.24 Lemma. *A pair (X, A) is n-connected if and only if $\pi_i(A) \xrightarrow{i_*} \pi_i(X)$ is an isomorphism for $i < n$ and a surjection for $i = n$.*

Proof. Use the long exact sequence in homotopy. □

Exercise 5

Let X be a path-connected space, and CX the cone on X. Show that

$$\pi_n(CX, X, X_0) \cong \pi_{n-1}(X, x_0)$$

for $n \geq 1$.

1.3 Cofibrations and the homotopy extension property

1.3.1 Definition. Let $\mathcal{C}$ be a class of topological spaces. A map $i\colon A \to X$ has the homotopy extension property (HEP) if, for every $Y \in \mathcal{C}$, the following extension property has a solution[3]

$$\begin{tikzcd} A \arrow[r, hook, "i_0"] \arrow[d, "i"'] & A \times I \arrow[d, "i\times\mathrm{id}"] \arrow[ddr, bend left, "H"] & \\ X \arrow[r, hook, "i_0"'] \arrow[drr, bend right, "f"'] & X \times I \arrow[dr, dashed, "\exists\tilde{H}"] & \\ & & Y \end{tikzcd}$$

[3] Here $i_0(x) = (x, 0)$.

A map $f\colon A \to X$ is a cofibration if it has the HEP with respect to all spaces Y.[4]

[4] We will see later that cofibrations are always inclusions, and, if X is Hausdorff, are always closed maps.

1.3.2 Remark. Note that we do not ask that $\tilde{H}$ is unique.

1.3.3 Remark. If we are in a 'nice' category of topological spaces (see CREF), which we always assume, then we have an adjunction

$$\operatorname{Hom}(X, \operatorname{Hom}(Y, Z)) \cong \operatorname{Hom}(X \otimes Y, Z)$$

of topological spaces, where $\operatorname{Hom}(Y, Z)$ is given the compact open topology. Writing, $Z^Y := \operatorname{Hom}(Y, Z)$, the homotopy extension property admits a reformulation in the following diagram

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "i"'] & Y^I \arrow[d, "p"] \\ X \arrow[r, "f"'] \arrow[ur, dashed, "\exists\tilde{h}"] & Y \end{tikzcd}$$

where $p\colon Y^I \to Y$ is the evaluation at 0 map. It is often easier to work with this equivalent diagram.

Exercise 6

Let (X, A) have the HEP, and assume moreover that $i\colon A \to X$ is a retract up to homotopy. Show that A is a retract of X.

1.3.4 Lemma. *Let $J = [0, 1]$.*

(i) The inclusion $i_0\colon X \to X \times J$ has the homotopy extension property for all Y.

(ii) The inclusion $i_0\colon X \to CX$ has the homotopy extension property for all Y.

Proof. The proof in both cases is very similar; we do the first case in some detail. We are claiming there exists a lift $\tilde{H}$ in the following diagram:

$$\begin{tikzcd}
X \arrow[r, hook, "i_0"] \arrow[d, "i"'] & X \times I \arrow[d, "i \times \mathrm{id}"] \arrow[ddr, bend left, "H"] & \\
X \times J \arrow[r, hook, "i_0"'] \arrow[drr, bend right, "f"'] & X \times J \times I \arrow[dr, dashed, "\exists \tilde{H}"] & \\
 & & Y
\end{tikzcd}$$

Geometrically, we will do this in two parts: we will define a map that "stacks" the two intervals on top of each other, i.e., we construct a map $G\colon X \times J \times I \to X \times [0, 2]$. We will then do H on one part of the cylinder, and f on the remaining part.

For the first part, let $G\colon X \times J \times I \to X \times [0, 2]$ be defined as[5]

$$G(x, t, s) = (x, t(1 + s)).$$

We then define $F\colon X \times [0, 2] \to Y$ by

$$F(x, k) = \begin{cases} f(x, k) & 0 \le k \le 1 \\ H(x, k/2) & 1 \le k \le 2. \end{cases}$$

Putting these together and defining $\tilde{H} := F \circ G$, we see that[6]

$$\tilde{H}((x, t), s) = \begin{cases} f(x, 1 - (1 - t)(1 + s)), & (1 - t)(1 + s) \le 1 \\ H(x, (1 - t)(1 + s) - 1), & (1 - t)(1 + s) \ge 1. \end{cases}$$

One verifies directly that this gives the required extension. $\square$

[5] To see what is going on it is worth testing some cases and drawing pictures. For example, when $t = 0$ we have $G(x, 0, s) = (x, 0)$. When $t = 1$ we have $G(x, 1, s) = (x, 1 + s)$. When $s = 0$ we have $G(x, t, 0) = (x, t)$ and when $s = 1$ we have $G(x, t, 1) = (x, 2t)$.

[6] Again, it is worthwhile to consider some cases. For example, if $t = 0$, then $(1 - t)(1 + s) = (1 + s) \ge 1$ for all s, so $\tilde{H}((x, 0), s) = H(x, s)$. At the other extreme, if $t = 1$, then $(1 - t)(1 + s) = 0 \le 1$ for all s, so $\tilde{H}((x, 1), s) = f(x, 1)$.

1.3.5 Remark. We recall that given a map $f\colon X \to Y$, the mapping cylinder (see Figure 1.4) is the pushout

$$\begin{tikzcd}
X \arrow[r, "i_0"] \arrow[d, "f"'] & X \times I \arrow[d] \\
Y \arrow[r] \arrow[ur, phantom, "\lrcorner"] & M_f
\end{tikzcd}$$

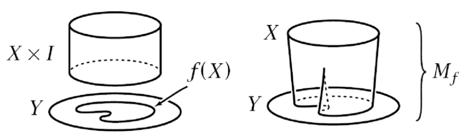


Figure 1.4: The mapping cylinder.

In formulas,

$$M_f = ((X \times I) \coprod Y)/((0, x) \sim f(x), \forall x \in X)$$

Note that M_f deformation retracts on Y by sliding each point $(x, t) \in M_f$ to the end-point. Note that we have a natural map $j\colon X \to M_f$ sending x to $(x, 1)$.

1.3.6 Lemma. *The map $j\colon X \to M_f$ has the HEP for all spaces Y.*

Proof. The proof is similar to the previous lemma; one just has to modify the end point by defining

$$\tilde{H}|_{Y\times I}(y, s) = f(y, 0).$$

$\square$

1.3.7 Corollary. *The inclusion $S^{n-1} \to D^n$ is a cofibration.*

Proof. Simply note that $D^n \simeq CS^{n-1}$. $\square$

There is a universal test space for cofibrations.

1.3.8 Proposition. *Let $i\colon A \to X$, and let M_i be the mapping cylinder. Then $i\colon A \to X$ is a cofibration if and only if there exists a map $r\colon X \times I \to M_i$ making the diagram*

$$\begin{tikzcd}
A \arrow[rr, "i_0"] \arrow[dd, "i"'] & & A \times I \arrow[dl, "i\times id"'] \arrow[dd] \\
 & X \times I \arrow[dr, dashed, "\exists r"] & \\
X \arrow[ur, "i_0"'] \arrow[rr] & & M_i
\end{tikzcd}$$

commute.

Proof. If i is a cofibration, then the map r exists as a consequence of the HEP.

For the other direction, if r exists, then for any maps $f\colon X \to Y$ and $H\colon A \times I \to Y$ making the obvious diagram commute, the universal property of the pushout gives us a map $H'\colon M_i \to Y$. Then let $\tilde{H} = H' \circ r$, and we are done. $\square$

1.3.9 Corollary. *If $A \subseteq X$, then $I\colon A \to X$ is a cofibration if and only if $X \times I$ is a retract of $M_i = X \times \{0\} \cup A \times I$.*

1.3.10 Corollary. *A cofibration $i\colon A \to X$ is an injection. If X is Hausdorff, then $i(A)$ is closed in X.*

Proof. Let $J\colon A \times I \to M_i$ be the canonical map (arising from the definition of M_i as a pushout). Then, $J(a, 1) = r(i(a), 1)$, and observe that $J\mid_{A\times\{1\}}$ is the identity, as it is the top of the mapping cylinder. So, $i(a) \neq i(a')$ if $a \neq a'$, i.e., i is injective.

Because $i\colon A \to X$ is a cofibration, so is $i(A) \to X$. Hence $X \times I$ retracts onto $X \times \{0\} \cup i(A) \times I$ (Corollary 1.3.9). For a Hausdorff space, the image of a retract is closed, and so $X \times \{0\} \cup i(A) \times I$ is a closed subspace of $X \times I$. Intersecting with $X \times \{1\}$, we see that $i(A) \times \{1\}$ is closed in $X \times \{1\}$, i.e, $i(A)$ is closed in X. $\square$

The following (rather pathological) example shows that i is not always a closed map if X is not Hausdorff.

Exercise 7

Let $A = \{a\}$ and $X = \{a, b\}$ with the trivial topology. Show that the inclusion $A \to X$ is a cofibration whose image is not closed.

1.3.11 Remark. The next goal is to show that CW-complexes (X, A) are always cofibrations. The key is the following exercise.

Exercise 8

(a) Suppose $\{(X_i, A_i)\}$ are a collection of spaces satisfying the HEP, then so does $\{(\coprod X_i, \coprod A_i)\}$.

(b) Suppose (X, A) satisfies the HEP, and $f\colon A \to B$ is a continuous map. Let $Y = X \cup_f B$ be the pushout, then (Y, B) satisfies the HEP.

(c) Suppose $A = X_0 \subseteq X_1 \subseteq \cdots \subseteq X_n \subseteq X_{n+1} \subseteq \cdots$.

Let $X = \operatorname{colim} X_i$. If each (X_i, X_{i-1}) satisfies the HEP, then so does (X, A).

1.3.12 Theorem. *A relative CW-complex (X, A) satisfies the HEP.*

Proof. Using Corollary 1.3.7 and the previous exercise we see that (S^{n-1}, D^n) satisfies the HEP $\implies$ $(\coprod S^{n-1}, \coprod D^n)$ satisfies the HEP. Inductively, (X_{n-1}, A) satisfies the HEP and by the exercise (X, A) satisfies the HEP. □

1.3.13 Remark. One can also prove this directly by constructing a deformation retract $r\colon X \times I \to X \times \{0\} \cup A \times I$.

1.3.14 Remark. One can consider the following question: Suppose that $A \subset X$ with A contractible, then is $X \simeq X/A$? Surprisingly, this is not true in general. Indeed, let $A = S^1 \setminus \{(1,0)\}$ and consider $A \to S^1$. Then $S^1/A \cong T$, the $T = \{a, b\}$ the two point space with open sets $\emptyset, \{a\}, \{a, b\}$ (this is the Sierpiński space). One can check that this space is contractible.[7] The exact condition we need is that $A \to X$ is a cofibration.

[7] See https://math.stackexchange.com/a/264789/64273.

1.3.15 Definition. A contracting homotopy is a map $H\colon X \times I \to X$ such that $H(x, 0) = \mathrm{id}_X$ and $H(x, 1) = c_{x_0}$, the constant map at x_0.

1.3.16 Proposition. *Suppose $A \subseteq X$ and $x_0 \in A$. Suppose there exists a map $H\colon X \times I \to X$ such that $H|_{X \times \{0\}} = id_X$ and $H|_{A \times I}$ has image in A and is a contacting homotopy for A. Then $q\colon X \to X/A$ is a homotopy equivalence.*

Proof. We need to find $p\colon X/A \to X$ such that $q \circ p \simeq \mathrm{id}_{X/A}$ and $p \circ q \simeq \mathrm{id}_X$. The quotient map has a set-theoretic section given by

$$s(\overline{x}) = \begin{cases} x & x \notin A \\ x_0 & x \in A \end{cases}$$

Define $p\colon X/A \to X$ by the following diagram

$$\begin{tikzcd} X \arrow[r, "q"] & X/A \arrow[r, "s"] \arrow[dr, "p"'] & X \arrow[d, "H|_{X\times\{1\}}"] \\ & & X \end{tikzcd}$$

Assume for a moment that p is continuous. Then $p \circ q = H|_{X\times\{1\}}$, and so H gives a homotopy between id_X and $p \circ q = H|_{X\times\{1\}}$. Likewise, if we define G by

$$\begin{tikzcd} X/A \times I \arrow[r, "s\times\mathrm{id}"] \arrow[drr, "G"'] & X \times I \arrow[r, "H"] & X \arrow[d, "q"] \\ & & X/A \end{tikzcd}$$

and assume that G is continuous, then

$$G(\bar{x}, 1) = q \circ (H|_{X\times\{1\}} \circ s) = q \circ p,$$

so that G is a homotopy between $\mathrm{id}_{X/A}$ and $q \circ p$. To see that p is continuous, let $U \subset X$ be open, then

$$q^{-1}p^{-1}(U) = (p \circ q)^{-1}(U) = (H|_{X\times\{1\}})^{-1}(U)$$

is open in X by the continuity of $H|_{X\times\{1\}}$, hence $p^{-1}(U)$ is open in X/A by the definition of the quotient topology, and so p is continuous. We leave the proof of continuity of G to the reader. $\square$

1.3.17 Theorem. *Let $A \subseteq X$ be a subspace with A contractible. Suppose that the inclusion $i\colon A \to X$ is a cofibration, then $X \to X/A$ is a homotopy equivalence.*

Proof. Let $h\colon A \to I \to A$ be a contracting homotopy. Let $H\colon A \times I \to X$ be the composition of h with the inclusion map of A into X, i.e., the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, hook, "i_0"] \arrow[d, "i"'] & A \times I \arrow[d, "i\times\mathrm{id}"] \arrow[ddr, bend left, "H"] & \\ X \arrow[r, hook, "i_0"'] \arrow[drr, bend right, "\mathrm{id}_X"'] & X \times I \arrow[dr, dashed, "\exists\tilde{H}"] & \\ & & X \end{tikzcd}$$

By the HEP, the dotted map $\tilde{H}$ exists as in the diagram. This map satisfies the conditions of Proposition 1.3.16:

(i) $\tilde{H}\colon X \times \{0\} \to X$ is the identity.

(ii) $\tilde{H}(A \times I) = H(A \times I) = h(A \times I) \subseteq A$.

(iii) $\tilde{H}(A \times \{1\}) = x_0$.

Therefore, $q\colon X \to X/A$ is a homotopy equivalence, as claimed. $\square$

Exercise 9: Cofibrations are pushout closed.

Let $i\colon A \to X$ be a cofibration, and $g\colon A \to B$ any map ,then the induced map $B \to B \cup_g X$ is a fibration.