1
Characteristic classes

In the end of the previous chapter we saw how two cohomology classes, the first Chern class, and the first Stiefel–Whitney class completely characterize complex and real line bundles respectively. In this section we develop a general theory of Chern and Stiefel–Whitney classes for higher rank bundles.

1.1 Chern classes of complex vector bundles

1.1.1 Remark. We recall that we have a bijection

$$\left\{\text{principal } GL_n(\mathbb{C})\text{-bundles over } X\right\} \longleftrightarrow \left\{\text{Rank } n \text{ complex vector bundles over } X\right\}$$

for any CW-complex X. We will freely use this to pass between complex vector bundles and principal bundles. Moreover, by Gram–Schmidt we have $GL_n(\mathbb{C}) \simeq U(n)$. We begin by computing the cohomology of the classifying space $BU(n)$.

1.1.2 Proposition. *We have*

$$H^*(BU(n);\mathbb{Z}) \cong \mathbb{Z}[c_1, c_2, \ldots, c_n]$$

with $|c_i| = 2i$. Moreover, the map

$$i\colon BU(n-1) \to BU(n)$$

induces a map $i^*\colon H^*(BU(n);\mathbb{Z}) \to H^*(BU(n-1);\mathbb{Z})$ sending c_i to c_i for $i < n$.

Proof. There are any number of ways to do this. For example, we can do this by induction on n. When $n = 1$ we have $BU(1) \simeq \mathbb{C}P^\infty$ and $H^*(\mathbb{C}P^\infty;\mathbb{Z}) \cong \mathbb{Z}[c_1]$ as we have already seen. In general, we have a fibration

$$S^{2n-1} \cong U(n)/U(n-1) \to BU(n-1) \to BU(n)$$

so the Gysin sequence (Proposition 1.1.2) is of the form

$$\cdots \to H^{k-1}(BU(n-1)) \to H^{k-2n}(BU(n)) \xrightarrow{\smile e} H^k(BU(n)) \xrightarrow{p^*} H^k(BU(n-1)) \to H^{k-2n+1}(BU(n)) \to \cdots$$

Inductively we note that $H^*(BU(n-1))$ is concentrated in even degrees, so we get short exact sequences

$$0 \to H^{k-2n}(BU(n)) \xrightarrow{\smile e} H^k(BU(n)) \xrightarrow{p^*} H^k(BU(n-1)) \to 0$$

It follows that $H^k(BU(n)) = 0$ for k odd as well.[1] Moreover, there is an isomorphism $H^*(BU(n))/(e) \xrightarrow{\sim} H^*(BU(n-1))$. Because $H^*(BU(n-1))$ is a polynoimal algebra, we can lift the generators to generators of $H^*(BU(n))$, and produce an algebra map $H^*(BU(n-1))[e] \to H^*(BU(n))$, which we claim is an isomorphism. Indeed, we can filter both sides by powers of e, and note that this gives an isomorphism on associated gradeds. A five-lemma argument shows that the map induces an isomorphism modulo e^k for any k, but the powers of e increase in dimension, so we obtain an isomorphism in each dimension. Finally, we can define $c_n = (-1)^n e \in H^{2n}(BU(n))$.

If you don't like that argument, another way is to first prove that $H^*(U(n)) \cong \Lambda_{\mathbb{Z}}[x_1, \dots, x_{2n-1}]$ using the Serre spectral sequence. A second application of the Serre spectral sequence for the fibration $U(n) \to EU(n) \to BU(n)$ gives the result. We leave the details to the reader. □

[1] This is clear for $k < 2n$, but note that we can then feed this into the leftmost term and use induction to see it for all k.

1.1.3 Definition. The generators $c_1, \dots, c_n$ are called the universal Chern classes of $U(n)$-bundles.

1.1.4 Remark. Recall that given a principal $U(n)$-bundle $\pi\colon E \to X$, there exists a map $f_\pi\colon X \to BU(n)$ such that $\pi \cong f_\pi^*(\pi_{U(n)})$.

1.1.5 Definition. The i-th Chern class of the $U(n)$-bundle $\pi\colon E \to X$ is defined as $c_i(\pi) := f_\pi^*(c_i) \in H^{2i}(X;\mathbb{Z})$.

1.1.6 Proposition (Functoriality of Chern classes)**.** *If $f\colon Y \to X$ is a continuous map, and $\pi\colon E \to X$ is a $U(n)$-bundle, then $c_i(f^*\pi) \cong f^*(c_i(\pi))$ for any i.*[2]

[2] Note that f^* has a dual role here: once as a pullback bundle, and once as the pullback of a cohomology class.

Exercise 1. *Prove Proposition 1.1.6.*

1.1.7 Corollary. *If ϵ is the trivial $U(n)$-bundle on a space X, then $c_i(\epsilon) = 0$ for all $i > 0$.*

Proof. The bundle ϵ is the pullback of the bundle $\nu\colon G \to *$ along the canonical map $q\colon X \to *$:

$$\begin{tikzcd} X \times G \arrow[r] \arrow[d, "\epsilon"'] \arrow[dr, phantom, "\lrcorner", very near start] & G \arrow[d, "\nu"] \\ X \arrow[r, "q"'] & * \end{tikzcd}$$

So we have

$$c_i(\epsilon) \cong c_i(q^*(\nu)) \cong q^* c_i(\nu).$$

But $c_i(\nu) \in H^{2i}(*) = 0$ when $i > 0$. □

1.1.8 Definition. The total Chern class of a $U(n)$-bundle $\pi\colon E \to X$ is defined by[3]

$$c(\pi) = c_0(\pi) + c_1(\pi) + \cdots + c_n(\pi) \in H^*(X;\mathbb{Z})$$

as an element in the cohomology ring of the base space.

[3] Note that if π is a $U(n)$-bundle, then $c_i(\pi) = 0$ for $i > n$ by definition.

1.1.9 Definition (Whitney Sum). Let $\pi_1\colon E_1 \to X$ and $\pi_2\colon E_2 \to X$ be principal $U(n)$ and $U(m)$-bundles respectively. Consider the product bundle $\pi_1 \times \pi_2 \colon E_1 \times E_2 \to X \times X$ which is a principal $U(n+m)$-bundle, via the inclusion $U(n) \times U(m) \to U(n+m)$. The Whitney sum of π_1 and π_2 is defined as

$$\pi_1 \oplus \pi_2 := \Delta^*(\pi_1 \times \pi_2)$$

where $\Delta \colon X \to X \times X$ is the diagonal.

1.1.10 Proposition (Whitney sum formula). *If $\pi_1 \colon E_1 \to X$ and $\pi_2 \colon E_2 \to X$ are principal $U(n)$ and $U(m)$-bundles respectively, then*

$$c(\pi_1 \oplus \pi_2) \cong c(\pi_1) \smile c(\pi_2),$$

or equivalently,

$$c_k(\pi_1 \oplus \pi_2) = \sum_{i+j=k} c_i(\pi_1) \smile c_j(\pi_2).$$

Proof. First observe that by the exercises we have $B(U(n) \times U(m)) \simeq BU(n+m)$. We then consider the map

$$\omega \colon B(U(n) \times U(m)) \simeq BU(n) \times BU(m) \to BU(n+m)$$

induced by $U(n) \times U(m) \to U(n+m)$. One can show that[4]

$$\omega^*(c_k) = \sum_{i+j=k} c_i \otimes c_j.$$

It follows that

$$\begin{aligned} c_k(\pi_1 \oplus \pi_2) &= c_k(\Delta^*(\pi_1 \times \pi_2)) \\ &\cong \Delta^* c_k(\pi_1 \times \pi_2) \\ &= \Delta^*(f^*_{\pi_1 \times \pi_2}(c_k)) \end{aligned}$$

Now we note that the classifying map for $\pi_1 \times \pi_2$ regarded as a $U(n+m)$-bundle is $\omega \circ (f_{\pi_1} \times f_{\pi_2})$. Therefore, we continue:

$$\begin{aligned} c_k(\pi_1 \oplus \pi_2) &\cong \Delta^*(f^*_{\pi_1} \times f^*_{\pi_2})(\omega^*(c_k)) \\ &\cong \sum_{i+j=k} \Delta^*(f^*_{\pi_1}(c_i) \times f^*_{\pi_2}(c_j)) \\ &\cong \sum_{i+j=k} \Delta^*(c_i(\pi_1) \times c_j(\pi_2)) \\ &\cong \sum_{i+j=k} c_i(\pi_1) \smile c_j(\pi_2) \end{aligned}$$

as required. □

[4] Here is an idea of one way to do this. Let $T(n) = U(1) \times \cdots U(1)$, a product of n-copies of S^1. The canonical map $T(n) \to U(n)$ induces $\mu_n \colon BT(n) \to BU(n)$. We have $H^*(BT(n)) \cong \mathbb{Z}[x_1, \ldots, x_n]$ for $|x_i| = 2$, and μ^* is a monomorphism determined by $\mu_n^*(c_k) \cong \sigma_k(x_1, \ldots, x_n)$, the k-th elementary symmetric polynomial in $x_1, \ldots, x_n$. This allows us to reduce to a computation with $BT(n)$, and some diagram chasing. The details can be found, for example, in Corollary 2.44 in Kochman's book 'Bordism, stable homotopy, and the Adams spectral sequence.'

1.1.11 Corollary (Stability of Chern classes). *Let ϵ^1 denote the trivial $U(1)$-bundle, then $c(\pi \oplus \epsilon^1) \cong c(\pi)$.*

Proof. This follows from the proposition and Corollary 1.1.7. □

1.1.12 Remark. It turns out that Chern classes are completely determined by four axioms:

A1. To each principal $U(n)$-bundle $\pi \colon E \to X$ there exists a sequence of classes $c_i(\pi) \in H^{2i}(X; \mathbb{Z})$ such that $c_0(\pi) = 1 \in H^0(X; \mathbb{Z})$ and $c_i(\pi) = 0$ for $i > n$.

A2. Naturality: If $f\colon Y \to X$ is a continuous map, then $c_k(f^*(\pi)) \cong f^*(c_k(\pi))$.

A3. Whitney sum formula: $c(\pi_1 \oplus \pi_2) = c(\pi_1) \smile c(\pi_2)$.

A4. Normalization: Let x be the generator of $H^2(\mathbb{C}P^n;\mathbb{Z}) \cong \mathbb{Z}$, then the total Chern class of the tautological line bundle (see **??**) over $S^2 \cong \mathbb{C}P^1$ is $1+x$.[5]

[5] Or $1-x$, depending on convention.

1.1.13 Theorem. *There exists at most one correspondence $\pi \mapsto c(\pi)$ which assigns to each complex vector bundle over a paracompact base space a sequence of cohomology classes satisfying the above four axioms.*

The proof uses the following (very important) splitting principle for complex vector bundles.[6]

[6] For example, the splitting principle can also reduce the proof of the Whitney sum formula to line bundles.

1.1.14 Proposition. *For each complex vector bundle $\pi\colon E \to X$ there exists a space $F(E)$ and a map $p\colon F(E) \to X$ such that the pull-back $p^*F(E) \to F(E)$ splits as a direct (Whitney) sum of line bundles and $p^*\colon H^*(X;\mathbb{Z}) \to H^*(F(E);\mathbb{Z})$ is injective.*

1.1.15 Remark. There is a similar result for real vector bundles if we use $\mathbb{Z}/2$ coefficients.

Sketch proof. By induction, it will suffice to find a map $p'\colon F'(E) \to X$ such that $(p')^*(\pi) \cong E' \oplus L$ with L a complex line bundle, and $p^*\colon H^*(X;\mathbb{Z}) \to H^*(F'(E);\mathbb{Z})$ injective, as we can then inductively apply the same argument to E'.

We use the projective bundle construction of **??**, and set $F'(E) := P(E)$ with $p'\colon P(E) \to X$. Then there is an injective map

$$\phi\colon L_E \to (p')^*(E), \quad (\ell, v) \mapsto (\ell, v)$$

where L_E is a line bundle. Because X is compact, we can choose a Hermitian inner product on E inducing one on $(p')^*$, and hence take E' to be the orthogonal complement of $\phi(L_E)$ in $(p')^*(E)$. Therefore, $(p')^*(E) \cong L_E \oplus E'$, as required. The claim about cohomology follows from the Leray–Hirsch theorem [7] - $H^*(P(E);\mathbb{Z})$ is the free $H^*(X;\mathbb{Z})$-module with basis $1, x, \ldots, x^{n-1}$; in particular, the map $H^*(X;\mathbb{Z}) \to H^*(P(E);\mathbb{Z})$ is injective since one of the basis elements is 1. □

[7] `https://en.wikipedia.org/wiki/Leray%E2%80%93Hirsch_theorem`

Proof of theorem 1.1.13. Let $\pi \mapsto c(\pi), \tilde{c}(\pi)$ be two sets of Chern classes. By Axioms A1 and A4 for the canonical line bundle γ_1^1 over $\mathbb{C}P^1$ we have

$$c(\gamma_1^1) = \tilde{c}(\gamma_1^1) = 1 + x.$$

Using the embedding $\mathbb{C}P^1 \to \mathbb{C}P^\infty$ we deduce that

$$c(\gamma_1) = \tilde{c}(\gamma_1) = 1 + x.$$

for γ_1 the canonical line bundle over $\mathbb{C}P^\infty$ by Axioms A1 and A2. Then, for $\xi = \gamma_1 \oplus \cdots \oplus \gamma_1$ we deduce that

$$c(\xi) = \tilde{c}(\xi)$$

by Axiom A3.

Now, let $\pi\colon E \to X$ be arbitrary, and $p\colon F(E) \to X$ the map that exists by the splitting principle (Proposition 1.1.14). Then we have

$$\begin{aligned} p^*c(\pi) &\cong c(p^*\pi) && (\textit{Axiom A2}) \\ &\cong c(\lambda_1 \oplus \cdots \oplus \lambda_n) && (\textit{Proposition } 1.1.14) \\ &\cong \tilde{c}(\lambda_1 \oplus \cdots \oplus \lambda_n) \\ &\cong \tilde{c}(p^*\pi) \\ &\cong p^*\tilde{c}(\pi) \end{aligned}$$

Because p^* is injective, we deuce that $c(\pi) \cong \tilde{c}(\pi)$, as required. □

1.1.16 Remark. This shows that there is *at most* one theory of Chern classes. We omit the proof that Chern classes do actually exist with the required properties (we are almost there; we have just not shown Item A4).

1.1.17 Example (Chern classes of the dual bundle). Given a complex vector bundle $\pi\colon E \to M$ its dual bundle is the Hom bundle (**??**) $\mathrm{Hom}(\pi, \mathbb{C} \times M)$, i.e. the hom bundle from π to the trivial bundle $\mathbb{C} \times M \to M$. We denote this bundle by $\pi^*\colon E^* \to M$. The fibers of this bundle are the dual spaces to the fiber of π. Let L be a complex line bundle, then one can check that $L \otimes L^* = \mathrm{Hom}(L, L)$ is a trivial bundle. Moreover, $c_1(L \otimes L^*) = c_1(L) + c_1(L^*)$, so that $c_1(L) = -C_1(L^*)$.

Now suppose that $E = L_1 \oplus \cdots \oplus L_n$ is a sum of line bundles. By the Whitney sum formula

$$c(E^*) = c(L_1) \smile \cdots \smile c(L_n) = (1 + c_1(L_1)) \cdots (1 + c_n(L_n)).$$

Similarly, $E^* = L_1^* \oplus \cdots \oplus L_n^*$, and

$$c(E) = c(L_1^*) \smile \cdots \smile c(L_n^*) = (1 - c_1(L_1)) \cdots (1 - c_n(L_n)).$$

By comparing coefficients,[8] we have $c_q(E^*) = (-1)^q c_q(E)$. By the splitting principle, this holds for all complex vector bundles.

[8] Use the bimonial formula if you need

1.2 Stiefel–Whitney classes for real vector bundles

Analogous to Chern classes for complex vector bundles, we have a good theory of Stiefel–Whitney classes for real vector bundles, where we replace $BU(n)$ with $BO(n)$.

1.2.1 Proposition.

$$H^*(BO(n); \mathbb{Z}/2) \cong \mathbb{Z}/2[w_1, \ldots, w_n]$$

with $|w_i| = i$.

Proof. This is very similar to Proposition 1.1.2. For example, we can use induction using the Serre spectral sequence of the fibration

$$O(n)/O(n-1) \cong S^{n-1} \to BO(n-1) \to BO(n)$$

and $BO(1) \simeq \mathbb{R}P^\infty$ with $H^*(\mathbb{R}P^\infty) \cong \mathbb{Z}/2[w_1]$. □

1.2.2 Definition. The generators $w_1, \ldots, w_n$ are called the universal Stiefel–Whitney classes of $O(n)$-bundles.

1.2.3 Remark. Recall that given a principal $O(n)$-bundle $\pi\colon E \to X$, there exists a map $f_\pi\colon X \to BO(n)$ such that $\pi \cong f_\pi^*(\pi_{U(n)})$.

1.2.4 Definition. The i-th Stiefel–Whitney class of the $O(n)$-bundle $\pi\colon E \to X$ is defined as $w_i(\pi) := f_\pi^*(w_i) \in H^i(X;\mathbb{Z}/2)$.

Using identical proofs as in the complex case, Stiefel–Whitney classes are characterized by four axioms:

A1. To each principal $O(n)$-bundle $\pi\colon E \to X$ there exists a sequence of classes $c_i(\pi) \in H^i(X;\mathbb{Z}/2)$ such that $w_0(\pi) = 1 \in H^0(X;\mathbb{Z})$ and $w_i(\pi) = 0$ for $i > n$.

A2. Naturality: If $f\colon Y \to X$ is a continuous map, then $w_k(f^*(\pi)) \cong f^*(w_k(\pi))$.

A3. Whitney sum formula: $w(\pi_1 \oplus \pi_2) = w(\pi_1) \smile w(\pi_2)$.

A4. Normalization: Let x be the non-zero element of of $H^2(\mathbb{R}P^n;\mathbb{Z}/2) \cong \mathbb{Z}/2$, then the total Chern class of the tautological line bundle (see **??**) over $S^1 \cong \mathbb{R}P^1$ is $1 + x$.

1.2.5 Theorem. *There exists at most one correspondence $\pi \mapsto w(\pi)$ which assigns to each real vector bundle over a paracompact base space a sequence of cohomology classes satisfying the above four axioms.*

1.2.6 Remark. Given a real vector bundle $\pi\colon E \to X$ we can consider its complexificaiton $\pi \otimes \mathbb{C}$, the complex vector bundles with transition functions $\Phi_{\alpha\beta}\colon U_\alpha \cap U_\beta \to O(n) \subseteq U(n)$ and fiber $\mathbb{R}^n \otimes \mathbb{C} \cong \mathbb{C}^n$.

Now if $\pi'\colon E' \to X$ is a complex vector bundle, we can always make a conjugate bundle $\overline{\pi'}$. Note that the dual bundle of π' is isomorphic to the conjugate bundle, but the choice of isomorphism is non-canonical unless E' has a hermitian product. The transition functions of the conjugate bundle are given as the composite

$$\overline{\Phi}_{\alpha\beta}\colon U_\alpha \cap U_\beta \xrightarrow{\Phi_{\alpha\beta}} U(n) \xrightarrow{(-)^\dagger} U(n)$$

where the last map takes the complex conjugate of a unitary matrix. In any case, we have the following.

1.2.7 Lemma. *Let π be a real vector bundle, then $\overline{\pi \otimes \mathbb{C}} \cong \pi \otimes \mathbb{C}$.*

Proof. Just observe that the transition functions for $\pi \otimes \mathbb{C}$ are real-valued; they land in $O(n) \subseteq U(n)$, and so they are also the transition functions for $\overline{\pi \otimes \mathbb{C}}$. □

1.2.8 Proposition.

$$c_k(\pi \otimes \mathbb{C}) \cong c_k(\overline{\pi \otimes \mathbb{C}}) \cong (-1)^k c_k(\pi \otimes \mathbb{C})$$

In particular, if k is odd, then $c_k(\pi \otimes \mathbb{C})$ is an integral cohomology class of order 2.

Proof. This follows from Example 1.1.17 and lemma 1.2.7. □

Given a complex vector bundle ω, we let $\omega_{\mathbb{R}}$ denote the underlying real vector bundle.

1.2.9 Proposition. *If ω is a complex vector bundle, then*

$$\omega_{\mathbb{R}} \otimes \mathbb{C} \cong \omega \oplus \overline{\omega}.$$

Proof. We prove the statement at the level of vector-spaces; the proof passes to vector bundles as well. To that end, let L be a complex vector space, and $L_{\mathbb{R}}$ its underlying real vector space, then we claim that $L_{\mathbb{R}} \otimes \mathbb{C} \cong L \oplus \overline{L}$. To see this, let

$$J \colon L_{\mathbb{R}} \otimes \mathbb{C} \to L_{\mathbb{R}} \otimes \mathbb{C}$$

by given by multiplication by i. Because $J^2 = -\mathrm{id}$, we have an eigenvalue decomposition

$$L_{\mathbb{R}} \otimes \mathbb{C} \cong \mathrm{eigen}(i) \oplus \mathrm{eigen}(-i).$$

We have a map

$$P_i \colon L \to L_{\mathbb{R}} \to L_{\mathbb{R}} \otimes \mathbb{C} \to \mathrm{eigen}(i),$$

which is $\mathbb{R}$-linear, but is in fact $\mathbb{C}$-linear because $P_i J(\ell) = iP_i(\ell)$ for all $\ell \in L$. This composite is therefore an isomorphism by a dimension count. Similarly,

$$P_{-i} \colon L \to \mathrm{eigen}(-i)$$

is a $\mathbb{C}$-*anti*linear isomorphism, and so $\mathrm{eigen}(i) \cong \overline{L}$. $\square$

1.2.10 Corollary. *For a complex vector bundle ω we have*

$$c(\omega_R \otimes \mathbb{C}) \cong c(\omega) \cdot c(\overline{\omega}),$$

or equivalently,

$$c_k(\omega_R \otimes \mathbb{C}) = \sum_{i+j=k} (-1)^j c_i(\omega) \cdot c_j(\omega).$$

1.2.11 Remark. Note that if k is odd, then this sum is always zero.

Exercise 2. *Let $\pi_{\mathbb{R}}$ denote the underlying real bundle of a complex bundle; π note that if π has rank n as a complex bundle, then $\pi_{\mathbb{R}}$ has rank $2n$ as a real bundle. Via the map $\mathbb{Z} \to \mathbb{Z}/2$ the class $c_i(\pi) \in H^{2i}(X;\mathbb{Z})$ determines a cohomology class $\overline{c}_i(\pi) \in H^{2i}(X;\mathbb{Z}/2)$. Show that the Stiefel–Whitney classes of $\pi_{\mathbb{R}}$ are computed as follows:*

1. $\omega_{2i}(\pi_{\mathbb{R}}) = \overline{c}_i(\pi)$ *for* $0 \leq i \leq n$.
2. $\omega_{2i+1}(\pi_{\mathbb{R}}) = 0$ *for any integer* i.

1.2.12 Remark. Here is a hint: Let $\mu_n \colon U(n) \to O(2n)$ be the inclusion, then the classifying map of π_R is the composite $X \xrightarrow{f} BU(n) \xrightarrow{\mu_n} BO(2n)$, where f is the classifying map for π. So you should try and compute

$$\mu_n^* \colon H^*(BO(2n);\mathbb{Z}/2) \cong \mathbb{Z}/2[w_1, w_2, \ldots, w_{2n}] \to H^*(BU(n);\mathbb{Z}/2) \cong \mathbb{Z}/2[\overline{c}_1, \overline{c}_2, \ldots, \overline{c}_n].$$

1.3 Applications of Stiefel–Whitney classes

Stiefel–Whitney classes are useful in the study of smooth manifolds. Indeed, if M is smooth, then we recall (**??**) that the tangent bundle $\pi\colon TM \to M$ is a real vector bundle, and hence corresponds to an $O(n)$-bundle (which by our conventions, we use the same notation for).

1.3.1 Definition. The Stiefel–Whitney classes of a smooth manifold M are defined as the Stiefel–Whitney classes of the corresponding $O(n) - bundle$: $w_i(M) := w_i(TM)$.

1.3.2 Remark. In order, for this to be a reasonable notion, we should prove that these are homotopy invariants. Fortunately, we have the following theorem.[9]

[9] The proof of the following is beyond the scope of this course, but here is the idea: One can give an alternative construction of the Stiefel–Whitney classes in terms of Steenrod operations; this implies that the the Stiefel–Whitney classes of M are determined entirely in terms of the mod 2 cohomology ring along with its structure under the Steenrod algebra (which is preserved by homotopy equivalences). So in fact, the theorem doesn't even need homotopy equivalence, but only a mod 2 cohomology isomorphism over the Steenrod algebra.

1.3.3 Theorem (Wu). *Stiefel–Whitney classes are homotopy invariants, i.e., if $h\colon M_1 \to M_2$ is a homotopy equivalence, then $h^*w_i(M_2) = w_i(M_1)$ for any $i \geq 0$.*

We now turn to an application of Stiefel–Whitney classes to the embedding problem. We begin with the following algebraic lemma.

1.3.4 Lemma. *Suppose that $E \oplus E' \simeq \epsilon^n$ is a trivial bundle, then there exists a unique polynomial q_i such that*

$$\omega_i(E') = q_i(\omega_1(E), \omega_2(E), \ldots, \omega_i(E)).$$

Proof. Induction on i. When $i = 1$ we have

$$\begin{aligned} 0 = \omega_1(\epsilon^n) &= \omega_0(E) \smile \omega_1(E') + \omega_1(E) \smile \omega_0(E') \\ &= 1 \smile \omega_1(E') + \omega_1(E) \smile 1, \end{aligned}$$

and hence

$$\omega_1(E') = -\omega_1(E) = \omega_1(E),$$

since we work over $\mathbb{Z}/2$.

Supposing we have proved the claim up to $i-1$. Then,

$$\begin{aligned} 0 = \omega_i(\epsilon^n) &= \sum_{k+j=i} \omega_k(E) \smile \omega_j(E') \\ &= \omega_i(E') + \sum_{k+j=1, j<i} \omega_k(E) \smile \omega_j(E') \\ &= \omega_i(E') + \sum_{k+j=1, j<i} \omega_k(E) \smile q_j(\omega_1(E), \ldots, \omega_j(E)). \end{aligned}$$

Therefore,

$$\omega_i(E') = q_i(\omega_1(E), \ldots, \omega_i(E)) := \sum_{k+j=i, j<i} \omega_k(E) \smile q_j(\omega_1(E), \ldots, \omega_j(E)).$$

□

1.3.5 Definition. We write $\overline{w}_i(E)$ for $q_i(\omega_1(E), \ldots, \omega_i(E))$. These are the dual Stiefel–Whitney classes.

1.3.6 Remark. Let $f\colon M^m \to N^{m+k}$ be an embedding of smooth manifolds.[10] Let f^*TN denote the pullback of the tangent bundle $TN \to N$ along f. The normal bundle is defined by the short exact sequence

$$0 \to TM \to f^*TN \to \nu \to 0,$$

which splits, i.e., $f^*TN \simeq TM \oplus \nu$ where ν has rank k. So, using the Whitney sum formula we have

$$f^*\omega(N) = \omega(M) \smile \omega(\nu).$$

[10] When we use superscripts, we refer to the dimension of the manifolds. So, we may also write $f\colon M \to N$.

1.3.7 Example. Let $S^n \subseteq \mathbb{R}^{n+1}$, then the normal bundle ν is trivial. Indeed, if we write

$$\nu(S^n) = \bigcup_{p\in S^n} T_p\mathbb{R}^{n+1}/T_pS^n$$

then an explicit isomorphism is given by the map Φ sending

$$[v] \in \nu_p(S^n) \text{ to } (p, \langle v, p\rangle) \in S^n \times \mathbb{R},$$

with inverse Ψ sending $(q,t) \mapsto [tq] \in \nu_q(S^n)$.[11] In other words,

$$TS^n \oplus \nu \simeq \epsilon^{n+1} \simeq TS^n \oplus \epsilon^1 \simeq \epsilon^{n+1}.$$

[11] For example, $(\Phi \circ \Psi)(p,t) = \Phi([tp]) = (p, \langle tp, p\rangle) = (p, t\langle p, p\rangle) = (p,t)$.

Since trivial bundles do not change Stiefel–Whitney classes, we deduce that $\omega_i(S^n) = 0$ for all $i > 0$, and the Stiefel–Whitney classes of TS^n are the same as the trivial bundle (and recall that we have seen that $p\colon TS^2 \to S^2$ is not a trivial bundle - in fact this is true for every even sphere).[12]

[12] There is a 'moral' reason for this: the only possible Stiefel–Whitney classes in positive degrees is the top one, as $H^i(S^n;\mathbb{Z}/2)$ is non-zero only when $i = 0, n$. This top class is the image of the *Euler class* in $H^n(S^n;\mathbb{Z})$ under the natural homomorphism $H^i(S^n;\mathbb{Z}) \to H^i(S^n;\mathbb{Z}/2)$ - see Milnor–Stasheff, Property 9.5. But the Euler class of the sphere is $e(TS^n) = 2[S^n]$ is 0 modulo 2.

1.3.8 Example. Suppose that $N = \mathbb{R}^{m+k}$, then using Lemma 1.3.4 (and the hopefully obvious observation that the tangent bundle is trivial: $T\mathbb{R}^{m+k} \simeq \mathbb{R}^{m+k} \times \mathbb{R}^{m+k}$), then we deduce that

$$\omega_i(\nu) = \overline{\omega}_i(TM). \tag{1.3.9}$$

The following calculation is important for our applications. We postpone the proof until after the applications.

1.3.10 Theorem.

$$\omega(\mathbb{R}P^m) \cong (1+x)^{m+1}$$

where $x \in H^1(\mathbb{R}P^m;\mathbb{Z}/2)$ is a generator.

1.3.11 Example. Can we give an embedding of $\mathbb{R}P^9$ into $\mathbb{R}^{9+k}$? Let us note that

$$\omega(\mathbb{R}P^9) = (1+x)^{10} = (1+x)^8(1+x)^2 = (1+x^8)(1+x^2) = 1+x^2+x^8$$

because $H^*(\mathbb{R}P^m;\mathbb{Z}/2) \cong \mathbb{Z}/2[x]/(x^{m+1})$. Therefore,[13]

$$\overline{\omega}(\mathbb{R}P^9) = 1+x^2+x^4+x^6$$

[13] We can check this: $(1+x^2+x^8)(1+x^2+x^4+x^6) = 1+2x^2+2x^4+2x^6+2x^8+x^{10}+x^{12}+x^{14} \equiv 1$ in the ring $\mathbb{Z}/2[x]/(x^{10})$.

Therefore by (1.3.9) we must have,

$$\omega(\nu) = 1+x^2+x^4+x^6,$$

and in particular, $\omega_6(\nu) \neq 0$. Now note that ν is a bundle of rank k, and hence we have $\omega_i(\nu) = 0$ for $i > k$. Therefore, we must have $k \geq 6$. We deduce that $\mathbb{R}P^9$ cannot be embedded into $\mathbb{R}^{14}$. Note that this doesn't say anything about when it *can* be embedded into $\mathbb{R}^{9+k}$. In fact, this bound is sharp: $\mathbb{R}P^9$ can be embedded into $\mathbb{R}^{15}$.[14]

[14] Sanderson, B. J. Immersions and embeddings of projective spaces. Proc. London Math. Soc. (3) 14 (1964), 137–153.

1.3.12 Example. Let $m = 2^r$, then

$$\omega(\mathbb{R}P^{2^r}) = (1+x)^{2^r+1} = (1+x)^{2^r}(1+x) = (1+x^{2^r})(1+x) = 1+x+x^{2^r}$$

Arguing as in the previous example, we have

$$\omega(\nu) = \overline{\omega}(\mathbb{R}P^{2^r}) = 1 + x + x^2 + \cdots + x^{2^r-1},$$

and so $k \geq 2^r - 1 = m - 1$, i.e., $\mathbb{R}P^{2^r}$ cannot be embedded into $\mathbb{R}^{2^{r+1}-1}$. In particular, $\mathbb{R}P^8$ cannot be embedded into $\mathbb{R}^{15}$. Again, this bound is sharp: there exists an embedding of $\mathbb{R}P^8$ into $\mathbb{R}^{16}$, by the Whitney embedding theorem.

We now return to the proof of Theorem 1.3.10.

Proof of Theorem 1.3.10. Let $[x] \in \mathbb{R}P^m$ and $\nu \in [x]$. As usual, we let γ_1 denote the canonical line bundle over $\mathbb{R}P^m$, i.e., $\gamma_1 = \{([x], \nu) \in \mathbb{R}P^m \times \mathbb{R}^{m+1} \mid [x] \in \mathbb{R}P^m, \nu \in [x]\}$. Define L_x to to be the line in $\mathbb{R}^{m+1}$ joining x and $-x$, and let $L_x^\perp$ be its orthogonal complement in $\mathbb{R}P^m \times \mathbb{R}^{m+1}$.

For each $(x, \nu) \in T\mathbb{R}P^n$ we have a linear map

$$\ell(x, \mu) \colon L_x \to L_x^\perp$$

defined by $\ell(x,\mu)(x) = \nu$, which is well defined because $\ell(x,\mu)(-x) = -\nu$. This gives us a fiberwise isomorphism $T_{[x]}\mathbb{R}P^m \to \mathrm{Hom}(L_x, L_x^\perp)$, by sending (x, ν) to $T(x, \nu)$. Now a continuous map between vector bundles over the same base space B is an isomorphism if it is a fiberwise linear isomorphism.[15] Therefore, we have $T\mathbb{R}P^m \cong \mathrm{Hom}(\gamma_1, \gamma_1^\perp)$.

[15] See, Lemma 1.1 of https://pi.math.cornell.edu/~hatcher/VBKT/VB.pdf for example.

Now we make the following observation: the bundle $\mathrm{Hom}(\gamma_1, \gamma_1)$ is just the trivial line bundle ϵ^1. Indeed, the transition map is $\phi_{\alpha\beta}\phi_{\alpha\beta}^{-1} = \mathrm{id}$ (this is special about line bundles: the transpose of a 1×1 matrix is the same matrix!). Therefore,

$$\begin{aligned} T\mathbb{R}P^m \oplus \epsilon^1 &\cong \mathrm{Hom}(\gamma_1, \gamma_1^\perp) \oplus \mathrm{Hom}(\gamma_1, \gamma_1) \\ &\cong \mathrm{Hom}(\gamma_1, \gamma_1^\perp \oplus \gamma_1) \\ &\cong \mathrm{Hom}(\gamma_1, \epsilon^{m+1}) \\ &\cong \mathrm{Hom}(\gamma_1, \epsilon^1)^{m+1} \end{aligned}$$

However, $\mathrm{Hom}(\gamma_1, \epsilon^1) \cong \gamma_1$, and so

$$T\mathbb{R}P^m \oplus \epsilon^1 \cong \gamma_1^{\oplus m+1}.$$

Therefore, we have

$$\begin{aligned} \omega(\mathbb{R}P^m) \cong \omega(T\mathbb{R}P^m \oplus \epsilon^1) &\cong \omega(\gamma_1^{\oplus m+1}) \\ &\cong \omega(\gamma_1)^{\smile(m+1)} \\ &\cong (1+x)^{m+1}. \end{aligned}$$

Here we have used the stability of Stiefel–Whitney classes, the previous discussion, the Whitney sum formula, and the normalization axiom. □

1.3.13 Remark. In other words, we have

$$\omega_i(\mathbb{R}P^m) = \binom{m+1}{i} x^i,$$

where the binomial coefficient is taken modulo 2.

1.3.14 Definition. A manifold is parallelizable if its tangent bundle is trivial.

1.3.15 Corollary. *The total Stiefel–Whitney class $\omega(\mathbb{R}P^m) = 1$ if and only if $m+1 = 2^r$ for some r. In particular, if $\mathbb{R}P^m$ is parallelizable, then $m+1 = 2^r$ for some r.*[16]

[16] A deeper theorem of Adams is that it is parallelizable only when $m = 1, 3, 7$.

Proof. If $m+1 = 2^r$, then

$$\omega(\mathbb{R}P^m) = (1+x)^{2^r} = 1 + x^{2^r} = 1 + x^{m+1} = 1,$$

since $x^{m+1} = 0$. Conversely, if $m+1 = 2^r k$ where $k > 1$ is odd, then

$$\omega(\mathbb{R}P^m) = [(1+x)^{2^r}]^k = (1 + x^{2^r})^k = 1 + kx^{2^r} + \cdots \neq 1,$$

since $x^{2^r} \neq 0$. The final statement follows, as $\omega(\mathbb{R}P^m) = 1$ whenever $\mathbb{R}P^m$ is parallelizable, by definition. □

1.4 Pontryagin classes

1.4.1 Notation. Let us fix some notation throughout this section: we let π denote a real vector bundle (equivalently, a principal $O(n)$-bundle), while ω will denote a complex vector bundle (equivalently, a $U(n)$-bundle).

The reader may want to refresh the statements of Proposition 1.2.8 and Proposition 1.2.9 in order to appreciate the following definitions.

1.4.2 Definition. Let $\pi \colon E \to X$ be a real vector bundle of rank n. The i-th Pontryagin class of π is defined as

$$p_i(\pi) := (-1)^i c_{2i}(\pi \otimes \mathbb{C}) \in H^{4i}(X; \mathbb{Z})$$

If ω is a complex vector bundle of rank n we define its i-th Pontryagin class as

$$p_i(\omega) := p_i(\omega_{\mathbb{R}}) = (-1)^i c_{2i}(\omega \oplus \overline{\omega})$$

1.4.3 Remark. Note that $p_i(\pi) = 0$ for all $i > n/2$.

1.4.4 Definition. The total Pontryagin class is

$$p(\pi) = 1 + p_1(\pi) + \cdots \in H^*(X; \mathbb{Z})$$

1.4.5 Remark. We would Pontryagin classes to satisfy a product formula. Since we have ignored odd degree classes, this is a bit more complicated to state.

1.4.6 Theorem. *If π_1 and π_2 are real vector bundles on a space X, then*

$$p(\pi_1 \oplus \pi_2) = p(\pi_1) \smile p(\pi_2) \text{ mod 2-torsion.}$$

Proof. We have

$$(\pi_1 \oplus \pi_2) \otimes \mathbb{C} \cong (\pi_1 \otimes \mathbb{C}) \oplus (\pi_2 \otimes \mathbb{C}).$$

Therefore,

$$\begin{aligned} p_i(\pi_1 \oplus \pi_2) &= (-1)^i c_{2i}((\pi_1 \oplus \pi_2) \otimes \mathbb{C}) \\ &= (-1)^i c_{2i}((\pi_1 \otimes \mathbb{C}) \oplus (\pi_2 \otimes \mathbb{C})). \end{aligned}$$

Now we compute that

$$\begin{aligned} c_{2i}((\pi_1 \otimes \mathbb{C}) \oplus (\pi_2 \otimes \mathbb{C})) &= \sum_{k+\ell=2i} c_k(\pi_1 \otimes \mathbb{C}) \smile c_\ell(\pi_2 \otimes \mathbb{C}) \\ &= \sum_{a+b=i} c_{2a}(\pi_1 \otimes \mathbb{C}) \smile c_{2b}(\pi_2 \otimes \mathbb{C}) \end{aligned}$$

where both statements hold modulo 2-torsion. The result follows. □

1.4.7 Definition. If M is a real smooth manifold we define

$$p(M) := p(TM)$$

If M is a complex manifold, we define

$$p(M) := p((TM)_{\mathbb{R}}).$$

1.4.8 Theorem. *The total Chern classes and Pontryagin classes of the complex projective space $\mathbb{C}P^n$ are given by*

$$c(\mathbb{C}P^n) = (1+c)^{n+1}$$

and

$$p(\mathbb{C}P^n) = (1+c^2)^{n+1}$$

where $c \in H^2(\mathbb{C}P^n; \mathbb{Z})$ is a generator.

Proof. The computation of $c(\mathbb{C}P^n)$ is very similar to that of $w(\mathbb{R}P^n)$ (Theorem 1.3.10): we first show that

$$T\mathbb{C}P^n \oplus \epsilon^1 \simeq \gamma_1^{\oplus n+1}$$

where ϵ^1 is the trivial complex line bundle on $\mathbb{C}P^n$ and γ_1 is the canonical line bundle over $\mathbb{C}P^n$. This map is classified by the inclusion map $\mathbb{C}P^n \to \mathbb{C}P^\infty$ and so $c_1(\gamma_1) = c$, the generator of $H^2(\mathbb{C}P^\infty; \mathbb{Z}) = H^2(\mathbb{C}P^n; \mathbb{Z})$. Using the Whitney sum formula we have

$$c(\mathbb{C}P^n) = c(T\mathbb{C}P^n \oplus \epsilon^1) = c(T\mathbb{C}P^n) = c(\gamma_1)^{n+1} = (1+c)^{n+1}$$

or in other words, that

$$c_i(\mathbb{C}P^n) = \binom{n+1}{i} c^i.$$

It follows that
$$c(\overline{\mathbb{C}P^n}) = (1-c)^{n+1}.$$
Therefore,
$$\begin{aligned} c((T\mathbb{C}P^n)_{\mathbb{R}} \otimes \mathbb{C}) &= c(T\mathbb{C}P^n \oplus \overline{T\mathbb{C}P^n}) \\ &= c(T\mathbb{C}P^n) \smile c(\overline{T\mathbb{C}P^n}) \\ &= (1-c^2)^{n+1}. \end{aligned}$$
In particular,
$$p_i(\mathbb{C}P^n) = (-1)^i c_{2i}(T\mathbb{C}P^n \oplus \overline{T\mathbb{C}P^n}) = \binom{n+1}{i} c^{2i}.$$
so that
$$p(\mathbb{C}P^n) = (1+c^2)^{n+1}. \qquad \square$$

We now return to the embedding problem.

1.4.9 Proposition. *There is no embedding of $\mathbb{C}P^2$ into $\mathbb{R}^5$.*

Proof. Note that after forgetting the complex structure, $\mathbb{C}P^2$ is a 4-dimensional real dimensional manifold. We will use Pontryagin classes to find a minimal k for which there can be an emedding $\mathbb{C}P^2 \to \mathbb{R}^{4+k}$. Let $T(\mathbb{C}P^2)_{\mathbb{R}}$ be the realization of the tangent bundle for $\mathbb{C}P^2$, then then any embedding would give a normal real bundle ν^k of rank k such that
$$T(\mathbb{C}P^2)_{\mathbb{R}} \oplus \nu^k \cong \epsilon^{4+k}$$
By Theorem 1.4.8 we have
$$p(\mathbb{C}P^2) = (1+c^2)^3 = 1 + 3c^2 \in H^*(\mathbb{C}P^2; \mathbb{Z}) \cong \mathbb{Z}[c]/(c^3).$$
Using that $H^*(\mathbb{C}P^2; \mathbb{Z})$ has no 2-torsion, we see from Theorem 1.4.6
$$p(\mathbb{C}P^2) \cdot p(\nu^k) = 1,$$
so that
$$p(\nu^k) = 1 - 3c^2.$$
In particular, $p_1(\nu^k) \neq 0$. Finally, we observe that if $p_1(\nu^k) = 0$, then $1 \leq k/2$, i.e., $k \geq 2$, so that the minimal possible embedding is $\mathbb{C}P^2 \to \mathbb{R}^6$. $\square$