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MA3408 - ALGEBRAIC TOPOLOGY II

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Homotopy theory

1.1 Review of basics on homotopy theory

We begin with a recollection of some facts that have been covered in Algebraic Topology I and Introduction to Topology.

1.1.1 Notation. We let $I = [0,1]$ denote the unit interval. For a pointed topological space X we will denote the basepoint by x_0 or $*$.

We recall the following definition.

1.1.2 Definition. A homotopy between $f, g\colon X \to Y$ is a continuous function $H\colon X \times I \to Y$ such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$ and $H(x_0, t) = y_0$ for all $t \in I$. We will write $f \simeq g$, or $f \simeq_H g$, if we need to make the choice of homotopy clear.

For a subspace $A \subseteq X$, a relative homotopy is a homotopy with $H(a,t) = f(a) = g(a)$ for all $a \in A, t \in I$.

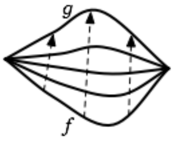

Figure 1.1: A homotopy between f and g.

1.1.3 Remark. Equivalently, we can specify a family of continuous maps $h_t\colon X \to Y$ such that $h_0 = f, h_1 = g$ and

$$\begin{aligned} H\colon X \times I &\to Y \\ (x,t) &\mapsto h_t(x) \end{aligned}$$

is continuous. We will switch between the two equivalent definitions without comment, using whatever is more convenient.

1.1.4 Proposition. *For all spaces X and Y, homotopy is an equivalence relation on the set of maps from X to Y. Furthermore, if we are given $k\colon A \to X, \ell\colon Y \to B$ and homotopic maps $f \simeq g\colon X \to Y$, then $fk \simeq gk\colon A \to Y$ and $\ell f \simeq \ell g\colon X \to B$.*

Proof. Let $f, g\colon X \to Y$, then

1. $f \simeq_F f$ via $F(x,t) = f(x)$ for all $x \in X, t \in I$.
2. If $f \simeq_F g$, then $g \simeq_G f$ where $G(x,t) = F(x, 1-t)$.
3. If $f \simeq_F g$ and $g \simeq_G h$, then $f \simeq_H h$ via

$$H(x,t) = \begin{cases} F(x,2t) & \text{if } 0 \leq t \leq 1/2 \\ G(x,2t-1) & \text{if } 1/2 \leq t \leq 1. \end{cases}$$

For the last part of the proposition let f_t be a homotopy between f and g, then $f_t k$ and ℓf_t give the required homotopy. ☐

1.1.5 Definition. For a map $f\colon X \to Y$, we let $[f]$ denote the equivalence class containing f. The collection of all homotopy classes of maps from X to Y is denoted $[X, Y]$.[1]

1.1.6 Remark. Note that if $\alpha = [f] \in [Y, Z]$ and $\beta = [g] \in [X, Y]$, then $\alpha\beta = [f \circ g] \in [X, Z]$, i.e., we can form the category $hTop_*$ whose objects are topological spaces, and whose morphisms are homotopy classes of maps.

1.1.7 Remark. We now very quickly review a number of standard topological constructions.

- Let X be a space and $A \subseteq X$. A map $r\colon X \to A$ such that $ri(a) = a$ for all $a \in A$ is called a retraction of X onto A, and A is called a retract of X.
- Let $i\colon A \hookrightarrow X$ be the inclusion, so that $ri = \mathrm{id}_A$. If $ir \simeq \mathrm{id}_X$, we call this a deformation retraction, and say that A is a deformation retract of X.
- If $f\colon X \to Y$, then a section of f is a map $s\colon Y \to X$ such that $f \circ s = \mathrm{id}_Y$. We can also ask for a *homotopy* section by requiring only that $f \circ s \simeq \mathrm{id}_Y$.

1.1.8 Definition. A map $f\colon X \to Y$ is called null-homotopic if $f\colon c_y\colon X \to Y$ where $c_y X \to Y$ is the constant map sending all of X to the point $y \in Y$. A homotopy between f and c_y is called a null-homotopy. A space X is contractible if id_X is null-homotopic.

1.1.9 Definition. Let (X, x_0) be a based topological space and $X \times I$ the cylinder on X. The quotient

$$CX = (X \times I)/(X \times \{1\} \cup \{x_0\} \times I)$$

with the base-point the equivalence class of $(x_0, 1)$ is called the (reduced) cone on X. Note that we have a natural inclusion $X \to CX$ of based maps given by $x \mapsto [x, 0]$.

1.1.10 Lemma. *The cone CX is contractible.*

Proof. Define $F\colon CX \times I \to CX$ by

$$F([x,t],s) = [x, s + (1-s)t].$$

Note then that we have

$$F([x,t],0) = [x,t] \quad \text{and} \quad F([x,t],1) = [x,1]. \qquad \square$$

1.1.11 Lemma. *The following are equivalent:*

(i) *$f\colon X \to Y$ is null-homotopic.*

(ii) *f can be extended to CX:*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, hook, "i"'] & Y \\ CX \arrow[ur, dashed, "\exists \tilde{f}"'] & \end{tikzcd}$$

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[1] If our spaces are based, then these should be homotopy classes of *based* maps.

Proof. $(i) \implies (ii)$: Suppose f is null-homotopic, so $f \simeq_F *$. Then $F(X \times \{1\} \cup \{*\} \times I) = *$, so by the universal property of the quotient, we can find $\tilde{F}\colon CX \to Y$ such that $\tilde{f} \circ i = f$.

$(ii) \implies (i)$: Suppose $\tilde{f} \circ i = f$, then because CX is contractible (Lemma 1.1.10), we have $f = \tilde{f} \circ \mathrm{id}_{CX} \circ i \simeq \tilde{f} \circ (*_{CX}) \circ i \simeq *$, so that f is null-homotopic. ☐

1.1.12 Definition. A map $f\colon X \to Y$ is a homotopy equivalence if there exists $g\colon Y \to X$ such that $fg \simeq \mathrm{id}_Y$ and $gf \simeq \mathrm{id}_X$. We write $X \simeq Y$.

1.1.13 Example. (i) X is contractible if and only if $X \simeq *$.

(ii) If $i\colon A \hookrightarrow X$, and $r\colon X \to A$ is a deformation retract, then i and r are homotopy equivalences, and $A \simeq X$.

1.2 *Higher homotopy groups*

1.2.1 Notation. We will let $I_n = I^{\times n}, \partial I^n$ be the boundary of I^n, and write $[-,-]$ for homotopy classes of maps (if our spaces are based, these fix the base point).

1.2.2 Definition. For each $n \geq 0$ and X a topological space with $x_0 \in X$, we define

$$\pi_n(X, x_0) = [(I^n, \partial I^n), (X, x_0)].$$

1.2.3 Remark. (i) When $n = 0$, we have $I^0 = \mathrm{pt}$ and $\partial I^0 = \emptyset$, therefore $\pi_0(X)$ is the set of path components of X.

(ii) When $n = 1$, this is a group, but need not be abelian (for example, consider the wedge of two circles).

(iii) Note that $I^n / \partial I^n \simeq S^n$ and $\partial I^n / \partial I^n \simeq s_0$. By the universal property of the quotient map, we see that

$$\pi_n(X, x_0) \cong [(S_n, s_0), (X, x_0)].$$

1.2.4 Definition. A maps of pairs $(X, A) \to (Y, B)$ is a map $f\colon X \to Y$ with $f(A) \subseteq B$, i.e., the diagram:

$$\begin{tikzcd} A \arrow[r] \arrow[d, hook] & B \arrow[d, hook] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

commutes.

1.2.5 Proposition. *If $n \geq 1$, then $\pi_n(X, x_0)$ is a group with respect to the operation*

$$(f+g)(t_1, \ldots, t_n) = \begin{cases} f(2t_1, t_2, \ldots, t_n) & 0 \leq t_1 \leq 1/2 \\ g(2t_1 - 1, t_2, \ldots, t_n) & 1/2 \leq t_1 \leq 1. \end{cases}$$

Proof. The identity is given by the constant map taking all of I^n to x_0 and the inverse of f is given by

$$-f(t_1, \ldots, t_n) = f(1 - t_1, t_2, \ldots, t_n). \qquad \square$$

1.2.6 Remark. Call the group operation $+_1$. Note that we can also define an operation $+_i$ for $1 \le i \le n$ by the same formula on the i-th coordinate.

1.2.7 Theorem. *All of these operations agree, and for $n \geq 2$, these give $\pi_n(X, x_0)$ the structure of an abelian group.*

This is a consequence of the following exercise, known as the Eckmann–Hilton lemma.

Exercise 1 (Eckmann–Hilton lemma)**.** *Let M be a set and let $*, \bullet$ be two binary operations on M, $*, \bullet \colon M \times M \to M$, both with unit elements. Suppose that*

$$(a * b) \bullet (c * d) = (a \bullet c) * (b \bullet d)$$

for all $a, b, c, d \in M$. Show that the units agree, these two operations agree, and that the multiplication is commutative and associative.

1.2.8 Remark. Let use show that

$$(f +_1 g) +_2 (h +_1 i) \simeq (f +_2 h) +_1 (g +_2 i).$$

Indeed, both of these are the following map

$$(t_1, t_2, \ldots,) \mapsto \begin{cases} f(2t_1, 2t_2, \ldots,) & [1/2, 0] \times [1/2, 0] \\ g(2t_1 - 2, 2t_2, \ldots,) & [1/2, 1] \times [0, 1/2] \\ h(2t_1, 2t_2 - 2, \ldots) & [0, 1/2] \times [1/2, 1] \\ i(2t_1 - 1, 2t_2 - 2, \ldots) & [1/2, 1] \times [1/2, 1]. \end{cases}$$

1.2.9 Remark. Another approach is given by the following visualization: That is, so long as $n \geq 2$, we can shrink the domain of f and g to smaller cubes (mapping the remaining region to the base point), slide f and g past each other, and then increase the domains back again.

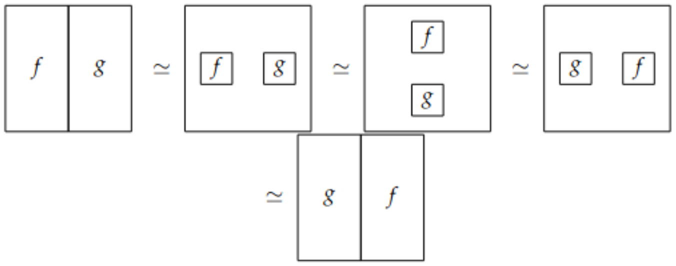


Figure 1.2: $f + g \simeq g + f$.

Exercise 2. *Let G be a topological group with identity element e, then $\pi_1(G, e)$ is abelian.*[2]

[2] **Hint:** Use Eckmann–Hilton, or note the following: A topological group is a group object in the category of topological spaces. What is a group object in the category of groups?

1.2.10 Proposition. *If $n \geq 1$ and X is path connected, then there is an isomorphism $\beta_\gamma : \pi_n(X, x_0) \xrightarrow{\simeq} \pi_n(X, x_0)$ given by $\beta_\gamma([f]) = [\gamma \circ f]$ where γ is a path in X from x_1 to x_0 and $\gamma \circ f$ is constructed by first shrinking the domain of f to a smaller cube inside of I^n, and then inserting the path γ radially from x_1 to x_0 on the boundaries of these cubes.*

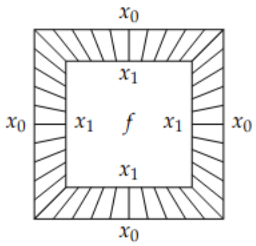
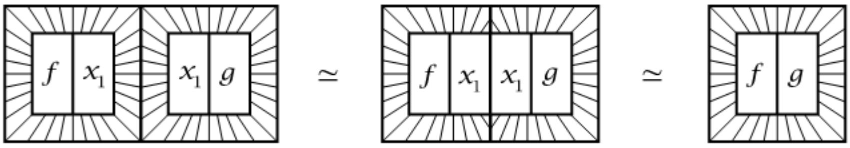


Figure 1.3: β_γ.

Proof. Observe the following:

1. $\gamma \circ (f + g) \simeq \gamma \circ f + \gamma \circ g$, i.e., β_γ is a group homomorphism.
2. $(\gamma \circ \eta) \circ f \simeq \gamma \circ (\eta \circ f)$, for η a path from x_0 to x_1.
3. $c_{x_0} \circ f \simeq f$, where c_{x_0} denotes the constant path based at x_0.
4. β_γ is well-defined with respect to homotopies of f or γ.

The only point that is perhaps not clear is (i). For this, we deform f and g to be constant on the right and left halves of I^n, respectively, producing maps we call $f + 0$ and $0 + g$. We then excise a wider symmetric middle slab of $\gamma(f + 0)$ and $\gamma(0 + g)$ until it becomes $\gamma(f + g)$: □



1.2.11 Remark. Therefore if X is path-connected, different choices of base point x_0 yield isomorphic groups $\pi_n(X, x_0)$, which may then simply be written as $\pi_n(X)$.

1.2.12 Lemma. *If $\{X_\alpha\}$ is a collection of path-connected spaces, then $\pi_n(\prod_\alpha X_\alpha) \cong \prod_\alpha \pi_n(X_\alpha)$.*

Proof. Note that $\mathrm{Hom}(Y, \prod_\alpha X_\alpha) \simeq \prod_\alpha \mathrm{Hom}(Y, X_\alpha)$. In particular, a map $S^n \to \mathrm{Hom}(Y, \prod_\alpha X_\alpha)$ is determined by a collection of maps $S^n \to X_\alpha$. Likewise, a homotopy $S^n \times I \to \prod_\alpha X_\alpha$ is determined by a colletion of homotopies $S^n \times I \to X_\alpha$. This implies the result. □

1.2.13 Proposition. *Homotopy groups are functorial: given a map $\phi\colon X \to Y$ we get group homomorphisms $\phi_*\colon \pi_n(X, x_0) \to \pi_n(X, \phi(x_0))$ given by $[f] \mapsto [\phi \circ f]$ for all $n \geq 1$.*

Proof. We have the following:

1. ϕ_* is well-defined: if $f \simeq g$ via ψ_t, then $\phi \circ \psi_t$ defines a homotopy between $\phi \circ f$ and $\phi \circ g$.

2. This is a group homomorphism: $\phi \circ (f+g) \simeq \phi \circ g + \phi \circ g$ by the definition of the addition operation. Therefore.
$$\phi_*[f+g] = \phi_*[f] + \phi_*[g].$$

□

Exercise 3. *If $\phi\colon X \to Y$ is homotopy equivalence (not necessarily base-point preserving), then $\pi_*\colon \pi_n(X, x_0) \to \pi_n(Y, \phi(y_0))$ is an isomorphism.*

1.2.14 Remark. We recall the following lifting property: Suppose $p\colon (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ is a covering, and there is a map $f\colon (Y, y_0) \to (X, x_0)$ with Y path-connected and locally path-connected. Then a lift $\tilde{f}$ exists if and only if $f_*\pi_1(Y, y_0) \subseteq p_*\pi_1(\tilde{X}, \tilde{x}_0)$.

$$\begin{tikzcd} & (\tilde{X}, \tilde{x}_0) \arrow[d, "p"] \\ (Y, y_0) \arrow[r, "f"] \arrow[ur, dashed, "\tilde{f}"] & (X, x_0) \end{tikzcd}$$

1.2.15 Proposition. *If p is a covering, then $p_*\colon \pi_n(\tilde{X}, \tilde{x}_0) \to \pi_n(X, x_0)$ is an isomorphism for all $n \geq 2$.*

Proof. Let us first show surjectivity. To that end, suppose we have a map $f\colon (S^n, s_0) \to (X, x_0)$ where $n \geq 2$. The assumption on n gives $\pi_1(S^n) = 0$, so $f_*\pi_1(S^n, s_0) \subseteq p_*\pi_1(\tilde{X}, \tilde{x}_0)$ holds. We therefore find a lift in the following:

$$\begin{tikzcd} & (\tilde{X}, \tilde{x}_0) \arrow[d, "p"] \\ (S^n, s_0) \arrow[r, "f"] \arrow[ur, dashed, "\tilde{f}"] & (X, x_0) \end{tikzcd}$$

Then $p_*[\tilde{f}] = [f]$, and p_* is surjective.

To see that p_* is injective, let $[\tilde{f}] \in \ker(p_*)$, i.e., $p_*[\tilde{f}] = [p \circ \tilde{f}] = 0$. Let $f = p \circ \tilde{f}$, then this is homotopic to the constant map $f \simeq c_{x_0}$ via a homotopy $\phi_t\colon (S^n, s_0) \to (X, x_0)$ with $\phi_1 = f$ and $\phi_0 = c_{x_0}$. By the same argument as above, the homotopy ϕ_t can be lifted to $\tilde{\phi}_t$. This satisfies $p \circ \tilde{\phi}_1 \simeq \phi_1$ and $p \circ \tilde{\phi}_0 \simeq \phi_0$. By the uniqueness of lifts, we must have $\tilde{\phi}_1 \simeq \tilde{f}$ and $\tilde{\phi}_0 \simeq c_{x_0}$. In other words, $\tilde{\phi}_t$ gives a homotopy between $\tilde{f}$ and c_{x_0}, so that $[\tilde{f}] = 0$, and p_* is injective. □

1.2.16 Example. S^1 has universal cover $p\colon \mathbb{R} \to S^1$, $p(t) = e^{2\pi i t}$. Then $\pi_n(S^1) \cong \pi_n(\mathbb{R}) \cong 0$ for $n \geq 2$.

Exercise 4. *Find two spaces X, Y with $\pi_n X \cong \pi_n Y$ but $X \not\simeq Y$.*

Hint: *What is the universal cover of $\mathbb{R}P^n$?*

1.2.17 *Remark* (Relative homotopy groups). Suppose we have (X, x_0) and a subspace A containing x_0. We note that $i_*\colon \pi_n(A, x_0) \to \pi_n(X, x_0)$ is not injective in general (example, take S^1 into $\mathbb{R}^2$). An element in the kernel of i_* is a map $f\colon (I^n, \partial I^n) \to (A, x_0)$ such that

$$(I^n, \partial I^n) \xrightarrow{f} (A, x_0) \xrightarrow{i} (X, x_0)$$

is homotopic to c_{x_0}. This means there exists a homotopy

$$H\colon I^n \times I \to X$$

such that $H(-,1) = f, H(-,0) = c_{x_0}$ and $H\mid_{\partial I^n \times I} = c_{x_0}$.

If we define $J^n = I^n \times \{0\} \cup \partial I^n \times I \subseteq I^n \times I$, then this is a map of triples

$$H\colon (I^{n+1}, \partial I^{n+1}, J^n) \to (X, A, x_0).$$

1.2.18 *Definition.*

$$\pi_n(X, A, x_0) = [(I^n, \partial I^n, J^{n-1}), (X, A, x_0)]$$

1.2.19 *Remark.* Equivalently,

$$\pi_n(X, A, x_0) = [(D^n, S^{n-1}, s_0), (X, A, x_0)]$$

1.2.20 Proposition. *If $n \geq 2$, then $\pi_n(X, A, x_0)$ is a group, and if $n \geq 3$, then it is abelian.*

For all $n \geq 2$, a map of pairs $\phi\colon (X, A, x_0) \to (Y, B, y_0)$ induces homomorphisms $\phi_*\colon \pi_n(X, A, x_0) \to \pi_n(Y, B, y_0)$ for all $n \geq 2$.

Proof. This is similar to the case of $\pi_n(X)$ itself, and the details are left to the reader. □

1.2.21 Theorem. *The relative homotopy groups (X, A, x_0) fit into a long exact sequence*

$$\cdots \to \pi_n(A, x_0) \xrightarrow{i_*} \pi_n(X, x_0) \xrightarrow{j_*} \pi_n(X, A, x_0) \xrightarrow{\partial_n} \pi_{n-1}(A, x_0) \to \cdots$$

where the map ∂_n is defined by $\partial_n([f]) = [f\mid_{I^{n-1}}]$.

The proof relies on the following.

1.2.22 Lemma (Compression criterion). *A map $f\colon (D^n, S^{n-1}, x_0) \to (X, A, x_0)$ represents 0 in $\pi_n(X, A, x_0)$ if and only if $f \sim g$ rel S^{n-1}, where g is a map whose image is contained entirely in A.*

Proof. Suppose $[f] = [g]$ with g as in the statement of the lemma. Note that there is a deformation of D^n onto x_0, and so $[f] = [g] = 0$ in $\pi_n(X, A, x_0)$.

Conversely, suppose that $[f]$ represents 0 in $\pi_n(X, A, x_0)$. This means there exists a homotopy, relative to S^{n-1}, $F\colon D^n \times I \to X$ with $F\mid_{D^n \times \{0\}} = f$, $F\mid_{D^n \times 1} = c_{x_0}$ and $F\mid_{S^{n-1} \times I} \subseteq A$. We can restrict F to a family of n-disks in $D^n \times I$ starting with $D^n \times \{0\}$ and ending with the disk $D^n \times \{1\} \cup S^{n-1} \times \{1\}$, all the disks in the family having the same boundary, then we get a homotopy from f to a map in A, stationary on S^{n-1} (said in other words, we can deformation retract $D^n \times [0,1]$ onto $D^n \times \{1\} \cup S^{n-1} \times I$). □

We now prove the existence of the long exact sequence.[3]

Proof of Theorem 1.2.21. **Step 1.** Let us first show exactness at $\pi_n(X, x_0)$.

We first show $\mathrm{im}(i_*) \subseteq \ker(j_*)$. Note that $j_* i_*$ is induced by the composition $j \circ i$ and that these are both inclusion maps. Therefore, for $[f] \in \pi_n(A, x_0)$ we have $j_* i_*[f] = [j \circ i \circ f]$, but this has image contained in A, and so $j_* i_*[f] = 0$. This shows $\mathrm{im}(i_*) \subseteq \ker(j_*)$.

To see the converse (namely, $\ker(j_*) \subseteq \mathrm{im}(i_*)$) let $[f] \in \ker(j_*)$, i.e. $[j \circ f] = 0$. Note that again j is an inclusion map, and by the compression criteria $f \simeq g'$ relative to S^{n-1}, where g' has image contained in A. Since $x_0 \in S^{n-1}$, the homotopy fixes the basepoint, i.e, $[f] = [g'] \in \pi_n(X, x_0)$. But because g' has image in A, $[g'] \in \pi_n(A, x_0)$ and $i_*[g'] = [i \circ g'] = [f]$, so $[f] \in \mathrm{im}(i_*)$.

Step 2. Let us now show exactness at $\pi_n(X, A, x_0)$.

Note that the composite $\partial \circ j_* = 0$ since the restriction of a map $(I^n, \partial I^n, J^{n-1}) \to (X, x_0, x_0)$ to I^{n-1} has image x_0 and so represents 0 in $\pi_{n-1}(A, x_0)$. Therefore, $\mathrm{im}(j_*) \subseteq \ker(\partial)$. For the converse, suppose $[f] \in \ker(\partial)$. This means there exists a basepoint preserving homotopy $H\colon I^{n-1} \times I \to A$ (relative to ∂I^{n-1}) from $f\mid_{I^{n-1} \times \{0\}}$ to the constant map where the image of H is contained entirely in A. We can then define another homotopy H, such that $G_0 = f$, $G_t \mid_{I^{n-1}} = H_t$ and the rest of the image of G_t is $f[I^n]$ union with the image of H_s for $0 \le s \le t$. This homotopy maps S^{n-1} into A at all times, so $[f] = [G_1]$. Moreover, G_1 maps the boundary of I^n to x_0, so $[G_1] \in \pi_n(X, x_0)$. Altogether,

$$j_*[G_1] = [j \circ G_1] = [G_1] = [f]$$

so $\ker(\partial) \subseteq \mathrm{im}(j_*)$.

Step 3: Exactness at $\pi_n(A, x_0)$.

Let $[f] \in \pi_n(X, A, x_0)$ then $i_*\partial \in \pi_{n-1}(X, x_0)$ is the class represented by $f \mid_{I^{n-1}}$ and this is homotopic relative J^{n-2} to the constant map to x_0, via f viewed as a homotopy. So this implies $\mathrm{im}(\partial_*) \subseteq \ker(i_*)$. Conversely, let $[f] \in \ker(i_*)$ i.e., $i_*[f] = [i \circ f] = 0$. Therefore, there exists a homotopy H between f and a constant map through a homotopy that has image in X and preserves x_0. Since $H_0 = f$ has image in A and H_1 has image $\{x_0\}$, and H_0 takes the boundary to $\{x_0\}$, we see that $[H] \in \pi_n(X, A, x_0)$, and moreover $\partial([H]) \simeq f$. Therefore, $[f] \in \mathrm{im}(\partial)$, and $\mathrm{im}(\partial) = \ker(i_*)$. □

1.2.23 Definition. A pair (X, A) with basepoint x_0 is said to be n-connected if $\pi_i(X, A) = 0$ for all $i \le n$.[4]

1.2.24 Lemma. *A pair (X, A) is n-connected if and only if $\pi_i(A) \xrightarrow{i_*} \pi_i(X)$ is an isomorphism for $i < n$ and a surjection for $i = n$.*

Proof. Use the long exact sequence in homotopy. □

Exercise 5. *Let X be a path-connected space, and CX the cone on X. Show that*

$$\pi_n(CX, X, X_0) \cong \pi_{n-1}(X, x_0)$$

for $n \ge 1$.

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[3] This is the type of proof that is best done by the reader themselves.

[4] A 0-connected space is exactly a path-connected space.

1.3 Cofibrations and the homotopy extension property

1.3.1 Definition. Let $\mathcal{C}$ be a class of topological spaces. A map $i\colon A \to X$ has the homotopy extension property (HEP) if, for every $Y \in \mathcal{C}$, the following extension property has a solution[5]

$$\begin{tikzcd}
A \arrow[r, hook, "i_0"] \arrow[d, "i"'] & A \times I \arrow[d, "i\times\mathrm{id}"] \arrow[ddr, bend left, "H"] & \\
X \arrow[r, hook, "i_0"'] \arrow[drr, bend right, "f"'] & X \times I \arrow[dr, dashed, "\exists\tilde{H}"] & \\
 & & Y
\end{tikzcd}$$

A map $f\colon A \to X$ is a cofibration if it has the HEP with respect to all spaces Y.[6]

1.3.2 Remark. Note that we do not ask that $\tilde{H}$ is unique.

1.3.3 Remark. If we are in a 'nice' category of topological spaces (see CREF), which we always assume, then we have an adjunction

$$\mathrm{Hom}(X, \mathrm{Hom}(Y, Z)) \cong \mathrm{Hom}(X \otimes Y, Z)$$

of topological spaces, where $\mathrm{Hom}(Y, Z)$ is given the compact open topology. Writing, $Z^Y := \mathrm{Hom}(Y, Z)$, the homotopy extension property admits a reformulation in the following diagram

$$\begin{tikzcd}
A \arrow[r, "h"] \arrow[d, "i"'] & Y^I \arrow[d, "p"] \\
X \arrow[r, "f"'] \arrow[ur, dashed, "\exists\tilde{h}"] & Y
\end{tikzcd}$$

where $p\colon Y^I \to Y$ is the evaluation at 0 map. It is often easier to work with this equivalent diagram.

Exercise 6. *Let (X, A) have the HEP, and assume moreover that $i\colon A \to X$ is a retract up to homotopy. Show that A is a retract of X.*

1.3.4 Lemma. *Let $J = [0, 1]$.*

(i) The inclusion $i_0\colon X \to X \times J$ has the homotopy extension property for all Y.

(ii) The inclusion $i_0\colon X \to CX$ has the homotopy extension property for all Y.

Proof. The proof in both cases is very similar; we do the first case in some detail. We are claiming there exists a lift $\tilde{H}$ in the following diagram:

$$\begin{tikzcd}
X \arrow[r, hook, "i_0"] \arrow[d, "i"'] & X \times I \arrow[d, "i\times\mathrm{id}"] \arrow[ddr, bend left, "H"] & \\
X \times J \arrow[r, hook, "i_0"'] \arrow[drr, bend right, "f"'] & X \times J \times I \arrow[dr, dashed, "\exists\tilde{H}"] & \\
 & & Y
\end{tikzcd}$$

---

[5] Here $i_0(x) = (x, 0)$.

[6] We will see later that cofibrations are always inclusions, and, if X is Hausdorff, are always closed maps.

Geometrically, we will do this in two parts: we will define a map that "stacks" the two intervals on top of each other, i.e., we construct a map $G\colon X \times J \times I \to X \times [0,2]$. We will then do H on one part of the cylinder, and f on the remaining part.

For the first part, let $G\colon X \times J \times I \to X \times [0,2]$ be defined as[7]

$$G(x,t,s) = (x, t(1+s)).$$

We then define $F\colon X \times [0,2] \to Y$ by

$$F(x,k) = \begin{cases} f(x,k) & 0 \le k \le 1 \\ H(x,k/2) & 1 \le k \le 2. \end{cases}$$

Putting these together and defining $\tilde{H} := F \circ G$, we see that[8]

$$\tilde{H}((x,t),s) = \begin{cases} f(x, 1-(1-t)(1+s)), & (1-t)(1+s) \le 1 \\ H(x,(1-t)(1+s)-1), & (1-t)(1+s) \ge 1. \end{cases}$$

One verifies directly that this gives the required extension. □

1.3.5 Remark. We recall that given a map $f\colon X \to Y$, the mapping cylinder (see Figure 1.4) is the pushout

$$\begin{tikzcd} X \arrow[r, "i_0"] \arrow[d, "f"'] \arrow[dr, phantom, "\lrcorner", very near end] & X \times I \arrow[d] \\ Y \arrow[r] & M_f \end{tikzcd}$$

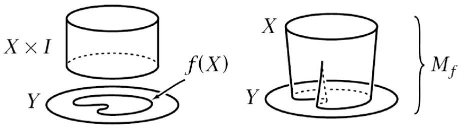

Figure 1.4: The mapping cylinder.

In formulas,

$$M_f = ((X \times I) \coprod Y)/((0,x) \sim f(x),\ \forall x \in X)$$

Note that M_f deformation retracts on Y by sliding each point $(x,t) \in M_f$ to the end-point. Note that we have a natural map $j\colon X \to M_f$ sending x to $(x,1)$.

1.3.6 Lemma. *The map $j\colon X \to M_f$ has the HEP for all spaces Y.*

Proof. The proof is similar to the previous lemma; one just has to modify the end point by defining

$$\tilde{H}|_{Y \times I}(y,s) = f(y,0).$$

□

1.3.7 Corollary. *The inclusion $S^{n-1} \to D^n$ is a cofibration.*

Proof. Simply note that $D^n \simeq CS^{n-1}$. □

There is a universal test space for cofibrations.

1.3.8 Proposition. *Let $i\colon A \to X$, and let M_i be the mapping cylinder. Then $i\colon A \to X$ is a cofibration if and only if there exists a map $r\colon X \times$*

[7] To see what is going on it is worth testing some cases and drawing pictures. For example, when $t = 0$ we have $G(x,0,s) = (x,0)$. When $t = 1$ we have $G(x,1,s) = (x, 1+s)$. When $s = 0$ we have $G(x,t,0) = (x,t)$ and when $s = 1$ we have $G(x,t,1) = (x,2t)$.

[8] Again, it is worthwhile to consider some cases. For example, if $t = 0$, then $(1-t)(1+s) = (1+s) \ge 1$ for all s, so $\tilde{H}((x,0),s) = H(x,s)$. At the other extreme, if $t = 1$, then $(1-t)(1+s) = 0 \le 1$ for all s, so $\tilde{H}((x,1),s) = f(x,1)$.

$I \to M_i$ making the diagram

$$\begin{tikzcd}
A \arrow[rr, "i_0"] \arrow[dd, "i"'] & & A \times I \arrow[dl, "i \times id"'] \arrow[dd] \\
& X \times I \arrow[dr, dashed, "\exists r"] & \\
X \arrow[ur, "i_0"] \arrow[rr] & & M_i
\end{tikzcd}$$

commute.

Proof. If i is a cofibration, then the map r exists as a consequence of the HEP.

For the other direction, if r exists, then for any maps $f\colon X \to Y$ and $H\colon A \times I \to Y$ making the obvious diagram commute, the universal property of the pushout gives us a map $H'\colon M_i \to Y$. Then let $\tilde{H} = H' \circ r$, and we are done. □

1.3.9 Corollary. *If $A \subseteq X$, then $I\colon A \to X$ is a cofibration if and only if $X \times I$ is a retract of $M_i = X \times \{0\} \cup A \times I$.*

1.3.10 Corollary. *A cofibration $i\colon A \to X$ is an injection. If X is Hausdorff, then $i(A)$ is closed in X.*

Proof. Let $J\colon A \times I \to M_i$ be the canonical map (arising from the definition of M_i as a pushout). Then, $J(a, 1) = r(i(a), 1)$, and observe that $J\mid_{A \times \{1\}}$ is the identity, as it is the top of the mapping cylinder. So, $i(a) \neq i(a')$ if $a \neq a'$, i.e., i is injective.

Because $i\colon A \to X$ is a cofibration, so is $i(A) \to X$. Hence $X \times I$ retracts onto $X \times \{0\} \cup i(A) \times I$ (Corollary 1.3.9). For a Hausdorff space, the image of a retract is closed, and so $X \times \{0\} \cup i(A) \times I$ is a closed subspace of $X \times I$. Intersecting with $X \times \{1\}$, we see that $i(A) \times \{1\}$ is closed in $X \times \{1\}$, i.e, $i(A)$ is closed in X. □

The following (rather pathological) example shows that i is not always a closed map if X is not Hausdorff.

Exercise 7. *Let $A = \{a\}$ and $X = \{a, b\}$ with the trivial topology. Show that the inclusion $A \to X$ is a cofibration whose image is not closed.*

1.3.11 Remark. The next goal is to show that CW-complexes (X, A) are always cofibrations. The key is the following exercise.

Exercise 8.*(a) Suppose $\{(X_i, A_i)\}$ are a collection of spaces satisfying the HEP, then so does $\{(\coprod X_i, \coprod A_i)\}$.*

(b) Suppose (X, A) satisfies the HEP, and $f\colon A \to B$ is a continuous map. Let $Y = X \cup_f B$ be the pushout, then (Y, B) satisfies the HEP.

(c) Suppose $A = X_0 \subseteq X_1 \subseteq \cdots \subseteq X_n \subseteq X_{n+1} \subseteq \cdots$.

Let $X = \operatorname{colim} X_i$. If each (X_i, X_{i-1}) satisfies the HEP, then so does (X, A).

1.3.12 Theorem. *A relative CW-complex (X, A) satisfies the HEP.*

Proof. Using Corollary 1.3.7 and the previous exercise we see that (S^{n-1}, D^n) satisfies the HEP $\implies$ $(\coprod S^{n-1}, \coprod D^n)$ satisfies the HEP. Inductively, (X_{n-1}, A) satisfies the HEP and by the exercise (X, A) satisfies the HEP. ☐

1.3.13 Remark. One can also prove this directly by constructing a deformation retract $r \colon X \times I \to X \times \{0\} \cup A \times I$.

1.3.14 Remark. One can consider the following question: Suppose that $A \subset X$ with A contractible, then is $X \simeq X/A$? Surprisingly, this is not true in general. Indeed, let $A = S^1 \setminus \{(1,0)\}$ and consider $A \to S^1$. Then $S^1/A \cong T$, the $T = \{a, b\}$ the two point space with open sets $\emptyset, \{a\}, \{a, b\}$ (this is the Sierpiński space). One can check that this space is contractible.[9] The exact condition we need is that $A \to X$ is a cofibration.

1.3.15 Definition. A contracting homotopy is a map $H \colon X \times I \to X$ such that $H(x, 0) = \mathrm{id}_X$ and $H(x, 1) = c_{x_0}$, the constant map at x_0.

1.3.16 Proposition. *Suppose $A \subseteq X$ and $x_0 \in A$. Suppose there exists a map $H \colon X \times I \to X$ such that $H\,|_{X \times \{0\}} = id_X$ and $H\,|_{A \times I}$ has image in A and is a contacting homotopy for A. Then $q \colon X \to X/A$ is a homotopy equivalence.*

Proof. We need to find $p \colon X/A \to X$ such that $q \circ p \simeq \mathrm{id}_{X/A}$ and $p \circ q \simeq \mathrm{id}_X$. The quotient map has a set-theoretic section given by

$$s(\overline{x}) = \begin{cases} x & x \notin A \\ x_0 & x \in A \end{cases}$$

Define $p \colon X/A \to X$ by the following diagram

$$\begin{tikzcd} X \arrow[r, "q"] & X/A \arrow[r, "s"] \arrow[dr, "p"'] & X \arrow[d, "H|_{X\times\{1\}}"] \\ & & X \end{tikzcd}$$

Assume for a moment that p is continuous. Then $p \circ q = H\,|_{X \times \{1\}}$, and so H gives a homotopy between id_X and $p \circ q = H\,|_{X \times \{1\}}$. Likewise, if we define G by

$$\begin{tikzcd} X/A \times I \arrow[r, "s \times \mathrm{id}"] \arrow[drr, "G"'] & X \times I \arrow[r, "H"] & X \arrow[d, "q"] \\ & & X/A \end{tikzcd}$$

and assume that G is continuous, then

$$G(\overline{x}, 1) = q \circ (H\,|_{X \times \{1\}} \circ s) = q \circ p,$$

so that G is a homotopy between $\mathrm{id}_{X/A}$ and $q \circ p$. To see that p is continuous, let $U \subset X$ be open, then

$$q^{-1} p^{-1}(U) = (p \circ q)^{-1}(U) = (H\,|_{X \times \{1\}})^{-1}(U)$$

is open in X by the continuity of $H\,|_{X \times \{1\}}$, hence $p^{-1}(U)$ is open in X/A by the definition of the quotient topology, and so p is continuous. We leave the proof of continuity of G to the reader. ☐

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[9] See `https://math.stackexchange.com/a/264789/64273`.

1.3.17 Theorem. *Let $A \subseteq X$ be a subspace with A contractible. Suppose that the inclusion $i\colon A \to X$ is a cofibration, then $X \to X/A$ is a homotopy equivalence.*

Proof. Let $h\colon A \to I \to A$ be a contracting homotopy. Let $H\colon A \times I \to X$ be the composition of h with the inclusion map of A into X, i.e., the following diagram commutes:

$$\begin{tikzcd}
A \arrow[r, hook, "i_0"] \arrow[d, "i"'] & A \times I \arrow[d, "i\times\mathrm{id}"] \arrow[ddr, bend left, "H"] & \\
X \arrow[r, hook, "i_0"'] \arrow[drr, bend right, "\mathrm{id}_X"'] & X \times I \arrow[dr, dashed, "\exists\tilde{H}"] & \\
 & & X
\end{tikzcd}$$

By the HEP, the dotted map $\tilde{H}$ exists as in the diagram. This map satisfies the conditions of Proposition 1.3.16:

(i) $\tilde{H}\colon X \times \{0\} \to X$ is the identity.

(ii) $\tilde{H}(A \times I) = H(A \times I) = h(A \times I) \subseteq A$.

(iii) $\tilde{H}(A \times \{1\}) = x_0$.

Therefore, $q\colon X \to X/A$ is a homotopy equivalence, as claimed. □

Exercise 9 (label=ex:cofibratonpushout)**.** *Cofibrations are pushout closed. Let $i\colon A \to X$ be a cofibration, and $g\colon A \to B$ any map, then the induced map $B \to B \cup_g X$ is a cofibration.*

1.4 *Fibrations and the homotopy lifting property*

The dual notion of a cofibration is a fibration, where the homotopy extension property is replaced by the homotopy lifting property.

1.4.1 Definition. Let $\mathcal{E}$ be a class of topological spaces. Assume that $p\colon E \to B$ is a continuous map, then we say that p has the homotopy lifting property (with respect to $\mathcal{E}$) if for every $X \in \mathcal{E}$, and map $f\colon X \to E$ and every homotopy $H\colon X \times I \to B$ that begins with $p \circ f$, we can lift it to a homotopy $\tilde{H}\colon X \times I \to E$ that begins with f, i.e., $p \circ \tilde{H} = H$ and $\tilde{H}(x, 0) = f(x)$. In a diagram, we require the lift $\tilde{H}$ in the following:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "i_0"'] & E \arrow[d, "p"] \\
X \times I \arrow[r, "H"'] \arrow[ur, dashed, "\exists\tilde{H}"] & B
\end{tikzcd}$$

If $\mathcal{E}$ is the class of all topological spaces, then p is called a (Hurewicz) fibration, while if $\mathcal{E} = \{I^n\}$ (or equivalently, the class of CW-complexes), then p is called a Serre fibration.

1.4.2 Remark. As in Remark 1.3.3, there is an equivalent way to present the homotopy lifting property: we ask for the lift $\tilde{h}$ as shown in the following

$$\begin{tikzcd}
X \arrow[drr, bend left, "f"] \arrow[dr, dashed, "\exists \tilde{h}"] \arrow[ddr, bend right, "h"'] & & \\
& E^I \arrow[r, "ev_0"] \arrow[d, "p_*"'] & E \arrow[d, "p"] \\
& B^I \arrow[r, "ev_0"'] & B
\end{tikzcd}$$

This makes it clear how the homotopy lifting property is dual to the homotopy extension property.

1.4.3 Remark. We can also talk about the homotopy lifting property with respect to a pair (X, A): namely, a map $p\colon E \to B$ has the homotopy lifting property with respect to a pair (X, A) if each homotopy $H\colon X \times I \to B$ lifts to a homotopy $\tilde{H}\colon X \times I \to E$ which agrees with a given homotopy H_A on $A \times I$. In a diagram, we ask for the lift $\tilde{H}$ in the following:

$$\begin{tikzcd}
X \cup (A \times I) \arrow[r, "f \cup H_A"] \arrow[d, "i_0"'] & E \arrow[d, "p"] \\
X \times I \arrow[r, "H"'] \arrow[ur, dashed, "\exists \tilde{H}"] & B
\end{tikzcd}$$

1.4.4 Theorem. *The following are equivalent:*

(i) p is a Serre fibration.

(ii) p has the homotopy lifting property with respect to all n-discs D^n.

(iii) p has relative homotopy property with respect to all pairs (D^n, S^{m-1})

(iv) p has the relative homotopy property with respect to all CW-pairs (X, A).

Proof sketch. $(i) \implies (ii)$ is immediate from the definitions.

$(ii) \implies (iii)$ follows because the pairs $(D^n \times I, D^n \times \{0\})$ and $(D^n \times I, D^n \times \{0\} \cup S^{n-1} \times I)$ are homeomorphic.

$(iii) \implies (iv)$ by induction over the skeleton of X; one reduces to the case (iii).

$(iv) \implies (i)$ by taking $A = \emptyset$. □

Exercise 10. *Show that the composition of fibrations is a fibration.*

1.4.5 Definition. We recall the construction of pullbacks in topological spaces: given maps $p\colon E \to B$ and $f\colon B' \to B$, we let

$$E' = \{(b', e) \in B' \times E \mid p(e) = f(b')\}.$$

This comes with natural projection maps $f'\colon E' \to E$ and $p'\colon E' \to B'$. Then E' is the pull-back in topological spaces, and so we often also denote it by f^*E.

The following is dual to **??**.

1.4.6 Lemma. *If $p\colon E \to B$ satisfies the HLP with respect to the class $\mathcal{E}$, then so does $p'\colon E' \to B'$.*

Proof. Consider the following diagram:

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "i_0"'] & E' \arrow[r, "f'"] \arrow[d, "p'"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "p"] \\
X \times I \arrow[r] & B' \arrow[r, "f"'] & B
\end{tikzcd}$$

Because $p\colon E \to B$ satisfies the HLP, there is a lift $\tilde{H}'\colon X \times I \to E$ of $X \times I \to B$. Then, by the universal property of the pullback, we get a map $\tilde{H}\colon X \times I \to E'$ satisfying the desired properties. □

1.4.7 Definition. If $p\colon E \to B$ is a fibration, then $F := p^{-1}(*)$ is called the fiber, E is called the total space, and B is the base space. We write this as

$$F \to E \to B.$$

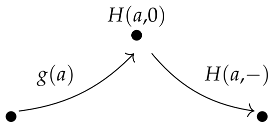
1.4.8 Example. Given a based space X, let

$$PX = \mathrm{Hom}_*(I, X) = \{f\colon I \to X \mid f(0) = *\}$$

be the space of paths starting at the base-point. Then $PX \xrightarrow{p_1} X$ is a fibration with fiber ΩX, the loop space in X (i.e., $f(0) = f(1) = *$). To see this, consider our test diagram, where we must show that $\tilde{H}$ exists:

$$\begin{tikzcd}
A \arrow[r, "g"] \arrow[d, "i_0"'] & PX \arrow[d, "p_1"] \\
A \times I \arrow[r, "H"'] \arrow[ur, dashed, "\exists \tilde{H}"] & X
\end{tikzcd}$$

Note that for each $a \in A$, $g(a)$ is a path in X which ends at $p_1 g(a) = H(a, 0)$. This point is the start of the path $H(a, -)$.



We will define $\tilde{H}(a,s)(t)$ to be a path running along $g(a)$ and then part-way along $H(a, -)$ ending at $H(a, s)$. In symbols,

$$\tilde{H}(a,s)(t) = \begin{cases} g(a)((1+s)t) & 0 \le t \le 1/(1+s) \\ H(a, (1+s)t - 1) & 1/(1+s) \le t \le 1. \end{cases}$$

Then $\tilde{H}(a, 0) = g(a)$ and $p_1 \tilde{H}(a, s) = \tilde{H}(a, s)(1) = H(a, s)$, as required.

The same argument shows that there is a fibration

$$p_* Y \to Y^I \xrightarrow{p_1} Y$$

where $p_* Y$ is the space of paths with end-point $*$.

1.4.9 Remark (The path-space fibration). The fibration $\Omega X \to PX \to B$ is known as the path-space fibration. Note that the space PX is contractible.

1.4.10 Definition. Given $f\colon X \to Y$ the mapping path space P_f (or mapping cocylinder), is the pullback of f along $Y^I \xrightarrow{p_1} Y$, i.e.,

$$\begin{tikzcd}
P_f \arrow[r] \arrow[d, "p'"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y^I \arrow[d, "p_1"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

Note that $P_f \simeq X$.

1.4.11 Proposition. *The map $p\colon P_f \to Y$ given by $p(x, \alpha) = \alpha(1)$ is a fibration.*

Proof. This is very similar to Example 1.4.8. Our test diagram is the following:

$$\begin{tikzcd}
A \arrow[r, "g"] \arrow[d, hook, "i_0"'] & P_f \arrow[d, "p"] \\
A \times I \arrow[r, "H"'] \arrow[ur, dashed, "\exists \tilde{H}"] & Y
\end{tikzcd}$$

Note that $g(a) \in P_f \subset X \times Y^I$, so we can write $g(a) = (g_1(a), g_2(a))$. Here $g_1(a)$ maps via f to the starting point of the path $g_2(a)$ and the commutativity of the diagram implies that the endpoint of the path $g_2(a)$ is the starting point of $H(a, -)$. The lift $\tilde{H}$ will have two components. The x component will be constant in s, i.e., $\tilde{H}_1(a, s) = g_1(a)$. Overall, we define

$$\tilde{H}(a, s) = (g_1(a), \tilde{H}_2(a, s)(-)) \in P_f$$

where[10]

$$\tilde{H}_2(a, s)(t) = \begin{cases} g_2(a)((1+s)t) & 0 \le t \le 1/(1+s) \\ H(a, (1+s)t - 1) & 1/(1+s) \le t \le 1. \end{cases}$$

One check directly that $\tilde{H}(a, s)$ has the required properties. $\square$

As with the homotopy extension property, we have a universal test space. The details (which are dual to Proposition 1.3.8) are left to the reader.

1.4.12 Proposition. *Let $f\colon E \to B$ be a continuous map, then f is a fibration if and only if there exists $s\colon P_f \to E^I$ making the following diagram commute:*

$$\begin{tikzcd}
P_f \arrow[dr, dashed, "\exists s"] \arrow[drr, bend left, "\pi_E"] \arrow[ddr, bend right, "\pi_{B_I}"'] & & \\
& E^I \arrow[r, "ev_0"] \arrow[d, "f_*"'] & E \arrow[d, "f"] \\
& B^I \arrow[r, "ev_0"'] & B
\end{tikzcd}$$

where π_{B_I} and π_E are the projection maps coming from the construction of P_f as a pullback.

---

[10] Compare this to the formula in Example 1.4.8.

1.4.13 Remark. One property of cofibrations that does not dualize to fibrations is that cofibrations are inclusions, but fibrations need not be surjective. Indeed, given $p\colon E \to B$ a fibration, then the composite

$$E \xrightarrow{p} B \hookrightarrow B \coprod *$$

is also a fibration, but is not surjective.

1.4.14 Remark. We will want to talk about exact sequences where the terms appearing may not have a group structure, but are rather only sets with base-points. Therefore, given a sequence of functions

$$A \xrightarrow{f} B \xrightarrow{g} C$$

of sets with base-points, we say that this is exact at B if $f(A) = g^{-1}(c_0)$ where c_0 is the base-point of C. Note that if A, B, C are groups with base-points the identity elements of the group, then exactness of sets corresponds to exactness of groups.

1.4.15 Theorem. *Let $p\colon E \to B$ be a fibration with fiber F and B path-connected. Let Y be any space, then*

$$[Y, F] \xrightarrow{i_*} [Y, E] \xrightarrow{p_*} [Y, B]$$

is exact.

Proof. For one direction, it is clear that $p_*(i_*[g])) = 0$.

Suppose $f \in [Y, E]$ is such that $p_*[f] = [\text{const}]$, i.e., $p \circ f$ is null-homotopic. Let $G\colon Y \times I \to B$ be a null-homotopy, and let $H\colon Y \times I \to E$ be a solution to the lifting problem indicated in the following diagram, using that p is a fibration:

$$\begin{tikzcd}
Y \times \{0\} \arrow[r, "f"] \arrow[d, hook, "i_0"'] & E \arrow[d, "p"] \\
Y \times I \arrow[r, "G"'] \arrow[ur, dashed, "H"] & B
\end{tikzcd}$$

Note now that $p \circ H(y, 1) = G(y, 1) = b_0$, so that $H(y, 1) \in F := p^{-1}(b_0)$. It follows that $[f] = i_*[H(-, 1)]$. □

We have an analogous result for cofibration.

1.4.16 Theorem. *Let $i\colon A \to X$ be a cofibration, and $q\colon X \to X/A$ the quotient map. Let Y be any path-connected space, then the sequence of pointed sets*

$$[X/A, Y] \xrightarrow{q^*} [X, Y] \xrightarrow{i^*} [A, Y]$$

is exact.

Proof. Again, one inclusion is clear: we have $i^*(g^*([g])) = [g \circ q \circ i] = [\text{const}]$.

Now suppose that $f\colon X \to Y$ is a map with $f\mid_A\colon A \to Y$ null-homotopic. Let $h\colon A \times I \to Y$ be a hull-homotopy, and let

$F\colon X\times I\to Y$ be the extension as shown in the following diagram:

$$\begin{tikzcd}
A \arrow[r, hook, "i_0"] \arrow[d, "i"'] & A\times I \arrow[d, "i\times\mathrm{id}"] \arrow[rd, bend left, "H"] & \\
X \arrow[r, hook, "i_0"'] \arrow[rr, bend right, "f"'] & X\times I \arrow[r, dashed, "F"] & Y
\end{tikzcd}$$

Let $f' := F(-,1)$. Then, $f\sim f'$ and $f'(A)=F(A,1)=y_0$. By the universal property of the quotient, we can find $g\colon X/A\to Y$ making the following diagram commute:

$$\begin{tikzcd}
X \arrow[r, "f'"] \arrow[d, "q"'] & Y \\
X/A \arrow[ru, dashed, "g'"'] &
\end{tikzcd}$$

Therefore $[f]=[f']=q^*[g']$. ☐

As an extension of Theorem 1.4.15 we have the following.

1.4.17 Theorem. *Given a (Serre) fibration $p\colon E\to B$, and base points $b\in B$ and $e\in F:=f^{-1}(b)$, then there is an isomorphism $p_*\colon \pi_n(E,F,e)\xrightarrow{\simeq}\pi_n(B,b)$ for all $n\geq 1$. Hence, if B is path-connected, there is a long exact sequence of homotopy groups*

$$\cdots\pi_n(F,e)\to\pi_n(E,e)\xrightarrow{p_*}\pi_n(B,b)\to\pi_{n-1}(F,e)\to\cdots$$
$$\cdots\to\pi_0(E,e)\to 0.$$

Proof. We first show that p_* is surjective. Let $[f]\in\pi_n(B,b)$, represented by a map $f\colon(I^n,\partial I^n)\to(B,b)$. Note that $I^{n-1}\times\{0\}\subseteq\partial I^n$, so we can form the diagram

$$\begin{tikzcd}
I^{n-1}\times\{0\} \arrow[r, "*"] \arrow[d] & E \arrow[d, "p"] \\
I^n \arrow[r, "f"'] \arrow[ru, dashed, "\tilde{f}"] & B
\end{tikzcd}$$

where the lift $\tilde{f}$ exists because p is a Serre fibration. Because $f(\partial I^n)=b$, we have $\tilde{f}(\partial I^n)\subseteq F$. So $\tilde{f}$ represents an element of $\pi_n(E,F,e)$ with $p_*([\tilde{f}])=[p\circ\tilde{f}]=[f]$.

To show injectivity, let $\tilde{f}_0,\tilde{f}_1\colon(I^n,\partial I^n,J^{n-1})\to(E,F,e)$ be such that $p_*(\tilde{f}_0)=p_*(\tilde{f}_1)$. Let $H\colon(I^n\times I,\partial I^n\times I)\to(B,b)$ be a homotopy from $p\tilde{f}_0$ to $p\tilde{f}_1$. We can find a lift in the following diagram:

$$\begin{tikzcd}
W \arrow[r, "f"] \arrow[d] & E \arrow[d, "p"] \\
I^n\times I \arrow[r, "H"'] \arrow[ru, dashed, "\tilde{H}"] & B
\end{tikzcd}$$

where $W=I^n\times\{0\}\cup I^n\times\{1\}\cup\partial I^n\times I$, and f is $\tilde{f}_0$ on $I^n\times\{0\}$, $\tilde{f}_1$ on $I^n\times\{1\}$ and f is constant on $\partial I^n\times I$. The homotopy lifting property gives $\tilde{H}$ defining a homotopy between $\tilde{f}_0$ and $\tilde{f}_1$.

The result then follows (modulo some noise in the low homotopy groups, which can be checked by hand) from Theorem 1.2.21. ☐

1.4.18 Example (Hopf fibrations). Let $\mathbb{F} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ and fix an integer $d = 1, 2$ or 4, respectively.

Let

$$\mathbb{F}^{n+1} = \begin{cases} \mathbb{R}^{n+1} & \mathbb{F} = \mathbb{R} \\ \mathbb{C}^{n+1} \cong \mathbb{R}^{2(n+1)} & \mathbb{F} = \mathbb{C} \\ \mathbb{H}^{n+1} \cong \mathbb{R}^{4(n+1)} & \mathbb{F} = \mathbb{H}. \end{cases}$$

In other words, $\mathbb{F}^{n+1} \cong \mathbb{R}^{d(n+1)}$. We define the $d(n+1)-1$ dimensional sphere inside $\mathbb{F}^{n+1}$:

$$S^{d(n+1)-1} = \{(u_0, \dots, u_n) \mid u_i \in \mathbb{F}, \sum_{k=0}^{n} |u_k|^2 = 1\}.$$

We define the $\mathbb{F}$-projective space by

$$\mathbb{F}P^n := \mathbb{F}^{n+1} \setminus \{0\} / \sim$$

where $(u_0, \dots, u_n) \simeq (v_0, \dots, v_n)$ if and only if there exists $\lambda \in \mathbb{F} \setminus \{0\}$ such that $v_i = \lambda u_i$ for $i = 0, \dots, n$.

Now we have a map $\phi\colon S^{d(n+1)-1} \to \mathbb{F}P^n$ that sends $(u_0, \dots, u_n)$ to its equivalence class $[u_0, \dots, u_n]$. Let $F = \phi^{-1}[1, \dots, 0] = \{(\lambda, 0, \dots, 0) \mid \lambda \in \mathbb{F}, |\lambda| = 1\} \cong S^{d-1}$.

We will see later in the course that $S^{d-1} \to S^{d(n+1)-1} \to \mathbb{R}P^n$ is a fibration. Explicitly, the fibrations are

$$\begin{aligned} &S^0 \to S^n \to \mathbb{R}P^n \\ &S^1 \to S^{2n+1} \to \mathbb{C}P^n \\ &S^3 \to S^{4n+3} \to \mathbb{H}P^n. \end{aligned}$$

The case $n = 1$ is of interest, as then projective spaces are just spheres, and we obtain the following Hopf fibrations

$$\begin{aligned} &S^0 \to S^1 \to S^1 \\ &S^1 \to S^3 \to S^2 \\ &S^3 \to S^7 \to S^4. \end{aligned}$$

There is also a fibration $S^7 \to S^{15} \to S^8$. It is a difficult theorem of Adams that these are the only fibrations between spheres.

1.5 *The homotopy extension and lifting property*

We recall that given $f\colon X \to Y$ we defined the mapping path space P_f in Definition 1.4.10, and that $P_f \to Y$ is a fibration.

1.5.1 Definition. The homotopy fiber F_f of $f\colon X \to Y$ is the fiber of the fibration $P_f \to Y$. This is well-defined up to homotopy.

The following is an extremely useful definition in homotopy theory; as we will see later, any weak equivalence between CW-complexes is in fact a homotopy equivalence.

1.5.2 Definition. A map $f\colon (X,x_0) \to (Y,y_0)$ is a weak equivalence if $f_0\colon \pi_0(X,x_0) \to \pi_0(Y,y_0)$ is a bijection and $f_*\colon \pi_k(X,x_0) \to \pi_k(Y,y_i)$ is an isomorphism for all $k \geq 1$.

1.5.3 Lemma. *If $f\colon X \to Y$ is a weak-equivalence, then $\pi_k(F_f) = 0$ for all $k \geq 0$.*

Proof. This follows from the long exact sequence of a fibration (Theorem 1.4.17). □

1.5.4 Remark. We now make a series of remarks about a map $f\colon X \to Y$ with homotopy fiber F_f.

(i) A map $\phi\colon S^{n-1} \to F_f$ corresponds to a diagram

$$\begin{tikzcd}
S^{n-1} \arrow[r, "g"] \arrow[d, hook] & X \arrow[d, "f"] \\
C(S^{n-1}) \cong D^n \arrow[r, "h"'] & Y
\end{tikzcd} \tag{1.5.5}$$

where g is the composite $S^{n-1} \xrightarrow{\phi} F_f \to X$ (use Lemma 1.1.11).

(ii) The boundary map $\pi_n(Y) \to \pi_{n-1}(F_f)$ in the long exact sequence corresponds to the map sending the class of $\bar{h}\colon S^n \to Y$ to the class of $\pi_{n-1}(F_f)$ represented by the diagram

$$\begin{tikzcd}
S^{n-1} \arrow[r, "c"] \arrow[d, hook] & X \arrow[d, "f"] \\
D^n \arrow[r, "h"'] & Y
\end{tikzcd}$$

where $c = c_{x_0}$ is the constant map, and h is the composite $D^n \to D^n/S^{n-1} \cong S^n \xrightarrow{\bar{h}} Y$.

(iii) Similarly, the map $\pi_{n-1}(F_f) \to \pi_{n-1}(X)$ corresponds to sending the diagram (1.5.5) to the class $[g]$.

(iv) In particular, $\pi_{n-1}(F_f) = 0$ is equivalent to completing the diagram (1.5.5) in the following way:

$$\begin{tikzcd}
S^{n-1} \arrow[rr, "g"] \arrow[dr] \arrow[ddd, hook] & & X \arrow[ddd, "f"] \\
& C(S^{n-1}) \arrow[ur, dashed, "G"] \arrow[d, hook] & \\
& C(D^n) \arrow[dr, dashed, "H"] & \\
D^n \arrow[ur] \arrow[rr, "h"'] & & Y
\end{tikzcd}$$

We can restate the last remark in the following lemma.

1.5.6 Lemma. *Suppose $f\colon X \to Y$ is a map with homotopy fiber F_f. Then $\pi_{n-1}(F_f) = 0$ if and only if each diagram*

$$\begin{tikzcd}
S^{n-1} \times \{1\} \arrow[r, "g"] \arrow[d, hook] & X \arrow[d, "f"] \\
S^{n-1} \times I \cup D^n \times \{0\} \arrow[r, "h"'] & Y
\end{tikzcd}$$

can be completed to a diagram

$$\begin{tikzcd}
S^{n-1} \times \{1\} \arrow[rr, "g"] \arrow[dddd, hook] \arrow[dr, hook] & & X \arrow[dddd, "f"] \\
& D^n \times \{1\} \arrow[ur, dashed, "G"] \arrow[d, hook] & \\
& D^n \times I \arrow[ddr, dashed, "H"] & \\
& & \\
S^{n-1} \times I \cup D^n \cup \{0\} \arrow[rr, "h"'] \arrow[uur, hook] & & Y
\end{tikzcd}$$

Proof. For any disk we have a homeomorphism $CD^n \cong D^n \times I$ which sends the cone point to the center of $D^n \times \{1\}, D^n$ to $S^{n-1} \times I \cup D^n\{0\}, S^{n-1}$ to $S^{n-1} \times \{1\}$ and CS^{n-1} to $D^n \times \{1\}$. Thus the statement follows from the last part of the remark. □

This extends to relative CW-complexes.[11]

1.5.7 Theorem (Homotopy extension and lifting property (HELP))**.** *Let (X, A) be a relative CW-pair and $f\colon Y \to Z$ a weak equivalence. Then every diagram*

$$\begin{tikzcd}
A \times \{1\} \arrow[r, "g"] \arrow[d, hook] & Y \arrow[d, "f"] \\
A \times I \cup X \times \{0\} \arrow[r, "h"'] & Z
\end{tikzcd}$$

can be completed to a diagram

$$\begin{tikzcd}
A \times \{1\} \arrow[rr, "g"] \arrow[dddd, hook] \arrow[dr, hook] & & Y \arrow[dddd, "f"] \\
& X \times \{1\} \arrow[ur, dashed, "G"] \arrow[d, hook] & \\
& X \times I \arrow[ddr, dashed, "H"] & \\
& & \\
A \times I \cup X \times \{0\} \arrow[rr, "h"'] \arrow[uur, hook] & & Z
\end{tikzcd}$$

Proof. The proof is by induction over the n-skeleton, with the base case being straightforward. For the inductive step, one reduces to

[11] The following result is perhaps difficult to remember, but very useful!

attaching a single cell using the diagram

$$\begin{tikzcd}
X_{n-1} \times \{1\} \arrow[rr, "G_{n-1}"] \arrow[dr, hook] \arrow[ddd, hook] & & Y \arrow[ddd, "f"] \\
& X_n \times \{1\} \arrow[ur, dashed, "G_n"] \arrow[d, hook] & \\
& X_n \times I \arrow[dr, dashed, "H_n"] & \\
X_{n-1} \times I \cup X_n \times \{0\} \arrow[ur, hook] \arrow[rr, "H_{n-1} \cup h"'] & & Z
\end{tikzcd}$$

□

1.5.8 Remark. If (X, A) is a relative CW-complex of dimension n, and $f\colon Y \to Z$ is an n-equivalence[12], the same argument goes through to show that the conclusion of HELP also holds in this case.

Exercise 11. *Show that if $f = id$ in HELP, then we recover the homotopy extension property.*

Our first application of this will be Whitehead's theorem. We start with the folliwing lemma.

1.5.9 Lemma. *For any weak equivalence $f\colon Y \to Z$ and any CW-complex X, the induced map $f_*\colon [X, Y] \to [X, Z]$ is a bijection.*

Proof. We first show surjectivity. The pair $X = (X, \emptyset)$ is a relative CW-complex, and so we can apply HELP. Then, for any $h\colon X \to Z$ we have a diagram

$$\begin{tikzcd}
* \arrow[rr] \arrow[dr, hook] \arrow[ddd, hook] & & Y \arrow[ddd, "f"] \\
& X \times \{1\} \arrow[ur, dashed, "G"] \arrow[d, hook] & \\
& X \times I \arrow[dr, dashed, "H"] & \\
X \times \{0\} \arrow[ur, hook] \arrow[rr, "h"'] & & Z
\end{tikzcd}$$

The homotopy $H\colon X \times I \to Z$ satisfies $H_0 = h$ and $H_1 = f \circ G$. Therefore, $[h] = f_*[G]$.

Now assume that $g_0, g_1 \in [X, Y]$ with $f_*[g_0] = f_*[g_1]$. Let $F\colon X \times I \to Z$ be a homotopy between $f \circ g_0$ and $f \circ g_1$. Consider the pair $(X \times I, X \times \partial I)$. This is a relative CW-pair, and HELP gives a

---

[12] That is, the homotopy fiber of f is $(n-1)$-conneceted

diagram

$$\begin{tikzcd}
X \times \partial I \times \{1\} \arrow[rr, "g"] \arrow[dr, hook] \arrow[ddd, hook] & & Y \arrow[ddd, "f"] \\
& X \times I \times \{1\} \arrow[ur, dashed, "G"] \arrow[d, hook] & \\
& X \times I \times J \arrow[dr, dashed, "H"] & \\
X \times \partial I \times J \cup X \times I \times \{0\} \arrow[ur, hook] \arrow[rr, "h"'] & & Z
\end{tikzcd}$$

Here $g \colon X \times \partial I \to Y$ sends (X, ν) to $g_\nu(x)$ for $\nu = 1, 2$ and $h \colon X \times \partial I \times J \to Z$ sends (x, ν, s) to $f \circ g_\nu(x)$. The lift $G \colon X \times I \to Y$ gives a homotopy between g_0 and g_1, i.e., $[g_0] = [g_1]$, and so f_* is injective. ☐

1.5.10 Remark. Using Remark 1.5.8 we have the following variant of Lemma 1.5.9: If $f \colon Y \to Z$ is n-connected, then for any CW-complex X, the induced map $f_* \colon [X, Y] \to [X, Z]$ is an isomorphism if $n < \dim(X)$ and is surjective if $n = \dim(X)$.

1.5.11 Theorem (Whitehead theorem). *If $f \colon X \to Y$ is weak-equivalence between CW-complexes, then it is a homotopy equivalence.*

Proof. Suppose $f \colon X \to Y$ is a weak equivalence, so $f_* \colon [Y, X] \xrightarrow{\simeq} [Y, Y]$. In other words, there exists a $g \colon Y \to X$ such that $f_*[g] = [f \circ g] = [\mathrm{id}_Y]$, i.e., $f \circ g \simeq \mathrm{id}_Y$. Then, $f \circ g \circ f \simeq f$ as well. But, we also have $f_* \colon [X, X] \xrightarrow{\simeq} [X, Y]$, which sends id_X to f and $g \circ f$ to $f \circ g \circ f \simeq f$. Therefore, $\mathrm{id}_X \simeq g \circ f$, and so $X \simeq Y$. ☐

1.5.12 Corollary. *If X is a CW-complex with $\pi_i(X) = 0$ for all i, then X is contractible.*

Proof. Apply Whitehead's theorem to the unique map $X \to *$. ☐

1.5.13 Remark. We cannot drop any assumptions from this theorem, as the following examples show:

(i) We must have a map inducing the weak equivalence; the homotopy groups cannot be abstractly isomorphism, e.g., consider $\mathbb{R}P^2 \times S^3$ and $\mathbb{R}P^3 \times S^2$.

(ii) The Warsaw circle[13] is an example of a space with $\pi_n X = 0$ for all n, but for which X is not contractible.

Exercise 12. Use Whitehead's theorem to show that a CW complex is contractible if it is the union of an increasing sequence of sub-complexes $X_1 \subseteq X_2 \subseteq \cdots$ such that each inclusion $X_i \to X_{i+1}$ is null-homotopic.

Exercise 13. Let $f \colon X \to Y$ be a weak homotopy equivalence. Assuming X is a CW-complex, and Y has the homotopy type of a CW-complex, show that f is a homotopy equivalence.

[13] See, for example, `https://wildtopology.com/bestiary/warsaw-circle/`

1.6 *The cellular approximation theorem*

The next important theorem is the cellular approximation theorem.

1.6.1 Definition. If X and Y are CW-complexes, and $g\colon X \to Y$ a map, then g is cellular if g carries the n-skeleton of X into the n-skeleton of Y, i.e., $f(X^n) \subseteq Y^n$ for all $n \geq 0$. Similarly, for relative CW-complexes (X, A) and (Y, B) a map $g\colon (X, A) \to (Y, B)$ is cellular if $g((X, A)^n) \subseteq (Y, B)^n$ for all $n \geq 0$.

The main result of this section is the following.

1.6.2 Theorem (Cellular approximation theorem)**.** *Suppose $f\colon (X, A) \to (Y, B)$ is a map of relative CW-complexes, then f is homotopic rel A to a cellular map of pairs.*

We will use the following lemma.

1.6.3 Lemma. *If Z is obtained from Y by attaching cells of dimension $> n$, then $\pi_k(Z, Y) = 0$ for all $k \leq n$, i.e., (Z, Y) is n-connected.*

Proof. We can reduce to the case where $Z = Y \cup_\alpha D^r$ for $\alpha\colon S^{r-1} \to Y, r \geq n+1$. Then $\pi_k(Z, Y)$ corresponds to a map of pairs $g\colon (D^k, S^{k-1}) \to (Z, Y)$. By smooth or simplicial approximation,[14] we can find a map $g'\colon D^k \to Z$ such that $g' = g$ on S^{k-1} and g' misses a point p in the interior of D^k. We can deform $X \setminus \{p\}$ onto A and so deform g' to a map in Y. □

Proof of Theorem 1.6.2. The proof is by induction, with the base case left to the reader. So, by induction, we have $g_{n-1}\colon X^{n-1} \to Y^{n-1}$ and a homotopy $H_{n-1}\colon X^{n-1} \times I \to Y$ such that $H_0 = f$ and $H_1 = g_{n-1}$. Now consider the diagram

$$\begin{tikzcd}
X^{n-1} \times \{1\} \arrow[rr, "\iota_n \circ g_{n-1}"] \arrow[dr, hook] \arrow[ddd] & & Y^n \arrow[ddd] \\
& X^n \times \{1\} \arrow[ur, dashed, "g^n"] \arrow[d, hook] & \\
& X^n \times I \arrow[dr, dashed, "H_n"] & \\
X^{n-1} \times I \cup X^n \times \{0\} \arrow[ur, hook] \arrow[rr, "H_{n-1} \cup f"] & & Y
\end{tikzcd}$$

Here $\iota_n\colon Y^{n-1} \to Y^n$ is the inclusion map. We can apply the version of HELP given in Remark 1.5.8 since $Y^n \hookrightarrow Y$ is an n-equivalence by Lemma 1.6.3, which gives the required extensions g_n and H_n. □

There is also a relative version, whose proof we omit.

1.6.4 Theorem. *Suppose $f\colon (X, A) \to (Y, B)$ is a map of relative CW-complexes which is cellular on a subspace (X', A') of (X, A), then there is a cellular map $g\colon (X, A) \to (Y, B)$ homotopic to f relative to Y such that $g\,|_{X'} = f$.*

[14] See `https://ncatlab.org/nlab/show/simplicial+approximation+theorem`

1.6.5 Example. Suppose $i < n$. Taking the standard CW structure on the k-sphere with one 0-cell and one k-cell, we see that any map $S^i \to S^n$ can be made cellular. Because the i-skeleton of S^n is a point, we see that such a map is null-homotopic, and deduce that $\pi_i(S^n) = 0$ for $i < n$.

1.6.6 Corollary. *Let $A \subseteq X$ be CW-complexes, and suppose that all cells of $X \setminus A$ have dimension $> n$. Then $\pi_i(X, A) = 0$ for $i \leq n$.*

Proof. Let $[f] \in \pi_i(X, A)$, i.e., $f\colon (D^i, S^{i-1}) \to (X, A)$. We can use cellular approximation to replace f with a cellular map g with $g(D^i) \subseteq X^i$. But for $i \leq n$ we have $X^i \subseteq A$, so the image of g is contained in A. By the compression criterion (Lemma 1.2.22) we have $[f] = [g] = 0$. □

1.6.7 Corollary. *If X is a CW-complex, then $\pi_i(X, X_n) = 0$ for all $i \leq n$.*

Exercise 14. *Use cellular approximation to show that the n-skeletons of homotopy equivalent CW-complexes without cells of dimension $n+1$ are also homotopy equivalent.*

1.7 *Excision and the Freudenthal suspension theorem*

On of the most powerful results in (co)homology is excision. As we will see in this section, things are more complicated for homotopy groups. This is one of the reasons why homotopy groups are (generally) more complicated to compute than homology groups.

1.7.1 Definition. An excisive triad $(X; A, B)$ consists of a space X along with two subspaces $A, B \subseteq X$ such that $X = A^\circ \cup B^\circ$.

1.7.2 Remark. In homology $(A, A \cap B) \to (X, B)$ induces an isomorphism in homology (by excision). This fails in homotopy, as the following example shows.

1.7.3 Example. Let $X = S^2 \vee S^2$ and let $A = C_+$ and $B = C_-$, the southern and northern hemispheres, with a small overlap between the two hemispheres (see Figure 1.5).

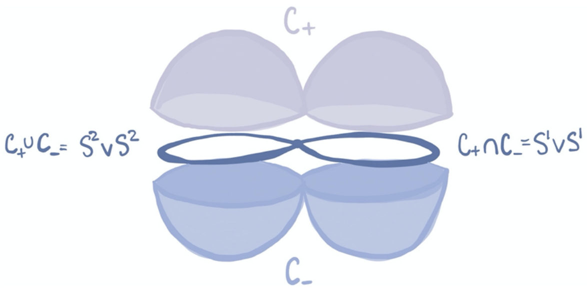


Figure 1.5: The decomposition of $X = S^2 \vee S^2$ into the upper/lower hemispheres $C_\pm$, which intersect along the equator $S^1 \vee S^1$.

Then $C_+ \cap C_- \simeq S^1 \vee S^1$ and $C_+ \cup C_- = S^2 \vee S^2$. Note that both C_+ and C_- are contractible (they have the homotopy type of a wedge of two discs). By the long exact sequence in homotopy we have

$$\pi_i(S^2 \vee S^2, C_-) \cong \pi_i(S^2 \vee S^2) \quad \text{and} \quad \pi_i(C_+, S^1 \vee S^2) \cong \pi_{i-1}(S^1 \vee S^2).$$

In particular, when $i = 2$ we have $\pi_2(S^2 \vee S^2, C_-) \cong \pi_2(S^2 \vee S^2)$ is the free abelian group on two generators while $\pi_2(C_+, S^1 \vee S^1) \cong \pi_1(S^1 \vee S^1)$ is the free group on two generators. Therefore $(C_+, S^1 \vee S^1) \to (S^2 \vee S^2, C_-)$ does not induce an isomorphism on homotopy.

1.7.4 Remark. The following is the homotopy theoretic version of excision. We will not give a full proof as it is quite involved. The full details can be found in May's book, for example.

1.7.5 Theorem (Homotopy excision/Blakers–Massey theorem). *Let $(X; A, B)$ be an excisive triad such that $C = A \cap B$ is non-empty and (A, C) and (B, C) are relative CW-complexes. Suppose $(A, C, *)$ is n-connected and $(B, C, *)$ is m-connected for every choice of base-point $* \in C$. Then the map*

$$\pi_i(A, C) \to \pi_i(X, B)$$

induced by the inclusion is an isomorphism for $i < n + m$ and a surjection for $i = n + m$, i.e., it is an $(n + m)$-equivalence.

Sketch of proof. The proof proceeds by a number of reductions.

Reduction 1: It suffices to prove this when A is built from C by attaching cells of dimension greater than n and B is built by attaching cells of dimension greater than m. Indeed, we claim we can replace the pair (A, C) with an n-connected pair (A', C) such that the following diagram commutes:

$$\begin{tikzcd}
C \arrow[r, hook] \arrow[d] & A' \arrow[dl, "\sim"] \\
A &
\end{tikzcd}$$

and A' is built from C by attaching cells of dimension greater than n only. To show this, we build up a CW complex from C by adding cells which represent elements of $\pi_i(A)$ or gets rid of elements which should not be there.[15] Since $\pi_i(C) \cong \pi_i(A)$ for all $i < n$, we only need to add cells of dimension greater than n to make this work. This procedure can be carried out for (B, C) as well.

Reduction 2: It suffices to prove excision when each of A and B is built from C by attaching one cell apiece. To see this, let us say that a pair of extensions $C \to A$ and $C \to B$ is of size (p, q) if A is obtained by attaching p-cells (of dimension greater than n) and B is obtained by attaching q-cells (of dimension greater than m). The claim is that excision holds for size $(1, 1)$ if it holds for size (p, q). The proof is inductive, via a long each sequence of 'triad homotopy groups' and the 5-lemma.

The following lemma, whose proof is omitted, then completes the proof of homotopy excision. □

1.7.6 Lemma. *Suppose that $X = A \cup_C B$ where $A = C \cup e$ and $B = C \cup e'$ are built from C by attaching cells of dimension $> n$ and $> m$, respectively. Then, $\pi_i(A, C) \to \pi_i(X, B)$ is an isomorphism for $i < n + m$ and a surjection for $i = n + m$.*

---

[15] For example, the first step is to kill the kernel of the surjection $\pi_n(C) \to \pi_n(A)$ by attaching cells to A. We will discuss this procedure in more detail when we discuss Eilenberg–Maclane spaces.

Our main application will be the Freudenthal suspension theorem. We first make a definition.

1.7.7 Definition. Let (X, x_0) be a based space. The suspension homomorphism is the map $\Sigma_*\colon \pi_i(X) \to \pi_{i+1}(\Sigma X)$ which sends $[f]$ to $[\Sigma f]$, where $\Sigma f\colon S^{i+1} \to \Sigma X$ sends $[s,t]$ to $[f(s),t]$.

1.7.8 Theorem. *Let X be an $(n-1)$-connected CW-complex, then the suspension homomorphism*

$$\Sigma_*\colon \pi_i(X) \to \pi_{i+1}(\Sigma X)$$

is an isomorphism for $i < 2n-1$ and a surjection for $i = 2n-1$.

Proof. Write $\Sigma X = C_+X \cup C_-X$ for the decomposition of ΣX into its upper and lower cone. Now consider the diagram

$$\begin{tikzcd}
\pi_{i+1}(C_+X, X) \arrow[r] \arrow[d, "\partial", "\cong"'] & \pi_{i+1}(\Sigma X, C_-X) \arrow[d, "\cong", "\partial"'] \\
\pi_i(X) \arrow[r, "\Sigma_*"'] & \pi_{i+1}(\Sigma X)
\end{tikzcd}$$

which can be shown to commute. Then it suffices to show that the upper diagram is an isomorphism/surjection in the appropriate range. To see this, note that if X is $(n-1)$-connected, then $(C_\pm X, X)$ are n-connected (use the long exact sequence and contractibility of $C_\pm X$). By excision,

$$\pi_{i+1}(C_+, X) \to \pi_{i+1}(\Sigma X, C_-X)$$

is an isomorphism for $i+1 < 2n$ and a surjection for $i+1 = 2n$, and the result follows. □

1.7.9 Example. The n-sphere has $\pi_i(S^n) = 0$ for $i < n$ (Example 1.6.5). So by the Freudenthal suspension theorem

$$\Sigma_*\colon \pi_i(S^n) \to \pi_{i+1}(S^{n+1})$$

is an isomorphism for $i < 2n-1$. In particular, $\pi_n(S^n) \to \pi_{n+1}(S^{n+1})$ is an isomorphism for $n < 2n-1$, i.e., for $n \geq 2$. In particular, there is a surjection $\mathbb{Z} \cong \pi_1(S^1) \to \pi_2(S^2)$ and isomorphisms $\pi_2(S^2) \cong \pi_3(S^3) \cong \cdots \pi_n(S^n)$. In fact, the Hopf fibration $S^1 \to S^2 \to S^3$ shows that $\pi_2(S^2) \cong \mathbb{Z}$, and so we have $\pi_n(S^n) \cong \mathbb{Z}$ for all $n \geq 1$.

1.7.10 Remark. Let X be a CW-complex. By the suspension theorem $\Sigma^n X$ is always $(n+1)$-connected. Thus,

$$\Sigma_*\colon \pi_i(\Sigma n X) \to \pi_{i+1}(\Sigma^{n+1}X)$$

is an isomorphism for $i < 2n-1$. This means that for a fixed value of k, the maps in the sequence

$$\pi_k(X) \to \pi_{k+1}(\Sigma X) \to \pi_{k+2}(\Sigma^2 X) \to \cdots \Sigma_{k+i}(\Sigma^i X)$$

eventually become isomorphisms. This is known as the k-th stable homotopy group of X.

1.7.11 Remark. There is an equivalent way to state homotopy excision. Suppose $f\colon A \to X$ is an m-equivalence and $g\colon A \to Y$ is an n-equivalence. We can form the following diagram

$$\begin{tikzcd}
F_f \arrow[r] \arrow[d, dashed, "\tilde{g}"'] & A \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, phantom, "\lrcorner", very near end] & X \arrow[d] \\
F_z \arrow[r] & Y \arrow[r, "z"'] & Z
\end{tikzcd}$$

Then $\tilde{f}\colon F_f \to F_z$ is an $(n+m-1)$-equivalence. This follows because $\pi_n(X,A) \cong \pi_{n-1}F_f$ (use Theorem 1.4.17 and the 5-lemma, for example).

Exercise 15. *Show that if $f\colon X \to Y$ is an n-connected map between spaces with X an $(m-1)$-connected CW-complex, then the comparison map $F(f) \to \Omega C(f)$ is $(m+n-1)$-connected.*

1.8 The CW-approximation theorem

In this next section we show that, up to weak homotopy equivalence, every space is a CW-complex. We begin with the following.

1.8.1 Lemma. *Let X be any space. Then there exists a space Y and a map $i\colon X \to Y$ such that i induces isomorphisms $\pi_q\colon \pi_q X \xrightarrow{\simeq} \pi_q Y$ for $0 \le q \le n$ and $\pi_{n+1}Y = 0$.*

Proof. The idea of the proof is to attach cells to X to kill the classes we don't want to exist. To that end, let J be a set of representatives for each $[j] \in \pi_{n+1}X$. Form the pushout diagram

$$\begin{tikzcd}
\coprod_{j\in J} S^{n+1} \arrow[r] \arrow[d] & X \arrow[d] \\
\coprod_{j\in J} D^{n+2} \arrow[r] \arrow[ur, phantom, "\lrcorner", very near end] & Y
\end{tikzcd}$$

Note that (Y,X) is a relative CW-complex with $Y^{n+1} = X$ so $X \to Y$ is an $(n+1)$-equivalence. We claim that $\pi_{n+1}Y = 0$. Indeed, let $f\colon (S^{n+1},*) \to (Y,*)$ be a representative of $\pi_{n+1}(Y)$. By cellular approximation we can assume that f factors through the $(n+1)$-skeleton of Y, which is just X. In other words, $f \simeq j$ for some $j \in J$. But $j\colon S^{n+1} \to X \to Y$ is null-homotopic by assumption, and so $\pi_{n+1}Y = 0$. □

1.8.2 Proposition (CW-approximation)**.** *Given any topological space X there exists a CW complex ΓX and a weak equivalence $\Gamma X \xrightarrow{\gamma} X$.*[16]

Proof. We can assume that X is path-connected, or we can work one path component at a time. Choose a set of representatives $J = \{j_q \mid [j] \in \pi_q X, q \ge 1\}$. Let $X_1 = \bigvee_{j_q \in J} S^q$ (with its standard CW structure) and $\gamma_1 := \vee j_q\colon X_1 \to X$. By construction, γ_1 is a 1-equivalence. Suppose by induction that we have constructed a CW complex X_n and an n-equivalence $X_n \to X$. Once again, let

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[16] In slightly fancy language, this is cofibrant replacement in a certain model structure on the category of topological spaces.

$J = \{j \mid [j] \in \pi_n X_n, [\gamma_n \circ j] = 0 \in \pi_n X\}$. By construction, for each $j \in J$ there exists an extension h_j of $S^n \xrightarrow{\gamma_n \circ j} X$ to D^{n+1}. We then construct X_{n+1} by the pushout

$$\begin{tikzcd}
\coprod_{j\in J} S^n \arrow[r, "\coprod j"] \arrow[d] & X_n \arrow[d] \arrow[ddr, bend left, "\gamma_n"] & \\
\coprod_{j\in J} D^{n+1} \arrow[r] \arrow[drr, bend right, "\coprod h_j"'] & X_{n+1} \arrow[ul, phantom, "\lrcorner", very near start] \arrow[dr, dashed, "\gamma_{n+1}"] & \\
& & X
\end{tikzcd}$$

Any map $S^q \to X_{n+1}$ for $q \leq n$ factors through X_n, so $\pi_q \gamma_{n+1}$ factors through $\pi_q \gamma_n$ for $q \leq n$. Since $X^n \to X^{n+1}$ is an n-equivalence, we have $\pi_q \gamma_{n+1} = \pi_q \gamma_n$ is an isomorphism for $0 \leq q < n$. For $q = n$ we have a commutative diagram

$$\begin{tikzcd}
\pi_n X^n \arrow[r, two heads] \arrow[dr, two heads, "\pi_n \gamma_n"'] & \pi_n X^{n+1} \arrow[d, "\pi_n \gamma_{n+1}"] \\
& \pi_n X
\end{tikzcd}$$

so that $\pi_n \gamma_{n+1}$ is a surjection. Moreover, any map $j \in \pi_n X^n$ such that $[\gamma_n \circ j] = 0 \in \pi_{n+1}$ extends to D^{n+1} in X_{n+1}, hence maps to zero in $\pi_{n+1} X_{n_1}$. Therefore, $\pi_n \gamma_{n+1}$ is also injective, and thus an isomorphism.

Finally, setting $\Gamma X = \operatorname{colim}_n X_n$ and $\gamma = \operatorname{colim}_n \gamma_n$ gives the required CW-approximation. ☐

1.8.3 Remark. There is also a version of CW-approximation for pairs. Namely, if (X, A) is pair then one can product a CW-pair $(\Gamma X, \Gamma A)$ weakly-homotopic to (X, A) such that ΓA is a sub-complex of ΓX.

Exercise 16. *Let $f\colon X \to Y$ be a map of topological spaces. Show that there is an induced map, $\Gamma f\colon \Gamma X \to \Gamma Y$, unique up to homotopy, between the CW-approximations to X and Y, such that the following diagram commutes:*

$$\begin{tikzcd}
\Gamma X \arrow[r, "\Gamma f"] \arrow[d, "\gamma'"'] & \Gamma Y \arrow[d, "\gamma"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd} .$$

Deduce that CW-approximations are unique up to homotopy.

Exercise 17. *Assume given maps $f\colon X \to Y$ and $g\colon Y \to X$ such that $g \circ f$ is homotopic to the identity. (We say that Y "dominates" X.) Suppose that Y is a CW complex. Prove that X has the homotopy type of a CW complex.*

1.9 Eilenberg–Maclane spaces

1.9.1 Definition. A space X having just one non-trivial homotopy group $\pi_n(X) = G$ is called an Eilenberg–MacLane space $K(G, n)$.

1.9.2 Example. We have seen that S^1 is a $K(\mathbb{Z}, 1)$ (Example 1.2.16). We will see later that $\mathbb{C}P^\infty$ is a $K(\mathbb{Z}, 2)$.

1.9.3 Remark. We have not yet shown that Eilenberg–MacLane spaces exist in any generality. In fact, since π_n is abelian for $n \geq 2$ we see that for $n \geq 2$ Eilenberg–Maclane spaces can only exist for G abelian. The goal of this section is to show that this is the only obstruction: for any group G, abelian if $n \geq 2$, the Eilenberg–MacLane space $K(G, n)$ exists. We begin with a lemma.

1.9.4 Lemma. *For $n \geq 2$ we have $\pi_n(\bigvee_\alpha S^n_\alpha)$ is free abelian, generated by the inclusion of the factors.*

Proof. Suppose first that we have only finitely many factors. Then we can regard $\bigvee_\alpha S^n_\alpha$ as the n-skeleton of $\prod_\alpha S^n_\alpha$. Taking the usual cell structure on S^n we see that $\prod_\alpha S^n_\alpha$ has a cell structure with one zero cell and the n-cells

$$\bigcup_\alpha (\prod_{\beta \neq \alpha} D^0_\beta) \times D^n_\alpha$$

and together these form the n-skeleton of $\prod_\alpha S^n_\alpha$. Hence $\prod_\alpha S^n_\alpha \setminus \bigvee_\alpha S^n_\alpha$ has only cells of dimension at least $2n$, so that the pair $(\prod_\alpha S^n_\alpha, \bigvee_\alpha S^n_\alpha)$ is $(2n-1)$-connected by Corollary 1.6.7. Therefore, we have (recall we fix $n \geq 2$)

$$\pi_n(\bigvee_\alpha S^n_\alpha) \cong \pi_n(\prod_\alpha S^n_\alpha) \cong \prod_\alpha \pi_n(S^n_\alpha) \cong \bigoplus_\alpha \mathbb{Z}.$$

This handles the case of finitely many summands. The infinite case can be reduced to this case in the following way: Let $\Phi \colon \bigoplus_\alpha \pi_n(S^n_\alpha) \to \pi_n(\bigvee_\alpha S^n_\alpha)$ be induced by the inclusions. Then, any map $f \colon S^n \to \bigvee_\alpha S^n_\alpha$ has compact image contained in the wedge of finitely many summands, so such that $[f]$ is in the image of Φ by the finite case, and Φ is surjective. Similarly, a null-homotopy of f has compact image and again by the finite case Φ must be injective. ☐

1.9.5 Remark. If $n = 1$ then the Seifert–Van Kampen theorem shows that $\pi_1(\bigvee_\alpha S^1_\alpha)$ is the free group on the components; as soon as we have more than one sphere, this group is not abelian.

1.9.6 Lemma. *If a CW-pair (X, A) is r-connected $(r \geq 1)$ and A is s-connected $(s \geq 0)$, then the map $\pi_i(X, A) \to \pi_i(X/A)$ induced by the quotient map $X \to X/A$ is an isomorphism if $i \leq r + s$ and onto if $i = r + s - 1$.*

Proof. Let $i \colon A \to X$ be the inclusion and $C(i)$ the mapping cone, $C(i) = X \cup_A CA$. Since CA is contractible and $(CA, C(i))$ is a cofibration the quotient map

$$q \colon C(i) \to C(i)/CA \simeq X/A$$

is a homotopy equivalence (Theorem 1.3.17). So we have a sequence of homomorphisms

$$\pi_i(X, A) \to \pi_i(C(i), CA) \xleftarrow{\cong} \pi_i(C(i)) \xrightarrow{\cong} \pi_i(X/A),$$

where the first and second maps are induced by the inclusion of pairs and the third map is the isomorphism q_*. The second map is an isomorphism by the long exact sequence of the pair $(C(i), CA)$.

Now we know that (X, A) is r-connected and $CA, A)$ is $(s+1)$-connected which follows from the assumption on A on the long exact sequence in homotopy. The result now follows from excision, Theorem 1.7.5. ☐

1.9.7 Remark. Suppose $n \geq 2$, and we are given maps $\phi_\beta \colon S^n_\beta \to \bigvee_\alpha S^n_\alpha$. Then we construct a space X as the pushout

$$\begin{tikzcd}
\bigvee_\beta S^n_\beta \arrow[r, "(\phi_\beta)"] \arrow[d, hook] & \bigvee_\alpha S^n_\alpha \arrow[d] \\
\bigvee_\beta D^{n+1}_\beta \arrow[r, "(\Phi_\beta)"'] & X \arrow[ul, phantom, "\lrcorner", very near start]
\end{tikzcd}$$

1.9.8 Lemma. *We have*

$$\pi_n(X) \cong \pi_n(\bigvee_\alpha S^n_\alpha)/\langle \phi_\beta \rangle \cong (\bigoplus_\alpha \mathbb{Z})/\langle \phi_\beta \rangle.$$

Proof. The pair $(X, \bigvee_\alpha S^n_\alpha)$ is n-connected and fits in a long exact sequence

$$\pi_{n+1}(X, \bigvee_\alpha S^n_\alpha) \xrightarrow{\partial} \pi_n(\bigvee_\alpha S^n_\alpha) \to \pi_n(X) \to \pi_n(X, \bigvee_\alpha S^n_\alpha) = 0,$$

where the final equality follows as a consequence of cellular approximation, see Corollary 1.6.7. It follows that $\pi_n(X) \cong \pi_n(\bigvee_\alpha S^n_\alpha)/\operatorname{im}(\partial)$.

We have $X/\bigvee_\alpha S^n_\alpha \simeq \bigvee S^{n+1}_\beta$, and so by Lemma 1.9.4 and Lemma 1.9.6 we have $\pi_{n+1}(X, \bigvee_\alpha S^n_\alpha) \cong \pi_{n+1}(\bigvee_\beta S^{n+1}_\beta)$ is free with a basis consisting of the (restriction of the) characteristic maps Φ_β of the attaching cells. Since $\partial([\Phi_\beta]) = [\phi_\beta]$, the claim follows. ☐

1.9.9 Example. Using the previous results, any abelian group G can be realized as $\pi_n(X)$ for $n \geq 2$ of some space X. Indeed, choosing a presentation $G = \langle g_\alpha \mid r_\beta \rangle$ we can take

$$X = \left(\bigvee_\alpha S^n_\alpha \right) \cup \bigcup_\beta D^{n+1}_\beta,$$

with the S^n_α's corresponding to the generators, and the discs are attached by maps $f \colon S^n_\beta \to \bigvee_\alpha S^n_\alpha$ satisfying $[f] = r_\beta$. Then, the previous lemma says that $\pi_n(X) \cong G$. In fact, $\pi_i(X) = 0$ for $i < n$ by cellular approximation, but we have no control over the higher homotopy groups. In order to construct Eilenberg–Maclane spaces, we have to kill higher homotopy groups.

1.9.10 Theorem. *For any $n \geq 1$ and any group G (abelian if $n \geq 2$) there exists an Eilenberg–Maclane space $K(G, n)$.*

Proof. Let $X_{n+1} = (\bigvee_\alpha S^n_\alpha) \cup \bigcup_\beta D^{n+1}_\beta$ be as in Example 1.9.9. We want to build a space X_{n+2} which agrees with X_{n+1} in homotopy up to π_n but has $\pi_{n+1}X_{n+2} = 0$. We have seen exactly how to do this in Lemma 1.8.1. Repeating this procedure inductively and taking colimits, we product a space X with the correct homotopy groups. □

Exercise 18. *Exhibit a fibration $F \to E \to B$ where, up to weak homotopy equivalence, F is a $K(G, n-1)$, B is a $K(G, n)$ and E is contractible.*

1.9.11 Remark. Eilenberg–Maclane spaces represent cohomology in the following sense.[17]

1.9.12 Theorem. *There are natural bijections $T \colon [X, K(G,n)] \xrightarrow{\simeq} H^n(X;G)$ for all CW-complexes and all $n > 0$ with G any abelian group. Such a T has the form $T([f]) = f^*(\alpha)$ for a distinguished class $\alpha \in H^n(K(G,n);G)$.*

Proof. Recall that if h^* is an unreduced cohomology theory on the category of CW-pairs, and $h^n(*) = 0$ for $n \neq 0$, then there exists a natural isomorphism $h^n(X,A) \cong H^n(X,A;h^0(*))$ for all CW-pairs (X,A) and all n.

Now define $h^n(X) = [X, K(G,n)]$. This defines a reduced cohomology theory and the coefficient groups $h^n(S^i) = \pi_i(K(G,n)$ are the same as $\tilde{H}^i(S^i;G)$. Therefore, the previous paragraph (translated into reduced cohomology) gives the representability result. It remains to be seen that T has the claimed form. This is formal: let $\alpha = T(\mathrm{id})$ for $\mathrm{id} \colon K(G,n) \to K(G,n)$ the identity map. Then,

$$T([f]) = T(f^*(\mathrm{id})) \cong f^*(T(\mathrm{id})) = f^*(\alpha). \qquad \square$$

We can use this to prove a uniqueness theorem for Eilenberg–MacLane spaces. We begin with a lemma.

1.9.13 Lemma. *If X is $(n-1)$-connected, then $H^n(X;H) \cong \mathrm{Hom}(H_n(X);G)$.*

Proof. This follows from a form of the Universal Coefficient theorem: there is an exact sequence

$$0 \to \mathrm{Ext}(H_{n-1}(X), G) \to H^n(X;G) \to \mathrm{Hom}(H_n(X), G) \to 0,$$

but the first term is zero by assumption. □

1.9.14 Remark. Let $G = \pi_n(X)$, then for $(n-1)$-connected X the lemma gives $H^n(X;\pi_n(X)) \cong \mathrm{Hom}(H_n(X), \pi_n(X))$. The fundamental class ι is the class in $H^n(X;\pi_n(X))$ which corresponds to the inverse of a certain isomorphism $h_n \colon \pi_n(X) \cong H_n(X)$ we will construct in Section 1.10. In particular, to $K(G,n)$ we can associate a fundamental class $\iota_n \in H^n(K(G,n);G)$.

The following is a consequence of the representability of cohomology (Theorem 1.9.12).

1.9.15 Corollary. *There is a bijection*

$$[K(G,n), K(G',n)] \xrightarrow{\simeq} \mathrm{Hom}(G, G').$$

[17] We will talk about representability more when we discuss Brown representability

Proof. We have bijections

$$\begin{aligned}[K(G,n),K(G',n)] &\cong H^n(K(G,n);G') \\ &\cong \mathrm{Hom}(H_n(K(G,n)),G') \\ &\cong \mathrm{Hom}(G,G').\end{aligned}$$

□

1.9.16 Corollary. *The homotopy type of a CW-complex $K(G,n)$ is uniquely determined by G and n.*

Proof. If $G \cong G'$ then this can be realized by a map $K(G,n) \to K(G',n)$ by the previous corollary, and since all other homotopy groups are trivial, it follows from Whitehead's theorem that this map is a homotopy equivalence. □

1.9.17 Remark (Moore spaces). There is a homology version of an Eilenberg–MacLane space, known as a Moore space: there exists a space $M(G,n)$ for an abelian group G such that

$$\widetilde{H}_k(M(G,n)) \cong \begin{cases} G & k=n \\ 0 & \text{else.} \end{cases}$$

These can in fact be used to construct Eilenberg–MacLane spaces:[18] Let $\mathrm{Sp}^n(X) := X^{\times n}/\Sigma_n$, and let the infinite symmetric product of a space X, denoted $\mathrm{Sp}^\infty(X)$, be the colimit of the $\mathrm{Sp}^n(X)$. Then, for any connected space X, we have $\widetilde{H}_n(X) \cong \pi_n(\mathrm{Sp}^\infty(X)$. In particular, $\mathrm{Sp}^\infty(M(G,n))$ is the Eilenberg–MacLane space $K(G,n)$.

1.10 The Hurewicz theorem

Define a homomorphism $h\colon \pi_i(X) \to H_i(X)$ by choosing a generator u of $H_i(S^i) \cong \mathbb{Z}$ and defining $h([f]) = f_*(u)$. This is the Hurewicz homomorphism, and extends the Hurewicz homomorphism studied previously for π_1.

1.10.1 Theorem. *If a space X is $(n-1)$-connected and $n \geq 2$, then $\tilde{H}_i(X) = 0$ for $i < n$ and $\pi_n(X) \cong H_n(X)$. Moreover, if a pair (X,A) is $(n-1)$-connected with $n \geq 2$ and $\pi_i(A) = 0$, then $H_i(X,A) = 0$ for all $i \leq n$ and $\pi_n(X,A) \cong H_n(X,A)$.*

1.10.2 Remark. The isomorphism in the theorem is induced by h_n, but we do not show this.

Proof. For note that the statements only involve homology and homotopy groups, so it suffices to use a CW approximation for X, or for (X,A).[19] Secondly, the relative case can be reduced to the absolute case: (X,A) is $(n-1)$-connected and A is 1-connected, so Lemma 1.5.9 applies to show that $\pi_i(X,A) \cong \pi_i(X/A)$ for $i \leq n$, while $H_i(X,A) \cong \tilde{H}_i(X/A)$ always holds for CW-pairs.

By a version of CW-approximation, we can assume X has $(n-1)$-skeleton a point. In particular, $\tilde{H}_i(X) = 0$ for $i < n$. For showing

[18] This is the Dold–Thom theorem.

[19] Note that a weak equivalence in homotopy also induces a weak equivalence in (co)homology.

$\pi_n(X) \cong H_n(X)$ we may ignore cells of dimension $> n+1$ since these do not change π_n or H_n. So we can assume $X = (\bigvee_\alpha S^n_\alpha) \cup \bigcup_\beta D^{n+1}_\beta$. As we have seen, we then have $\pi_n(X) \cong \oplus_\alpha \mathbb{Z}/\langle \phi_\beta \rangle$, and the same is true for $H_n(X)$ by cellular homology. ☐

1.10.3 Corollary (The homology Whitehead theorem). *If X and Y are simply connected and $f\colon X \to Y$ a map inducing an isomorphism on homology, then f is a homotopy equivalence.*

Proof. By using the mapping cylinder M_f we can assume f is an inclusion. Then by the long exact sequence in homology we have $H_n(Y, X) = 0$ for all n. Since X and Y are simply-connected $\pi_1(Y, X) = 0$. So by the relative Hurewicz theorem, the first non-zero $\pi_n(Y, X)$ is equal to the first non-zero $H_n(YmX)$. Therefore, $\pi_n(Y, X) = 0$ for all n. Therefore, $f_*\colon \pi_n(X) \to \pi_n(Y)$ is an isomorphism for all n. By Whitehead's theorem, f is a homotopy equivalence. ☐

Exercise 19. *Show that a map between simply-connected CW complexes is a homotopy equivalence if its mapping cone is contractible.*

Exercise 20. *Show that a simply-connected closed 3-manifold is homotopy equivalent to S^3. (You may assume that every closed manifold is homotopy equivalent to a CW-complex.)*

1.10.4 Remark. This exercise is a little bit tricky, but you should try and compute the homology groups of such a manifold, and then use the homology Whitehead theorem. You should begin by noting that simply-connected manifolds are orientable.

1.11 Brown representability

In this section we explain Brown's representability theorem, which one can use to give an abstract proof of Theorem 1.9.12.[20]

1.11.1 Definition. Let $F\colon \mathcal{C} \to \text{Set}$ be a functor. Then F is representable if it is naturally isomorphic to $\text{Hom}_{\mathcal{C}}(A, -)$ for some $A \in \mathcal{C}$.

1.11.2 Remark. If such an A exists, then it is unique up to unique isomorphism.

1.11.3 Example.

- The forgetful functor Ab → Set is representable. Indeed, $\text{Hom}_{\text{Ab}}(\mathbb{Z}, M) \to M$ sending f to $f(1)$ is an isomorphism.

- The forgetful functor Ring → Set is represented by $\mathbb{Z}[x]$: the map $\text{Hom}_{\text{Ring}}(\mathbb{Z}[x], M) \to M$ sending f to $f(x)$ is a bijection.

- The functor $\text{Top}^{\text{op}} \to \text{Set}$ that sends a topological space (X, τ_X) to its topology (the family of open sets) is representable. Indeed, let $Y = \{0, 1\}$ with topology $\{\emptyset, \{1\}, \{0, 1\}\}$. Then, consider $\text{Hom}_{\text{Top}}(X, Y) \to \tau_X$ sending f to $f^{-1}(\{1\})$. This map is a bijection: the inverse is given by sending an open set U to the

[20] There are no proofs given in this section.

characteristic function $\mathbb{1}_U$, i.e., the functor $\mathbb{1}_U \colon X \to \{0,1\}$ defined by

$$\mathbb{1}_U(x) = \begin{cases} 1, & x \in U \\ 0, & x \notin U. \end{cases}$$

1.11.4 Remark. Note that representable functors preserve limits, because $\mathrm{Hom}_{\mathcal{C}}(A, -)$ preserves limits. This, if we want a functor to be representable it should at least preserve limits. Brown's representability theorem gives sufficient additional conditions for a functor from CW-complexes to Sets to be representable.

Let CW denote the homotopy category of based connected CW complexes.

1.11.5 Theorem. *Let $F \colon CW^{\mathrm{op}} \to \mathrm{Set}$ be a functor satisfying:*

1. $F(\bigvee_\alpha X_\alpha) = \prod_\alpha F(X_\alpha)$
2. *Let X be an object of CW. Consider a cover $X = Y \cup Z$ by sub-complexes such that $Y, Z, Y \cap Z \in CW$. Then, for all $y \in F(Y)$ and $z \in F(Z)$ that restrict to the same elect of $F(Y \cap Z)$, there exists some $x \in F(x)$ that restricts to $z \in F(Z)$ and $y \in F(Y)$.*

Then F is representable: there exists some $C \in CW$ and $c \in F(c)$ such that for all $X \in CW$, the map

$$[X, C] \to F(x), f \mapsto f^*(c)$$

is a bijection.

1.11.6 Example. Consider the functor $\tilde{H}^n(-;G) \colon CW^{\mathrm{op}} \to \mathrm{Ab} \to \mathrm{Set}$. This satisfies the above conditions (the second condition is essentially Mayer–Vietoris), and so there exists some $C \in CW$ and $c \in \widetilde{H}^n(C;G)$ such that

$$[X, C] \to \widetilde{H}^n(X;G), f \mapsto f^*(c).$$

is a bijection. In particular, if we take $X = S^i$, then

$$[S^i, C] = \pi_i(C) \cong \widetilde{H}^n(S^i;G) \cong \begin{cases} G & i = n \\ 0 & \text{otherwise.} \end{cases}$$

In other words, C is a $K(G, n)$, and we recover Theorem 1.9.12.

1.11.7 Example. Let X be *any* space, and consider the functor $F_X \colon CW^{\mathrm{op}} \to \mathrm{Set}$ given by $Z \mapsto [Z, X]$. This is representable, and so there exists $Y \in CW$ and a map $f \colon Y \to X$ such that $[Z, Y] \xrightarrow{\cong} [Z, X]$. Taking Z to be the spheres, we see that $Y \to X$ is a CW-approximation.

2

Spectral sequences

Spectral sequences are a powerful computation tool in topology. Computing with spectral sequences is a bit like computing integral in calculus; it is helpful to have ingenuity and a big bag of tricks - and even that may not be enough!

2.1 Filtered complexes

We begin our discussion on spectral sequences by discussing filtered complexes.

2.1.1 Remark. Let $C_\bullet$ be a chain complex and $F_0C_\bullet$ a sub-complex. Then we have a short exact sequence

$$0 \to F_0C_\bullet \to C_\bullet \to C_\bullet/F_0C_\bullet \to 0$$

which gives rise to a long exact sequence in homology

$$\cdots \to H_i(F_0C_\bullet) \to H_i(C_\bullet) \to H_i(C_\bullet/F_0C_\bullet) \xrightarrow{\partial} H_{i-1}(F_0C_\bullet) \to \cdots$$

Suppose we know $H_*(F_0C_\bullet)$ and $H_*(C_\bullet/F_0C_\bullet)$. Can we compute $H_*(C_\bullet)$? We can split the long exact sequence into short exact sequences

$$0 \to \operatorname{coker}(\partial) \to H_*(C_\bullet) \to \ker(\partial) \to 0$$

which gives the following procedure for computing $H_*(C_\bullet)$:

1. Compute $H_*(F_0C_\bullet)$ and $H_*(C_\bullet/F_0C_\bullet)$
2. Consider the two-term chain complex

   $$H_*(C_\bullet/F_0C_\bullet) \xrightarrow{\partial} H_*(F_0C_\bullet).$$

   Denote its homology groups by G_1H_* and G_0H_*.
3. There is a short exact sequence

   $$0 \to G_0H_* \to H_*(C_\bullet) \to G_1H_* \to 0.$$

   This determines $H_*(C_\bullet)$ up to extension.[1]

How would we handle the situation if we have a longer filtration: $\cdots F_pC_\bullet \subseteq F_{p+1}C_\bullet \subseteq \cdots$?

[1] This is a common phenomenon for a spectral sequence. For example, if we have a short exact sequence $0 \to \mathbb{Z}/2 \to M \to \mathbb{Z}/2 \to 0$, can you say what the middle group is? Not without further information!

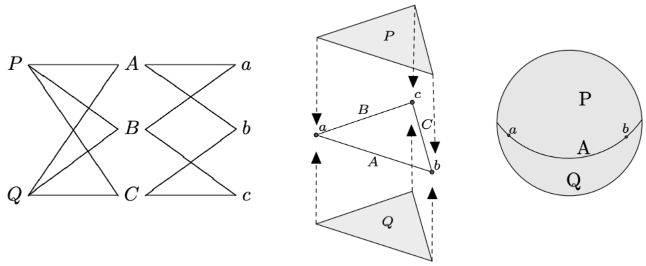


Figure 2.1: Simplicial model of S^2

2.1.2 Example. Consider a (semi-simplicial) model of the 2-sphere S^2 with vertices $\{a, b, c\}$, edges $\{A, B, C\}$ and solid triangles $\{P, Q\}$ and with inclusions as shown in Figure 2.1.[2] The associated chain complex is $C_\bullet$

$$0 \to \mathbb{Z}\{P, Q\} \xrightarrow{d} \mathbb{Z}\{A, B, C\} \xrightarrow{d} \mathbb{Z}\{a, b, c\} \to 0$$

with

$$d(P) = C - B + A \quad d(Q) = C - B + A$$

and

$$d(A) = b - a \quad d(B) = c - a \quad d(C) = c - b.$$

One can check directly that $H_i(C_\bullet; \mathbb{Z}) \cong \mathbb{Z}$ for $i = 0, 2$ and is zero otherwise. Alternatively, we use the following filtration:

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}\{P, Q\} \arrow[r] & \mathbb{Z}\{A, B, C\} \arrow[r] & \mathbb{Z}\{a, b, c\} \arrow[r] & 0 \\
0 \arrow[r] & 0 \arrow[r] & \mathbb{Z}\{A, B\} \arrow[r] & \mathbb{Z}\{a, b, c\} \arrow[r] & 0 \\
0 \arrow[r] & 0 \arrow[r] & \mathbb{Z}\{A\} \arrow[r] & \mathbb{Z}\{a, b\} \arrow[r] & 0.
\end{tikzcd}$$

The differentials are induced from d_1 and d_2 and a direct check shows that they are still chain complexes. Passing to the quotient, we get a chain complex we call E_0:

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}\{P, Q\} \arrow[r] & \mathbb{Z}\{C\} \arrow[r] & 0 \arrow[r] & 0 & d_0(P) = C, d_0(Q) = C \\
0 \arrow[r] & 0 \arrow[r] & \mathbb{Z}\{B\} \arrow[r] & \mathbb{Z}\{c\} \arrow[r] & 0 & d_0(B) = c \\
0 \arrow[r] & 0 \arrow[r] & \mathbb{Z}\{A\} \arrow[r] & \mathbb{Z}\{a, b\} \arrow[r] & 0 & d_0(A) = b - a.
\end{tikzcd}$$

Taking homology with respect to d_0 we obtain E^1:

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}\{P - Q\} \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \\
0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \\
0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \mathbb{Z}\{\bar{a}\} \arrow[r] & 0.
\end{tikzcd}$$

The general theory of spectral sequences will tell us that we have computed the homology of $H_*(C_\bullet)$; there is a $\mathbb{Z}$ in degree 2, generated by $P - Q$ and a $\mathbb{Z}$ in degree 0, generated by $\bar{a}$.

This leads us to the theory of filtered modules.

2.1.3 Definition. A filtered R-module is an R-module A together with an increasing sequence of submodules $F_pA \subseteq F_{p+1}A$ indexed by $p \in \mathbb{Z}$ such that $\cup_p F_pA = A$ and $\cap_p F_pA = \{0\}$. The filtration is

---

[2] This example comes from Example 2.1 of `https://arxiv.org/pdf/1702.00666.pdf`.

bounded if $F_pA = \{0\}$ for p sufficiently small, and $F_pA = A$ for p sufficiently large. The associated graded module is defined by

$$G_pA = F_pA/F_{p-1}A.$$

2.1.4 Definition. A filtered chain complex is a chain complex $(C_\bullet, \partial)$ together with a filtration $\{F_pC_i\}$ of each C_i such that the differential preserves the filtration: $\partial(F_pC_i) \subseteq F_pC_{i-1}$. Then, ∂ induces $\partial\colon G_pC_i \to G_pC_{i-1}$ on the associated graded modules.

2.1.5 Remark. The filtration on $C_\bullet$ induces a filtration on the homology of $C_\bullet$ by

$$F_pH_i(C_\bullet) = \{\alpha \in H_i(C_\bullet) \mid \exists x \in F_pC_i, \alpha = [x]\}.$$

This has associated graded pieces $G_pH_i(C_\bullet)$.

2.1.6 Remark. Suppose we want to compute $H_*(C_\bullet)$ and that we can compute the homology of the associated graded pieces $H_*(G_pC_\bullet)$. Does this determine $G_pH_*(C_\bullet)$? This leads to the idea of the spectral sequence of a filtered complex.

2.2 *The spectral sequence of a filtered complex*

2.2.1 Definition. Let $(F_pC_\bullet, \partial)$ be a filtered chain complex. Let us write

$$E^0_{p,q} := G_pC_{p+q} = F_pC_{p+q}/F_{p-1}C_{p+q}.$$

The differential ∂ induces a differential on E^0,

$$\partial_0\colon E^0_{p,q} \to E^0_{p,q-1}.$$

We denote the homology of the associated graded by

$$E^1_{p,q} := H_{p+q}(G_pC_\bullet, \partial_0).$$

2.2.2 Remark. We can think of $E^1_{p,q}$ as a "first order approximation" to $H_*(C_\bullet)$. We can also define a differential

$$\partial_1\colon E^1_{p,q} \to E^1_{p-1,q}$$

as follows: a homology class $\alpha \in E^1_{p,q}$ can be represented by a chain $x \in F_pC_{p+1}$ such that $\partial x \in F_{p-1}C_{p+q-1}$. We define $\partial_1(\alpha) = [\partial x]$. Because $\partial^2 = 0$, we can check that $\partial_1^2 = 0$ and that ∂_1 is well defined.

2.2.3 Definition. With notation as above, we define

$$E_2^{p,q} = \ker(\partial_1\colon E^1_{p,q} \to E^1_{p-1,q})/\operatorname{im}(\partial_1\colon E^1_{p+1,q} \to E^1_{p,q}).$$

2.2.4 Remark. We can continue this procedure, and define an "r"-th order approximation to $G_pH_{p+q}(C_\bullet)$ by

$$E^r_{p,q} = \frac{x \in F_pC_{p+q} \mid \partial x \in F_{p-r}C_{p+q-1}}{F_{p-1}C_{p+q} + \partial(F_{p+r-1}C_{p+q+1}}).$$

The notation denotes the quotient of the numerator by the intersection with the denominator.

So instead of considering cycles, we consider chains in F_p whose differentials vanishes "to order r", and instead of modding out by the entire image, we only mod out by $\partial(F_{p+r-1})$.

The main result regarding these groups is the following.

2.2.5 Lemma. *Let* $(F_pC_\bullet, \partial)$ *denote a filtered chain complex, and define* $E^r_{p,q}$ *as above. Then,*

1. ∂ *induces a map*
$$\partial_r \colon E^r_{p,q} \to E^r_{p-r,q+r-1}$$
satisfying $\partial_r^2 = 0$.
2. E^{r+1} *is the homology of the chain complex* (E^r, ∂_r), *i.e.,*
$$E^{r+1}_{p,q} = \ker(\partial_r \colon E^r_{p,q} \to E^r_{p-r,q+r-1}) / \operatorname{im}(\partial_r \colon E^r_{p+r,q+r-1} \to E^r_{p,q}).$$
3. $E^1_{p,q} = H_{p+q}(G_pC_\bullet)$.
4. *If the filtration of* C_i *is bounded for each* i*, then for every* p, q *if* r *is sufficiently large, then*
$$E^r_{p,q} = G_pH_{p+q}(C_\bullet).$$

Proof. This is a rather tedious diagram chase,[3] which generalizes the argument that a short exact sequence of chain complexes induces a long exact sequence on homology. □

2.2.6 Example. In this example[4] we show that the singular and cellular homology groups of a CW-complex X agree. To that end, let $C_*(X)$ denote the singular chain complex of X. We filter this by

$$F_pC_*(X) := C_*(X^p)$$

where X^p denotes the p-skeleton of X. The associated graded is

$$E^0_{p,q} = C_{p+q}(X^p)/C_{p+q}(X^{p-1}).$$

By definition, the homology is

$$E^1_{p,q} = H_{p+q}(X^p, X^{p-1}),$$

the relative homology of the pair (X^p, X^{p-1}). We have

$$H_{p+q}(X^p, X^{p-1}) \cong \begin{cases} C^{cell}_p(X) & q = 0 \\ 0, & q \neq 0 \end{cases}$$

where $C^{cell}_p(X)$ is the cellular chains on X, the free $\mathbb{Z}$-module with one generator for each p-cell. The cellular differential $\partial \colon C^{cell}_p(X) \to C^{cell}_{p-1}(X)$ is exactly the boundary map $E^1_{p,0} \to E^1_{p-1,0}$. Therefore, we have

$$E^2_{p,q} = \begin{cases} H^{cell}_p(X), & q = 0 \\ 0, & q \neq 0. \end{cases}$$

We must have $\partial_r = 0$ for $r \geq 2$ as either the domain or the range is zero. So, $E^{p,q}_r = E^2_{p,q}$ for all $r \geq 2$. If X is finite-dimensional, then the filtration is bounded and so $H_p(X) = H^{cell}_p(X)$ by Lemma 2.2.5.[5]

[3] For example, see `http://www.math.uchicago.edu/~may/MISC/SpecSeqPrimer.pdf`

[4] See page 67 of Mosher–Tangor, *Cohomology Operations and Applications in Homotopy Theory*

[5] One can allow arbitrary X by, for example, using colimits.

2.3 *Homological spectral sequences*

We have managed to so far avoid defining exactly what a spectral sequence is. Let us change that now.

2.3.1 Definition. A (homological) spectral sequence is a sequence

$$\{E^r_{*,*}, d^r_{*,*}\}_{r\geq 0}$$

of chain complexes of abelian groups, such that

$$E^{r+1}_{*,*} = H_*(E^r_{*,*})$$

where the homology is taken with respect to maps (called differentials)

$$d^r_{p,q} \colon E^r_{p,q} \to E^r_{p-r,q+r-1}$$

such that $(d^r)^2 = 0$.

2.3.2 Remark. We say that a spectral sequence is first quadrant if $E^r_{p,q} = 0$ whenever $p < 0$ or $q < 0$. Note that this implies that $d^r_{p,q} = 0$ for $r \gg 0$ (as either the source or the target is zero). In particular,

$$E^r_{p,q} = E^{r+1}_{p,q} = \cdots = E^\infty_{p,q}.$$

We say that the spectral sequence collapses or degenerates at E^r.

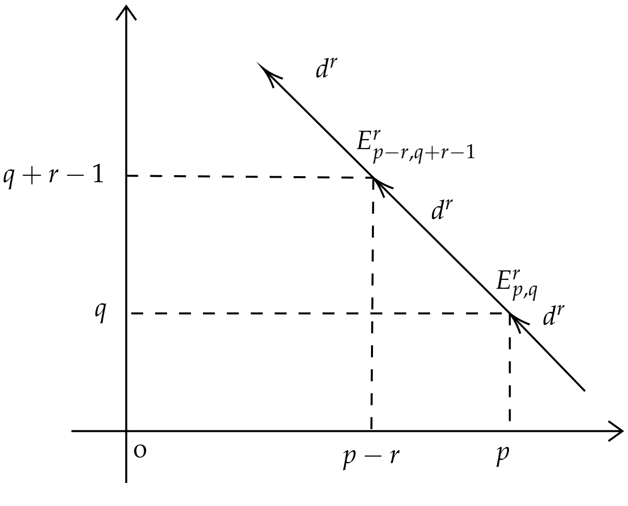


Figure 2.2: The E^r-page of a homological spectral sequence

2.3.3 Definition. If $\{H_n\}_n$ are groups, then we say that the spectral sequence converges, or abuts, to H_*, denoted $E^2_{*,*} \implies H_*$, if for each n there is a filtration

$$H_n = D_{n,0} \subseteq D_{n-1,1} \subseteq \cdots \subseteq D_{1,n-1} \subseteq D_{0,n} \subseteq 0$$

such that, for all p, q,

$$E^\infty_{p,q} = D_{p,q}/D_{p-1,q+1}.$$

2.3.4 Remark. In more straightforward terms: the if we look along the n-th diagonal of the spectral sequence, then the E_∞-page computes the associated graded of the filtration on H_n. For example, if $E^\infty_{p,q} = 0$ for all $p+q=n$, then $H_n = 0$. If there is only a single non-zero term, say $E^\infty_{p,n-p}$, then the filtration is trivial, and $H_n = E^\infty_{p,n-p}$. If we have two non-zero terms, then H_n fits into a short exact sequence, and so on.

2.3.5 *Example.* We have previously discussed the spectral sequence of a filtered complex without explicitly mentioning it. Indeed, if $C_\bullet$ is a filtered chain complex, then there is a spectral sequence with $E^1_{p,q} = H_{p+q}(G_pC_\bullet)$, such that if the filtration of C_i is bounded for each i the spectral sequence converges to $H_{p+q}(C_\bullet)$.[6]

2.4 *The spectral sequence of a double complex*

An important example where a filtered complex arises is from a double complex.

2.4.1 *Definition.* A double complex is a bi-indexed family $\{C_{p,q}\}$ of abelian groups, with two differentials

$$d' \colon C_{p,q} \to C_{p-1,q}, \quad d'' \colon C_{p,q} \to C_{p,q-1}$$

such that $d'd' = 0, d''d'' = 0$, and $d'd'' + d''d' = 0$. For simplicity, we also assume that $C_{p,q} = 0$ for $p < 0$ or $q < 0$.

2.4.2 *Example.* Suppose that (A, d_A) and (B, d_B) are chain complexes. If we define $C_{p,q} = A_p \otimes B_q$ and define $d' = d_A \otimes 1$ and $d'' = (-1)^p 1 \otimes d_B$, then $C_{p,q}$ is a double complex.[7]

2.4.3 *Construction .* A double complex gives rise to a chain complex (the total complex), defined by $C_n = \sum_{p+q=n} C_{p,q}$ and $d = d' + d''$. This has two obvious filtrations, by row and by column:

1. ${}'C^p_n = \sum_{j+q=n, j \le p} C_{j,q}$.
2. ${}''C^p_n = \sum_{p+q=n, k \le q} C_{p,k}$.

The spectral sequence of a filtered complex (Example 2.3.5) gives us two spectral sequences:

1. ${}'E^1_{p,q} = H_{p+q}({}'C^p/{}'C^{p-1}) = C_{p,n-p}$.
2. ${}''E^1_{p,q} = H_{p+q}({}''C^q/{}''C^{q-1}) = C_{q,n-q}$.

One checks that ${}'E^1$ is computed via means of d'' and that d^1 is induced by d', while in ${}''E^1$ the role of the two indices are exchanged. We can therefore write:

1. ${}'E^2_{p,q} = H'_p H''_q(C)$.
2. ${}''E^2_{p,q} = H''_q H'_p(C)$.

Moreover, both spectral sequences converge to $H_*(C)$, and the idea is to compare the two spectral sequences.

It is constructive to do an example.

[6] Recall what this means: we have $E^\infty_{p,q} = G_pH_{p+q}(C_\bullet)$.

[7] Try and verify this to make sure you understand the definitions.

2.4.4 Example. Let ${}'\,\mathrm{Tor}(A, B)$ be defined as follows: take a free resolution of A, $0 \to R' \to F' \to A \to 0$, then ${}'\,\mathrm{Tor}(A, B)$ is defined by

$$0 \to' \mathrm{Tor}(A, B) \to R' \otimes B \to F' \otimes B \to A \otimes B \to 0.$$

Similarly, let ${}''\,\mathrm{Tor}(A, B)$ be defined as follows: take a free resolution of B, $0 \to R'' \to F'' \to B \to 0$, then ${}''\,\mathrm{Tor}(A, B)$ is defined by

$$0 \to {}''\,\mathrm{Tor}(A, B) \to A \otimes R'' \to A \otimes F'' \to A \otimes B \to 0.$$

It is a classical theorem of homological algebra that $\mathrm{Tor}(A, B) = {}''\,\mathrm{Tor}(A, B)$. Let us prove this via a spectral sequence argument.

Let X be the chain complex $0 \to R' \xrightarrow{d'} F' \to 0$ and let Y be the chain complex $0 \to R'' \xrightarrow{d''} F'' \to 0$. We can build a double complex $C_{*,*}$ as in Example 2.4.2, which we write as a matrix:

$$\begin{bmatrix} C_{p,q} \end{bmatrix} = \begin{bmatrix} F' \otimes R'' & R' \otimes R'' \\ F' \otimes F'' & R' \otimes F'' \end{bmatrix}$$

We have two spectral sequences: the first is take vertical and then horizontal homology:

$$H_q''(C_{p,q}) = \begin{bmatrix} {}''\,\mathrm{Tor}(F', B) & {}'\,\mathrm{Tor}(R', B) \\ F' \otimes B & R' \otimes B \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ F' \otimes B & R' \otimes B \end{bmatrix}$$

and

$$H_p H_q''(C_{p,q}) = \begin{bmatrix} 0 & 0 \\ A \otimes B & {}'\,\mathrm{Tor}(A, B) \end{bmatrix}$$

In other words, the total complex has $H_0(C) = A \otimes B$ and $H_1(C) =' \mathrm{Tor}(A, B)$.

However, we can use the second spectral sequence, which first takes horizontal and then vertical homology:

$$H_p(C_{p,q}) \begin{bmatrix} A \otimes R'' & {}'\,\mathrm{Tor}(A, R'') \\ A \otimes F'' & {}'\,\mathrm{Tor}(A, F'') \end{bmatrix} = \begin{bmatrix} A \otimes R'' & 0 \\ A \otimes F'' & 0 \end{bmatrix}$$

and then

$$H''_q H_p(C_{p,q}) = \begin{bmatrix} {}''\,\mathrm{Tor}(A, B) & 0 \\ A \otimes B & 0 \end{bmatrix}$$

In this case we see that $H_0(C) = A \otimes B$ and $H_1(C) = {}''\,\mathrm{Tor}(A, B)$. Therefore, ${}'\,\mathrm{Tor}(A, B) = {}''\,\mathrm{Tor}(A, B)$.

Exercise 21 (The snake lemma). *Show, using spectral sequences, the following result in homological algebra (the snake lemma):*

Given a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "f"] & B \arrow[r] \arrow[d, "g"] & C \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0
\end{tikzcd}$$

in an abelian category with exact rows, there is a long exact sequence

$$\begin{aligned} 0 \to \ker(f) \to \ker(g) \to \ker(h) \\ \to \mathrm{coker}(f) \to \mathrm{coker}(g) \to \mathrm{coker}(h) \to 0. \end{aligned}$$

Exercise 22. *(1) Suppose we have a commutative triangle*

$$\begin{tikzcd} A \arrow[rr, "i"] \arrow[dr, "f"'] & & C \arrow[dl, "q"] \\ & B & \end{tikzcd}$$

Show using the snake lemma that

$$\ker(\operatorname{coker} f \to \operatorname{coker} q) \cong \operatorname{im}(q) / \operatorname{im}(f)$$

and

$$\operatorname{coker}(\operatorname{coker} f \to \operatorname{coker} q) = 0.$$

(2) Using Part (1), prove the following 'butterfly lemma': given a commutative diagram

$$\begin{tikzcd} A \arrow[dd, "f"'] \arrow[dr, "i"] & & D \arrow[dl, "j"'] \arrow[dd, "g"] \\ & C \arrow[dl, "q"] \arrow[dr, "p"'] & \\ B & & E \end{tikzcd}$$

of abelian groups, in which the diagonals pi and qj are exact at C, there is an isomorphism

$$\frac{\operatorname{im} q}{\operatorname{im} f} \cong \frac{\operatorname{im} p}{\operatorname{im} g}.$$

2.5 *The Serre spectral sequence*

For us the most important example of a spectral sequence will be the Serre spectral sequence. We will state the theorem now and then return to the proof after some examples and applications.

2.5.1 Theorem (The Serre spectral sequence)**.** *Let $\pi \colon E \to F$ be a fibration with fiber F and assume that $\pi_1(B) = 0$ and $\pi_0(F) = 0$, then there is a first quadrant spectral sequence*

$$E^2_{p,q} = H_p(B; H_q(F)) \implies H_{p+q}(E).$$

In particular, this means there is a filtration

$$H_n(E) = D_{n,0} \supseteq D_{n-1,1} \supseteq \ldots \supseteq D_{0,n} \supseteq D_{-1,n+1} = 0$$

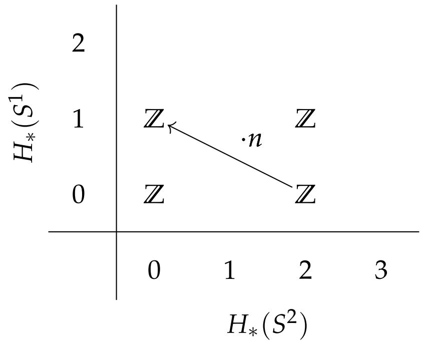
such that $E^\infty_{p,q} = D_{p,q}/D_{p-1,q+1}$.

2.5.2 Remark. There is a version of this spectral sequence where $\pi_1(B) \neq 0$; the E_2-page is then given by the cohomology of B with local coefficients $\mathcal{H}_q(F)$. This will not play a role in this course.

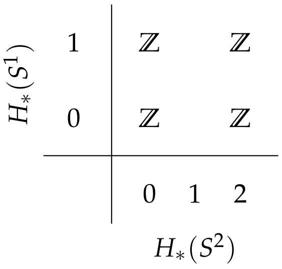
2.5.3 Example. Consider the Hopf fibration $S^1 \to S^3 \to S^2$. We have

$$E^2_{p,q} = H_p(S^2; H_q(S^1)) \cong \begin{cases} \mathbb{Z} & p = 0,2 \text{ and } q = 0,1 \\ 0 & \text{otherwise.} \end{cases}$$

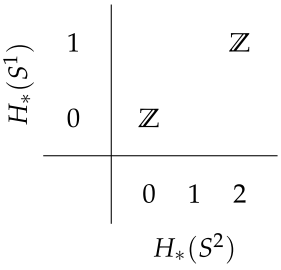
The E^2-term is as follows (and we have $E^3 = E^\infty$ for degree reasons):



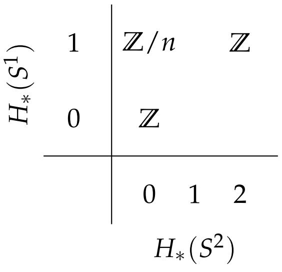
There are three possibilities for the d_2-differential (which is multiplication by $n \in \mathbb{Z}$ as indicated): either $n = 0, n = \pm 1$ or $n \neq 0, \pm 1$, which lead to the following $E^3 = E^\infty$-page:



$n = 0$



$n = \pm 1$



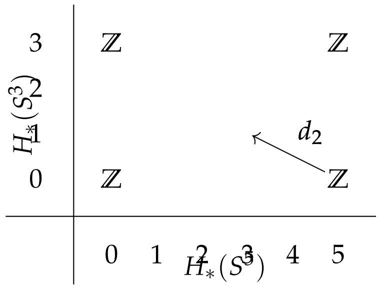
$n \neq 0, \pm 1$

We see that taking $n = \pm 1$ computes the correct answer for $H_*(S^3)$; we have a copy of $\mathbb{Z}$ in the $p + q = 0$ and $p + q = 3$ columns, as required.

2.5.4 Example. There is a fibration $SU(n-1) \to SU(n) \to S^{2n-1}$. Taking $n = 3$ and using $SU(2) \cong S^3$, we obtain a fibration $S^3 \to SU(3) \to S^5$. We have

$$E^2_{p,q} \cong \begin{cases} \mathbb{Z} & p = 0,5 \text{ and } q = 0,3 \\ 0 & \text{otherwise.} \end{cases}$$

The E^2-term is as follows:



Note that there are no differentials for degree reasons, as shown for d_2. Therefore, the spectral sequence collapses and we see that

$$H_i(SU(3)) \cong \begin{cases} \mathbb{Z}, & i = 0,3,5,8 \\ 0, & \text{otherwise.} \end{cases}$$

2.5.5 Example. We can continue the previous example and take $n = 4$ to get a fibration $SU(3) \to SU(4) \to S^7$. We can compute the

E^2-term using the previous example

$$E^2_{p,q} = H_p(S^7; H_q(SU(3))) \cong \begin{cases} \mathbb{Z}, & p = 0,7, q = 0,3,5,8 \\ 0, & \text{otherwise.} \end{cases}$$

The E^2-term is as follows:

$H_*(SU(3))$

8	$\mathbb{Z}$							$\mathbb{Z}$
7								
6								
5	$\mathbb{Z}$							$\mathbb{Z}$
4								
3	$\mathbb{Z}$							$\mathbb{Z}$
2								
1								
0	$\mathbb{Z}$							$\mathbb{Z}$
	0	1	2	3	4	5	6	7

$H_*(S^7)$

Note that there are no differentials for degree reasons, and we compute

$$H_i(SU(4)) \cong \begin{cases} \mathbb{Z}, & i = 0,3,5,7,8,10,12,15 \\ 0, & \text{otherwise.} \end{cases}$$

2.5.6 Remark. If one tries the same argument for $SU(5)$ there are possible differentials. We will see later that it is easier to use cohomology, where one can use multiplicative structures to rule out differentials.

2.5.7 Remark (Naturality of the Serre spectral sequence). The Serre spectral sequence is natural in the following sense. Suppose we are given two fibrations satisfying the hypothesis of the Serre spectral sequence, and a map between them:

$$\begin{tikzcd} F \arrow[r, hook] \arrow[d] & E \arrow[r] \arrow[d, "\tilde{f}"] & B \arrow[d, "f"] \\ F' \arrow[r, hook] & E' \arrow[r] & B' \end{tikzcd}$$

Then the following hold:

1. There are induced maps $f^r_* \colon E^r_{p,q} \to' E^r_{p,q}$ commuting with differentials, i,e, the diagram

$$\begin{tikzcd} E^r_{p,q} \arrow[r, "d_r"] \arrow[d, "f^r_*"'] & E^r_{p-r,q+r-1} \arrow[d, "f^r_*"] \\ {}'E^r_{p,q} \arrow[r, "{}'d_r"'] & {}'E^r_{p-r,q+r-1} \end{tikzcd}$$

   commutes, and moreover f^{r+1}_* is the map induced on homology by f^r_*.

2. The map $\tilde{f}_*\colon H_*(E) \to H_*(E')$ preserves filtrations, inducing a map on associated graded which is exactly f_*^∞.
3. Under the isomorphisms $E^2_{p,q} \cong H_p(B; H_q(F))$ and $'E^2_{p,q} \cong H_p(B'; H_q(F'))$ the map f_*^2 corresponds to the map induced by the maps $B \to B'$ and $F \to F'$.

Once again, we can demonstrate this with an example.

2.5.8 Example. We recall the Hopf fibration $S^1 \to S^3 \to S^2$. This factors through $\mathbb{R}P^3 = S^3/\{\pm 1\}$ as in the following diagram:

$$\begin{tikzcd}
S^1 \arrow[r, hook] \arrow[d, two heads, "q"'] & S^3 \arrow[r] \arrow[d, two heads, "q"'] & S^2 \arrow[d, equal] \\
S^1/\{\pm 1\} \arrow[r, hook] & S^3/\{\pm 1\} \arrow[r] & S^2
\end{tikzcd}$$

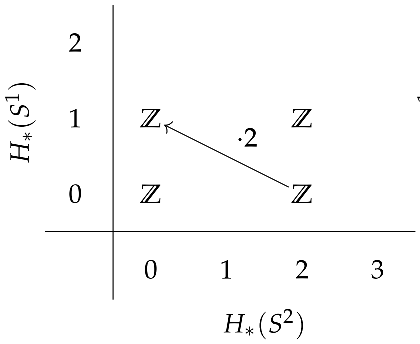
We see that we have a fibration $S^1 \to \mathbb{R}P^3 \to S^2$. The $'E^2$-term of this spectral sequence is as for the Hopf fibration:

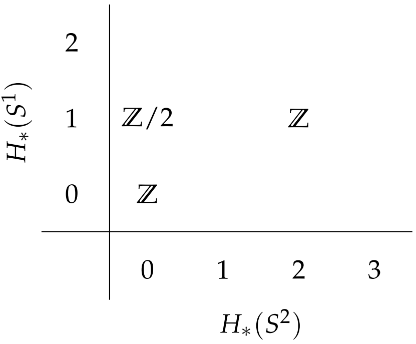
$$E^2_{p,q} = H_p(S^2; H_q(S^1)) \cong \begin{cases} \mathbb{Z} & p = 0,2 \text{ and } q = 0,1 \\ 0 & \text{otherwise.} \end{cases}$$

As in Example 2.5.3 there is only one possible differential, which is $'d_2\colon {'E}^2_{2,0} \to' E^2_{0,1}$, and this is given by multiplication by an integer n. We use naturality to determine what this is. We note that we have a commutative diagram

$$\begin{tikzcd}
H_2(S^2; H_0(S^1)) \arrow[r, "d_2", "\cong"'] \arrow[d, "q_*"', "\cong"] & H_0(S^2; H_1(S^1)) \arrow[d, "\cdot 2"] \\
H_2(S^2; H_0(S^1/\{\pm 1\})) \arrow[r, "'d_2"'] & H_0(S^2; H_1(S^1/\{\pm 1\}))
\end{tikzcd}$$

The right hand arrow is multiplication by 2 because the map induced on homology by $S^1 \to S^1/\{\pm 1\}$ has degree 2 (it is the attaching map for the top cell of $\mathbb{R}P^2$). Commutativity of the diagram implies that $'d_2$ is multiplication by 2. Therefore, the E^2 and $E^3 = E^\infty$-terms are as follows:





We deduce that

$$H_i(\mathbb{R}P^3) \cong \begin{cases} \mathbb{Z}, & i = 0,3 \\ \mathbb{Z}/2, & i = 1 \\ 0, & \text{otherwise.} \end{cases}$$

2.5.9 Remark. It is also possible to deduce some information about $H^*(F)$ or $H^*(B)$ in certain cases, as the following example demonstrates.

2.5.10 Example. There is a fibration $S^1 \to S^\infty \to \mathbb{C}P^\infty$. Note that $\pi_1(\mathbb{C}P^\infty) = 0$, so we can run the Serre spectral sequence. We have

$$E^2_{p,q} = \begin{cases} H_p(\mathbb{C}P^\infty), & q = 0, 1 \\ 0, & \text{otherwise.} \end{cases}$$

We also know that the spectral sequence converges to $H_{p+q}(S^\infty, \mathbb{Z})$, which is only non-zero when $p + q = 0$. In particular, the E^∞ page should be zero except for $E^\infty_{0,0}$. Now consider the E_2-page of the spectral sequence:

$$\begin{tikzcd}[column sep=small]
 & 2 & & & & \\
H_*(S^1) & 1 & \mathbb{Z} & H_1(\mathbb{C}P^\infty) & H_2(\mathbb{C}P^\infty) & H_3(\mathbb{C}P^\infty) \\
 & 0 & \mathbb{Z} & H_1(\mathbb{C}P^\infty) & H_2(\mathbb{C}P^\infty) \arrow[llu] & H_3(\mathbb{C}P^\infty) \arrow[llu] \\
 & & 0 & 1 & 2 & 3 \\
 & & & H_*(\mathbb{C}P^\infty) & &
\end{tikzcd}$$

Note that for degree reasons $E^2_{1,0} \cong H_1(\mathbb{C}P^\infty)$ survives the spectral sequences, and so must be 0. So the E^2-page is as follows:

$$\begin{tikzcd}[column sep=small]
 & 2 & & & & \\
H_*(S^1) & 1 & \mathbb{Z} & 0 & H_2(\mathbb{C}P^\infty) & H_3(\mathbb{C}P^\infty) \\
 & 0 & \mathbb{Z} & 0 & H_2(\mathbb{C}P^\infty) \arrow[llu] & H_3(\mathbb{C}P^\infty) \arrow[llu] \\
 & & 0 & 1 & 2 & 3 \\
 & & & H_*(\mathbb{C}P^\infty) & &
\end{tikzcd}$$

By the same argument $E^2_{3,0} \cong H_3(\mathbb{C}P^\infty)$ survives the spectral sequences, and so must be 0. Inductively, we deduce that $H_n(\mathbb{C}P^\infty) = 0$ for all n odd. Since $E^2_{0,1} \cong \mathbb{Z}$ must also die in the spectral sequence, we see that we must have $H_2(\mathbb{C}P^\infty) \cong \mathbb{Z}$, and that d^2 must be an isomorphism. Continuing inductively, we get

$$H_n(\mathbb{C}P^\infty) \cong \begin{cases} \mathbb{Z}, & n \text{ even} \\ \mathbb{Z}, & 0 \text{ odd.} \end{cases}$$

2.5.11 Example. In our next example, we compute $H_*(\Omega S^n)$ for $n > 1$. We use the path-space fibration of Remark 1.4.9. In this case, this takes the form

$$\Omega S^n \to PS^n \to S^n$$

where we recall that PS^n is contractible, i.e. $H_0(PS^n) = \mathbb{Z}$ and is zero otherwise. In particular, the only non-zero term on the E^∞-

page of the spectral sequence is a copy of $\mathbb{Z}$ when $p+q=0$. Now consider a small portion of the E_2-term:

$$\begin{tikzcd}
3 & H_3(\Omega S^n) & H_3(\Omega S^n) \\
2 & H_2(\Omega S^n) & H_2(\Omega S^n) \\
1 & H_1(\Omega S^n) & H_1(\Omega S^n) \\
0 & \mathbb{Z} & \mathbb{Z} \\
 & 0 & n
\end{tikzcd}$$

Note that the only possible differential is a d_n, and so goes $n-1$-terms upwards. We immediately see that $H_i(\Omega S^n) = 0$ for $0 < i < n-1$. Moreover, the only way to get rid of the $\mathbb{Z}$ in $E^2_{n,0} = E^n_{n,0}$ is that $H_{n-1}(\Omega S^n) \cong \mathbb{Z}$, and that d_n is an isomorphism. We can inductively repeat this argument, getting the following, where all the differentials shown are isomorphisms:

$$\begin{tikzcd}
3n-3 & H_{3n-3}(\Omega S^n) & H_{3n-3}(\Omega S^n) \\
2n-2 & H_{2n-2}(\Omega S^n) & H_{2n-2}(\Omega S^n) \arrow[lu] \\
n-1 & H_{n-1}(\Omega S^n) & H_{n-1}(\Omega S^n) \arrow[lu] \\
0 & \mathbb{Z} & \mathbb{Z} \arrow[lu] \\
 & 0 & n
\end{tikzcd}$$

We conclude that

$$H_i(\Omega S^n) \cong \begin{cases} \mathbb{Z}, & i = k(n-1) \\ 0, & \text{otherwise.} \end{cases}$$

2.5.12 Remark. So far we have only considered examples where the extension problem is trivial; we have had at most one non-zero term in each diagonal on the E^∞-page. The following gives an example where this is not the case.

2.5.13 Example. Consider the Serre spectral sequence of the fibration

$$S^1 \to U(2) \to \mathbb{R}P^3$$

where we identify $S^1 \cong U(1)$ and the first map is given by

$$\lambda \mapsto \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$$

The E^2-page is given by

$$H_p(\mathbb{R}P^3; H_q(S^1)) \cong \begin{cases} \mathbb{Z}, & p = 0, 3, q = 0, 1 \\ \mathbb{Z}/2 & p = 1, q = 0, 1 \\ 0 & \text{else.} \end{cases}$$

The E^2-page looks as follows:

$$\begin{tikzcd} & 1 & \mathbb{Z} & \mathbb{Z}/2 & & \mathbb{Z} \\ H_*(S^1) & 0 & \mathbb{Z} & \mathbb{Z}/2 & & \mathbb{Z} \arrow[ull] \\ & & 0 & 1 & 2 & 3 \\ & & & H_*(\mathbb{R}P^3) & & \end{tikzcd}$$

We know (or take it as fact) that $H_2(U(2)) = 0$; the only way that this is compatible with the spectral sequence is if the differential shown is a surjection, and we get the following $E^3 = E^\infty$-page:

$$\begin{tikzcd} & 1 & \mathbb{Z} & & & \mathbb{Z} \\ H_*(S^1) & 0 & \mathbb{Z} & \mathbb{Z}/2 & & \\ & & 0 & 1 & 2 & 3 \\ & & & H_*(\mathbb{R}P^3) & & \end{tikzcd}$$

Now, in fact we have that[8]

$$H_i(U(2)) \begin{cases} \mathbb{Z}, & i = 0, 1, 3, 4 \\ 0, & \text{else.} \end{cases}$$

Note that in the E^∞-page shown we have two non-zero terms in the $p + q = 1$ column, a $\mathbb{Z}$ in $(0, 1)$ and $\mathbb{Z}/2$ in $(1, 0)$. This means there is an extension[9]

$$0 \to \mathbb{Z} \to H_1(U(2)) \to \mathbb{Z}/2 \to 0.$$

From the calculations above we know that this extension must be non-trivial. Yet, if we didn't know another way to compute $H_1(U(2))$ we could not determine (without more information) if $H_1(U(2))$ was $\mathbb{Z}$ or $\mathbb{Z} \oplus \mathbb{Z}/2$.

2.5.14 Remark. We now return to the Hurewicz theorem, giving a second proof of Theorem 1.10.1.

2.5.15 Theorem. *If X is $(n-1)$-connected, $n \geq 2$, then $\widetilde{H}_i(X) = 0$ for $i \leq n - 1$ and $\pi_n(X) \cong H_n(X)$.*

Proof. We use the path-space fibration

$$\Omega X \to PX \to X,$$

and the fact that PX is contractible. The E^2-page of the Serre spectral sequence is

$$E^2_{p,q} = H_p(X; H_q(\Omega X)) \implies H_{p+q}(PX).$$

---

[8] For example, note that $U(2) \cong SU(2) \times U(1)$

[9] In fact, the spectral sequence shows that we have filtered $H_1(U(2))$ as follows $0 \subseteq 2\mathbb{Z} \subseteq \mathbb{Z} \subseteq \mathbb{Z} = H_1(U(2))$.

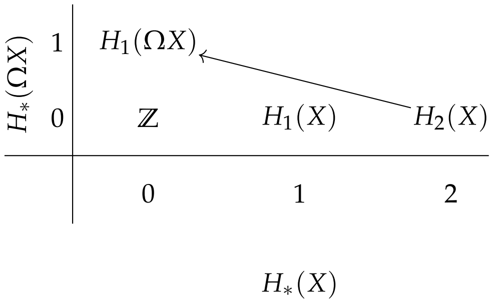
We prove the theorem by induction on n. When $n = 2$, we have $H_1(X) = 0$ because X is simply connected by assumption. Moreover, we have

$$\pi_2(X) \cong \pi_1(\Omega X) \cong H_1(\Omega X)$$

where the first isomorphism follows by the long exact sequence of the fibration, and the second follows from the fact that $\pi_1(\Omega X)$ is abelian, so that $H_1(\Omega X) \cong \pi_1(\Omega X)^{ab} \cong \pi_1(\Omega X)$. It remains to show that $H_1(\Omega X) \cong H_2(X)$. We will use the Serre spectral sequence to show this. Note that $E^2_{2,0} = H_2(X)$ and $E^2_{0,1} = H_1(\Omega X)$, so it suffices to show that

$$d^2\colon E^2_{2,0} = H_2(X) \to E^2_{0,1} = H_1(\Omega X)$$

is an isomorphism. We consider then then a portion of the E^2-page:



Note that if d_2 is not an isomorphism, then both of these groups will persist to the E^∞-page, giving a contradiction to the fact that PX is contractible. So, d_2 must be an isomorphism, as required. This gives the base case of the induction.

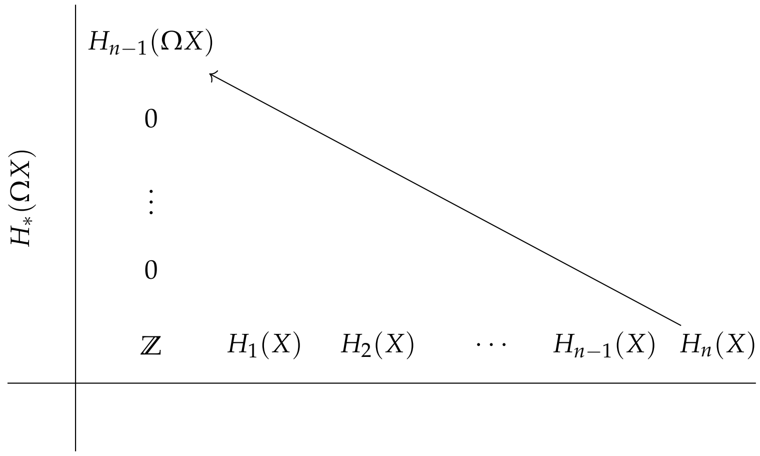
We now assume the statement of the theorem holds for $n-1$ and deduce it for n. Since X is $(n-1)$-connected, ΩX is $(n-2)$-connected, and so by the inductive hypothesis, we have that $\widetilde{H}_i(\Omega X) = 0$ for $i < n-1$ and $\pi_{n-1}(\Omega X) \cong H_{n-1}(\Omega X)$. In particular, we get isomorphisms

$$\pi_n(X) \cong \pi_{n-1}(\Omega X) \cong H_{n-1}(\Omega X),$$

and so it suffices to show that $H_{n-1}(\Omega X) \cong H_n(X)$. We do this via the Serre spectral sequence. We have

$$\begin{aligned} E^2_{p,q} &= H_p(X; H_q(\Omega)) \\ &\cong H_p(X) \otimes H_q(X) \oplus \operatorname{Tor}(H_{p-1}(X), H_q(\Omega X)) \\ 0 & \end{aligned}$$

for $0 < q < n-1$ by the inductive hypothesis. Now consider the Serre spectral sequence:



The only differentials that interact with $H_n(X)$ and $H_{n-1}(\Omega X)$ is the d_n differential shown, and so this must be an isomorphism in order for these terms to die in the spectral sequence. Moreover, the terms $H_i(X)$ for $1 \leq i \leq n-1$ have no differentials at all in the spectral sequence; in particular, we must have $H_i(X) = 0$ for $1 \leq i \leq n-1$ and $d_n \colon H_n(X) \to H_{n-1}(\Omega X)$ is an isomorphism. □

Exercise 23. *Show, using the Serre spectral sequence, that if* $S^k \to S^m \to S^n$ *is a fibration with* $n \geq 2$*, then* $k = n-1$ *and* $m = 2n-1$.

2.6 *The Serre spectral sequence in cohomology*

The Serre spectral sequence in cohomology looks much like the homology version:

2.6.1 Theorem (The Serre spectral sequence in cohomology)**.** *Let* $\pi \colon E \to F$ *be a fibration with fiber* F *and assume that* $\pi_1(B) = 0$ *and* $\pi_0(F) = 0$*, then there is a first quadrant spectral sequence*

$$E_2^{p,q} = H^p(B; H^q(F)) \implies H^{p+q}(E).$$

In particular, this means there is a filtration

$$H^n(E) = D^{0,n} \supseteq D^{1,n-1} \supseteq \ldots \supseteq D^{n,0} \supseteq D^{n+1,-1} = 0$$

such that $E_\infty^{p,q} = D^{p,q}/D^{p+1,q-1}$.

The differentials run $d_r^{p,q} \colon E_r^{p,q} \to E_r^{p+r,q+r-1}$.

2.6.2 *Remark.* Apart from the direction of differentials, this looks much like the Serre spectral sequence in homology. However, there is one major difference: each E_r page has a bilinear product, i.e., a map

$$\bullet \colon E_r^{p,q} \times E_r^{p',q'} \to E_r^{p+p',q+q'}$$

or equivalently,

$$E_r^{p,q} \otimes E_r^{p',q'} \to E_r^{p+p',q+q'}$$

satisfying the Leibniz rule

$$d_r(x \bullet y) = d_r(x) \bullet y + (-1)^{\deg(x)} x \bullet d_r(y).$$

where $\deg(x) = p + q$. Moreover, on the E_2-page, this product is induced by the cup product.

Once again, it is instructive to do an example.

2.6.3 Example. Consider the fibration

$$S^1 \to S^\infty \simeq * \to \mathbb{C}P^\infty$$

The E_2-page looks as follows

$$\begin{tikzcd}
 & 2 & & & & \\
H^*(S^1) & 1 & \mathbb{Z} \arrow[rrd] & H^1(\mathbb{C}P^\infty) \arrow[rrd] & H^2(\mathbb{C}P^\infty) & H^3(\mathbb{C}P^\infty) \\
 & 0 & \mathbb{Z} & H^1(\mathbb{C}P^\infty) & H^2(\mathbb{C}P^\infty) & H^3(\mathbb{C}P^\infty) \\
 & & 0 & 1 & 2 & 3 \\
 & & & & H^*(\mathbb{C}P^\infty) &
\end{tikzcd}$$

Running an argument similar to Example 2.5.10 it is not too hard to compute the additive structure: we must have $E_2^{2k+1,0} = 0$, and $d_2 \colon E_2^{p,1} \to E_2^{p+2,0}$ is an isomorphism $\mathbb{Z} \to \mathbb{Z}$. In particular, we have

$$H^i(\mathbb{C}P^\infty) \begin{cases} \mathbb{Z}, & i = \text{ even} \\ 0, & i = \text{ odd.} \end{cases}$$

Now we wish to compute the multiplicative structure. Let use note that by the universal coefficient theorem in cohomology[10] we have

$$E_2^{p,q} = H^p(\mathbb{C}P^\infty) \otimes H^q(S^1).$$

Let $\mathbb{Z} = \langle x \rangle = H^1(S^1)$ and let $\mathbb{Z} = \langle y \rangle = H^2(\mathbb{C}P^\infty)$, chosen so that $d_2(x) = y$. Then we have

$$E_2^{2,1} = H^2(\mathbb{C}P^\infty) \otimes H^1(S^1) = \mathbb{Z} \otimes \mathbb{Z} \cong \mathbb{Z}.$$

The pairing

$$\bullet \colon E_2^{2,0} \times E_2^{0,1} \to E_2^{2,1}$$

is induced by the cup product, and unwinding the definitions, sends (x, y) to xy, i.e., xy generates $E_2^{2,1}$.

Let z be a generator of $H^4(\mathbb{C}P^\infty)$. We want to show that $z = y^2$. By the Leibniz rule,

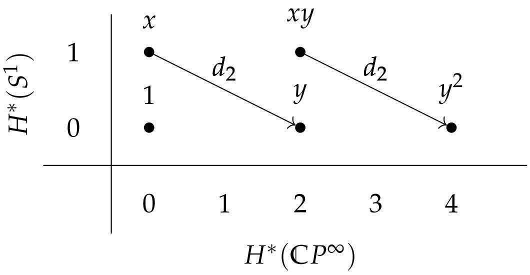
$$d_2(xy) = d_2(x)y + (-1)^{\deg(x)} x d_2(y) = y^2.$$

Noting that d_2 is an isomorphism, we see that $d_2(xy) = y^2 = z$, as needed. Arguing inductively, we see that $d_2(xy^{n-1}) = y^n$ is a generator of $H^{2n}(\mathbb{C}P^\infty)$ and we deduce that $H^*(\mathbb{C}P^\infty) \cong \mathbb{Z}[y]$ with $\deg(y) = 2$.

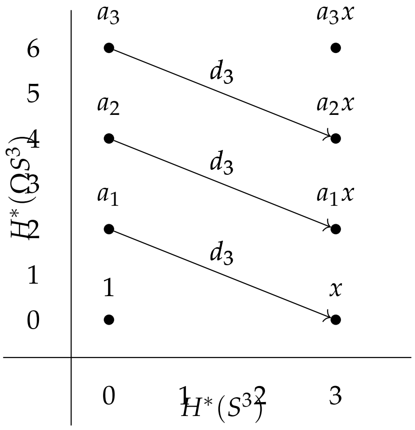
---

[10] In case this was not covered or you need a reminder, this states there is a natural short exact sequence

$$0 \to H^n(X; \mathbb{Z}) \otimes M \to H^n(X; M) \to \operatorname{Tor}(H^{n+1}(X; \mathbb{Z}), M) \to 0.$$



2.6.4 Example. We now consider the cohomology ring $H^*(\Omega S^3)$, leaving the general case of $H^*(\Omega S^n)$ as an exercise. To do this, we use the Serre spectral sequence of the fibration $\Omega S^3 \to PS^3 \simeq * \to S^3$. The additive structure can be determined much as in Example 2.5.11,[11] and the spectral sequence looks as follows:



That is, additively, we have

$$H^i(\Omega S^3) \cong \begin{cases} \mathbb{Z}, & i = 2k \\ 0, \text{else.} \end{cases}$$

In order to work out the multiplicative structure, we need to work out how the classes a_i relate to each other. For example, is $a_1^2 = a_2$? We have chosen the generators such that $d_3(a_i) = a_{i-1}x$ where $a_0 = 1$. Now we use the Leibniz rule to see that[12]

$$d_3(a_1^2) = d_3(a_1)a_1 + a_1 d_3(a_1) = 2a_1x = d_3(2a_2).$$

Because d_3 is an isomorphism, we deduce that $a_1^2 = 2a_2$. What about a_3? Note that

$$\begin{aligned} d_3(a_1a_2) &= d_3(a_1)a_2 + a_1d_3(a_2) = xa_2 + a_1^2x \\ &= xa_2 + 2xa_2 = 3xa_2 \\ &= d_3(3a_3). \end{aligned}$$

Because d_3 is an isomorphism, we deduce that $a_1a_2 = 3a_3$. Said another way, $a_1^3 = a_1a_1^2 = 2a_1a_2 = 3 \cdot 2 \cdot a_3$. By an inductive argument, we deduce that $a_1^n = n!a_n$, where a_n generates $E_2^{0,2n}$. We see that $H^*(\Omega S^3) \cong \Gamma_{\mathbb{Z}}[a_1]$, the divided polynomial algebra on a class a_1 in degree 2.[13]

You should now attempt the following exercise.[14]

---

[11] Convince yourself of this!

[12] Here it is important that all our classes are in even total degrees!

[13] In general, the divided polynomial algebra on a ring R, denoted $\Gamma_R[\alpha]$ where α has (even) degree n is the algebra with additive generators α_i in degree ni and multiplication $\alpha_1^k = k!\alpha_k$ (and hence $\alpha_i\alpha_j = \binom{i+j}{i}\alpha_{i_j}$). Note that if $R = \mathbb{Q}$, then $\Gamma_{\mathbb{Q}}[\alpha] \cong \mathbb{Q}[\alpha]$, but in general it is more complex. For example, if $R = \mathbb{F}_p$, then $\Gamma_{\mathbb{F}_p}[\alpha] \cong \otimes_{i \geq 0}\mathbb{F}_p[\alpha_{p_i}]/(\alpha_{p^i}^p)$, a tensor product of truncated polynomial rings.

[14] Here $\Lambda_{\mathbb{Z}}[x] \cong \mathbb{Z}[x]/(x^2)$ is the exterior algebra

Exercise 24. *Use the cohomological Serre spectral sequence associated to the path fibration*

$$\Omega S^n \to PS^n \to S^n$$

to show the following: If n is odd, then

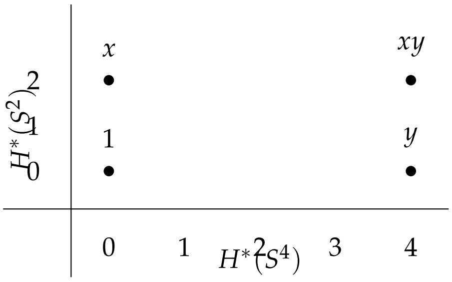
$$H^*(\Omega S^n) \cong \Gamma_{\mathbb{Z}}[x]$$

where $|x| = n - 1$. *If n is even, then*

$$H^*(\Omega S^n) \cong \Lambda_{\mathbb{Z}}[x] \otimes \Gamma_{\mathbb{Z}}[y]$$

where $|x| = n - 1$ *and* $|y| = 2n - 2$.

2.6.5 Example. Much like in the additive case, we can sometimes have multiplicative extensions that we cannot solve without additional information. For example, there is a fibration $S^2 \to \mathbb{C}P^3 \to S^4$, and the associated spectral sequence looks as follows:



There is no room for differentials, and so

$$H^i(\mathbb{C}P^\infty) \cong \begin{cases} \mathbb{Z}, & i = 0, 2, 4, 6 \\ 0, & \text{else.} \end{cases}$$

Yet from the spectral sequence, we cannot deduce (without further information) that $y = x_2^2$, which we know holds.[15]

2.6.6 Remark. A useful way to compute multiplicative extensions is the following theorem:[16] If there is a spectral sequence converging to H_* as an algebra and the E_∞-term is a free, graded-commutative, bigraded algebra, then H_* is a free, graded commutative algebra isomorphic to the total complex $E_\infty^{*,*}$, i.e.,

$$H_i \cong \bigoplus_{p+q=i} E_\infty^{p,q}.$$

2.6.7 Example. Recall that we have a fiber sequence

$$SU(n-1) \to SU(n) \to S^{2n-1}$$

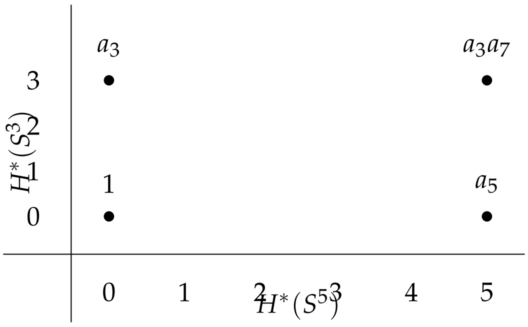
Taking $n = 3$, this has the form $S^3 \to SU(3) \to S^5$. The E^2-term is

$$E_2^{p,q} = H^p(S^5; H^q(S^3)) \cong H^p(S^5) \otimes H^q(S^3).$$

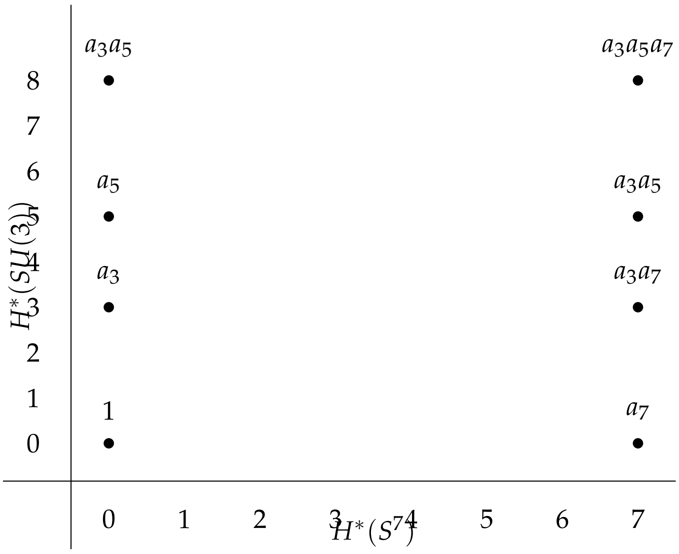
The $E^2 = E^\infty$-page looks as follows:

[15] Recall that $H^*(\mathbb{C}P^3) \cong \mathbb{Z}[x]/(x^4)$ for $|x| = 2$.

[16] See Example 1.K of McCleary's "A user's guide to spectral sequences"



Unlike in the previous example, there are no possible multiplicative extension problems, and we deduce that $H^*(SU(3)) \cong \Gamma_{\mathbb{Z}}(a_3, a_5)$, the exterior algebra on two generators a_3 and a_5. Now we consider the case $n = 4$. We leave it for the reader to deduce the following $E^2 = E^\infty$-page



The extension problem can be solved by Remark 2.6.6, and we get $H^*(SU(4)) \cong \Gamma_{\mathbb{Z}}(a_3, a_5, a_7)$. These two examples suggest a general pattern: is $H^*(SU(n)) \cong \Gamma_{\mathbb{Z}}(a_3, a_5, \ldots, a_{2n-1})$? The next exercise is to show that this is true.[17]

Exercise 25. *Show, using induction, that*

$$H^*(SU(n)) \cong \Gamma_{\mathbb{Z}}(a_3, a_5, \ldots, a_{2n-1}).$$

2.7 *The Gysin and Wang sequences*

The Gysin and Wang sequences are long exact sequences derived from special cases of the Serre spectral sequence. We will prove the following.

2.7.1 Theorem (The Gysin sequence). *Let $S^n \to E \to B$ be a fibration with $\pi_1 B = 0$ and $n \geq 1$. Then, there exists an exact sequence*

$$\cdots H_r(E) \to H_r(B) \to H_{r-n-1}(B) \to H_{r-1}(E) \to \cdots$$

We begin with two algebraic lemmas, whose proof we leave as exercises for the reader.

[17] However, unlike the previous two examples, there will be differentials to deal with. The hint is to use the multiplicative structure, in particular the Leibniz rule.

2.7.2 Lemma. *Let $A \to B \xrightarrow{f} C$ and $D \to E \xrightarrow{g}$ be exact sequences of abelian groups. Suppose there exists an isomorphism $\phi\colon \operatorname{coker}(f) \cong \ker(g)$, then there is an exact sequence*

$$A \to B \xrightarrow{f} C \xrightarrow{\phi} D \to E \xrightarrow{g} F,$$

where $c \mapsto \phi(\bar{c})$, for $\bar{c}$ the class of c in $\operatorname{coker}(f)$.

2.7.3 Lemma. *Given the following diagram of abelian groups:*

$$\begin{tikzcd} & A \arrow[d, "f"] & & \\ & B \arrow[d, "g"] \arrow[dr, "hg"] & & \\ 0 \arrow[r] & C \arrow[r, "h"'] & D \arrow[r, "k"'] & E \end{tikzcd}$$

with rows and columns exact, then the sequence $A \xrightarrow{f} B \xrightarrow{hg} D \xrightarrow{k} E$ is exact.

We now return to the Gysin sequence.

Proof of Theorem 2.7.1. We consider the Serre spectral sequence of the fibration. This has E_2-term

$$E^2_{p,q} \cong H_p(B; H_q(S^n)) \cong \begin{cases} H_p(B) & q = 0, n \\ 0 & \text{else.} \end{cases}$$

and so is as follows:

$$\begin{array}{c|cccc} n & H_0(B) & H_1(B) & H_2(B) & H_3(B) \\ & & & & \\ 0 & H_0(B) & H_1(B) & H_2(B) & H_3(B) \\ \hline & & & & \end{array}$$

$H_*(S^n)$

$H_*(B)$

We observe that there is only one possible differential, namely $d_{n+1}\colon E^{n+1}_{p,0} \to E^{n+1}_{p-n-1,n}$, and so $E^2 = E^{n+1}$ and $E^{n+2} = E^\infty$. Therefore, using Lemma 2.7.2 we get a short exact sequence

$$0 \to E^\infty_{p,0} \to E^{n+1}_{p,0} \xrightarrow{d_{n+1}} E^{n+1}_{p-n-1,n} \to E^\infty_{p-n-1,n} \to 0. \tag{2.7.4}$$

The filtration on $H_i(E)$ is $0 \subseteq E^\infty_{i-n,n} = D_{i-n,n} \subseteq D_{i,0} = H_i(E)$, i.e., we have a short exact sequence:

$$0 \to E^\infty_{i-n,n} \to H_i(E) \to E^\infty_{i,0} \to 0. \tag{2.7.5}$$

Pasting (2.7.4) and (2.7.5) together we get a diagram of the form:

$$\begin{tikzcd}
{} \arrow[dr, dashed] & {} \arrow[d, dashed] & & & & \\
& H_r(E) \arrow[d] \arrow[dr, dashed] & & & 0 \arrow[d] & \\
0 \arrow[r] & E^\infty_{r,0} \arrow[r] \arrow[d] & \underbrace{E^2_{r,0}}_{=H_r(B)} \arrow[r, "d_{n+1}"] & \underbrace{E^2_{r-n-1,n}}_{H_{r-n-1}(B)} \arrow[r] \arrow[dr, dashed] & E^\infty_{r-n-1,n} \arrow[r] \arrow[d] & 0 \\
& 0 & & & H_{r-1}(E) \arrow[d] \arrow[dr, dashed] & \\
& & & 0 \arrow[r] & E^\infty_{r-1,0} \arrow[r] \arrow[d] & \cdots \\
& & & & 0 &
\end{tikzcd}$$

and Lemma 2.7.3 implies that the sequence in red is exact. ☐

2.7.6 *Example.* Consider the fiber sequence $S^1 \to S^{2n-1} \to \mathbb{C}P^n$ for $n \geq 1$. One recalls that $H_p(\mathbb{C}P^n) = 0$ for $p > 2n$ using, for example, cellular homology. We will show that

$$H_p(\mathbb{C}P^n) \cong \begin{cases} \mathbb{Z} & p \text{ even} \\ 0 & p \text{ odd} \end{cases}$$

using the Gysin sequence. The sequence tell us that

$$0 = H_{2n+2}(\mathbb{C}P^n) \to H_n(\mathbb{C}P^n) \to H_{2n+1}(S^{2n+1}) \cong \mathbb{Z} \to H_{2n+1}(\mathbb{C}P^n) = 0$$

is exact, and so $H_n(\mathbb{C}P^n) \cong \mathbb{Z}$. Next, observe that we have an exact sequence

$$0 = H_{2n}(S^{2n+1}) \to H_{2n}(\mathbb{C}P^n) \cong \mathbb{Z} \to H_{2n-2}(\mathbb{C}P^n) \to H_{2n-1}(S^{2n-1})$$

so that $H_{2n-2}(\mathbb{C}P^n) \cong \mathbb{Z}$. Moreover, the exact sequence

$$0 = H_{2n+1}(\mathbb{C}P^n) \to H_{2n-1}(\mathbb{C}P^n) \to H_{2n}(S^{2n+1}) = 0$$

shows that $H_{2n-1}(\mathbb{C}P^n) = 0$. Inductively continuing, we get the claimed result.

Exercise 26. *Wang sequence Use the Serre spectral sequence to prove the following:*

If $F \to E \to S^n$ with $n \geq 2$ is a fibration, then there is an exact sequence

$$\cdots \to H_i(F) \to H_i(E) \to H_{i-n}(F) \to H_{i-1}(F) \to H_{i-1}(E) \to \cdots$$

2.7.7 *Example.* Let us return to the homology of ΩS^n for $n \geq 2$. We will show that

$$H_r(\Omega S^n) \cong \begin{cases} \mathbb{Z} & r = k(n-1) \\ 0 & \text{else.} \end{cases}$$

We do this via the Wang sequence of the fibration $\Omega S^n \to PS^n \to S^n$. Since PS^n is contractible, every third term vanishes except for $H_0(PS^n) \cong \mathbb{Z}$. So,

$$H_{r-n}(\Omega S^n) \cong H_{r-1}(\Omega S^n).$$

Because $H_0(\Omega S^n) \cong \mathbb{Z}$ (by path-connectedness), we get the claimed result inductively.

We also have a Wang sequence in cohomology.

2.7.8 Theorem (Wang sequence in cohomology). *Let $F \xrightarrow{i} E \xrightarrow{p} S^n$ be a fiber sequence with $n \geq 1$, then there is a long exact sequence*

$$\cdots \to H^i(E) \xrightarrow{i^*} H^i(F) \xrightarrow{\theta} H^{q-n+1}(F) \to H^{q+1}(E) \to \cdots$$

where

$$\theta(u \smile v) = \theta(u) \smile v + (-1)^{(n-1)\deg(u)} u \smile \theta(v).$$

Proof sketch. The additive structure is determined as in the exercise. The fact that θ is a derivation follows because it is identified with the d^{n+1} differential in the spectral sequence. □

We can use this to recover the ring structure on $H^*(\Omega S^n)$.

2.7.9 Theorem. *If u is odd, then $H^*(\Omega S^u) \cong \Gamma_{\mathbb{Z}}(x), |x| = u - 1$.*

Proof. The Wang sequence gives isomorphisms

$$H^n(\Omega S^u) \xrightarrow[\cong]{\theta} H^{n-u-1}(\Omega S^u).$$

This determines the additive structure. To work out the multiplicative structure, let $\gamma_0(x) = 1$ and inductively let $\gamma_i(x) \in H^{i(i-1)}(\Omega S^u)$ for $i \geq 1$ so that $\theta(\gamma_i(x)) = \gamma_{i-1}(x)$. By induction on i and j and using that θ is a derivation, we have[18]

$$\begin{aligned}\theta(\gamma_i(x) \smile \gamma_j(x)) &= \gamma_{i-1}(x) \smile \gamma_j(x) + \gamma_i(x) \smile \gamma_{j-1}(x) \\ &= \binom{i-1+j}{j}\gamma_{i-1+j}(x) + \binom{i-1+j}{j-1}\gamma_{i-1+j}(x) \\ &= \binom{j+i}{j}\gamma_{i-1+j}(x) \\ &= \binom{j+i}{j}\theta(\gamma_{i+j}(x)).\end{aligned}$$

Because θ is an isomorphism, we deduce that $\gamma_i(x) \smile \gamma_j(x) = \gamma_{i+j}(x)$, and the result follows. □

2.7.10 Theorem. *If $u \geq 2$ is even, then $H^*(\Omega S^u) \cong \Lambda_{\mathbb{Z}}(x) \otimes \Gamma_{\mathbb{Z}}(y)$, $|x| = u - 1$ and $|y| = 2(u-1)$.*

Proof. Let $x \in H^{u-1}(\Omega S^u)$ be such that $\theta(x) = 1$. By graded commutativity we have $x^2 = 0$. Let $\gamma_0(y) = 1$ and inductively define $\gamma_i(y) \in H^{2i(u-1)}(\Omega S^u)$ so that $\theta(\gamma_i(y)) = x\gamma_{i-1}(y)$. Then,

$$\theta(x\gamma_i(y)) = 1 \smile \gamma_i(y) - x \smile x\gamma_{i-1}(y) = 1 \smile \gamma_i(y) = \gamma_i(y),$$

---

[18] Try and spot where we use that x is is even degree.

so that $\gamma_i(y)$ generates $H^{2i(u-1)}(\Omega S^u)$ and $x\gamma_i(y)$ generates $H^{(2i+1)(u-1)}(\Omega S^u)$. Then

$$\begin{aligned}\theta(\gamma_i(y) \smile \gamma_j(y)) &= x\gamma_{i-1}(y) \smile \gamma_j(y) + \gamma_i(y) \smile x\gamma_{j-1}(y) \\ &= \binom{i-1+j}{j} x\gamma_{i-1+j}(y) + \binom{i-1+j}{j-1} x\gamma_{i-1+j}(y) \\ &= \binom{j+i}{j} x\gamma_{i-1+j}(t) \\ &= \binom{j+i}{j} \theta(\gamma_{i+j}(y)).\end{aligned}$$

Because θ is an isomorphism, we deduce that $\gamma_i(y) \smile \gamma_j(y) = \gamma_{i+j}(y)$, and the result follows. $\square$

2.7.11 Remark. Of course, we also have a cohomological Gysin sequence: Suppose $S^{n-1} \to E \xrightarrow{p} B$ is a fibration with $\pi_1 B = 0$ and $n \geq 0$. Then, there exists an exact sequence

$$\cdots \to H^{i-n}(B) \xrightarrow{\smile e} H^i(B) \xrightarrow{p^*} H^i(E) \to H^{i-n+1}(B) \to \cdots$$

for a certain Euler class $e \in H^n(B)$ Moreover, if n is odd, then $2e = 0$.

2.8 *Serre mod $\mathcal{C}$ theory*

Our next goal is to prove the following theorem of Serre.

2.8.1 Theorem. *If X is a finite CW complex with $\pi_1(X) = 0$, then the homotopy groups $\pi_i(X)$ are finitely generated abelian groups for $i \geq 2$.*

2.8.2 Remark. How do we begin to prove such a theorem? Computing it directly is not possible. The ingenious idea of Serre is to use what are now known as Serre classes.

2.8.3 Definition. A class $\mathcal{C}$ of abelian groups is a Serre class if $0 \in \mathcal{C}$ and if for any short exact sequence $0 \to A \to B \to C \to 0$ we have $A, C \in \mathcal{C} \implies B \in \mathcal{C}$.

2.8.4 Remark. The following are a consequence of the definitions:[19]

(i) A Serre class is closed under isomorphisms.

(ii) A Serre class is closed under the formation of subgroups and quotient groups.

(iii) Let $A \xrightarrow{i} B \xrightarrow{p} C$ be exact at B. IF $A, C \in \mathcal{C}$, then $B \in \mathcal{C}$. This follows from the previous two points and the diagram

$$\begin{tikzcd} & A \arrow[d, two heads] \arrow[dr, "i"] & & & \\ 0 \arrow[r] & \ker(p) \arrow[r] & B \arrow[r] \arrow[dr, "p"'] & \operatorname{coker}(i) \arrow[r] \arrow[d, hook] & 0 \\ & & & C & \end{tikzcd}$$

---

[19] You should check these.

2.8.5 Example. Examples of Serre classes include finite abelian groups, finitely generated abelian groups, and torsion abelian groups.

2.8.6 Definition. Given a morphism of abelian groups $\phi\colon A \to B$, we say ϕ is a monomorphism mod $\mathcal{C}$ if $\ker(\phi) \in \mathcal{C}$, an epimorphism mod $\mathcal{C}$ if $\operatorname{coker}\phi \in \mathcal{C}$, and an isomorphism mod $\mathcal{C}$ if both $\ker\phi, \operatorname{coker}\phi \in \mathcal{C}$.

2.8.7 Remark. We note the following:

(i) Let $C_\bullet$ be a chain complex. If $C_n \in \mathcal{C}$, then $H_n(C_\bullet) \in \mathcal{C}$.

(ii) Suppose $F_\bullet A$ is a filtration on an abelian group. If $A \in \mathcal{C}$, then $G_s A \in \mathcal{C}$ for all s. If the filtration is finite ($F_m = 0, F_n = A$, for some m, n) and $G_s(A) \in \mathcal{C}$ for all a, then $A \in \mathcal{C}$.

(iii) Suppose we have a spectral sequence $\{E^r_{S,t}\}$. If $E^2_{s,t} \in \mathcal{C}$, then $E^r_{s,t} \in \mathcal{C}$ for all $r \geq 2$. If $\{E^r\}$ is a first quadrant spectral sequence, then $E^\infty \in \mathcal{C}$. If the spectral sequence comes from a finite filtered complex C and $E^2_{s,t} \in \mathcal{C}$ for all $s+t=n$, then $H_n(C) \in \mathcal{C}$.

2.8.8 Definition. A Serre class is called a Serre ring if $A, B \in \mathcal{C} \implies A \otimes B \in \mathcal{C}$ and $\operatorname{Tor}(A, B) \in \mathcal{C}$, and a Serre ideal if only one of A or B is required to be in $\mathcal{C}$. Finally, a Serre class is called acyclic if $A \in \mathcal{C}$ implies $\tilde{H}_*(K(A,1), \mathbb{Z}) \in \mathcal{C}$. [20]

2.8.9 Theorem (Hurewicz mod $\mathcal{C}$). *Assume that $\mathcal{C}$ is an acyclic Serre ring, and let X be a simply connected space. If $\pi_q(X) \in \mathcal{C}$ for all $q < n$, then $\widetilde{H}_q(X) \in \mathcal{C}$ for all $q < n$, and in that case the Hurewicz map $\pi_n(X) \to H_n(X)$ is a mod $\mathcal{C}$ isomorphism.*

Proof. Simply observe that the same argument as Theorem 2.5.15 works replacing 'isomorphism' with 'isomorphism mod $\mathcal{C}$' where necessary. ☐

We return to Serre's theorem.

Proof of Theorem 2.8.1. Take $\mathcal{C}$ to be the class of finitely generated abelian groups, which is an acyclic Serre ring. Because X is a finite CW-complex $\widetilde{H}_i(X) \in \mathcal{C}$ since X is a finite CW-complex. Now suppose there exists a minimal i such that $\pi_i(X)$ is not finitely generated. Then, by the relative Hurewicz theorem, $h\colon \pi_i(X) \to H_n(X)$ is an isomorphism mod finitely generated abelian groups, and so $H_i(X)$ is also not finitely generated, a contradiction. ☐

2.9 *Final notes on spectral sequences*

We finish this section on spectral sequences by including some observations which we have so far omitted, beginning with the construction of the Serre spectral sequence.

2.9.1 Remark (Construction of the Serre spectral sequence). Let us sketch the construction of the Serre spectral sequence. Let $F \to E \xrightarrow{\pi} B$ be a fibration with B a simply-connected CW complex, and $\pi_0(F) = 0$.[21] Let $C_*(E)$ be the singular chain complex of E, and

[20] For example, the class of all finitely generated abelian groups is an acyclic Serre ring, while the class of torsion abelian groups is an acyclic Serre ideal.

[21] By cellular approximation, we can always replace B a CW-complex, , and replace $\pi\colon E \to B$ with $\pi'\colon E' \to B'$, the pullback of the fibration along $B' \to B$.

filter this by

$$F^pC_*(E) := C_*(\pi^{-1}(B_p)),$$

where B_p is the p-skeleton of B. Then,

$$\begin{aligned} F^pC_*(E)/F^{p-1}C_*(E) &\simeq C_*(\pi^{-1}(B_p))/C_*(\pi^{-1}(B_{p-1})) \\ &\simeq C_*(\pi^{-1}(B_p), \pi^{-1}(C_{p-1})) \end{aligned}$$

and (using excision)

$$\begin{aligned} H_*(F^pC_*(E)/F^{p-1}C_*(E)) &\cong H_*(\pi^{-1}(B_p), \pi^{-1}(B_{p-1})) \\ &\cong \oplus_{e^p} H_*(\pi^{-1}(e^p), \pi^{-1}(\partial e^p)) \end{aligned}$$

where the direct sum is over the p-cells e^p in B. Since e^p is contractible, locally the fibration looks like the trivial fibration, i.e., $\pi^{-1}(e^p) \cong e^p \times F$. Then,

$$\begin{aligned} H_*(\pi^{-1}(e^p), \pi^{-1}(\partial e^p)) &\cong H_*(e^p \times F, \partial e^p \times F) \\ &\cong H_*(D^p \times F, S^{p-1} \times F) \\ &\cong H_{*-p}(F) \\ &\cong H_p(D^p, S^{p-1}; H_{*-p}(F)) \end{aligned}$$

where we have use the Künneth formula. The general machinery of spectral sequence tells us we have a spectral sequence with E_1 term

$$\begin{aligned} E^1_{p,q} &= H_{p+q}(F^pC_*(E)/F^{p-1}C_*(E)) \\ &\cong \oplus_{e^p} H_p(D^p, S^{p-1}; H_q(F)) \end{aligned}$$

The d^1-differential can be checked to be exactly the cellular boundary map of the CW-chain complex of B with coefficients in $H_q(F)$, i.e.,

$$E^1_{p,q} = C^{cell}_p(B; H_q(F))$$

and

$$E^2_{p,q} = H_p(B; H_q(F)).$$

2.9.2 *Remark.* There are several natural generalizations of the Serre spectral sequence:

1. We can take homology with coefficients in an abelian group, i.e.,

$$E^2_{s,t} = H_p(B; H_q(F; G)) \implies H_{p+q}(E; G).$$

2. There is a relative form of the Serre spectral sequence: Suppose we have $F \to E \xrightarrow{\pi} B$, and $B' \subseteq B$. If we set $E' \colon \pi^{-1}(B')$, then there is a spectral sequence

$$E^2_{s,t} = H_p(B, B'; H_q(F; G)) \implies H_{p+q}(E, E'; G).$$

3. If $\pi_1(B)$ is not 0, but acts trivially on $H_*(F)$, then the spectral sequence takes the same form. If $\pi_1(B)$ does not act trivially then there is still a Serre spectral sequence, but then one must use homology with local coefficients.

We finish this section with another theorem of Serre.

2.9.3 Theorem (Serre). *The group $\pi_i(S^n)$ are finite for $i > n$ except for $\pi_{4k-1}(S^{2k})$, which is the direct sum of $\mathbb{Z}$ with a finite group.*

Proof. We can assume $n > 1$, in which case we can use the Serre spectral sequence. Note that we already know that these are finitely generated abelian groups by Theorem 2.8.1. Recall that $H_n(S^n) \cong \mathbb{Z}$, and that $H^n(S^n) \cong [S^n, K(\mathbb{Z}, n)]$. Moreover, the Hurewicz map $h_n \colon \pi_n(S^n) \to H_n(S^n)$ is an isomorphism, so together we can find a homotopy class of map $f \colon S^n \to K(\mathbb{Z}, n)$ inducing an isomorphism on π_n. We can assume that this is a fibration. By the long exact sequence in homotopy, the fiber F is $(n-1)$-connected, and $\pi_i(F) \cong \pi_i(S^n)$ for $i > n$. Converting $F \to S^n$ into a fibration, and taking the fiber (which is then a $K(\mathbb{Z}, n-1)$), we have a fibration

$$K(\mathbb{Z}, n-1) \to F \to S^n.$$

We will study the Serre spectral sequence associated to this rationally, i.e., with $\mathbb{Q}$-coefficients. This takes the form

$$\begin{aligned} E_2^{p,q} &\cong H^p(S^n; H^q(K(\mathbb{Z}, n-1); \mathbb{Q})) \\ &\cong H^p(S^n; \mathbb{Q}) \otimes H^q(K(\mathbb{Z}, n-1); \mathbb{Q})). \end{aligned}$$

We have[22]

$$H^*(K(\mathbb{Z}, n); \mathbb{Q}) \cong \begin{cases} \mathbb{Q}[x], & n \text{ even} \\ \Lambda_{\mathbb{Q}}(x), & n \text{ odd,} \end{cases}$$

where $|x| = n$.

We begin with the case n odd. The Serre spectral sequence looks as follows:

$$\begin{tikzcd}
3n-3 & \mathbb{Q}\{a^3\} \arrow[dr] & \mathbb{Q}\{a^3x\} \\
2n-2 & \mathbb{Q}\{a^2\} \arrow[dr] & \mathbb{Q}\{a^2x\} \\
n-1 & \mathbb{Q}\{a\} \arrow[dr] & \mathbb{Q}\{ax\} \\
0 & \mathbb{Q}\{1\} & \mathbb{Q}\{x\} \\
 & 0 & n
\end{tikzcd}$$

The differential $\mathbb{Q}\{a\} \to \mathbb{Q}\{x\}$ must be an isomorphism, as otherwise it would be zero, and $\mathbb{Q}\{a\}$ would survive to E^∞, a contradiction to the fact that F is $(n-1)$-connected. By the Leibniz rule, all the differentials are isomorphisms, and so $\widetilde{H}_*(F; \mathbb{Q}) = 0$, and so $\pi_i(F) \otimes \mathbb{Q} = 0$ for all i. By a Serre class argument $\pi_i(F)$ is a finitely-generated abelian group, and so it must in fact be finite for all i. Therefore, $\pi_i(S^n)$ is finite for $i > n$.

The case n is a bit trickier. The spectral sequence now looks as follows:

---

[22] This is another good exercise. The hint is to use induction, starting from $K(\mathbb{Z}, 1) = S^1$.

$$\begin{tikzcd}
n-1 & \mathbb{Q}\{a\} \arrow[dr] & \mathbb{Q}\{ax\} \\
0 & \mathbb{Q}\{1\} & \mathbb{Q}\{x\} \\
 & 0 & n
\end{tikzcd}$$

So we deduce

$$\widetilde{H}^*(F;\mathbb{Q}) \cong \begin{cases} \mathbb{Q} & * = 0, 2n-1 \\ 0 & \text{else.} \end{cases} .$$

By the Hurewicz theorem mod finitely generated abelian groups,we conclude that $\pi_i(S^n)$ is finite for $n < i < 2n-1$ and $\pi_{2n-1} \cong \mathbb{Z}$ plus a finite abelian group. To deal with $\pi_i(S^n)$ for $i > 2n-1$, we let Y be the space obtained from F by attaching cells of dimension $2n+1$ and greater to kill $\pi_i(F)$ for $i \geq 2n-1$. We can assume $F \to Y$ is fibration, with fiber Z. Then Z is $(2n-2)$-connected and has $\pi_i(Z) \cong \pi_i(Y)$ for $i < 2n-1$ and so all the homotopy groups of Y are finite. Thus $\widetilde{H}^*(Y;\mathbb{Q}) = 0$ and from the Serre spectral sequence associated to $Z \to F \to Y$ we get

$$\widetilde{H}^*(Z;\mathbb{Q}) \cong \widetilde{H}^*(F;\mathbb{Q}) \cong \widetilde{H}^*(S^{2n-1};\mathbb{Q})$$

Using the earlier argument for n odd with Z instead of S^n, we conclude that $\pi_i(Z)$ is finite for $i > 2n-1$, but $\pi_i(S^n) \cong \pi_i(Z)$ for $i > 2n-1$ and we are done. ☐

Exercise 27. *Let $\pi\colon E \to B$ be a fibration with fiber F, let k be a field, and suppose $\pi_1(B) = \pi_0(F) = 0$. Assume that the Euler characteristics $\chi(B), \chi(F)$ are defined over the field k. (Recall that for a chain complex C, the Euler characteristic is the alternating sum of the ranks of the homology of the chain complex, assuming these ranks are all finite.) Then $\chi(E)$ is defined, and*[23]

$$\chi(E) = \chi(B) \cdot \chi(F).$$

[23] **Hint:** Construct an 'Euler characteristic' for the E_r-page of the Serre spectral sequence.

3
Bundle theory

3.1 Locally trivial bundles

We begin with what we will call a locally trivially bundle.[1]

3.1.1 Definition. A map $\pi\colon E \to B$ is a locally trivial bundle with fiber F if the following conditions hold:

1. Each point $b \in B$ has a neighborhood U such that $\pi^{-1}(U_b) \xrightarrow{h_b} U_b \times F$.
2. The following diagram commutes

$$\begin{tikzcd}
\pi^{-1}(U_b) \arrow[rr, "h_b"] \arrow[dr, "\pi"'] & & U_b \times F \arrow[dl, "\mathrm{pr}_1"] \\
& U_b &
\end{tikzcd}$$

The maps h_b are called the *local trivializations* of the bundle.

3.1.2 Example. Let $E = B \times F$ and $\pi\colon E = B \times F \to B$ the projection map. This is called the trivial bundle.

3.1.3 Example. If F is discrete, then a locally trivial bundle with fiber F is a covering map.

3.1.4 Example. The Möbius band is a locally trivial bundle with fiber S^1, see Figure 3.1. We will return to this example in due course.

3.1.5 Remark. We can write a locally trivial bundle as $F \to E \to B$, which is reminiscent of the notation for a fibration. In fact, fiber bundles over paracompact base spaces are always fibrations.[2] More generally, any locally trivial bundle is a Serre fibration.

3.1.6 Remark. Let us unwind the definition of a locally trivial bundle a little more. Let $\pi\colon E \to B$ be a locally trivial bundle with fiber F. From the definition we can cover B by a family of open sets $\{U_\alpha\}$ such that each inverse image $\pi^{-1}(U_\alpha)$ is fiberwise homeomorphic to $U_\alpha \times F$. This gives a system of homeomorphisms

$$\phi_\alpha\colon U_\alpha \times F \to \pi^{-1}(U_\alpha).$$

Observe that if $V \subseteq U_\alpha$ then the restriction of ϕ_α to $V \times F$ gives the homeomorphism with $\pi^{-1}(V)$. Hence on $U_\alpha \cap U_\beta$ there are two

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[1] Confusingly, some books will also call this a fiber bundle. We will see why later.

[2] A space is paracompact if every open cover has an open refinement that is locally finite.

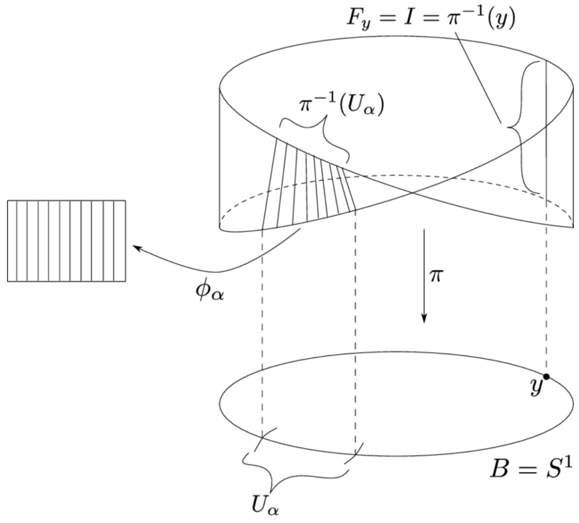

Figure 3.1: The Möbius band

fiberwise homeomorphisms

$$\phi_\alpha \colon (U_\alpha \cap U_\beta) \times F \to \pi^{-1}(U_\alpha \cap U_\beta)$$
$$\phi_\beta \colon (U_\alpha \cap U_\beta) \times F \to \pi^{-1}(U_\alpha \cap U_\beta)$$

Consider the following commutative diagram

$$\begin{tikzcd}
(U_\alpha \cap U_\beta) \times F \arrow[r, "\phi_\alpha", "\cong"'] \arrow[dr] & \pi^{-1}(U_\alpha \cap U_\beta) \arrow[r, "\phi_\beta^{-1}", "\cong"'] \arrow[d] & (U_\alpha \cap U_\beta) \times F \arrow[dl] \\
& U_\alpha \cap U_\beta &
\end{tikzcd}$$

Let $\phi_{\alpha\beta}$ denote the top composite $\phi_\beta^{-1}\phi_\alpha$. Then the locally trivially bundle is completed determined by the base B, the fiber F, the covering U_α and the homeomorphisms $U_{\alpha\beta}$. Roughly speaking, E should be thought of as the cartesian product of the $U_\alpha \times F$ with some identifications by the $\phi_{\alpha\beta}$.

3.1.7 Definition. The open sets U_α are called *charts*, the family U_α the *atlas of charts*, the homeomorphisms ϕ_α are called the *coordinate homeomorphisms* and the $\phi_{\alpha\beta}$ are called the *transition functions.*

3.1.8 Remark. In order for homeomorphisms $\phi_{\alpha\beta}$ to be the transition functions of a locally trivial bundle, they must satisfy a number of conditions. For example,

$$\phi_{\alpha\alpha} = \mathrm{id}$$

and

$$\phi_{\gamma\alpha}\phi_{\beta\gamma}\psi_{\alpha\beta} = \mathrm{id}$$

on any triple $(U_\alpha \cap U_\beta \cap U_\gamma) \cap F$. Taking $\gamma = \alpha$ we get

$$\phi_{\alpha\beta}\phi_{\beta\alpha} = \mathrm{id}.$$

In fact, these conditions suffices to reconstruct the locally trivial bundle for the base, the fiber, atlas and homeomorphisms. Indeed, set $E = E'/\sim$ where

$$E' = \bigcup_\alpha (U_\alpha \times F)$$

and for $(x, f) \in U_\alpha \times F$ and $(y, g) \in U_\beta \times F$ we have $(x, f) \sim (y, g) \iff x = y \in (U_\alpha \cap U_\beta)$ and $(y, g) = \phi_{\alpha\beta}(x, f)$. It is a rather tedious exercise to show that this determines a locally trivial bundle.

3.1.9 Definition. Two locally trivial bundles $\pi \colon E \to B$ and $\pi' \colon E' \to B'$ are isomorphic if there is a homeomorphism $\psi \colon E \to E'$ such that the diagram

$$\begin{tikzcd} E \arrow[rr, "\psi"] \arrow[dr, "\pi"'] & & E' \arrow[dl, "\pi'"] \\ & B & \end{tikzcd}$$

commutes. (Note that this implies that there is a homeomorphism $F \to F'$ between the fibers as well).

3.1.10 Theorem. *Two systems of transition functors $\phi_{\beta\alpha}$ and $\phi'_{\beta\alpha}$ define isomorphic locally trivial bundles iff there exists fiber preserving homeomorphisms*

$$h_\alpha \colon U_\alpha \times F \to U_\alpha \times F$$

such that $\phi_{\beta\alpha} = h_\beta^{-1} \phi'_{\beta\alpha} h_\alpha$.

Proof. First, we suppose that the two bundles are isomorphic, so in particular there is a homeomorphism $\psi \colon E \to E'$. We let

$$h_\alpha := \phi_\alpha^{'-1} \psi^{-1} \phi_\alpha \colon U_\alpha \times F \to U_\alpha \times F.$$

Then we have

$$\begin{aligned} h_\beta^{-1} \phi'_{\beta\alpha} h_\alpha &= \phi_\beta^{-1} \psi^{-1} \phi'_\beta \phi'_{\beta\alpha} \phi_\alpha^{-1} \psi^{-1} \phi_\alpha \\ &= \phi_\beta^{-1} \psi^{-1} \phi_{\beta'} \phi_\beta^{'-1} \phi'_\alpha \phi_\alpha^{-1} \psi^{-1} \phi_\alpha = \phi_{\beta\alpha}. \end{aligned}$$

Conversely, if the relations hold, then we set $\psi = \phi_\alpha h_\alpha^{-1} \phi_\alpha^{'-1}$. A similar argument then shows that $\phi_\beta h_\beta^{-1} \phi_\beta^{'-1} = \phi_\alpha h_\alpha^{-1} \phi' - 1_\alpha$. ☐

3.1.11 Remark. If π is (isomorphic to) a trivial bundle, then all transition functions can be chosen to be the identity. One can use the previous theorem to show that a bundle is not isomorphic to a trivial bundle.

3.1.12 Example. After this discussion, let us return to the example of the Möbius bundle (Example 3.1.4). One can think of this as the space

$$E = \{(x, y) \colon 0 \leq x \leq 1, 0 \leq y \leq 1\} / \sim$$

where we identify $(0, y)$ and $(1, 1 - y)$ for each $y \in [0, 1]$. The projection maps E to $I_x = \{0 \leq x \leq 1\}$ with the endpoints identified, that is, onto the circle. To see that this is a bundle we use the atlas

$$U_\alpha = \{0 \leq x \leq 1\}, \text{ and } U_\beta = \{0 \leq x < 1/2\} \cup \{1/2 < x \leq 1\}.$$

We define

$$\phi_\alpha \colon U_\alpha \times I_y \to E, \quad \phi_\alpha(x, y) = (x, y),$$

and

$$\phi_\beta \colon U_\beta \times I_y \to E$$

by

$$\phi_\beta = \begin{cases} (x,y) & \text{for } 0 \leq x \leq /1/2, \\ (x, 1-y) & \text{for } 1/2 < x \leq 1. \end{cases}$$

The intersection of these two charts is the union $(0,1/2) \cup (1/2,1)$, and the transition functions have the form

$$\phi_{\beta\alpha} = (x,y) \text{ for } 0 < x < 1/2$$

and

$$\phi_{\beta\alpha} = (x, 1-y) \text{ for } 1/2 < x < 1.$$

One can check from Remark 3.1.11 that the Möbius bundle is not isomorphic to a trivial bundle.

We now give some more examples which will be useful in our study of characteristic classes.

3.1.13 Definition. For $n < k$ the n-th Stiefel manifold associated to $\mathbb{R}^k$ is defined as

$$V_n(\mathbb{R}^k) = \{n - \text{frames in } \mathbb{R}^k\}$$

where an n-frame in $\mathbb{R}^k$ is a tuple $\{v_1, \ldots, v_n\}$ of orthonormal vectors in $\mathbb{R}^k$, i.e., $v_1, \ldots, v_n$ are pairwise orthonormal, $\langle v_i, v_j \rangle = \delta_{ij}$. We given $V_n(\mathbb{R}^k)$ the subspace topology induced by thinking of it as a subspace of $S^{k-1} \times \ldots S^{k-1}$ (n-copies of S^{k-1}).

3.1.14 Example. A 1-frame is nothing but a unit vector, so the Stiefel manifold $V_1(\mathbb{R}^k)$ is the unit sphere in $\mathbb{R}^k$, i.e., $V_1(\mathbb{R}^k) \cong S^{k-1}$. On the other hand, an n-frame is an ordered basis, so $V_n(\mathbb{R}^n) \cong O(n)$.

3.1.15 Definition. The n-th Grassmannian associated to $\mathbb{R}^k$ is defined as

$$G_n(\mathbb{R}^k) = \{n - \text{dimensional vector subspaces in } \mathbb{R}^k\}$$

There is a map $p \colon V_n(\mathbb{R}^k) \to G_n(\mathbb{R}^k)$ sending $\{v_1, \ldots, v_n\}$ to the span, which is surjective by Gram–Schmidt, and we given $G_n(\mathbb{R}^k)$ the quotient topology.

3.1.16 Example. We have $G_1(\mathbb{R}^k)$ is the space of lines through the origin in k-space, so $G_1(\mathbb{R}^k) \simeq \mathbb{R}P^{k-1}$.

3.1.17 Lemma. *For $k > n$ the quotient map $p \colon V_n(\mathbb{R}^k) \to G_n(\mathbb{R}^k)$ is a locally trivial bundle with fiber $V_n(\mathbb{R}^n) \cong O(n)$, i.e., we have a locally trivial bundle*

$$O(n) \to V_n(\mathbb{R}^k) \xrightarrow{p} G_n(\mathbb{R}^k). \tag{3.1.18}$$

Similarly, for $m < n \leq k$ there are locally trivial bundles

$$V_{n-m}(\mathbb{R}^k) \to V_n(\mathbb{R}^k) \xrightarrow{p} V_m(\mathbb{R}^k). \tag{3.1.19}$$

where the map p takes $\{v_1, \ldots, v_n\}$ to $\{v_1, \ldots, v_m\}$. Taking $k = n$ we get a locally trivial bundle

$$O(n-m) \to O(n) \xrightarrow{p} V_m(\mathbb{R}^n). \tag{3.1.20}$$

3.1.21 Example. Taking $m = 1$ in (3.1.20) we get a locally trivial bundle

$$O(n-1) \to O(n) \to S^{n-1}.$$

Here the first map takes A to $\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$ and the second takes B to Bu for $u \in S^{n-1}$ some unit vector. In particular, this identifies S^{n-1} as an orbit space $S^{n-1} \cong O(n)/O(n-1)$.

Exercise 28. *Use the fibrations*

$$O(n-k) \to O(n) \to V_k(\mathbb{R}^n)$$

to show that

$$\pi_i(O(n-1)) \simeq \pi_i(O(n)) \text{ for } i < n-2$$

and

$$\pi_i(V_k(\mathbb{R}^n)) = 0$$

for $i < n-k-1$.

3.1.22 Definition. We have infinite versions of the Stiefel manifold and Grassmanian:

$$V_n(\mathbb{R}^\infty) := \bigcup_{k=1}^{\infty} V_n(\mathbb{R}^k) \qquad G_n(\mathbb{R}^\infty) := \bigcup_{k=1}^{\infty} G_n(\mathbb{R}^k)$$

3.1.23 Remark. We get a fiber sequence

$$O(n) \to V_n(\mathbb{R}^\infty) \to G_n(\mathbb{R}^\infty).$$

3.1.24 Proposition. $V_n(\mathbb{R}^\infty)$ *is contractible.*

Proof. As in the exercise, we deduce that $\pi_i(V_n(\mathbb{R}^\infty) = 0$ for all i. We can give $V_n(\mathbb{R}^\infty)$ the structure of a CW-complex, and so the claim follows from Corollary 1.5.12. ☐

3.1.25 Remark. One can repeat the same story using $\mathbb{C}$ or $\mathbb{H}$ instead of $\mathbb{R}$. In the first case, all instances of $O(n)$ get replaced by $U(n)$, and in the second case by $\mathrm{Sp}(n)$.

3.1.26 Example (The tangent bundle to S^2). Let $S^2 = \{(x_0, x_1, x_2) \in \mathbb{R}^3 \mid x_0^2 + x_1^2 + x_2^2 = 1\}$. Recall that the tangent space at a point $x \in S^2$ is defined by $T_xS^2 = \{\xi \in \mathbb{R}^3 \mid x \perp \xi\}$. We then define $TS^2 = \coprod_{x \in S^2} T_xS^2$. This can be topologized as a subspace of $\mathbb{R}^3 \times \mathbb{R}^3$ when we write

$$TS^2 = \{(x, \xi) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid x \in S^2, x \perp \xi\}.$$

There is a natural projection map $p \colon TS^2 \to S^2$ sending the pair (x, ξ) to x, which we claim is a locally trivial bundle with fiber $\mathbb{R}^2$. To see this is a locally trivial bundle, let U be the open subset of S^2 defined by $x_3 > 0$. We will show how to construct the local trivialization on this open subset.

If $\xi = (\xi_1, \xi_2, \xi_3)$ then we have the relation

$$x_1\xi_1 + x_2\xi_2 + x_3\xi_3 = 0$$

or

$$\xi_3 = -(x_1\xi_1 + x_2\xi_2)/x_3.$$

We define

$$\phi\colon U \times \mathbb{R}^2 \to p^{-1}(U)$$

by

$$\phi(x_1, x_2, x_3, \xi_1, \xi_2) = (x_1, x_2, x_3, \xi_1, \xi_2, -(x_1\xi_1 + x_2\xi_2)/x_3),$$

which gives the required chart for this open subset.

3.1.27 Remark. More generally, for any smooth manifold X of dimension n, we have a locally trivial bundle $\pi\colon TX \to X$ with fiber $\mathbb{R}^n$.

3.2 *The structure group of locally trivial bundles*

We recall that the on the intersection of two local trivializations we constructed a homeomorphism

$$\phi_{\beta\alpha}\colon (U_\alpha \cap U_\beta) \times F \to \pi^{-1}(U_\alpha \cap U_\beta) \to (U_\alpha \cap U_\beta) \times F$$

Unwinding the definition, the map ϕ is completely determine by a map $\Phi\colon U \to \mathrm{Homeo}(F)$, where $\mathrm{Homeo}(F)$ denotes the group of all homeomorphisms of the fiber F (we call Φ coordinate transformations).[3]. Indeed, we have

$$\phi_{\alpha\beta}(x, f) = (x, \Phi(x)(f))$$

In other words, instead of $\phi_{\alpha\beta}$ to determine a bundle we can instead specify a family of functions

$$\Phi_{\alpha\beta}(x, f)\colon U_\alpha \cap U_\beta \to \mathrm{Homeo}(F),$$

having values in the group $\mathrm{Homeo}(F)$. Of course these are not arbitrary, but need to satisfy various compatibility conditions:

$$\Phi_{\alpha\alpha}(x) = \mathrm{id}$$

and

$$\Phi_{\alpha\gamma}(x)\Phi_{\gamma\beta}(x)\Phi_{\beta\alpha}(x) = \mathrm{id},$$

for $x \in U_\alpha \cap U_\beta \cap U_\gamma$.

3.2.1 Definition. Let E, B, F be topological spaces and G a topological group which acts freely on the space F. A continuous map $p\colon E \to B$ is a locally trivial bundle with fiber F and structure group G if there is an atlas $\{U_\alpha\}$ and the coordinate homeomorphisms

$$\phi_\alpha\colon U_\alpha \times F \to p^{-1}(U_\alpha)$$

[3] If we choose the correct topology on $\mathrm{Homeo}(F)$, namely the compact-open topology (for reasonable spaces at least), then this map is even continuous

such that the transition functions

$$\phi_{\beta\alpha} = \phi_\beta^{-1}\phi_\alpha \colon (U_\alpha \cap U_\beta) \times F \to (U_\alpha \cap U_\beta) \times F$$

have the form

$$\phi_{\beta\alpha}(x, f) = (x, \Phi_{\beta\alpha}(x) f)$$

where $\Phi_{\beta\alpha} \colon (U_\alpha \cap U_\beta) \to G$ are continuous functions satisfying

$$\Phi_{\alpha\alpha}(x) = \mathrm{id}$$

and

$$\Phi_{\alpha\gamma}(x)\Phi_{\gamma\beta}(x)\Phi_{\beta\alpha}(x) = \mathrm{id},$$

3.2.2 *Remark.* Some words on terminology are useful. What we defined as a locally trivial bundle with fiber F, is exactly a locally trivial bundle with fiber F and structure group $\mathrm{Homeo}(F)$. Either of these may also be called a *fiber bundle* or a *fiber bundle with structure group G.*

3.2.3 *Remark.* In diagrammatic form, to have structure group G means that we can find transition maps $\Phi'_{\alpha\beta}$ making the diagram commute:

$$\begin{tikzcd} G \arrow[rr] & & \mathrm{Homeo}(F) \\ & U_\alpha \cap U_\beta \arrow[ul, dashed, "\Phi'_{\alpha\beta}"] \arrow[ur, "\Phi_{\alpha\beta}"'] & \end{tikzcd}$$

3.2.4 *Remark.* Note that the structure group is not unique. For example, a bundle with structure group G may admit transition functions with values in a subgroup $H \leq G$. We say that the structure group G is reduced to subgroup H. More generally, if $\rho \colon G \to G'$ is a continuous homeomorphism of topological groups, and we are given a locally trivial bundle with structure group G and the transition functors $\alpha_{\alpha\beta} \colon U_\alpha \cap U_\beta \to G$, then a new locally trivial bundle with structure group G' may be constructed by

$$\phi'_{\alpha\beta}(x) = \rho(\phi_{\alpha\beta}(x)).$$

This operation is called the change of structure group.

Exercise 29. *Show that a trivial bundle has trivial structure group. Conversely, if the structure group can be reduced to the trivial group then the bundle is (isomorphic to) a trivial bundle.*

3.2.5 *Example.* Let us return to the Möbius bundle (Examples 3.1.4 and 3.1.12). We have that

$$\Phi_{\alpha\beta}(y) = y \quad \text{and } \Phi_{\beta\alpha}(y) = 1 - y.$$

Note that $\Phi_{\beta\alpha} \circ \Phi_{\beta\alpha}(y) = 1 - (1 - y) = y = \Phi_{\alpha\beta}(y)$. Therefore, the group generated by $\Phi_{\alpha\beta}$ and $\Phi_{\beta\alpha}$ has order 2. In other words, the Möbius bundle has (or can be reduced to) structure group $\mathbb{Z}/2$.

3.2.6 Example. The tangent bundle $TS^2 \to S^2$ was considered in Example 3.1.26. The coordinate homeomorphisms

$$\phi \colon U \times \mathbb{R}^2 \to \mathbb{R}^3 \times \mathbb{R}^3$$

are defined by formulas that are linear with respect to the second argument. Hence that transition functions have values in the group of linear translations of the fiber $F = \mathbb{R}^2$, that is $G = GL_2(\mathbb{R})$. In fact, it can be shown that the structure group can be reduced to the subgroup $O(n)$ of orthonormal rotations.

3.3 *Principal bundles*

The most important example of a bundle for a us is a *principal bundle*. We postpone the definition to talk a little about group actions.

3.3.1 Definition. Let G be a topological group, and X a topological space. A left action of G on X is a continuous map $\mu \colon G \times X \to X$, satisfying $\mu(e, x) = x$ and $\mu(h, \mu(g, x)) = \mu(hg, x)$.

3.3.2 Example. The multiplication map $\mu \colon G \times G \to G$ defines a left action of G on itself. Similarly, if $H \subseteq G$, then $\mu|_{H \times G} \colon H \times G \to G$ defines an action of H on G.

3.3.3 Remark. The map $\mu \colon G \times X \to X$ is adjoint to a map $ad(u) \colon G \to \text{Homeo}(X)$, where $\text{Homeo}(X)$ has the compact-open topology. If X is nice (specifically, locally compact Hausdorff) then a continuous map $G \to \text{Homeo}(X)$ gives rise to a group action $G \times X \to X$.

3.3.4 Example. The adjoint to the multiplication $\mu \colon G \times G \to G$ is the map $G \to \text{Homeo}(G)$ given by $g \mapsto f_g$, where $f_g \colon G \to G$ is given by $f_g(x) = gx$.

3.3.5 Remark. We recall some standard terminology associated to a group action of G on X:

(i) The orbit of a point $x \in X$ is the set $Gx = \{g \cdot xmidg \in G\}$.

(ii) The orbit space X/G is the quotient space $X/\sim$ where $x \sim g \cdot x$.

(iii) The fixed set $X^G := \{x \in X | g \cdot x \text{ for all } g \in G\}$.

(iv) An action if free[4] if $g \cdot x \neq x$ for all $x \in X$ and all $g \neq e$. (i.e., $gx = x$ for all x implies $g = e$).

(v) The stabilizer (or isotropy) group at x is $G_x = \{g \in G \mid g \cdot x = g\} \subseteq G$.

(vi) An action is effective is the adjoint $G \to \text{Homeo}(X)$ is injective, or equivalently if $\cap_{x \in X} G_x = \{e\}$.[5]

(vii) A group action is transitive if and only if it has exactly one orbit, i.e., there exists $x \in X$ such that $Gx = X$.

3.3.6 Example. To motivate the definition of a principal G-bundle we first consider an example. We recall that Hopf fibration $S^1 \to$

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[4] Here, meaning fixed-point free

[5] Note that a free action is effective, but not vice-versa.

$S^3 \to S^2$. By what we have already discussed, this is a locally trivial bundle.

Let us show this directly. Consider the sphere S^3 as a subset of $\mathbb{C}^2$ defined by the equation

$$|z_0|^2 + |z_1|^2 = 1.$$

Then the map $S^3 \to S^2 = \mathbb{C}P^1$ sends (z_0, z_1) to the projective coordinate $[z_0 : z_1]$. We take the atlas consisting of the charts

$$\begin{aligned} U_0 &= \{[z_0, z_1] : z_0 \neq 0\} \\ U_1 &= \{[z_0, z_1] : z_1 \neq 0\} \end{aligned}$$

The points of the chart U_0 are parametrized by the complex parameter $w_0 = z_1/z_0 \in \mathbb{C}$, while points of the chart U_1 are parameterized by $w_1 = z_0/z_1 \in \mathbb{C}$. The homeomorphisms

$$\phi_0 \colon p^{-1}(U_0) \to S^1 \times \mathbb{C} = S^1 \times U_0 \text{ and } \phi_0 \colon p^{-1}(U_1) \to S^1 \times \mathbb{C} = S^1 \times U_1$$

are given by

$$\begin{aligned} \phi_0(z_0, z_1) &= \left(\frac{z_0}{|z_0|}, \frac{z_1}{z_0} \right) = \left(\frac{z_0}{|z_0|}, [z_0 : z_1] \right) \\ \phi_0(z_0, z_1) &= \left(\frac{z_1}{|z_1|}, \frac{z_0}{z_1} \right) = \left(\frac{z_1}{|z_1|}, [z_0 : z_1] \right) \end{aligned}$$

One can explicitly write down inverses for these maps and check that they are homeomorphisms. The transition function

$$\phi_{01} \colon (U_0 \cap U_1) \times S^1 \to (U_0 \cap U_1) \times S^1$$

is defined by the formula

$$\phi_{01}([z_0 : z_1], \lambda) = \left([z_0 : z_1], \frac{z_1 |z_0|}{z_0 |z_1|} \right).$$

In other words, $\Phi_{01} \colon U_0 \cap U_1 \to \mathrm{Homeo}(S^1)$ is defined by

$$[z_0 : z_1] \mapsto (\lambda \mapsto \frac{z_1 |z_0|}{z_0 |z_1|} \lambda),$$

where $\frac{z_1|z_0|}{z_0|z_1|} \in S^1$. We can then define a map $\Phi'_{01} \colon U_0 \cap U_1 \to S^1$ by $[z_0, z_1] \mapsto \frac{z_1|z_0|}{z_0|z_1|}$. Unwinding the definitions, the diagram

$$\begin{tikzcd} S^1 \arrow[rr, "ad(\mu)"] & & \mathrm{Homeo}(S^1) \\ & U_0 \cap U_1 \arrow[ul, "\Phi'_{01}"] \arrow[ur, "\Phi_{01}"'] & \end{tikzcd}$$

commutes, where $ad(\mu)$ is adjoint to the multiplication map $\mu \colon S^1 \times S^1 \to S^1$.[6]

In other words, the structure group of the Hopf bundle can be reduced to S^1, where the action of S^1 on the fiber, namely S^1 again, is given by the multiplication map $\mu \colon S^1 \times S^1 \to S^1$.

This leads us to the following definition.

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[6] Here we use Example 3.3.4.

3.3.7 Definition. A locally trivial bundle with structure group G is called a principal G-bundle if $F = G$ and the action of the group G on F is defined by left translations, i,e, the multiplication map $\mu\colon G \times G \to G$ as in Example 3.3.4.

3.3.8 Example. Restated, Example 3.3.6 shows that $S^1 \to S^3 \to S^2$ is a principal S^1-bundle.

3.3.9 Theorem. *Let $p\colon E \to B$ be a principal G-bundle, with*

$$\phi_\alpha\colon U_\alpha \times G \to p^{-1}(U_\alpha)$$

coordinate homeomorphisms. Then there is a right action[7] *of the group G on the total space E such that*

(i) The right action is fiberwise,[8] i.e., $p(x) = p(xg)$ for $x \in E, g \in G$.

(ii) The homeomorphism ϕ_α^{-1} transformations the right action of the group G on the total space into right translations on the second factor, i.e.,

$$\phi_\alpha(x, f)g = \phi_\alpha(x, fg), \quad x \in U_\alpha, f, g \in G \tag{3.3.10}$$

(iii) G acts freely and transitively on the right of E.

Proof. Unwinding the definitions, the transition functions $\phi_{\beta\alpha}$ are given by

$$\phi_{\beta\alpha}(x, f) = (x, \Phi_{\beta\alpha}(x)f),$$

for some

$$\Phi\beta\alpha\colon U_\alpha \cap U_\beta \to G.$$

Note that any arbitrary $e \in E$ is of the form $e = \phi_\alpha(x, f)$ for some α, then we define a right action of G on E via the formula (3.3.10). We need to check that this is well defined, i.e., does not depend on the choice α. So assume that $e = \phi_\beta(x, f')$ as well, then we are required to show that

$$\phi_\alpha(x, fg) = \phi_\beta(x, f'g).$$

This says that

$$(x, f'g) = \phi_\beta^{-1}\phi_\alpha(x, fg) = \phi_{\beta\alpha}(x, fg) = (x, \Phi_{\beta\alpha}(x)fg).$$

So equivalently, we are required to show

$$f'g = \Phi_{\beta\alpha}(x)fg.$$

But this follows because $f' = \phi_{\beta\alpha}(x)f$. □

We omit the proof of the following difficult theorem, which gives many more examples of principal G-bundles.

3.3.11 Theorem. *Suppose X is a compact Hausdroff space and G is a compact Lie group acting freely on X. Then the orbit map $X \to X/G$ is a principal G bundle.*

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[7] Why on the right? This is just because the action of G on E is given by *left* translation. If we made the opposite convention, then E would get a left action of G.

[8] In other words, it preserves fibers.

3.3.12 Example. Let $H \subseteq G$ be a closed subgroup. Then H acts freely on G, and $G \to G/H$ is a principal H-bundle. Moreover, if $K \subseteq H$ is a subgroup, then H acts on H/K, and $G/K \to G/H$ is a principal H/K-bundle.

3.3.13 Example. Let $G = O(n)$, and consider the subgroup $H = O(k) \times O(n-k)$ defined by $(A, B) \mapsto \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ We also consider the subgroup the subgroup $K = O(n-k) \subseteq O(n)$, determine by the inclusion $A \mapsto \begin{pmatrix} I & 0 \\ 0 & A \end{pmatrix}$. We then have the principal $O(k) \times O(n-k)$-bundle

$$O(k) \times O(n-k) \to O(n) \to G_k(\mathbb{R}^n)$$

and the principal $O(n-k)$-bundle

$$O(n-k) \to O(n) \to V_k(\mathbb{R}^n)$$

as well as the principal $O(k)$-bundle

$$O(k) \to V_k(\mathbb{R}^n) \to G_k(\mathbb{R}^n).$$

3.4 *The associated bundle construction*

3.4.1 Remark. Let us recall that a locally trivial bundle $p\colon E \to B$ with structure group G and fiber F is determined by the following data:

1. An open covering $\{U_\alpha\}$ of B.
2. Maps $\phi_{\alpha\beta}\colon U_\alpha \cap U_\beta \to G$, and
3. An action of G on F, or equivalently a map $G \to \text{Homeo}(F)$.

Note that if p is a a principal G-bundle, then the last data in the above list is already included: the action of G on G is given by left multiplication. Thus a principal G-bundle $p\colon P \to B$ is determined by the following data:

1. An open covering $\{U_\alpha\}$ of B.
2. Maps $\phi_{\alpha\beta}\colon U_\alpha \cap U_\beta \to G$, and

Therefore, by removing the third data from a fiber bundle, we should get a principal G-bundle, and conversely given a principal G-bundle by adding the data of an action of a space F on G, we should be able to construct a locally trivial bundle with fiber F on the same base space. This is essentially true; we have the following theorem:

3.4.2 Theorem. *Let G be a topological group and fix an action of G on a space F. Suppose that an open cover $\{U_\alpha\}$ of B and families of maps $\Phi = \{\Phi_{\alpha\beta}\colon U_\alpha \cap U_\beta \to G\}$ are given. Suppose that that action of G on F is effective, then there is a bijection*

$$\left\{ \begin{matrix} \textit{principal } G\textit{-bundles over } B \\ \textit{with coordinate transformations } \Phi \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \textit{Locally trivial bundles over } B \\ \textit{with fiber } F\textit{, structure group } G\textit{,} \\ \textit{and coordinate transformations } \Phi \end{matrix} \right\}$$

It is not too hard to define a map in the → direction.

3.4.3 Construction . Let G be a topological group and assume that G acts on topological spaces P and F continuously from the right and left respectively. Define a left action of G on $P \times F$ by

$$g(p, f) = (x \cdot g^{-1}, gy).$$

Let $E \times_G F := E \times F/G$ denote the quotient (orbit) space, and $\omega\colon E \times_G F \to E/G$ the projection map.

3.4.4 Definition. Let $p\colon E \to B$ be a principal G-bundle and fixed a G-space F. The projection map $\omega\colon E \times_G F \to B$ sending $[x, y] \mapsto p(x)$ is called the associated bundle with fiber F.

We should of course verify that this is actually a locally trivial bundle.

3.4.5 Theorem. *The map $\omega\colon E \times_G F \to B$ defines a locally trivial bundle with structure group G and fiber F.*

Proof. Let $\{\Phi_\alpha\colon p^{-1}(U_\alpha) \to U_\alpha \times G\}$ be local trivializations. We wish to construct local trivializations

$$\Psi_\alpha\colon \omega^{-1}(U_\alpha) \to U_\alpha \times F.$$

Note that

$$\begin{aligned}\omega^{-1}(U_\alpha) &= \{[x, y] \in E \times_G F \mid \omega([x, y]) \in U_\alpha\}\\ &= \{[x, y] \in E \times_G F \mid p(x) \in U_\alpha\}\\ &= p^{-1}(U_\alpha) \times_G F\end{aligned}$$

We then define the required map Ψ_α as the composite

$$\omega^{-1}(U_\alpha) = p^{-1}(U_\alpha) \times_G F \xrightarrow{\Phi_\alpha \times_G 1_F} (U_\alpha \times G) \times_G F \cong U_\alpha \times F$$

where the homeomorphism $(U_\alpha \times G) \times_G F \cong U_\alpha \times F$ is given by $[(x, g), y] \mapsto (x, gy)$.[9] As a composite of homeomorphisms, this map is also a homeomorphism. A short diagram chase shows that it defines a locally trivial bundle. In order to see that the structure group is G, we must compute $\Psi_\beta \Psi_\alpha^{-1}(x, y)$ for $(x, y) \in U_\alpha \cap U_\beta \times F$. A rather tedious computation shows that

$$\Psi_\beta \Psi_\alpha^{-1}(x, y) = (x, \Phi_{\alpha\beta}(x)y)$$

where $\Phi_{\alpha\beta}\colon U_\alpha \cap U_\beta \to G$ is a coordinate transformation for $p\colon E \to B$. Therefore, the associated bundle has structure group G, as claimed. □

3.4.6 Example. Let $\pi\colon S^1 \to S^1, z \mapsto z^2$ be regarded as a principal $\mathbb{Z}/2$-bundle.[10] Let $F = [-1, 1]$, and let $\mathbb{Z}/2 = \{-1, 1\}$ act on F by multiplication. The associated bundle is then

$$S^1 \times_{\mathbb{Z}/2} [-1, 1] = S^1 \times [-1, 1]/(x, t) \sim (a(x), -t)$$

for $a\colon S^1 \to S^1$ the antipodal map. This is the Möbius bundle.

[9] With inverse, $(x, y) \mapsto [(x, e), y]$.

[10] Note that any regular cover is a principal bundle.

3.4.7 *Remark.* You might be wondering about the inverse map in Theorem 3.4.2. Let $p\colon E \to B$ be a principal bundle with fiber F and structure group G. The associated principal bundle is constructed using a similar procedure as in Remark 3.1.8; we keep the base space B, the open chart $\{U_\alpha\}$ and the transition functions $\phi_{\alpha\beta}$, but we replace all instances of the fiber F by G, and allow G to act on itself by left translation. In simple terms: we forget the fiber F, and built a bundle which is principal out of the remaining data.

3.5 *Operations on locally trivial bundles*

The following is a fundamental operation on bundles.

3.5.1 Proposition. *Let $\xi = (p\colon E \to B)$ and $\xi' = (p'\colon E' \to B')$ be locally trivial bundles with fibers F and F' respectively. Then the product*

$$p \times p'\colon E \times E' \to B \times B'$$

is a locally trivial bundle bundle with fiber $F \times F'$ and structure group $G \times G'$.

Proof. Let $\phi_\alpha\colon p^{-1}(U_\alpha) \to U_\alpha \times F$ and $\psi_\beta\colon p'^{-1}(V_\beta) \to V_\beta \times F'$ be local trivializations of ξ and ξ' respectively. Note that $\{U_\alpha \times V_\beta\}$ is an open covering of $B \times B'$. Note also that $(U_\alpha \times V_\beta) \cap (U_{\alpha'} \times V_{\beta'}) = (U_\alpha \cap U_{\alpha'}) \times (V_\beta \cap V_{\beta'})$, and using this identification we obtain maps

$$\Phi_{\alpha\alpha'} \times \Psi_{\beta\beta'}\colon (U_\alpha \cap U_{\alpha'}) \times (V_\beta \cap V_{\beta'}) \to G \times G'$$

which form coordinate transforms of the product bundle. □

3.5.2 Corollary. *Let $p\colon E \to B$ be a locally trivial bundle with fiber F and structure group G, then for any topological space X*

$$p \times 1_X\colon E \times X \to B \times X$$

is a locally trivial bundle with fiber F and structure group G.

3.5.3 *Remark.* We recall the pullback construction for topological spaces: given $f\colon A \to C$ and $g\colon B \to C$, the pullback is

$$A \times_C B := \{(a, b) \in A \times B \mid f(a) = g(b)\}$$

along with the projection maps $A \times_C B \to A$ and $A \times_C B \to B$. This is a pullback in the categorical sense: given maps $q_1\colon Q \to A$ and $q_2\colon Q \to B$ as in the following diagram:

$$\begin{tikzcd}
Q \arrow[rd, dashed, "u"] \arrow[rrd, bend left, "q_1"] \arrow[rdd, bend right, "q_2"'] & & \\
& A \times_C B \arrow[r] \arrow[d] & A \arrow[d, "f"] \\
& B \arrow[r, "g"'] & C
\end{tikzcd}$$

there exists a map $u\colon Q \to A \times_C B$ making the diagram commute.

3.5.4 Proposition. *Let $\xi = (p\colon E \to B)$ be a locally trivial bundle with fiber F and structure group G. For any continuous map $f\colon X \to B$, the pullback $f^*(p)\colon E \times_B X \to X$ is a locally trivial bundle with fiber F and structure group G.*[11]

Proof. Let $\{U_\alpha\}_{\alpha \in A}$ be an open cover of B, then $\{f^{-1}(U_\alpha)\}_{\alpha\in A}$ forms an open cover of X.

Let $\phi_\alpha\colon p^{-1}(U_\alpha) \to U_\alpha \times F$ denote a local trivialization of ξ. We write $\phi_\alpha(e) = (p(e), \overline{\phi}_\alpha(e))$. We define a map $\psi_\alpha\colon f^*(p)^{-1}(V_\alpha) \to V_\alpha \times F$ as the composite

$$f^*(p)^{-1}(V_\alpha) \hookrightarrow V_\alpha \times p^{-1}(U_\alpha) \xrightarrow{1\times\overline{\phi}_\alpha(e)} V_\alpha \times F.$$

We will show this is a homeomorphism by constructing a continuous inverse. Define γ_α as the composite

$$V_\alpha \times F \xrightarrow{\Delta\times 1} V_\alpha \times V_\alpha \times F \xrightarrow{1\times f\times 1} V_\alpha \times U_\alpha \times F \xrightarrow{1\times\phi_\alpha^{-1}} V_\alpha \times p^{-1}(U_\alpha),$$

i.e., $\gamma_\alpha(x,y) = (x, \phi_\alpha^{-1}(f(x),y))$.

Note that $p(\phi_\alpha^{-1}(f(x),y)) = f(x)$, and so $\gamma_\alpha(x,y) \in f^*(E)$. To see that γ_α and ψ_α are inverse, simply compute:

$$\begin{aligned} \gamma_\alpha \circ \psi_\alpha(x,e) &= \gamma_\alpha(x, \overline{\phi}_\alpha(e)) \\ &= (x, \phi_\alpha^{-1}(f(x), \overline{\phi}_\alpha(e))) \\ &= (x, \phi_\alpha^{-1}(p(e), \overline{\phi}_\alpha(e))) \\ &= (x, \phi_\alpha^{-1} \circ \phi_\alpha(e)) \\ &= (x,e) \end{aligned}$$

and

$$\begin{aligned} \psi_\alpha \circ \gamma_\alpha(x,y)\circ &= \psi_\alpha(x, \psi_\alpha^{-1}(f(x),y)) \\ &= (x, \overline{\phi}_\alpha \circ \phi_\alpha^{-1}(f(x),y)) \\ &= (x,y). \end{aligned}$$

We leave it to the reader to verify the $f^*(E) \to X$ has structure group G.[12] ☐

3.5.5 Remark. Suitably interpreted, this pullback is actually a categorical pullback in a category of locally trivial bundles.

3.6 Vector bundles

A vector bundle is a special case of a locally trivial bundle.

3.6.1 Definition. A locally trivial bundle is called a real (respectively, complex) vector bundle of rank n if its fiber is a vector space V of dimension n over $\mathbb{R}$ (respectively, $\mathbb{C}$) and the structure group is $GL(V)$.

3.6.2 Example. For a manifold of dimension n, the tangent bundle $TM \to M$ is a real vector bundle of rank n.

[11] For brevity, we write $f^*(E) := E \times_B X$.

[12] Hint: you should end up showing that the coordinate transformations of $f^*(E) \to X$ are given by $\{\Phi_{\alpha\beta} \circ f\}$.

3.6.3 Remark. If the following, when we write K we mean either $K = \mathbb{R}$ if the vector bundle is real, and $K = \mathbb{C}$ if the vector bundle is complex.

3.6.4 Definition. Let $\xi = (p\colon E \to B)$ and $\xi' = (p'\colon E' \to B')$ be vector bundles of rank m and n respectively. A map of vector bundles from $\xi \to \xi'$ is a fiber preserving map $\mathbf{f} = (\tilde{f}, f)\colon (E, B) \to (E', B')$ satisfying the following condition: for local trivializations $\phi_\alpha\colon p^{-1}(U_\alpha) \to U_\alpha \times K^m$ and $\psi_\beta\colon p'^{-1}(V_\beta) \to V_\beta \times K^n$ of ξ and ξ' respectively satisfying $U_\alpha \cap f^{-1}(V_\beta) \neq \varnothing$, the map

$$L^f_{\alpha,\beta}\colon U_\alpha \cap f^{-1}(V_\beta) \to \mathrm{Map}(K^m, K^n)$$

which is adjoint to

$$U_\alpha \cap f^{-1}(V_\beta) \times F \xrightarrow{\phi_\alpha^{-1}} p^{-1}(U_\alpha \cap f^{-1}(V_\beta)) \xrightarrow{\tilde{f}} p'^{-1}(U_\alpha \cap f^{-1}(V_\beta)) \xrightarrow{\Psi_\beta} (f(U_\alpha) \cap V_\beta) \times F' \xrightarrow{\mathrm{pr}_2} F'$$

takes values in the set of *linear* maps $\mathrm{Hom}_K(K^m, K^n)$

3.6.5 Remark. Note that $\mathrm{Hom}_K(K^m, K^n)$ inherits the structure of a K-vector space, by sum and scalar multiplication of linear maps.

3.6.6 Remark. Given two vector bundles $p\colon E \to B$ and $p'\colon E' \to B'$ of rank m and n respectively, the product $p \times p'\colon E \times E' \to B \times B'$ is a vector bundle of rank $m + n$ (Proposition 3.5.1). Let us denote this bundle by $\xi \times \xi'$. This has structure group $GL_m(K) \times GL_n(K)$. Note that we can regard $GL_m(K) \times GL_n(K)$ as a subgroup of $GL_{m+n}(K)$ via the map

$$(A, B) \mapsto \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}.$$

We use this to define a direct sum for vector bundles.

3.6.7 Definition. Let $\xi = (p\colon E \to B)$ and $\xi' = (p'\colon E' \to B)$ be vector bundles of rank m and n over a fixed space B. The pullback of $\xi \times \xi'$ along the diagonal map $\Delta\colon B \to B \times B$ is denoted $\xi \oplus \xi'$ and is called the direct sum (or Whitney sum) of ξ and ξ'. The total space of this bundle is $E \times_B E'$. This is a vector bundle of rank $m + n$.

3.6.8 Definition. An inner product on a vector bundle ξ is a homomorphism of vector bundles $g\colon \xi \oplus \xi \to K \times B$[13] such that the map on each fiber gives rise to an inner product $K^n \times K^n \to K$ when translated by local trivializations.

3.6.9 Remark. Any vector bundle over a paracompact Hausdorff base space has an inner product. This uses a partition of unity argument to glue inner products over local trivializations. By using the Gram–Schmidt process, one can show that any vector bundle of rank n over a paracompact Hausdorff space has structure group that can be reduced to $O(n)$ in the real case, or $U(n)$ in the complex case.

There are a number of other natural constructions for vector bundles. We outline some now.

3.6.10 Definition (Tensor product bundle). We have a external tensor product: given two vector bundles $\xi = (p\colon E \to B)$ and

[13] here $K \times B$ denotes the trivial K-vector bundle over B.

$\xi' = (p'\colon E' \to B')$ of rank m and n, we can construct a bundle $\xi\widehat{\otimes}\xi'$ with coordinate transformations

$$\Phi_{\alpha\alpha'} \otimes \Psi_{\beta\beta'}\colon (U_\alpha \times V_\beta) \cap (U_{\alpha'} \times V_{\beta'}) = (U_\alpha \cap U'_\alpha) \times (V_\beta \cap V_{\beta'}) \to GL_{mn}(K)$$

as the composite

$$(U_\alpha \cap U'_\alpha) \times (V_\beta \cap V_{\beta'}) \xrightarrow{\Phi_{\alpha\alpha'} \times \Psi_{\beta\beta'}} GL_m(K) \times GL_n(K) \xrightarrow{\otimes} GL_{mn}(K)$$

where the last map is given by

$$(A, B) \mapsto (\mathbb{R}^{mn} \cong \mathbb{R}^m \otimes \mathbb{R}^n \xrightarrow{A \otimes B} \mathbb{R}^m \otimes \mathbb{R}^n \cong \mathbb{R}^{mn}).$$

Note that this is a vector bundle over $B \times B'$. In the case that $B = B'$, the pullback along the diagonal map gives a bundle $\xi \otimes \xi'$ over B called the tensor product bundle.

3.6.11 Definition (Hom bundle). Let $\xi = (p\colon E \to B)$ and $\xi' = (p'\colon E' \to B)$ be two vector bundles over the same base space of rank m and n respectively. We can assume that local trivializations are defined over the same cover by taking a subdivision if necessary. Let

$$\Phi_{\alpha\alpha'}\colon U_\alpha \cap U_{\alpha'} \to GL_m(K)$$

and

$$\Psi_{\alpha\alpha'}\colon U_\alpha \cap U_{\alpha'} \to GL_n(K)$$

be coordinate transformations of ξ and ξ'. We define a new bundle with coordinate transformations the composite

$${}^t\Phi_{\alpha\alpha'} \otimes \Psi_{\alpha\alpha'} : U_\alpha \cap U_{\alpha'} \xrightarrow{\Phi_{\alpha\alpha'} \times \Psi_{\alpha\alpha'}} GL_m(K) \times GL_n(K) \xrightarrow{{}^t(-)\otimes 1} GLm(K) \times GL_n(K) \xrightarrow{\otimes} GL_{mn}K$$

where ${}^t(-)$ denotes the transpose.[14] We define a bundle $\mathrm{Hom}(E, E')$ by gluing $U_\alpha \times \mathrm{Hom}_K(K^m, K^n)$ using these coordinate transformations.

3.6.12 Example. Let $\underline{K}_B$ denote the trivial K-bundle over a base B. Then $E^* := \mathrm{Hom}(E, \underline{K}_B)$ is called the dual vector bundle, and has fibers dual to those of E. We note that if $K = \mathbb{R}$ and we work over a paracompact Hausdorff base space then a finite rank vector bundle and its dual are isomorphic as vector bundles (but not canonically).

3.6.13 Definition. Associated to a real or complex vector bundle $\pi\colon E \to B$ is the *projective bundle* $P(E) \to B$, where $P(E)$ is the space of all lines through the origin in all of the fibers of E. This is a locally trivial bundle with fiber $\mathbb{R}P^{n-1}$ or $\mathbb{C}P^{n-1}$ respectively.

3.7 Morphism of bundles

Let us now formalize the notion of a morphism of locally trivial bundles. There are various possibilities depending on how much structure we wish to preserve.

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[14] To justify the use of the transpose we note the following: Let $f\colon V \to W$ be a linear map between finite dimensional vector spaces with bases $\mathcal{B}_V$ and $\mathcal{B}_W$ with a matrix A representing f with respect to these bases. Then the map $f^* : W^* \to V^*$ has matrix ${}^tA$ with respect to the dual bases $\mathcal{B}^*_V$ and $\mathcal{B}^*_W$.

3.7.1 *Definition* (Bundle homomorphism). Let $\xi = (p\colon E \to B)$ and $\xi' = (p'\colon E' \to B')$ be two locally trivial bundles with fiber F and structure group G and G action given by a common map $\mu_G\colon G \times F \to F$. A morphism $(\tilde{f}, f)\colon (E, B) \to (E', B')$ is a bundle map if:

1. The following diagram commutes:

$$\begin{tikzcd}
E' \arrow[r, "\tilde{f}"] \arrow[d, "p'"'] & E \arrow[d, "p"] \\
B' \arrow[r, "f"'] & B
\end{tikzcd}$$

2. For $x \in B$, let $\phi_\alpha\colon p^{-1}(U_\alpha) \to U_\alpha \times F$ and $\psi_\beta\colon p'^{-1}(V_\beta) \to V_\beta \times F'$ be local trivilizations around x and $f(x)$ respectively. Then we ask that the maps

$$L^f_{\alpha,\beta}\colon U_\alpha \cap f^{-1}(V_\beta) \to \mathrm{Map}(F, F')$$

which are adjoint to

$$U_\alpha \cap f^{-1}(V_\beta) \times F \xrightarrow{\phi_\alpha^{-1}} p^{-1}(U_\alpha \cap f^{-1}(V_\beta)) \xrightarrow{\tilde{f}} p'^{-1}(U_\alpha \cap f^{-1}(V_\beta)) \xrightarrow{\psi_\beta} (f(U_\alpha) \cap V_\beta) \times F' \xrightarrow{\mathrm{pr}_2} F'$$

take values in G, i.e., there exists a dashed arrow making the following diagram commute:

$$\begin{tikzcd}
G \arrow[rr, "ad(\mu_G)"] & & \mathrm{Map}(F, F) \\
& U_\alpha \cap f^{-1}(V_\beta) \arrow[ul, dashed, "\overline{L}^f_{\alpha,\beta}"] \arrow[ur, "L^f_{\alpha,\beta}"'] &
\end{tikzcd}$$

We also have a notion of isomorphic bundles.[15]

3.7.2 *Definition.* Let $\xi = (p\colon E \to B)$ and $\xi' = (p'\colon E' \to B)$ be bundles with the same fiber F, the same structure group G and the same base space B. We say that ξ and ξ' are isomorphic if there exists a bundle map $(f, \tilde{f})$ with $f = \mathrm{id}_B$.

3.7.3 *Example.* In Proposition 3.5.4 we showed that for a locally trivially bundle $p\colon E \to B$ and a map $f\colon X \to B$, the pullback $f^*(E)\colon E \times_B X \to X$ is a locally trivial fiber bundle. More is true - the induced map $\phi\colon f^*(E) \to E$ is a morphism of locally trivial bundles. Indeed, commutativity of the diagram is clear from the definition of the pullback. Moreover, it is straightforward from the definitions to see that ϕ carries the fiber over a point $x \in X$ to the fiber over $f(x)$. Finally, the coordinate transformations of the bundle $f^*(E)$ are given by

$$V_\alpha \cap V_\beta \xrightarrow{f} U_\alpha \cap U_\beta \xrightarrow{\Phi_{\alpha\beta}} G \xrightarrow{\mathrm{ad}(\mu)} \mathrm{Homeo}(F)$$

where

$$U_\alpha \cap U_\beta \xrightarrow{\Phi_{\alpha\beta}} G \xrightarrow{\mathrm{ad}(\mu)} \mathrm{Homeo}(F)$$

is the coordinate transformations for $p\colon E \to B$.

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[15] The categorically minded reader will complain that this is not the correct definition, as an isomorphism should be defined as a bundle homomorphism which admits a map in the reverse direction so that both composites are the identity. Fortunately, it is a theorem that any bundle map over the identity map of a fixed space B is an isomorphism in this sense, thus justifying our definition.

3.7.4 Remark. The converse to the previous remark is also true.

3.7.5 Theorem. *Let $p\colon E \to B$ and $p'\colon E' \to X$ be locally trivial bundles having the same fiber and structure group. Suppose there is a bundle map*

$$\begin{tikzcd} E' \arrow[r, "\tilde{f}"] \arrow[d, "p'"'] & E \arrow[d, "p"] \\ X \arrow[r, "f"'] & B \end{tikzcd}$$

then the bundle $E' \to X$ is isomorphic to the pullback bundle $f^*(E) \to X$.

Proof. By the universal property of the pullback we have a map $\overline{f}\colon E' \to f^*(E)$ which is given by $\overline{f}(e) = (p'(e), \tilde{f}(e))$. We claim that this is a bundle isomorphism. It is a rather lengthy and tedious exercise to show this, so we omit the proof. □

We also note the following two lemmas, which are straightforward to show using the definition of the pullback, and shows that the pullback is functorial in this category.

3.7.6 Lemma. *Let $\pi\colon E \to B$ be a locally trivial bundle. For continuous map $f\colon X \to B$ and $g\colon Y \to X$, we have a bundle isomorphism $(f \circ g)^*(\pi) \cong g^*(f^*(\pi))$.*

3.7.7 Lemma. *Let $p\colon E \to B$ be a locally trivial bundle, then $id_B^*(E) \simeq E$.*

Exercise 30. *Show that the pullback of a trivial bundle is a trivial bundle.*

3.7.8 Remark. The definition of a bundle morphism is very technical. Suppose however, that our bundles are principal G-bundles, so that E and E' come with right G-actions (Theorem 3.3.9). One can check that for a bundle morphism $(\tilde{f}, f)$ between two principal G-bundles, the map $\tilde{f}$ is G-equivariant, i.e., $\tilde{f}(e \cdot g) = \tilde{f}(e) \cdot g$.

3.7.9 Remark. Given a bundle $\pi\colon E \to B$ and two maps $f, g\colon X \to B$, one can ask when the two bundles $f^*(\pi)$ and $g^*(\pi)$ are isomorphic. The following is an important result in this direction.

3.7.10 Theorem. *Let $\pi\colon E \to B$ be a bundle with B compact, and suppose that $f \simeq g\colon X \to B$ are homotopic, then there is an isomorphism $f^*(\pi) \cong g^*(\pi)$ of bundles over B.*

3.7.11 Remark. We will prove this in the case that π is a principal G-bundle (the general case can be reduced to this case). We will do this by constructing a bundle map

$$\begin{tikzcd} f^*E \arrow[rr, dashed, "?"] \arrow[dr] & & g^*\pi \arrow[dl] \\ & X & \end{tikzcd}$$

We will make use of the following definition.

3.7.12 Definition. A section of a bundle $\pi\colon E \to B$ is a continuous map $s\colon B \to E$ such that $\pi \circ s \simeq \mathrm{id}_B$.

3.7.13 Remark. We wish to describe the set of bundle maps between two principal G-bundles $\pi_1 \colon E_1 \to X$ and $\pi_2 \colon E_2 \to Y$. Note that G acts on the right of E_1 and E_2, and so on the left of E_2 via $g \cdot e_2 := e_2 \cdot g^{-1}$. Then, the associated bundle construction (Definition 3.4.4) gives an associated bundle of π_1 with fiber E_2, namely

$$\omega := \pi_1 \times_G E_2 \colon E_1 \times_G E_2 \to X.$$

3.7.14 Theorem. *Bundle maps from π_1 to π_2 are in bijection with sections of ω.*

Proof. We first assume that $\pi_1 \colon X \times G \to G$ and $\pi_2 \colon Y \times G \to G$ are trivial bundles. Suppose that we are given a bundle homomorphism, then we must define a section s in the associated bundle

$$\begin{tikzcd}
(X \times G) \times_G (Y \times G) \arrow[d, shift left, "\omega"] \\
X \arrow[u, shift left, hook, "s"]
\end{tikzcd}$$

Let $e_1 \in X \times G$ with $x = \pi_1(e_1) \in X$, we set

$$s(x) = [e_1, \tilde{f}(e_1)].$$

Note that this is well-defined, as

$$[e_1 \cdot g, \tilde{f}(e_1 \cdot g)] = [e_1 \cdot g, \tilde{f}(e_1) \cdot g] = [e_1 \cdot g, g^{-1} \cdot, \tilde{f}(e_1)] = [e_1, \tilde{f}(e_1)],$$

where we have used Remark 3.7.8. Moreover, it is continuous, and provides a section:

$$\pi \circ s(x) = \pi_1[e_1, \tilde{f}(e_1)] = \pi_1(e_1) = x.$$

The general case can be reduced to the case of a trivial bundle by working locally and gluing.

Conversely, suppose we have been given a section of $E_1 \times_G E_2 \xrightarrow{\omega} X$. We define a map $\tilde{f} \colon E_1 \to E_2$ by $\tilde{f}(e_1) = e_2$ where $s(\pi_1(E_1)) = [(e_1, e_2)]$. We note that this map is G-equivariant:

$$[e_1 \cdot g, e_2 \cdot g] = [e_1 \cdot g, g^{-1} \cdot e_2] = [e_1, e_2]$$

and so descends to a map on orbit spaces $f \colon X \to Y$. As usual, we omit the tedious verification that this actually defines a bundle map. □

3.7.15 Remark. For a *principal* G-bundle, the condition of having a section is extremely strong:

3.7.16 Lemma. *Let $\pi \colon E \to X \times I$ be a principal G-bundle, and let $\pi_0 := i_0^* \pi \colon E_0 \to X$ be the pullback of π under the map $i_0 \colon X \to X \times I$. Then $\pi \cong (pr_1)^* \pi_0 \cong \pi_0 \times id_I$, where $pr_1 \colon X \times I \to X$ is the projection map.*

$$\begin{tikzcd}
pr_1^*(E_0) \arrow[r] \arrow[d] & E_0 \arrow[r] \arrow[d, "\pi_0"] & E \arrow[d, "\pi"] \\
X \times I \arrow[r, "pr_1"'] & X \arrow[r, hook, "i_0"'] & X \times I
\end{tikzcd}$$

Proof. It suffices to find a bundle map as indicated:

$$\begin{tikzcd}
E_0 \arrow[r] \arrow[d, "\pi_0"'] & E \arrow[r, dashed, "\widetilde{\mathrm{pr}_1}"] \arrow[d, "\pi"'] & E_0 \arrow[d, "\pi_0"] \\
X \arrow[r, hook, "i_0"'] & X \times I \arrow[r, "\mathrm{pr}_1"'] & X
\end{tikzcd}$$

By Theorem 3.7.14 this is equivalent to finding a section s of $\omega\colon E \times_G E_0 \to X \times I$. Note that there exists a section s_0 of $\omega_0\colon E_0 \times_G E_0 \to X = X \times \{0\}$ corresponding to the identity bundle map. Then composing s_0 with with the inclusion into $E \times_G E_0$ we get the following diagram:

$$\begin{tikzcd}
X \times \{0\} \arrow[r, "s_0"] \arrow[d, hook] & E \times_G E_0 \arrow[d, "\omega"] \\
X \times I \arrow[r, equal] \arrow[ur, dashed, "s"] & X \times I
\end{tikzcd}$$

Because ω is a fibration (Remark 3.1.5) we can apply the homotopy lifting property (Definition 1.4.1) to produce a map $s\colon X \times I \to E \times_G E_0$, which is a section of ω. □

Proof of Theorem 3.7.10. Let $H\colon X \times I \to B$ be a homotopy between f and g, and consider the following diagram:

$$\begin{tikzcd}
f^*E \arrow[rr] \arrow[dd, "f^*\pi"'] & & H^*E \arrow[r, "\tilde{H}"] \arrow[dd, "H^*\pi"] & E \arrow[dd, "\pi"] \\
& g^*E \arrow[ur] \arrow[dd, "g^*\pi"'] & & \\
X \times \{0\} \arrow[rr, hook, "i_0"] & & X \times I \arrow[r, "H"'] \arrow[dr, "\mathrm{pr}_1"] & B \\
& X \times \{1\} \arrow[ur, "i_1"'] & & X
\end{tikzcd}$$

By Lemma 3.7.6 we have $f^*\pi \cong i_0^* H^*\pi$. By Lemma 3.7.16 $H^*\pi \cong \mathrm{pr}_1^*(f^*\pi) \cong \mathrm{pr}_1^*(g^*\pi)$, and so $f^*\pi \cong i_0^* H^*\pi \cong i_0^*\mathrm{pr}_1^*(g^*\pi) \cong g^*\pi$. □

Exercise 31. *Show that a principal G-bundle is trivial if and only if it has a section.*

3.8 Classification of principal G-bundles

We now move to the most important part of this part of the course, which is the classification theorem for principal G-bundles.

3.8.1 Definition. A principal G-bundle $\pi_G\colon EG \to BG$ is called *universal* if the total space EG is (weakly) contractible. The space $BG \simeq EG/G$ is called the classifying space of G.

The main theorem of this part of the course is the following. We let $\mathcal{P}(X, G)$ denote the set of principal G-bundles over a space X.[16]

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[16] The following theorem can also be proved abstractly by Brown representability. But we give a direct proof here.

3.8.2 Theorem. *Let X be a CW-complex, then there is a bijection*

$$\Phi\colon [X, BG] \xrightarrow{\cong} \mathcal{P}(X, G), \quad f \mapsto f^*\pi_G$$

Proof. We first show note that Φ is well-defined by Theorem 3.7.10. Let us first show that Φ is onto. Let $\pi\colon E \to X$ be a principal G-bundle, then we are required to find $f\colon X \to G$ such that $\pi \cong f^*\pi_G$. Equivalently, we are required to find a bundle map $(f, \tilde{f})\colon \pi \to \pi_G$. By Theorem 3.7.14 this is equivalent to finding a section of the bundle $E \times_G EG \to X$ with fiber EG. Because EG is contactible this following from the following lemma.

3.8.3 Lemma. *Let X be a trivial bundle and $\pi\colon E \to X$ a locally trival bundle with fiber G and structure group G with $\pi_i(F) = 0$ for all $i \geq 0$. If $A \subseteq X$ is a subcomplex, then every section of π over A extends to a section defined on all of X. In particular, π has a section. Moreover, any two sections of π are homotopic.*

Proof. Let $s_0\colon A \to E$ be a section of π over A, then we will extend it to a section $s\colon X \to E$ using induction over the cells in $X \setminus A$. In other words, we can assume that $X = A \cup_\phi e^n$ for x^n an n-cell in $X \setminus A$ and attaching map $\phi\colon \partial e^n \to A$. The bundle is trivial over e^n (as e^n is contractible), so we have a commutative diagram

$$\begin{tikzcd}
 & \pi^{-1}(e^n) \arrow[r, "\cong", "h"'] \arrow[d, "\pi"] & e^n \times F \arrow[dl, "\mathrm{pr}_1"'] \\
\partial e^n \arrow[ur, "s_0"] \arrow[r, hook] & e^n \arrow[ur, dashed, bend right, "s"'] &
\end{tikzcd}$$

where h is the chart for π over e^n and s is the seciton we wish to define.

The composite $h \circ s_0\colon \partial e^n \to e^n \times F$ is of the form

$$s_0(x) = (x, \tau_0(x)) \in e_n \times F,$$

with $\tau_0\colon \partial e_n \cong S^{n-1} \to F$. Because $\pi_{n-1}F = 0$ by assumption, τ_0 extends to a map $\tau\colon e^n \cong D^n \to F$ which we use to define $s\colon e^n \to e^n \times F$ by $s(x) = (x, \tau(x))$. After composing with h^{-1} we get the desired extension of s_0 over e^n.

To see that the section is unique up to homotopy, assume that we are given another section s'. Consider the bundle $\pi \times \mathrm{id}\colon E \times I \to X \times I$. We can define sections of this bundle over $X \times \{0\}$ and $X \times \{1\}$ using s and s', which together define a section over $X \times \{0, 1\}$. Arguing as in the first part, we can extend this section to a section Σ over $X \times I$. This section is of the form $\Sigma(x, t) = (s_t(x), t)\colon X \times I \to E \times I$, and map s_t provides the desired homotopy between s and s'. □

We now return to the proof of Theorem 3.8.2. It remains to show the injectivity of Φ. That is, if $\pi_0 \cong f^*\pi_G \cong f^*\pi_G \cong \pi_1$, then $f \simeq g$.

Consider the two defining diagrams:

$$\begin{tikzcd}
E_0 = f^*EG \arrow[r, "\tilde{f}"] \arrow[d, "\pi_0"'] & EG \arrow[d, "\pi"] \\
X = X \times \{0\} \arrow[r, "f"'] & BG
\end{tikzcd}
\qquad
\begin{tikzcd}
E_0 \cong E_1 = g^*EG \arrow[r, "\tilde{g}"] \arrow[d, "\pi_1"'] & EG \arrow[d, "\pi"] \\
X = X \times \{1\} \arrow[r, "g"'] & BG
\end{tikzcd}$$

We can combine them to make the diagram:

$$\begin{tikzcd}
E_0 \times I \arrow[d, "\pi_0 \times \mathrm{id}"'] & E_0 \times \{0,1\} \arrow[l, hook'] \arrow[r, "\tilde{\alpha}=(\tilde{f},0)\cup(\tilde{g},1)"] \arrow[d, "\pi_0 \times \{0,1\}"] & EG \arrow[d, "\pi"] \\
X \times I & X \times \{0,1\} \arrow[l, hook'] \arrow[r, "\alpha=(f,0)\cup(g,1)"] & BG
\end{tikzcd}$$

We will extend the map $(\alpha, \tilde{\alpha})$ to a bundle map $(H, \tilde{H})\colon \pi_0 \times \mathrm{id} \to \pi_G$; then H will give the desired homotopy. Using Theorem 3.7.14 again this corresponds to a section s of the bundle $\omega\colon (E_0 \times I) \times_G EG \to X \times I$. But the map $(\alpha, \tilde{\alpha})$ gives a section s_0 of the bundle $\omega_0\colon (E_0 \times \{0,1\}) \times_G EG \to X \times \{0,1\} \subseteq (E_0 \times I) \times_G EG \to X \times I$. Since EG is contractible, we can use Lemma 3.8.3 to extend the section s_0 to the desired section s. □

3.8.4 Example. We have $\mathcal{P}(S^n, G) \simeq [S^n, BG] \cong \pi_n(BG)$. The long exact sequence in homotopy shows that $\pi_n(BG) \cong \pi_{n-1}(G)$.

3.8.5 Remark. So far we have made no claim about the existence of universal principal G-bundles. Nonetheless, we have the following result of Milnor.

3.8.6 Theorem. *Let G be a locally compact topological group. Then a universal princiapl G-bundle exists, and is functorial in the sense that a continuous group homomorphism $f\colon G \to H$ induces a bundle map $(B\mu, E\mu)\colon \pi_G \to \pi_H$. Moreover, the classifying space BG is unique up to homotopy.*

Proof. Let us explain why BG is unique up to homotopy, before commenting on the construction. Suppose $\pi_G\colon EG \to BG$ and $\pi'_G\colon EG' \to BG'$ are universal principal G-bundles. Using the universal properties of π_G and π'_G we can find maps $f\colon BG' \to BG$ and $g\colon BG \to BG'$ such that $\pi'_G \cong f^*\pi_G$ and $\pi_G \cong g^*\pi'_G$. Then,

$$\pi_G \cong g^*\pi'_G \cong g^*f^*\pi_G \cong (f \circ g)^*(\pi_G) \simeq (\mathrm{id}_B)^*\pi_G.$$

By Theorem 3.8.2 we have $f \simeq g \simeq \mathrm{id}_{BG}$. Similarly, we deduce that $g \circ f \simeq \mathrm{id}_{BG'}$. Therefore, $f \simeq g$. □

3.8.7 Remark. There are several different ways to construct the bundle π_G. Here is Milnor's construction. We recall that the join of X and Y is the space[17]

$$X * Y := X \times I \times Y / \sim$$

where $(x, 0, y_1) \sim (x, 0, y_2)$ for all $y_1, y_2 \in Y$ and $(x_1, 1, y) \sim (x_2, 1, y)$ for all $x_1, x_2 \in X$. For example,

$$X * \{y\} = (X \times I)/(X \times \{1\}) \cong CX,$$

[17] Technically, for Milnor's construction, we need to equip the join with coarsest topology which makes $t\colon X * Y \to I$ (given on $X \times I \times Y$ by $(x, t, y) \mapsto t$, constantly 0 on X, and constantly 1 on Y) and the projections $\pi_1\colon t^{-1}(I) \to X$ and $\pi_2\colon t^{-1}(I) \to Y$ continuous.

the cone on X. If $Y = \{y_1, y_2\} = S^0$ is two points, then $X * Y \cong \Sigma X$, the suspension. There is also a reduced version of the join for pointed spaces. For CW-complexes, we have a homotopy equivalence $A * B \simeq \Sigma(A \wedge B)$. For example, $S^n * S^m \simeq S^{n+m-1}$ (in this case, this is actually a homeomorphism).

Now, we let $G^{*(k+1)} := G * \cdots * G$, the join of $k+1$-copies of G. This has a free G-action, given by the diagonal action on the copies of G, and trivial action on I. Let $\mathcal{J}(G) := \operatorname{colim}_k G^{*(k+1)}$. Then, in fact $\mathcal{J}(G)$ has a free G-action and $\mathcal{J}(G) \to \mathcal{J}(G)/G$ is a universal principal G-bundle. The rough idea is that as we join more and more copies of G, the space becomes more and more connective, and in the colimit, (weakly) contractible.

Exercise 32. *Show that $B(G \times H) \simeq BG \times BH$ (whenever this makes sense).*

3.8.8 Remark. In practice, for us we will construct the universal bundles we need by hand.

3.8.9 Example. We recall from (3.1.18) that we have a principal $O(n)$-bundle

$$O(n) \to V_n(\mathbb{R}^k) \to G_n(\mathbb{R}^k)$$

with $V_n(\mathbb{R}^k)$ $(k-n-1)$-connected. Letting $k \to \infty$, we get the bundle

$$O(n) \to V_n(\mathbb{R}^\infty) \to G_n(\mathbb{R}^\infty)$$

with $V_n(\mathbb{R}^\infty)$ contractible. This is a model for the universal bundle $EO(n) \to BO(n)$, i.e., $BO(n) \simeq G_n(\mathbb{R}^\infty)$.

Now recall that a real vector bundle is a locally trivial bundle with fiber a vector space V of dimension n over $\mathbb{R}$. Using the associated bundle construction and Theorem 3.4.2 we see that there is a bijection

$$\Big\{\text{principal } GL_n(\mathbb{R})\text{-bundles over } X\Big\} \longleftrightarrow \Big\{\text{Rank } n \text{ vector bundles over } X\Big\}$$

By Gram–Schmidt we have $GL_n(\mathbb{R}) \simeq O(n)$, and so we deduce the following from Theorem 3.8.2

$$\mathrm{Vect}_n^{\mathbb{R}}(X) \simeq \mathcal{P}(GL_n(\mathbb{R}), X) \simeq \mathcal{P}(O(n), X) \simeq [x, BO(n)] \simeq [X, G_n(\mathbb{R}^\infty)],$$

and we say that $G_n(\mathbb{R}^\infty)$ is a classifying space for real vector bundles of rank n.

Similarly, $BU(n) \simeq G_n(\mathbb{C}^\infty)$ and this is the classifying space for rank n complex vector bundles.

3.8.10 Example (Classification of real line bundles). Consider the (real) case $n = 1$ in the previous example, so that we are classifying real line bundles. In this case, we have a principal $\mathbb{Z}/2$-bundle $\mathbb{Z}/2 \to S^\infty \to \mathbb{R}P^\infty$, and so $B\mathbb{Z}/2 \simeq \mathbb{R}P^\infty$. But, note that $\mathbb{R}P^\infty \simeq K(\mathbb{Z}/2, 1)$, so Theorem 1.9.12 gives

$$\mathrm{Vect}_1^{\mathbb{R}}(X) \simeq \mathcal{P}(X, \mathbb{Z}/2) \simeq [X, \mathbb{R}P^\infty] \simeq H^1(X; \mathbb{Z}/2)$$

for any CW-complex X.

Recall that $H^*(\mathbb{R}P^\infty, \mathbb{Z}/2) \cong \mathbb{Z}/2[w]$ for $|w| = 1$. In particular, if π is a real line bundle on X with classifying map $f_\pi \colon X \to \mathbb{R}P^\infty$, we get a well-defined degree one cohomology class

$$w_1(\pi) := f_\pi^\infty(w)$$

called the first Stiefel–Whitney class of π. The bijection $P(X, \mathbb{Z}/2) \simeq H^1(X; \mathbb{Z}/2)$ sends π to $\omega_1(\pi)$ and so real line bundles are completely classified by their first Stiefel–Whitney classes.

3.8.11 Example (Classification of complex line bundles). Consider the complex case $n = 1$ in Example 3.8.9, so that we are classifying complex line bundles. In this case, we have a principal S^1-bundle $S^1 \to S^\infty \to \mathbb{C}P^\infty$, and so $BS^1 \simeq \mathbb{C}P^\infty$. But, note that $\mathbb{C}P^\infty \simeq K(\mathbb{Z}, 2)$, so Theorem 1.9.12 gives

$$\mathrm{Vect}_1^{\mathbb{C}}(X) \simeq \mathcal{P}(X, BS^1) \simeq [X, \mathbb{C}P^\infty] \simeq H^2(X; \mathbb{Z})$$

for any CW-complex X.

Recall that $H^*(\mathbb{C}P^\infty, \mathbb{Z}) \cong \mathbb{Z}[c]$ for $|c| = 2$. In particular, if π is a complex line bundle on X with classifying map $f_\pi \colon X \to \mathbb{C}P^\infty$, we get a well-defined degree two cohomology class

$$c_1(\pi) := f_\pi^\infty(c)$$

called the first Chern class of π. The bijection $P(X, S^1) \simeq H^2(X; \mathbb{Z})$ sends π to $c_1(\pi)$ and so real line bundles are completely classified by their first Chern classes.

3.8.12 Example. How many (real) vector bundles over $\mathbb{R}P^n$ are there? We have

$$\mathrm{Vect}_1^{\mathbb{R}}(\mathbb{R}^n) \cong H^1(\mathbb{R}P^n; \mathbb{Z}/2) \cong \mathbb{Z}/2,$$

so there are two (up to equivalence). One is the trivial bundle. What is the non-trivial bundle?

Let $x \in S^n$ and $[x] \in \mathbb{R}P^n \simeq S^n/\sim$ the class represented by x. Let $E = \{([x], v) \colon [x] \in \mathbb{R}P^n, v \in [x]\} \subseteq \mathbb{R}P^n \times \mathbb{R}^{n+1}$. Then, we have a bundle γ_1 defined by[18]

$$\gamma_1 \colon E \to \mathbb{R}P^n, \quad ([x], v) \mapsto [x].$$

To see that this is non-trivial note that when $n = 1$ this is exactly the Möbius bundle, which is non-trivial. In general, if γ_1 was trivial, then pull-back along $\mathbb{R}P^1 \to \mathbb{R}P^n$ would also be trivial, but this is the non-trivial Möbius bundle again, a contradiction. So γ_1 must be non-trivial.

3.8.13 Example. Isomorphism classes of principal S^1-bundles over S^2 are given by $[S^2, BS^1] \cong \pi_2(BS^1) \cong \pi_1(S^1) \cong \mathbb{Z} \cong H^2(\mathbb{C}P^\infty; \mathbb{Z})$. The Hopf bundle $H \colon S^1 \to S^3 \to S^2$ is a principal S^1-bundle. In particular, it is given as the pullback along a map $f \colon S^2 \cong \mathbb{C}P^1 \to \mathbb{C}P^\infty$:

$$\begin{tikzcd}
S^3 \arrow[r] \arrow[d, "H"'] & S^\infty \arrow[d, "\pi_{S^1}"] \\
S^2 \cong \mathbb{C}P^1 \arrow[r, "f"'] & \mathbb{C}P^\infty
\end{tikzcd}$$

---

[18] This bundle is known as the tautological, or canonical, bundle over $\mathbb{R}P^n$.

It turns out that this map is the inclusion $\mathbb{C}P^1 \hookrightarrow \mathbb{C}P^\infty$, and so $H^*(f)\colon H^*(\mathbb{C}P^\infty) \to H^*(\mathbb{C}P^1)$ sends ω to ω. In particular, $c_1(H) \neq 0$, and $c_1(H)$ generates $H^2(\mathbb{C}P^1)$ as a cyclic group.

The following exercise gives another way to compute this Chern class (using Examples 2.5.3 and 2.5.8)

Exercise 33. *Let $\pi\colon E \to X$ be a principal S^1-bundle over a simply-connected space X. Let $a \in H^1(S^1;\mathbb{Z})$ be a generator. Show that*

$$c_1(\pi) = d_2(a)$$

where d_2 is the differential on the E_2-page of the Leray–Serre spectral sequence associated to π, i.e., $E_2^{p,q} \cong H^p(X; H^1(S^1)) \implies H^{p+q}(E,\mathbb{Z})$.

4

Characteristic classes

In the end of the previous chapter we saw how two cohomology classes, the first Chern class, and the first Stiefel–Whitney class completely characterize complex and real line bundles respectively. In this section we develop a general theory of Chern and Stiefel–Whitney classes for higher rank bundles.

4.1 Chern classes of complex vector bundles

4.1.1 Remark. We recall that we have a bijection

$$\Big\{\text{principal } GL_n(\mathbb{C})\text{-bundles over } X\Big\} \longleftrightarrow \Big\{\text{Rank } n \text{ complex vector bundles over } X\Big\}$$

for any CW-complex X. We will freely use this to pass between complex vector bundles and principal bundles. Moreover, by Gram–Schmidt we have $GL_n(\mathbb{C}) \simeq U(n)$. We begin by computing the cohomology of the classifying space $BU(n)$.

4.1.2 Proposition. *We have*

$$H^*(BU(n);\mathbb{Z}) \cong \mathbb{Z}[c_1, c_2, \ldots, c_n]$$

with $|c_i| = 2i$*. Moreover, the map*

$$i\colon BU(n-1) \to BU(n)$$

induces a map $i^*\colon H^*(BU(n);\mathbb{Z}) \to H^*(BU(n-1);\mathbb{Z})$ *sending* c_i *to* c_i *for* $i < n$*.*

Proof. There are any number of ways to do this. For example, we can do this by induction on n. When $n = 1$ we have $BU(1) \simeq \mathbb{C}P^\infty$ and $H^*(\mathbb{C}P^\infty;\mathbb{Z}) \cong \mathbb{Z}[c_1]$ as we have already seen. In general, we have a fibration

$$S^{2n-1} \cong U(n)/U(n-1) \to BU(n-1) \to BU(n)$$

so the Gysin sequence (Proposition 4.1.2) is of the form

$$\cdots \to H^{k-1}(BU(n-1)) \to H^{k-2n}(BU(n)) \xrightarrow{\smile e} H^k(BU(n)) \xrightarrow{p^*} H^k(BU(n-1)) \to H^{k-2n+1}(BU(n)) \to \cdots$$

Inductively we note that $H^*(BU(n-1))$ is concentrated in even degrees, so we get short exact sequences

$$0 \to H^{k-2n}(BU(n)) \xrightarrow{\smile e} H^k(BU(n)) \xrightarrow{p^*} H^k(BU(n-1)) \to 0$$

It follows that $H^k(BU(n)) = 0$ for k odd as well.[1] Moreover, there is an isomorphism $H^*(BU(n))/(e) \xrightarrow{\sim} H^*(BU(n-1))$. Because $H^*(BU(n-1))$ is a polynoimal algebra, we can lift the generators to generators of $H^*(BU(n))$, and produce an algebra map $H^*(BU(n-1))[e] \to H^*(BU(n))$, which we claim is an isomorphism. Indeed, we can filter both sides by powers of e, and note that this gives an isomorphism on associated gradeds. A five-lemma argument shows that the map induces an isomorphism modulo e^k for any k, but the powers of e increase in dimension, so we obtain an isomorphism in each dimension. Finally, we can define $c_n = (-1)^n e \in H^{2n}(BU(n))$.

If you don't like that argument, another way is to first prove that $H^*(U(n)) \cong \Lambda_{\mathbb{Z}}[x_1, \ldots, x_{2n-1}]$ using the Serre spectral sequence. A second application of the Serre spectral sequence for the fibration $U(n) \to EU(n) \to BU(n)$ gives the result. We leave the details to the reader. □

4.1.3 Definition. The generators $c_1, \ldots, c_n$ are called the universal Chern classes of $U(n)$-bundles.

4.1.4 Remark. Recall that given a principal $U(n)$-bundle $\pi\colon E \to X$, there exists a map $f_\pi\colon X \to BU(n)$ such that $\pi \cong f_\pi^*(\pi_{U(n)})$.

4.1.5 Definition. The i-th Chern class of the $U(n)$-bundle $\pi\colon E \to X$ is defined as $c_i(\pi) := f_\pi^*(c_i) \in H^{2i}(X;\mathbb{Z})$.

4.1.6 Proposition (Functoriality of Chern classes)**.** *If $f\colon Y \to X$ is a continuous map, and $\pi\colon E \to X$ is a $U(n)$-bundle, then $c_i(f^*\pi) \cong f^*(c_i(\pi))$ for any i.*[2]

Exercise 34. *Prove Proposition 4.1.6.*

4.1.7 Corollary. *If ϵ is the trivial $U(n)$-bundle on a space X, then $c_i(\epsilon) = 0$ for all $i > 0$.*

Proof. The bundle ϵ is the pullback of the bundle $\nu\colon G \to *$ along the canonical map $q\colon X \to *$:

$$\begin{tikzcd}
X \times G \arrow[r] \arrow[d, "\epsilon"'] \arrow[dr, phantom, "\lrcorner", very near start] & G \arrow[d, "\nu"] \\
X \arrow[r, "q"'] & *
\end{tikzcd}$$

So we have

$$c_i(\epsilon) \cong c_i(q^*(\nu)) \cong q^* c_i(\nu).$$

But $c_i(\nu) \in H^{2i}(*) = 0$ when $i > 0$. □

4.1.8 Definition. The total Chern class of a $U(n)$-bundle $\pi\colon E \to X$ is defined by[3]

$$c(\pi) = c_0(\pi) + c_1(\pi) + \cdots + c_n(\pi) \in H^*(X;\mathbb{Z})$$

as an element in the cohomology ring of the base space.

4.1.9 Definition (Whitney Sum). Let $\pi_1\colon E_1 \to X$ and $\pi_2\colon E_2 \to X$ be principal $U(n)$ and $U(m)$-bundles respectively. Consider the

[1] This is clear for $k < 2n$, but note that we can then feed this into the leftmost term and use induction to see it for all k.

[2] Note that f^* has a dual role here: once as a pullback bundle, and once as the pullback of a cohomology class.

[3] Note that if π is a $U(n)$-bundle, then $c_i(\pi) = 0$ for $i > n$ by definition.

product bundle $\pi_1 \times \pi_2 \colon E_1 \times E_2 \to X \times X$ which is a principal $U(n+m)$-bundle, via the inclusion $U(n) \times U(m) \to U(n+m)$. The Whitney sum of π_1 and π_2 is defined as

$$\pi_1 \oplus \pi_2 := \Delta^*(\pi_1 \times \pi_2)$$

where $\Delta \colon X \to X \times X$ is the diagonal.

4.1.10 Proposition (Whitney sum formula). *If $\pi_1 \colon E_1 \to X$ and $\pi_2 \colon E_2 \to X$ are principal $U(n)$ and $U(m)$-bundles respectively, then*

$$c(\pi_1 \oplus \pi_2) \cong c(\pi_1) \smile c(\pi_2),$$

or equivalently,

$$c_k(\pi_1 \oplus \pi_2) = \sum_{i+j=k} c_i(\pi_1) \smile c_j(\pi_2).$$

Proof. First observe that by the exercises we have $B(U(n) \times U(m)) \simeq BU(n+m)$. We then consider the map

$$\omega \colon B(U(n) \times U(m)) \simeq BU(n) \times BU(m) \to BU(n+m)$$

induced by $U(n) \times U(m) \to U(n+m)$. One can show that[4]

$$\omega^*(c_k) = \sum_{i+j=k} c_i \otimes c_j.$$

It follows that

$$\begin{aligned} c_k(\pi_1 \oplus \pi_2) &= c_k(\Delta^*(\pi_1 \times \pi_2)) \\ &\cong \Delta^* c_k(\pi_1 \times \pi_2) \\ &= \Delta^*(f^*_{\pi_1 \times \pi_2}(c_k)) \end{aligned}$$

Now we note that the classifying map for $\pi_1 \times \pi_2$ regarded as a $U(n+m)$-bundle is $\omega \circ (f_{\pi_1} \times f_{\pi_2})$. Therefore, we continue:

$$\begin{aligned} c_k(\pi_1 \oplus \pi_2) &\cong \Delta^*(f^*_{\pi_1} \times f^*_{\pi_2})(\omega^*(c_k)) \\ &\cong \sum_{i+j=k} \Delta^*(f^*_{\pi_1}(c_i) \times f^*_{\pi_2}(c_j)) \\ &\cong \sum_{i+j=k} \Delta^*(c_i(\pi_1) \times c_j(\pi_2)) \\ &\cong \sum_{i+j=k} c_i(\pi_1) \smile c_j(\pi_2) \end{aligned}$$

as required. □

4.1.11 Corollary (Stability of Chern classes). *Let ϵ^1 denote the trivial $U(1)$-bundle, then $c(\pi \oplus \epsilon^1) \cong c(\pi)$.*

Proof. This follows from the proposition and Corollary 4.1.7. □

4.1.12 Remark. It turns out that Chern classes are completely determined by four axioms:

A1. To each principal $U(n)$-bundle $\pi \colon E \to X$ there exists a sequence of classes $c_i(\pi) \in H^{2i}(X; \mathbb{Z})$ such that $c_0(\pi) = 1 \in H^0(X;\mathbb{Z})$ and $c_i(\pi) = 0$ for $i > n$.

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[4] Here is an idea of one way to do this. Let $T(n) = U(1) \times \cdots U(1)$, a product of n-copies of S^1. The canonical map $T(n) \to U(n)$ induces $\mu_n \colon BT(n) \to BU(n)$. We have $H^*(BT(n)) \cong \mathbb{Z}[x_1, \ldots, x_n]$ for $|x_i| = 2$, and μ^* is a monomorphism determined by $\mu_n^*(c_k) \cong \sigma_k(x_1, \ldots, x_n)$, the k-th elementary symmetric polynomial in $x_1, \ldots, x_n$. This allows us to reduce to a computation with $BT(n)$, and some diagram chasing. The details can be found, for example, in Corollary 2.44 in Kochman's book 'Bordism, stable homotopy, and the Adams spectral sequence.'

A2. Naturality: If $f\colon Y \to X$ is a continuous map, then $c_k(f^*(\pi)) \cong f^*(c_k(\pi))$.

A3. Whitney sum formula: $c(\pi_1 \oplus \pi_2) = c(\pi_1) \smile c(\pi_2)$.

A4. Normalization: Let x be the generator of $H^2(\mathbb{C}P^n;\mathbb{Z}) \cong \mathbb{Z}$, then the total Chern class of the tautological line bundle (see Example 3.8.12) over $S^2 \cong \mathbb{C}P^1$ is $1+x$.[5]

4.1.13 Theorem. *There exists at most one correspondence $\pi \mapsto c(\pi)$ which assigns to each complex vector bundle over a paracompact base space a sequence of cohomology classes satisfying the above four axioms.*

The proof uses the following (very important) splitting principle for complex vector bundles.[6]

4.1.14 Proposition. *For each complex vector bundle $\pi\colon E \to X$ there exists a space $F(E)$ and a map $p\colon F(E) \to X$ such that the pull-back $p^*F(E) \to F(E)$ splits as a direct (Whitney) sum of line bundles and $p^*\colon H^*(X;\mathbb{Z}) \to H^*(F(E);\mathbb{Z})$ is injective.*

4.1.15 Remark. There is a similar result for real vector bundles if we use $\mathbb{Z}/2$ coefficients.

Sketch proof. By induction, it will suffice to find a map $p'\colon F'(E) \to X$ such that $(p')^*(\pi) \cong E' \oplus L$ with L a complex line bundle, and $p^*\colon H^*(X;\mathbb{Z}) \to H^*(F'(E);\mathbb{Z})$ injective, as we can then inductively apply the same argument to E'.

We use the projective bundle construction of Definition 3.6.13, and set $F'(E) := P(E)$ with $p'\colon P(E) \to X$. Then there is an injective map

$$\phi\colon L_E \to (p')^*(E), \quad (\ell, v) \mapsto (\ell, v)$$

where L_E is a line bundle. Because X is compact, we can choose a Hermitian inner product on E inducing one on $(p')^*$, and hence take E' to be the orthogonal complement of $\phi(L_E)$ in $(p')^*(E)$. Therefore, $(p')^*(E) \cong L_E \oplus E'$, as required. The claim about cohomology follows from the Leray–Hirsch theorem [7] - $H^*(P(E);\mathbb{Z})$ is the free $H^*(X;\mathbb{Z})$-module with basis $1, x, \dots, x^{n-1}$; in particular, the map $H^*(X;\mathbb{Z}) \to H^*(P(E);\mathbb{Z})$ is injective since one of the basis elements is 1. ☐

Proof of theorem 4.1.13. Let $\pi \mapsto c(\pi), \tilde{c}(\pi)$ be two sets of Chern classes. By Axioms A1 and A4 for the canonical line bundle γ_1^1 over $\mathbb{C}P^1$ we have

$$c(\gamma_1^1) = \tilde{c}(\gamma_1^1) = 1 + x.$$

Using the embedding $\mathbb{C}P^1 \to \mathbb{C}P^\infty$ we deduce that

$$c(\gamma_1) = \tilde{c}(\gamma_1) = 1 + x.$$

for γ_1 the canonical line bundle over $\mathbb{C}P^\infty$ by Axioms A1 and A2. Then, for $\xi = \gamma_1 \oplus \cdots \oplus \gamma_1$ we deduce that

$$c(\xi) = \tilde{c}(\xi)$$

---

[5] Or $1 - x$, depending on convention.

[6] For example, the splitting principle can also reduce the proof of the Whitney sum formula to line bundles.

[7] `https://en.wikipedia.org/wiki/Leray%E2%80%93Hirsch_theorem`

by Axiom A3.

Now, let $\pi\colon E \to X$ be arbitrary, and $p\colon F(E) \to X$ the map that exists by the splitting principle (Proposition 4.1.14). Then we have

$$\begin{aligned} p^*c(\pi) &\cong c(p^*\pi) && (\textit{Axiom A2}) \\ &\cong c(\lambda_1 \oplus \cdots \oplus \lambda_n) && (\textit{Proposition 4.1.14}) \\ &\cong \tilde{c}(\lambda_1 \oplus \cdots \oplus \lambda_n) \\ &\cong \tilde{c}(p^*\pi) \\ &\cong p^*\tilde{c}(\pi) \end{aligned}$$

Because p^* is injective, we deuce that $c(\pi) \cong \tilde{c}(\pi)$, as required. □

4.1.16 Remark. This shows that there is *at most* one theory of Chern classes. We omit the proof that Chern classes do actually exist with the required properties (we are almost there; we have just not shown Item A4).

4.1.17 Example (Chern classes of the dual bundle). Given a complex vector bundle $\pi\colon E \to M$ its dual bundle is the Hom bundle (Definition 3.6.11) $\mathrm{Hom}(\pi, \mathbb{C} \times M)$, i.e. the hom bundle from π to the trivial bundle $\mathbb{C} \times M \to M$. We denote this bundle by $\pi^*\colon E^* \to M$. The fibers of this bundle are the dual spaces to the fiber of π. Let L be a complex line bundle, then one can check that $L \otimes L^* = \mathrm{Hom}(L, L)$ is a trivial bundle. Moreover, $c_1(L \otimes L^*) = c_1(L) + c_1(L^*)$, so that $c_1(L) = -C_1(L^*)$.

Now suppose that $E = L_1 \oplus \cdots \oplus L_n$ is a sum of line bundles. By the Whitney sum formula

$$c(E^*) = c(L_1) \smile \cdots \smile c(L_n) = (1 + c_1(L_1)) \cdots (1 + c_n(L_n)).$$

Similarly, $E^* = L_1^* \oplus \cdots \oplus L_n^*$, and

$$c(E) = c(L_1^*) \smile \cdots \smile c(L_n^*) = (1 - c_1(L_1)) \cdots (1 - c_n(L_n)).$$

By comparing coefficients,[8] we have $c_q(E^*) = (-1)^q c_q(E)$. By the splitting principle, this holds for all complex vector bundles.

4.2 Stiefel–Whitney classes for real vector bundles

Analogous to Chern classes for complex vector bundles, we have a good theory of Stiefel–Whitney classes for real vector bundles, where we replace $BU(n)$ with $BO(n)$.

4.2.1 Proposition.

$$H^*(BO(n); \mathbb{Z}/2) \cong \mathbb{Z}/2[w_1, \ldots, w_n]$$

with $|w_i| = i$.

Proof. This is very similar to Proposition 4.1.2. For example, we can use induction using the Serre spectral sequence of the fibration

$$O(n)/O(n-1) \cong S^{n-1} \to BO(n-1) \to BO(n)$$

and $BO(1) \simeq \mathbb{R}P^\infty$ with $H^*(\mathbb{R}P^\infty) \cong \mathbb{Z}/2[w_1]$. □

---

[8] Use the bimonial formula if you need

4.2.2 Definition. The generators $w_1, \dots, w_n$ are called the universal Stiefel–Whitney classes of $O(n)$-bundles.

4.2.3 Remark. Recall that given a principal $O(n)$-bundle $\pi\colon E \to X$, there exists a map $f_\pi\colon X \to BO(n)$ such that $\pi \cong f_\pi^*(\pi_{U(n)})$.

4.2.4 Definition. The i-th Stiefel–Whitney class of the $O(n)$-bundle $\pi\colon E \to X$ is defined as $w_i(\pi) := f_\pi^*(w_i) \in H^i(X; \mathbb{Z}/2)$.

Using identical proofs as in the complex case, Stiefel–Whitney classes are characterized by four axioms:

A1. To each principal $O(n)$-bundle $\pi\colon E \to X$ there exists a sequence of classes $c_i(\pi) \in H^i(X; \mathbb{Z}/2)$ such that $w_0(\pi) = 1 \in H^0(X; \mathbb{Z})$ and $w_i(\pi) = 0$ for $i > n$.

A2. Naturality: If $f\colon Y \to X$ is a continuous map, then $w_k(f^*(\pi)) \cong f^*(w_k(\pi))$.

A3. Whitney sum formula: $w(\pi_1 \oplus \pi_2) = w(\pi_1) \smile w(\pi_2)$.

A4. Normalization: Let x be the non-zero element of of $H^2(\mathbb{R}P^n; \mathbb{Z}/2) \cong \mathbb{Z}/2$, then the total Chern class of the tautological line bundle (see Example 3.8.12) over $S^1 \cong \mathbb{R}P^1$ is $1 + x$.

4.2.5 Theorem. *There exists at most one correspondence $\pi \mapsto w(\pi)$ which assigns to each real vector bundle over a paracompact base space a sequence of cohomology classes satisfying the above four axioms.*

4.2.6 Remark. Given a real vector bundle $\pi\colon E \to X$ we can consider its complexificaiton $\pi \otimes \mathbb{C}$, the complex vector bundles with transition functions $\Phi_{\alpha\beta}\colon U_\alpha \cap U_\beta \to O(n) \subseteq U(n)$ and fiber $\mathbb{R}^n \otimes \mathbb{C} \cong \mathbb{C}^n$.

Now if $\pi'\colon E' \to X$ is a complex vector bundle, we can always make a conjugate bundle $\overline{\pi'}$. Note that the dual bundle of π' is isomorphic to the conjugate bundle, but the choice of isomorphism is non-canonical unless E' has a hermitian product. The transition functions of the conjugate bundle are given as the composite

$$\overline{\Phi}_{\alpha\beta}\colon U_\alpha \cap U_\beta \xrightarrow{\Phi_{\alpha\beta}} U(n) \xrightarrow{(-)^\dagger} U(n)$$

where the last map takes the complex conjugate of a unitary matrix. In any case, we have the following.

4.2.7 Lemma. *Let π be a real vector bundle, then $\overline{\pi \otimes \mathbb{C}} \cong \pi \otimes \mathbb{C}$.*

Proof. Just observe that the transition functions for $\pi \otimes \mathbb{C}$ are real-valued; they land in $O(n) \subseteq U(n)$, and so they are also the transition functions for $\overline{\pi \otimes \mathbb{C}}$. □

4.2.8 Proposition.

$$c_k(\pi \otimes \mathbb{C}) \cong c_k(\overline{\pi \otimes \mathbb{C}}) \cong (-1)^k c_k(\pi \otimes \mathbb{C})$$

In particular, if k is odd, then $c_k(\pi \otimes \mathbb{C})$ is an integral cohomology class of order 2.

Proof. This follows from Example 4.1.17 and lemma 4.2.7. □

Given a complex vector bundle ω, we let $\omega_{\mathbb{R}}$ denote the underlying real vector bundle.

4.2.9 Proposition. *If ω is a complex vector bundle, then*

$$\omega_{\mathbb{R}} \otimes \mathbb{C} \cong \omega \oplus \overline{\omega}.$$

Proof. We prove the statement at the level of vector-spaces; the proof passes to vector bundles as well. To that end, let L be a complex vector space, and $L_{\mathbb{R}}$ its underlying real vector space, then we claim that $L_{\mathbb{R}} \otimes \mathbb{C} \cong L \oplus \overline{L}$. To see this, let

$$J \colon L_{\mathbb{R}} \otimes \mathbb{C} \to L_{\mathbb{R}} \otimes \mathbb{C}$$

by given by multiplication by i. Because $J^2 = -\text{id}$, we have an eigenvalue decomposition

$$L_{\mathbb{R}} \otimes \mathbb{C} \cong \text{eigen}(i) \oplus \text{eigen}(-i).$$

We have a map

$$P_i \colon L \to L_{\mathbb{R}} \to L_{\mathbb{R}} \otimes \mathbb{C} \to \text{eigen}(i),$$

which is $\mathbb{R}$-linear, but is in fact $\mathbb{C}$-linear because $P_i J(\ell) = iP_i(\ell)$ for all $\ell \in L$. This composite is therefore an isomorphism by a dimension count. Similarly,

$$P_{-i} \colon L \to \text{eigen}(-i)$$

is a $\mathbb{C}$-*anti*linear isomorphism, and so $\text{eigen}(i) \cong \overline{L}$. ☐

4.2.10 Corollary. *For a complex vector bundle ω we have*

$$c(\omega_R \otimes \mathbb{C}) \cong c(\omega) \cdot c(\overline{\omega}),$$

or equivalently,

$$c_k(\omega_R \otimes \mathbb{C}) = \sum_{i+j=k} (-1)^j c_i(\omega) \cdot c_j(\omega).$$

4.2.11 Remark. Note that if k is odd, then this sum is always zero.

Exercise 35. *Let $\pi_{\mathbb{R}}$ denote the underlying real bundle of a complex bundle; π note that if π has rank n as a complex bundle, then $\pi_{\mathbb{R}}$ has rank $2n$ as a real bundle. Via the map $\mathbb{Z} \to \mathbb{Z}/2$ the class $c_i(\pi) \in H^{2i}(X;\mathbb{Z})$ determines a cohomology class $\overline{c}_i(\pi) \in H^{2i}(X;\mathbb{Z}/2)$. Show that the Stiefel–Whitney classes of $\pi_{\mathbb{R}}$ are computed as follows:*

1. $\omega_{2i}(\pi_{\mathbb{R}}) = \overline{c}_i(\pi)$ *for* $0 \leq i \leq n$.
2. $\omega_{2i+1}(\pi_{\mathbb{R}}) = 0$ *for any integer i.*

4.2.12 Remark. Here is a hint: Let $\mu_n \colon U(n) \to O(2n)$ be the inclusion, then the classifying map of π_R is the composite $X \xrightarrow{f} BU(n) \xrightarrow{\mu_n} BO(2n)$, where f is the classifying map for π. So you should try and compute

$$\mu_n^* \colon H^*(BO(2n);\mathbb{Z}/2) \cong \mathbb{Z}/2[w_1, w_2, \dots, w_{2n}] \to H^*(BU(n);\mathbb{Z}/2) \cong \mathbb{Z}/2[\overline{c}_1, \overline{c}_2, \dots, \overline{c}_n].$$

4.3 *Applications of Stiefel–Whitney classes*

Stiefel–Whitney classes are useful in the study of smooth manifolds. Indeed, if M is smooth, then we recall (Example 3.6.2) that the tangent bundle $\pi\colon TM \to M$ is a real vector bundle, and hence corresponds to an $O(n)$-bundle (which by our conventions, we use the same notation for).

4.3.1 Definition. The Stiefel–Whitney classes of a smooth manifold M are defined as the Stiefel–Whitney classes of the corresponding $O(n) - bundle$: $w_i(M) := w_i(TM)$.

4.3.2 Remark. In order, for this to be a reasonable notion, we should prove that these are homotopy invariants. Fortunately, we have the following theorem.[9]

4.3.3 Theorem (Wu)**.** *Stiefel–Whitney classes are homotopy invariants, i.e., if* $h\colon M_1 \to M_2$ *is a homotopy equivalence, then* $h^* w_i(M_2) = w_i(M_1)$ *for any* $i \geq 0$.

We now turn to an application of Stiefel–Whitney classes to the embedding problem. We begin with the following algebraic lemma.

4.3.4 Lemma. *Suppose that* $E \oplus E' \simeq \epsilon^n$ *is a trivial bundle, then there exists a unique polynomial* q_i *such that*

$$\omega_i(E') = q_i(\omega_1(E), \omega_2(E), \dots, \omega_i(E)).$$

Proof. Induction on i. When $i = 1$ we have

$$\begin{aligned} 0 = \omega_1(\epsilon^n) &= \omega_0(E) \smile \omega_1(E') + \omega_1(E) \smile \omega_0(E') \\ &= 1 \smile \omega_1(E') + \omega_1(E) \smile 1, \end{aligned}$$

and hence

$$\omega_1(E') = -\omega_1(E) = \omega_1(E),$$

since we work over $\mathbb{Z}/2$.

Supposing we have proved the claim up to $i-1$. Then,

$$\begin{aligned} 0 = \omega_i(\epsilon^n) &= \sum_{k+j=i} \omega_k(E) \smile \omega_j(E') \\ &= \omega_i(E') + \sum_{k+j=1, j<i} \omega_k(E) \smile \omega_j(E') \\ &= \omega_i(E') + \sum_{k+j=1, j<i} \omega_k(E) \smile q_j(\omega_1(E), \dots, \omega_j(E)). \end{aligned}$$

Therefore,

$$\omega_i(E') = q_i(\omega_1(E), \dots, \omega_i(E)) := \sum_{k+j=i, j<i} \omega_k(E) \smile q_j(\omega_1(E), \dots, \omega_j(E)).$$

□

4.3.5 Definition. We write $\overline{w}_i(E)$ for $q_i(\omega_1(E), \dots, \omega_i(E))$. These are the dual Stiefel–Whitney classes.

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[9] The proof of the following is beyond the scope of this course, but here is the idea: One can give an alternative construction of the Stiefel–Whitney classes in terms of Steenrod operations; this implies that the the Stiefel–Whitney classes of M are determined entirely in terms of the mod 2 cohomology ring along with its structure under the Steenrod algebra (which is preserved by homotopy equivalences). So in fact, the theorem doesn't even need homotopy equivalence, but only a mod 2 cohomology isomorphism over the Steenrod algebra.

4.3.6 Remark. Let $f\colon M^m \to N^{m+k}$ be an embedding of smooth manifolds.[10] Let f^*TN denote the pullback of the tangent bundle $TN \to N$ along f. The normal bundle is defined by the short exact sequence

$$0 \to TM \to f^*TN \to \nu \to 0,$$

which splits, i.e., $f^*TN \simeq TM \oplus \nu$ where ν has rank k. So, using the Whitney sum formula we have

$$f^*\omega(N) = \omega(M) \smile \omega(\nu).$$

4.3.7 Example. Let $S^n \subseteq \mathbb{R}^{n+1}$, then the normal bundle ν is trivial. Indeed, if we write

$$\nu(S^n) = \bigcup_{p \in S^n} T_p\mathbb{R}^{n+1}/T_pS^n$$

then an explicit isomorphism is given by the map Φ sending

$$[v] \in \nu_p(S^n) \text{ to } (p, \langle v, p\rangle) \in S^n \times \mathbb{R},$$

with inverse Ψ sending $(q,t) \mapsto [tq] \in \nu_q(S^n)$.[11] In other words,

$$TS^n \oplus \nu \simeq \epsilon^{n+1} \simeq TS^n \oplus \epsilon^1 \simeq \epsilon^{n+1}.$$

Since trivial bundles do not change Stiefel–Whitney classes, we deduce that $\omega_i(S^n) = 0$ for all $i > 0$, and the Stiefel–Whitney classes of TS^n are the same as the trivial bundle (and recall that we have seen that $p\colon TS^2 \to S^2$ is not a trivial bundle - in fact this is true for every even sphere).[12]

4.3.8 Example. Suppose that $N = \mathbb{R}^{m+k}$, then using Lemma 4.3.4 (and the hopefully obvious observation that the tangent bundle is trivial: $T\mathbb{R}^{m+k} \simeq \mathbb{R}^{m+k} \times \mathbb{R}^{m+k}$), then we deduce that

$$\omega_i(\nu) = \overline{\omega}_i(TM). \tag{4.3.9}$$

The following calculation is important for our applications. We postpone the proof until after the applications.

4.3.10 Theorem.

$$\omega(\mathbb{R}P^m) \cong (1+x)^{m+1}$$

where $x \in H^1(\mathbb{R}P^m; \mathbb{Z}/2)$ is a generator.

4.3.11 Example. Can we give an embedding of $\mathbb{R}P^9$ into $\mathbb{R}^{9+k}$? Let us note that

$$\omega(\mathbb{R}P^9) = (1+x)^{10} = (1+x)^8(1+x)^2 = (1+x^8)(1+x^2) = 1 + x^2 + x^8$$

because $H^*(\mathbb{R}P^m; \mathbb{Z}/2) \cong \mathbb{Z}/2[x]/(x^{m+1})$. Therefore,[13]

$$\overline{\omega}(\mathbb{R}P^9) = 1 + x^2 + x^4 + x^6$$

Therefore by (4.3.9) we must have,

$$\omega(\nu) = 1 + x^2 + x^4 + x^6,$$

---

[10] When we use superscripts, we refer to the dimension of the manifolds. So, we may also write $f\colon M \to N$.

[11] For example, $(\Phi \circ \Psi)(p,t) = \Phi([tp]) = (p, \langle tp, p\rangle) = (p, t\langle p, p\rangle) = (p,t)$.

[12] There is a 'moral' reason for this: the only possible Stiefel–Whitney classes in positive degrees is the top one, as $H^i(S^n; \mathbb{Z}/2)$ is non-zero only when $i = 0, n$. This top class is the image of the *Euler class* in $H^n(S^n; \mathbb{Z})$ under the natural homomorphism $H^i(S^n; \mathbb{Z}) \to H^i(S^n; \mathbb{Z}/2)$ - see Milnor–Stasheff, Property 9.5. But the Euler class of the sphere is $e(TS^n) = 2[S^n]$ is 0 modulo 2.

[13] We can check this: $(1 + x^2 + x^8)(1 + x^2 + x^4 + x^6) = 1 + 2x^2 + 2x^4 + 2x^6 + 2x^8 + x^{10} + x^{12} + x^{14} \equiv 1$ in the ring $\mathbb{Z}/2[x]/(x^{10})$.

and in particular, $\omega_6(\nu) \neq 0$. Now note that ν is a bundle of rank k, and hence we have $\omega_i(\nu) = 0$ for $i > k$. Therefore, we must have $k \geq 6$. We deduce that $\mathbb{R}P^9$ cannot be embedded into $\mathbb{R}^{14}$. Note that this doesn't say anything about when it *can* be embedded into $\mathbb{R}^{9+k}$. In fact, this bound is sharp: $\mathbb{R}P^9$ can be embedded into $\mathbb{R}^{15}$.[14]

4.3.12 Example. Let $m = 2^r$, then

$$\omega(\mathbb{R}P^{2^r}) = (1+x)^{2^r+1} = (1+x)^{2^r}(1+x) = (1+x^{2^r})(1+x) = 1+x+x^{2^r}$$

Arguing as in the previous example, we have

$$\omega(\nu) = \overline{\omega}(\mathbb{R}P^{2^r}) = 1 + x + x^2 + \cdots + x^{2^r-1},$$

and so $k \geq 2^r - 1 = m - 1$, i.e., $\mathbb{R}P^{2^r}$ cannot be embedded into $\mathbb{R}^{2^{r+1}-1}$. In particular, $\mathbb{R}P^8$ cannot be embedded into $\mathbb{R}^{15}$. Again, this bound is sharp: there exists an embedding of $\mathbb{R}P^8$ into $\mathbb{R}^{16}$, by the Whitney embedding theorem.

We now return to the proof of Theorem 4.3.10.

Proof of Theorem 4.3.10. Let $[x] \in \mathbb{R}P^m$ and $\nu \in [x]$. As usual, we let γ_1 denote the canonical line bundle over $\mathbb{R}P^m$, i.e., $\gamma_1 = \{([x], \nu) \in \mathbb{R}P^m \times \mathbb{R}^{m+1} \mid [x] \in \mathbb{R}P^m, \nu \in [x]\}$. Define L_x to to be the line in $\mathbb{R}^{m+1}$ joining x and $-x$, and let $L_x^\perp$ be its orthogonal complement in $\mathbb{R}P^m \times \mathbb{R}^{m+1}$.

For each $(x, \nu) \in T\mathbb{R}P^n$ we have a linear map

$$\ell(x,\mu)\colon L_x \to L_x^\perp$$

defined by $\ell(x,\mu)(x) = \nu$, which is well defined because $\ell(x,\mu)(-x) = -\nu$. This gives us a fiberwise isomorphism $T_{[x]}\mathbb{R}P^m \to \operatorname{Hom}(L_x, L_x^\perp)$, by sending (x,ν) to $T(x,\nu)$. Now a continuous map between vector bundles over the same base space B is an isomorphism if it is a fiberwise linear isomorphism.[15] Therefore, we have $T\mathbb{R}P^m \cong \operatorname{Hom}(\gamma_1, \gamma_1^\perp)$.

Now we make the following observation: the bundle $\operatorname{Hom}(\gamma_1, \gamma_1)$ is just the trivial line bundle ϵ^1. Indeed, the transition map is $\phi_{\alpha\beta}\phi_{\alpha\beta}^{-1} = \mathrm{id}$ (this is special about line bundles: the transpose of a 1×1 matrix is the same matrix!). Therefore,

$$\begin{aligned} T\mathbb{R}P^m \oplus \epsilon^1 &\cong \operatorname{Hom}(\gamma_1, \gamma_1^\perp) \oplus \operatorname{Hom}(\gamma_1, \gamma_1) \\ &\cong \operatorname{Hom}(\gamma_1, \gamma_1^\perp \oplus \gamma_1) \\ &\cong \operatorname{Hom}(\gamma_1, \epsilon^{m+1}) \\ &\cong \operatorname{Hom}(\gamma_1, \epsilon^1)^{m+1} \end{aligned}$$

However, $\operatorname{Hom}(\gamma_1, \epsilon^1) \cong \gamma_1$, and so

$$T\mathbb{R}P^m \oplus \epsilon^1 \cong \gamma_1^{\oplus m+1}.$$

Therefore, we have

$$\begin{aligned} \omega(\mathbb{R}P^m) \cong \omega(T\mathbb{R}P^m \oplus \epsilon^1) &\cong \omega(\gamma_1^{\oplus m+1}) \\ &\cong \omega(\gamma_1)^{\smile(m+1)} \\ &\cong (1+x)^{m+1}. \end{aligned}$$

[14] Sanderson, B. J. Immersions and embeddings of projective spaces. Proc. London Math. Soc. (3) 14 (1964), 137–153.

[15] See, Lemma 1.1 of `https://pi.math.cornell.edu/~hatcher/VBKT/VB.pdf` for example.

Here we have used the stability of Stiefel–Whitney classes, the previous discussion, the Whitney sum formula, and the normalization axiom. □

4.3.13 Remark. In other words, we have

$$\omega_i(\mathbb{R}P^m) = \binom{m+1}{i} x^i,$$

where the binomial coefficient is taken modulo 2.

4.3.14 Definition. A manifold is parallelizable if its tangent bundle is trivial.

4.3.15 Corollary. *The total Stiefel–Whitney class $\omega(\mathbb{R}P^m) = 1$ if and only if $m+1 = 2^r$ for some r. In particular, if $\mathbb{R}P^m$ is parallelizable, then $m+1 = 2^r$ for some r.*[16]

Proof. If $m+1 = 2^r$, then

$$\omega(\mathbb{R}P^m) = (1+x)^{2^r} = 1 + x^{2^r} = 1 + x^{m+1} = 1,$$

since $x^{m+1} = 0$. Conversely, if $m+1 = 2^r k$ where $k > 1$ is odd, then

$$\omega(\mathbb{R}P^m) = [(1+x)^{2^r}]^k = (1+x^{2^r})^k = 1 + kx^{2^r} + \cdots \neq 1,$$

since $x^{2^r} \neq 0$. The final statement follows, as $\omega(\mathbb{R}P^m) = 1$ whenever $\mathbb{R}P^m$ is parallelizable, by definition. □

4.4 *Pontryagin classes*

4.4.1 Notation. Let us fix some notation throughout this section: we let π denote a real vector bundle (equivalently, a principal $O(n)$-bundle), while ω will denote a complex vector bundle (equivalently, a $U(n)$-bundle).

The reader may want to refresh the statements of Proposition 4.2.8 and Proposition 4.2.9 in order to appreciate the following definitions.

4.4.2 Definition. Let $\pi \colon E \to X$ be a real vector bundle of rank n. The i-th Pontryagin class of π is defined as

$$p_i(\pi) := (-1)^i c_{2i}(\pi \otimes \mathbb{C}) \in H^{4i}(X;\mathbb{Z})$$

If ω is a complex vector bundle of rank n we define its i-th Pontryagin class as

$$p_i(\omega) := p_i(\omega_{\mathbb{R}}) = (-1)^i c_{2i}(\omega \oplus \overline{\omega})$$

4.4.3 Remark. Note that $p_i(\pi) = 0$ for all $i > n/2$.

4.4.4 Definition. The total Pontryagin class is

$$p(\pi) = 1 + p_1(\pi) + \cdots \in H^*(X;\mathbb{Z})$$

4.4.5 Remark. We would Pontryagin classes to satisfy a product formula. Since we have ignored odd degree classes, this is a bit more complicated to state.

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[16] A deeper theorem of Adams is that it is parallelizable only when $m = 1, 3, 7$.

4.4.6 Theorem. *If π_1 and π_2 are real vector bundles on a space X, then*

$$p(\pi_1 \oplus \pi_2) = p(\pi_1) \smile p(\pi_2) \text{ mod 2-torsion.}$$

Proof. We have

$$(\pi_1 \oplus \pi_2) \otimes \mathbb{C} \cong (\pi_1 \otimes \mathbb{C}) \oplus (\pi_2 \otimes \mathbb{C}).$$

Therefore,

$$\begin{aligned} p_i(\pi_1 \oplus \pi_2) &= (-1)^i c_{2i}((\pi_1 \oplus \pi_2) \otimes \mathbb{C}) \\ &= (-1)^i c_{2i}((\pi_1 \otimes \mathbb{C}) \oplus (\pi_2 \otimes \mathbb{C})). \end{aligned}$$

Now we compute that

$$\begin{aligned} c_{2i}((\pi_1 \otimes \mathbb{C}) \oplus (\pi_2 \otimes \mathbb{C})) &= \sum_{k+\ell=2i} c_k(\pi_1 \otimes \mathbb{C}) \smile c_\ell(\pi_2 \otimes \mathbb{C}) \\ &= \sum_{a+b=i} c_{2a}(\pi_1 \otimes \mathbb{C}) \smile c_{2b}(\pi_2 \otimes \mathbb{C}) \end{aligned}$$

where both statements hold modulo 2-torsion. The result follows.

□

4.4.7 Definition. If M is a real smooth manifold we define

$$p(M) := p(TM)$$

If M is a complex manifold, we define

$$p(M) := p((TM)_\mathbb{R}).$$

4.4.8 Theorem. *The total Chern classes and Pontryagin classes of the complex projective space $\mathbb{C}P^n$ are given by*

$$c(\mathbb{C}P^n) = (1+c)^{n+1}$$

and

$$p(\mathbb{C}P^n) = (1+c^2)^{n+1}$$

where $c \in H^2(\mathbb{C}P^n;\mathbb{Z})$ is a generator.

Proof. The computation of $c(\mathbb{C}P^n)$ is very similar to that of $w(\mathbb{R}P^n)$ (Theorem 4.3.10): we first show that

$$T\mathbb{C}P^n \oplus \epsilon^1 \simeq \gamma_1^{\oplus n+1}$$

where ϵ^1 is the trivial complex line bundle on $\mathbb{C}P^n$ and γ_1 is the canonical line bundle over $\mathbb{C}P^n$. This map is classified by the inclusion map $\mathbb{C}P^n \to \mathbb{C}P^\infty$ and so $c_1(\gamma_1) = c$, the generator of $H^2(\mathbb{C}P^\infty;\mathbb{Z}) = H^2(\mathbb{C}P^n;\mathbb{Z})$. Using the Whitney sum formula we have

$$c(\mathbb{C}P^n) = c(T\mathbb{C}P^n \oplus \epsilon^1) = c(T\mathbb{C}P^n) = c(\gamma_1)^{n+1} = (1+c)^{n+1}$$

or in other words, that

$$c_i(\mathbb{C}P^n) = \binom{n+1}{i} c^i.$$

It follows that

$$c(\overline{\mathbb{C}P^n}) = (1-c)^{n+1}.$$

Therefore,

$$\begin{aligned} c((T\mathbb{C}P^n)_{\mathbb{R}} \otimes \mathbb{C}) &= c(T\mathbb{C}P^n \oplus \overline{T\mathbb{C}P^n}) \\ &= c(T\mathbb{C}P^n) \smile c(\overline{T\mathbb{C}P^n}) \\ &= (1-c^2)^{n+1}. \end{aligned}$$

In particular,

$$p_i(\mathbb{C}P^n) = (-1)^i c_{2i}(T\mathbb{C}P^n \oplus \overline{T\mathbb{C}P^n}) = \binom{n+1}{i} c^{2i}.$$

so that

$$p(\mathbb{C}P^n) = (1+c^2)^{n+1}.$$ □

We now return to the embedding problem.

4.4.9 Proposition. *There is no embedding of* $\mathbb{C}P^2$ *into* $\mathbb{R}^5$.

Proof. Note that after forgetting the complex structure, $\mathbb{C}P^2$ is a 4-dimensional real dimensional manifold. We will use Pontryagin classes to find a minimal k for which there can be an emedding $\mathbb{C}P^2 \to \mathbb{R}^{4+k}$. Let $T(\mathbb{C}P^2)_{\mathbb{R}}$ be the realization of the tangent bundle for $\mathbb{C}P^2$, then then any embedding would give a normal real bundle ν^k of rank k such that

$$T(\mathbb{C}P^2)_{\mathbb{R}} \oplus \nu^k \cong \epsilon^{4+k}$$

By Theorem 4.4.8 we have

$$p(\mathbb{C}P^2) = (1+c^2)^3 = 1 + 3c^2 \in H^*(\mathbb{C}P^2;\mathbb{Z}) \cong \mathbb{Z}[c]/(c^3).$$

Using that $H^*(\mathbb{C}P^2;\mathbb{Z})$ has no 2-torsion, we see from Theorem 4.4.6

$$p(\mathbb{C}P^2) \cdot p(\nu^k) = 1,$$

so that

$$p(\nu^k) = 1 - 3c^2.$$

In particular, $p_1(\nu^k) \neq 0$. Finally, we observe that if $p_1(\nu^k) = 0$, then $1 \leq k/2$, i.e., $k \geq 2$, so that the minimal possible embedding is $\mathbb{C}P^2 \to \mathbb{R}^6$. □

A

A nice category of topological spaces

A.1 The compact open topology

In this appendix we briefly discuss how to give the set of continuous maps between topological spaces X and Y a topology, such that the product is left adjoint to the Hom functor. To begin, we fix some notation.

A.1.1 Remark. Let X and Y be topological spaces. Let $M(X, Y)$ denote the *set* of continuous homomorphisms from X to Y. There is an evaluation map

$$e' \colon \operatorname{Hom}_{Sets}(X, Y) \times X \to Y$$

given by $e'(f, x) = f(x)$. This restricts to a function

$$e \colon M(X, Y) \times X \to Y.$$

A.1.2 Definition. A topology on $M(X, Y)$ is called admissible if e is continuous with respect to this topology.

A.1.3 Remark. It is possible that $M(X, Y)$ has no admissible topologies.

A.1.4 Definition. The compact-open topology on $M(X, Y)$ has as a sub-base the family of sets

$$U^K = \{f \in M(X, Y) \mid f(K) \subseteq U\}$$

where $K \subseteq U$ is compact and U is open in Y.

A.1.5 Proposition. *If X is a locally compact[1] Hausdorff space, the the compact-open topology on $M(X, Y)$ is admissible.*

A.1.6 Remark. The compact-open topology is the coarsest admissible topology: for any admissible topology τ we have $\tau_{\mathrm{co}} \subseteq \tau$.

A.1.7 Remark. Suppose we have sets X, Y, X. Then there is an adjoint equivalence

$$\operatorname{Hom}_{Sets}(X \times Y, Z) \underset{\psi}{\overset{\phi}{\underset{\sim}{\rightleftarrows}}} \operatorname{Hom}_{Sets}(X, \operatorname{Hom}_{Sets})(Y, Z)$$

given by

$$\phi(f)(x)(y) = f(x, y) \quad \text{and} \quad \psi(g)(x, y) = g(x)(y).$$

---

[1] i.e., every point in X has a compact neighborhood

A.1.8 Proposition. *If X, Y, X are topological spaces with Y Hausdorff, locally compact, then*

$$\phi\colon M(X \times Y, Z) \xrightarrow{\cong} M(X, M(Y, Z))$$

is an isomorphism of sets. If X is Hausdorff, then it is a homeomorphism (using the compact-open topology).