

1

Bundle theory

1.1 Locally trivial bundles

We begin with what we will call a locally trivially bundle.¹

1.1.1 Definition. A map $\pi: E \rightarrow B$ is a locally trivial bundle with fiber F if the following conditions hold:

1. Each point $b \in B$ has a neighborhood U such that $\pi^{-1}(U_b) \xrightarrow{h_b} U_b \times F$.
2. The following diagram commutes

$$\begin{array}{ccc} \pi^{-1}(U_b) & \xrightarrow{h_b} & U_b \times F \\ & \searrow \pi & \swarrow \text{pr}_1 \\ & U_b & \end{array}$$

The maps h_b are called the *local trivializations* of the bundle.

1.1.2 Example. Let $E = B \times F$ and $\pi: E = B \times F \rightarrow B$ the projection map. This is called the trivial bundle.

1.1.3 Example. If F is discrete, then a locally trivial bundle with fiber F is a covering map.

1.1.4 Example. The Möbius band is a locally trivial bundle with fiber S^1 , see Figure 1.1. We will return to this example in due course.

1.1.5 Remark. We can write a locally trivial bundle as $F \rightarrow E \rightarrow B$, which is reminiscent of the notation for a fibration. In fact, fiber bundles over paracompact base spaces are always fibrations.² More generally, any locally trivial bundle is a Serre fibration.

1.1.6 Remark. Let us unwind the definition of a locally trivial bundle a little more. Let $\pi: E \rightarrow B$ be a locally trivial bundle with fiber F . From the definition we can cover B by a family of open sets $\{U_\alpha\}$ such that each inverse image $\pi^{-1}(U_\alpha)$ is fiberwise homeomorphic to $U_\alpha \times F$. This gives a system of homeomorphisms

$$\phi_\alpha: U_\alpha \times F \rightarrow \pi^{-1}(U_\alpha).$$

Observe that if $V \subseteq U_\alpha$ then the restriction of ϕ_α to $V \times F$ gives the homeomorphism with $\pi^{-1}(V)$. Hence on $U_\alpha \cap U_\beta$ there are two

¹ Confusingly, some books will also call this a fiber bundle. We will see why later.

² A space is paracompact if every open cover has an open refinement that is locally finite.

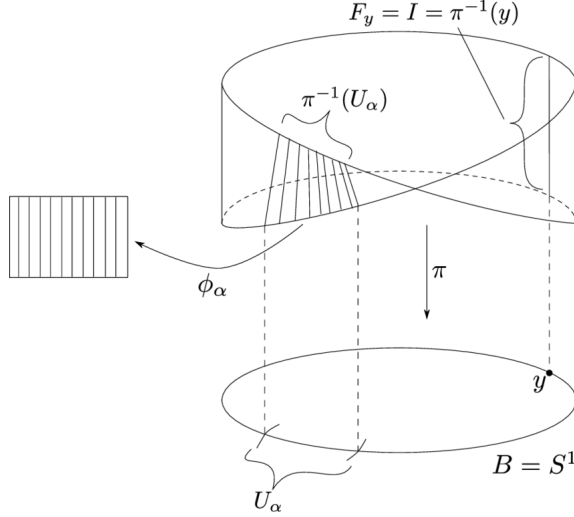


Figure 1.1: The Möbius band

fiberwise homeomorphisms

$$\phi_\alpha: (U_\alpha \cap U_\beta) \times F \rightarrow \pi^{-1}(U_\alpha \cap U_\beta)$$

$$\phi_\beta: (U_\alpha \cap U_\beta) \times F \rightarrow \pi^{-1}(U_\alpha \cap U_\beta)$$

Consider the following commutative diagram

$$\begin{array}{ccccc}
 (U_\alpha \cap U_\beta) \times F & \xrightarrow[\cong]{\phi_\alpha} & \pi^{-1}(U_\alpha \cap U_\beta) & \xrightarrow[\cong]{\phi_\beta^{-1}} & (U_\alpha \cap U_\beta) \times F \\
 & \searrow & \downarrow & \swarrow & \\
 & & U_\alpha \cap U_\beta & &
 \end{array}$$

Let $\phi_{\alpha\beta}$ denote the top composite $\phi_\beta^{-1}\phi_\alpha$. Then the locally trivially bundle is completely determined by the base B , the fiber F , the covering U_α and the homeomorphisms $\phi_{\alpha\beta}$. Roughly speaking, E should be thought of as the cartesian product of the $U_\alpha \times F$ with some identifications by the $\phi_{\alpha\beta}$.

1.1.7 Definition. The open sets U_α are called *charts*, the family U_α the *atlas of charts*, the homeomorphisms ϕ_α are called the *coordinate homeomorphisms* and the $\phi_{\alpha\beta}$ are called the *transition functions*.

1.1.8 Remark. In order for homeomorphisms $\phi_{\alpha\beta}$ to be the transition functions of a locally trivial bundle, they must satisfy a number of conditions. For example,

$$\phi_{\alpha\alpha} = \text{id}$$

and

$$\phi_{\gamma\alpha}\phi_{\beta\gamma}\phi_{\alpha\beta} = \text{id}$$

on any triple $(U_\alpha \cap U_\beta \cap U_\gamma) \cap F$. Taking $\gamma = \alpha$ we get

$$\phi_{\alpha\beta}\phi_{\beta\alpha} = \text{id}.$$

In fact, these conditions suffice to reconstruct the locally trivial bundle for the base, the fiber, atlas and homeomorphisms. Indeed, set $E = E' / \sim$ where

$$E' = \bigcup_{\alpha} (U_\alpha \times F)$$

and for $(x, f) \in U_\alpha \times F$ and $(y, g) \in U_\beta \times F$ we have $(x, f) \sim (y, g) \iff x = y \in (U_\alpha \cap U_\beta)$ and $(y, g) = \phi_{\alpha\beta}(x, f)$. It is a rather tedious exercise to show that this determines a locally trivial bundle.

1.1.9 Definition. Two locally trivial bundles $\pi: E \rightarrow B$ and $\pi': E' \rightarrow B'$ are isomorphic if there is a homeomorphism $\psi: E \rightarrow E'$ such that the diagram

$$\begin{array}{ccc} E & \xrightarrow{\psi} & E' \\ & \searrow \pi & \swarrow \pi' \\ & B & \end{array}$$

commutes. (Note that this implies that there is a homeomorphism $F \rightarrow F'$ between the fibers as well).

1.1.10 Theorem. Two systems of transition functors $\phi_{\beta\alpha}$ and $\phi'_{\beta\alpha}$ define isomorphic locally trivial bundles iff there exists fiber preserving homeomorphisms

$$h_\alpha: U_\alpha \times F \rightarrow U_\alpha \times F$$

such that $\phi_{\beta\alpha} = h_\beta^{-1} \phi'_{\beta\alpha} h_\alpha$.

Proof. First, we suppose that the two bundles are isomorphic, so in particular there is a homeomorphism $\psi: E \rightarrow E'$. We let

$$h_\alpha := \phi_\alpha'^{-1} \psi^{-1} \phi_\alpha: U_\alpha \times F \rightarrow U_\alpha \times F.$$

Then we have

$$\begin{aligned} h_\beta^{-1} \phi'_{\beta\alpha} h_\alpha &= \phi_\beta^{-1} \psi^{-1} \phi_\beta' \phi_{\beta\alpha}' \phi_\alpha^{-1} \psi^{-1} \phi_\alpha \\ &= \phi_\beta^{-1} \psi^{-1} \phi_\beta' \phi_\beta'^{-1} \phi_\alpha' \phi_\alpha^{-1} \psi^{-1} \phi_\alpha = \phi_{\beta\alpha}. \end{aligned}$$

Conversely, if the relations hold, then we set $\psi = \phi_\alpha h_\alpha^{-1} \phi_\alpha'^{-1}$. A similar argument then shows that $\phi_\beta h_\beta^{-1} \phi_\beta'^{-1} = \phi_\alpha h_\alpha^{-1} \phi_\alpha' - 1_\alpha$. \square

1.1.11 Remark. If π is (isomorphic to) a trivial bundle, then all transition functions can be chosen to be the identity. One can use the previous theorem to show that a bundle is not isomorphic to a trivial bundle.

1.1.12 Example. After this discussion, let us return to the example of the Möbius bundle (Example 1.1.4). One can think of this as the space

$$E = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1\} / \sim$$

where we identify $(0, y)$ and $(1, 1 - y)$ for each $y \in [0, 1]$. The projection maps E to $I_x = \{0 \leq x \leq 1\}$ with the endpoints identified, that is, onto the circle. To see that this is a bundle we use the atlas

$$U_\alpha = \{0 \leq x \leq 1\}, \text{ and } U_\beta = \{0 \leq x < 1/2\} \cup \{1/2 < x \leq 1\}.$$

We define

$$\phi_\alpha: U_\alpha \times I_y \rightarrow E, \quad \phi_\alpha(x, y) = (x, y),$$

and

$$\phi_\beta: U_\beta \times I_y \rightarrow E$$

by

$$\phi_\beta = \begin{cases} (x, y) & \text{for } 0 \leq x \leq 1/2, \\ (x, 1 - y) & \text{for } 1/2 < x \leq 1. \end{cases}$$

The intersection of these two charts is the union $(0, 1/2) \cup (1/2, 1)$, and the transition functions have the form

$$\phi_{\beta\alpha} = (x, y) \text{ for } 0 < x < 1/2$$

and

$$\phi_{\beta\alpha} = (x, 1 - y) \text{ for } 1/2 < x < 1.$$

One can check from Remark 1.1.11 that the Möbius bundle is not isomorphic to a trivial bundle.

We now give some more examples which will be useful in our study of characteristic classes.

1.1.13 Definition. For $n < k$ the n -th Stiefel manifold associated to \mathbb{R}^k is defined as

$$V_n(\mathbb{R}^k) = \{n\text{-frames in } \mathbb{R}^k\}$$

where an n -frame in \mathbb{R}^k is a tuple $\{v_1, \dots, v_n\}$ of orthonormal vectors in \mathbb{R}^k , i.e., v_1, \dots, v_n are pairwise orthonormal, $\langle v_i, v_j \rangle = \delta_{ij}$. We give $V_n(\mathbb{R}^k)$ the subspace topology induced by thinking of it as a subspace of $S^{k-1} \times \dots \times S^{k-1}$ (n -copies of S^{k-1}).

1.1.14 Example. A 1-frame is nothing but a unit vector, so the Stiefel manifold $V_1(\mathbb{R}^k)$ is the unit sphere in \mathbb{R}^k , i.e., $V_1(\mathbb{R}^k) \cong S^{k-1}$. On the other hand, an n -frame is an ordered basis, so $V_n(\mathbb{R}^n) \cong O(n)$.

1.1.15 Definition. The n -th Grassmannian associated to \mathbb{R}^k is defined as

$$G_n(\mathbb{R}^k) = \{n\text{-dimensional vector subspaces in } \mathbb{R}^k\}$$

There is a map $p: V_n(\mathbb{R}^k) \rightarrow G_n(\mathbb{R}^k)$ sending $\{v_1, \dots, v_n\}$ to the span, which is surjective by Gram-Schmidt, and we give $G_n(\mathbb{R}^k)$ the quotient topology.

1.1.16 Example. We have $G_1(\mathbb{R}^k)$ is the space of lines through the origin in k -space, so $G_1(\mathbb{R}^k) \simeq \mathbb{R}P^{k-1}$.

1.1.17 Lemma. For $k > n$ the quotient map $p: V_n(\mathbb{R}^k) \rightarrow G_n(\mathbb{R}^k)$ is a locally trivial bundle with fiber $V_n(\mathbb{R}^n) \cong O(n)$, i.e., we have a locally trivial bundle

$$O(n) \rightarrow V_n(\mathbb{R}^k) \xrightarrow{p} G_n(\mathbb{R}^k). \quad (1.1.18)$$

Similarly, for $m < n \leq k$ there are locally trivial bundles

$$V_{n-m}(\mathbb{R}^k) \rightarrow V_n(\mathbb{R}^k) \xrightarrow{p} V_m(\mathbb{R}^k). \quad (1.1.19)$$

where the map p takes $\{v_1, \dots, v_n\}$ to $\{v_1, \dots, v_m\}$. Taking $k = n$ we get a locally trivial bundle

$$O(n - m) \rightarrow O(n) \xrightarrow{p} V_m(\mathbb{R}^n). \quad (1.1.20)$$

1.1.21 Example. Taking $m = 1$ in (1.1.20) we get a locally trivial bundle

$$O(n-1) \rightarrow O(n) \rightarrow S^{n-1}.$$

Here the first map takes A to $\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$ and the second takes B to Bu for $u \in S^{n-1}$ some unit vector. In particular, this identifies S^{n-1} as an orbit space $S^{n-1} \cong O(n)/O(n-1)$.

Exercise 1. Use the fibrations

$$O(n-k) \rightarrow O(n) \rightarrow V_k(\mathbb{R}^n)$$

to show that

$$\pi_i(O(n-1)) \simeq \pi_i(O(n)) \text{ for } i < n-2$$

and

$$\pi_i(V_k(\mathbb{R}^n)) = 0$$

for $i < n-k-1$.

1.1.22 Definition. We have infinite versions of the Stiefel manifold and Grassmanian:

$$V_n(\mathbb{R}^\infty) := \bigcup_{k=1}^{\infty} V_n(\mathbb{R}^k) \quad G_n(\mathbb{R}^\infty) := \bigcup_{k=1}^{\infty} G_n(\mathbb{R}^k)$$

1.1.23 Remark. We get a fiber sequence

$$O(n) \rightarrow V_n(\mathbb{R}^\infty) \rightarrow G_n(\mathbb{R}^\infty).$$

1.1.24 Proposition. $V_n(\mathbb{R}^\infty)$ is contractible.

Proof. As in the exercise, we deduce that $\pi_i(V_n(\mathbb{R}^\infty)) = 0$ for all i . We can give $V_n(\mathbb{R}^\infty)$ the structure of a CW-complex, and so the claim follows from ?? \square

1.1.25 Remark. One can repeat the same story using \mathbb{C} or \mathbb{H} instead of \mathbb{R} . In the first case, all instances of $O(n)$ get replaced by $U(n)$, and in the second case by $\text{Sp}(n)$.

1.1.26 Example (The tangent bundle to S^2). Let $S^2 = \{(x_0, x_1, x_2) \in \mathbb{R}^3 \mid x_0^2 + x_1^2 + x_2^2 = 1\}$. Recall that the tangent space at a point $x \in S^2$ is defined by $T_x S^2 = \{\xi \in \mathbb{R}^3 \mid x \perp \xi\}$. We then define $TS^2 = \bigsqcup_{x \in S^2} T_x S^2$. This can be topologized as a subspace of $\mathbb{R}^3 \times \mathbb{R}^3$ when we write

$$TS^2 = \{(x, \xi) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid x \in S^2, x \perp \xi\}.$$

There is a natural projection map $p: TS^2 \rightarrow S^2$ sending the pair (x, ξ) to x , which we claim is a locally trivial bundle with fiber \mathbb{R}^2 . To see this is a locally trivial bundle, let U be the open subset of S^2 defined by $x_3 > 0$. We will show how to construct the local trivialization on this open subset.

If $\xi = (\xi_1, \xi_2, \xi_3)$ then we have the relation

$$x_1\xi_1 + x_2\xi_2 + x_3\xi_3 = 0$$

or

$$\xi_3 = -(x_1\xi_1 + x_2\xi_2)/x_3.$$

We define

$$\phi: U \times \mathbb{R}^2 \rightarrow p^{-1}(U)$$

by

$$\phi(x_1, x_2, x_3, \xi_1, \xi_2) = (x_1, x_2, x_3, \xi_1, \xi_2, -(x_1\xi_1 + x_2\xi_2)/x_3),$$

which gives the required chart for this open subset.

1.1.27 Remark. More generally, for any smooth manifold X of dimension n , we have a locally trivial bundle $\pi: TX \rightarrow X$ with fiber \mathbb{R}^n .

1.2 The structure group of locally trivial bundles

We recall that the on the intersection of two local trivializations we constructed a homeomorphism

$$\phi_{\beta\alpha}: (U_\alpha \cap U_\beta) \times F \rightarrow \pi^{-1}(U_\alpha \cap U_\beta) \rightarrow (U_\alpha \cap U_\beta) \times F$$

Unwinding the definition, the map ϕ is completely determine by a map $\Phi: U \rightarrow \text{Homeo}(F)$, where $\text{Homeo}(F)$ denotes the group of all homeomorphisms of the fiber F (we call Φ coordinate transformations).³ Indeed, we have

$$\phi_{\alpha\beta}(x, f) = (x, \Phi(x)(f))$$

In other words, instead of $\phi_{\alpha\beta}$ to determine a bundle we can instead specify a family of functions

$$\Phi_{\alpha\beta}(x, f): U_\alpha \cap U_\beta \rightarrow \text{Homeo}(F),$$

having values in the group $\text{Homeo}(F)$. Of course these are not arbitrary, but need to satisfy various compatibility conditions:

$$\Phi_{\alpha\alpha}(x) = \text{id}$$

and

$$\Phi_{\alpha\gamma}(x)\Phi_{\gamma\beta}(x)\Phi_{\beta\alpha}(x) = \text{id},$$

for $x \in U_\alpha \cap U_\beta \cap U_\gamma$.

1.2.1 Definition. Let E, B, F be topological spaces and G a topological group which acts freely on the space F . A continuous map $p: E \rightarrow B$ is a locally trivial bundle with fiber F and structure group G if there is an atlas $\{U_\alpha\}$ and the coordinate homeomorphisms

$$\phi_\alpha: U_\alpha \times F \rightarrow p^{-1}(U_\alpha)$$

³ If we choose the correct topology on $\text{Homeo}(F)$, namely the compact-open topology (for reasonable spaces at least), then this map is even continuous

such that the transition functions

$$\phi_{\beta\alpha} = \phi_{\beta}^{-1} \phi_{\alpha} : (U_{\alpha} \cap U_{\beta}) \times F \rightarrow (U_{\alpha} \cap U_{\beta}) \times F$$

have the form

$$\phi_{\beta\alpha}(x, f) = (x, \Phi_{\beta\alpha}(x)f)$$

where $\Phi_{\beta\alpha} : (U_{\alpha} \cap U_{\beta}) \rightarrow G$ are continuous functions satisfying

$$\Phi_{\alpha\alpha}(x) = \text{id}$$

and

$$\Phi_{\alpha\gamma}(x)\Phi_{\gamma\beta}(x)\Phi_{\beta\alpha}(x) = \text{id},$$

1.2.2 Remark. Some words on terminology are useful. What we defined as a locally trivial bundle with fiber F , is exactly a locally trivial bundle with fiber F and structure group $\text{Homeo}(F)$. Either of these may also be called a *fiber bundle* or a *fiber bundle with structure group* G .

1.2.3 Remark. In diagrammatic form, to have structure group G means that we can find transition maps $\Phi'_{\alpha\beta}$ making the diagram commute:

$$\begin{array}{ccc} G & \xrightarrow{\quad} & \text{Homeo}(F) \\ & \nwarrow \Phi'_{\alpha\beta} \quad \nearrow \Phi_{\alpha\beta} & \\ & U_{\alpha} \cap U_{\beta} & \end{array}$$

1.2.4 Remark. Note that the structure group is not unique. For example, a bundle with structure group G may admit transition functions with values in a subgroup $H \leq G$. We say that the structure group G is reduced to subgroup H . More generally, if $\rho : G \rightarrow G'$ is a continuous homeomorphism of topological groups, and we are given a locally trivial bundle with structure group G and the transition functions $\alpha_{\alpha\beta} : U_{\alpha} \cap U_{\beta} \rightarrow G$, then a new locally trivial bundle with structure group G' may be constructed by

$$\phi'_{\alpha\beta}(x) = \rho(\phi_{\alpha\beta}(x)).$$

This operation is called the change of structure group.

Exercise 2. Show that a trivial bundle has trivial structure group. Conversely, if the structure group can be reduced to the trivial group then the bundle is (isomorphic to) a trivial bundle.

1.2.5 Example. Let us return to the Möbius bundle (Examples 1.1.4 and 1.1.12). We have that

$$\Phi_{\alpha\beta}(y) = y \quad \text{and} \quad \Phi_{\beta\alpha}(y) = 1 - y.$$

Note that $\Phi_{\beta\alpha} \circ \Phi_{\beta\alpha}(y) = 1 - (1 - y) = y = \Phi_{\alpha\beta}(y)$. Therefore, the group generated by $\Phi_{\alpha\beta}$ and $\Phi_{\beta\alpha}$ has order 2. In other words, the Möbius bundle has (or can be reduced to) structure group $\mathbb{Z}/2$.

1.2.6 *Example.* The tangent bundle $TS^2 \rightarrow S^2$ was considered in Example 1.1.26. The coordinate homeomorphisms

$$\phi: U \times \mathbb{R}^2 \rightarrow \mathbb{R}^3 \times \mathbb{R}^3$$

are defined by formulas that are linear with respect to the second argument. Hence that transition functions have values in the group of linear translations of the fiber $F = \mathbb{R}^2$, that is $G = GL_2(\mathbb{R})$. In fact, it can be shown that the structure group can be reduced to the subgroup $O(n)$ of orthonormal rotations.

1.3 Principal bundles

The most important example of a bundle for a us is a *principal bundle*. We postpone the definition to talk a little about group actions.

1.3.1 *Definition.* Let G be a topological group, and X a topological space. A left action of G on X is a continuous map $\mu: G \times X \rightarrow X$, satisfying $\mu(e, x) = x$ and $\mu(h, \mu(g, x)) = \mu(hg, x)$.

1.3.2 *Example.* The multiplication map $\mu: G \times G \rightarrow G$ defines a left action of G on itself. Similarly, if $H \subseteq G$, then $\mu|_{H \times G}: H \times G \rightarrow G$ defines an action of H on G .

1.3.3 *Remark.* The map $\mu: G \times X \rightarrow X$ is adjoint to a map $ad(u): G \rightarrow \text{Homeo}(X)$, where $\text{Homeo}(X)$ has the compact-open topology. If X is nice (specifically, locally compact Hausdorff) then a continuous map $G \rightarrow \text{Homeo}(X)$ gives rise to a group action $G \times X \rightarrow X$.

1.3.4 *Example.* The adjoint to the multiplication $\mu: G \times G \rightarrow G$ is the map $G \rightarrow \text{Homeo}(G)$ given by $g \mapsto f_g$, where $f_g: G \rightarrow G$ is given by $f_g(x) = gx$.

1.3.5 *Remark.* We recall some standard terminology associated to a group action of G on X :

- (i) The orbit of a point $x \in X$ is the set $Gx = \{g \cdot x \mid g \in G\}$.
- (ii) The orbit space X/G is the quotient space X/\sim where $x \sim g \cdot x$.
- (iii) The fixed set $X^G := \{x \in X \mid g \cdot x = x \text{ for all } g \in G\}$.
- (iv) An action is free⁴ if $g \cdot x \neq x$ for all $x \in X$ and all $g \neq e$. (i.e., $gx = x$ for all x implies $g = e$).
- (v) The stabilizer (or isotropy) group at x is $G_x = \{g \in G \mid g \cdot x = x\} \subseteq G$.
- (vi) An action is effective if the adjoint $G \rightarrow \text{Homeo}(X)$ is injective, or equivalently if $\bigcap_{x \in X} G_x = \{e\}$.⁵
- (vii) A group action is transitive if and only if it has exactly one orbit, i.e., there exists $x \in X$ such that $Gx = X$.

⁴ Here, meaning fixed-point free

⁵ Note that a free action is effective, but not vice-versa.

1.3.6 *Example.* To motivate the definition of a principal G -bundle we first consider an example. We recall that Hopf fibration $S^1 \rightarrow$

$S^3 \rightarrow S^2$. By what we have already discussed, this is a locally trivial bundle.

Let us show this directly. Consider the sphere S^3 as a subset of \mathbb{C}^2 defined by the equation

$$|z_0|^2 + |z_1|^2 = 1.$$

Then the map $S^3 \rightarrow S^2 = \mathbb{C}P^1$ sends (z_0, z_1) to the projective coordinate $[z_0 : z_1]$. We take the atlas consisting of the charts

$$U_0 = \{[z_0, z_1] : z_0 \neq 0\}$$

$$U_1 = \{[z_0, z_1] : z_1 \neq 0\}$$

The points of the chart U_0 are parametrized by the complex parameter $w_0 = z_1/z_0 \in \mathbb{C}$, while points of the chart U_1 are parameterized by $w_1 = z_0/z_1 \in \mathbb{C}$. The homeomorphisms

$$\phi_0: p^{-1}(U_0) \rightarrow S^1 \times \mathbb{C} = S^1 \times U_0 \text{ and } \phi_0: p^{-1}(U_1) \rightarrow S^1 \times \mathbb{C} = S^1 \times U_1$$

are given by

$$\begin{aligned} \phi_0(z_0, z_1) &= \left(\frac{z_0}{|z_0|}, \frac{z_1}{z_0} \right) = \left(\frac{z_0}{|z_0|}, [z_0 : z_1] \right) \\ \phi_0(z_0, z_1) &= \left(\frac{z_1}{|z_1|}, \frac{z_0}{z_1} \right) = \left(\frac{z_1}{|z_1|}, [z_0 : z_1] \right) \end{aligned}$$

One can explicitly write down inverses for these maps and check that they are homeomorphisms. The transition function

$$\phi_{01}: (U_0 \cap U_1) \times S^1 \rightarrow (U_0 \cap U_1) \times S^1$$

is defined by the formula

$$\phi_{01}([z_0 : z_1], \lambda) = \left([z_0 : z_1], \frac{z_1|z_0|}{z_0|z_1|} \lambda \right).$$

In other words, $\Phi_{01}: U_0 \cap U_1 \rightarrow \text{Homeo}(S^1)$ is defined by

$$[z_0 : z_1] \mapsto (\lambda \mapsto \frac{z_1|z_0|}{z_0|z_1|} \lambda),$$

where $\frac{z_1|z_0|}{z_0|z_1|} \in S^1$. We can then define a map $\Phi'_{01}: U_0 \cap U_1 \rightarrow S^1$ by $[z_0, z_1] \mapsto \frac{z_1|z_0|}{z_0|z_1|}$. Unwinding the definitions, the diagram

$$\begin{array}{ccc} S^1 & \xrightarrow{ad(\mu)} & \text{Homeo}(S^1) \\ & \swarrow \Phi'_{01} \quad \searrow \Phi_{01} & \\ & U_0 \cap U_1 & \end{array}$$

commutes, where $ad(\mu)$ is adjoint to the multiplication map $\mu: S^1 \times S^1 \rightarrow S^1$.⁶

⁶ Here we use Example 1.3.4.

In other words, the structure group of the Hopf bundle can be reduced to S^1 , where the action of S^1 on the fiber, namely S^1 again, is given by the multiplication map $\mu: S^1 \times S^1 \rightarrow S^1$.

This leads us to the following definition.

1.3.7 Definition. A locally trivial bundle with structure group G is called a principal G -bundle if $F = G$ and the action of the group G on F is defined by left translations, i.e., the multiplication map $\mu: G \times G \rightarrow G$ as in Example 1.3.4.

1.3.8 Example. Restated, Example 1.3.6 shows that $S^1 \rightarrow S^3 \rightarrow S^2$ is a principal S^1 -bundle.

1.3.9 Theorem. Let $p: E \rightarrow B$ be a principal G -bundle, with

$$\phi_\alpha: U_\alpha \times G \rightarrow p^{-1}(U_\alpha)$$

coordinate homeomorphisms. Then there is a right action⁷ of the group G on the total space E such that

- (i) The right action is fiberwise,⁸ i.e., $p(x) = p(xg)$ for $x \in E, g \in G$.
- (ii) The homeomorphism ϕ_α^{-1} transforms the right action of the group G on the total space into right translations on the second factor, i.e.,

$$\phi_\alpha(x, f)g = \phi_\alpha(x, fg), \quad x \in U_\alpha, f, g \in G \quad (1.3.10)$$

- (iii) G acts freely and transitively on the right of E .

Proof. Unwinding the definitions, the transition functions $\phi_{\beta\alpha}$ are given by

$$\phi_{\beta\alpha}(x, f) = (x, \Phi_{\beta\alpha}(x)f),$$

for some

$$\Phi_{\beta\alpha}: U_\alpha \cap U_\beta \rightarrow G.$$

Note that any arbitrary $e \in E$ is of the form $e = \phi_\alpha(x, f)$ for some α , then we define a right action of G on E via the formula (1.3.10). We need to check that this is well defined, i.e., does not depend on the choice α . So assume that $e = \phi_\beta(x, f')$ as well, then we are required to show that

$$\phi_\alpha(x, fg) = \phi_\beta(x, f'g).$$

This says that

$$(x, f'g) = \phi_\beta^{-1}\phi_\alpha(x, fg) = \phi_{\beta\alpha}(x, fg) = (x, \Phi_{\beta\alpha}(x)fg).$$

So equivalently, we are required to show

$$f'g = \Phi_{\beta\alpha}(x)fg.$$

But this follows because $f' = \phi_{\beta\alpha}(x)f$. □

We omit the proof of the following difficult theorem, which gives many more examples of principal G -bundles.

1.3.11 Theorem. Suppose X is a compact Hausdorff space and G is a compact Lie group acting freely on X . Then the orbit map $X \rightarrow X/G$ is a principal G bundle.

⁷ Why on the right? This is just because the action of G on E is given by left translation. If we made the opposite convention, then E would get a left action of G .

⁸ In other words, it preserves fibers.

1.3.12 Example. Let $H \subseteq G$ be a closed subgroup. Then H acts freely on G , and $G \rightarrow G/H$ is a principal H -bundle. Moreover, if $K \subseteq H$ is a subgroup, then H acts on H/K , and $G/K \rightarrow G/H$ is a principal H/K -bundle.

1.3.13 Example. Let $G = O(n)$, and consider the subgroup $H = O(k) \times O(n-k)$ defined by $(A, B) \mapsto \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$. We also consider the subgroup $K = O(n-k) \subseteq O(n)$, determine by the inclusion $A \mapsto \begin{pmatrix} I & 0 \\ 0 & A \end{pmatrix}$. We then have the principal $O(k) \times O(n-k)$ -bundle

$$O(k) \times O(n-k) \rightarrow O(n) \rightarrow G_k(\mathbb{R}^n)$$

and the principal $O(n-k)$ -bundle

$$O(n-k) \rightarrow O(n) \rightarrow V_k(\mathbb{R}^n)$$

as well as the principal $O(k)$ -bundle

$$O(k) \rightarrow V_k(\mathbb{R}^n) \rightarrow G_k(\mathbb{R}^n).$$

1.4 The associated bundle construction

1.4.1 Remark. Let us recall that a locally trivial bundle $p: E \rightarrow B$ with structure group G and fiber F is determined by the following data:

1. An open covering $\{U_\alpha\}$ of B .
2. Maps $\phi_{\alpha\beta}: U_\alpha \cap U_\beta \rightarrow G$, and
3. An action of G on F , or equivalently a map $G \rightarrow \text{Homeo}(F)$.

Note that if p is a principal G -bundle, then the last data in the above list is already included: the action of G on G is given by left multiplication. Thus a principal G -bundle $p: P \rightarrow B$ is determined by the following data:

1. An open covering $\{U_\alpha\}$ of B .
2. Maps $\phi_{\alpha\beta}: U_\alpha \cap U_\beta \rightarrow G$, and

Therefore, by removing the third data from a fiber bundle, we should get a principal G -bundle, and conversely given a principal G -bundle by adding the data of an action of a space F on G , we should be able to construct a locally trivial bundle with fiber F on the same base space. This is essentially true; we have the following theorem:

1.4.2 Theorem. Let G be a topological group and fix an action of G on a space F . Suppose that an open cover $\{U_\alpha\}$ of B and families of maps $\Phi = \{\phi_{\alpha\beta}: U_\alpha \cap U_\beta \rightarrow G\}$ are given. Suppose that that action of G on F is effective, then there is a bijection

$$\left\{ \begin{array}{l} \text{principal } G\text{-bundles over } B \\ \text{with coordinate transformations } \Phi \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{Locally trivial bundles over } B \\ \text{with fiber } F, \text{ structure group } G, \\ \text{and coordinate transformations } \Phi \end{array} \right\}$$

It is not too hard to define a map in the \rightarrow direction.

1.4.3 Construction . Let G be a topological group and assume that G acts on topological spaces P and F continuously from the right and left respectively. Define a left action of G on $P \times F$ by

$$g(p, f) = (x \cdot g^{-1}, gy).$$

Let $E \times_G F := E \times F / G$ denote the quotient (orbit) space, and $\omega: E \times_G F \rightarrow E/G$ the projection map.

1.4.4 Definition. Let $p: E \rightarrow B$ be a principal G -bundle and fixed a G -space F . The projection map $\omega: E \times_G F \rightarrow B$ sending $[x, y] \mapsto p(x)$ is called the associated bundle with fiber F .

We should of course verify that this is actually a locally trivial bundle.

1.4.5 Theorem. *The map $\omega: E \times_G F \rightarrow B$ defines a locally trivial bundle with structure group G and fiber F .*

Proof. Let $\{\Phi_\alpha: p^{-1}(U_\alpha) \rightarrow U_\alpha \times G\}$ be local trivializations. We wish to construct local trivializations

$$\Psi_\alpha: \omega^{-1}(U_\alpha) \rightarrow U_\alpha \times F.$$

Note that

$$\begin{aligned} \omega^{-1}(U_\alpha) &= \{[x, y] \in E \times_G F \mid \omega([x, y]) \in U_\alpha\} \\ &= \{[x, y] \in E \times_G F \mid p(x) \in U_\alpha\} \\ &= p^{-1}(U_\alpha) \times_G F \end{aligned}$$

We then define the required map Ψ_α as the composite

$$\omega^{-1}(U_\alpha) = p^{-1}(U_\alpha) \times_G F \xrightarrow{\Phi_\alpha \times_G 1_F} (U_\alpha \times G) \times_G F \cong U_\alpha \times F$$

where the homeomorphism $(U_\alpha \times G) \times_G F \cong U_\alpha \times F$ is given by $[(x, g), y] \mapsto (x, gy)$.⁹ As a composite of homeomorphisms, this map is also a homeomorphism. A short diagram chase shows that it defines a locally trivial bundle. In order to see that the structure group is G , we must compute $\Psi_\beta \Psi_\alpha^{-1}(x, y)$ for $(x, y) \in U_\alpha \cap U_\beta \times F$. A rather tedious computation shows that

$$\Psi_\beta \Psi_\alpha^{-1}(x, y) = (x, \Phi_{\alpha\beta}(x)y)$$

where $\Phi_{\alpha\beta}: U_\alpha \cap U_\beta \rightarrow G$ is a coordinate transformation for $p: E \rightarrow B$. Therefore, the associated bundle has structure group G , as claimed. \square

1.4.6 Example. Let $\pi: S^1 \rightarrow S^1, z \mapsto z^2$ be regarded as a principal $\mathbb{Z}/2$ -bundle.¹⁰ Let $F = [-1, 1]$, and let $\mathbb{Z}/2 = \{-1, 1\}$ act on F by multiplication. The associated bundle is then

$$S^1 \times_{\mathbb{Z}/2} [-1, 1] = S^1 \times [-1, 1] / (x, t) \sim (a(x), -t)$$

for $a: S^1 \rightarrow S^1$ the antipodal map. This is the Möbius bundle.

⁹ With inverse, $(x, y) \mapsto [(x, e), y]$.

¹⁰ Note that any regular cover is a principal bundle.

1.4.7 Remark. You might be wondering about the inverse map in Theorem 1.4.2. Let $p: E \rightarrow B$ be a principal bundle with fiber F and structure group G . The associated principal bundle is constructed using a similar procedure as in Remark 1.1.8; we keep the base space B , the open chart $\{U_\alpha\}$ and the transition functions $\phi_{\alpha\beta}$, but we replace all instances of the fiber F by G , and allow G to act on itself by left translation. In simple terms: we forget the fiber F , and build a bundle which is principal out of the remaining data.

1.5 Operations on locally trivial bundles

The following is a fundamental operation on bundles.

1.5.1 Proposition. Let $\xi = (p: E \rightarrow B)$ and $\xi' = (p': E' \rightarrow B')$ be locally trivial bundles with fibers F and F' respectively. Then the product

$$p \times p': E \times E' \rightarrow B \times B'$$

is a locally trivial bundle with fiber $F \times F'$ and structure group $G \times G'$.

Proof. Let $\phi_\alpha: p^{-1}(U_\alpha) \rightarrow U_\alpha \times F$ and $\psi_\beta: p'^{-1}(V_\beta) \rightarrow V_\beta \times F'$ be local trivializations of ξ and ξ' respectively. Note that $\{U_\alpha \times V_\beta\}$ is an open covering of $B \times B'$. Note also that $(U_\alpha \times V_\beta) \cap (U_{\alpha'} \times V_{\beta'}) = (U_\alpha \cap U_{\alpha'}) \times (V_\beta \cap V_{\beta'})$, and using this identification we obtain maps

$$\Phi_{\alpha\alpha'} \times \Psi_{\beta\beta'}: (U_\alpha \cap U_{\alpha'}) \times (V_\beta \cap V_{\beta'}) \rightarrow G \times G'$$

which form coordinate transforms of the product bundle. \square

1.5.2 Corollary. Let $p: E \rightarrow B$ be a locally trivial bundle with fiber F and structure group G , then for any topological space X

$$p \times 1_X: E \times X \rightarrow B \times X$$

is a locally trivial bundle with fiber F and structure group G .

1.5.3 Remark. We recall the pullback construction for topological spaces: given $f: A \rightarrow C$ and $g: B \rightarrow C$, the pullback is

$$A \times_C B := \{(a, b) \in A \times B \mid f(a) = g(b)\}$$

along with the projection maps $A \times_C B \rightarrow A$ and $A \times_C B \rightarrow B$. This is a pullback in the categorical sense: given maps $q_1: Q \rightarrow A$ and $q_2: Q \rightarrow B$ as in the following diagram:

$$\begin{array}{ccccc} Q & & & & \\ & \searrow^{q_1} & & \searrow^{f} & \\ & & A \times_C B & \longrightarrow & A \\ & \searrow^{q_2} & \downarrow & & \downarrow \\ & & B & \xrightarrow{g} & C \end{array}$$

there exists a map $u: Q \rightarrow A \times_C B$ making the diagram commute.

1.5.4 Proposition. Let $\xi = (p: E \rightarrow B)$ be a locally trivial bundle with fiber F and structure group G . For any continuous map $f: X \rightarrow B$, the pullback $f^*(p): E \times_B X \rightarrow X$ is a locally trivial bundle with fiber F and structure group G .¹¹

¹¹ For brevity, we write $f^*(E) := E \times_B X$.

Proof. Let $\{U_\alpha\}_{\alpha \in A}$ be an open cover of B , then $\{f^{-1}(U_\alpha)\}_{\alpha \in A}$ forms an open cover of X .

Let $\phi_\alpha: p^{-1}(U_\alpha) \rightarrow U_\alpha \times F$ denote a local trivialization of ξ . We write $\phi_\alpha(e) = (p(e), \bar{\phi}_\alpha(e))$. We define a map $\psi_\alpha: f^*(p)^{-1}(V_\alpha) \rightarrow V_\alpha \times F$ as the composite

$$f^*(p)^{-1}(V_\alpha) \hookrightarrow V_\alpha \times p^{-1}(U_\alpha) \xrightarrow{1 \times \bar{\phi}_\alpha(e)} V_\alpha \times F.$$

We will show this is a homeomorphism by constructing a continuous inverse. Define γ_α as the composite

$$V_\alpha \times F \xrightarrow{\Delta \times 1} V_\alpha \times V_\alpha \times F \xrightarrow{1 \times f \times 1} V_\alpha \times U_\alpha \times F \xrightarrow{1 \times \phi_\alpha^{-1}} V_\alpha \times p^{-1}(U_\alpha),$$

i.e., $\gamma_\alpha(x, y) = (x, \phi_\alpha^{-1}(f(x), y))$.

Note that $p(\phi_\alpha^{-1}(f(x), y)) = f(x)$, and so $\gamma_\alpha(x, y) \in f^*(E)$. To see that γ_α and ψ_α are inverse, simply compute:

$$\begin{aligned} \gamma_\alpha \circ \psi_\alpha(x, e) &= \gamma_\alpha(x, \bar{\phi}_\alpha(e)) \\ &= (x, \phi_\alpha^{-1}(f(x), \bar{\phi}_\alpha(e))) \\ &= (x, \phi_\alpha^{-1}(p(e), \bar{\phi}_\alpha(e))) \\ &= (x, \phi_\alpha^{-1} \circ \phi_\alpha(e)) \\ &= (x, e) \end{aligned}$$

and

$$\begin{aligned} \psi_\alpha \circ \gamma_\alpha(x, y) &= \psi_\alpha(x, \phi_\alpha^{-1}(f(x), y)) \\ &= (x, \bar{\phi}_\alpha \circ \phi_\alpha^{-1}(f(x), y)) \\ &= (x, y). \end{aligned}$$

We leave it to the reader to verify the $f^*(E) \rightarrow X$ has structure group G .¹² □

¹² Hint: you should end up showing that the coordinate transformations of $f^*(E) \rightarrow X$ are given by $\{\Phi_{\alpha\beta} \circ f\}$.

1.5.5 Remark. Suitably interpreted, this pullback is actually a categorical pullback in a category of locally trivial bundles.

1.6 Vector bundles

A vector bundle is a special case of a locally trivial bundle.

1.6.1 Definition. A locally trivial bundle is called a real (respectively, complex) vector bundle of rank n if its fiber is a vector space V of dimension n over \mathbb{R} (respectively, \mathbb{C}) and the structure group is $GL(V)$.

1.6.2 Example. For a manifold of dimension n , the tangent bundle $TM \rightarrow M$ is a real vector bundle of rank n .

1.6.3 Remark. If the following, when we write K we mean either $K = \mathbb{R}$ if the vector bundle is real, and $K = \mathbb{C}$ if the vector bundle is complex.

1.6.4 Definition. Let $\xi = (p: E \rightarrow B)$ and $\xi' = (p': E' \rightarrow B')$ be vector bundles of rank m and n respectively. A map of vector bundles from $\xi \rightarrow \xi'$ is a fiber preserving map $\mathbf{f} = (\tilde{f}, f): (E, B) \rightarrow (E', B')$ satisfying the following condition: for local trivializations $\phi_\alpha: p^{-1}(U_\alpha) \rightarrow U_\alpha \times K^m$ and $\psi_\beta: p'^{-1}(V_\beta) \rightarrow V_\beta \times K^n$ of ξ and ξ' respectively satisfying $U_\alpha \cap f^{-1}(V_\beta) \neq \emptyset$, the map

$$L_{\alpha,\beta}^f: U_\alpha \cap f^{-1}(V_\beta) \rightarrow \text{Map}(K^m, K^n)$$

which is adjoint to

$$U_\alpha \cap f^{-1}(V_\beta) \times F \xrightarrow{\phi_\alpha^{-1}} p^{-1}(U_\alpha \cap f^{-1}(V_\beta)) \xrightarrow{\tilde{f}} p'^{-1}(U_\alpha \cap f^{-1}(V_\beta)) \xrightarrow{\Psi_\beta} (f(U_\alpha) \cap V_\beta) \times F' \xrightarrow{\text{pr}_2} F'$$

takes values in the set of linear maps $\text{Hom}_K(K^m, K^n)$

1.6.5 Remark. Note that $\text{Hom}_K(K^m, K^n)$ inherits the structure of a K -vector space, by sum and scalar multiplication of linear maps.

1.6.6 Remark. Given two vector bundles $p: E \rightarrow B$ and $p': E' \rightarrow B'$ of rank m and n respectively, the product $p \times p': E \times E' \rightarrow B \times B'$ is a vector bundle of rank $m + n$ (Proposition 1.5.1). Let us denote this bundle by $\xi \times \xi'$. This has structure group $GL_m(K) \times GL_n(K)$. Note that we can regard $GL_m(K) \times GL_n(K)$ as a subgroup of $GL_{m+n}(K)$ via the map

$$(A, B) \mapsto \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}.$$

We use this to define a direct sum for vector bundles.

1.6.7 Definition. Let $\xi = (p: E \rightarrow B)$ and $\xi' = (p': E' \rightarrow B)$ be vector bundles of rank m and n over a fixed space B . The pullback of $\xi \times \xi'$ along the diagonal map $\Delta: B \rightarrow B \times B$ is denoted $\xi \oplus \xi'$ and is called the direct sum (or Whitney sum) of ξ and ξ' . The total space of this bundle is $E \times_B E'$. This is a vector bundle of rank $m + n$.

1.6.8 Definition. An inner product on a vector bundle ξ is a homomorphism of vector bundles $g: \xi \oplus \xi \rightarrow K \times B$ ¹³ such that the map on each fiber gives rise to an inner product $K^n \times K^n \rightarrow K$ when translated by local trivializations.

¹³ here $K \times B$ denotes the trivial K -vector bundle over B .

1.6.9 Remark. Any vector bundle over a paracompact Hausdorff base space has an inner product. This uses a partition of unity argument to glue inner products over local trivializations. By using the Gram-Schmidt process, one can show that any vector bundle of rank n over a paracompact Hausdorff space has structure group that can be reduced to $O(n)$ in the real case, or $U(n)$ in the complex case.

There are a number of other natural constructions for vector bundles. We outline some now.

1.6.10 Definition (Tensor product bundle). We have an external tensor product: given two vector bundles $\xi = (p: E \rightarrow B)$ and

$\zeta' = (p': E' \rightarrow B')$ of rank m and n , we can construct a bundle $\zeta \hat{\otimes} \zeta'$ with coordinate transformations

$$\Phi_{\alpha\alpha'} \otimes \Psi_{\beta\beta'}: (U_\alpha \times V_\beta) \cap (U_{\alpha'} \times V_{\beta'}) = (U_\alpha \cap U_{\alpha'}) \times (V_\beta \cap V_{\beta'}) \rightarrow GL_{mn}(K)$$

as the composite

$$(U_\alpha \cap U_{\alpha'}) \times (V_\beta \cap V_{\beta'}) \xrightarrow{\Phi_{\alpha\alpha'} \times \Psi_{\beta\beta'}} GL_m(K) \times GL_n(K) \xrightarrow{\otimes} GL_{mn}(K)$$

where the last map is given by

$$(A, B) \mapsto (\mathbb{R}^{mn} \cong \mathbb{R}^m \otimes \mathbb{R}^n \xrightarrow{A \otimes B} \mathbb{R}^m \otimes \mathbb{R}^n \cong \mathbb{R}^{mn}).$$

Note that this is a vector bundle over $B \times B'$. In the case that $B = B'$, the pullback along the diagonal map gives a bundle $\zeta \otimes \zeta'$ over B called the tensor product bundle.

1.6.11 Definition (Hom bundle). Let $\zeta = (p: E \rightarrow B)$ and $\zeta' = (p': E' \rightarrow B)$ be two vector bundles over the same base space of rank m and n respectively. We can assume that local trivializations are defined over the same cover by taking a subdivision if necessary. Let

$$\Phi_{\alpha\alpha'}: U_\alpha \cap U_{\alpha'} \rightarrow GL_m(K)$$

and

$$\Psi_{\alpha\alpha'}: U_\alpha \cap U_{\alpha'} \rightarrow GL_n(K)$$

be coordinate transformations of ζ and ζ' . We define a new bundle with coordinate transformations the composite

$${}^t\Phi_{\alpha\alpha'} \otimes \Psi_{\alpha\alpha'}: U_\alpha \cap U_{\alpha'} \xrightarrow{\Phi_{\alpha\alpha'} \times \Psi_{\alpha\alpha'}} GL_m(K) \times GL_n(K) \xrightarrow{{}^t(-) \otimes 1} GL_m(K) \times GL_n(K) \xrightarrow{\otimes} GL_{mn}(K)$$

where ${}^t(-)$ denotes the transpose.¹⁴ We define a bundle $\text{Hom}(E, E')$ by gluing $U_\alpha \times \text{Hom}_K(K^m, K^n)$ using these coordinate transformations.

1.6.12 Example. Let \underline{K}_B denote the trivial K -bundle over a base B . Then $E^* := \text{Hom}(E, \underline{K}_B)$ is called the dual vector bundle, and has fibers dual to those of E . We note that if $K = \mathbb{R}$ and we work over a paracompact Hausdorff base space then a finite rank vector bundle and its dual are isomorphic as vector bundles (but not canonically).

1.6.13 Definition. Associated to a real or complex vector bundle $\pi: E \rightarrow B$ is the *projective bundle* $P(E) \rightarrow B$, where $P(E)$ is the space of all lines through the origin in all of the fibers of E . This is a locally trivial bundle with fiber $\mathbb{R}P^{n-1}$ or $\mathbb{C}P^{n-1}$ respectively.

1.7 Morphism of bundles

Let us now formalize the notion of a morphism of locally trivial bundles. There are various possibilities depending on how much structure we wish to preserve.

¹⁴ To justify the use of the transpose we note the following: Let $f: V \rightarrow W$ be a linear map between finite dimensional vector spaces with bases \mathcal{B}_V and \mathcal{B}_W with a matrix A representing f with respect to these bases. Then the map $f^*: W^* \rightarrow V^*$ has matrix tA with respect to the dual bases \mathcal{B}_V^* and \mathcal{B}_W^* .

1.7.1 Definition (Bundle homomorphism). Let $\xi = (p: E \rightarrow B)$ and $\xi' = (p': E' \rightarrow B')$ be two locally trivial bundles with fiber F and structure group G and G action given by a common map $\mu_G: G \times F \rightarrow F$. A morphism $(\tilde{f}, f): (E, B) \rightarrow (E', B')$ is a bundle map if:

1. The following diagram commutes:

$$\begin{array}{ccc} E' & \xrightarrow{\tilde{f}} & E \\ p' \downarrow & & \downarrow p \\ B' & \xrightarrow{f} & B \end{array}$$

2. For $x \in B$, let $\phi_\alpha: p^{-1}(U_\alpha) \rightarrow U_\alpha \times F$ and $\psi_\beta: p'^{-1}(V_\beta) \rightarrow V_\beta \times F'$ be local trivialisations around x and $f(x)$ respectively. Then we ask that the maps

$$L_{\alpha,\beta}^f: U_\alpha \cap f^{-1}(V_\beta) \rightarrow \text{Map}(F, F')$$

which are adjoint to

$$U_\alpha \cap f^{-1}(V_\beta) \times F \xrightarrow{\phi_\alpha^{-1}} p^{-1}(U_\alpha \cap f^{-1}(V_\beta)) \xrightarrow{\tilde{f}} p'^{-1}(U_\alpha \cap f^{-1}(V_\beta)) \xrightarrow{\psi_\beta} (f(U_\alpha) \cap V_\beta) \times F' \xrightarrow{\text{Pr}_2} F'$$

take values in G , i.e., there exists a dashed arrow making the following diagram commute:

$$\begin{array}{ccc} G & \xrightarrow{\text{ad}(\mu_G)} & \text{Map}(F, F') \\ \swarrow \tilde{L}_{\alpha,\beta}^f & & \searrow L_{\alpha,\beta}^f \\ & U_\alpha \cap f^{-1}(V_\beta) & \end{array}$$

We also have a notion of isomorphic bundles.¹⁵

1.7.2 Definition. Let $\xi = (p: E \rightarrow B)$ and $\xi' = (p': E' \rightarrow B)$ be bundles with the same fiber F , the same structure group G and the same base space B . We say that ξ and ξ' are isomorphic if there exists a bundle map (f, \tilde{f}) with $f = \text{id}_B$.

1.7.3 Example. In Proposition 1.5.4 we showed that for a locally trivial bundle $p: E \rightarrow B$ and a map $f: X \rightarrow B$, the pullback $f^*(E): E \times_B X \rightarrow X$ is a locally trivial fiber bundle. More is true - the induced map $\phi: f^*(E) \rightarrow E$ is a morphism of locally trivial bundles. Indeed, commutativity of the diagram is clear from the definition of the pullback. Moreover, it is straightforward from the definitions to see that ϕ carries the fiber over a point $x \in X$ to the fiber over $f(x)$. Finally, the coordinate transformations of the bundle $f^*(E)$ are given by

$$V_\alpha \cap V_\beta \xrightarrow{f} U_\alpha \cap U_\beta \xrightarrow{\Phi_{\alpha\beta}} G \xrightarrow{\text{ad}(\mu)} \text{Homeo}(F)$$

where

$$U_\alpha \cap U_\beta \xrightarrow{\Phi_{\alpha\beta}} G \xrightarrow{\text{ad}(\mu)} \text{Homeo}(F)$$

is the coordinate transformations for $p: E \rightarrow B$.

¹⁵ The categorically minded reader will complain that this is not the correct definition, as an isomorphism should be defined as a bundle homomorphism which admits a map in the reverse direction so that both composites are the identity. Fortunately, it is a theorem that any bundle map over the identity map of a fixed space B is an isomorphism in this sense, thus justifying our definition.

1.7.4 *Remark.* The converse to the previous remark is also true.

1.7.5 Theorem. Let $p: E \rightarrow B$ and $p': E' \rightarrow X$ be locally trivial bundles having the same fiber and structure group. Suppose there is a bundle map

$$\begin{array}{ccc} E' & \xrightarrow{\tilde{f}} & E \\ p' \downarrow & & \downarrow p \\ X & \xrightarrow{f} & B \end{array}$$

then the bundle $E' \rightarrow X$ is isomorphic to the pullback bundle $f^*(E) \rightarrow X$.

Proof. By the universal property of the pullback we have a map $\bar{f}: E' \rightarrow f^*(E)$ which is given by $\bar{f}(e) = (p'(e), \tilde{f}(e))$. We claim that this is a bundle isomorphism. It is a rather lengthy and tedious exercise to show this, so we omit the proof. \square

We also note the following two lemmas, which are straightforward to show using the definition of the pullback, and shows that the pullback is functorial in this category.

1.7.6 Lemma. Let $\pi: E \rightarrow B$ be a locally trivial bundle. For continuous map $f: X \rightarrow B$ and $g: Y \rightarrow X$, we have a bundle isomorphism $(f \circ g)^*(\pi) \cong g^*(f^*(\pi))$.

1.7.7 Lemma. Let $p: E \rightarrow B$ be a locally trivial bundle, then $\text{id}_B^*(E) \simeq E$.

Exercise 3. Show that the pullback of a trivial bundle is a trivial bundle.

1.7.8 *Remark.* The definition of a bundle morphism is very technical. Suppose however, that our bundles are principal G -bundles, so that E and E' come with right G -actions (Theorem 1.3.9). One can check that for a bundle morphism (\tilde{f}, f) between two principal G -bundles, the map \tilde{f} is G -equivariant, i.e., $\tilde{f}(e \cdot g) = \tilde{f}(e) \cdot g$.

1.7.9 *Remark.* Given a bundle $\pi: E \rightarrow B$ and two maps $f, g: X \rightarrow B$, one can ask when the two bundles $f^*(\pi)$ and $g^*(\pi)$ are isomorphic. The following is an important result in this direction.

1.7.10 Theorem. Let $\pi: E \rightarrow B$ be a bundle with B compact, and suppose that $f \simeq g: X \rightarrow B$ are homotopic, then there is an isomorphism $f^*(\pi) \cong g^*(\pi)$ of bundles over B .

1.7.11 *Remark.* We will prove this in the case that π is a principal G -bundle (the general case can be reduced to this case). We will do this by constructing a bundle map

$$\begin{array}{ccc} f^*E & \overset{?}{\dashrightarrow} & g^*\pi \\ & \searrow & \swarrow \\ & X & \end{array}$$

We will make use of the following definition.

1.7.12 *Definition.* A section of a bundle $\pi: E \rightarrow B$ is a continuous map $s: B \rightarrow E$ such that $\pi \circ s \simeq \text{id}_B$.

1.7.13 Remark. We wish to describe the set of bundle maps between two principal G -bundles $\pi_1: E_1 \rightarrow X$ and $\pi_2: E_2 \rightarrow Y$. Note that G acts on the right of E_1 and E_2 , and so on the left of E_2 via $g \cdot e_2 := e_2 \cdot g^{-1}$. Then, the associated bundle construction (Definition 1.4.4) gives an associated bundle of π_1 with fiber E_2 , namely

$$\omega := \pi_1 \times_G E_2: E_1 \times_G E_2 \rightarrow X.$$

1.7.14 Theorem. Bundle maps from π_1 to π_2 are in bijection with sections of ω .

Proof. We first assume that $\pi_1: X \times G \rightarrow G$ and $\pi_2: Y \times G \rightarrow G$ are trivial bundles. Suppose that we are given a bundle homomorphism, then we must define a section s in the associated bundle

$$\begin{array}{c} (X \times G) \times_G (Y \times G) \\ \begin{array}{c} \uparrow s \\ \downarrow \omega \end{array} \\ X \end{array}$$

Let $e_1 \in X \times G$ with $x = \pi_1(e_1) \in X$, we set

$$s(x) = [e_1, \tilde{f}(e_1)].$$

Note that this is well-defined, as

$$[e_1 \cdot g, \tilde{f}(e_1 \cdot g)] = [e_1 \cdot g, \tilde{f}(e_1) \cdot g] = [e_1 \cdot g, g^{-1} \cdot \tilde{f}(e_1)] = [e_1, \tilde{f}(e_1)],$$

where we have used Remark 1.7.8. Moreover, it is continuous, and provides a section:

$$\pi \circ s(x) = \pi_1[e_1, \tilde{f}(e_1)] = \pi_1(e_1) = x.$$

The general case can be reduced to the case of a trivial bundle by working locally and gluing.

Conversely, suppose we have been given a section of $E_1 \times_G E_2 \xrightarrow{\omega} X$. We define a map $\tilde{f}: E_1 \rightarrow E_2$ by $\tilde{f}(e_1) = e_2$ where $s(\pi_1(e_1)) = [(e_1, e_2)]$. We note that this map is G -equivariant:

$$[e_1 \cdot g, e_2 \cdot g] = [e_1 \cdot g, g^{-1} \cdot e_2] = [e_1, e_2]$$

and so descends to a map on orbit spaces $f: X \rightarrow Y$. As usual, we omit the tedious verification that this actually defines a bundle map. \square

1.7.15 Remark. For a principal G -bundle, the condition of having a section is extremely strong:

Exercise 4. Show that a principal G -bundle is trivial if and only if it has a section.

1.7.16 Lemma. Let $\pi: E \rightarrow X \times I$ be a principal G -bundle, and let $\pi_0 := i_0^* \pi: E_0 \rightarrow X$ be the pullback of π under the map $i_0: X \rightarrow X \times I$. Then $\pi \cong (pr_1)^* \pi_0 \cong \pi_0 \times id_I$, where $pr_1: X \times I \rightarrow X$ is the projection map.

$$\begin{array}{ccccc} pr_1^*(E_0) & \longrightarrow & E_0 & \longrightarrow & E \\ \downarrow & & \downarrow \pi_0 & & \downarrow \pi \\ X \times I & \xrightarrow{pr_1} & X & \xrightarrow{i_0} & X \times I \end{array}$$

Proof. It suffices to find a bundle map as indicated:

$$\begin{array}{ccccc}
 E_0 & \longrightarrow & E & \xrightarrow{\widetilde{\text{pr}}_1} & E_0 \\
 \pi_0 \downarrow & & \pi \downarrow & & \downarrow \pi_0 \\
 X & \xrightarrow{i_0} & X \times I & \xrightarrow{\text{pr}_1} & X
 \end{array}$$

By Theorem 1.7.14 this is equivalent to finding a section s of $\omega: E \times_G E_0 \rightarrow X \times I$. Note that there exists a section s_0 of $\omega_0: E_0 \times_G E_0 \rightarrow X = X \times \{0\}$ corresponding to the identity bundle map. Then composing s_0 with the inclusion into $E \times_G E_0$ we get the following diagram:

$$\begin{array}{ccc}
 X \times \{0\} & \xrightarrow{s_0} & E \times_G E_0 \\
 \downarrow & \nearrow s & \downarrow \omega \\
 X \times I & \xlongequal{\quad} & X \times I
 \end{array}$$

Because ω is a fibration (Remark 1.1.5) we can apply the homotopy lifting property (??) to produce a map $s: X \times I \rightarrow E \times_G E_0$, which is a section of ω . \square

Proof of Theorem 1.7.10. Let $H: X \times I \rightarrow B$ be a homotopy between f and g , and consider the following diagram:

$$\begin{array}{ccccc}
 f^*E & \xrightarrow{\quad} & H^*E & \xrightarrow{\tilde{H}} & E \\
 \downarrow f^*\pi & & \nearrow g^*E & \downarrow H^*\pi & \downarrow \pi \\
 X \times \{0\} & \xleftarrow{i_0} & X \times I & \xrightarrow{H} & B \\
 & \downarrow g^*\pi & \nearrow i_1 & \searrow \text{pr}_1 & \\
 & X \times \{1\} & & X &
 \end{array}$$

By Lemma 1.7.6 we have $f^*\pi \cong i_0^*H^*\pi$. By Lemma 1.7.16 $H^*\pi \cong \text{pr}_1^*(f^*\pi) \cong \text{pr}_1^*(g^*\pi)$, and so $f^*\pi \cong i_0^*H^*\pi \cong i_0^*\text{pr}_1^*(g^*\pi) \cong g^*\pi$. \square

Exercise 5. Show that a fibre bundle over a contractible base space is trivial.

1.8 Classification of principal G -bundles

We now move to the most important part of this part of the course, which is the classification theorem for principal G -bundles.

1.8.1 Definition. A principal G -bundle $\pi_G: EG \rightarrow BG$ is called *universal* if the total space EG is (weakly) contractible. The space $BG \simeq EG/G$ is called the classifying space of G .

The main theorem of this part of the course is the following. We let $\mathcal{P}(X, G)$ denote the set of principal G -bundles over a space X .¹⁶

¹⁶ The following theorem can also be proved abstractly by Brown representability. But we give a direct proof here.

1.8.2 Theorem. *Let X be a CW-complex, then there is a bijection*

$$\Phi: [X, BG] \xrightarrow{\cong} \mathcal{P}(X, G), \quad f \mapsto f^* \pi_G$$

Proof. We first show note that Φ is well-defined by Theorem 1.7.10. Let us first show that Φ is onto. Let $\pi: E \rightarrow X$ be a principal G -bundle, then we are required to find $f: X \rightarrow G$ such that $\pi \cong f^* \pi_G$. Equivalently, we are required to find a bundle map $(f, \tilde{f}): \pi \rightarrow \pi_G$. By Theorem 1.7.14 this is equivalent to finding a section of the bundle $E \times_G EG \rightarrow X$ with fiber EG . Because EG is contractible this following from the following lemma.

1.8.3 Lemma. *Let X be a trivial bundle and $\pi: E \rightarrow X$ a locally trivial bundle with fiber G and structure group G with $\pi_i(F) = 0$ for all $i \geq 0$. If $A \subseteq X$ is a subcomplex, then every section of π over A extends to a section defined on all of X . In particular, π has a section. Moreover, any two sections of π are homotopic.*

Proof. Let $s_0: A \rightarrow E$ be a section of π over A , then we will extend it to a section $s: X \rightarrow E$ using induction over the cells in $X \setminus A$. In other words, we can assume that $X = A \cup_\phi e^n$ for x^n an n -cell in $X \setminus A$ and attaching map $\phi: \partial e^n \rightarrow A$. The bundle is trivial over e^n (as e^n is contractible), so we have a commutative diagram

$$\begin{array}{ccc} & \pi^{-1}(e^n) & \xrightarrow[\cong]{h} e^n \times F \\ s_0 \nearrow & \downarrow \pi & \nwarrow \text{Pr}_1 \\ \partial e^n & \xrightarrow{\quad} e^n & \end{array} \quad \begin{array}{c} \nearrow s \\ \nwarrow \end{array}$$

where h is the chart for π over e^n and s is the section we wish to define.

The composite $h \circ s_0: \partial e^n \rightarrow e^n \times F$ is of the form

$$s_0(x) = (x, \tau_0(x)) \in e^n \times F,$$

with $\tau_0: \partial e^n \cong S^{n-1} \rightarrow F$. Because $\pi_{n-1}F = 0$ by assumption, τ_0 extends to a map $\tau: e^n \cong D^n \rightarrow F$ which we use to define $s: e^n \rightarrow e^n \times F$ by $s(x) = (x, \tau(x))$. After composing with h^{-1} we get the desired extension of s_0 over e^n .

To see that the section is unique up to homotopy, assume that we are given another section s' . Consider the bundle $\pi \times \text{id}: E \times I \rightarrow X \times I$. We can define sections of this bundle over $X \times \{0\}$ and $X \times \{1\}$ using s and s' , which together define a section over $X \times \{0, 1\}$. Arguing as in the first part, we can extend this section to a section Σ over $X \times I$. This section is of the form $\Sigma(x, t) = (s_t(x), t): X \times I \rightarrow E \times I$, and map s_t provides the desired homotopy between s and s' . \square

We now return to the proof of Theorem 1.8.2. It remains to show the injectivity of Φ . That is, if $\pi_0 \cong f^* \pi_G \cong f^* \pi_G \cong \pi_1$, then $f \simeq g$.

Consider the two defining diagrams:

$$\begin{array}{ccc} E_0 = f^*EG & \xrightarrow{\tilde{f}} & EG \\ \pi_0 \downarrow & & \downarrow \pi \\ X = X \times \{0\} & \xrightarrow{f} & BG \end{array} \quad \begin{array}{ccc} E_0 \cong E_1 = g^*EG & \xrightarrow{\tilde{g}} & EG \\ \pi_1 \downarrow & & \downarrow \pi \\ X = X \times \{1\} & \xrightarrow{g} & BG \end{array}$$

We can combine them to make the diagram:

$$\begin{array}{ccccc} E_0 \times I & \longleftarrow & E_0 \times \{0, 1\} & \xrightarrow{\tilde{\alpha} = (\tilde{f}, 0) \cup (\tilde{g}, 1)} & EG \\ \pi_0 \times \text{id} \downarrow & & \downarrow \pi_0 \times \{0, 1\} & & \downarrow \pi \\ X \times I & \longleftarrow & X \times \{0, 1\} & \xrightarrow{\alpha = (f, 0) \cup (g, 1)} & BG \end{array}$$

We will extend the map $(\alpha, \tilde{\alpha})$ to a bundle map $(H, \tilde{H}): \pi_0 \times \text{id} \rightarrow \pi_G$; then H will give the desired homotopy. Using Theorem 1.7.14 again this corresponds to a section s of the bundle $\omega: (E_0 \times I) \times_G EG \rightarrow X \times I$. But the map $(\alpha, \tilde{\alpha})$ gives a section s_0 of the bundle $\omega_0: (E_0 \times \{0, 1\}) \times_G EG \rightarrow X \times \{0, 1\} \subseteq (E_0 \times I) \times_G EG \rightarrow X \times I$. Since EG is contractible, we can use Lemma 1.8.3 to extend the section s_0 to the desired section s . \square

1.8.4 Example. We have $\mathcal{P}(S^n, G) \simeq [S^n, BG] \cong \pi_n(BG)$. The long exact sequence in homotopy shows that $\pi_n(BG) \cong \pi_{n-1}(G)$.

1.8.5 Remark. So far we have made no claim about the existence of universal principal G -bundles. Nonetheless, we have the following result of Milnor.

1.8.6 Theorem. *Let G be a locally compact topological group. Then a universal principal G -bundle exists, and is functorial in the sense that a continuous group homomorphism $f: G \rightarrow H$ induces a bundle map $(B\mu, E\mu): \pi_G \rightarrow \pi_H$. Moreover, the classifying space BG is unique up to homotopy.*

Proof. Let us explain why BG is unique up to homotopy, before commenting on the construction. Suppose $\pi_G: EG \rightarrow BG$ and $\pi'_G: EG' \rightarrow BG'$ are universal principal G -bundles. Using the universal properties of π_G and π'_G we can find maps $f: BG' \rightarrow BG$ and $g: BG \rightarrow BG'$ such that $\pi'_G \cong f^* \pi_G$ and $\pi_G \cong g^* \pi'_G$. Then,

$$\pi_G \cong g^* \pi'_G \cong g^* f^* \pi_G \cong (f \circ g)^* (\pi_G) \simeq (\text{id}_B)^* \pi_G.$$

By Theorem 1.8.2 we have $f \simeq g \simeq \text{id}_{BG}$. Similarly, we deduce that $g \circ f \simeq \text{id}_{BG'}$. Therefore, $f \simeq g$. \square

1.8.7 Remark. There are several different ways to construct the bundle π_G . Here is Milnor's construction. We recall that the join of X and Y is the space¹⁷

$$X * Y := X \times I \times Y / \sim$$

where $(x, 0, y_1) \sim (x, 0, y_2)$ for all $y_1, y_2 \in Y$ and $(x_1, 1, y) \sim (x_2, 1, y)$ for all $x_1, x_2 \in X$. For example,

$$X * \{y\} = (X \times I) / (X \times \{1\}) \cong CX,$$

¹⁷ Technically, for Milnor's construction, we need to equip the join with coarsest topology which makes $t: X * Y \rightarrow I$ (given on $X \times I \times Y$ by $(x, t, y) \mapsto t$, constantly 0 on X , and constantly 1 on Y) and the projections $\pi_1: t^{-1}(I) \rightarrow X$ and $\pi_2: t^{-1}(I) \rightarrow Y$ continuous.

the cone on X . If $Y = \{y_1, y_2\} = S^0$ is two points, then $X * Y \cong \Sigma X$, the suspension. There is also a reduced version of the join for pointed spaces. For CW-complexes, we have a homotopy equivalence $A * B \simeq \Sigma(A \wedge B)$. For example, $S^n * S^m \simeq S^{n+m-1}$ (in this case, this is actually a homeomorphism).

Now, we let $G^{*(k+1)} := G * \cdots * G$, the join of $k+1$ -copies of G . This has a free G -action, given by the diagonal action on the copies of G , and trivial action on I . Let $\mathcal{J}(G) := \operatorname{colim}_k G^{*(k+1)}$. Then, in fact $\mathcal{J}(G)$ has a free G -action and $\mathcal{J}(G) \rightarrow \mathcal{J}(G)/G$ is a universal principal G -bundle. The rough idea is that as we join more and more copies of G , the space becomes more and more connective, and in the colimit, (weakly) contractible.

Exercise 6. Show that $B(G \times H) \simeq BG \times BH$ (whenever this makes sense).

1.8.8 Remark. In practice, for us we will construct the universal bundles we need by hand.

1.8.9 Example. We recall from (1.1.18) that we have a principal $O(n)$ -bundle

$$O(n) \rightarrow V_n(\mathbb{R}^k) \rightarrow G_n(\mathbb{R}^k)$$

with $V_n(\mathbb{R}^k)$ $(k - n - 1)$ -connected. Letting $k \rightarrow \infty$, we get the bundle

$$O(n) \rightarrow V_n(\mathbb{R}^\infty) \rightarrow G_n(\mathbb{R}^\infty)$$

with $V_n(\mathbb{R}^\infty)$ contractible. This is a model for the universal bundle $EO(n) \rightarrow BO(n)$, i.e., $BO(n) \simeq G_n(\mathbb{R}^\infty)$.

Now recall that a real vector bundle is a locally trivial bundle with fiber a vector space V of dimension n over \mathbb{R} . Using the associated bundle construction and Theorem 1.4.2 we see that there is a bijection

$$\{\text{principal } GL_n(\mathbb{R})\text{-bundles over } X\} \longleftrightarrow \{\text{Rank } n \text{ vector bundles over } X\}$$

By Gram-Schmidt we have $GL_n(\mathbb{R}) \simeq O(n)$, and so we deduce the following from Theorem 1.8.2

$$\operatorname{Vect}_n^{\mathbb{R}}(X) \simeq \mathcal{P}(GL_n(\mathbb{R}), X) \simeq \mathcal{P}(O(n), X) \simeq [x, BO(n)] \simeq [X, G_n(\mathbb{R}^\infty)],$$

and we say that $G_n(\mathbb{R}^\infty)$ is a classifying space for real vector bundles of rank n .

Similarly, $BU(n) \simeq G_n(\mathbb{C}^\infty)$ and this is the classifying space for rank n complex vector bundles.

1.8.10 Example (Classification of real line bundles). Consider the (real) case $n = 1$ in the previous example, so that we are classifying real line bundles. In this case, we have a principal $\mathbb{Z}/2$ -bundle $\mathbb{Z}/2 \rightarrow S^\infty \rightarrow \mathbb{R}P^\infty$, and so $B\mathbb{Z}/2 \simeq \mathbb{R}P^\infty$. But, note that $\mathbb{R}P^\infty \simeq K(\mathbb{Z}/2, 1)$, so ?? gives

$$\operatorname{Vect}_1^{\mathbb{R}}(X) \simeq \mathcal{P}(X, \mathbb{Z}/2) \simeq [X, \mathbb{R}P^\infty] \simeq H^1(X; \mathbb{Z}/2)$$

for any CW-complex X .

Recall that $H^*(\mathbb{R}P^\infty, \mathbb{Z}/2) \cong \mathbb{Z}/2[w]$ for $|w| = 1$. In particular, if π is a real line bundle on X with classifying map $f_\pi: X \rightarrow \mathbb{R}P^\infty$, we get a well-defined degree one cohomology class

$$w_1(\pi) := f_\pi^*(w)$$

called the first Stiefel–Whitney class of π . The bijection $P(X, \mathbb{Z}/2) \simeq H^1(X; \mathbb{Z}/2)$ sends π to $w_1(\pi)$ and so real line bundles are completely classified by their first Stiefel–Whitney classes.

1.8.11 Example (Classification of complex line bundles). Consider the complex case $n = 1$ in Example 1.8.9, so that we are classifying complex line bundles. In this case, we have a principal S^1 -bundle $S^1 \rightarrow S^\infty \rightarrow \mathbb{C}P^\infty$, and so $BS^1 \simeq \mathbb{C}P^\infty$. But, note that $\mathbb{C}P^\infty \simeq K(\mathbb{Z}, 2)$, so ?? gives

$$\text{Vect}_1^{\mathbb{C}}(X) \simeq \mathcal{P}(X, BS^1) \simeq [X, \mathbb{C}P^\infty] \simeq H^2(X; \mathbb{Z})$$

for any CW-complex X .

Recall that $H^*(\mathbb{C}P^\infty, \mathbb{Z}) \cong \mathbb{Z}[c]$ for $|c| = 2$. In particular, if π is a complex line bundle on X with classifying map $f_\pi: X \rightarrow \mathbb{C}P^\infty$, we get a well-defined degree two cohomology class

$$c_1(\pi) := f_\pi^*(c)$$

called the first Chern class of π . The bijection $P(X, S^1) \simeq H^2(X; \mathbb{Z})$ sends π to $c_1(\pi)$ and so real line bundles are completely classified by their first Chern classes.

1.8.12 Example. How many (real) vector bundles over $\mathbb{R}P^n$ are there? We have

$$\text{Vect}_1^{\mathbb{R}}(\mathbb{R}^n) \cong H^1(\mathbb{R}P^n; \mathbb{Z}/2) \cong \mathbb{Z}/2,$$

so there are two (up to equivalence). One is the trivial bundle.

What is the non-trivial bundle?

Let $x \in S^n$ and $[x] \in \mathbb{R}P^n \simeq S^n / \sim$ the class represented by x . Let $E = \{([x], \nu) : [x] \in \mathbb{R}P^n, \nu \in [x]\} \subseteq \mathbb{R}P^n \times \mathbb{R}^{n+1}$. Then, we have a bundle γ_1 defined by¹⁸

$$\gamma_1: E \rightarrow \mathbb{R}P^n, \quad ([x], \nu) \mapsto [x].$$

To see that this is non-trivial note that when $n = 1$ this is exactly the Möbius bundle, which is non-trivial. In general, if γ_1 was trivial, then pull-back along $\mathbb{R}P^1 \rightarrow \mathbb{R}P^n$ would also be trivial, but this is the non-trivial Möbius bundle again, a contradiction. So γ_1 must be non-trivial.

1.8.13 Example. Isomorphism classes of principal S^1 -bundles over S^2 are given by $[S^2, BS^1] \cong \pi_2(BS^1) \cong \pi_1(S^1) \cong \mathbb{Z} \cong H^2(\mathbb{C}P^\infty; \mathbb{Z})$. The Hopf bundle $H: S^3 \rightarrow S^2 \rightarrow \mathbb{C}P^1$ is a principal S^1 -bundle. In particular, it is given as the pullback along a map $f: S^2 \cong \mathbb{C}P^1 \rightarrow \mathbb{C}P^\infty$:

$$\begin{array}{ccc} S^3 & \longrightarrow & S^\infty \\ H \downarrow & & \downarrow \pi_{S^1} \\ S^2 \cong \mathbb{C}P^1 & \xrightarrow{f} & \mathbb{C}P^\infty \end{array}$$

¹⁸ This bundle is known as the tautological, or canonical, bundle over $\mathbb{R}P^n$.

It turns out that this map is the inclusion $\mathbb{C}P^1 \hookrightarrow \mathbb{C}P^\infty$, and so $H^*(f): H^*(\mathbb{C}P^\infty) \rightarrow H^*(\mathbb{C}P^1)$ sends ω to ω . In particular, $c_1(H) \neq 0$, and $c_1(H)$ generates $H^2(\mathbb{C}P^1)$ as a cyclic group.

The following exercise gives another way to compute this Chern class (using ????)

Exercise 7. Let $\pi: E \rightarrow X$ be a principal S^1 -bundle over a simply-connected space X . Let $a \in H^1(S^1; \mathbb{Z})$ be a generator. Show that

$$c_1(\pi) = d_2(a)$$

where d_2 is the differential on the E_2 -page of the Leray–Serre spectral sequence associated to π , i.e., $E_2^{p,q} \cong H^p(X; H^q(S^1)) \implies H^{p+q}(E, \mathbb{Z})$.