1
Spectral sequences

Spectral sequences are a powerful computation tool in topology. Computing with spectral sequences is a bit like computing integral in calculus; it is helpful to have ingenuity and a big bag of tricks - and even that may not be enough!

1.1 Filtered complexes

We begin our discussion on spectral sequences by discussing filtered complexes.

1.1.1 Remark. Let $C_\bullet$ be a chain complex and $F_0C_\bullet$ a sub-complex. Then we have a short exact sequence

$$0 \to F_0C_\bullet \to C_\bullet \to C_\bullet/F_0C_\bullet \to 0$$

which gives rise to a long exact sequence in homology

$$\cdots \to H_i(F_0C_\bullet) \to H_i(C_\bullet) \to H_i(C_\bullet/F_0C_\bullet) \xrightarrow{\partial} H_{i-1}(F_0C_\bullet) \to \cdots$$

Suppose we know $H_*(F_0C_\bullet)$ and $H_*(C_\bullet/F_0C_\bullet)$. Can we compute $H_*(C_\bullet)$? We can split the long exact sequence into short exact sequences

$$0 \to \mathrm{coker}(\partial) \to H_*(C_\bullet) \to \ker(\partial) \to 0$$

which gives the following procedure for computing $H_*(C_\bullet)$:

1. Compute $H_*(F_0C_\bullet)$ and $H_*(C_\bullet/F_0C_\bullet)$
2. Consider the two-term chain complex

   $$H_*(C_\bullet/F_0C_\bullet) \xrightarrow{\partial} H_*(F_0C_\bullet).$$

   Denote its homology groups by G_1H_* and G_0H_*.
3. There is a short exact sequence

   $$0 \to G_0H_* \to H_*(C_\bullet) \to G_1H_* \to 0.$$

   This determines $H_*(C_\bullet)$ up to extension.[1]

How would we handle the situation if we have a longer filtration: $\cdots F_pC_\bullet \subseteq F_{p+1}C_\bullet \subseteq \cdots$?

[1] This is a common phenomenon for a spectral sequence. For example, if we have a short exact sequence $0 \to \mathbb{Z}/2 \to M \to \mathbb{Z}/2 \to 0$, can you say what the middle group is? Not without further information!

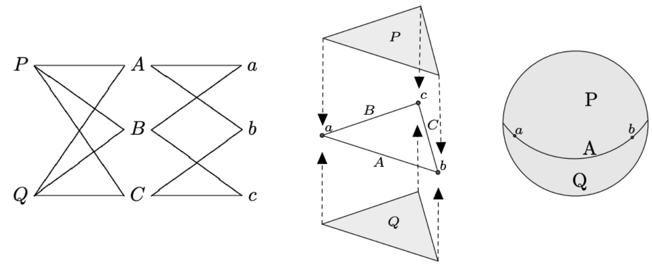

Figure 1.1: Simplicial model of S^2

1.1.2 Example. Consider a (semi-simplicial) model of the 2-sphere S^2 with vertices $\{a, b, c\}$, edges $\{A, B, C\}$ and solid triangles $\{P, Q\}$ and with inclusions as soon in Figure 1.1.[2] The associated chain complex is $C_\bullet$

$$0 \to \mathbb{Z}\{P, Q\} \xrightarrow{d} \mathbb{Z}\{A, B, C\} \xrightarrow{d} \mathbb{Z}\{a, b, c\} \to 0$$

with

$$d(P) = C - B + A \quad d(Q) = C - B + A$$

and

$$d(A) = b - a \quad d(B) = c - a \quad d(C) = c - b.$$

One can check directly that $H_i(C_\bullet; \mathbb{Z}) \cong \mathbb{Z}$ for $i = 0, 2$ and is zero otherwise. Alternatively, we use the following filtration:

$$\begin{array}{ccccccccc}
0 & \to & \mathbb{Z}\{P, Q\} & \to & \mathbb{Z}\{A, B, C\} & \to & \mathbb{Z}\{a, b, c\} & \to & 0 \\
0 & \to & 0 & \to & \mathbb{Z}\{A, B\} & \to & \mathbb{Z}\{a, b, c\} & \to & 0 \\
0 & \to & 0 & \to & \mathbb{Z}\{A\} & \to & \mathbb{Z}\{a, b\} & \to & 0.
\end{array}$$

The differentials are induced from d_1 and d_2 and a direct check shows that they are still chain complexes. Passing to the quotient, we get a chain complex we call E_0:

$$\begin{array}{ccccccccc}
0 & \to & \mathbb{Z}\{P, Q\} & \to & \mathbb{Z}\{C\} & \to & 0 & \to & 0 \\
0 & \to & 0 & \to & \mathbb{Z}\{B\} & \to & \mathbb{Z}\{c\} & \to & 0 \\
0 & \to & 0 & \to & \mathbb{Z}\{A\} & \to & \mathbb{Z}\{a, b\} & \to & 0
\end{array}
\qquad
\begin{array}{c}
d_0(P) = C, d_0(Q) = C \\
d_0(B) = c \\
d_0(A) = b - a.
\end{array}$$

Taking homology with respect to d_0 we obtain E^1:

$$\begin{array}{ccccccccc}
0 & \to & \mathbb{Z}\{P - Q\} & \to & 0 & \to & 0 & \to & 0 \\
0 & \to & 0 & \to & 0 & \to & 0 & \to & 0 \\
0 & \to & 0 & \to & 0 & \to & \mathbb{Z}\{\bar{a}\} & \to & 0.
\end{array}$$

The general theory of spectral sequences will tell us that we have computed the homology of $H_*(C_\bullet)$; there is a $\mathbb{Z}$ in degree 2, generated by $P - Q$ and a $\mathbb{Z}$ in degree 0, generated by $\bar{a}$.

This leads us to the theory of filtered modules.

1.1.3 Definition. A filtered R-module is an R-module A together with an increasing sequence of submodules $F_pA \subseteq F_{p+1}A$ indexed by $p \in \mathbb{Z}$ such that $\cup_p F_pA = A$ and $\cap_p F_pA = \{0\}$. The filtration is

[2] This example comes from Example 2.1 of https://arxiv.org/pdf/1702.00666.pdf.

bounded if $F_pA = \{0\}$ for p sufficiently small, and $F_pA = A$ for p sufficiently large. The associated graded module is defined by

$$G_pA = F_pA/F_{p-1}A.$$

1.1.4 Definition. A filtered chain complex is a chain complex $(C_\bullet, \partial)$ together with a filtration $\{F_pC_i\}$ of each C_i such that the differential preserves the filtration: $\partial(F_pC_i) \subseteq F_pC_{i-1}$. Then, ∂ induces $\partial\colon G_pC_i \to G_pC_{i-1}$ on the associated graded modules.

1.1.5 Remark. The filtration on $C_\bullet$ induces a filtration on the homology of $C_\bullet$ by

$$F_pH_i(C_\bullet) = \{\alpha \in H_i(C_\bullet) \mid \exists x \in F_pC_i, \alpha = [x]\}.$$

This has associated graded pieces $G_pH_i(C_\bullet)$.

1.1.6 Remark. Suppose we want to compute $H_*(C_\bullet)$ and that we can compute the homology of the associated graded pieces $H_*(G_pC_\bullet)$. Does this determine $G_pH_*(C_\bullet)$? This leads to the idea of the spectral sequence of a filtered complex.

1.2 *The spectral sequence of a filtered complex*

1.2.1 Definition. Let $(F_pC_\bullet, \partial)$ be a filtered chain complex. Let us write

$$E^0_{p,q} := G_pC_{p+q} = F_pC_{p+q}/F_{p-1}C_{p+q}.$$

The differential ∂ induces a differential on E^0,

$$\partial_0\colon E^0_{p,q} \to E^0_{p,q-1}.$$

We denote the homology of the associated graded by

$$E^1_{p,q} := H_{p+q}(G_pC_\bullet, \partial_0).$$

1.2.2 Remark. We can think of $E^1_{p,q}$ as a "first order approximation" to $H_*(C_\bullet)$. We can also define a differential

$$\partial_1\colon E^1_{p,q} \to E^1_{p-1,q}$$

as follows: a homology class $\alpha \in E^1_{p,q}$ can be represented by a chain $x \in F_pC_{p+1}$ such that $\partial x \in F_{p-1}C_{p+q-1}$. We define $\partial_1(\alpha) = [\partial x]$. Because $\partial^2 = 0$, we can check that $\partial_1^2 = 0$ and that ∂_1 is well defined.

1.2.3 Definition. With notation as above, we define

$$E_2^{p,q} = \ker(\partial_1\colon E^1_{p,q} \to E^1_{p-1,q}) / \operatorname{im}(\partial_1\colon E^1_{p+1,q} \to E^1_{p,q}).$$

1.2.4 Remark. We can continue this procedure, and define an "r"-th order approximation to $G_pH_{p+q}(C_\bullet)$ by

$$E^r_{p,q} = \frac{x \in F_pC_{p+q} \mid \partial x \in F_{p-r}C_{p+q-1}}{F_{p-1}C_{p+q} + \partial(F_{p+r-1}C_{p+q+1}}).$$

The notation denotes the quotient of the numerator by the intersection with the denominator.

So instead of considering cycles, we consider chains in F_p whose differentials vanishes "to order r", and instead of modding out by the entire image, we only mod out by $\partial(F_{p+r-1})$.

The main result regarding these groups is the following.

1.2.5 Lemma. *Let $(F_pC_\bullet, \partial)$ denote a filtered chain complex, and define $E^r_{p,q}$ as above. Then,*

1. *∂ induces a map*
$$\partial_r \colon E^r_{p,q} \to E^r_{p-r,q+r-1}$$
satisfying $\partial_r^2 = 0$.

2. *E^{r+1} is the homology of the chain complex (E^r, ∂_r), i.e.,*
$$E^{r+1}_{p,q} = \ker(\partial_r \colon E^r_{p,q} \to E^r_{p-r,q+r-1}) / \operatorname{im}(\partial_r \colon E^r_{p+r,q+r-1} \to E^r_{p,q}).$$

3. *$E^1_{p,q} = H_{p+q}(G_pC_\bullet)$.*

4. *If the filtration of C_i is bounded for each i, then for every p, q if r is sufficiently large, then*
$$E^r_{p,q} = G_pH_{p+q}(C_\bullet).$$

Proof. This is a rather tedious diagram chase,[3] which generalizes the argument that a short exact sequence of chain complexes induces a long exact sequence on homology. □

[3] For example, see `http://www.math.uchicago.edu/~may/MISC/SpecSeqPrimer.pdf`

1.2.6 Example. In this example[4] we show that the singular and cellular homology groups of a CW-complex X agree. To that end, let $C_*(X)$ denote the singular chain complex of X. We filter this by

[4] See page 67 of Mosher–Tangor, *Cohomology Operations and Applications in Homotopy Theory*

$$F_pC_*(X) := C_*(X^p)$$

where X^p denotes the p-skeleton of X. The associated graded is

$$E^0_{p,q} = C_{p+q}(X^p)/C_{p+q}(X^{p-1}).$$

By definition, the homology is

$$E^1_{p,q} = H_{p+q}(X^p, X^{p-1}),$$

the relative homology of the pair (X^p, X^{p-1}). We have

$$H_{p+q}(X^p, X^{p-1}) \cong \begin{cases} C^{cell}_p(X) & q = 0 \\ 0, & q \neq 0 \end{cases}$$

where $C^{cell}_p(X)$ is the cellular chains on X, the free $\mathbb{Z}$-module with one generator for each p-cell. The cellular differential $\partial \colon C^{cell}_p(X) \to C^{cell}_{p-1}(X)$ is exactly the boundary map $E^1_{p,0} \to E^1_{p-1,0}$. Therefore, we have

$$E^2_{p,q} = \begin{cases} H^{cell}_p(X), & q = 0 \\ 0, & q \neq 0. \end{cases}$$

We must have $\partial_r = 0$ for $r \geq 2$ as either the domain or the range is zero. So, $E^{p,q}_r = E^2_{p,q}$ for all $r \geq 2$. If X is finite-dimensional, then the filtration is bounded and so $H_p(X) = H^{cell}_p(X)$ by Lemma 1.2.5.[5]

[5] One can allow arbitrary X by, for example, using colimits.

1.3 *Homological spectral sequences*

We have managed to so far avoid defining exactly what a spectral sequence is. Let us change that now.

1.3.1 Definition. A (homological) spectral sequence is a sequence

$$\{E^r_{*,*}, d^r_{*,*}\}_{r\geq 0}$$

of chain complexes of abelian groups, such that

$$E^{r+1}_{*,*} = H_*(E^r_{*,*})$$

where the homology is taken with respect to maps (called differentials)

$$d^r_{p,q} \colon E^r_{p,q} \to E^r_{p-q,q+r-1}$$

such that $(d^r)^2 = 0$.

1.3.2 Remark. We say that a spectral sequence is first quadrant if $E^r_{p,q} = 0$ whenever $p < 0$ or $q < 0$. Note that this implies that $d^r_{p,q} = 0$ for $r \gg 0$ (as either the source or the target is zero). In particular,

$$E^r_{p,q} = E^{r+1}_{p,q} = \cdots = E^\infty_{p,q}.$$

We say that the spectral sequence collapses or degenerates at E^r.

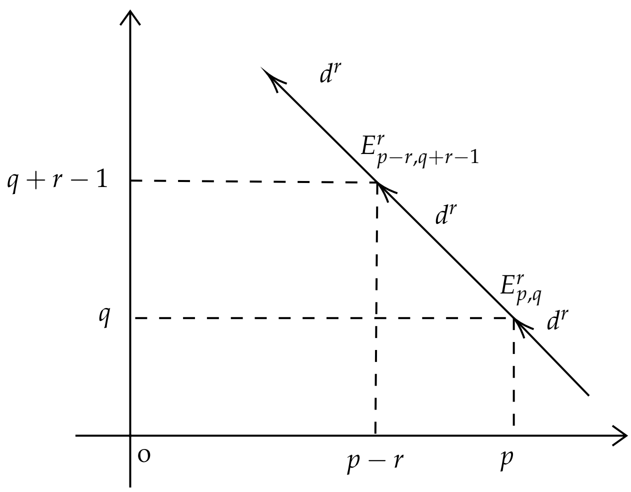

Figure 1.2: The E_r-page of the spectral sequence

1.3.3 Definition. If $\{H_n\}_n$ are groups, then we say that the spectral sequence converges, or abuts, to H_*, denoted $E^r_{*,*} \implies H_*$, if for each n there is a filtration

$$H_n = D_{n,0} \subseteq D_{n-1,1} \subseteq \cdots \subseteq D_{1,n-1} \subseteq D_{0,n} \subseteq 0$$

such that, for all p, q,

$$E^\infty_{p,q} = D_{p,q}/D_{p-1,q+1}.$$

1.3.4 Remark. In more straightforward terms: the if we look along the n-th diagonal of the spectral sequence, then the E_∞-page computes the associated graded of the filtration on H_n. For example, if there is only a single non-zero term, say $E^\infty_{p,n-p}$, then the filtration is trivial, and $H_n = E^\infty_{p,n-p}$. If we have two non-zero terms, then H_n fits into a short exact sequence, and so on.

1.3.5 Example. We have previously discussed the spectral sequence of a filtered complex without explicitly mentioning it. Indeed, if $C_\bullet$ is a filtered chain complex, then there is a spectral sequence with $E^1_{p,q} = H_{p+q}(G_pC_\bullet)$, such that if the filtration of C_i is bounded for each i the spectral sequence converges to $H_{p+q}(C_\bullet)$.[6]

[6] Recall what this means: we have $E^\infty_{p,q} = G_pH_{p+q}(C_\bullet)$.

1.4 *The spectral sequence of a double complex*

An important example where a filtered complex arises is from a double complex.

1.4.1 Definition. A double complex is a bi-indexed family $\{C_{p,q}\}$ of abelian groups, with two differentials

$$d' \colon C_{p,q} \to C_{p-1,q}, \quad d'' \colon C_{p,q} \to C_{p,q-1}$$

such that $d'd' = 0, d''d'' = 0$, and $d'd'' + d''d' = 0$. For simplicity, we also assume that $C_{p,q} = 0$ for $p < 0$ or $q < 0$.

1.4.2 Example. Suppose that (A, d_A) and (B, d_B) are chain complexes. If we define $C_{p,q} = A_p \otimes B_q$ and define $d' = d_A \otimes 1$ and $d'' = (-1)^p 1 \otimes d_B$, then $C_{p,q}$ is a double complex.[7]

[7] Try and verify this to make sure you understand the definitions.

1.4.3 Construction . A double complex gives rise to a chain complex (the total complex), defined by $C_n = \sum_{p+q=n} C_{p,q}$ and $d = d' + d''$. This has two obvious filtrations, by row and by column:

1. $'C^p_n = \sum_{j+q=n, j \leq p} C_{j,q}$.

2. $''C^p_n = \sum_{p+q=n, k \leq q} C_{p,k}$.

The spectral sequence of a filtered complex (Example 1.3.5) gives us two spectral sequences:

1. $'E^1_{p,q} = H_{p+q}('C^p/'C^{p-1}) = C_{p,n-p}$.

2. $''E^1_{p,q} = H_{p+q}(''C^q/''C^{q-1}) = C_{q,n-q}$.

One checks that $'E^1$ is computed via means of d'' and that d^1 is induced by d', while in $''E^1$ the role of the two indices are exchanged. We can therefore write:

1. $'E^2_{p,q} = H'_pH''_q(C)$.

2. $''E^2_{p,q} = H''_qH'_p(C)$.

Moreover, both spectral sequences converge to $H_*(C)$, and the idea is to compare the two spectral sequences.

It is constructive to do an example.

1.4.4 Example. Let ${}'\operatorname{Tor}(A,B)$ be defined as follows: take a free resolution of A, $0 \to R' \to F' \to A \to 0$, then ${}'\operatorname{Tor}(A,B)$ is defined by

$$0 \to {}' \operatorname{Tor}(A,B) \to R' \otimes B \to F' \otimes B \to A \otimes B \to 0.$$

Similarly, let ${}''\operatorname{Tor}(A,B)$ be defined as follows: take a free resolution of B, $0 \to R'' \to F'' \to B \to 0$, then ${}''\operatorname{Tor}(A,B)$ is defined by

$$0 \to {}''\operatorname{Tor}(A,B) \to A \otimes R'' \to A \otimes F'' \to A \otimes B \to 0.$$

It is a classical theorem of homological algebra that $\operatorname{Tor}(A,B) = {}''\operatorname{Tor}(A,B)$. Let us prove this via a spectral sequence argument.

Let X be the chain complex $0 \to R' \xrightarrow{d'} F' \to 0$ and let Y be the chain complex $0 \to R'' \xrightarrow{d''} F'' \to 0$. We can build a double complex $C_{*,*}$ as in Example 1.4.2, which we write as a matrix:

$$\begin{bmatrix} C_{p,q} \end{bmatrix} = \begin{bmatrix} F' \otimes R'' & R' \otimes R'' \\ F' \otimes F'' & R' \otimes F'' \end{bmatrix}$$

We have two spectral sequences: the first is take vertical and then horizontal homology:

$$H_q''(C_{p,q}) = \begin{bmatrix} {}''\operatorname{Tor}(F',B) & {}'\operatorname{Tor}(R',B) \\ F' \otimes B & R' \otimes B \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ F' \otimes B & R' \otimes B \end{bmatrix}$$

and

$$H_p H_q''(C_{p,q}) = \begin{bmatrix} 0 & 0 \\ A \otimes B & {}'\operatorname{Tor}(A,B) \end{bmatrix}$$

In other words, the total complex has $H_0(C) = A \otimes B$ and $H_1(C) = {}'\operatorname{Tor}(A,B)$.

However, we can use the second spectral sequence, which first takes horizontal and then vertical homology:

$$H_p(C_{p,q}) \begin{bmatrix} A \otimes R'' & {}'\operatorname{Tor}(A,R'') \\ A \otimes F'' & {}'\operatorname{Tor}(A,F'') \end{bmatrix} = \begin{bmatrix} A \otimes R'' & 0 \\ A \otimes F'' & 0 \end{bmatrix}$$

and then

$$H''_q H_p(C_{p,q}) = \begin{bmatrix} {}''\operatorname{Tor}(A,B) & 0 \\ A \otimes B & 0 \end{bmatrix}$$

In this case we see that $H_0(C) = A \otimes B$ and $H_1(C) = {}''\operatorname{Tor}(A,B)$. Therefore, ${}'\operatorname{Tor}(A,B) = {}''\operatorname{Tor}(A,B)$.

Exercise 1: The snake lemma

Show, using spectral sequences, the following result in homological algebra (the snake lemma):
Given a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, "f"'] & B \arrow[r] \arrow[d, "g"'] & C \arrow[r] \arrow[d, "h"'] & 0 \\ 0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0 \end{tikzcd}$$

in an abelian category with exact rows, there is a long exact sequence

$$0 \to \ker(f) \to \ker(g) \to \ker(h) \\ \to \operatorname{coker}(f) \to \operatorname{coker}(g) \to \operatorname{coker}(h) \to 0.$$

Exercise 2

(1) Suppose we have a commutative triangle

$$\begin{tikzcd} A \arrow[rr, "i"] \arrow[dr, "f"'] & & C \arrow[dl, "q"] \\ & B & \end{tikzcd}$$

Show using the snake lemma that

$$\ker(\operatorname{coker} f \to \operatorname{coker} q) \cong \operatorname{im}(q) / \operatorname{im}(f)$$

and

$$\operatorname{coker}(\operatorname{coker} f \to \operatorname{coker} q) = 0.$$

(2) Using Part (1), prove the following 'butterfly lemma': given a commutative diagram

$$\begin{tikzcd} A \arrow[dd, "f"'] \arrow[dr, "i"] & & D \arrow[dl, "j"'] \arrow[dd, "g"] \\ & C \arrow[dl, "q"] \arrow[dr, "p"'] & \\ B & & E \end{tikzcd}$$

of abelian groups, in which the diagonals pi and qj are exact at C, there is an isomorphism

$$\frac{\operatorname{im} q}{\operatorname{im} f} \cong \frac{\operatorname{im} p}{\operatorname{im} g}.$$