1

Homotopy theory

1.1 Review of basics on homotopy theory

We begin with a recollection of some facts that have been covered in Algebraic Topology I and Introduction to Topology.

1.1.1 Notation. We let $I = [0, 1]$ denote the unit interval. For a pointed topological space X we will denote the basepoint by x_0 or $*$.

We recall the following definition.

1.1.2 Definition. A homotopy between $f, g \colon X \to Y$ is a continuous function $H \colon X \times I \to Y$ such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ and $H(x_0, t) = y_0$ for all $t \in I$. We will write $f \simeq g$, or $f \simeq_H g$, if we need to make the choice of homotopy clear.

For a subspace $A \subseteq X$, a relative homotopy is a homotopy with $H(a, t) = f(a) = g(a)$ for all $a \in A, t \in I$.

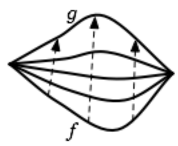


Figure 1.1: A homotopy between f and g.

1.1.3 Remark. Equivalently, we can specify a family of continuous maps $h_t \colon X \to Y$ such that $h_0 = f, h_1 = g$ and

$$\begin{aligned} H \colon X \times I &\to Y \\ (x, t) &\mapsto h_t(x) \end{aligned}$$

is continuous. We will switch between the two equivalent definitions without comment, using whatever is more convenient.

1.1.4 Proposition. *For all spaces X and Y, homotopy is an equivalence relation on the set of maps from X to Y. Furthermore, if we are given $k \colon A \to X, \ell \colon Y \to B$ and homotopic maps $f \simeq g \colon X \to Y$, then $fk \simeq gk \colon A \to Y$ and $\ell f \simeq \ell g \colon X \to B$.*

Proof. Let $f, g \colon X \to Y$, then

1. $f \simeq_F f$ via $F(x, t) = f(x)$ for all $x \in X, t \in I$.
2. If $f \simeq_F g$, then $g \simeq_G f$ where $G(x, t) = F(x, 1 - t)$.
3. If $f \simeq_F g$ and $g \simeq_G h$, then $f \simeq_H h$ via

$$H(x, t) = \begin{cases} F(x, 2t) & \text{if } 0 \le t \le 1/2 \\ G(x, 2t - 1) & \text{if } 1/2 \le t \le 1. \end{cases}$$

For the last part of the proposition let f_t be a homotopy between f and g, then $f_t k$ and ℓf_t give the required homotopy. $\square$

1.1.5 Definition. For a map $f\colon X \to Y$, we let $[f]$ denote the equivalence class containing f. The collection of all homotopy classes of maps from X to Y is denoted $[X, Y]$.[1]

[1] If our spaces are based, then these should be homotopy classes of *based* maps.

1.1.6 Remark. Note that if $\alpha = [f] \in [Y, Z]$ and $\beta = [g] \in [X, Y]$, then $\alpha\beta = [f \circ g] \in [X, Z]$, i.e., we can form the category $hTop_*$ whose objects are topological spaces, and whose morphisms are homotopy classes of maps.

1.1.7 Remark. We now very quickly review a number of standard topological constructions.

- Let X be a space and $A \subseteq X$. A map $r\colon X \to A$ such that $ri(a) = a$ for all $a \in A$ is called a retraction of X onto A, and A is called a retract of X.
- Let $i\colon A \hookrightarrow X$ be the inclusion, so that $ri = \mathrm{id}_A$. If $ir \simeq \mathrm{id}_X$, we call this a deformation retraction, and say that A is a deformation retract of X.
- If $f\colon X \to Y$, then a section of f is a map $s\colon Y \to X$ such that $f \circ s = \mathrm{id}_Y$. We can also ask for a *homotopy* section by requiring only that $f \circ s \simeq \mathrm{id}_Y$.

1.1.8 Definition. A map $f\colon X \to Y$ is called null-homotopic if $f\colon c_y\colon X \to Y$ where $c_y X \to Y$ is the constant map sending all of X to the point $y \in Y$. A homotopy between f and c_y is called a null-homotopy. A space X is contractible if id_X is null-homotopic.

1.1.9 Definition. Let (X, x_0) be a based topological space and $X \times I$ the cylinder on X. The quotient

$$CX = (X \times I)/(X \times \{1\} \cup \{x_0\} \times I)$$

with the base-point the equivalence class of $(x_0, 1)$ is called the (reduced) cone on X. Note that we have a natural inclusion $X \to CX$ of based maps given by $x \mapsto [x, 0]$.

1.1.10 Lemma. *The cone CX is contractible.*

Proof. Define $F\colon CX \times I \to CX$ by

$$F([x, t], s) = [x, s + (1 - s)t].$$

Note then that we have

$$F([x, t], 0) = [x, t] \quad \text{and} \quad F([x, t], 1) = [x, 1]. \qquad \square$$

1.1.11 Lemma. *The following are equivalent:*

(i) $f\colon X \to Y$ *is null-homotopic.*

(ii) f *can be extended to* CX*:*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, hook, "i"'] & Y \\ CX \arrow[ur, dashed, "\exists \tilde{f}"'] & \end{tikzcd}$$

Proof. $(i) \implies (ii)$: Suppose f is null-homotopic, so $f \simeq_F *$. Then $F(X \times \{1\} \cup \{*\} \times I) = *$, so by the universal property of the quotient, we can find $\tilde{F}\colon CX \to Y$ such that $\tilde{f} \circ i = f$.

$(ii) \implies (i)$: Suppose $\tilde{f} \circ i = f$, then because CX is contractible (Lemma 1.1.10), we have $f = \tilde{f} \circ \mathrm{id}_{CX} \circ i \simeq \tilde{f} \circ (*_{CX}) \circ i \simeq *$, so that f is null-homotopic. □

1.1.12 *Definition.* A map $f\colon X \to Y$ is a homotopy equivalence if there exists $g\colon Y \to X$ such that $fg \simeq \mathrm{id}_Y$ and $gf \simeq \mathrm{id}_X$. We write $X \simeq Y$.

1.1.13 *Example.* (i) X is contractible if and only if $X \simeq *$.

(ii) If $i\colon A \hookrightarrow X$, and $r\colon X \to A$ is a deformation retract, then i and r are homotopy equivalences, and $A \simeq X$.

1.2 *Higher homotopy groups*

1.2.1 *Notation.* We will let $I_n = I^{\times n}, \partial I^n$ be the boundary of I^n, and write $[-,-]$ for homotopy classes of maps (if our spaces are based, these fix the base point).

1.2.2 *Definition.* For each $n \geq 0$ and X a topological space with $x_0 \in X$, we define

$$\pi_n(X, x_0) = [(I^n, \partial I^n), (X, x_0)].$$

1.2.3 *Remark.* (i) When $n = 0$, we have $I^0 = \mathrm{pt}$ and $\partial I^0 = \emptyset$, therefore $\pi_0(X)$ is the set of path components of X.

(ii) When $n = 1$, this is a group, but need not be abelian (for example, consider the wedge of two circles).

(iii) Note that $I^n / \partial I^n \simeq S^n$ and $\partial I^n / \partial I^n \simeq s_0$. By the universal property of the quotient map, we see that

$$\pi_n(X, x_0) \cong [(S_n, s_0), (X, x_0)].$$

1.2.4 *Definition.* A maps of pairs $(X, A) \to (Y, B)$ is a map $f\colon X \to Y$ with $f(A) \subseteq B$, i.e., the diagram:

$$\begin{tikzcd} A \arrow[r] \arrow[d, hook] & B \arrow[d, hook] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

commutes.

1.2.5 Proposition. *If $n \geq 1$, then $\pi_n(X, x_0)$ is a group with respect to the operation*

$$(f+g)(t_1, \ldots, t_n) = \begin{cases} f(2t_1, t_2, \ldots, t_n) & 0 \leq t_1 \leq 1/2 \\ g(2t_1 - 1, t_2, \ldots, t_n) & 1/2 \leq t_1 \leq 1. \end{cases}$$

Proof. The identity is given by the constant map taking all of I^n to x_0 and the inverse of f is given by

$$-f(t_1, \dots, t_n) = f(1 - t_1, t_2, \dots, t_n). \qquad \square$$

1.2.6 Remark. Call the group operation $+_1$. Note that we can also define an operation $+_i$ for $1 \le i \le n$ by the same formula on the i-th coordinate.

1.2.7 Theorem. *All of these operations agree, and for $n \ge 2$, these give $\pi_n(X, x_0)$ the structure of an abelian group.*

This is a consequence of the following exercise, known as the Eckmann–Hilton lemma.

Exercise 1: Eckmann–Hilton lemma

Let M be a set and let $*, \bullet$ be two binary operations on M, $*, \bullet \colon M \times M \to M$, both with unit elements. Suppose that

$$(a * b) \bullet (c * d) = (a \bullet c) * (b \bullet d)$$

for all $a, b, c, d \in M$. Show that the units agree, these two operations agree, and that the multiplication is commutative and associative.

1.2.8 Remark. Let use show that

$$(f +_1 g) +_2 (h +_1 i) \simeq (f +_2 h) +_1 (g +_2 i).$$

Indeed, both of these are the following map

$$(t_1, t_2, \dots,) \mapsto \begin{cases} f(2t_1, 2t_2, \dots,) & [1/2, 0] \times [1/2, 0] \\ g(2t_1 - 2, 2t_2, \dots,) & [1/2, 1] \times [0, 1/2] \\ h(2t_1, 2t_2 - 2, \dots) & [0, 1/2] \times [1/2, 1] \\ i(2t_1 - 1, 2t_2 - 2, \dots) & [1/2, 1] \times [1/2, 1]. \end{cases}$$

1.2.9 Remark. Another approach is given by the following visualization: That is, so long as $n \ge 2$, we can shrink the domain of f and g

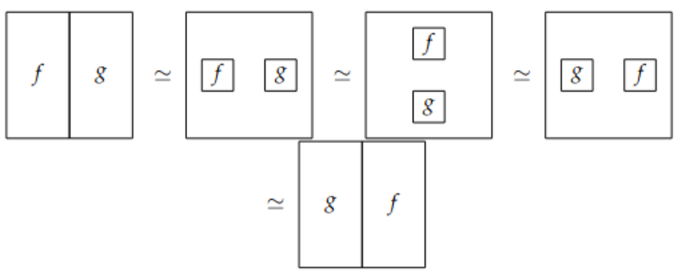


Figure 1.2: $f + g \simeq g + f$.

to smaller cubes (mapping the remaining region to the base point), slide f and g past each other, and then increase the domains back again.

Exercise 2

Let G be a topological group with identity element e, then $\pi_1(G, e)$ is abelian.
Hint: Use Eckmann–Hilton, or note the following: A topological group is a group object in the category of topological spaces. What is a group object in the category of groups?

1.2.10 Proposition. *If $n \geq 1$ and X is path connected, then there is an isomorphism $\beta_\gamma : \pi_n(X, x_0) \xrightarrow{\simeq} \pi_n(X, x_0)$ given by $\beta_\gamma([f]) = [\gamma \circ f]$ where γ is a path in X from x_1 to x_0 and $\gamma \circ f$ is constructed by first shrinking the domain of f to a smaller cube inside of I^n, and then inserting the path γ radially from x_1 to x_0 on the boundaries of these cubes.*

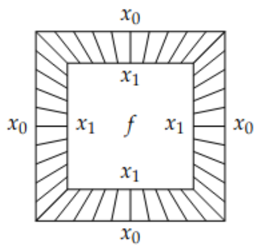
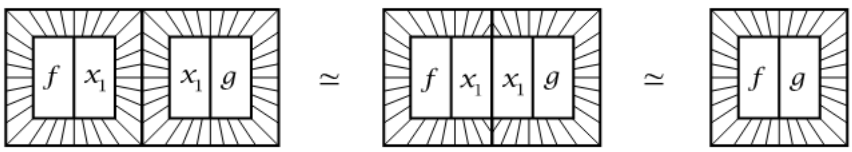


Figure 1.3: β_γ.

Proof. Observe the following:

1. $\gamma \circ (f + g) \simeq \gamma \circ f + \gamma \circ g$, i.e., β_γ is a group homomorphism.
2. $(\gamma \circ \eta) \circ f \simeq \gamma \circ (\eta \circ f)$, for η a path from x_0 to x_1.
3. $c_{x_0} \circ f \simeq f$, where c_{x_0} denotes the constant path based at x_0.
4. β_γ is well-defined with respect to homotopies of f or γ.

The only point that is perhaps not clear is (i). For this, we deform f and g to be constant on the right and left halves of I^n, respectively, producing maps we call $f + 0$ and $0 + g$. We then excise a wider symmetric middle slab of $\gamma(f + 0)$ and $\gamma(0 + g)$ until it becomes $\gamma(f + g)$: $\square$



1.2.11 Remark. Therefore if X is path-connected, different choices of base point x_0 yield isomorphic groups $\pi_n(X, x_0)$, which may then simply be written as $\pi_n(X)$.

1.2.12 Lemma. *If $\{X_\alpha\}$ is a collection of path-connected spaces, then $\pi_n(\prod_\alpha X_\alpha) \cong \prod_\alpha \pi_n(X_\alpha)$.*

Proof. Note that $\mathrm{Hom}(Y, \prod_\alpha X_\alpha) \simeq \prod_\alpha \mathrm{Hom}(Y, X_\alpha)$. In particular, a map $S^n \to \mathrm{Hom}(Y, \prod_\alpha X_\alpha)$ is determined by a collection of maps $S^n \to X_\alpha$. Likewise, a homotopy $S^n \times I \to \prod_\alpha X_\alpha$ is determined by a colletion of homotopies $S^n \times I \to X_\alpha$. This implies the result. □

1.2.13 Proposition. *Homotopy groups are functorial: given a map $\phi\colon X \to Y$ we get group homomorphisms $\phi_*\colon \pi_n(X, x_0) \to \pi_n(X, \phi(x_0))$ given by $[f] \mapsto [\phi \circ f]$ for all $n \geq 1$.*

Proof. We have the following:

1. ϕ_* is well-defined: if $f \simeq g$ via ψ_t, then $\phi \circ \psi_t$ defines a homotopy between $\phi \circ f$ and $\phi \circ g$.

2. This is a group homomorphism: $\phi \circ (f + g) \simeq \phi \circ g + \phi \circ g$ by the definition of the addition operation. Therefore.

$$\phi_*[f + g] = \phi_*[f] + \phi_*[g].$$

□

Exercise 3

If $\phi\colon X \to Y$ is homotopy equivalence (not necessarily base-point preserving), then $\pi_*\colon \pi_n(X, x_0) \to \pi_n(Y, \phi(y_0))$ is an isomorphism.

1.2.14 Remark. We recall the following lifting property: Suppose $p\colon (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ is a covering, and there is a map $f\colon (Y, y_0) \to (X, x_0)$ with Y path-connected and locally path-connected. Then a lift $\tilde{f}$ exists if and only if $f_*\pi_1(Y, y_0) \subseteq p_*\pi_1(\tilde{X}, \tilde{x}_0)$.

$$\begin{tikzcd}
 & (\tilde{X}, \tilde{x}_0) \arrow[d, "p"] \\
(Y, y_0) \arrow[r, "f"] \arrow[ur, dashed, "\tilde{f}"] & (X, x_0)
\end{tikzcd}$$

1.2.15 Proposition. *If p is a covering, then $p_*\colon \pi_n(\tilde{X}, \tilde{x}_0) \to \pi_n(X, x_0)$ is an isomorphism for all $n \geq 2$.*

Proof. Let us first show surjectivity. To that end, suppose we have a map $f\colon (S^n, s_0) \to (X, x_0)$ where $n \geq 2$. The assumption on n gives $\pi_1(S^n) = 0$, so $f_*\pi_1(S^n, s_0) \subseteq p_*\pi_1(\tilde{X}, \tilde{x}_0)$ holds. We therefore find a lift in the following:

$$\begin{tikzcd}
 & (\tilde{X}, \tilde{x}_0) \arrow[d, "p"] \\
(S^n, s_0) \arrow[r, "f"] \arrow[ur, dashed, "\tilde{f}"] & (X, x_0)
\end{tikzcd}$$

Then $p_*[\tilde{f}] = [f]$, and p_* is surjective.

To see that p_* is injective, let $[\tilde{f}] \in \ker(p_*)$, i.e., $p_*[\tilde{f}] = [p \circ \tilde{f}] = 0$. Let $f = p \circ \tilde{f}$, then this is homotopic to the constant map $f \simeq c_{x_0}$

via a homotopy $\phi_t\colon (S^n, s_0) \to (X, x_0)$ with $\phi_1 = f$ and $\phi_0 = c_{x_0}$. By the same argument as above, the homotopy ϕ_t can be lifted to $\tilde{\phi}_t$. This satisfies $p \circ \tilde{\phi}_1 \simeq \phi_1$ and $p \circ \tilde{\phi}_0 \simeq \phi_0$. By the uniqueness of lifts, we must have $\tilde{\phi}_1 \simeq \tilde{f}$ and $\tilde{\phi}_0 \simeq c_{x_0}$. In other words, $\tilde{\phi}_t$ gives a homotopy between $\tilde{f}$ and c_{x_0}, so that $[\tilde{f}] = 0$, and p_* is injective. □

1.2.16 Example. S^1 has universal cover $p\colon \mathbb{R} \to S^1$, $p(t) = e^{2\pi i t}$. Then $\pi_n(S^1) \cong \pi_n(\mathbb{R}) \cong 0$ for $n \geq 2$.

Exercise 4

Find two spaces X, Y with $\pi_n X \cong \pi_n Y$ but $X \not\simeq Y$.
Hint: What is the universal cover of $\mathbb{R}P^n$?

1.2.17 Remark (Relative homotopy groups). Suppose we have (X, x_0) and a subspace A containing x_0. We note that $i_*\colon \pi_n(A, x_0) \to \pi_n(X, x_0)$ is not injective in general (example, take S^1 into $\mathbb{R}^2$). An element in the kernel of i_* is a map $f\colon (I^n, \partial I^n) \to (A, x_0)$ such that

$$(I^n, \partial I^n) \xrightarrow{f} (A, x_0) \xrightarrow{i} (X, x_0)$$

is homotopic to c_{x_0}. This means there exists a homotopy

$$H\colon I^n \times I \to X$$

such that $H(-,1) = f, H(-,0) = c_{x_0}$ and $H\mid_{\partial I^n \times I} = c_{x_0}$.

If we define $J^n = I^n \times \{0\} \cup \partial I^n \times I \subseteq I^n \times I$, then this is a map of triples

$$H\colon (I^{n+1}, \partial I^{n+1}, J^n) \to (X, A, x_0).$$

1.2.18 Definition.

$$\pi_n(X, A, x_0) = [(I^n, \partial I^n, J^{n-1}), (X, A, x_0)]$$

1.2.19 Remark. Equivalently,

$$\pi_n(X, A, x_0) = [(D^n, S^{n-1}, s_0), (X, A, x_0)]$$

1.2.20 Proposition. *If $n \geq 2$, then $\pi_n(X, A, x_0)$ is a group, and if $n \geq 3$, then it is abelian.*

For all $n \geq 2$, a map of pairs $\phi\colon (X, A, x_0) \to (Y, B, y_0)$ induces homomorphisms $\phi_*\colon \pi_n(X, A, x_0) \to \pi_n(Y, B, y_0)$ for all $n \geq 2$.

Proof. This is similar to the case of $\pi_n(X)$ itself, and the details are left to the reader. □

1.2.21 Theorem. *The relative homotopy groups (X, A, x_0) fit into a long exact sequence*

$$\cdots \to \pi_n(A, x_0) \xrightarrow{i_*} \pi_n(X, x_0) \xrightarrow{j_*} \pi_n(X, A, x_0) \xrightarrow{\partial_n} \pi_{n-1}(A, x_0) \to \cdots$$

where the map ∂_n is defined by $\partial_n([f]) = [f\mid_{I^{n-1}}]$.

The proof relies on the following.

1.2.22 Lemma (Compression criterion)**.** *A map $f\colon (D^n, S^{n-1}, x_0) \to (X, A, x_0)$ represents 0 in $\pi_n(X, A, x_0)$ if and only if $f \sim g$ rel S^{n-1}, where g is a map whose image is contained entirely in A.*

Proof. Suppose $[f] = [g]$ with g as in the statement of the lemma. Note that there is a deformation of D^n onto x_0, and so $[f] = [g] = 0$ in $\pi_n(X, A, x_0)$.

Conversely, suppose that $[f]$ represents 0 in $\pi_n(X, A, x_0)$. This means there exists a homotopy, relative to S^{n-1}, $F\colon D^n \times I \to X$ with $F|_{D^n \times \{0\}} = f$, $F|_{D^n \times 1} = c_{x_0}$ and $F|_{S^{n-1} \times I} \subseteq A$. We can restrict F to a family of n-disks in $D^n \times I$ starting with $D^n \times \{0\}$ and ending with the disk $D^n \times \{1\} \cup S^{n-1} \times \{1\}$, all the disks in the family having the same boundary, then we get a homotopy from f to a map in A, stationary on S^{n-1} (said in other words, we can deformation retract $D^n \times [0,1]$ onto $D^n \times \{1\} \cup S^{n-1} \times I$). □

We now prove the existence of the long exact sequence.[2]

[2] This is the type of proof that is best done by the reader themselves.

Proof of Theorem 1.2.21. **Step 1.** Let us first show exactness at $\pi_n(X, x_0)$.

We first show $\mathrm{im}(i_*) \subseteq \ker(j_*)$. Note that $j_* i_*$ is induced by the composition $j \circ i$ and that these are both inclusion maps. Therefore, for $[f] \in \pi_n(A, x_0)$ we have $j_* i_*[f] = [j \circ i \circ f]$, but this has image contained in A, and so $j_* i_*[f] = 0$. This shows $\mathrm{im}(i_*) \subseteq \ker(j_*)$.

To see the converse (namely, $\ker(j_*) \subseteq \mathrm{im}(i_*)$) let $[f] \in \ker(j_*)$, i.e. $[j \circ f] = 0$. Note that again j is an inclusion map, and by the compression criteria $f \simeq g'$ relative to S^{n-1}, where g' has image contained in A. Since $x_0 \in S^{n-1}$, the homotopy fixes the basepoint, i.e, $[f] = [g'] \in \pi_n(X, x_0)$. But because g' has image in A, $[g'] \in \pi_n(A, x_0)$ and $i_*[g'] = [i \circ g'] = [f]$, so $[f] \in \mathrm{im}(i_*)$.

Step 2. Let us now show exactness at $\pi_n(X, A, x_0)$.

Note that the composite $\partial \circ j_* = 0$ since the restriction of a map $(I^n, \partial I^n, J^{n-1}) \to (X, x_0, x_0)$ to I^{n-1} has image x_0 and so represents 0 in $\pi_{n-1}(A, x_0)$. Therefore, $\mathrm{im}(j_*) \subseteq \ker(\partial)$. For the converse, suppose $[f] \in \ker(\partial)$. This means there exists a basepoint preserving homotopy $H\colon I^{n-1} \times I \to A$ (relative to ∂I^{n-1}) from $f|_{I^{n-1} \times \{0\}}$ to the constant map where the image of H is contained entirely in A. We can then define another homotopy H, such that $G_0 = f$, $G_t|_{I^{n-1}} = H_t$ and the rest of the image of G_t is $f[I^n]$ union with the image of H_s for $0 \leq s \leq t$. This homotopy maps S^{n-1} into A at all times, so $[f] = [G_1]$. Moreover, G_1 maps the boundary of I^n to x_0, so $[G_1] \in \pi_n(X, x_0)$. Altogether,

$$j_*[G_1] = [j \circ G_1] = [G_1] = [f]$$

so $\ker(\partial) \subseteq \mathrm{im}(j_*)$.

Step 3: Exactness at $\pi_n(A, x_0)$.

Let $[f] \in \pi_n(X, A, x_0)$ then $i_* \partial \in \pi_{n-1}(X, x_0)$ is the class represented by $f|_{I^{n-1}}$ and this is homotopic relative J^{n-2} to the constant map to x_0, via f viewed as a homotopy. So this implies $\mathrm{im}(\partial_*) \subseteq \ker(i_*)$. Conversely, let $[f] \in \ker(i_*)$ i.e., $i_*[f] = [i \circ f] = 0$.

Therefore, there exists a homotopy H between f and a constant map through a homotopy that has image in X and preserves x_0. Since $H_0 = f$ has image in A and H_1 has image $\{x_0\}$, and H_0 takes the boundary to $\{x_0\}$, we see that $[H] \in \pi_n(X, A, x_0)$, and moreover $\partial([H]) \simeq f$. Therefore, $[f] \in \operatorname{im}(\partial)$, and $\operatorname{im}(\partial) = \ker(i_*)$. □

1.2.23 Definition. A pair (X, A) with basepoint x_0 is said to be n-connected if $\pi_i(X, A) = 0$ for all $i \leq n$.[3]

[3] A 0-connected space is exactly a path-connected space.

1.2.24 Lemma. *A pair (X, A) is n-connected if and only if $\pi_i(A) \xrightarrow{i_*} \pi_i(X)$ is an isomorphism for $i < n$ and a surjection for $i = n$.*

Proof. Use the long exact sequence in homotopy. □

Exercise 5

Let X be a path-connected space, and CX the cone on X. Show that

$$\pi_n(CX, X, X_0) \cong \pi_{n-1}(X, x_0)$$

for $n \geq 1$.

1.3 *Cofibrations and the homotopy extension property*

1.3.1 Definition. Let $\mathcal{C}$ be a class of topological spaces. A map $i \colon A \to X$ has the homotopy extension property (HEP) if, for every $Y \in \mathcal{C}$, the following extension property has a solution[4]

[4] Here $i_0(x) = (x, 0)$.

$$\begin{tikzcd}
A \arrow[r, hook, "i_0"] \arrow[d, "i"'] & A \times I \arrow[d, "i \times \mathrm{id}"] \arrow[ddr, bend left, "H"] & \\
X \arrow[r, hook, "i_0"'] \arrow[drr, bend right, "f"'] & X \times I \arrow[dr, dashed, "\exists \tilde{H}"] & \\
& & Y
\end{tikzcd}$$

A map $f \colon A \to X$ is a cofibration if it has the HEP with respect to all spaces Y.[5]

[5] We will see later that cofibrations are always inclusions, and, if X is Hausdorff, are always closed maps.

1.3.2 Remark. Note that we do not ask that $\tilde{H}$ is unique.

1.3.3 Remark. If we are in a 'nice' category of topological spaces (see CREF), which we always assume, then we have an adjunction

$$\operatorname{Hom}(X, \operatorname{Hom}(Y, Z)) \cong \operatorname{Hom}(X \otimes Y, Z)$$

of topological spaces, where $\operatorname{Hom}(Y, Z)$ is given the compact open topology. Writing, $Z^Y := \operatorname{Hom}(Y, Z)$, the homotopy extension property admits a reformulation in the following diagram

$$\begin{tikzcd}
A \arrow[r, "h"] \arrow[d, "i"'] & Y^I \arrow[d, "p"] \\
X \arrow[r, "f"'] \arrow[ur, dashed, "\exists \tilde{h}"] & Y
\end{tikzcd}$$

where $p \colon Y^I \to Y$ is the evaluation at 0 map. It is often easier to work with this equivalent diagram.

Exercise 6

Let (X, A) have the HEP, and assume moreover that $i\colon A \to X$ is a retract up to homotopy. Show that A is a retract of X.

1.3.4 Lemma. *Let $J = [0, 1]$.*

(i) The inclusion $i_0\colon X \to X \times J$ has the homotopy extension property for all Y.

(ii) The inclusion $i_0\colon X \to CX$ has the homotopy extension property for all Y.

Proof. The proof in both cases is very similar; we do the first case in some detail. We are claiming there exists a lift $\tilde{H}$ in the following diagram:

$$\begin{tikzcd}
X \arrow[r, hook, "i_0"] \arrow[d, "i"'] & X \times I \arrow[d, "i\times\mathrm{id}"] \arrow[ddr, bend left, "H"] & \\
X \times J \arrow[r, hook, "i_0"'] \arrow[drr, bend right, "f"'] & X \times J \times I \arrow[dr, dashed, "\exists\tilde{H}"] & \\
 & & Y
\end{tikzcd}$$

Geometrically, we will do this in two parts: we will define a map that "stacks" the two intervals on top of each other, i.e., we construct a map $G\colon X \times J \times I \to X \times [0, 2]$. We will then do H on one part of the cylinder, and f on the remaining part.

For the first part, let $G\colon X \times J \times I \to X \times [0, 2]$ be defined as[6]

$$G(x, t, s) = (x, t(1 + s)).$$

[6] To see what is going on it is worth testing some cases and drawing pictures. For example, when $t = 0$ we have $G(x, 0, s) = (x, 0)$. When $t = 1$ we have $G(x, 1, s) = (x, 1 + s)$. When $s = 0$ we have $G(x, t, 0) = (x, t)$ and when $s = 1$ we have $G(x, t, 1) = (x, 2t)$.

We then define $F\colon X \times [0, 2] \to Y$ by

$$F(x, k) = \begin{cases} f(x, k) & 0 \le k \le 1 \\ H(x, k/2) & 1 \le k \le 2. \end{cases}$$

Putting these together and defining $\tilde{H} := F \circ G$, we see that[7]

$$\tilde{H}((x, t), s) = \begin{cases} f(x, 1 - (1 - t)(1 + s)), & (1 - t)(1 + s) \le 1 \\ H(x, (1 - t)(1 + s) - 1), & (1 - t)(1 + s) \ge 1. \end{cases}$$

[7] Again, it is worthwhile to consider some cases. For example, if $t = 0$, then $(1 - t)(1 + s) = (1 + s) \ge 1$ for all s, so $\tilde{H}((x, 0), s) = H(x, s)$. At the other extreme, if $t = 1$, then $(1 - t)(1 + s) = 0 \le 1$ for all s, so $\tilde{H}((x, 1), s) = f(x, 1)$.

One verifies directly that this gives the required extension. $\square$

1.3.5 Remark. We recall that given a map $f\colon X \to Y$, the mapping cylinder (see Figure 1.4) is the pushout

$$\begin{tikzcd}
X \arrow[r, "i_0"] \arrow[d, "f"'] & X \times I \arrow[d] \\
Y \arrow[r] \arrow[ur, phantom, "\lrcorner"] & M_f
\end{tikzcd}$$

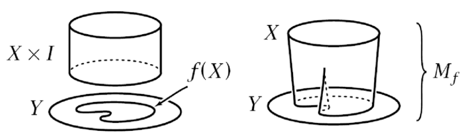


Figure 1.4: The mapping cylinder.

In formulas,

$$M_f = ((X \times I) \coprod Y)/((0, x) \sim f(x),\ \forall x \in X)$$

Note that M_f deformation retracts on Y by sliding each point $(x,t) \in M_f$ to the end-point. Note that we have a natural map $j\colon X \to M_f$ sending x to $(x,1)$.

1.3.6 Lemma. *The map $j\colon X \to M_f$ has the HEP for all spaces Y.*

Proof. The proof is similar to the previous lemma; one just has to modify the end point by defining

$$\tilde{H}|_{Y\times I}(y,s) = f(y,0).$$

$\square$

1.3.7 Corollary. *The inclusion $S^{n-1} \to D^n$ is a cofibration.*

Proof. Simply note that $D^n \simeq CS^{n-1}$. $\square$

There is a universal test space for cofibrations.

1.3.8 Proposition. *Let $i\colon A \to X$, and let M_i be the mapping cylinder. Then $i\colon A \to X$ is a cofibration if and only if there exists a map $r\colon X \times I \to M_i$ making the diagram*

$$\begin{tikzcd}
A \arrow[rr, "i_0"] \arrow[dd, "i"'] & & A \times I \arrow[dl, "i\times id"'] \arrow[dd] \\
& X \times I \arrow[dr, dashed, "\exists r"] & \\
X \arrow[ur, "i_0"'] \arrow[rr] & & M_i
\end{tikzcd}$$

commute.

Proof. If i is a cofibration, then the map r exists as a consequence of the HEP.

For the other direction, if r exists, then for any maps $f\colon X \to Y$ and $H\colon A \times I \to Y$ making the obvious diagram commute, the universal property of the pushout gives us a map $H'\colon M_i \to Y$. Then let $\tilde{H} = H' \circ r$, and we are done. $\square$

1.3.9 Corollary. *If $A \subseteq X$, then $I\colon A \to X$ is a cofibration if and only if $X \times I$ is a retract of $M_i = X \times \{0\} \cup A \times I$.*

1.3.10 Corollary. *A cofibration $i\colon A \to X$ is an injection. If X is Hausdorff, then $i(A)$ is closed in X.*

Proof. Let $J\colon A \times I \to M_i$ be the canonical map (arising from the definition of M_i as a pushout). Then, $J(a,1) = r(i(a),1)$, and observe that $J\mid_{A\times\{1\}}$ is the identity, as it is the top of the mapping cylinder. So, $i(a) \neq i(a')$ if $a \neq a'$, i.e., i is injective.

Because $i\colon A \to X$ is a cofibration, so is $i(A) \to X$. Hence $X \times I$ retracts onto $X \times \{0\} \cup i(A) \times I$ (Corollary 1.3.9). For a Hausdorff space, the image of a retract is closed, and so $X \times \{0\} \cup i(A) \times I$ is a closed subspace of $X \times I$. Intersecting with $X \times \{1\}$, we see that $i(A) \times \{1\}$ is closed in $X \times \{1\}$, i.e, $i(A)$ is closed in X. $\square$

The following (rather pathological) example shows that i is not always a closed map if X is not Hausdorff.

Exercise 7

Let $A = \{a\}$ and $X = \{a, b\}$ with the trivial topology. Show that the inclusion $A \to X$ is a cofibration whose image is not closed.

1.3.11 Remark. The next goal is to show that CW-complexes (X, A) are always cofibrations. The key is the following exercise.

Exercise 8

(a) Suppose $\{(X_i, A_i)\}$ are a collection of spaces satisfying the HEP, then so does $\{(\coprod X_i, \coprod A_i)\}$.

(b) Suppose (X, A) satisfies the HEP, and $f\colon A \to B$ is a continuous map. Let $Y = X \cup_f B$ be the pushout, then (Y, B) satisfies the HEP.

(c) Suppose $A = X_0 \subseteq X_1 \subseteq \cdots \subseteq X_n \subseteq X_{n+1} \subseteq \cdots$.

Let $X = \operatorname{colim} X_i$. If each (X_i, X_{i-1}) satisfies the HEP, then so does (X, A).

1.3.12 Theorem. *A relative CW-complex (X, A) satisfies the HEP.*

Proof. Using Corollary 1.3.7 and the previous exercise we see that (S^{n-1}, D^n) satisfies the HEP $\implies$ $(\coprod S^{n-1}, \coprod D^n)$ satisfies the HEP. Inductively, (X_{n-1}, A) satisfies the HEP and by the exercise (X, A) satisfies the HEP. □

1.3.13 Remark. One can also prove this directly by constructing a deformation retract $r\colon X \times I \to X \times \{0\} \cup A \times I$.

1.3.14 Remark. One can consider the following question: Suppose that $A \subset X$ with A contractible, then is $X \simeq X/A$? Surprisingly, this is not true in general. Indeed, let $A = S^1 \setminus \{(1, 0)\}$ and consider $A \to S^1$. Then $S^1/A \cong T$, the $T = \{a, b\}$ the two point space with open sets $\emptyset, \{a\}, \{a, b\}$ (this is the Sierpiński space). One can check that this space is contractible.[8] The exact condition we need is that $A \to X$ is a cofibration.

[8] See https://math.stackexchange.com/a/264789/64273.

1.3.15 Definition. A contracting homotopy is a map $H\colon X \times I \to X$ such that $H(x, 0) = \mathrm{id}_X$ and $H(x, 1) = c_{x_0}$, the constant map at x_0.

1.3.16 Proposition. *Suppose $A \subseteq X$ and $x_0 \in A$. Suppose there exists a map $H\colon X \times I \to X$ such that $H\,|_{X\times\{0\}} = \mathrm{id}_X$ and $H\,|_{A\times I}$ has image in A and is a contacting homotopy for A. Then $q\colon X \to X/A$ is a homotopy equivalence.*

Proof. We need to find $p\colon X/A \to X$ such that $q \circ p \simeq \mathrm{id}_{X/A}$ and $p \circ q \simeq \mathrm{id}_X$. The quotient map has a set-theoretic section given by

$$s(\overline{x}) = \begin{cases} x & x \notin A \\ x_0 & x \in A \end{cases}$$

Define $p\colon X/A \to X$ by the following diagram

$$\begin{tikzcd} X \arrow[r, "q"] & X/A \arrow[r, "s"] \arrow[dr, "p"'] & X \arrow[d, "H|_{X\times\{1\}}"] \\ & & X \end{tikzcd}$$

Assume for a moment that p is continuous. Then $p \circ q = H\,|_{X\times\{1\}}$, and so H gives a homotopy between id_X and $p \circ q = H\,|_{X\times\{1\}}$. Likewise, if we define G by

$$\begin{tikzcd} X/A \times I \arrow[r, "s\times\mathrm{id}"] \arrow[drr, "G"'] & X \times I \arrow[r, "H"] & X \arrow[d, "q"] \\ & & X/A \end{tikzcd}$$

and assume that G is continuous, then

$$G(\overline{x}, 1) = q \circ (H\,|_{X\times\{1\}} \circ s) = q \circ p,$$

so that G is a homotopy between $\mathrm{id}_{X/A}$ and $q \circ p$. To see that p is continuous, let $U \subset X$ be open, then

$$q^{-1}p^{-1}(U) = (p \circ q)^{-1}(U) = (H\,|_{X\times\{1\}})^{-1}(U)$$

is open in X by the continuity of $H\,|_{X\times\{1\}}$, hence $p^{-1}(U)$ is open in X/A by the definition of the quotient topology, and so p is continuous. We leave the proof of continuity of G to the reader. $\square$

1.3.17 Theorem. *Let $A \subseteq X$ be a subspace with A contractible. Suppose that the inclusion $i\colon A \to X$ is a cofibration, then $X \to X/A$ is a homotopy equivalence.*

Proof. Let $h\colon A \to I \to A$ be a contracting homotopy. Let $H\colon A \times I \to X$ be the composition of h with the inclusion map of A into X, i.e., the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, hook, "i_0"] \arrow[d, "i"'] & A \times I \arrow[d, "i\times\mathrm{id}"] \arrow[ddr, bend left, "H"] & \\ X \arrow[r, hook, "i_0"'] \arrow[drr, bend right, "\mathrm{id}_X"'] & X \times I \arrow[dr, dashed, "\exists\tilde{H}"] & \\ & & X \end{tikzcd}$$

By the HEP, the dotted map $\tilde{H}$ exists as in the diagram. This map satisfies the conditions of Proposition 1.3.16:

(i) $\tilde{H}\colon X \times \{0\} \to X$ is the identity.

(ii) $\tilde{H}(A \times I) = H(A \times I) = h(A \times I) \subseteq A$.

(iii) $\tilde{H}(A \times \{1\}) = x_0$.

Therefore, $q\colon X \to X/A$ is a homotopy equivalence, as claimed. $\square$

Exercise 9: Cofibrations are pushout closed.

Let $i\colon A \to X$ be a cofibration, and $g\colon A \to B$ any map, then the induced map $B \to B \cup_g X$ is a fibration.

1.4 *Fibrations and the homotopy lifting property*

The dual notion of a cofibration is a fibration, where the homotopy extension property is replaced by the homotopy lifting property.

1.4.1 Definition. Let $\mathcal{E}$ be a class of topological spaces. Assume that $p\colon E \to B$ is a continuous map, then we say that p has the homotopy lifting property (with respect to $\mathcal{E}$) if for every $X \in \mathcal{E}$, and map $f\colon X \to E$ and every homotopy $H\colon X \times I \to B$ that begins with $p \circ f$, we can lift it to a homotopy $\tilde{H}\colon X \times I \to E$ that begins with f, i.e., $p \circ \tilde{H} = H$ and $\tilde{H}(x, 0) = f(x)$. In a diagram, we require the lift $\tilde{H}$ in the following:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "i_0"'] & E \arrow[d, "p"] \\ X \times I \arrow[r, "H"'] \arrow[ur, dashed, "\exists \tilde{H}"] & B \end{tikzcd}$$

If $\mathcal{E}$ is the class of all topological spaces, then p is called a (Hurewicz) fibration, while if $\mathcal{E} = \{I^n\}$ (or equivalently, the class of CW-complexes), then p is called a Serre fibration.

1.4.2 Remark. As in Remark 1.3.3, there is an equivalent way to present the homotopy lifting property: we ask for the lift $\tilde{h}$ as shown in the following

$$\begin{tikzcd} X \arrow[drr, bend left, "f"] \arrow[dr, dashed, "\exists \tilde{h}"] \arrow[ddr, bend right, "h"'] & & \\ & E^I \arrow[r, "ev_0"] \arrow[d, "p_*"'] & E \arrow[d, "p"] \\ & B^I \arrow[r, "ev_0"'] & B \end{tikzcd}$$

This makes it clear how the homotopy lifting property is dual to the homotopy extension property.

1.4.3 Remark. We can also talk about the homotopy lifting property with respect to a pair (X, A): namely, a map $p\colon E \to B$ has the homotopy lifting property with respect to a pair (X, A) if each homotopy $H\colon X \times I \to B$ lifts to a homotopy $\tilde{H}\colon X \times I \to E$ which agrees with a given homotopy H_A on $A \times I$. In a diagram, we ask for the lift $\tilde{H}$ in the following:

$$\begin{tikzcd} X \cup (A \times I) \arrow[r, "f \cup H_A"] \arrow[d, "i_0"'] & E \arrow[d, "p"] \\ X \times I \arrow[r, "H"'] \arrow[ur, dashed, "\exists \tilde{H}"] & B \end{tikzcd}$$

1.4.4 Theorem. *The following are equivalent:*

(i) *p is a Serre fibration.*

(ii) *p has the homotopy lifting property with respect to all n-discs D^n.*

(iii) *p has relative homotopy property with respect to all pairs (D^n, S^{m-1})*

(iv) *p has the relative homotopy property with respect to all CW-pairs (X, A).*

Proof sketch. $(i) \implies (ii)$ is immediate from the definitions.

$(ii) \implies (iii)$ follows because the pairs $(D^n \times I, D^n \times \{0\})$ and $(D^n \times I, D^n \times \{0\} \cup S^{n-1} \times I)$ are homeomorphic.

$(iii) \implies (iv)$ by induction over the skeleton of X; one reduces to the case (iii).

$(iv) \implies (i)$ by taking $A = \emptyset$. □

Exercise 10

Show that the composition of fibrations is a fibration.

1.4.5 Definition. We recall the construction of pullbacks in topological spaces: given maps $p\colon E \to B$ and $f\colon B' \to B$, we let

$$E' = \{(b', e) \in B' \times E \mid p(e) = f(b')\}.$$

This comes with natural projection maps $f'\colon E' \to E$ and $p'\colon E' \to B'$. Then E' is the pull-back in topological spaces, and so we often also denote it by f^*E.

The following is dual to Exercise 9.

1.4.6 Lemma. *If $p\colon E \to B$ satisfies the HLP with respect to the class $\mathcal{E}$, then so does $p'\colon E' \to B'$.*

Proof. Consider the following diagram:

$$\begin{tikzcd} X \arrow[r] \arrow[d, "i_0"'] & E' \arrow[r, "f'"] \arrow[d, "p'"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "p"] \\ X \times I \arrow[r] & B' \arrow[r, "f"'] & B \end{tikzcd}$$

Because $p\colon E \to B$ satisfies the HLP, there is a lift $\tilde{H}'\colon X \times I \to E$ of $X \times I \to B$. Then, by the universal property of the pullback, we get a map $\tilde{H}\colon X \times I \to E'$ satisfying the desired properties. □

1.4.7 Definition. If $p\colon E \to B$ is a fibration, then $F := p^{-1}(*)$ is called the fiber, E is called the total space, and B is the base space. We write this as

$$F \to E \to B.$$

1.4.8 Example. Given a based space X, let

$$PX = \mathrm{Hom}_*(I, X) = \{f\colon I \to X \mid f(0) = *\}$$

be the space of paths starting at the base-point. Then $PX \xrightarrow{p_1} X$ is a fibration with fiber ΩX, the loop space in X (i.e., $f(0) = f(1) = *$). To see this, consider our test diagram, where we must show that $\tilde{H}$ exists:

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "i_0"'] & PX \arrow[d, "p_1"] \\ A \times I \arrow[r, "H"'] \arrow[ur, dashed, "\exists \tilde{H}"] & X \end{tikzcd}$$

Note that for each $a \in A$, $g(a)$ is a path in X which ends at $p_1 g(a) = H(a,0)$. This point is the start of the path $H(a,-)$.

$$\begin{tikzcd}
 & \underset{}{\overset{H(a,0)}{\bullet}} \arrow[dr, bend left, "H(a,-)"] & \\
\bullet \arrow[ur, bend right, "g(a)"] & & \bullet
\end{tikzcd}$$

We will define $\tilde{H}(a,s)(t)$ to be a path running along $g(a)$ and then part-way along $H(a,-)$ ending at $H(a,s)$. In symbols,

$$\tilde{H}(a,s)(t) = \begin{cases} g(a)((1+s)t) & 0 \le t \le 1/(1+s) \\ H(a,(1+s)t-1) & 1/(1+s) \le t \le 1. \end{cases}$$

Then $\tilde{H}(a,0) = g(a)$ and $p_1\tilde{H}(a,s) = \tilde{H}(a,s)(1) = H(a,s)$, as required.

The same argument shows that there is a fibration

$$p_*Y \to Y^I \xrightarrow{p_1} Y$$

where p_*Y is the space of paths with end-point $*$.

1.4.9 Definition. Given $f\colon X \to Y$ the mapping path space P_f (or mapping cocylinder), is the pullback of f along $Y^I \xrightarrow{p_1} Y$, i.e.,

$$\begin{tikzcd}
P_f \arrow[r] \arrow[d, "p'"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y^I \arrow[d, "p_1"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

Note that $P_f \simeq X$.

1.4.10 Proposition. *The map $p\colon P_f \to Y$ given by $p(x,\alpha) = \alpha(1)$ is a fibration.*

Proof. This is very similar to Example 1.4.8. Our test diagram is the following:

$$\begin{tikzcd}
A \arrow[r, "g"] \arrow[d, hook, "i_0"'] & P_f \arrow[d, "p"] \\
A \times I \arrow[r, "H"'] \arrow[ur, dashed, "\exists \tilde{H}"] & Y
\end{tikzcd}$$

Note that $g(a) \in P_f \subset X \times Y^I$, so we can write $g(a) = (g_1(a), g_2(a))$. Here $g_1(a)$ maps via f to the starting point of the path $g_2(a)$ and the commutativity of the diagram implies that the endpoint of the path $g_2(a)$ is the starting point of $H(a,-)$. The lift $\tilde{H}$ will have two components. The x component will be constant in s, i.e., $\tilde{H}_1(a,s) = g_1(a)$. Overall, we define

$$\tilde{H}(a,s) = (g_1(a), \tilde{H}_2(a,s)(-)) \in P_f$$

where[9]

$$\tilde{H}_2(a,s)(t) = \begin{cases} g_2(a)((1+s)t) & 0 \le t \le 1/(1+s) \\ H(a,(1+s)t-1) & 1/(1+s) \le t \le 1. \end{cases}$$

One check directly that $\tilde{H}(a,s)$ has the required properties. □

[9] Compare this to the formula in Example 1.4.8.

As with the homotopy extension property, we have a universal test space. The details (which are dual to Proposition 1.3.8) are left to the reader.

1.4.11 Proposition. *Let $f\colon E \to B$ be a continuous map, then f is a fibration if and only if there exists $s\colon P_f \to E^I$ making the following diagram commute:*

$$\begin{tikzcd}
P_f \arrow[dr, dashed, "\exists s"] \arrow[drr, bend left, "\pi_E"] \arrow[ddr, bend right, "\pi_{B_I}"'] & & \\
& E^I \arrow[r, "ev_0"] \arrow[d, "f_*"'] & E \arrow[d, "f"] \\
& B^I \arrow[r, "ev_0"'] & B
\end{tikzcd}$$

where π_{B_I} and π_E are the projection maps coming from the construction of P_f as a pullback.

1.4.12 Remark. One property of cofibrations that does not dualize to fibrations is that cofibrations are inclusions, but fibrations need not be surjective. Indeed, given $p\colon E \to B$ a fibration, then the composite

$$E \xrightarrow{p} B \hookrightarrow B \coprod *$$

is also a fibration, but is not surjective.

1.4.13 Remark. We will want to talk about exact sequences where the terms appearing may not have a group structure, but are rather only sets with base-points. Therefore, given a sequence of functions

$$A \xrightarrow{f} B \xrightarrow{g} C$$

of sets with base-points, we say that this is exact at B if $f(A) = g^{-1}(c_0)$ where c_0 is the base-point of C. Note that if A, B, C are groups with base-points the identity elements of the group, then exactness of sets corresponds to exactness of groups.

1.4.14 Theorem. *Let $p\colon E \to B$ be a fibration with fiber F and B path-connected. Let Y be any space, then*

$$[Y, F] \xrightarrow{i_*} [Y, E] \xrightarrow{p_*} [Y, B]$$

is exact.

Proof. For one direction, it is clear that $p_*(i_*[g])) = 0$.

Suppose $f \in [Y, E]$ is such that $p_*[f] = [\text{const}]$, i.e., $p \circ f$ is null-homotopic. Let $G\colon Y \times I \to B$ be a null-homotopy, and let $H\colon Y \times I \to E$ be a solution to the lifting problem indicated in the following diagram, using that p is a fibration:

$$\begin{tikzcd}
Y \times \{0\} \arrow[r, "f"] \arrow[d, "i_0"'] & E \arrow[d, "p"] \\
Y \times I \arrow[r, "G"'] \arrow[ur, dashed, "H"] & B
\end{tikzcd}$$

Note now that $p \circ H(y, 1) = G(y, 1) = b_0$, so that $H(y, 1) \in F := p^{-1}(b_0)$. It follows that $[f] = i_*[H(-, 1)]$. $\square$

We have an analogous result for cofibration.

1.4.15 Theorem. *Let $i\colon A \to X$ be a cofibration, and $q\colon X \to X/A$ the quotient map. Let Y be any path-connected space, then the sequence of pointed sets*

$$[X/A, Y] \xrightarrow{q^*} [X, Y] \xrightarrow{i^*} [A, Y]$$

is exact.

Proof. Again, one inclusion is clear: we have $i^*(g^*([g])) = [g \circ q \circ i] = [\text{const}]$.

Now suppose that $f\colon X \to Y$ is a map with $f\ |_A\colon A \to Y$ null-homotopic. Let $h\colon A \times I \to Y$ be a hull-homotopy, and let $F\colon X \times I \to Y$ be the extension as shown in the following diagram:

$$\begin{tikzcd}
A \arrow[r, hook, "i_0"] \arrow[d, "i"'] & A \times I \arrow[d, "i\times\mathrm{id}"] \arrow[ddr, bend left, "H"] & \\
X \arrow[r, hook, "i_0"'] \arrow[drr, bend right, "f"'] & X \times I \arrow[dr, dashed, "F"] & \\
& & Y
\end{tikzcd}$$

Let $f' := F(-, 1)$. Then, $f \sim f'$ and $f'(A) = F(A, 1) = y_0$. By the universal property of the quotient, we can find $g\colon X/A \to Y$ making the following diagram commute:

$$\begin{tikzcd}
X \arrow[r, "f'"] \arrow[d, "q"'] & Y \\
X/A \arrow[ur, dashed, "g'"'] &
\end{tikzcd}$$

Therefore $[f] = [f'] = q^*[g']$. □

As an extension of Theorem 1.4.14 we have the following.

1.4.16 Theorem. *Given a (Serre) fibration $p\colon E \to B$, and base points $b \in B$ and $e \in F := f^{-1}(b)$, then there is an isomorphism $p_*\colon \pi_n(E, F, e) \xrightarrow{\cong} \pi_n(B, b)$ for all $n \geq 1$. Hence, if B is path-connected, there is a long exact sequence of homotopy groups*

$$\cdots \pi_n(F, e) \to \pi_n(E, e) \xrightarrow{p_*} \pi_n(B, b) \to \pi_{n-1}(F, e) \to \cdots$$
$$\cdots \to \pi_0(E, e) \to 0.$$

Proof. We first show that p_* is surjective. Let $[f] \in \pi_n(B, b)$, represented by a map $f\colon (I^n, \partial I^n) \to (B, b)$. Note that $I^{n-1} \times \{0\} \subseteq \partial I^n$, so we can form the diagram

$$\begin{tikzcd}
I^{n-1} \times \{0\} \arrow[r, "*"] \arrow[d] & E \arrow[d, "p"] \\
I^n \arrow[r, "f"'] \arrow[ur, dashed, "\tilde{f}"] & B
\end{tikzcd}$$

where the lift $\tilde{f}$ exists because p is a Serre fibration. Because $f(\partial I^n) = b$, we have $\tilde{f}(\partial I^n) \subseteq F$. So $\tilde{f}$ represents an element of $\pi_n(E, F, e)$ with $p_*([\tilde{f}]) = [p \circ \tilde{f}] = [f]$.

To show injectivity, let $\tilde{f}_0, \tilde{f}_1 \colon (I^n, \partial I^n, J^{n-1}) \to (E, F, e)$ be such that $p_*(\tilde{f}_0) = p_*(\tilde{f}_1)$. Let $H \colon (I^n \times I, \partial I^n \times I) \to (B, b)$ be a homotopy from $p\tilde{f}_0$ to $p\tilde{f}_1$. We can find a lift in the following diagram:

$$\begin{tikzcd} W \arrow[r, "f"] \arrow[d] & E \arrow[d, "p"] \\ I^n \times I \arrow[r, "H"'] \arrow[ur, dashed, "\tilde{H}"] & B \end{tikzcd}$$

where $W = I^n \times \{0\} \cup I^n \times \{1\} \cup \partial I^n \times I$, and f is $\tilde{f}_0$ on $I^n \times \{0\}$, $\tilde{f}_1$ on $I^n \times \{1\}$ and f is constant on $\partial I^n \times I$. The homotopy lifting property gives $\tilde{H}$ defining a homotopy between $\tilde{f}_0$ and $\tilde{f}_1$.

The result then follows (modulo some noise in the low homotopy groups, which can be checked by hand) from Theorem 1.2.21. □

1.4.17 Example (Hopf fibrations). Let $\mathbb{F} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ and fix an integer $d = 1, 2$ or 4, respectively.

Let

$$\mathbb{F}^{n+1} = \begin{cases} \mathbb{R}^{n+1} & \mathbb{F} = \mathbb{R} \\ \mathbb{C}^{n+1} \cong \mathbb{R}^{2(n+1)} & \mathbb{F} = \mathbb{C} \\ \mathbb{H}^{n+1} \cong \mathbb{R}^{4(n+1)} & \mathbb{F} = \mathbb{H}. \end{cases}$$

In other words, $\mathbb{F}^{n+1} \cong \mathbb{R}^{d(n+1)}$. We define the $d(n+1) - 1$ dimensional sphere inside $\mathbb{F}^{n+1}$:

$$S^{d(n+1)-1} = \{(u_0, \ldots, u_n) \mid u_i \in \mathbb{F}, \sum_{k=0}^{n} |u_k|^2 = 1\}.$$

We define the $\mathbb{F}$-projective space by

$$\mathbb{F}P^n := \mathbb{F}^{n+1} \setminus \{0\} / \sim$$

where $(u_0, \ldots, u_n) \simeq (v_0, \ldots, v_n)$ if and only if there exists $\lambda \in \mathbb{F} \setminus \{0\}$ such that $v_i = \lambda u_i$ for $i = 0, \ldots, n$.

Now we have a map $\phi \colon S^{d(n+1)-1} \to \mathbb{F}P^n$ that sends $(u_0, \ldots, u_n)$ to its equivalence class $[u_0, \ldots, u_n]$. Let $F = \phi^{-1}[1, \ldots, 0] = \{(\lambda, 0, \ldots, 0) \mid \lambda \in \mathbb{F}, |\lambda| = 1\} \cong S^{d-1}$.

We will see later in the course that $S^{d-1} \to S^{d(n+1)-1} \to \mathbb{R}P^n$ is a fibration. Explicitly, the fibrations are

$$\begin{aligned} S^0 &\to S^n \to \mathbb{R}P^n \\ S^1 &\to S^{2n+1} \to \mathbb{C}P^n \\ S^3 &\to S^{4n+3} \to \mathbb{H}P^n. \end{aligned}$$

The case $n = 1$ is of interest, as then projective spaces are just spheres, and we obtain the following Hopf fibrations

$$\begin{aligned} S^0 &\to S^1 \to S^1 \\ S^1 &\to S^3 \to S^2 \\ S^3 &\to S^7 \to S^4. \end{aligned}$$

There is also a fibration $S^7 \to S^{15} \to S^8$. It is a difficult theorem of Adams that these are the only fibrations between spheres.

1.5 The homotopy extension and lifting property

We recall that given $f\colon X \to Y$ we defined the mapping path space P_f in Definition 1.4.9, and that $P_f \to Y$ is a fibration.

1.5.1 Definition. The homotopy fiber F_f of $f\colon X \to Y$ is the fiber of the fibration $P_f \to Y$. This is well-defined up to homotopy.

The following is an extremely useful definition in homotopy theory; as we will see later, any weak equivalence between CW-complexes is in fact a homotopy equivalence.

1.5.2 Definition. A map $f\colon (X, x_0) \to (Y, y_0)$ is a weak equivalence if $f_0\colon \pi_0(X, x_0) \to \pi_0(Y, y_0)$ is a bijection and $f_*\colon \pi_k(X, x_0) \to \pi_k(Y, y_i)$ is an isomorphism for all $k \geq 1$.

1.5.3 Lemma. *If $f\colon X \to Y$ is a weak-equivalence, then $\pi_k(F_f) = 0$ for all $k \geq 0$.*

Proof. This follows from the long exact sequence of a fibration (Theorem 1.4.16). □

1.5.4 Remark. We now make a series of remarks about a map $f\colon X \to Y$ with homotopy fiber F_f.

(i) A map $\phi\colon S^{n-1} \to F_f$ corresponds to a diagram

$$\begin{tikzcd} S^{n-1} \arrow[r, "g"] \arrow[d, hook] & X \arrow[d, "f"] \\ C(S^{n-1}) \cong D^n \arrow[r, "h"'] & Y \end{tikzcd} \tag{1.5.5}$$

where g is the composite $S^{n-1} \xrightarrow{\phi} F_f \to X$ (use Lemma 1.1.11).

(ii) The boundary map $\pi_n(Y) \to \pi_{n-1}(F_f)$ in the long exact sequence corresponds to the map sending the class of $\bar{h}\colon S^n \to Y$ to the class of $\pi_{n-1}(F_f)$ represented by the diagram

$$\begin{tikzcd} S^{n-1} \arrow[r, "c"] \arrow[d, hook] & X \arrow[d, "f"] \\ D^n \arrow[r, "h"'] & Y \end{tikzcd}$$

where $c = c_{x_0}$ is the constant map, and h is the composite $D^n \to D^n/S^{n-1} \cong S^n \xrightarrow{\bar{h}} Y$.

(iii) Similarly, the map $\pi_{n-1}(F_f) \to \pi_{n-1}(X)$ corresponds to sending the diagram (1.5.5) to the class $[g]$.

(iv) In particular, $\pi_{n-1}(F_f) = 0$ is equivalent to completing the diagram (1.5.5) in the following way:

$$\begin{tikzcd}
S^{n-1} \arrow[rr, "g"] \arrow[dddd, hook] \arrow[dr] & & X \arrow[dddd, "f"] \\
& C(S^{n-1}) \arrow[ur, dashed, "G"] \arrow[d, hook] & \\
& C(D^n) \arrow[ddr, dashed, "H"] & \\
& & \\
D^n \arrow[rr, "h"'] \arrow[uur] & & Y
\end{tikzcd}$$

We can restate the last remark in the following lemma.

1.5.6 Lemma. *Suppose $f\colon X \to Y$ is a map with homotopy fiber F_f. Then $\pi_{n-1}(F_f) = 0$ if and only if each diagram*

$$\begin{tikzcd}
S^{n-1} \times \{1\} \arrow[r, "g"] \arrow[d, hook] & X \arrow[d, "f"] \\
S^{n-1} \times I \cup D^n \times \{0\} \arrow[r, "h"'] & Y
\end{tikzcd}$$

can be completed to a diagram

$$\begin{tikzcd}
S^{n-1} \times \{1\} \arrow[rr, "g"] \arrow[dddd, hook] \arrow[dr, hook] & & X \arrow[dddd, "f"] \\
& D^n \times \{1\} \arrow[ur, dashed, "G"] \arrow[d, hook] & \\
& D^n \times I \arrow[ddr, dashed, "H"] & \\
& & \\
S^{n-1} \times I \cup D^n \cup \{0\} \arrow[rr, "h"'] \arrow[uur, hook] & & Y
\end{tikzcd}$$

Proof. For any disk we have a homeomorphism $CD^n \cong D^n \times I$ which sends the cone point to the center of $D^n \times \{1\}$, D^n to $S^{n-1} \times I \cup D^n\{0\}$, S^{n-1} to $S^{n-1} \times \{1\}$ and CS^{n-1} to $D^n \times \{1\}$. Thus the statement follows from the last part of the remark. $\square$

This extends to relative CW-complexes.[10]

[10] The following result is perhaps difficult to remember, but very useful!

1.5.7 Theorem (Homotopy extension and lifting property (HELP))**.** *Let (X, A) be a relative CW-pair and $f\colon Y \to Z$ a weak equivalence. Then every diagram*

$$\begin{tikzcd}
A \times \{1\} \arrow[r, "g"] \arrow[d, hook] & Y \arrow[d, "f"] \\
A \times I \cup X \times \{0\} \arrow[r, "h"'] & Z
\end{tikzcd}$$

can be completed to a diagram

$$\begin{tikzcd}
A \times \{1\} \arrow[rr, "g"] \arrow[dr, hook] \arrow[ddd, hook] & & Y \arrow[ddd, "f"] \\
& X \times \{1\} \arrow[ur, dashed, "G"] \arrow[d, hook] & \\
& X \times I \arrow[dr, dashed, "H"] & \\
A \times I \cup X \times \{0\} \arrow[ur, hook] \arrow[rr, "h"'] & & Z
\end{tikzcd}$$

Proof. The proof is by induction over the n-skeleton, with the base case being straightforward. For the inductive step, one reduces to attaching a single cell using the diagram

$$\begin{tikzcd}
X_{n-1} \times \{1\} \arrow[rr, "G_{n-1}"] \arrow[dr, hook] \arrow[ddd, hook] & & Y \arrow[ddd, "f"] \\
& X_n \times \{1\} \arrow[ur, dashed, "G_n"] \arrow[d, hook] & \\
& X_n \times I \arrow[dr, dashed, "H_n"] & \\
X_{n-1} \times I \cup X_n \times \{0\} \arrow[ur, hook] \arrow[rr, "H_{n-1} \cup h"'] & & Z
\end{tikzcd}$$

□

1.5.8 Remark. If (X, A) is a relative CW-complex of dimension n, and $f \colon Y \to Z$ is an n-equivalence[11], the same argument goes through to show that the conclusion of HELP also holds in this case.

[11] That is, the homotopy fiber of f is $(n-1)$-conneceted

Exercise 11

Show that if $f = \mathrm{id}$ in HELP, then we recover the homotopy extension property.

Our first application of this will be Whitehead's theorem. We start with the folliwing lemma.

1.5.9 Lemma. *For any weak equivalence $f \colon Y \to Z$ and any CW-complex X, the induced map $f_* \colon [X, Y] \to [X, Z]$ is a bijection.*

Proof. We first show surjectivity. The pair $X = (X, \emptyset)$ is a relative CW-complex, and so we can apply HELP. Then, for any $h \colon X \to Z$

we have a diagram

$$\begin{tikzcd}
* \arrow[rr] \arrow[dr, hook] \arrow[ddd, hook] & & Y \arrow[ddd, "f"] \\
& X \times \{1\} \arrow[ur, dashed, "G"] \arrow[d, hook] & \\
& X \times I \arrow[dr, dashed, "H"] & \\
X \times \{0\} \arrow[ur, hook] \arrow[rr, "h"'] & & Z
\end{tikzcd}$$

The homotopy $H\colon X \times I \to Z$ satisfies $H_0 = h$ and $H_1 = f \circ G$. Therefore, $[h] = f_*[G]$.

Now assume that $g_0, g_1 \in [X, Y]$ with $f_*[g_0] = f_*[g_1]$. Let $F\colon X \times I \to Z$ be a homotopy between $f \circ g_0$ and $f \circ g_1$. Consider the pair $(X \times I, X \times \partial I)$. This is a relative CW-pair, and HELP gives a diagram

$$\begin{tikzcd}
X \times \partial I \times \{1\} \arrow[rr, "g"] \arrow[dr, hook] \arrow[ddd, hook] & & Y \arrow[ddd, "f"] \\
& X \times I \times \{1\} \arrow[ur, dashed, "G"] \arrow[d, hook] & \\
& X \times I \times J \arrow[dr, dashed, "H"] & \\
X \times \partial I \times J \cup X \times I \times \{0\} \arrow[ur, hook] \arrow[rr, "h"'] & & Z
\end{tikzcd}$$

Here $g\colon X \times \partial I \to Y$ sends (X, ν) to $g_\nu(x)$ for $\nu = 1, 2$ and $h\colon X \times \partial I \times J \to Z$ sends (x, ν, s) to $f \circ g_\nu(x)$. The lift $G\colon X \times I \to Y$ gives a homotopy between g_0 and g_1, i.e., $[g_0] = [g_1]$, and so f_* is injective. $\square$

1.5.10 Remark. Using Remark 1.5.8 we have the following variant of Lemma 1.5.9: If $f\colon Y \to Z$ is n-connected, then for any CW-complex X, the induced map $f_*\colon [X, Y] \to [X, Z]$ is an isomorphism if $n < \dim(X)$ and is surjective if $n = \dim(X)$.

1.5.11 Theorem (Whitehead theorem)**.** *If $f\colon X \to Y$ is weak-equivalence between CW-complexes, then it is a homotopy equivalence.*

Proof. Suppose $f\colon X \to Y$ is a weak equivalence, so $f_*\colon [Y, X] \xrightarrow{\simeq} [Y, Y]$. In other words, there exists a $g\colon Y \to X$ such that $f_*[g] = [f \circ g] = [\mathrm{id}_Y]$, i.e., $f \circ g \simeq \mathrm{id}_Y$. Then, $f \circ g \circ f \simeq f$ as well. But, we also have $f_*\colon [X, X] \xrightarrow{\simeq} [X, Y]$, which sends id_X to f and $g \circ f$ to $f \circ g \circ f \simeq f$. Therefore, $\mathrm{id}_X \simeq g \circ f$, and so $X \simeq Y$. $\square$

1.5.12 Corollary. *If X is a CW-complex with $\pi_i(X) = 0$ for all i, then X is contractible.*

Proof. Apply Whitehead's theorem to the unique map $X \to *$. $\square$

1.5.13 Remark. We cannot drop any assumptions from this theorem, as the following examples show:

(i) We must have a map inducing the weak equivalence; the homotopy groups cannot be abstractly isomorphism, e.g., consider $\mathbb{R}P^2 \times S^3$ and $\mathbb{R}P^3 \times S^2$.

(ii) The Warsaw circle[12] is an example of a space with $\pi_n X = 0$ for all n, but for which X is not contractible.

[12] See, for example, https://wildtopology.com/bestiary/warsaw-circle/

Exercise 12

Use Whitehead's theorem to show that a CW complex is contractible if it is the union of an increasing sequence of sub-complexes $X_1 \subseteq X_2 \subseteq \cdots$ such that each inclusion $X_i \to X_{i+1}$ is null-homotopic.

Exercise 13

Let $f\colon X \to Y$ be a weak homotopy equivalence. Assuming X is a CW-complex, and Y has the homotopy type of a CW-complex, show that f is a homotopy equivalence.

1.6 *The cellular approximation theorem*

The next important theorem is the cellular approximation theorem.

1.6.1 Definition. If X and Y are CW-complexes, and $g\colon X \to Y$ a map, then g is cellular if g carries the n-skeleton of X into the n-skeleton of Y, i.e., $f(X^n) \subseteq Y^n$ for all $n \geq 0$. Similarly, for relative CW-complexes (X, A) and (Y, B) a map $g\colon (X, A) \to (Y, B)$ is cellular if $g((X, A)^n) \subseteq (Y, B)^n$ for all $n \geq 0$.

The main result of this section is the following.

1.6.2 Theorem (Cellular approximation theorem)**.** *Suppose $f\colon (X, A) \to (Y, B)$ is a map of relative CW-complexes, then f is homotopic rel A to a cellular map of pairs.*

We will use the following lemma.

1.6.3 Lemma. *If Z is obtained from Y by attaching cells of dimension $> n$, then $\pi_k(Z, Y) = 0$ for all $k \leq n$, i.e., (Z, Y) is n-connected.*

Proof. We can reduce to the case where $Z = Y \cup_\alpha D^r$ for $\alpha\colon S^{r-1} \to Y, r \geq n+1$. Then $\pi_k(Z, Y)$ corresponds to a map of pairs $g\colon (D^k, S^{k-1}) \to (Z, Y)$. By smooth or simplicial approximation,[13] we can find a map $g'\colon D^k \to Z$ such that $g' = g$ on S^{k-1} and g' misses a point p in the interior of D^k. We can deform $X \setminus \{p\}$ onto A and so deform g' to a map in Y. □

[13] See https://ncatlab.org/nlab/show/simplicial+approximation+theorem

Proof of Theorem 1.6.2. The proof is by induction, with the base case left to the reader. So, by induction, we have $g_{n-1}\colon X^{n-1} \to Y^{n-1}$

and a homotopy $H_{n-1}\colon X^{n-1} \times I \to Y$ such that $H_0 = f$ and $H_1 = g_{n-1}$. Now consider the diagram

$$\begin{tikzcd}
X^{n-1} \times \{1\} \arrow[rr, "\iota_n \circ g_{n-1}"] \arrow[dr, hook] \arrow[ddd] & & Y^n \arrow[ddd] \\
& X^n \times \{1\} \arrow[ur, dashed, "g^n"] \arrow[d, hook] & \\
& X^n \times I \arrow[dr, dashed, "H_n"] & \\
X^{n-1} \times I \cup X^n \times \{0\} \arrow[ur, hook] \arrow[rr, "H_{n-1} \cup f"] & & Y
\end{tikzcd}$$

Here $\iota_n\colon Y^{n-1} \to Y^n$ is the inclusion map. We can apply the version of HELP given in Remark 1.5.8 since $Y^n \hookrightarrow Y$ is an n-equivalence by Lemma 1.6.3, which gives the required extensions g_n and H_n. $\square$

There is also a relative version, whose proof we omit.

1.6.4 Theorem. *Suppose $f\colon (X, A) \to (Y, B)$ is a map of relative CW-complexes which is cellular on a subspace (X', A') of (X, A), then there is a cellular map $g\colon (X, A) \to (Y, B)$ homotopic to f relative to Y such that $g|_{X'} = f$.*

1.6.5 Example. Suppose $i < n$. Taking the standard CW structure on the k-sphere with one 0-cell and one k-cell, we see that any map $S^i \to S^n$ can be made cellular. Because the i-skeleton of S^n is a point, we see that such a map is null-homotopic, and deduce that $\pi_i(S^n) = 0$ for $i < n$.

1.6.6 Corollary. *Let $A \subseteq X$ be CW-complexes, and suppose that all cells of $X \setminus A$ have dimension $> n$. Then $\pi_i(X, A) = 0$ for $i \leq n$.*

Proof. Let $[f] \in \pi_i(X, A)$, i.e., $f\colon (D^i, S^{i-1}) \to (X, A)$. We can use cellular approximation to replace f with a cellular map g with $g(D^i) \subseteq X^i$. But for $i \leq n$ we have $X^i \subseteq A$, so the image of g is contained in A. By the compression criterion (Lemma 1.2.22) we have $[f] = [g] = 0$. $\square$

Exercise 14

Use cellular approximation to show that the n-skeletons of homotopy equivalent CW-complexes without cells of dimension $n+1$ are also homotopy equivalent.

1.7 *Excision and the Freudenthal suspension theorem*

On of the most powerful results in (co)homology is excision. As we will see in this section, things are more complicated for homotopy groups. This is one of the reasons why homotopy groups are (generally) more complicated to compute than homology groups.

1.7.1 Definition. An excisive triad $(X; A, B)$ consists of a space X along with two subspaces $A, B \subseteq X$ such that $X = A^\circ \cup B^\circ$.

1.7.2 Remark. In homology $(A, A \cap B) \to (X, B)$ induces an isomorphism in homology (by excision). This fails in homotopy, as the following example shows.

1.7.3 Example. Let $X = S^2 \vee S^2$ and let $A = C_+$ and $B = C_-$, the southern and northern hemispheres, with a small overlap between the two hemispheres (see Figure 1.5).

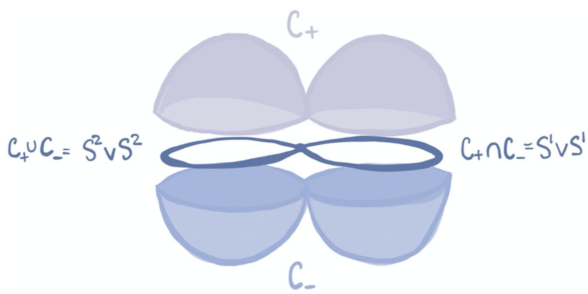


Figure 1.5: The decomposition of $X = S^2 \vee S^2$ into the upper/lower hemispheres $C_\pm$, which intersect along the equator $S^1 \vee S^1$.

Then $C_+ \cap C_- \simeq S^1 \vee S^1$ and $C_+ \cup C_- = S^2 \vee S^2$. Note that both C_+ and C_- are contractible (they have the homotopy type of a wedge of two discs). By the long exact sequence in homotopy we have

$$\pi_i(S^2 \vee S^2, C_-) \cong \pi_i(S^2 \vee S^2) \quad \text{and} \quad \pi_i(C_+, S^1 \vee S^2) \cong \pi_{i-1}(S^1 \vee S^2).$$

In particular, when $i = 2$ we have $\pi_2(S^2 \vee S^2, C_-) \cong \pi_2(S^2 \vee S^2)$ is the free abelian group on two generators while $\pi_2(C_+, S^1 \vee S^1) \cong \pi_1(S^1 \vee S^1)$ is the free group on two generators. Therefore $(C_+, S^1 \vee S^1) \to (S^2 \vee S^2, C_-)$ does not induce an isomorphism on homotopy.

1.7.4 Remark. The following is the homotopy theoretic version of excision. We will not give a full proof as it is quite involved. The full details can be found in May's book, for example.

1.7.5 Theorem (Homotopy excision/Blakers–Massey theorem)**.** *Let $(X; A, B)$ be an excisive triad such that $C = A \cap B$ is non-empty and (A, C) and (B, C) are relative CW-complexes. Suppose $(A, C, *)$ is n-connected and $(B, C, *)$ is m-connected for every choice of base-point $* \in C$. Then the map*

$$\pi_i(A, C) \to \pi_i(X, B)$$

induced by the inclusion is an isomorphism for $i < n + m$ and a surjection for $i = n + m$, i.e., it is an $(n + m)$-equivalence.

Sketch of proof. The proof proceeds by a number of reductions.

Reduction 1: It suffices to prove this when A is built from C by attaching cells of dimension greater than n and B is built by attaching cells of dimension greater than m. Indeed, we claim we can replace the pair (A, C) with an n-connected pair (A', C) such that the following diagram commutes:

$$\begin{tikzcd} C \arrow[r, hook] \arrow[d] & A' \arrow[dl, "\sim"] \\ A & \end{tikzcd}$$

and A' is built from C by attaching cells of dimension greater than n only. To show this, we build up a CW complex from C by adding cells which represent elements of $\pi_i(A)$ or gets rid of elements which should not be there.[14] Since $\pi_i(C) \cong \pi_i(A)$ for all $i < n$, we only need to add cells of dimension greater than n to make this work. This procedure can be carried out for (B, C) as well.

[14] For example, the first step is to kill the kernel of the surjection $\pi_n(C) \to \pi_n(A)$ by attaching cells to A. We will discuss this procedure in more detail when we discuss Eilenberg–Maclane spaces.

Reduction 2: It suffices to prove excision when each of A and B is built from C by attaching one cell apiece. To see this, let us say that a pair of extensions $C \to A$ and $C \to B$ is of size (p, q) if A is obtained by attaching p-cells (of dimension greater than n) and B is obtained by attaching q-cells (of dimension greater than m). The claim is that excision holds for size $(1, 1)$ if it holds for size (p, q). The proof is inductive, via a long each sequence of 'triad homotopy groups' and the 5-lemma.

The following lemma, whose proof is omitted, then completes the proof of homotopy excision. □

1.7.6 Lemma. *Suppose that $X = A \cup_C B$ where $A = C \cup e$ and $B = C \cup e'$ are built from C by attaching cells of dimension $> n$ and $> m$, respectively. Then, $\pi_i(A, C) \to \pi_i(X, B)$ is an isomorphism for $i < n + m$ and a surjection for $i = n + m$.*

Our main application will be the Freudenthal suspension theorem. We first make a definition.

1.7.7 Definition. Let (X, x_0) be a based space. The suspension homomorphism is the map $\Sigma_* \colon \pi_i(X) \to \pi_{i+1}(\Sigma X)$ which sends $[f]$ to $[\Sigma f]$, where $\Sigma f \colon S^{i+1} \to \Sigma X$ sends $[s, t]$ to $[f(s), t]$.

1.7.8 Theorem. *Let X be an $(n-1)$-connected CW-complex, then the suspension homomorphism*

$$\Sigma_* \colon \pi_i(X) \to \pi_{i+1}(\Sigma X)$$

is an isomorphism for $i < 2n - 1$ and a surjection for $i = 2n - 1$.

Proof. Write $\Sigma X = C_+ X \cup C_- X$ for the decomposition of ΣX into its upper and lower cone. Now consider the diagram

$$\begin{tikzcd} \pi_{i+1}(C_+X, X) \arrow[r] \arrow[d, "\partial", "\cong"'] & \pi_{i+1}(\Sigma X, C_-X) \arrow[d, "\partial"', "\cong"] \\ \pi_i(X) \arrow[r, "\Sigma_*"'] & \pi_{i+1}(\Sigma X) \end{tikzcd}$$

which can be shown to commute. Then it suffices to show that the upper diagram is an isomorphism/surjection in the appropriate range. To see this, note that if X is $(n-1)$-connected, then $(C_\pm X, X)$

are n-connected (use the long exact sequence and contractibility of $C_{\pm}X$). By excision,

$$\pi_{i+1}(C_+, X) \to \pi_{i+1}(\Sigma X, C_- X)$$

is an isomorphism for $i+1 < 2n$ and a surjection for $i+1 = 2n$, and the result follows. □

1.7.9 Example. The n-sphere has $\pi_i(S^n) = 0$ for $i < n$ (Example 1.6.5). So by the Freudenthal suspension theorem

$$\Sigma_* \colon \pi_i(S^n) \to \pi_{i+1}(S^{n+1})$$

is an isomorphism for $i < 2n-1$. In particular, $\pi_n(S^n) \to \pi_{n+1}(S^{n+1})$ is an isomorphism for $n < 2n-1$, i.e., for $n \geq 2$. In particular, there is a surjection $\mathbb{Z} \cong \pi_1(S^1) \to \pi_2(S^2)$ and isomorphisms $\pi_2(S^2) \cong \pi_3(S^3) \cong \cdots \pi_n(S^n)$. In fact, the Hopf fibration $S^1 \to S^2 \to S^3$ shows that $\pi_2(S^2) \cong \mathbb{Z}$, and so we have $\pi_n(S^n) \cong \mathbb{Z}$ for all $n \geq 1$.

1.7.10 Remark. Let X be a CW-complex. By the suspension theorem $\Sigma^n X$ is always $(n+1)$-connected. Thus,

$$\Sigma_* \colon \pi_i(\Sigma n X) \to \pi_{i+1}(\Sigma^{n+1} X)$$

is an isomorphism for $i < 2n-1$. This means that for a fixed value of k, the maps in the sequence

$$\pi_k(X) \to \pi_{k+1}(\Sigma X) \to \pi_{k+2}(\Sigma^2 X) \to \cdots \Sigma_{k+i}(\Sigma^i X)$$

eventually become isomorphisms. This is known as the k-th stable homotopy group of X.

1.7.11 Remark. There is an equivalent way to state homotopy excision. Suppose $f\colon A \to X$ is an m-equivalence and $g\colon A \to Y$ is an n-equivalence. We can form the following diagram

$$\begin{tikzcd}
F_f \arrow[r] \arrow[d, dashed, "\tilde{g}"'] & A \arrow[r, "f"] \arrow[d, "g"'] & X \arrow[d] \\
F_z \arrow[r] & Y \arrow[r, "z"'] \arrow[ur, phantom, "\lrcorner"] & Z
\end{tikzcd}$$

Then $\tilde{f}\colon F_f \to F_z$ is an $(n+m-1)$-equivalence. This follows because $\pi_n(X, A) \cong \pi_{n-1}F_f$ (use Theorem 1.4.16 and the 5-lemma, for example).

Exercise 15

Show that if $f\colon X \to Y$ is an n-connected map between spaces with X an $(m-1)$-connected CW-complex, then the comparison map $F(f) \to \Omega C(f)$ is $(m+n-1)$-connected.