1

Spectral sequences

Spectral sequences are a powerful computation tool in topology. Computing with spectral sequences is a bit like computing integral in calculus; it is helpful to have ingenuity and a big bag of tricks - and even that may not be enough!

1.1 Filtered complexes

We begin our discussion on spectral sequences by discussing filtered complexes.

1.1.1 Remark. Let $C_\bullet$ be a chain complex and $F_0C_\bullet$ a sub-complex. Then we have a short exact sequence

$$0 \to F_0C_\bullet \to C_\bullet \to C_\bullet/F_0C_\bullet \to 0$$

which gives rise to a long exact sequence in homology

$$\cdots \to H_i(F_0C_\bullet) \to H_i(C_\bullet) \to H_i(C_\bullet/F_0C_\bullet) \xrightarrow{\partial} H_{i-1}(F_0C_\bullet) \to \cdots$$

Suppose we know $H_*(F_0C_\bullet)$ and $H_*(C_\bullet/F_0C_\bullet)$. Can we compute $H_*(C_\bullet)$? We can split the long exact sequence into short exact sequences

$$0 \to \operatorname{coker}(\partial) \to H_*(C_\bullet) \to \ker(\partial) \to 0$$

which gives the following procedure for computing $H_*(C_\bullet)$:

1. Compute $H_*(F_0C_\bullet)$ and $H_*(C_\bullet/F_0C_\bullet)$
2. Consider the two-term chain complex

   $$H_*(C_\bullet/F_0C_\bullet) \xrightarrow{\partial} H_*(F_0C_\bullet).$$

   Denote its homology groups by G_1H_* and G_0H_*.
3. There is a short exact sequence

   $$0 \to G_0H_* \to H_*(C_\bullet) \to G_1H_* \to 0.$$

   This determines $H_*(C_\bullet)$ up to extension.[1]

How would we handle the situation if we have a longer filtration: $\cdots F_pC_\bullet \subseteq F_{p+1}C_\bullet \subseteq \cdots$?

[1] This is a common phenomenon for a spectral sequence. For example, if we have a short exact sequence $0 \to \mathbb{Z}/2 \to M \to \mathbb{Z}/2 \to 0$, can you say what the middle group is? Not without further information!

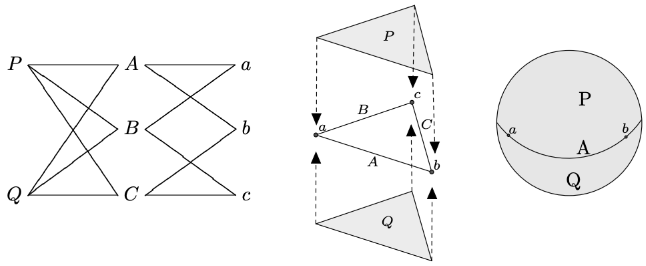

Figure 1.1: Simplicial model of S^2

1.1.2 Example. Consider a (semi-simplicial) model of the 2-sphere S^2 with vertices $\{a, b, c\}$, edges $\{A, B, C\}$ and solid triangles $\{P, Q\}$ and with inclusions as shown in Figure 1.1.[2] The associated chain complex is $C_\bullet$

$$0 \to \mathbb{Z}\{P, Q\} \xrightarrow{d} \mathbb{Z}\{A, B, C\} \xrightarrow{d} \mathbb{Z}\{a, b, c\} \to 0$$

with

$$d(P) = C - B + A \quad d(Q) = C - B + A$$

and

$$d(A) = b - a \quad d(B) = c - a \quad d(C) = c - b.$$

One can check directly that $H_i(C_\bullet; \mathbb{Z}) \cong \mathbb{Z}$ for $i = 0, 2$ and is zero otherwise. Alternatively, we use the following filtration:

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}\{P, Q\} \arrow[r] & \mathbb{Z}\{A, B, C\} \arrow[r] & \mathbb{Z}\{a, b, c\} \arrow[r] & 0 \\
0 \arrow[r] & 0 \arrow[r] & \mathbb{Z}\{A, B\} \arrow[r] & \mathbb{Z}\{a, b, c\} \arrow[r] & 0 \\
0 \arrow[r] & 0 \arrow[r] & \mathbb{Z}\{A\} \arrow[r] & \mathbb{Z}\{a, b\} \arrow[r] & 0.
\end{tikzcd}$$

The differentials are induced from d_1 and d_2 and a direct check shows that they are still chain complexes. Passing to the quotient, we get a chain complex we call E_0:

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}\{P, Q\} \arrow[r] & \mathbb{Z}\{C\} \arrow[r] & 0 \arrow[r] & 0 & d_0(P) = C, d_0(Q) = C \\
0 \arrow[r] & 0 \arrow[r] & \mathbb{Z}\{B\} \arrow[r] & \mathbb{Z}\{c\} \arrow[r] & 0 & d_0(B) = c \\
0 \arrow[r] & 0 \arrow[r] & \mathbb{Z}\{A\} \arrow[r] & \mathbb{Z}\{a, b\} \arrow[r] & 0 & d_0(A) = b - a.
\end{tikzcd}$$

Taking homology with respect to d_0 we obtain E^1:

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}\{P - Q\} \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \\
0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \\
0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \mathbb{Z}\{\bar{a}\} \arrow[r] & 0.
\end{tikzcd}$$

The general theory of spectral sequences will tell us that we have computed the homology of $H_*(C_\bullet)$; there is a $\mathbb{Z}$ in degree 2, generated by $P - Q$ and a $\mathbb{Z}$ in degree 0, generated by $\bar{a}$.

This leads us to the theory of filtered modules.

1.1.3 Definition. A filtered R-module is an R-module A together with an increasing sequence of submodules $F_pA \subseteq F_{p+1}A$ indexed by $p \in \mathbb{Z}$ such that $\cup_p F_pA = A$ and $\cap_p F_pA = \{0\}$. The filtration is

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[2] This example comes from Example 2.1 of `https://arxiv.org/pdf/1702.00666.pdf`.

bounded if $F_pA = \{0\}$ for p sufficiently small, and $F_pA = A$ for p sufficiently large. The associated graded module is defined by

$$G_pA = F_pA/F_{p-1}A.$$

1.1.4 Definition. A filtered chain complex is a chain complex $(C_\bullet, \partial)$ together with a filtration $\{F_pC_i\}$ of each C_i such that the differential preserves the filtration: $\partial(F_pC_i) \subseteq F_pC_{i-1}$. Then, ∂ induces $\partial\colon G_pC_i \to G_pC_{i-1}$ on the associated graded modules.

1.1.5 Remark. The filtration on $C_\bullet$ induces a filtration on the homology of $C_\bullet$ by

$$F_pH_i(C_\bullet) = \{\alpha \in H_i(C_\bullet) \mid \exists x \in F_pC_i, \alpha = [x]\}.$$

This has associated graded pieces $G_pH_i(C_\bullet)$.

1.1.6 Remark. Suppose we want to compute $H_*(C_\bullet)$ and that we can compute the homology of the associated graded pieces $H_*(G_pC_\bullet)$. Does this determine $G_pH_*(C_\bullet)$? This leads to the idea of the spectral sequence of a filtered complex.

1.2 *The spectral sequence of a filtered complex*

1.2.1 Definition. Let $(F_pC_\bullet, \partial)$ be a filtered chain complex. Let us write

$$E^0_{p,q} := G_pC_{p+q} = F_pC_{p+q}/F_{p-1}C_{p+q}.$$

The differential ∂ induces a differential on E^0,

$$\partial_0\colon E^0_{p,q} \to E^0_{p,q-1}.$$

We denote the homology of the associated graded by

$$E^1_{p,q} := H_{p+q}(G_pC_\bullet, \partial_0).$$

1.2.2 Remark. We can think of $E^1_{p,q}$ as a "first order approximation" to $H_*(C_\bullet)$. We can also define a differential

$$\partial_1\colon E^1_{p,q} \to E^1_{p-1,q}$$

as follows: a homology class $\alpha \in E^1_{p,q}$ can be represented by a chain $x \in F_pC_{p+1}$ such that $\partial x \in F_{p-1}C_{p+q-1}$. We define $\partial_1(\alpha) = [\partial x]$. Because $\partial^2 = 0$, we can check that $\partial_1^2 = 0$ and that ∂_1 is well defined.

1.2.3 Definition. With notation as above, we define

$$E_2^{p,q} = \ker(\partial_1\colon E^1_{p,q} \to E^1_{p-1,q}) / \operatorname{im}(\partial_1\colon E^1_{p+1,q} \to E^1_{p,q}).$$

1.2.4 Remark. We can continue this procedure, and define an "r"-th order approximation to $G_pH_{p+q}(C_\bullet)$ by

$$E^r_{p,q} = \frac{x \in F_pC_{p+q} \mid \partial x \in F_{p-r}C_{p+q-1}}{F_{p-1}C_{p+q} + \partial(F_{p+r-1}C_{p+q+1}}).$$

The notation denotes the quotient of the numerator by the intersection with the denominator.

So instead of considering cycles, we consider chains in F_p whose differentials vanishes "to order r", and instead of modding out by the entire image, we only mod out by $\partial(F_{p+r-1})$.

The main result regarding these groups is the following.

1.2.5 Lemma. *Let* $(F_pC_\bullet, \partial)$ *denote a filtered chain complex, and define* $E^r_{p,q}$ *as above. Then,*

1. ∂ *induces a map*
$$\partial_r\colon E^r_{p,q} \to E^r_{p-r,q+r-1}$$
satisfying $\partial_r^2 = 0$.
2. E^{r+1} *is the homology of the chain complex* (E^r, ∂_r)*, i.e.,*
$$E^{r+1}_{p,q} = \ker(\partial_r\colon E^r_{p,q} \to E^r_{p-r,q+r-1}) / \operatorname{im}(\partial_r\colon E^r_{p+r,q+r-1} \to E^r_{p,q}).$$
3. $E^1_{p,q} = H_{p+q}(G_pC_\bullet)$.
4. *If the filtration of* C_i *is bounded for each* i*, then for every* p, q *if* r *is sufficiently large, then*
$$E^r_{p,q} = G_pH_{p+q}(C_\bullet).$$

Proof. This is a rather tedious diagram chase,[3] which generalizes the argument that a short exact sequence of chain complexes induces a long exact sequence on homology. □

1.2.6 Example. In this example[4] we show that the singular and cellular homology groups of a CW-complex X agree. To that end, let $C_*(X)$ denote the singular chain complex of X. We filter this by

$$F_pC_*(X) := C_*(X^p)$$

where X^p denotes the p-skeleton of X. The associated graded is

$$E^0_{p,q} = C_{p+q}(X^p)/C_{p+q}(X^{p-1}).$$

By definition, the homology is

$$E^1_{p,q} = H_{p+q}(X^p, X^{p-1}),$$

the relative homology of the pair (X^p, X^{p-1}). We have

$$H_{p+q}(X^p, X^{p-1}) \cong \begin{cases} C^{cell}_p(X) & q = 0 \\ 0, & q \neq 0 \end{cases}$$

where $C^{cell}_p(X)$ is the cellular chains on X, the free $\mathbb{Z}$-module with one generator for each p-cell. The cellular differential $\partial\colon C^{cell}_p(X) \to C^{cell}_{p-1}(X)$ is exactly the boundary map $E^1_{p,0} \to E^1_{p-1,0}$. Therefore, we have

$$E^2_{p,q} = \begin{cases} H^{cell}_p(X), & q = 0 \\ 0, & q \neq 0. \end{cases}$$

We must have $\partial_r = 0$ for $r \geq 2$ as either the domain or the range is zero. So, $E^{p,q}_r = E^2_{p,q}$ for all $r \geq 2$. If X is finite-dimensional, then the filtration is bounded and so $H_p(X) = H^{cell}_p(X)$ by Lemma 1.2.5.[5]

[3] For example, see `http://www.math.uchicago.edu/~may/MISC/SpecSeqPrimer.pdf`

[4] See page 67 of Mosher–Tangor, *Cohomology Operations and Applications in Homotopy Theory*

[5] One can allow arbitrary X by, for example, using colimits.

1.3 Homological spectral sequences

We have managed to so far avoid defining exactly what a spectral sequence is. Let us change that now.

1.3.1 Definition. A (homological) spectral sequence is a sequence

$$\{E^r_{*,*}, d^r_{*,*}\}_{r\geq 0}$$

of chain complexes of abelian groups, such that

$$E^{r+1}_{*,*} = H_*(E^r_{*,*})$$

where the homology is taken with respect to maps (called differentials)

$$d^r_{p,q} \colon E^r_{p,q} \to E^r_{p-r,q+r-1}$$

such that $(d^r)^2 = 0$.

1.3.2 Remark. We say that a spectral sequence is first quadrant if $E^r_{p,q} = 0$ whenever $p < 0$ or $q < 0$. Note that this implies that $d^r_{p,q} = 0$ for $r \gg 0$ (as either the source or the target is zero). In particular,

$$E^r_{p,q} = E^{r+1}_{p,q} = \cdots = E^\infty_{p,q}.$$

We say that the spectral sequence collapses or degenerates at E^r.

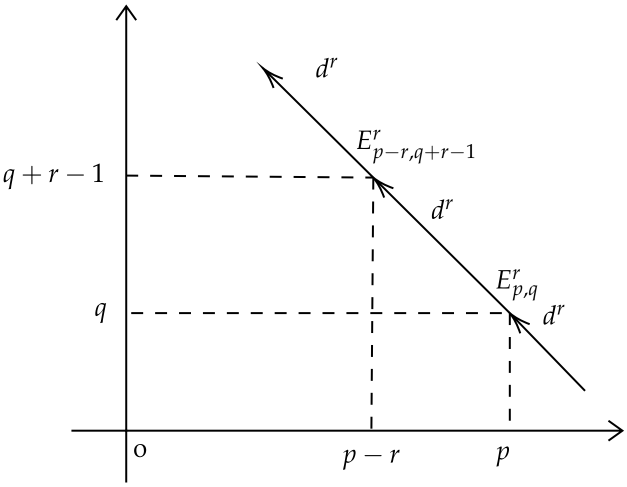

Figure 1.2: The E^r-page of a homological spectral sequence

1.3.3 Definition. If $\{H_n\}_n$ are groups, then we say that the spectral sequence converges, or abuts, to H_*, denoted $E^2_{*,*} \implies H_*$, if for each n there is a filtration

$$H_n = D_{n,0} \subseteq D_{n-1,1} \subseteq \cdots \subseteq D_{1,n-1} \subseteq D_{0,n} \subseteq 0$$

such that, for all p, q,

$$E^\infty_{p,q} = D_{p,q}/D_{p-1,q+1}.$$

1.3.4 Remark. In more straightforward terms: the if we look along the n-th diagonal of the spectral sequence, then the E_∞-page computes the associated graded of the filtration on H_n. For example, if $E^\infty_{p,q} = 0$ for all $p+q=n$, then $H_n = 0$. If there is only a single non-zero term, say $E^\infty_{p,n-p}$, then the filtration is trivial, and $H_n = E^\infty_{p,n-p}$. If we have two non-zero terms, then H_n fits into a short exact sequence, and so on.

1.3.5 Example. We have previously discussed the spectral sequence of a filtered complex without explicitly mentioning it. Indeed, if $C_\bullet$ is a filtered chain complex, then there is a spectral sequence with $E^1_{p,q} = H_{p+q}(G_pC_\bullet)$, such that if the filtration of C_i is bounded for each i the spectral sequence converges to $H_{p+q}(C_\bullet)$.[6]

1.4 *The spectral sequence of a double complex*

An important example where a filtered complex arises is from a double complex.

1.4.1 Definition. A double complex is a bi-indexed family $\{C_{p,q}\}$ of abelian groups, with two differentials

$$d'\colon C_{p,q} \to C_{p-1,q}, \quad d''\colon C_{p,q} \to C_{p,q-1}$$

such that $d'd' = 0, d''d'' = 0$, and $d'd'' + d''d' = 0$. For simplicity, we also assume that $C_{p,q} = 0$ for $p < 0$ or $q < 0$.

1.4.2 Example. Suppose that (A, d_A) and (B, d_B) are chain complexes. If we define $C_{p,q} = A_p \otimes B_q$ and define $d' = d_A \otimes 1$ and $d'' = (-1)^p 1 \otimes d_B$, then $C_{p,q}$ is a double complex.[7]

1.4.3 Construction . A double complex gives rise to a chain complex (the total complex), defined by $C_n = \sum_{p+q=n} C_{p,q}$ and $d = d' + d''$. This has two obvious filtrations, by row and by column:

1. $'C^p_n = \sum_{j+q=n, j\leq p} C_{j,q}$.
2. $''C^p_n = \sum_{p+q=n, k\leq q} C_{p,k}$.

The spectral sequence of a filtered complex (Example 1.3.5) gives us two spectral sequences:

1. $'E^1_{p,q} = H_{p+q}('C^p/'C^{p-1}) = C_{p,n-p}$.
2. $''E^1_{p,q} = H_{p+q}(''C^q/''C^{q-1}) = C_{q,n-q}$.

One checks that $'E^1$ is computed via means of d'' and that d^1 is induced by d', while in $''E^1$ the role of the two indices are exchanged. We can therefore write:

1. $'E^2_{p,q} = H'_p H''_q(C)$.
2. $''E^2_{p,q} = H''_q H'_p(C)$.

Moreover, both spectral sequences converge to $H_*(C)$, and the idea is to compare the two spectral sequences.

It is constructive to do an example.

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[6] Recall what this means: we have $E^\infty_{p,q} = G_pH_{p+q}(C_\bullet)$.

[7] Try and verify this to make sure you understand the definitions.

1.4.4 Example. Let $'\operatorname{Tor}(A,B)$ be defined as follows: take a free resolution of A, $0 \to R' \to F' \to A \to 0$, then $'\operatorname{Tor}(A,B)$ is defined by

$$0 \to' \operatorname{Tor}(A,B) \to R' \otimes B \to F' \otimes B \to A \otimes B \to 0.$$

Similarly, let $''\operatorname{Tor}(A,B)$ be defined as follows: take a free resolution of B, $0 \to R'' \to F'' \to B \to 0$, then $''\operatorname{Tor}(A,B)$ is defined by

$$0 \to {}''\operatorname{Tor}(A,B) \to A \otimes R'' \to A \otimes F'' \to A \otimes B \to 0.$$

It is a classical theorem of homological algebra that $\operatorname{Tor}(A,B) = {}''\operatorname{Tor}(A,B)$. Let us prove this via a spectral sequence argument.

Let X be the chain complex $0 \to R' \xrightarrow{d'} F' \to 0$ and let Y be the chain complex $0 \to R'' \xrightarrow{d''} F'' \to 0$. We can build a double complex $C_{*,*}$ as in Example 1.4.2, which we write as a matrix:

$$\begin{bmatrix} C_{p,q} \end{bmatrix} = \begin{bmatrix} F' \otimes R'' & R' \otimes R'' \\ F' \otimes F'' & R' \otimes F'' \end{bmatrix}$$

We have two spectral sequences: the first is take vertical and then horizontal homology:

$$H_q''(C_{p,q}) = \begin{bmatrix} ''\operatorname{Tor}(F',B) & '\operatorname{Tor}(R',B) \\ F' \otimes B & R' \otimes B \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ F' \otimes B & R' \otimes B \end{bmatrix}$$

and

$$H_p H_q''(C_{p,q}) = \begin{bmatrix} 0 & 0 \\ A \otimes B & '\operatorname{Tor}(A,B) \end{bmatrix}$$

In other words, the total complex has $H_0(C) = A \otimes B$ and $H_1(C) =' \operatorname{Tor}(A,B)$.

However, we can use the second spectral sequence, which first takes horizontal and then vertical homology:

$$H_p(C_{p,q}) \begin{bmatrix} A \otimes R'' & '\operatorname{Tor}(A,R'') \\ A \otimes F'' & '\operatorname{Tor}(A,F'') \end{bmatrix} = \begin{bmatrix} A \otimes R'' & 0 \\ A \otimes F'' & 0 \end{bmatrix}$$

and then

$$H''_q H_p(C_{p,q}) = \begin{bmatrix} ''\operatorname{Tor}(A,B) & 0 \\ A \otimes B & 0 \end{bmatrix}$$

In this case we see that $H_0(C) = A \otimes B$ and $H_1(C) = {}''\operatorname{Tor}(A,B)$. Therefore, $'\operatorname{Tor}(A,B) = {}''\operatorname{Tor}(A,B)$.

Exercise 1 (The snake lemma)**.** *Show, using spectral sequences, the following result in homological algebra (the snake lemma):*

Given a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, "f"] & B \arrow[r] \arrow[d, "g"] & C \arrow[r] \arrow[d, "h"] & 0 \\ 0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0 \end{tikzcd}$$

in an abelian category with exact rows, there is a long exact sequence

$$\begin{aligned} 0 \to \ker(f) \to \ker(g) \to \ker(h) \\ \to \operatorname{coker}(f) \to \operatorname{coker}(g) \to \operatorname{coker}(h) \to 0. \end{aligned}$$

Exercise 2. *(1) Suppose we have a commutative triangle*

$$\begin{tikzcd}
A \arrow[rr, "i"] \arrow[dr, "f"'] & & C \arrow[dl, "q"] \\
& B &
\end{tikzcd}$$

Show using the snake lemma that

$$\ker(\operatorname{coker} f \to \operatorname{coker} q) \cong \operatorname{im}(q) / \operatorname{im}(f)$$

and

$$\operatorname{coker}(\operatorname{coker} f \to \operatorname{coker} q) = 0.$$

(2) Using Part (1), prove the following 'butterfly lemma': given a commutative diagram

$$\begin{tikzcd}
A \arrow[dd, "f"'] \arrow[dr, "i"] & & D \arrow[dd, "g"] \arrow[dl, "j"'] \\
& C \arrow[dl, "q"] \arrow[dr, "p"'] & \\
B & & E
\end{tikzcd}$$

of abelian groups, in which the diagonals pi and qj are exact at C, there is an isomorphism

$$\frac{\operatorname{im} q}{\operatorname{im} f} \cong \frac{\operatorname{im} p}{\operatorname{im} g}.$$

1.5 The Serre spectral sequence

For us the most important example of a spectral sequence will be the Serre spectral sequence. We will state the theorem now and then return to the proof after some examples and applications.

1.5.1 Theorem (The Serre spectral sequence)**.** *Let $\pi\colon E \to F$ be a fibration with fiber F and assume that $\pi_1(B) = 0$ and $\pi_0(F) = 0$, then there is a first quadrant spectral sequence*

$$E^2_{p,q} = H_p(B; H_q(F)) \implies H_{p+q}(E).$$

In particular, this means there is a filtration

$$H_n(E) = D_{n,0} \supseteq D_{n-1,1} \supseteq \ldots \supseteq D_{0,n} \supseteq D_{-1,n+1} = 0$$

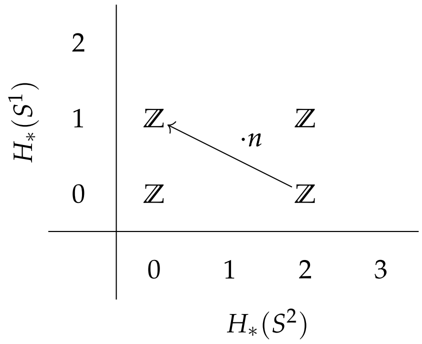
such that $E^\infty_{p,q} = D_{p,q}/D_{p-1,q+1}$.

1.5.2 Remark. There is a version of this spectral sequence where $\pi_1(B) \neq 0$; the E_2-page is then given by the cohomology of B with local coefficients $\mathcal{H}_q(F)$. This will not play a role in this course.

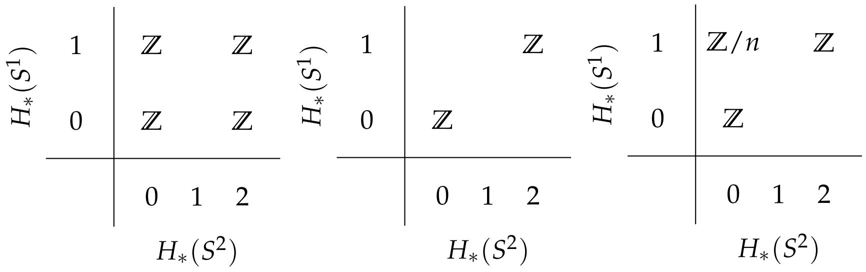
1.5.3 Example. Consider the Hopf fibration $S^1 \to S^3 \to S^2$. We have

$$E^2_{p,q} = H_p(S^2; H_q(S^1)) \cong \begin{cases} \mathbb{Z} & p = 0,2 \text{ and } q = 0,1 \\ 0 & \text{otherwise.} \end{cases}$$

The E^2-term is as follows (and we have $E^3 = E^\infty$ for degree reasons):



There are three possibilities for the d_2-differential (which is multiplication by $n \in \mathbb{Z}$ as indicated): either $n = 0, n = \pm 1$ or $n \neq 0, \pm 1$, which lead to the following $E^3 = E^\infty$-page:



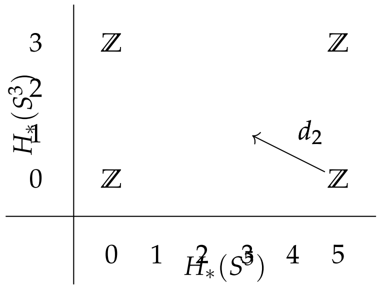
$n = 0$ $n = \pm 1$ $n \neq 0, \pm 1$

We see that taking $n = \pm 1$ computes the correct answer for $H_*(S^3)$; we have a copy of $\mathbb{Z}$ in the $p + q = 0$ and $p + q = 3$ columns, as required.

1.5.4 Example. There is a fibration $SU(n-1) \to SU(n) \to S^{2n-1}$. Taking $n = 3$ and using $SU(2) \cong S^3$, we obtain a fibration $S^3 \to SU(3) \to S^5$. We have

$$E^2_{p,q} \cong \begin{cases} \mathbb{Z} & p = 0,5 \text{ and } q = 0,3 \\ 0 & \text{otherwise.} \end{cases}$$

The E^2-term is as follows:



Note that there are no differentials for degree reasons, as shown for d_2. Therefore, the spectral sequence collapses and we see that

$$H_i(SU(3)) \cong \begin{cases} \mathbb{Z}, & i = 0,3,5,8 \\ 0, & \text{otherwise.} \end{cases}$$

1.5.5 Example. We can continue the previous example and take $n = 4$ to get a fibration $SU(3) \to SU(4) \to S^7$. We can compute the

E^2-term using the previous example

$$E^2_{p,q} = H_p(S^7; H_q(SU(3))) \cong \begin{cases} \mathbb{Z}, & p = 0,7, q = 0,3,5,8 \\ 0, & \text{otherwise.} \end{cases}$$

The E^2-term is as follows:

8	$\mathbb{Z}$							$\mathbb{Z}$
7								
6								
5	$\mathbb{Z}$							$\mathbb{Z}$
4								
3	$\mathbb{Z}$							$\mathbb{Z}$
2								
1								
0	$\mathbb{Z}$							$\mathbb{Z}$
	0	1	2	3	4	5	6	7

$H_*(SU(3))$

$H_*(S^7)$

Note that there are no differentials for degree reasons, and we compute

$$H_i(SU(4)) \cong \begin{cases} \mathbb{Z}, & i = 0,3,5,7,8,10,12,15 \\ 0, & \text{otherwise.} \end{cases}$$

1.5.6 Remark. If one tries the same argument for $SU(5)$ there are possible differentials. We will see later that it is easier to use cohomology, where one can use multiplicative structures to rule out differentials.

1.5.7 Remark (Naturality of the Serre spectral sequence). The Serre spectral sequence is natural in the following sense. Suppose we are given two fibrations satisfying the hypothesis of the Serre spectral sequence, and a map between them:

$$\begin{tikzcd} F \arrow[r, hook] \arrow[d] & E \arrow[r] \arrow[d, "\tilde{f}"] & B \arrow[d, "f"] \\ F' \arrow[r, hook] & E' \arrow[r] & B' \end{tikzcd}$$

Then the following hold:

1. There are induced maps $f^r_*\colon E^r_{p,q} \to' E^r_{p,q}$ commuting with differentials, i,e, the diagram

$$\begin{tikzcd} E^r_{p,q} \arrow[r, "d_r"] \arrow[d, "f^r_*"'] & E^r_{p-r,q+r-1} \arrow[d, "f^r_*"] \\ 'E^r_{p,q} \arrow[r, "'d_r"'] & 'E^r_{p-r,q+r-1} \end{tikzcd}$$

   commutes, and moreover f^{r+1}_* is the map induced on homology by f^r_*.

2. The map $\tilde{f}_*\colon H_*(E) \to H_*(E')$ preserves filtrations, inducing a map on associated graded which is exactly f_*^∞.
3. Under the isomorphisms $E^2_{p,q} \cong H_p(B; H_q(F))$ and $'E^2_{p,q} \cong H_p(B'; H_q(F'))$ the map f_*^2 corresponds to the map induced by the maps $B \to B'$ and $F \to F'$.

Once again, we can demonstrate this with an example.

1.5.8 Example. We recall the Hopf fibration $S^1 \to S^3 \to S^2$. This factors through $\mathbb{R}P^3 = S^3/\{\pm 1\}$ as in the following diagram:

$$\begin{tikzcd}
S^1 \arrow[r, hook] \arrow[d, two heads, "q"'] & S^3 \arrow[r] \arrow[d, two heads, "q"'] & S^2 \arrow[d, equal] \\
S^1/\{\pm 1\} \arrow[r, hook] & S^3/\{\pm 1\} \arrow[r] & S^2
\end{tikzcd}$$

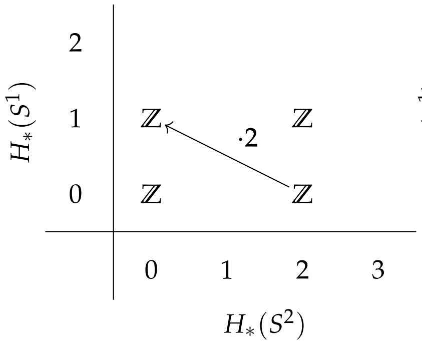
We see that we have a fibration $S^1 \to \mathbb{R}P^3 \to S^2$. The $'E^2$-term of this spectral sequence is as for the Hopf fibration:

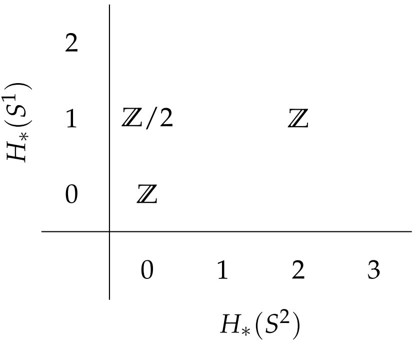
$$E^2_{p,q} = H_p(S^2; H_q(S^1)) \cong \begin{cases} \mathbb{Z} & p = 0,2 \text{ and } q = 0,1 \\ 0 & \text{otherwise.} \end{cases}$$

As in Example 1.5.3 there is only one possible differential, which is $'d_2\colon {}'E^2_{2,0} \to' E^2_{0,1}$, and this is given by multiplication by an integer n. We use naturality to determine what this is. We note that we have a commutative diagram

$$\begin{tikzcd}
H_2(S^2; H_0(S^1)) \arrow[r, "d_2", "\cong"'] \arrow[d, "q_*"', "\cong"] & H_0(S^2; H_1(S^1)) \arrow[d, "\cdot 2"] \\
H_2(S^2; H_0(S^1/\{\pm 1\})) \arrow[r, "'d_2"'] & H_0(S^2; H_1(S^1/\{\pm 1\}))
\end{tikzcd}$$

The right hand arrow is multiplication by 2 because the map induced on homology by $S^1 \to S^1/\{\pm 1\}$ has degree 2 (it is the attaching map for the top cell of $\mathbb{R}P^2$). Commutativity of the diagram implies that $'d_2$ is multiplication by 2. Therefore, the E^2 and $E^3 = E^\infty$-terms are as follows:





We deduce that

$$H_i(\mathbb{R}P^3) \cong \begin{cases} \mathbb{Z}, & i = 0,3 \\ \mathbb{Z}/2, & i = 1 \\ 0, & \text{otherwise.} \end{cases}$$

1.5.9 Remark. It is also possible to deduce some information about $H^*(F)$ or $H^*(B)$ in certain cases, as the following example demonstrates.

1.5.10 Example. There is a fibration $S^1 \to S^\infty \to \mathbb{C}P^\infty$. Note that $\pi_1(\mathbb{C}P^\infty) = 0$, so we can run the Serre spectral sequence. We have

$$E^2_{p,q} = \begin{cases} H_p(\mathbb{C}P^\infty), & q = 0, 1 \\ 0, & \text{otherwise.} \end{cases}$$

We also know that the spectral sequence converges to $H_{p+q}(S^\infty, \mathbb{Z})$, which is only non-zero when $p + q = 0$. In particular, the E^∞ page should be zero except for $E^\infty_{0,0}$. Now consider the E_2-page of the spectral sequence:

$$\begin{tikzcd}
& 2 & & & & \\
H_*(S^1) & 1 & \mathbb{Z} & H_1(\mathbb{C}P^\infty) & H_2(\mathbb{C}P^\infty) & H_3(\mathbb{C}P^\infty) \\
& 0 & \mathbb{Z} & H_1(\mathbb{C}P^\infty) & H_2(\mathbb{C}P^\infty) \arrow[ull] & H_3(\mathbb{C}P^\infty) \arrow[ull] \\
& & 0 & 1 & 2 & 3 \\
& & & H_*(\mathbb{C}P^\infty) & &
\end{tikzcd}$$

Note that for degree reasons $E^2_{1,0} \cong H_1(\mathbb{C}P^\infty)$ survives the spectral sequences, and so must be 0. So the E^2-page is as follows:

$$\begin{tikzcd}
& 2 & & & & \\
H_*(S^1) & 1 & \mathbb{Z} & 0 & H_2(\mathbb{C}P^\infty) & H_3(\mathbb{C}P^\infty) \\
& 0 & \mathbb{Z} & 0 & H_2(\mathbb{C}P^\infty) \arrow[ull] & H_3(\mathbb{C}P^\infty) \arrow[ull] \\
& & 0 & 1 & 2 & 3 \\
& & & H_*(\mathbb{C}P^\infty) & &
\end{tikzcd}$$

By the same argument $E^2_{3,0} \cong H_3(\mathbb{C}P^\infty)$ survives the spectral sequences, and so must be 0. Inductively, we deduce that $H_n(\mathbb{C}P^\infty) = 0$ for all n odd. Since $E^2_{0,1} \cong \mathbb{Z}$ must also die in the spectral sequence, we see that we must have $H_2(\mathbb{C}P^\infty) \cong \mathbb{Z}$, and that d^2 must be an isomorphism. Continuing inductively, we get

$$H_n(\mathbb{C}P^\infty) \cong \begin{cases} \mathbb{Z}, & n \text{ even} \\ \mathbb{Z}, & 0 \text{ odd.} \end{cases}$$

1.5.11 Example. In our next example, we compute $H_*(\Omega S^n)$ for $n > 1$. We use the path-space fibration of **??**. In this case, this takes the form

$$\Omega S^n \to PS^n \to S^n$$

where we recall that PS^n is contractible, i.e. $H_0(PS^n) = \mathbb{Z}$ and is zero otherwise. In particular, the only non-zero term on the E^∞-

page of the spectral sequence is a copy of $\mathbb{Z}$ when $p+q=0$. Now consider a small portion of the E_2-term:

$$\begin{tikzcd}
3 & H_3(\Omega S^n) & H_3(\Omega S^n) \\
2 & H_2(\Omega S^n) & H_2(\Omega S^n) \\
1 & H_1(\Omega S^n) & H_1(\Omega S^n) \\
0 & \mathbb{Z} & \mathbb{Z} \\
 & 0 & n
\end{tikzcd}$$

Note that the only possible differential is a d_n, and so goes $n-1$-terms upwards. We immediately see that $H_i(\Omega S^n) = 0$ for $0 < i < n-1$. Moreover, the only way to get rid of the $\mathbb{Z}$ in $E^2_{n,0} = E^n_{n,0}$ is that $H_{n-1}(\Omega S^n) \cong \mathbb{Z}$, and that d_n is an isomorphism. We can inductively repeat this argument, getting the following, where all the differentials shown are isomorphisms:

$$\begin{tikzcd}
3n-3 & H_{3n-3}(\Omega S^n) & H_{3n-3}(\Omega S^n) \\
2n-2 & H_{2n-2}(\Omega S^n) & H_{2n-2}(\Omega S^n) \arrow[ul] \\
n-1 & H_{n-1}(\Omega S^n) & H_{n-1}(\Omega S^n) \arrow[ul] \\
0 & \mathbb{Z} & \mathbb{Z} \arrow[ul] \\
 & 0 & n
\end{tikzcd}$$

We conclude that

$$H_i(\Omega S^n) \cong \begin{cases} \mathbb{Z}, & i = k(n-1) \\ 0, & \text{otherwise.} \end{cases}$$

1.5.12 Remark. So far we have only considered examples where the extension problem is trivial; we have had at most one non-zero term in each diagonal on the E^∞-page. The following gives an example where this is not the case.

1.5.13 Example. Consider the Serre spectral sequence of the fibration

$$S^1 \to U(2) \to \mathbb{R}P^3$$

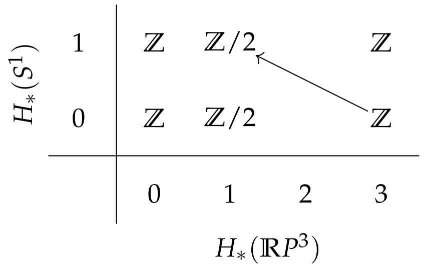
where we identify $S^1 \cong U(1)$ and the first map is given by

$$\lambda \mapsto \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$$

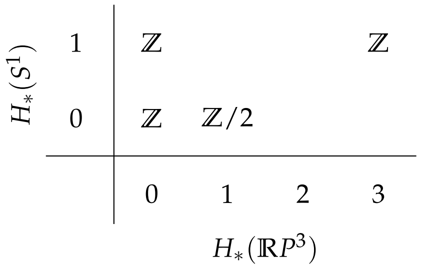
The E^2-page is given by

$$H_p(\mathbb{R}P^3; H_q(S^1)) \cong \begin{cases} \mathbb{Z}, & p = 0,3, q = 0,1 \\ \mathbb{Z}/2 & p = 1, q = 0,1 \\ 0 & \text{else.} \end{cases}$$

The E^2-page looks as follows:



We know (or take it as fact) that $H_2(U(2)) = 0$; the only way that this is compatible with the spectral sequence is if the differential shown is a surjection, and we get the following $E^3 = E^\infty$-page:



Now, in fact we have that[8]

$$H_i(U(2)) \begin{cases} \mathbb{Z}, & i = 0,1,3,4 \\ 0, & \text{else.} \end{cases}$$

Note that in the E^∞-page shown we have two non-zero terms in the $p+q=1$ column, a $\mathbb{Z}$ in $(0,1)$ and $\mathbb{Z}/2$ in $(1,0)$. This means there is an extension[9]

$$0 \to \mathbb{Z} \to H_1(U(2)) \to \mathbb{Z}/2 \to 0.$$

From the calculations above we know that this extension must be non-trivial. Yet, if we didn't know another way to compute $H_1(U(2))$ we could not determine (without more information) if $H_1(U(2))$ was $\mathbb{Z}$ or $\mathbb{Z} \oplus \mathbb{Z}/2$.

1.5.14 Remark. We now return to the Hurewicz theorem, giving a second proof of **??**.

1.5.15 Theorem. *If X is $(n-1)$-connected, $n \geq 2$, then $\widetilde{H}_i(X) = 0$ for $i \leq n-1$ and $\pi_n(X) \cong H_n(X)$.*

Proof. We use the path-space fibration

$$\Omega X \to PX \to X,$$

and the fact that PX is contractible. The E^2-page of the Serre spectral sequence is

$$E^2_{p,q} = H_p(X; H_q(\Omega X)) \implies H_{p+q}(PX).$$

---

[8] For example, note that $U(2) \cong SU(2) \times U(1)$

[9] In fact, the spectral sequence shows that we have filtered $H_1(U(2))$ as follows $0 \subseteq 2\mathbb{Z} \subseteq \mathbb{Z} \subseteq \mathbb{Z} = H_1(U(2))$.

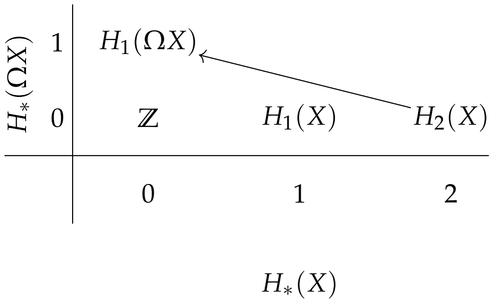
We prove the theorem by induction on n. When $n = 2$, we have $H_1(X) = 0$ because X is simply connected by assumption. Moreover, we have

$$\pi_2(X) \cong \pi_1(\Omega X) \cong H_1(\Omega X)$$

where the first isomorphism follows by the long exact sequence of the fibration, and the second follows from the fact that $\pi_1(\Omega X)$ is abelian, so that $H_1(\Omega X) \cong \pi_1(\Omega X)^{ab} \cong \pi_1(\Omega X)$. It remains to show that $H_1(\Omega X) \cong H_2(X)$. We will use the Serre spectral sequence to show this. Note that $E^2_{2,0} = H_2(X)$ and $E^2_{0,1} = H_1(\Omega X)$, so it suffices to show that

$$d^2 \colon E^2_{2,0} = H_2(X) \to E^2_{0,1} = H_1(\Omega X)$$

is an isomorphism. We consider then then a portion of the E^2-page:



Note that if d_2 is not an isomorphism, then both of these groups will persist to the E^∞-page, giving a contradiction to the fact that PX is contractible. So, d_2 must be an isomorphism, as required. This gives the base case of the induction.

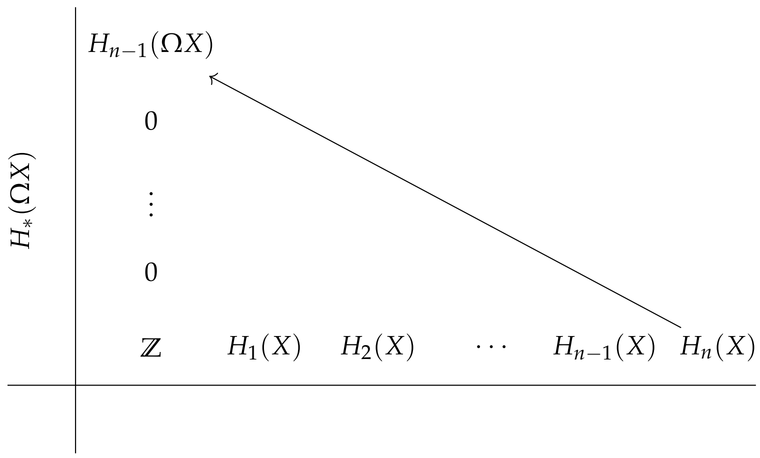
We now assume the statement of the theorem holds for $n-1$ and deduce it for n. Since X is $(n-1)$-connected, ΩX is $(n-2)$-connected, and so by the inductive hypothesis, we have that $\widetilde{H}_i(\Omega X) = 0$ for $i < n-1$ and $\pi_{n-1}(\Omega X) \cong H_{n-1}(\Omega X)$. In particular, we get isomorphisms

$$\pi_n(X) \cong \pi_{n-1}(\Omega X) \cong H_{n-1}(\Omega X),$$

and so it suffices to show that $H_{n-1}(\Omega X) \cong H_n(X)$. We do this via the Serre spectral sequence. We have

$$\begin{aligned} E^2_{p,q} &= H_p(X; H_q(\Omega)) \\ &\cong H_p(X) \otimes H_q(X) \oplus \operatorname{Tor}(H_{p-1}(X), H_q(\Omega X)) \\ 0 & \end{aligned}$$

for $0 < q < n-1$ by the inductive hypothesis. Now consider the Serre spectral sequence:



The only differentials that interact with $H_n(X)$ and $H_{n-1}(\Omega X)$ is the d_n differential shown, and so this must be an isomorphism in order for these terms to die in the spectral sequence. Moreover, the terms $H_i(X)$ for $1 \leq i \leq n-1$ have no differentials at all in the spectral sequence; in particular, we must have $H_i(X) = 0$ for $1 \leq i \leq n-1$ and $d_n\colon H_n(X) \to H_{n-1}(\Omega X)$ is an isomorphism. □

Exercise 3. *Show, using the Serre spectral sequence, that if $S^k \to S^m \to S^n$ is a fibration with $n \geq 2$, then $k = n-1$ and $m = 2n-1$.*

1.6 The Serre spectral sequence in cohomology

The Serre spectral sequence in cohomology looks much like the homology version:

1.6.1 Theorem (The Serre spectral sequence in cohomology)**.** *Let $\pi\colon E \to F$ be a fibration with fiber F and assume that $\pi_1(B) = 0$ and $\pi_0(F) = 0$, then there is a first quadrant spectral sequence*

$$E_2^{p,q} = H^p(B; H^q(F)) \implies H^{p+q}(E).$$

In particular, this means there is a filtration

$$H^n(E) = D^{0,n} \supseteq D^{1,n-1} \supseteq \ldots \supseteq D^{n,0} \supseteq D^{n+1,-1} = 0$$

such that $E_\infty^{p,q} = D^{p,q}/D^{p+1,q-1}$.

The differentials run $d_r^{p,q}\colon E_r^{p,q} \to E_r^{p+r,q+r-1}$.

1.6.2 Remark. Apart from the direction of differentials, this looks much like the Serre spectral sequence in homology. However, there is one major difference: each E_r page has a bilinear product, i.e., a map

$$\bullet\colon E_r^{p,q} \times E_r^{p',q'} \to E_r^{p+p',q+q'}$$

or equivalently,

$$E_r^{p,q} \otimes E_r^{p',q'} \to E_r^{p+p',q+q'}$$

satisfying the Leibniz rule

$$d_r(x \bullet y) = d_r(x) \bullet y + (-1)^{\deg(x)} x \bullet d_r(y).$$

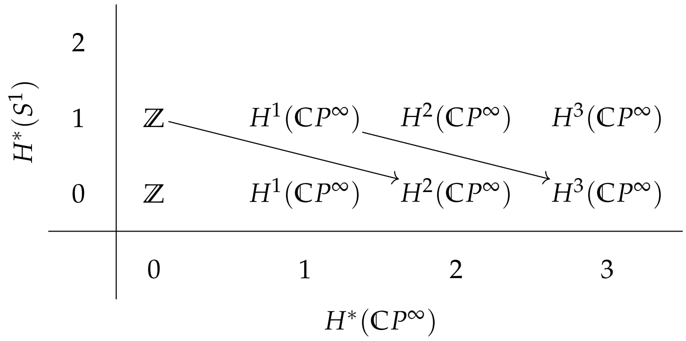
where $\deg(x) = p + q$. Moreover, on the E_2-page, this product is induced by the cup product.

Once again, it is instructive to do an example.

1.6.3 Example. Consider the fibration

$$S^1 \to S^\infty \simeq * \to \mathbb{C}P^\infty$$

The E_2-page looks as follows



Running an argument similar to Example 1.5.10 it is not too hard to compute the additive structure: we must have $E_2^{2k+1,0} = 0$, and $d_2 \colon E_2^{p,1} \to E_2^{p+2,0}$ is an isomorphism $\mathbb{Z} \to \mathbb{Z}$. In particular, we have

$$H^i(\mathbb{C}P^\infty) \begin{cases} \mathbb{Z}, & i = \text{ even} \\ 0, & i = \text{ odd.} \end{cases}$$

Now we wish to compute the multiplicative structure. Let use note that by the universal coefficient theorem in cohomology[10] we have

$$E_2^{p,q} = H^p(\mathbb{C}P^\infty) \otimes H^q(S^1).$$

Let $\mathbb{Z} = \langle x \rangle = H^1(S^1)$ and let $\mathbb{Z} = \langle y \rangle = H^2(\mathbb{C}P^\infty)$, chosen so that $d_2(x) = y$. Then we have

$$E_2^{2,1} = H^2(\mathbb{C}P^\infty) \otimes H^1(S^1) = \mathbb{Z} \otimes \mathbb{Z} \cong \mathbb{Z}.$$

The pairing

$$\bullet \colon E_2^{2,0} \times E_2^{0,1} \to E_2^{2,1}$$

is induced by the cup product, and unwinding the definitions, sends (x, y) to xy, i.e., xy generates $E_2^{2,1}$.

Let z be a generator of $H^4(\mathbb{C}P^\infty)$. We want to show that $z = y^2$. By the Leibniz rule,

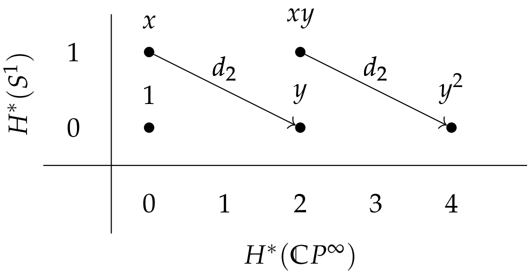
$$d_2(xy) = d_2(x)y + (-1)^{\deg(x)} x d_2(y) = y^2.$$

Noting that d_2 is an isomorphism, we see that $d_2(xy) = y^2 = z$, as needed. Arguing inductively, we see that $d_2(xy^{n-1}) = y^n$ is a generator of $H^{2n}(\mathbb{C}P^\infty)$ and we deduce that $H^*(\mathbb{C}P^\infty) \cong \mathbb{Z}[y]$ with $\deg(y) = 2$.

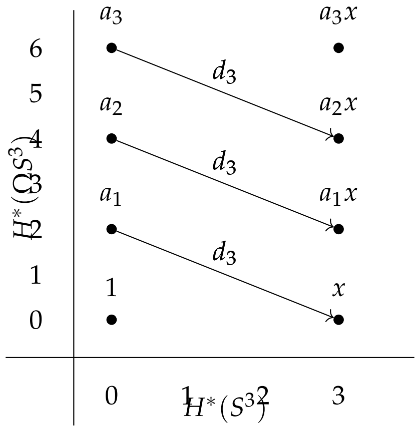
---

[10] In case this was not covered or you need a reminder, this states there is a natural short exact sequence

$$0 \to H^n(X; \mathbb{Z}) \otimes M \to H^n(X; M) \to \operatorname{Tor}(H^{n+1}(X; \mathbb{Z}), M) \to 0.$$



1.6.4 Example. We now consider the cohomology ring $H^*(\Omega S^3)$, leaving the general case of $H^*(\Omega S^n)$ as an exercise. To do this, we use the Serre spectral sequence of the fibration $\Omega S^3 \to PS^3 \simeq * \to S^3$. The additive structure can be determined much as in Example 1.5.11,[11] and the spectral sequence looks as follows:



That is, additively, we have

$$H^i(\Omega S^3) \cong \begin{cases} \mathbb{Z}, & i = 2k \\ 0, \text{else.} \end{cases}$$

In order to work out the multiplicative structure, we need to work out how the classes a_i relate to each other. For example, is $a_1^2 = a_2$? We have chosen the generators such that $d_3(a_i) = a_{i-1}x$ where $a_0 = 1$. Now we use the Leibniz rule to see that[12]

$$d_3(a_1^2) = d_3(a_1)a_1 + a_1 d_3(a_1) = 2a_1x = d_3(2a_2).$$

Because d_3 is an isomorphism, we deduce that $a_1^2 = 2a_2$. What about a_3? Note that

$$\begin{aligned} d_3(a_1a_2) &= d_3(a_1)a_2 + a_1d_3(a_2) = xa_2 + a_1^2x \\ &= xa_2 + 2xa_2 = 3xa_2 \\ &= d_3(3a_3). \end{aligned}$$

Because d_3 is an isomorphism, we deduce that $a_1a_2 = 3a_3$. Said another way, $a_1^3 = a_1a_1^2 = 2a_1a_2 = 3 \cdot 2 \cdot a_3$. By an inductive argument, we deduce that $a_1^n = n!a_n$, where a_n generates $E_2^{0,2n}$. We see that $H^*(\Omega S^3) \cong \Gamma_{\mathbb{Z}}[a_1]$, the divided polynomial algebra on a class a_1 in degree 2.[13]

You should now attempt the following exercise.[14]

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[11] Convince yourself of this!

[12] Here it is important that all our classes are in even total degrees!

[13] In general, the divided polynomial algebra on a ring R, denoted $\Gamma_R[\alpha]$ where α has (even) degree n is the algebra with additive generators α_i in degree ni and multiplication $\alpha_1^k = k!\alpha_k$ (and hence $\alpha_i\alpha_j = \binom{i+j}{i}\alpha_{i_j}$). Note that if $R = \mathbb{Q}$, then $\Gamma_{\mathbb{Q}}[\alpha] \cong \mathbb{Q}[\alpha]$, but in general it is more complex. For example, if $R = \mathbb{F}_p$, then $\Gamma_{\mathbb{F}_p}[\alpha] \cong \otimes_{i \geq 0}\mathbb{F}_p[\alpha_{p_i}]/(\alpha_{p^i}^p)$, a tensor product of truncated polynomial rings.

[14] Here $\Lambda_{\mathbb{Z}}[x] \cong \mathbb{Z}[x]/(x^2)$ is the exterior algebra

Exercise 4. *Use the cohomological Serre spectral sequence associated to the path fibration*

$$\Omega S^n \to PS^n \to S^n$$

to show the following: If n is odd, then

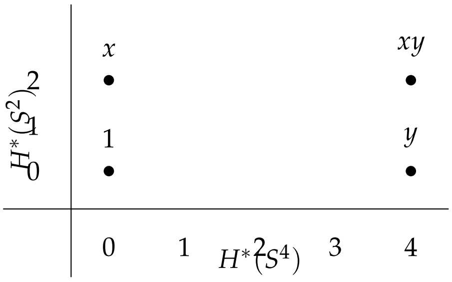
$$H^*(\Omega S^n) \cong \Gamma_{\mathbb{Z}}[x]$$

where $|x| = n - 1$*. If n is even, then*

$$H^*(\Omega S^n) \cong \Lambda_{\mathbb{Z}}[x] \otimes \Gamma_{\mathbb{Z}}[y]$$

where $|x| = n - 1$ *and* $|y| = 2n - 2$.

1.6.5 Example. Much like in the additive case, we can sometimes have multiplicative extensions that we cannot solve without additional information. For example, there is a fibration $S^2 \to \mathbb{C}P^3 \to S^4$, and the associated spectral sequence looks as follows:



There is no room for differentials, and so

$$H^i(\mathbb{C}P^\infty) \cong \begin{cases} \mathbb{Z}, & i = 0, 2, 4, 6 \\ 0, & \text{else.} \end{cases}$$

Yet from the spectral sequence, we cannot deduce (without further information) that $y = x_2^2$, which we know holds.[15]

1.6.6 Remark. A useful way to compute multiplicative extensions is the following theorem:[16] If there is a spectral sequence converging to H_* as an algebra and the E_∞-term is a free, graded-commutative, bigraded algebra, then H_* is a free, graded commutative algebra isomorphic to the total complex $E_\infty^{*,*}$, i.e.,

$$H_i \cong \bigoplus_{p+q=i} E_\infty^{p,q}.$$

1.6.7 Example. Recall that we have a fiber sequence

$$SU(n-1) \to SU(n) \to S^{2n-1}$$

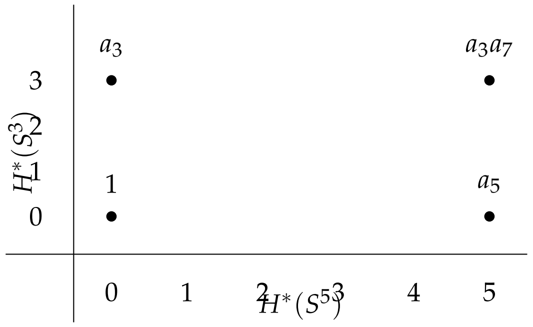
Taking $n = 3$, this has the form $S^3 \to SU(3) \to S^5$. The E^2-term is

$$E_2^{p,q} = H^p(S^5; H^q(S^3)) \cong H^p(S^5) \otimes H^q(S^3).$$

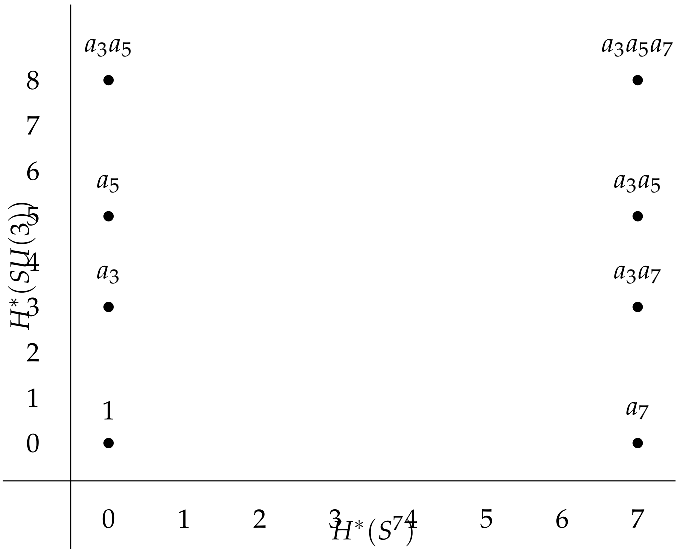
The $E^2 = E^\infty$-page looks as follows:

[15] Recall that $H^*(\mathbb{C}P^3) \cong \mathbb{Z}[x]/(x^4)$ for $|x| = 2$.

[16] See Example 1.K of McCleary's "A user's guide to spectral sequences"



Unlike in the previous example, there are no possible multiplicative extension problems, and we deduce that $H^*(SU(3)) \cong \Gamma_{\mathbb{Z}}(a_3, a_5)$, the exterior algebra on two generators a_3 and a_5. Now we consider the case $n = 4$. We leave it for the reader to deduce the following $E^2 = E^\infty$-page



The extension problem can be solved by Remark 1.6.6, and we get $H^*(SU(4)) \cong \Gamma_{\mathbb{Z}}(a_3, a_5, a_7)$. These two examples suggest a general pattern: is $H^*(SU(n)) \cong \Gamma_{\mathbb{Z}}(a_3, a_5, \ldots, a_{2n-1})$? The next exercise is to show that this is true.[17]

Exercise 5. *Show, using induction, that*

$$H^*(SU(n)) \cong \Gamma_{\mathbb{Z}}(a_3, a_5, \ldots, a_{2n-1}).$$

1.7 *The Gysin and Wang sequences*

The Gysin and Wang sequences are long exact sequences derived from special cases of the Serre spectral sequence. We will prove the following.

1.7.1 Theorem (The Gysin sequence). *Let $S^n \to E \to B$ be a fibration with $\pi_1 B = 0$ and $n \geq 1$. Then, there exists an exact sequence*

$$\cdots H_r(E) \to H_r(B) \to H_{r-n-1}(B) \to H_{r-1}(E) \to \cdots$$

We begin with two algebraic lemmas, whose proof we leave as exercises for the reader.

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[17] However, unlike the previous two examples, there will be differentials to deal with. The hint is to use the multiplicative structure, in particular the Leibniz rule.

1.7.2 Lemma. *Let $A \to B \xrightarrow{f} C$ and $D \to E \xrightarrow{g}$ be exact sequences of abelian groups. Suppose there exists an isomorphism $\phi\colon \operatorname{coker}(f) \cong \ker(g)$, then there is an exact sequence*

$$A \to B \xrightarrow{f} C \xrightarrow{\phi} D \to E \xrightarrow{g} F,$$

where $c \mapsto \phi(\bar{c})$, for $\bar{c}$ the class of c in $\operatorname{coker}(f)$.

1.7.3 Lemma. *Given the following diagram of abelian groups:*

$$\begin{tikzcd}
 & A \arrow[d, "f"] & & \\
 & B \arrow[d, "g"] \arrow[dr, "hg"] & & \\
0 \arrow[r] & C \arrow[r, "h"'] & D \arrow[r, "k"'] & E
\end{tikzcd}$$

with rows and columns exact, then the sequence $A \xrightarrow{f} B \xrightarrow{hg} D \xrightarrow{k} E$ is exact.

We now return to the Gysin sequence.

Proof of Theorem 1.7.1. We consider the Serre spectral sequence of the fibration. This has E_2-term

$$E^2_{p,q} \cong H_p(B; H_q(S^n)) \cong \begin{cases} H_p(B) & q = 0, n \\ 0 & \text{else.} \end{cases}$$

and so is as follows:

$H_*(S^n)$	n	$H_0(B)$	$H_1(B)$	$H_2(B)$	$H_3(B)$
	0	$H_0(B)$	$H_1(B)$	$H_2(B)$	$H_3(B)$

$H_*(B)$

We observe that there is only one possible differential, namely $d_{n+1}\colon E^{n+1}_{p,0} \to E^{n+1}_{p-n-1,n}$, and so $E^2 = E^{n+1}$ and $E^{n+2} = E^\infty$. Therefore, using Lemma 1.7.2 we get a short exact sequence

$$0 \to E^\infty_{p,0} \to E^{n+1}_{p,0} \xrightarrow{d_{n+1}} E^{n+1}_{p-n-1,n} \to E^\infty_{p-n-1,n} \to 0. \qquad (1.7.4)$$

The filtration on $H_i(E)$ is $0 \subseteq E^\infty_{i-n,n} = D_{i-n,n} \subseteq D_{i,0} = H_i(E)$, i.e., we have a short exact sequence:

$$0 \to E^\infty_{i-n,n} \to H_i(E) \to E^\infty_{i,0} \to 0. \qquad (1.7.5)$$

Pasting (1.7.4) and (1.7.5) together we get a diagram of the form:

$$\begin{tikzcd}
{} \arrow[dr, dashed] & {} \arrow[d, dashed] & & & & \\
& H_r(E) \arrow[d] \arrow[dr, dashed] & & & 0 \arrow[d] & \\
0 \arrow[r] & E^\infty_{r,0} \arrow[r] \arrow[d] & \underbrace{E^2_{r,0}}_{=H_r(B)} \arrow[r, "d_{n+1}"] & \underbrace{E^2_{r-n-1,n}}_{H_{r-n-1}(B)} \arrow[r] \arrow[dr, dashed] & E^\infty_{r-n-1,n} \arrow[r] \arrow[d] & 0 \\
& 0 & & & H_{r-1}(E) \arrow[d] \arrow[dr, dashed] & \\
& & & 0 \arrow[r] & E^\infty_{r-1,0} \arrow[r] \arrow[d] & \cdots \\
& & & & 0 &
\end{tikzcd}$$

and Lemma 1.7.3 implies that the sequence in red is exact. ☐

1.7.6 Example. Consider the fiber sequence $S^1 \to S^{2n-1} \to \mathbb{C}P^n$ for $n \geq 1$. One recalls that $H_p(\mathbb{C}P^n) = 0$ for $p > 2n$ using, for example, cellular homology. We will show that

$$H_p(\mathbb{C}P^n) \cong \begin{cases} \mathbb{Z} & p \text{ even} \\ 0 & p \text{ odd} \end{cases}$$

using the Gysin sequence. The sequence tell us that

$$0 = H_{2n+2}(\mathbb{C}P^n) \to H_n(\mathbb{C}P^n) \to H_{2n+1}(S^{2n+1}) \cong \mathbb{Z} \to H_{2n+1}(\mathbb{C}P^n) = 0$$

is exact, and so $H_n(\mathbb{C}P^n) \cong \mathbb{Z}$. Next, observe that we have an exact sequence

$$0 = H_{2n}(S^{2n+1}) \to H_{2n}(\mathbb{C}P^n) \cong \mathbb{Z} \to H_{2n-2}(\mathbb{C}P^n) \to H_{2n-1}(S^{2n-1})$$

so that $H_{2n-2}(\mathbb{C}P^n) \cong \mathbb{Z}$. Moreover, the exact sequence

$$0 = H_{2n+1}(\mathbb{C}P^n) \to H_{2n-1}(\mathbb{C}P^n) \to H_{2n}(S^{2n+1}) = 0$$

shows that $H_{2n-1}(\mathbb{C}P^n) = 0$. Inductively continuing, we get the claimed result.

Exercise 6. *Wang sequence Use the Serre spectral sequence to prove the following:*

If $F \to E \to S^n$ with $n \geq 2$ is a fibration, then there is an exact sequence

$$\cdots \to H_i(F) \to H_i(E) \to H_{i-n}(F) \to H_{i-1}(F) \to H_{i-1}(E) \to \cdots$$

1.7.7 Example. Let us return to the homology of ΩS^n for $n \geq 2$. We will show that

$$H_r(\Omega S^n) \cong \begin{cases} \mathbb{Z} & r = k(n-1) \\ 0 & \text{else.} \end{cases}$$

We do this via the Wang sequence of the fibration $\Omega S^n \to PS^n \to S^n$. Since PS^n is contractible, every third term vanishes except for $H_0(PS^n) \cong \mathbb{Z}$. So,

$$H_{r-n}(\Omega S^n) \cong H_{r-1}(\Omega S^n).$$

Because $H_0(\Omega S^n) \cong \mathbb{Z}$ (by path-connectedness), we get the claimed result inductively.

We also have a Wang sequence in cohomology.

1.7.8 Theorem (Wang sequence in cohomology)**.** *Let $F \xrightarrow{i} E \xrightarrow{p} S^n$ be a fiber sequence with $n \geq 1$, then there is a long exact sequence*

$$\cdots \to H^i(E) \xrightarrow{i^*} H^i(F) \xrightarrow{\theta} H^{q-n+1}(F) \to H^{q+1}(E) \to \cdots$$

where

$$\theta(u \smile v) = \theta(u) \smile v + (-1)^{(n-1)\deg(u)} u \smile \theta(v).$$

Proof sketch. The additive structure is determined as in the exercise. The fact that θ is a derivation follows because it is identified with the d^{n+1} differential in the spectral sequence. □

We can use this to recover the ring structure on $H^*(\Omega S^n)$.

1.7.9 Theorem. *If u is odd, then $H^*(\Omega S^u) \cong \Gamma_{\mathbb{Z}}(x)$, $|x| = u - 1$.*

Proof. The Wang sequence gives isomorphisms

$$H^n(\Omega S^u) \xrightarrow[\cong]{\theta} H^{n-u-1}(\Omega S^u).$$

This determines the additive structure. To work out the multiplicative structure, let $\gamma_0(x) = 1$ and inductively let $\gamma_i(x) \in H^{i(i-1)}(\Omega S^u)$ for $i \geq 1$ so that $\theta(\gamma_i(x)) = \gamma_{i-1}(x)$. By induction on i and j and using that θ is a derivation, we have[18]

$$\begin{aligned}\theta(\gamma_i(x) \smile \gamma_j(x)) &= \gamma_{i-1}(x) \smile \gamma_j(x) + \gamma_i(x) \smile \gamma_{j-1}(x) \\ &= \binom{i-1+j}{j}\gamma_{i-1+j}(x) + \binom{i-1+j}{j-1}\gamma_{i-1+j}(x) \\ &= \binom{j+i}{j}\gamma_{i-1+j}(x) \\ &= \binom{j+i}{j}\theta(\gamma_{i+j}(x)).\end{aligned}$$

Because θ is an isomorphism, we deduce that $\gamma_i(x) \smile \gamma_j(x) = \gamma_{i+j}(x)$, and the result follows. □

1.7.10 Theorem. *If $u \geq 2$ is even, then $H^*(\Omega S^u) \cong \Lambda_{\mathbb{Z}}(x) \otimes \Gamma_{\mathbb{Z}}(y)$, $|x| = u - 1$ and $|y| = 2(u-1)$.*

Proof. Let $x \in H^{u-1}(\Omega S^u)$ be such that $\theta(x) = 1$. By graded commutativity we have $x^2 = 0$. Let $\gamma_0(y) = 1$ and inductively define $\gamma_i(y) \in H^{2i(u-1)}(\Omega S^u)$ so that $\theta(\gamma_i(y)) = x\gamma_{i-1}(y)$. Then,

$$\theta(x\gamma_i(y)) = 1 \smile \gamma_i(y) - x \smile x\gamma_{i-1}(y) = 1 \smile \gamma_i(y) = \gamma_i(y),$$

---

[18] Try and spot where we use that x is is even degree.

so that $\gamma_i(y)$ generates $H^{2i(u-1)}(\Omega S^u)$ and $x\gamma_i(y)$ generates $H^{(2i+1)(u-1)}(\Omega S^u)$. Then

$$\begin{aligned}\theta(\gamma_i(y) \smile \gamma_j(y)) &= x\gamma_{i-1}(y) \smile \gamma_j(y) + \gamma_i(y) \smile x\gamma_{j-1}(y) \\ &= \binom{i-1+j}{j} x\gamma_{i-1+j}(y) + \binom{i-1+j}{j-1} x\gamma_{i-1+j}(y) \\ &= \binom{j+i}{j} x\gamma_{i-1+j}(t) \\ &= \binom{j+i}{j} \theta(\gamma_{i+j}(y)).\end{aligned}$$

Because θ is an isomorphism, we deduce that $\gamma_i(y) \smile \gamma_j(y) = \gamma_{i+j}(y)$, and the result follows. □

1.7.11 Remark. Of course, we also have a cohomological Gysin sequence: Suppose $S^{n-1} \to E \xrightarrow{p} B$ is a fibration with $\pi_1 B = 0$ and $n \geq 0$. Then, there exists an exact sequence

$$\cdots \to H^{i-n}(B) \xrightarrow{\smile e} H^i(B) \xrightarrow{p^*} H^i(E) \to H^{i-n+1}(B) \to \cdots$$

for a certain Euler class $e \in H^n(B)$ Moreover, if n is odd, then $2e = 0$.

1.8 Serre mod $\mathcal{C}$ theory

Our next goal is to prove the following theorem of Serre.

1.8.1 Theorem. *If X is a finite CW complex with $\pi_1(X) = 0$, then the homotopy groups $\pi_i(X)$ are finitely generated abelian groups for $i \geq 2$.*

1.8.2 Remark. How do we begin to prove such a theorem? Computing it directly is not possible. The ingenious idea of Serre is to use what are now known as Serre classes.

1.8.3 Definition. A class $\mathcal{C}$ of abelian groups is a Serre class if $0 \in \mathcal{C}$ and if for any short exact sequence $0 \to A \to B \to C \to 0$ we have $A, C \in \mathcal{C} \implies B \in \mathcal{C}$.

1.8.4 Remark. The following are a consequence of the definitions:[19]

(i) A Serre class is closed under isomorphisms.

(ii) A Serre class is closed under the formation of subgroups and quotient groups.

(iii) Let $A \xrightarrow{i} B \xrightarrow{p} C$ be exact at B. IF $A, C \in \mathcal{C}$, then $B \in \mathcal{C}$. This follows from the previous two points and the diagram

$$\begin{tikzcd}
& A \arrow[d, two heads] \arrow[dr, "i"] & & & \\
0 \arrow[r] & \ker(p) \arrow[r] & B \arrow[r] \arrow[dr, "p"'] & \operatorname{coker}(i) \arrow[r] \arrow[d, hook] & 0 \\
& & & C &
\end{tikzcd}$$

---

[19] You should check these.

1.8.5 Example. Examples of Serre classes include finite abelian groups, finitely generated abelian groups, and torsion abelian groups.

1.8.6 Definition. Given a morphism of abelian groups $\phi\colon A \to B$, we say ϕ is a monomorphism mod $\mathcal{C}$ if $\ker(\phi) \in \mathcal{C}$, an epimorphism mod $\mathcal{C}$ if $\operatorname{coker}\phi \in \mathcal{C}$, and an isomorphism mod $\mathcal{C}$ if both $\ker\phi, \operatorname{coker}\phi \in \mathcal{C}$.

1.8.7 Remark. We note the following:

(i) Let $C_\bullet$ be a chain complex. If $C_n \in \mathcal{C}$, then $H_n(C_\bullet) \in \mathcal{C}$.

(ii) Suppose $F_\bullet A$ is a filtration on an abelian group. If $A \in \mathcal{C}$, then $G_s A \in \mathcal{C}$ for all s. If the filtration is finite ($F_m = 0, F_n = A$, for some m, n) and $G_s(A) \in \mathcal{C}$ for all a, then $A \in \mathcal{C}$.

(iii) Suppose we have a spectral sequence $\{E^r_{S,t}\}$. If $E^2_{s,t} \in \mathcal{C}$, then $E^r_{s,t} \in \mathcal{C}$ for all $r \geq 2$. If $\{E^r\}$ is a first quadrant spectral sequence, then $E^\infty \in \mathcal{C}$. If the spectral sequence comes from a finite filtered complex C and $E^2_{s,t} \in \mathcal{C}$ for all $s + t = n$, then $H_n(C) \in \mathcal{C}$.

1.8.8 Definition. A Serre class is called a Serre ring if $A, B \in \mathcal{C} \implies A \otimes B \in \mathcal{C}$ and $\operatorname{Tor}(A, B) \in \mathcal{C}$, and a Serre ideal if only one of A or B is required to be in $\mathcal{C}$. Finally, a Serre class is called acyclic if $A \in \mathcal{C}$ implies $\tilde{H}_*(K(A, 1), \mathbb{Z}) \in \mathcal{C}$. [20]

1.8.9 Theorem (Hurewicz mod $\mathcal{C}$)**.** *Assume that $\mathcal{C}$ is an acyclic Serre ring, and let X be a simply connected space. If $\pi_q(X) \in \mathcal{C}$ for all $q < n$, then $\widetilde{H}_q(X) \in \mathcal{C}$ for all $q < n$, and in that case the Hurewicz map $\pi_n(X) \to H_n(X)$ is a mod $\mathcal{C}$ isomorphism.*

Proof. Simply observe that the same argument as Theorem 1.5.15 works replacing 'isomorphism' with 'isomorphism mod $\mathcal{C}$' where necessary. ☐

We return to Serre's theorem.

Proof of Theorem 1.8.1. Take $\mathcal{C}$ to be the class of finitely generated abelian groups, which is an acyclic Serre ring. Because X is a finite CW-complex $\widetilde{H}_i(X) \in \mathcal{C}$ since X is a finite CW-complex. Now suppose there exists a minimal i such that $\pi_i(X)$ is not finitely generated. Then, by the relative Hurewicz theorem, $h\colon \pi_i(X) \to H_n(X)$ is an isomorphism mod finitely generated abelian groups, and so $H_i(X)$ is also not finitely generated, a contradiction. ☐

1.9 *Final notes on spectral sequences*

We finish this section on spectral sequences by including some observations which we have so far omitted, beginning with the construction of the Serre spectral sequence.

1.9.1 Remark (Construction of the Serre spectral sequence). Let us sketch the construction of the Serre spectral sequence. Let $F \to E \xrightarrow{\pi} B$ be a fibration with B a simply-connected CW complex, and $\pi_0(F) = 0$.[21] Let $C_*(E)$ be the singular chain complex of E, and

[20] For example, the class of all finitely generated abelian groups is an acyclic Serre ring, while the class of torsion abelian groups is an acyclic Serre ideal.

[21] By cellular approximation, we can always replace B a CW-complex, , and replace $\pi\colon E \to B$ with $\pi'\colon E' \to B'$, the pullback of the fibration along $B' \to B$.

filter this by

$$F^pC_*(E) := C_*(\pi^{-1}(B_p)),$$

where B_p is the p-skeleton of B. Then,

$$\begin{aligned} F^pC_*(E)/F^{p-1}C_*(E) &\simeq C_*(\pi^{-1}(B_p))/C_*(\pi^{-1}(B_{p-1})) \\ &\simeq C_*(\pi^{-1}(B_p), \pi^{-1}(C_{p-1})) \end{aligned}$$

and (using excision)

$$\begin{aligned} H_*(F^pC_*(E)/F^{p-1}C_*(E)) &\cong H_*(\pi^{-1}(B_p), \pi^{-1}(B_{p-1})) \\ &\cong \oplus_{e^p} H_*(\pi^{-1}(e^p), \pi^{-1}(\partial e^p)) \end{aligned}$$

where the direct sum is over the p-cells e^p in B. Since e^p is contractible, locally the fibration looks like the trivial fibration, i.e., $\pi^{-1}(e^p) \cong e^p \times F$. Then,

$$\begin{aligned} H_*(\pi^{-1}(e^p), \pi^{-1}(\partial e^p)) &\cong H_*(e^p \times F, \partial e^p \times F) \\ &\cong H_*(D^p \times F, S^{p-1} \times F) \\ &\cong H_{*-p}(F) \\ &\cong H_p(D^p, S^{p-1}; H_{*-p}(F)) \end{aligned}$$

where we have use the Künneth formula. The general machinery of spectral sequence tells us we have a spectral sequence with E_1 term

$$\begin{aligned} E^1_{p,q} &= H_{p+q}(F^pC_*(E)/F^{p-1}C_*(E)) \\ &\cong \oplus_{e^p} H_p(D^p, S^{p-1}; H_q(F)) \end{aligned}$$

The d^1-differential can be checked to be exactly the cellular boundary map of the CW-chain complex of B with coefficients in $H_q(F)$, i.e.,

$$E^1_{p,q} = C^{cell}_p(B; H_q(F))$$

and

$$E^2_{p,q} = H_p(B; H_q(F)).$$

1.9.2 Remark. There are several natural generalizations of the Serre spectral sequence:

1. We can take homology with coefficients in an abelian group, i.e.,

$$E^2_{s,t} = H_p(B; H_q(F; G)) \implies H_{p+q}(E; G).$$

2. There is a relative form of the Serre spectral sequence: Suppose we have $F \to E \xrightarrow{\pi} B$, and $B' \subseteq B$. If we set $E' \colon \pi^{-1}(B')$, then there is a spectral sequence

$$E^2_{s,t} = H_p(B, B'; H_q(F; G)) \implies H_{p+q}(E, E'; G).$$

3. If $\pi_1(B)$ is not 0, but acts trivially on $H_*(F)$, then the spectral sequence takes the same form. If $\pi_1(B)$ does not act trivially then there is still a Serre spectral sequence, but then one must use homology with local coefficients.

We finish this section with another theorem of Serre.

1.9.3 Theorem (Serre). *The group $\pi_i(S^n)$ are finite for $i > n$ except for $\pi_{4k-1}(S^{2k})$, which is the direct sum of $\mathbb{Z}$ with a finite group.*

Proof. We can assume $n > 1$, in which case we can use the Serre spectral sequence. Note that we already know that these are finitely generated abelian groups by Theorem 1.8.1. Recall that $H_n(S^n) \cong \mathbb{Z}$, and that $H^n(S^n) \cong [S^n, K(\mathbb{Z}, n)]$. Moreover, the Hurewicz map $h_n \colon \pi_n(S^n) \to H_n(S^n)$ is an isomorphism, so together we can find a homotopy class of map $f \colon S^n \to K(\mathbb{Z}, n)$ inducing an isomorphism on π_n. We can assume that this is a fibration. By the long exact sequence in homotopy, the fiber F is $(n-1)$-connected, and $\pi_i(F) \cong \pi_i(S^n)$ for $i > n$. Converting $F \to S^n$ into a fibration, and taking the fiber (which is then a $K(\mathbb{Z}, n-1)$), we have a fibration

$$K(\mathbb{Z}, n-1) \to F \to S^n.$$

We will study the Serre spectral sequence associated to this rationally, i.e., with $\mathbb{Q}$-coefficients. This takes the form

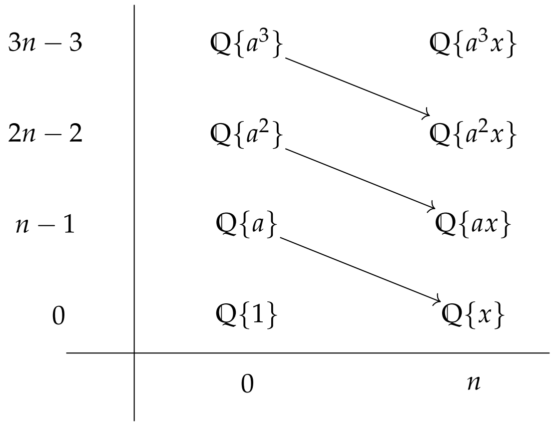
$$\begin{aligned} E_2^{p,q} &\cong H^p(S^n; H^q(K(\mathbb{Z}, n-1); \mathbb{Q})) \\ &\cong H^p(S^n; \mathbb{Q}) \otimes H^q(K(\mathbb{Z}, n-1); \mathbb{Q})). \end{aligned}$$

We have[22]

$$H^*(K(\mathbb{Z}, n); \mathbb{Q}) \cong \begin{cases} \mathbb{Q}[x], & n \text{ even} \\ \Lambda_{\mathbb{Q}}(x), & n \text{ odd}, \end{cases}$$

where $|x| = n$.

We begin with the case n odd. The Serre spectral sequence looks as follows:



The differential $\mathbb{Q}\{a\} \to \mathbb{Q}\{x\}$ must be an isomorphism, as otherwise it would be zero, and $\mathbb{Q}\{a\}$ would survive to E^∞, a contradiction to the fact that F is $(n-1)$-connected. By the Leibniz rule, all the differentials are isomorphisms, and so $\widetilde{H}_*(F; \mathbb{Q}) = 0$, and so $\pi_i(F) \otimes \mathbb{Q} = 0$ for all i. By a Serre class argument $\pi_i(F)$ is a finitely-generated abelian group, and so it must in fact be finite for all i. Therefore, $\pi_i(S^n)$ is finite for $i > n$.

The case n is a bit trickier. The spectral sequence now looks as follows:

[22] This is another good exercise. The hint is to use induction, starting from $K(\mathbb{Z}, 1) = S^1$.

$$\begin{tikzcd}
n-1 & \mathbb{Q}\{a\} \arrow[rd] & \mathbb{Q}\{ax\} \\
0 & \mathbb{Q}\{1\} & \mathbb{Q}\{x\} \\
 & 0 & n
\end{tikzcd}$$

So we deduce

$$\widetilde{H}^*(F;\mathbb{Q}) \cong \begin{cases} \mathbb{Q} & * = 0, 2n-1 \\ 0 & \text{else.} \end{cases}.$$

By the Hurewicz theorem mod finitely generated abelian groups,we conclude that $\pi_i(S^n)$ is finite for $n < i < 2n-1$ and $\pi_{2n-1} \cong \mathbb{Z}$ plus a finite abelian group. To deal with $\pi_i(S^n)$ for $i > 2n-1$, we let Y be the space obtained from F by attaching cells of dimension $2n+1$ and greater to kill $\pi_i(F)$ for $i \geq 2n-1$. We can assume $F \to Y$ is fibration, with fiber Z. Then Z is $(2n-2)$-connected and has $\pi_i(Z) \cong \pi_i(Y)$ for $i < 2n-1$ and so all the homotopy groups of Y are finite. Thus $\widetilde{H}^*(Y;\mathbb{Q}) = 0$ and from the Serre spectral sequence associated to $Z \to F \to Y$ we get

$$\widetilde{H}^*(Z;\mathbb{Q}) \cong \widetilde{H}^*(F;\mathbb{Q}) \cong \widetilde{H}^*(S^{2n-1};\mathbb{Q})$$

Using the earlier argument for n odd with Z instead of S^n, we conclude that $\pi_i(Z)$ is finite for $i > 2n-1$, but $\pi_i(S^n) \cong \pi_i(Z)$ for $i > 2n-1$ and we are done. ☐

Exercise 7. *Let $\pi\colon E \to B$ be a fibration with fiber F, let k be a field, and suppose $\pi_1(B) = \pi_0(F) = 0$. Assume that the Euler characteristics $\chi(B), \chi(F)$ are defined over the field k. (Recall that for a chain complex C, the Euler characteristic is the alternating sum of the ranks of the homology of the chain complex, assuming these ranks are all finite.) Then $\chi(E)$ is defined, and*[23]

$$\chi(E) = \chi(B) \cdot \chi(F).$$

[23] **Hint:** Construct an 'Euler characteristic' for the E_r-page of the Serre spectral sequence.