1
Fiber bundles

1.1 Locally trivial bundles

We begin with what we will call a locally trivially bundle.[1]

[1] Confusingly, some books will also call this a fiber bundle. We will see why later.

1.1.1 Definition. A map $\pi\colon E \to B$ is a locally trivial bundle with fiber F if the following conditions hold:

1. Each point $b \in B$ has a neighbourhood U such that $\pi^{-1}(U_b) \xrightarrow{h_b} U_b \times F$.
2. The following diagram commutes

$$\begin{tikzcd} \pi^{-1}(U_b) \arrow[rr, "h_b"] \arrow[dr, "\pi"'] & & U_b \times F \arrow[dl, "\mathrm{pr}_1"] \\ & U_b & \end{tikzcd}$$

The maps h_b are called the *local trivializations* of the bundle.

1.1.2 Example. Let $E = B \times F$ and $\pi\colon E = B \times F \to B$ the projection map. This is called the trivial bundle.

1.1.3 Example. If F is discrete, then a locally trivial bundle with fiber F is a covering map.

1.1.4 Example. The Möbius band is a locally trivial bundle with fiber S^1, see Figure 1.1. We will return to this example in due course.

1.1.5 Remark. We can write a locally trivial bundle as $F \to E \to B$, which is reminiscent of the notation for a fibration. In fact, fiber bundles over paracompact base spaces are always fibrations.[2] More generally, any locally trivial bundle is a Serre fibration.

[2] A space is paracompact if every open cover has an open refinement that is locally finite.

1.1.6 Remark. Let us unwind the definition of a locally trivial bundle a little more. Let $\pi\colon E \to B$ be a locally trivial bundle with fiber F. From the definition we can cover B by a family of open sets $\{U_\alpha\}$ such that each inverse image $\pi^{-1}(U_\alpha)$ is fiberwise homeomorphic to $U_\alpha \times F$. This gives a system of homeomorphisms

$$\phi_\alpha\colon U_\alpha \times F \to \pi^{-1}(U_\alpha).$$

Observe that if $V \subseteq U_\alpha$ then the restriction of ϕ_α to $V \times F$ gives the homeomorphism with $\pi^{-1}(V)$. Hence on $U_\alpha \cap U_\beta$ there are two

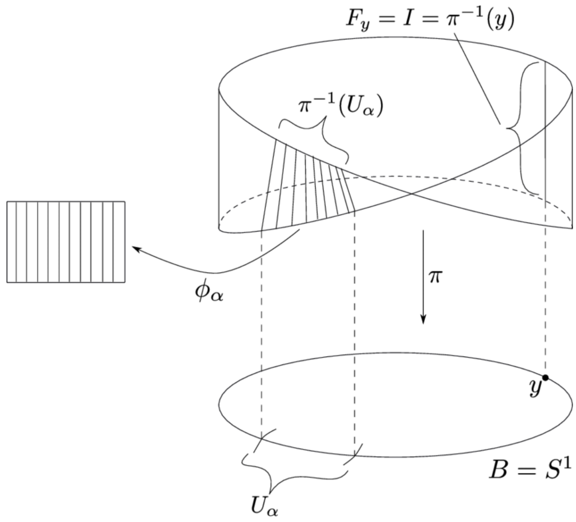

Figure 1.1: The Möbius band

fiberwise homeomorphisms

$$\phi_\alpha \colon (U_\alpha \cap U_\beta) \times F \to \pi^{-1}(U_\alpha \cap U_\beta)$$
$$\phi_\beta \colon (U_\alpha \cap U_\beta) \times F \to \pi^{-1}(U_\alpha \cap U_\beta)$$

Consider the following commutative diagram

$$\begin{tikzcd}
(U_\alpha \cap U_\beta) \times F \arrow[r, "\phi_\alpha", "\cong"'] \arrow[dr] & \pi^{-1}(U_\alpha \cap U_\beta) \arrow[r, "\phi_\beta^{-1}", "\cong"'] \arrow[d] & (U_\alpha \cap U_\beta) \times F \arrow[dl] \\
& U_\alpha \cap U_\beta &
\end{tikzcd}$$

Let $\phi_{\alpha\beta}$ denote the top composite $\phi_\beta^{-1}\phi_\alpha$. Then the locally trivially bundle is completed determined by the base B, the fiber F, the covering U_α and the homeomorphisms $U_{\alpha\beta}$. Roughly speaking, E should be thought of as the cartesian product of the $U_\alpha \times F$ with some identifications by the $\phi_{\alpha\beta}$.

1.1.7 Definition. The open sets U_α are called *charts*, the family U_α the *atlas of charts*, the homeomorphisms ϕ_α are called the *coordinate homeomorphisms* and the $\phi_{\alpha\beta}$ are called the *transition functions.*

1.1.8 Remark. In order for homeomorphisms $\phi_{\alpha\beta}$ to be the transition functions of a locally trivial bundle, they must satisfy, $\phi_{\alpha\beta} = \phi_\beta^{-1}\phi_\alpha$. For example,

$$\phi_{\alpha\alpha} = \mathrm{id}$$

and

$$\phi_{\gamma\alpha}\phi_{\beta\gamma}\psi_{\alpha\beta} = \mathrm{id}$$

on any triple $(U_\alpha \cap U_\beta \cap U_\gamma) \cap F$. Taking $\gamma = \alpha$ we get

$$\phi_{\alpha\beta}\phi_{\beta\alpha} = \mathrm{id}.$$

In fact, these conditions suffices to reconstruct the locally trivial bundle for the base, the fiber, atlas and homeomorphisms. Indeed, set $E = E'/\sim$ where

$$E' = \bigcup_\alpha (U_\alpha \times F)$$

and for $(x, f) \in U_\alpha \times F$ and $(y, g) \in U_\beta \times F$ we have $(x, f) \sim (y, g) \iff x = y \in (U_\alpha \cap U_\beta)$ and $(y, g) = \phi_{\alpha\beta}(x, f)$. It is a rather tedious exercise to show that this determines a locally trivial bundle.

1.1.9 Definition. Two locally trivial bundles $\pi \colon E \to B$ and $\pi' \colon E' \to B'$ are isomorphic if there is a homeomorphism $\psi \colon E \to E'$ such that the diagram

$$\begin{tikzcd} E \arrow[rr, "\psi"] \arrow[dr, "\pi"'] & & E' \arrow[dl, "\pi'"] \\ & B & \end{tikzcd}$$

commutes. (Note that this implies that there is a homeomorphism $F \to F'$ between the fibers as well).

1.1.10 Theorem. *Two systems of transition functors $\phi_{\beta\alpha}$ and $\phi'_{\beta\alpha}$ define isomorphic locally trivial bundles iff there exists fiber preserving homeomorphisms*

$$h_\alpha \colon U_\alpha \times F \to U_\alpha \times F$$

such that $\phi_{\beta\alpha} = h_\beta^{-1} \phi'_{\beta\alpha} h_\alpha$.

Proof. First, we suppose that the two bundles are isomorphic, so in particular there is a homeomorphism $\psi \colon E \to E'$. We let

$$h_\alpha := \phi_\alpha'^{-1} \psi^{-1} \phi_\alpha \colon U_\alpha \times F \to U_\alpha \times F.$$

Then we have

$$\begin{aligned} h_\beta^{-1} \phi'_{\beta\alpha} h_\alpha &= \phi_\beta^{-1} \psi^{-1} \phi'_\beta \phi'_{\beta\alpha} \phi_\alpha^{-1} \psi^{-1} \phi_\alpha \\ &= \phi_\beta^{-1} \psi^{-1} \phi_{\beta'} \phi_\beta'^{-1} \phi'_\alpha \phi_\alpha^{-1} \psi^{-1} \phi_\alpha = \phi_{\beta\alpha}. \end{aligned}$$

Conversely, if the relations hold, then we set $\psi = \phi_\alpha h_\alpha^{-1} \phi_\alpha'^{-1}$. A similar argument then shows that $\phi_\beta h_\beta^{-1} \phi_\beta'^{-1} = \phi_\alpha h_\alpha^{-1} \phi' - 1_\alpha$. $\square$

1.1.11 Remark. If π is (isomorphic to) a trivial bundle, then all transition functions can be chosen to be the identity. One can use the previous theorem to show that a bundle is not isomorphic to a trivial bundle.

1.1.12 Example. After this discussion, let us return to the example of the Möbius bundle (Example 1.1.4). One can think of this as the space

$$E = \{(x, y) \colon 0 \le x \le 1, 0 \le y \le 1\} / \sim$$

where we identify $(0, y)$ and $(1, 1 - y)$ for each $y \in [0, 1]$. The projection maps E to $I_x = \{0 \le x \le 1\}$ with the endpoints identified, that is, onto the circle. To see that this is a bundle we use the atlas

$$U_\alpha = \{0 \le x \le 1\}, \text{ and } U_\beta = \{0 \le x < 1/2\} \cup \{1/2 < x \le 1\}.$$

We define

$$\phi_\alpha \colon U_\alpha \times I_y \to E, \quad \phi_\alpha(x, y) = (x, y),$$

and

$$\phi_\beta : U_\beta \times I_y \to E$$

by

$$\phi_\beta = \begin{cases} (x,y) & \text{for } 0 \le x \le /1/2, \\ (x, 1-y) & \text{for } 1/2 < x \le 1. \end{cases}$$

The intersection of these two charts is the union $(0,1/2) \cup (1/2,1)$, and the transition functions have the form

$$\phi_{\beta\alpha} = (x,y) \text{ for } 0 < x < 1/2$$

and

$$\phi_{\beta\alpha} = (x, 1-y) \text{ for } 1/2 < x < 1.$$

One can check from Remark 1.1.11 that the Möbius bundle is not isomorphic to a trivial bundle.

We now give some more examples which will be useful in our study of characteristic classes.

1.1.13 Definition. For $n < k$ the n-th Stiefel manifold associated to $\mathbb{R}^k$ is defined as

$$V_n(\mathbb{R}^k) = \{n - \text{frames in } \mathbb{R}^k\}$$

where an n-frame in $\mathbb{R}^k$ is a tuple $\{v_1, \ldots, v_n\}$ of orthonormal vectors in $\mathbb{R}^k$, i.e., $v_1, \ldots, v_n$ are pairwise orthonormal, $\langle v_i, v_j \rangle = \delta_{ij}$. We given $V_n(\mathbb{R}^k)$ the subspace topology induced by thinking of it as a subspace of $S^{k-1} \times \ldots S^{k-1}$ (n-copies of S^{k-1}).

1.1.14 Example. A 1-frame is nothing but a unit vector, so the Stiefel manifold $V_1(\mathbb{R}^k)$ is the unit sphere in $\mathbb{R}^k$, i.e., $V_1(\mathbb{R}^k) \cong S^{k-1}$. On the other hand, an n-frame is an ordered basis, so $V_n(\mathbb{R}^n) \cong O(n)$.

1.1.15 Definition. The n-th Grassmannian associated to $\mathbb{R}^k$ is defined as

$$G_n(\mathbb{R}^k) = \{n - \text{dimensional vector subspaces in } \mathbb{R}^k\}$$

There is a map $p \colon V_n(\mathbb{R}^k) \to G_n(\mathbb{R}^k)$ sending $\{v_1, \ldots, v_n\}$ to the span, which is surjective by Gram–Schmidt, and we given $G_n(\mathbb{R}^k)$ the quotient topology.

1.1.16 Example. We have $G_1(\mathbb{R}^k)$ is the space of lines through the origin in k-space, so $G_1(\mathbb{R}^k) \simeq \mathbb{R}P^{k-1}$.

1.1.17 Lemma. *For $k > n$ the quotient map $p \colon V_n(\mathbb{R}^k) \to G_n(\mathbb{R}^k)$ is a locally trivial bundle with fiber $V_n(\mathbb{R}^n) \cong O(n)$, i.e., we have a locally trivial bundle*

$$O(n) \to V_n(\mathbb{R}^k) \xrightarrow{p} G_n(\mathbb{R}^k). \tag{1.1.18}$$

Similarly, for $m < n \le k$ there are locally trivial bundles

$$V_{n-m}(\mathbb{R}^k) \to V_n(\mathbb{R}^k) \xrightarrow{p} V_m(\mathbb{R}^k). \tag{1.1.19}$$

where the map p takes $\{v_1, \ldots, v_n\}$ to $\{v_1, \ldots, v_m\}$. Taking $k = n$ we get a locally trivial bundle

$$O(n-m) \to O(n) \xrightarrow{p} V_m(\mathbb{R}^n). \tag{1.1.20}$$

1.1.21 Example. Taking $m = 1$ in (1.1.20) we get a locally trivial bundle
$$O(n-1) \to O(n) \to S^{n-1}.$$
Here the first map takes A to $\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$ and the second takes B to Bu for $u \in S^{n-1}$ some unit vector. In particular, this identifies S^{n-1} as an orbit space $S^{n-1} \cong O(n)/O(n-1)$.

Exercise 1

Use the fibrations
$$O(n-k) \to O(n) \to V_k(\mathbb{R}^n)$$
to show that
$$\pi_i(O(n-1)) \simeq \pi_i(O(n)) \text{ for } i < n-2$$
and
$$\pi_i(V_k(\mathbb{R}^n)) = 0$$
for $i < n-k-1$.

1.1.22 Definition. We have infinite versions of the Stiefel manifold and Grassmanian:
$$V_n(\mathbb{R}^\infty) := \bigcup_{k=1}^{\infty} V_n(\mathbb{R}^k) \qquad G_n(\mathbb{R}^\infty) := \bigcup_{k=1}^{\infty} G_n(\mathbb{R}^k)$$

1.1.23 Remark. We get a fiber sequence
$$O(n) \to V_n(\mathbb{R}^\infty) \to G_n(\mathbb{R}^\infty).$$

1.1.24 Proposition. *$V_n(\mathbb{R}^\infty)$ is contractible.*

Proof. As in the exercise, we deduce that $\pi_i(V_n(\mathbb{R}^\infty) = 0$ for all i. We can give $V_n(\mathbb{R}^\infty)$ the structure of a CW-complex, and so the claim follows from **??**. □

1.1.25 Remark. One can repeat the same story using $\mathbb{C}$ or $\mathbb{H}$ instead of $\mathbb{R}$. In the first case, all instances of $O(n)$ get replaced by $U(n)$, and in the second case by $\mathrm{Sp}(n)$.

1.1.26 Example (The tangent bundle to S^2). Let $S^2 = \{(x_0, x_1, x_2) \in \mathbb{R}^3 \mid x_0^2 + x_1^2 + x_2^2 = 1\}$. Recall that the tangent space at a point $x \in S^2$ is defined by $T_xS^2 = \{\xi \in \mathbb{R}^3 \mid x \perp \xi\}$. We then define $TS^2 = \coprod_{x \in S^2} T_xS^2$. This can be toplogized as a subspace of $\mathbb{R}^3 \times \mathbb{R}^3$ when we write
$$TS^2 = \{(x, \xi) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid x \in S^2, x \perp \xi\}.$$
There is a natural projection map $p \colon TS^2 \to S^2$ sending the pair (x, ξ) to x, which we claim is a locally trivial bundle with fiber $\mathbb{R}^2$. To see this is a locally trivial bundle, let U be the open subset of

S^2 defined by $x_3 > 0$. We will show how to construct the local trivialization on this open subset.

If $\xi = (\xi_1, \xi_2, \xi_3)$ then we have the relation

$$x_1\xi_1 + x_2\xi_2 + x_3\xi_3 = 0$$

or

$$\xi_3 = -(x_1\xi_1 + x_2\xi_2)/x_3.$$

We define

$$\phi\colon U \times \mathbb{R}^2 \to p^{-1}(U)$$

by

$$\phi(x_1, x_2, x_3, \xi_1, \xi_2) = (x_1, x_2, x_3, \xi_1, \xi_2, -(x_1\xi_1 + x_2\xi_2)/x_3),$$

which gives the required chart for this open subset.

1.1.27 Remark. More generally, for any smooth manifold X of dimension n, we have a locally trivial bundle $\pi\colon TX \to X$ with fiber $\mathbb{R}^n$.

1.2 *The structure group of locally trivial bundles*

We recall that the on the intersection of two local trivializations we constructed a homeomorphism

$$\phi_{\beta\alpha}\colon (U_\alpha \cap U_\beta) \times F \to \pi^{-1}(U_\alpha \cap U_\beta) \to (U_\alpha \cap U_\beta) \times F$$

Unwinding the definition, the map ϕ is completely determine by a map $\overline{\phi}\colon U \to \mathrm{Homeo}(F)$, where $\mathrm{Homeo}(F)$ denotes the group of all homeomorphisms of the fiber F.[3]. Indeed, we have

$$\phi_{\alpha\beta}(x, f) = (x, \overline{\phi}(x)(f))$$

In other words, instead of $\phi_{\alpha\beta}$ to determine a bundle we can instead specify a family of functions

$$\overline{\phi}_{\alpha\beta}(x, f)\colon U_\alpha \cap U_\beta \to \mathrm{Homeo}(F),$$

having values in the group $\mathrm{Homeo}(F)$. Of course these are not arbitrary, but need to satisfy various compatibility conditions:

$$\overline{\phi}_{\alpha\alpha}(x) = \mathrm{id}$$

and

$$\overline{\phi}_{\alpha\gamma}(x)\overline{\phi}_{\gamma\beta}(x)\overline{\phi}_{\beta\alpha}(x) = \mathrm{id},$$

for $x \in U_\alpha \cap U_\beta \cap U_\gamma$.

1.2.1 Definition. Let E, B, F be topological spaces and G a topological group which acts freely on the space F. A continuous map $p\colon E \to B$ is a locally trivial bundle with fiber F and structure group G if there is an atlas $\{U_\alpha\}$ and the coordinate homeomorphisms

$$\phi_\alpha\colon U_\alpha \times F \to p^{-1}(U_\alpha)$$

[3] If we choose the correct topology on $\mathrm{Homeo}(F)$, namely the compact-open topology (for reasonable spaces at least), then this map is even continuous

such that the transition functions

$$\phi_{\beta\alpha} = \phi_\beta^{-1}\phi_\alpha \colon (U_\alpha \cap U_\beta \times F) \to (U_\alpha \cap U_\beta \times F)$$

have the form

$$\phi_{\beta\alpha}(x, f) = (x, \overline{\phi}_{\beta\alpha}(x)f)$$

where $\overline{\phi}_{\beta\alpha} \colon (U_\alpha \cap U_\beta) \to G$ are continuous functions satisfying

$$\overline{\phi}_{\alpha\alpha}(x) = \mathrm{id}$$

and

$$\overline{\phi}_{\alpha\gamma}(x)\overline{\phi}_{\gamma\beta}(x)\overline{\phi}_{\beta\alpha}(x) = \mathrm{id},$$

1.2.2 Remark. Some words on terminology are useful. What we defined as a locally trivial bundle with fiber F, is exactly a locally trivial bundle with fiber F and structure group $\mathrm{Homeo}(F)$. Either of these may also be called a *fiber bundle* or a *fiber bundle with structure group G*.

1.2.3 Remark.

1.2.4 Remark. Note that the structure group is not unique. For example, a bundle with structure group G may admit transition functions with values in a subgroup $H \leq G$. We say that the structure group G is reduced to subgroup H. More generally, if $\rho \colon G \to G'$ is a continuous homeomorphism of topological groups, and we are given a locally trivial bundle with structure group G and the transition functors $\alpha_{\alpha\beta} \colon U_\alpha \cap U_\beta \to G$, then a new locally trivial bundle with structure group G' may be constructed by

$$\phi'_{\alpha\beta}(x) = \rho(\phi_{\alpha\beta}(x)).$$

This operation is called the change of structure group.

Exercise 2

Show that a trivial bundle has trivial structure group. Conversely, if the structure group can be reduced to the trivial group then the bundle is (isomorphic to) a trivial bundle.

1.2.5 Example. Let us return to the Möbius bundle (Examples 1.1.4 and 1.1.12). We have that

$$\overline{\phi}_{\alpha\beta}(y) = y \quad \text{and } \overline{\phi}_{\beta\alpha}(y) = 1 - y.$$

Note that $\overline{\phi}_{\beta\alpha} \circ \overline{\phi}_{\beta\alpha}(y) = 1 - (1 - y) = y = \overline{\phi}_{\alpha\beta}(y)$. Therefore, the group generated by $\overline{\phi}_{\alpha\beta}$ and $\overline{\phi}_{\beta\alpha}$ has order 2. In other words, the Möbius bundle has (or can be reduced to) structure group $\mathbb{Z}/2$.

1.2.6 Example. The tangent bundle $TS^2 \to S^2$ was considered in Example 1.1.26. The coordinate homeomorphisms

$$\phi \colon U \times \mathbb{R}^2 \to \mathbb{R}^3 \times \mathbb{R}^3$$

are defined by formulas that are linear with respect to the second argument. Hence that transition functions have values in the group of linear translations of the fiber $F = \mathbb{R}^2$, that is $G = GL_2(\mathbb{R})$. In fact, it can be shown that the structure group can be reduced to the subgroup $O(n)$ of orthonormal rotations.

1.3 Principal bundles

The most important example of a bundle for a us is a *principal bundle*. We postpone the definition to talk a little about group actions.

1.3.1 Definition. Let G be a topological group, and X a topological space. A left action of G on X is a continuous map $\mu\colon G \times X \to X$, satisfying $\mu(e, x) = x$ and $\mu(h, \mu(g, x)) = \mu(hg, x)$.

1.3.2 Example. The multiplication map $\mu\colon G \times G \to G$ defines a left action of G on itself. Similarly, if $H \subseteq G$, then $\mu|_{H\times G}\colon H \times G \to G$ defines an action of H on G.

1.3.3 Remark. The map $\mu\colon G \times X \to X$ is adjoint to a map $ad(u)\colon G \to \text{Homeo}(X)$, where $\text{Homeo}(X)$ has the compact-open topology. If X is nice (specifically, locally compact Hausdorff) then a continuous map $G \to \text{Homeo}(X)$ gives rise to a group action $G \times X \to X$.

1.3.4 Example. The adjoint to the multiplication $\mu\colon G \times G \to G$ is the map $G \to \text{Homeo}(G)$ given by $g \mapsto f_g$, where $f_g\colon G \to G$ is given by $f_g(x) = gx$.

1.3.5 Remark. We recall some standard terminology associated to a group action of G on X:

1. The orbit of a point $x \in X$ is the set $Gx = \{g \cdot xmidg \in G\}$.
2. The orbit space X/G is the quotient space $X/\sim$ where $x \sim g \cdot x$.
3. The fixed set $X^G := \{x \in X | g \cdot x \text{ for all } g \in G\}$.
4. An action if free[4] if $g \cdot x \neq x$ for all $x \in X$ and all $g \neq e$. (i.e., $gx = x$ for all x implies $g = e$).
5. The stabilizer (or isotropy) group at x is $G_x = \{g \in G \mid g \cdot x = g\} \subseteq G$.
6. An action is effective is the adjoint $G \to \text{Homeo}(X)$ is injective, or equivalently if $\cap_{x\in X} G_x = \{e\}$.[5]
7. A group action is transitive if and only if it has exactly one orbit, i.e., there exists $x \in X$ such that $Gx = X$.

[4] Here, meaning fixed-point free

[5] Note that a free action is effective, but not vice-versa.

1.3.6 Example. To motivate the definition of a principal G-bundle we first consider an example.

1.3.7 Definition. A locally trivial bundle with structure group G is called a principal G-bundle if $F = G$ and the action of the group G on F is defined by left translations.