1
Bundle theory

1.1 Locally trivial bundles

We begin with what we will call a locally trivially bundle.[1]

1.1.1 Definition. A map $\pi\colon E \to B$ is a locally trivial bundle with fiber F if the following conditions hold:

1. Each point $b \in B$ has a neighborhood U such that $\pi^{-1}(U_b) \xrightarrow{h_b} U_b \times F$.

2. The following diagram commutes

$$\begin{tikzcd} \pi^{-1}(U_b) \arrow[rr, "h_b"] \arrow[dr, "\pi"'] & & U_b \times F \arrow[dl, "\mathrm{pr}_1"] \\ & U_b & \end{tikzcd}$$

The maps h_b are called the *local trivializations* of the bundle.

1.1.2 Example. Let $E = B \times F$ and $\pi\colon E = B \times F \to B$ the projection map. This is called the trivial bundle.

1.1.3 Example. If F is discrete, then a locally trivial bundle with fiber F is a covering map.

1.1.4 Example. The Möbius band is a locally trivial bundle with fiber S^1, see Figure 1.1. We will return to this example in due course.

1.1.5 Remark. We can write a locally trivial bundle as $F \to E \to B$, which is reminiscent of the notation for a fibration. In fact, fiber bundles over paracompact base spaces are always fibrations.[2] More generally, any locally trivial bundle is a Serre fibration.

1.1.6 Remark. Let us unwind the definition of a locally trivial bundle a little more. Let $\pi\colon E \to B$ be a locally trivial bundle with fiber F. From the definition we can cover B by a family of open sets $\{U_\alpha\}$ such that each inverse image $\pi^{-1}(U_\alpha)$ is fiberwise homeomorphic to $U_\alpha \times F$. This gives a system of homeomorphisms

$$\phi_\alpha\colon U_\alpha \times F \to \pi^{-1}(U_\alpha).$$

Observe that if $V \subseteq U_\alpha$ then the restriction of ϕ_α to $V \times F$ gives the homeomorphism with $\pi^{-1}(V)$. Hence on $U_\alpha \cap U_\beta$ there are two

[1] Confusingly, some books will also call this a fiber bundle. We will see why later.

[2] A space is paracompact if every open cover has an open refinement that is locally finite.

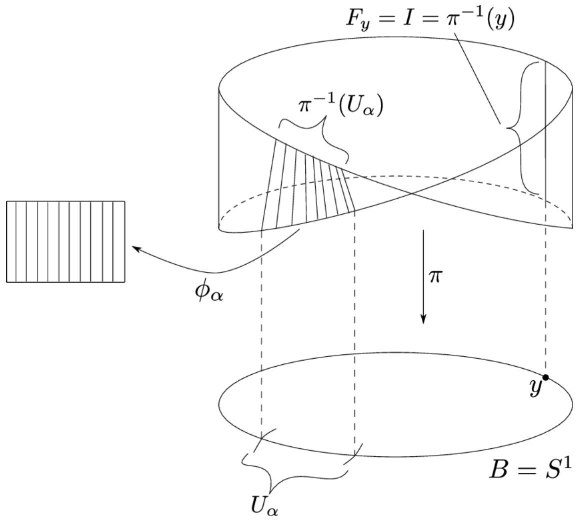

Figure 1.1: The Möbius band

fiberwise homeomorphisms

$$\phi_\alpha \colon (U_\alpha \cap U_\beta) \times F \to \pi^{-1}(U_\alpha \cap U_\beta)$$
$$\phi_\beta \colon (U_\alpha \cap U_\beta) \times F \to \pi^{-1}(U_\alpha \cap U_\beta)$$

Consider the following commutative diagram

$$\begin{tikzcd} (U_\alpha \cap U_\beta) \times F \arrow[r, "\phi_\alpha", "\cong"'] \arrow[dr] & \pi^{-1}(U_\alpha \cap U_\beta) \arrow[r, "\phi_\beta^{-1}", "\cong"'] \arrow[d] & (U_\alpha \cap U_\beta) \times F \arrow[dl] \\ & U_\alpha \cap U_\beta & \end{tikzcd}$$

Let $\phi_{\alpha\beta}$ denote the top composite $\phi_\beta^{-1}\phi_\alpha$. Then the locally trivially bundle is completed determined by the base B, the fiber F, the covering U_α and the homeomorphisms $U_{\alpha\beta}$. Roughly speaking, E should be thought of as the cartesian product of the $U_\alpha \times F$ with some identifications by the $\phi_{\alpha\beta}$.

1.1.7 Definition. The open sets U_α are called *charts*, the family U_α the *atlas of charts*, the homeomorphisms ϕ_α are called the *coordinate homeomorphisms* and the $\phi_{\alpha\beta}$ are called the *transition functions.*

1.1.8 Remark. In order for homeomorphisms $\phi_{\alpha\beta}$ to be the transition functions of a locally trivial bundle, they must satisfy a number of conditions. For example,

$$\phi_{\alpha\alpha} = \mathrm{id}$$

and

$$\phi_{\gamma\alpha}\phi_{\beta\gamma}\psi_{\alpha\beta} = \mathrm{id}$$

on any triple $(U_\alpha \cap U_\beta \cap U_\gamma) \cap F$. Taking $\gamma = \alpha$ we get

$$\phi_{\alpha\beta}\phi_{\beta\alpha} = \mathrm{id}.$$

In fact, these conditions suffices to reconstruct the locally trivial bundle for the base, the fiber, atlas and homeomorphisms. Indeed, set $E = E'/\sim$ where

$$E' = \bigcup_\alpha (U_\alpha \times F)$$

and for $(x, f) \in U_\alpha \times F$ and $(y, g) \in U_\beta \times F$ we have $(x, f) \sim (y, g) \iff x = y \in (U_\alpha \cap U_\beta)$ and $(y, g) = \phi_{\alpha\beta}(x, f)$. It is a rather tedious exercise to show that this determines a locally trivial bundle.

1.1.9 Definition. Two locally trivial bundles $\pi\colon E \to B$ and $\pi'\colon E' \to B'$ are isomorphic if there is a homeomorphism $\psi\colon E \to E'$ such that the diagram

$$\begin{tikzcd} E \arrow[rr, "\psi"] \arrow[dr, "\pi"'] & & E' \arrow[dl, "\pi'"] \\ & B & \end{tikzcd}$$

commutes. (Note that this implies that there is a homeomorphism $F \to F'$ between the fibers as well).

1.1.10 Theorem. *Two systems of transition functors $\phi_{\beta\alpha}$ and $\phi'_{\beta\alpha}$ define isomorphic locally trivial bundles iff there exists fiber preserving homeomorphisms*

$$h_\alpha\colon U_\alpha \times F \to U_\alpha \times F$$

such that $\phi_{\beta\alpha} = h_\beta^{-1}\phi'_{\beta\alpha}h_\alpha$.

Proof. First, we suppose that the two bundles are isomorphic, so in particular there is a homeomorphism $\psi\colon E \to E'$. We let

$$h_\alpha := \phi_\alpha'^{-1}\psi^{-1}\phi_\alpha\colon U_\alpha \times F \to U_\alpha \times F.$$

Then we have

$$\begin{aligned} h_\beta^{-1}\phi'_{\beta\alpha}h_\alpha &= \phi_\beta^{-1}\psi^{-1}\phi'_\beta\phi'_{\beta\alpha}\phi_\alpha^{-1}\psi^{-1}\phi_\alpha \\ &= \phi_\beta^{-1}\psi^{-1}\phi_{\beta'}\phi_\beta'^{-1}\phi'_\alpha\phi_\alpha^{-1}\psi^{-1}\phi_\alpha = \phi_{\beta\alpha}. \end{aligned}$$

Conversely, if the relations hold, then we set $\psi = \phi_\alpha h_\alpha^{-1}\phi_\alpha'^{-1}$. A similar argument then shows that $\phi_\beta h_\beta^{-1}\phi_\beta'^{-1} = \phi_\alpha h_\alpha^{-1}\phi' - 1_\alpha$. ☐

1.1.11 Remark. If π is (isomorphic to) a trivial bundle, then all transition functions can be chosen to be the identity. One can use the previous theorem to show that a bundle is not isomorphic to a trivial bundle.

1.1.12 Example. After this discussion, let us return to the example of the Möbius bundle (Example 1.1.4). One can think of this as the space

$$E = \{(x, y)\colon 0 \le x \le 1, 0 \le y \le 1\}/\sim$$

where we identify $(0, y)$ and $(1, 1 - y)$ for each $y \in [0, 1]$. The projection maps E to $I_x = \{0 \le x \le 1\}$ with the endpoints identified, that is, onto the circle. To see that this is a bundle we use the atlas

$$U_\alpha = \{0 \le x \le 1\}, \text{ and } U_\beta = \{0 \le x < 1/2\} \cup \{1/2 < x \le 1\}.$$

We define

$$\phi_\alpha\colon U_\alpha \times I_y \to E, \quad \phi_\alpha(x, y) = (x, y),$$

and

$$\phi_\beta \colon U_\beta \times I_y \to E$$

by

$$\phi_\beta = \begin{cases} (x, y) & \text{for } 0 \le x \le /1/2, \\ (x, 1-y) & \text{for } 1/2 < x \le 1. \end{cases}$$

The intersection of these two charts is the union $(0, 1/2) \cup (1/2, 1)$, and the transition functions have the form

$$\phi_{\beta\alpha} = (x, y) \text{ for } 0 < x < 1/2$$

and

$$\phi_{\beta\alpha} = (x, 1-y) \text{ for } 1/2 < x < 1.$$

One can check from Remark 1.1.11 that the Möbius bundle is not isomorphic to a trivial bundle.

We now give some more examples which will be useful in our study of characteristic classes.

1.1.13 Definition. For $n < k$ the n-th Stiefel manifold associated to $\mathbb{R}^k$ is defined as

$$V_n(\mathbb{R}^k) = \{n - \text{frames in } \mathbb{R}^k\}$$

where an n-frame in $\mathbb{R}^k$ is a tuple $\{v_1, \dots, v_n\}$ of orthonormal vectors in $\mathbb{R}^k$, i.e., $v_1, \dots, v_n$ are pairwise orthonormal, $\langle v_i, v_j \rangle = \delta_{ij}$. We given $V_n(\mathbb{R}^k)$ the subspace topology induced by thinking of it as a subspace of $S^{k-1} \times \dots S^{k-1}$ (n-copies of S^{k-1}).

1.1.14 Example. A 1-frame is nothing but a unit vector, so the Stiefel manifold $V_1(\mathbb{R}^k)$ is the unit sphere in $\mathbb{R}^k$, i.e., $V_1(\mathbb{R}^k) \cong S^{k-1}$. On the other hand, an n-frame is an ordered basis, so $V_n(\mathbb{R}^n) \cong O(n)$.

1.1.15 Definition. The n-th Grassmannian associated to $\mathbb{R}^k$ is defined as

$$G_n(\mathbb{R}^k) = \{n - \text{dimensional vector subspaces in } \mathbb{R}^k\}$$

There is a map $p \colon V_n(\mathbb{R}^k) \to G_n(\mathbb{R}^k)$ sending $\{v_1, \dots, v_n\}$ to the span, which is surjective by Gram–Schmidt, and we given $G_n(\mathbb{R}^k)$ the quotient topology.

1.1.16 Example. We have $G_1(\mathbb{R}^k)$ is the space of lines through the origin in k-space, so $G_1(\mathbb{R}^k) \simeq \mathbb{R}P^{k-1}$.

1.1.17 Lemma. *For $k > n$ the quotient map $p \colon V_n(\mathbb{R}^k) \to G_n(\mathbb{R}^k)$ is a locally trivial bundle with fiber $V_n(\mathbb{R}^n) \cong O(n)$, i.e., we have a locally trivial bundle*

$$O(n) \to V_n(\mathbb{R}^k) \xrightarrow{p} G_n(\mathbb{R}^k). \tag{1.1.18}$$

Similarly, for $m < n \le k$ there are locally trivial bundles

$$V_{n-m}(\mathbb{R}^k) \to V_n(\mathbb{R}^k) \xrightarrow{p} V_m(\mathbb{R}^k). \tag{1.1.19}$$

where the map p takes $\{v_1, \dots, v_n\}$ to $\{v_1, \dots, v_m\}$. Taking $k = n$ we get a locally trivial bundle

$$O(n-m) \to O(n) \xrightarrow{p} V_m(\mathbb{R}^n). \tag{1.1.20}$$

1.1.21 Example. Taking $m = 1$ in (1.1.20) we get a locally trivial bundle

$$O(n-1) \to O(n) \to S^{n-1}.$$

Here the first map takes A to $\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$ and the second takes B to Bu for $u \in S^{n-1}$ some unit vector. In particular, this identifies S^{n-1} as an orbit space $S^{n-1} \cong O(n)/O(n-1)$.

Exercise 1

Use the fibrations

$$O(n-k) \to O(n) \to V_k(\mathbb{R}^n)$$

to show that

$$\pi_i(O(n-1)) \simeq \pi_i(O(n)) \text{ for } i < n-2$$

and

$$\pi_i(V_k(\mathbb{R}^n)) = 0$$

for $i < n-k-1$.

1.1.22 Definition. We have infinite versions of the Stiefel manifold and Grassmanian:

$$V_n(\mathbb{R}^\infty) := \bigcup_{k=1}^{\infty} V_n(\mathbb{R}^k) \qquad G_n(\mathbb{R}^\infty) := \bigcup_{k=1}^{\infty} G_n(\mathbb{R}^k)$$

1.1.23 Remark. We get a fiber sequence

$$O(n) \to V_n(\mathbb{R}^\infty) \to G_n(\mathbb{R}^\infty).$$

1.1.24 Proposition. *$V_n(\mathbb{R}^\infty)$ is contractible.*

Proof. As in the exercise, we deduce that $\pi_i(V_n(\mathbb{R}^\infty) = 0$ for all i. We can give $V_n(\mathbb{R}^\infty)$ the structure of a CW-complex, and so the claim follows from **??**. □

1.1.25 Remark. One can repeat the same story using $\mathbb{C}$ or $\mathbb{H}$ instead of $\mathbb{R}$. In the first case, all instances of $O(n)$ get replaced by $U(n)$, and in the second case by $\mathrm{Sp}(n)$.

1.1.26 Example (The tangent bundle to S^2). Let $S^2 = \{(x_0, x_1, x_2) \in \mathbb{R}^3 \mid x_0^2 + x_1^2 + x_2^2 = 1\}$. Recall that the tangent space at a point $x \in S^2$ is defined by $T_xS^2 = \{\xi \in \mathbb{R}^3 \mid x \perp \xi\}$. We then define $TS^2 = \coprod_{x \in S^2} T_xS^2$. This can be topologized as a subspace of $\mathbb{R}^3 \times \mathbb{R}^3$ when we write

$$TS^2 = \{(x, \xi) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid x \in S^2, x \perp \xi\}.$$

There is a natural projection map $p\colon TS^2 \to S^2$ sending the pair (x, ξ) to x, which we claim is a locally trivial bundle with fiber $\mathbb{R}^2$. To see this is a locally trivial bundle, let U be the open subset of S^2 defined by $x_3 > 0$. We will show how to construct the local trivialization on this open subset.

If $\xi = (\xi_1, \xi_2, \xi_3)$ then we have the relation

$$x_1\xi_1 + x_2\xi_2 + x_3\xi_3 = 0$$

or

$$\xi_3 = -(x_1\xi_1 + x_2\xi_2)/x_3.$$

We define

$$\phi\colon U \times \mathbb{R}^2 \to p^{-1}(U)$$

by

$$\phi(x_1, x_2, x_3, \xi_1, \xi_2) = (x_1, x_2, x_3, \xi_1, \xi_2, -(x_1\xi_1 + x_2\xi_2)/x_3),$$

which gives the required chart for this open subset.

1.1.27 Remark. More generally, for any smooth manifold X of dimension n, we have a locally trivial bundle $\pi\colon TX \to X$ with fiber $\mathbb{R}^n$.

1.2 The structure group of locally trivial bundles

We recall that the on the intersection of two local trivializations we constructed a homeomorphism

$$\phi_{\beta\alpha}\colon (U_\alpha \cap U_\beta) \times F \to \pi^{-1}(U_\alpha \cap U_\beta) \to (U_\alpha \cap U_\beta) \times F$$

Unwinding the definition, the map ϕ is completely determine by a map $\Phi\colon U \to \mathrm{Homeo}(F)$, where $\mathrm{Homeo}(F)$ denotes the group of all homeomorphisms of the fiber F (we call Φ coordinate transformations).[3]. Indeed, we have

$$\phi_{\alpha\beta}(x, f) = (x, \Phi(x)(f))$$

[3] If we choose the correct topology on $\mathrm{Homeo}(F)$, namely the compact-open topology (for reasonable spaces at least), then this map is even continuous

In other words, instead of $\phi_{\alpha\beta}$ to determine a bundle we can instead specify a family of functions

$$\Phi_{\alpha\beta}(x, f)\colon U_\alpha \cap U_\beta \to \mathrm{Homeo}(F),$$

having values in the group $\mathrm{Homeo}(F)$. Of course these are not arbitrary, but need to satisfy various compatibility conditions:

$$\Phi_{\alpha\alpha}(x) = \mathrm{id}$$

and

$$\Phi_{\alpha\gamma}(x)\Phi_{\gamma\beta}(x)\Phi_{\beta\alpha}(x) = \mathrm{id},$$

for $x \in U_\alpha \cap U_\beta \cap U_\gamma$.

1.2.1 Definition. Let E, B, F be topological spaces and G a topological group which acts freely on the space F. A continuous map $p\colon E \to B$ is a locally trivial bundle with fiber F and structure group G if there is an atlas $\{U_\alpha\}$ and the coordinate homeomorphisms

$$\phi_\alpha\colon U_\alpha \times F \to p^{-1}(U_\alpha)$$

such that the transition functions

$$\phi_{\beta\alpha} = \phi_\beta^{-1}\phi_\alpha \colon (U_\alpha \cap U_\beta) \times F \to (U_\alpha \cap U_\beta) \times F$$

have the form

$$\phi_{\beta\alpha}(x, f) = (x, \Phi_{\beta\alpha}(x)f)$$

where $\Phi_{\beta\alpha} \colon (U_\alpha \cap U_\beta) \to G$ are continuous functions satisfying

$$\Phi_{\alpha\alpha}(x) = \mathrm{id}$$

and

$$\Phi_{\alpha\gamma}(x)\Phi_{\gamma\beta}(x)\Phi_{\beta\alpha}(x) = \mathrm{id},$$

1.2.2 Remark. Some words on terminology are useful. What we defined as a locally trivial bundle with fiber F, is exactly a locally trivial bundle with fiber F and structure group $\mathrm{Homeo}(F)$. Either of these may also be called a *fiber bundle* or a *fiber bundle with structure group G.*

1.2.3 Remark. In diagrammatic form, to have structure group G means that we can find transition maps $\Phi'_{\alpha\beta}$ making the diagram commute:

$$\begin{tikzcd}
G \arrow[rr] & & \mathrm{Homeo}(F) \\
& U_\alpha \cap U_\beta \arrow[ul, dashed, "\Phi'_{\alpha\beta}"] \arrow[ur, "\Phi_{\alpha\beta}"'] &
\end{tikzcd}$$

1.2.4 Remark. Note that the structure group is not unique. For example, a bundle with structure group G may admit transition functions with values in a subgroup $H \leq G$. We say that the structure group G is reduced to subgroup H. More generally, if $\rho \colon G \to G'$ is a continuous homeomorphism of topological groups, and we are given a locally trivial bundle with structure group G and the transition functors $\alpha_{\alpha\beta} \colon U_\alpha \cap U_\beta \to G$, then a new locally trivial bundle with structure group G' may be constructed by

$$\phi'_{\alpha\beta}(x) = \rho(\phi_{\alpha\beta}(x)).$$

This operation is called the change of structure group.

Exercise 2

Show that a trivial bundle has trivial structure group. Conversely, if the structure group can be reduced to the trivial group then the bundle is (isomorphic to) a trivial bundle.

1.2.5 Example. Let us return to the Möbius bundle (Examples 1.1.4 and 1.1.12). We have that

$$\Phi_{\alpha\beta}(y) = y \quad \text{and } \Phi_{\beta\alpha}(y) = 1 - y.$$

Note that $\Phi_{\beta\alpha} \circ \Phi_{\beta\alpha}(y) = 1 - (1 - y) = y = \Phi_{\alpha\beta}(y)$. Therefore, the group generated by $\Phi_{\alpha\beta}$ and $\Phi_{\beta\alpha}$ has order 2. In other words, the Möbius bundle has (or can be reduced to) structure group $\mathbb{Z}/2$.

1.2.6 Example. The tangent bundle $TS^2 \to S^2$ was considered in Example 1.1.26. The coordinate homeomorphisms

$$\phi\colon U \times \mathbb{R}^2 \to \mathbb{R}^3 \times \mathbb{R}^3$$

are defined by formulas that are linear with respect to the second argument. Hence that transition functions have values in the group of linear translations of the fiber $F = \mathbb{R}^2$, that is $G = GL_2(\mathbb{R})$. In fact, it can be shown that the structure group can be reduced to the subgroup $O(n)$ of orthonormal rotations.

1.3 Principal bundles

The most important example of a bundle for a us is a *principal bundle*. We postpone the definition to talk a little about group actions.

1.3.1 Definition. Let G be a topological group, and X a topological space. A left action of G on X is a continuous map $\mu\colon G \times X \to X$, satisfying $\mu(e, x) = x$ and $\mu(h, \mu(g, x)) = \mu(hg, x)$.

1.3.2 Example. The multiplication map $\mu\colon G \times G \to G$ defines a left action of G on itself. Similarly, if $H \subseteq G$, then $\mu|_{H \times G}\colon H \times G \to G$ defines an action of H on G.

1.3.3 Remark. The map $\mu\colon G \times X \to X$ is adjoint to a map $ad(u)\colon G \to \mathrm{Homeo}(X)$, where $\mathrm{Homeo}(X)$ has the compact-open topology. If X is nice (specifically, locally compact Hausdorff) then a continuous map $G \to \mathrm{Homeo}(X)$ gives rise to a group action $G \times X \to X$.

1.3.4 Example. The adjoint to the multiplication $\mu\colon G \times G \to G$ is the map $G \to \mathrm{Homeo}(G)$ given by $g \mapsto f_g$, where $f_g\colon G \to G$ is given by $f_g(x) = gx$.

1.3.5 Remark. We recall some standard terminology associated to a group action of G on X:

(i) The orbit of a point $x \in X$ is the set $Gx = \{g \cdot xmidg \in G\}$.

(ii) The orbit space X/G is the quotient space $X/\sim$ where $x \sim g \cdot x$.

(iii) The fixed set $X^G := \{x \in X | g \cdot x \text{ for all } g \in G\}$.

(iv) An action if free[4] if $g \cdot x \neq x$ for all $x \in X$ and all $g \neq e$. (i.e., $gx = x$ for all x implies $g = e$).

(v) The stabilizer (or isotropy) group at x is $G_x = \{g \in G \mid g \cdot x = g\} \subseteq G$.

(vi) An action is effective is the adjoint $G \to \mathrm{Homeo}(X)$ is injective, or equivalently if $\cap_{x \in X} G_x = \{e\}$.[5]

(vii) A group action is transitive if and only if it has exactly one orbit, i.e., there exists $x \in X$ such that $Gx = X$.

[4] Here, meaning fixed-point free

[5] Note that a free action is effective, but not vice-versa.

1.3.6 Example. To motivate the definition of a principal G-bundle we first consider an example. We recall that Hopf fibration $S^1 \to$

$S^3 \to S^2$. By what we have already discussed, this is a locally trivial bundle.

Let us show this directly. Consider the sphere S^3 as a subset of $\mathbb{C}^2$ defined by the equation

$$|z_0|^2 + |z_1|^2 = 1.$$

Then the map $S^3 \to S^2 = \mathbb{C}P^1$ sends (z_0, z_1) to the projective coordinate $[z_0 : z_1]$. We take the atlas consisting of the charts

$$\begin{aligned} U_0 &= \{[z_0, z_1] : z_0 \neq 0\} \\ U_1 &= \{[z_0, z_1] : z_1 \neq 0\} \end{aligned}$$

The points of the chart U_0 are parametrized by the complex parameter $w_0 = z_1/z_0 \in \mathbb{C}$, while points of the chart U_1 are parameterized by $w_1 = z_0/z_1 \in \mathbb{C}$. The homeomorphisms

$$\phi_0 \colon p^{-1}(U_0) \to S^1 \times \mathbb{C} = S^1 \times U_0 \text{ and } \phi_0 \colon p^{-1}(U_1) \to S^1 \times \mathbb{C} = S^1 \times U_1$$

are given by

$$\begin{aligned} \phi_0(z_0, z_1) &= \left(\frac{z_0}{|z_0|}, \frac{z_1}{z_0} \right) = \left(\frac{z_0}{|z_0|}, [z_0 : z_1] \right) \\ \phi_0(z_0, z_1) &= \left(\frac{z_1}{|z_1|}, \frac{z_0}{z_1} \right) = \left(\frac{z_1}{|z_1|}, [z_0 : z_1] \right) \end{aligned}$$

One can explicitly write down inverses for these maps and check that they are homeomorphisms. The transition function

$$\phi_{01} \colon (U_0 \cap U_1) \times S^1 \to (U_0 \cap U_1) \times S^1$$

is defined by the formula

$$\phi_{01}([z_0 : z_1], \lambda) = \left([z_0 : z_1], \frac{z_1 |z_0|}{z_0 |z_1|} \right).$$

In other words, $\Phi_{01} \colon U_0 \cap U_1 \to \mathrm{Homeo}(S^1)$ is defined by

$$[z_0 : z_1] \mapsto (\lambda \mapsto \frac{z_1 |z_0|}{z_0 |z_1|} \lambda),$$

where $\frac{z_1|z_0|}{z_0|z_1|} \in S^1$. We can then define a map $\Phi'_{01} \colon U_0 \cap U_1 \to S^1$ by $[z_0, z_1] \mapsto \frac{z_1|z_0|}{z_0|z_1|}$. Unwinding the definitions, the diagram

$$\begin{tikzcd} S^1 \arrow[rr, "ad(\mu)"] & & \mathrm{Homeo}(S^1) \\ & U_0 \cap U_1 \arrow[ul, "\Phi'_{01}"] \arrow[ur, "\Phi_{01}"'] & \end{tikzcd}$$

commutes, where $ad(\mu)$ is adjoint to the multiplication map $\mu \colon S^1 \times S^1 \to S^1$.[6]

[6] Here we use Example 1.3.4.

In other words, the structure group of the Hopf bundle can be reduced to S^1, where the action of S^1 on the fiber, namely S^1 again, is given by the multiplication map $\mu \colon S^1 \times S^1 \to S^1$.

This leads us to the following definition.

1.3.7 Definition. A locally trivial bundle with structure group G is called a principal G-bundle if $F = G$ and the action of the group G on F is defined by left translations, i,e, the multiplication map $\mu\colon G \times G \to G$ as in Example 1.3.4.

1.3.8 Example. Restated, Example 1.3.6 shows that $S^1 \to S^3 \to S^2$ is a principal S^1-bundle.

1.3.9 Theorem. *Let $p\colon E \to B$ be a principal G-bundle, with*

$$\phi_\alpha\colon U_\alpha \times G \to p^{-1}(U_\alpha)$$

coordinate homeomorphisms. Then there is a right action[7] *of the group G on the total space E such that*

(i) The right action is fiberwise,[8] i.e., $p(x) = p(xg)$ for $x \in E, g \in G$.

(ii) The homeomorphism ϕ_α^{-1} transformations the right action of the group G on the total space into right translations on the second factor, i.e.,

$$\phi_\alpha(x, f)g = \phi_\alpha(x, fg), \quad x \in U_\alpha, f, g \in G \tag{1.3.10}$$

(iii) G acts freely and transitively on the right of E.

Proof. Unwinding the definitions, the transition functions $\phi_{\beta\alpha}$ are given by

$$\phi_{\beta\alpha}(x, f) = (x, \Phi_{\beta\alpha}(x)f),$$

for some

$$\Phi\beta\alpha\colon U_\alpha \cap U_\beta \to G.$$

Note that any arbitrary $e \in E$ is of the form $e = \phi_\alpha(x, f)$ for some α, then we define a right action of G on E via the formula (1.3.10). We need to check that this is well defined, i.e., does not depend on the choice α. So assume that $e = \phi_\beta(x, f')$ as well, then we are required to show that

$$\phi_\alpha(x, fg) = \phi_\beta(x, f'g).$$

This says that

$$(x, f'g) = \phi_\beta^{-1}\phi_\alpha(x, fg) = \phi_{\beta\alpha}(x, fg) = (x, \Phi_{\beta\alpha}(x)fg).$$

So equivalently, we are required to show

$$f'g = \Phi_{\beta\alpha}(x)fg.$$

But this follows because $f' = \phi_{\beta\alpha}(x)f$. □

We omit the proof of the following difficult theorem, which gives many more examples of principal G-bundles.

1.3.11 Theorem. *Suppose X is a compact Hausdroff space and G is a compact Lie group acting freely on X. Then the orbit map $X \to X/G$ is a principal G bundle.*

[7] Why on the right? This is just because the action of G on E is given by *left* translation. If we made the opposite convention, then E would get a left action of G.

[8] In other words, it preserves fibers.

1.3.12 Example. Let $H \subseteq G$ be a closed subgroup. Then H acts freely on G, and $G \to G/H$ is a principal H-bundle. Moreover, if $K \subseteq H$ is a subgroup, then H acts on H/K, and $G/K \to G/H$ is a principal H/K-bundle.

1.3.13 Example. Let $G = O(n)$, and consider the subgroup $H = O(k) \times O(n-k)$ defined by $(A, B) \mapsto \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ We also consider the subgroup the subgroup $K = O(n-k) \subseteq O(n)$, determine by the inclusion $A \mapsto \begin{pmatrix} I & 0 \\ 0 & A \end{pmatrix}$. We then have the principal $O(k) \times O(n-k)$-bundle

$$O(k) \times O(n-k) \to O(n) \to G_k(\mathbb{R}^n)$$

and the principal $O(n-k)$-bundle

$$O(n-k) \to O(n) \to V_k(\mathbb{R}^n)$$

as well as the principal $O(k)$-bundle

$$O(k) \to V_k(\mathbb{R}^n) \to G_k(\mathbb{R}^n).$$

1.4 *The associated bundle construction*

1.4.1 Remark. Let us recall that a locally trivial bundle $p \colon E \to B$ with structure group G and fiber F is determined by the following data:

1. An open covering $\{U_\alpha\}$ of B.
2. Maps $\phi_{\alpha\beta} \colon U_\alpha \cap U_\beta \to G$, and
3. An action of G on F, or equivalently a map $G \to \mathrm{Homeo}(F)$.

Note that if p is a a principal G-bundle, then the last data in the above list is already included: the action of G on G is given by left multiplication. Thus a principal G-bundle $p \colon P \to B$ is determined by the following data:

1. An open covering $\{U_\alpha\}$ of B.
2. Maps $\phi_{\alpha\beta} \colon U_\alpha \cap U_\beta \to G$, and

Therefore, by removing the third data from a fiber bundle, we should get a principal G-bundle, and conversely given a principal G-bundle by adding the data of an action of a space F on G, we should be able to construct a locally trivial bundle with fiber F on the same base space. This is essentially true; we have the following theorem:

1.4.2 Theorem. *Let G be a topological group and fix an action of G on a space F. Suppose that an open cover $\{U_\alpha\}$ of B and families of maps $\Phi = \{\Phi_{\alpha\beta} \colon U_\alpha \cap U_\beta \to G\}$ are given. Suppose that that action of G on F is effective, then there is a bijection*

$$\left\{ \begin{array}{c} \textit{principal } G\textit{-bundles over } B \\ \textit{with coordinate transformations } \Phi \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \textit{Locally trivial bundles over } B \\ \textit{with fiber } F\textit{, structure group } G, \\ \textit{and coordinate transformations } \Phi \end{array} \right\}$$

It is not too hard to define a map in the $\to$ direction.

1.4.3 Construction . Let G be a topological group and assume that G acts on topological spaces P and F continuously from the right and left respectively. Define a left action of G on $P \times F$ by

$$g(p, f) = (x \cdot g^{-1}, gy).$$

Let $E \times_G F := E \times F/G$ denote the quotient (orbit) space, and $\omega \colon E \times_G F \to E/G$ the projection map.

1.4.4 Definition. Let $p \colon E \to B$ be a principal G-bundle and fixed a G-space F. The projection map $\omega \colon E \times_G F \to B$ sending $[x, y] \mapsto p(x)$ is called the associated bundle with fiber F.

We should of course verify that this is actually a locally trivial bundle.

1.4.5 Theorem. *The map $\omega \colon E \times_G F \to B$ defines a locally trivial bundle with structure group G and fiber F.*

Proof. Let $\{\Phi_\alpha \colon p^{-1}(U_\alpha) \to U_\alpha \times G\}$ be local trivializations. We wish to construct local trivializations

$$\Psi_\alpha \colon \omega^{-1}(U_\alpha) \to U_\alpha \times F.$$

Note that

$$\begin{aligned} \omega^{-1}(U_\alpha) &= \{[x, y] \in E \times_G F \mid \omega([x, y]) \in U_\alpha\} \\ &= \{[x, y] \in E \times_G F \mid p(x) \in U_\alpha\} \\ &= p^{-1}(U_\alpha) \times_G F \end{aligned}$$

We then define the required map Ψ_α as the composite

$$\omega^{-1}(U_\alpha) = p^{-1}(U_\alpha) \times_G F \xrightarrow{\Phi_\alpha \times_G 1_F} (U_\alpha \times G) \times_G F \cong U_\alpha \times F$$

where the homeomorphism $(U_\alpha \times G) \times_G F \cong U_\alpha \times F$ is given by $[(x, g), y] \mapsto (x, gy)$.[9] As a composite of homeomorphisms, this map is also a homeomorphism. A short diagram chase shows that it defines a locally trivial bundle. In order to see that the structure group is G, we must compute $\Psi_\beta \Psi_\alpha^{-1}(x, y)$ for $(x, y) \in U_\alpha \cap U_\beta \times F$. A rather tedious computation shows that

$$\Psi_\beta \Psi_\alpha^{-1}(x, y) = (x, \Phi_{\alpha\beta}(x) y)$$

where $\Phi_{\alpha\beta} \colon U_\alpha \cap U_\beta \to G$ is a coordinate transformation for $p \colon E \to B$. Therefore, the associated bundle has structure group G, as claimed. $\square$

[9] With inverse, $(x, y) \mapsto [(x, e), y]$.

1.4.6 Example. Let $\pi \colon S^1 \to S^1, z \mapsto z^2$ be regarded as a principal $\mathbb{Z}/2$-bundle.[10] Let $F = [-1, 1]$, and let $\mathbb{Z}/2 = \{-1, 1\}$ act on F by multiplication. The associated bundle is then

$$S^1 \times_{\mathbb{Z}/2} [-1, 1] = S^1 \times [-1, 1]/(x, t) \sim (a(x), -t)$$

for $a \colon S^1 \to S^1$ the antipodal map. This is the Möbius bundle.

[10] Note that any regular cover is a principal bundle.

1.4.7 Remark. You might be wondering about the inverse map in Theorem 1.4.2. Let $p\colon E \to B$ be a principal bundle with fiber F and structure group G. The associated principal bundle is constructed using a similar procedure as in Remark 1.1.8; we keep the base space B, the open chart $\{U_\alpha\}$ and the transition functions $\phi_{\alpha\beta}$, but we replace all instances of the fiber F by G, and allow G to act on itself by left translation. In simple terms: we forget the fiber F, and built a bundle which is principal out of the remaining data.

1.5 *Operations on locally trivial bundles*

The following is a fundamental operation on bundles.

1.5.1 Proposition. *Let $\xi = (p\colon E \to B)$ and $\xi' = (p'\colon E' \to B')$ be locally trivial bundles with fibers F and F' respectively. Then the product*

$$p \times p'\colon E \times E' \to B \times B'$$

is a locally trivial bundle bundle with fiber $F \times F'$ and structure group $G \times G'$.

Proof. Let $\phi_\alpha\colon p^{-1}(U_\alpha) \to U_\alpha \times F$ and $\psi_\beta\colon p'^{-1}(V_\beta) \to V_\beta \times F'$ be local trivializations of ξ and ξ' respectively. Note that $\{U_\alpha \times V_\beta\}$ is an open covering of $B \times B'$. Note also that $(U_\alpha \times V_\beta) \cap (U_{\alpha'} \times V_{\beta'}) = (U_\alpha \cap U_{\alpha'}) \times (V_\beta \cap V_{\beta'})$, and using this identification we obtain maps

$$\Phi_{\alpha\alpha'} \times \Psi_{\beta\beta'}\colon (U_\alpha \cap U_{\alpha'}) \times (V_\beta \cap V_{\beta'}) \to G \times G'$$

which form coordinate transforms of the product bundle. $\square$

1.5.2 Corollary. *Let $p\colon E \to B$ be a locally trivial bundle with fiber F and structure group G, then for any topological space X*

$$p \times 1_X\colon E \times X \to B \times X$$

is a locally trivial bundle with fiber F and structure group G.

1.5.3 Remark. We recall the pullback construction for topological spaces: given $f\colon A \to C$ and $g\colon B \to C$, the pullback is

$$A \times_C B := \{(a,b) \in A \times B \mid f(a) = g(b)\}$$

along with the projection maps $A \times_C B \to A$ and $A \times_C B \to B$. This is a pullback in the categorical sense: given maps $q_1\colon Q \to A$ and $q_2\colon Q \to B$ as in the following diagram:

$$\begin{tikzcd}
Q \arrow[dr, dashed, "u"] \arrow[drr, bend left, "q_1"] \arrow[ddr, bend right, "q_2"'] & & \\
& A \times_C B \arrow[r] \arrow[d] & A \arrow[d, "f"] \\
& B \arrow[r, "g"'] & C
\end{tikzcd}$$

there exists a map $u\colon Q \to A \times_C B$ making the diagram commute.

1.5.4 Proposition. *Let $\xi = (p\colon E \to B)$ be a locally trivial bundle with fiber F and structure group G. For any continuous map $f\colon X \to B$, the pullback $f^*(p)\colon E \times_B X \to X$ is a locally trivial bundle with fiber F and structure group G.*[11]

[11] For brevity, we write $f^*(E) := E \times_B X$.

Proof. Let $\{U_\alpha\}_{\alpha \in A}$ be an open cover of B, then $\{f^{-1}(U_\alpha)\}_{\alpha \in A}$ forms an open cover of X.

Let $\phi_\alpha\colon p^{-1}(U_\alpha) \to U_\alpha \times F$ denote a local trivialization of ξ. We write $\phi_\alpha(e) = (p(e), \overline{\phi}_\alpha(e))$. We define a map $\psi_\alpha\colon f^*(p)^{-1}(V_\alpha) \to V_\alpha \times F$ as the composite

$$f^*(p)^{-1}(V_\alpha) \hookrightarrow V_\alpha \times p^{-1}(U_\alpha) \xrightarrow{1 \times \overline{\phi}_\alpha(e)} V_\alpha \times F.$$

We will show this is a homeomorphism by constructing a continuous inverse. Define γ_α as the composite

$$V_\alpha \times F \xrightarrow{\Delta \times 1} V_\alpha \times V_\alpha \times F \xrightarrow{1 \times f \times 1} V_\alpha \times U_\alpha \times F \xrightarrow{1 \times \phi_\alpha^{-1}} V_\alpha \times p^{-1}(U_\alpha),$$

i.e., $\gamma_\alpha(x, y) = (x, \phi_\alpha^{-1}(f(x), y))$.

Note that $p(\phi_\alpha^{-1}(f(x), y)) = f(x)$, and so $\gamma_\alpha(x, y) \in f^*(E)$. To see that γ_α and ψ_α are inverse, simply compute:

$$\begin{aligned}
\gamma_\alpha \circ \psi_\alpha(x, e) &= \gamma_\alpha(x, \overline{\phi}_\alpha(e)) \\
&= (x, \phi_\alpha^{-1}(f(x), \overline{\phi}_\alpha(e))) \\
&= (x, \phi_\alpha^{-1}(p(e), \overline{\phi}_\alpha(e))) \\
&= (x, \phi_\alpha^{-1} \circ \phi_\alpha(e)) \\
&= (x, e)
\end{aligned}$$

and

$$\begin{aligned}
\psi_\alpha \circ \gamma_\alpha(x, y) \circ &= \psi_\alpha(x, \psi_\alpha^{-1}(f(x), y)) \\
&= (x, \overline{\phi}_\alpha \circ \phi_\alpha^{-1}(f(x), y)) \\
&= (x, y).
\end{aligned}$$

We leave it to the reader to verify the $f^*(E) \to X$ has structure group G.[12] □

[12] Hint: you should end up showing that the coordinate transformations of $f^*(E) \to X$ are given by $\{\Phi_{\alpha\beta} \circ f\}$.

1.5.5 Remark. Suitably interpreted, this pullback is actually a categorical pullback in a category of locally trivial bundles.

1.6 Vector bundles

A vector bundle is a special case of a locally trivial bundle.

1.6.1 Definition. A locally trivial bundle is called a real (respectively, complex) vector bundle of rank n if its fiber is a vector space V of dimension n over $\mathbb{R}$ (respectively, $\mathbb{C}$) and the structure group is $GL(V)$.

1.6.2 Example. For a manifold of dimension n, the tangent bundle $TM \to M$ is a real vector bundle of rank n.

1.6.3 Remark. If the following, when we write K we mean either $K = \mathbb{R}$ if the vector bundle is real, and $K = \mathbb{C}$ if the vector bundle is complex.

1.6.4 Definition. Let $\xi = (p\colon E \to B)$ and $\xi' = (p'\colon E' \to B')$ be vector bundles of rank m and n respectively. A map of vector bundles from $\xi \to \xi'$ is a fiber preserving map $\mathbf{f} = (\tilde{f}, f)\colon (E, B) \to (E', B')$ satisfying the following condition: for local trivializations $\phi_\alpha\colon p^{-1}(U_\alpha) \to U_\alpha \times K^m$ and $\psi_\beta\colon p'^{-1}(V_\beta) \to V_\beta \times K^n$ of ξ and ξ' respectively satisfying $U_\alpha \cap f^{-1}(V_\beta) \neq \varnothing$, the map

$$L^f_{\alpha,\beta}\colon U_\alpha \cap f^{-1}(V_\beta) \to \mathrm{Map}(K^m, K^n)$$

which is adjoint to

$$U_\alpha \cap f^{-1}(V_\beta) \times F \xrightarrow{\phi_\alpha^{-1}} p^{-1}(U_\alpha \cap f^{-1}(V_\beta)) \xrightarrow{\tilde{f}} p'^{-1}(U_\alpha \cap f^{-1}(V_\beta)) \xrightarrow{\Psi_\beta} (f(U_\alpha) \cap V_\beta) \times F' \xrightarrow{\mathrm{pr}_2} F'$$

takes values in the set of *linear* maps $\mathrm{Hom}_K(K^m, K^n)$

1.6.5 Remark. Note that $\mathrm{Hom}_K(K^m, K^n)$ inherits the structure of a K-vector space, by sum and scalar multiplication of linear maps.

1.6.6 Remark. Given two vector bundles $p\colon E \to B$ and $p'\colon E' \to B'$ of rank m and n respectively, the product $p \times p'\colon E \times E' \to B \times B'$ is a vector bundle of rank $m + n$ (Proposition 1.5.1). Let us denote this bundle by $\xi \times \xi'$. This has structure group $GL_m(K) \times GL_n(K)$. Note that we can regard $GL_m(K) \times GL_n(K)$ as a subgroup of $GL_{m+n}(K)$ via the map

$$(A, B) \mapsto \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}.$$

We use this to define a direct sum for vector bundles.

1.6.7 Definition. Let $\xi = (p\colon E \to B)$ and $\xi' = (p'\colon E' \to B)$ be vector bundles of rank m and n over a fixed space B. The pullback of $\xi \times \xi'$ along the diagonal map $\Delta\colon B \to B \times B$ is denoted $\xi \oplus \xi'$ and is called the direct sum (or Whitney sum) of ξ and ξ'. The total space of this bundle is $E \times_B E'$. This is a vector bundle of rank $m + n$.

1.6.8 Definition. An inner product on a vector bundle ξ is a homomorphism of vector bundles $g\colon \xi \oplus \xi \to K \times B$[13] such that the map on each fiber gives rise to an inner product $K^n \times K^n \to K$ when translated by local trivializations.

[13] here $K \times B$ denotes the trivial K-vector bundle over B.

1.6.9 Remark. Any vector bundle over a paracompact Hausdorff base space has an inner product. This uses a partition of unity argument to glue inner products over local trivializations. By using the Gram–Schmidt process, one can show that any vector bundle of rank n over a paracompact Hausdorff space has structure group that can be reduced to $O(n)$ in the real case, or $U(n)$ in the complex case.

There are a number of other natural constructions for vector bundles. We outline some now.

1.6.10 Definition (Tensor product bundle). We have a external tensor product: given two vector bundles $\xi = (p\colon E \to B)$ and

$\xi' = (p' \colon E' \to B')$ of rank m and n, we can construct a bundle $\xi \widehat{\otimes} \xi'$ with coordinate transformations

$$\Phi_{\alpha\alpha'} \otimes \Psi_{\beta\beta'} \colon (U_\alpha \times V_\beta) \cap (U_{\alpha'} \times V_{\beta'}) = (U_\alpha \cap U'_\alpha) \times (V_\beta \cap V_{\beta'}) \to GL_{mn}(K)$$

as the composite

$$(U_\alpha \cap U'_\alpha) \times (V_\beta \cap V_{\beta'}) \xrightarrow{\Phi_{\alpha\alpha'} \times \Psi_{\beta\beta'}} GL_m(K) \times GL_n(K) \xrightarrow{\otimes} GL_{mn}(K)$$

where the last map is given by

$$(A, B) \mapsto (\mathbb{R}^{mn} \cong \mathbb{R}^m \otimes \mathbb{R}^n \xrightarrow{A \otimes B} \mathbb{R}^m \otimes \mathbb{R}^n \cong \mathbb{R}^{mn}).$$

Note that this is a vector bundle over $B \times B'$. In the case that $B = B'$, the pullback along the diagonal map gives a bundle $\xi \otimes \xi'$ over B called the tensor product bundle.

1.6.11 Definition (Hom bundle). Let $\xi = (p \colon E \to B)$ and $\xi' = (p' \colon E' \to B)$ be two vector bundles over the same base space of rank m and n respectively. We can assume that local trivializations are defined over the same cover by taking a subdivision if necessary. Let

$$\Phi_{\alpha\alpha'} \colon U_\alpha \cap U_{\alpha'} \to GL_m(K)$$

and

$$\Psi_{\alpha\alpha'} \colon U_\alpha \cap U_{\alpha'} \to GL_n(K)$$

be coordinate transformations of ξ and ξ'. We define a new bundle with coordinate transformations the composite

$${}^t\Phi_{\alpha\alpha'} \otimes \Psi_{\alpha\alpha'} : U_\alpha \cap U_{\alpha'} \xrightarrow{\Phi_{\alpha\alpha'} \times \Psi_{\alpha\alpha'}} GL_m(K) \times GL_n(K) \xrightarrow{{}^t(-) \otimes 1} GLm(K) \times GL_n(K) \xrightarrow{\otimes} GL_{mn}K$$

where ${}^t(-)$ denotes the transpose.[14] We define a bundle $\operatorname{Hom}(E, E')$ by gluing $U_\alpha \times \operatorname{Hom}_K(K^m, K^n)$ using these coordinate transformations.

[14] To justify the use of the transpose we note the following: Let $f \colon V \to W$ be a linear map between finite dimensional vector spaces with bases $\mathcal{B}_V$ and $\mathcal{B}_W$ with a matrix A representing f with respect to these bases. Then the map $f^* : W^* \to V^*$ has matrix ${}^tA$ with respect to the dual bases $\mathcal{B}_V^*$ and $\mathcal{B}_W^*$.

1.6.12 Example. Let $\underline{K}_B$ denote the trivial K-bundle over a base B. Then $E^* := \operatorname{Hom}(E, \underline{K}_B)$ is called the dual vector bundle, and has fibers dual to those of E. We note that if $K = \mathbb{R}$ and we work over a paracompact Hausdorff base space then a finite rank vector bundle and its dual are isomorphic as vector bundles (but not canonically).

1.7 *Morphism of bundles*

Let us now formalize the notion of a morphism of locally trivial bundles. There are various possibilities depending on how much structure we wish to preserve.

1.7.1 Definition (Bundle homomorphism). Let $\xi = (p \colon E \to B)$ and $\xi' = (p' \colon E' \to B')$ be two locally trivial bundles with fiber F and structure group G and G action given by a common map $\mu_G \colon G \times F \to F$. A morphism $(\tilde{f}, f) \colon (E, B) \to (E', B')$ is a bundle map if:

1. The following diagram commutes:

$$\begin{tikzcd} E' \arrow[r, "\tilde{f}"] \arrow[d, "p'"'] & E \arrow[d, "p"] \\ B' \arrow[r, "f"'] & B \end{tikzcd}$$

2. For $x \in B$, let $\phi_\alpha \colon p^{-1}(U_\alpha) \to U_\alpha \times F$ and $\psi_\beta \colon p'^{-1}(V_\beta) \to V_\beta \times F'$ be local trivilizations around x and $f(x)$ respectively. Then we ask that the maps

$$L^f_{\alpha,\beta} \colon U_\alpha \cap f^{-1}(V_\beta) \to \mathrm{Map}(F, F')$$

which are adjoint to

$$U_\alpha \cap f^{-1}(V_\beta) \times F \xrightarrow{\phi_\alpha^{-1}} p^{-1}(U_\alpha \cap f^{-1}(V_\beta)) \xrightarrow{\tilde{f}} p'^{-1}(U_\alpha \cap f^{-1}(V_\beta)) \xrightarrow{\psi_\beta} (f(U_\alpha) \cap V_\beta) \times F' \xrightarrow{\mathrm{pr}_2} F'$$

take values in G, i.e., there exists a dashed arrow making the following diagram commute:

$$\begin{tikzcd} G \arrow[rr, "ad(\mu_G)"] & & \mathrm{Map}(F, F) \\ & U_\alpha \cap f^{-1}(V_\beta) \arrow[ul, dashed, "\overline{L}^f_{\alpha,\beta}"] \arrow[ur, "L^f_{\alpha,\beta}"'] & \end{tikzcd}$$

We also have a notion of isomorphic bundles.[15]

1.7.2 Definition. Let $\xi = (p \colon E \to B)$ and $\xi' = (p' \colon E' \to B)$ be bundles with the same fiber F, the same structure group G and the same base space B. We say that ξ and ξ' are isomorphic if there exists a bundle map $(f, \tilde{f})$ with $f = \mathrm{id}_B$.

1.7.3 Example. In Proposition 1.5.4 we showed that for a locally trivially bundle $p \colon E \to B$ and a map $f \colon X \to B$, the pullback $f^*(E) \colon E \times_B X \to X$ is a locally trivial fiber bundle. More is true - the induced map $\phi \colon f^*(E) \to E$ is a morphism of locally trivial bundles. Indeed, commutativity of the diagram is clear from the definition of the pullback. Moreover, it is straightforward from the definitions to see that ϕ carries the fiber over a point $x \in X$ to the fiber over $f(x)$. Finally, the coordinate transformations of the bundle $f^*(E)$ are given by

$$V_\alpha \cap V_\beta \xrightarrow{f} U_\alpha \cap U_\beta \xrightarrow{\Phi_{\alpha\beta}} G \xrightarrow{\mathrm{ad}(\mu)} \mathrm{Homeo}(F)$$

where

$$U_\alpha \cap U_\beta \xrightarrow{\Phi_{\alpha\beta}} G \xrightarrow{\mathrm{ad}(\mu)} \mathrm{Homeo}(F)$$

is the coordinate transformations for $p \colon E \to B$.

1.7.4 Remark. The converse to the previous remark is also true.

[15] The categorically minded reader will complain that this is not the correct definition, as an isomorphism should be defined as a bundle homomorphism which admits a map in the reverse direction so that both composites are the identity. Fortunately, it is a theorem that any bundle map over the identity map of a fixed space B is an isomorphism in this sense, thus justifying our definition.

1.7.5 Theorem. *Let $p\colon E \to B$ and $p'\colon E' \to X$ be locally trivial bundles having the same fiber and structure group. Suppose there is a bundle map*

$$\begin{tikzcd} E' \arrow[r, "\tilde{f}"] \arrow[d, "p'"'] & E \arrow[d, "p"] \\ X \arrow[r, "f"'] & B \end{tikzcd}$$

then the bundle $E' \to X$ is isomorphic to the pullback bundle $f^*(E) \to X$.

Proof. By the universal property of the pullback we have a map $\overline{f}\colon E' \to f^*(E)$ which is given by $\overline{f}(e) = (p'(e), \tilde{f}(e))$. We claim that this is a bundle isomorphism. It is a rather lengthy and tedious exercise to show this, so we omit the proof. □

We also note the following two lemmas, which are straightforward to show using the definition of the pullback, and shows that the pullback is functorial in this category.

1.7.6 Lemma. *Let $\pi\colon E \to B$ be a locally trivial bundle. For continuous map $f\colon X \to B$ and $g\colon Y \to X$, we have a bundle isomorphism $(f \circ g)^*(\pi) \cong g^*(f^*(\pi))$.*

1.7.7 Lemma. *Let $p\colon E \to B$ be a locally trivial bundle, then $id_B^*(E) \simeq E$.*

Exercise 3

Show that the pullback of a trivial bundle is a trivial bundle.

1.7.8 Remark. The definition of a bundle morphism is very technical. Suppose however, that our bundles are principal G-bundles, so that E and E' come with right G-actions (Theorem 1.3.9). One can check that for a bundle morphism $(\tilde{f}, f)$ between two principal G-bundles, the map $\tilde{f}$ is G-equivariant, i.e., $\tilde{f}(e \cdot g) = \tilde{f}(e) \cdot g$.

1.7.9 Remark. Given a bundle $\pi\colon E \to B$ and two maps $f, g\colon X \to B$, one can ask when the two bundles $f^*(\pi)$ and $g^*(\pi)$ are isomorphic. The following is an important result in this direction.

1.7.10 Theorem. *Let $\pi\colon E \to B$ be a bundle with B compact, and suppose that $f \simeq g\colon X \to B$ are homotopic, then there is an isomorphism $f^*(\pi) \cong g^*(\pi)$ of bundles over B.*

1.7.11 Remark. We will prove this in the case that π is a principal G-bundle (the general case can be reduced to this case). We will do this by constructing a bundle map

$$\begin{tikzcd} f^*E \arrow[rr, dashed, "?"] \arrow[dr] & & g^*\pi \arrow[dl] \\ & X & \end{tikzcd}$$

We will make use of the following definition.

1.7.12 Definition. A section of a bundle $\pi\colon E \to B$ is a continuous map $s\colon B \to E$ such that $\pi \circ s \simeq \mathrm{id}_B$.

1.7.13 Remark. We wish to describe the set of bundle maps between two principal G-bundles $\pi_1 \colon E_1 \to X$ and $\pi_2 \colon E_2 \to Y$. Note that G acts on the right of E_1 and E_2, and so on the left of E_2 via $g \cdot e_2 := e_2 \cdot g^{-1}$. Then, the associated bundle construction (Definition 1.4.4) gives an associated bundle of π_1 with fiber E_2, namely

$$\omega := \pi_1 \times_G E_2 \colon E_1 \times_G E_2 \to X.$$

1.7.14 Theorem. *Bundle maps from π_1 to π_2 are in bijection with sections of ω.*

Proof. We first assume that $\pi_1 \colon X \times G \to G$ and $\pi_2 \colon Y \times G \to G$ are trivial bundles. Suppose that we are given a bundle homomorphism, then we must define a section s in the associated bundle

$$\begin{tikzcd} (X \times G) \times_G (Y \times G) \arrow[d, shift left, "\omega"] \\ X \arrow[u, shift left, "s"] \end{tikzcd}$$

Let $e_1 \in X \times G$ with $x = \pi_1(e_1) \in X$, we set

$$s(x) = [e_1, \tilde{f}(e_1)].$$

Note that this is well-defined, as

$$[e_1 \cdot g, \tilde{f}(e_1 \cdot g)] = [e_1 \cdot g, \tilde{f}(e_1) \cdot g] = [e_1 \cdot g, g^{-1} \cdot, \tilde{f}(e_1)] = [e_1, \tilde{f}(e_1)],$$

where we have used Remark 1.7.8. Moreover, it is continuous, and provides a section:

$$\pi \circ s(x) = \pi_1[e_1, \tilde{f}(e_1)] = \pi_1(e_1) = x.$$

The general case can be reduced to the case of a trivial bundle by working locally and gluing.

Conversely, suppose we have been given a section of $E_1 \times_G E_2 \xrightarrow{\omega} X$. We define a map $\tilde{f} \colon E_1 \to E_2$ by $\tilde{f}(e_1) = e_2$ where $s(\pi_1(E_1)) = [(e_1, e_2)]$. We note that this map is G-equivariant:

$$[e_1 \cdot g, e_2 \cdot g] = [e_1 \cdot g, g^{-1} \cdot e_2] = [e_1, e_2]$$

and so descends to a map on orbit spaces $f \colon X \to Y$. As usual, we omit the tedious verification that this actually defines a bundle map. □

1.7.15 Remark. For a *principal* G-bundle, the condition of having a section is extremely strong:

1.7.16 Lemma. *Let $\pi \colon E \to X \times I$ be a principal G-bundle, and let $\pi_0 := i_0^* \pi \colon E_0 \to X$ be the pullback of π under the map $i_0 \colon X \to X \times I$. Then $\pi \cong (pr_1)^* \pi_0 \cong \pi_0 \times id_I$, where $pr_1 \colon X \times I \to X$ is the projection map.*

$$\begin{tikzcd} pr_1^*(E_0) \arrow[r] \arrow[d] & E_0 \arrow[r] \arrow[d, "\pi_0"] & E \arrow[d, "\pi"] \\ X \times I \arrow[r, "pr_1"'] & X \arrow[r, hook, "i_0"'] & X \times I \end{tikzcd}$$

Proof. It suffices to find a bundle map as indicated:

$$\begin{tikzcd}
E_0 \arrow[r] \arrow[d, "\pi_0"'] & E \arrow[r, dashed, "\widetilde{\mathrm{pr}_1}"] \arrow[d, "\pi"'] & E_0 \arrow[d, "\pi_0"] \\
X \arrow[r, hook, "i_0"'] & X \times I \arrow[r, "\mathrm{pr}_1"'] & X
\end{tikzcd}$$

By Theorem 1.7.14 this is equivalent to finding a section s of $\omega\colon E \times_G E_0 \to X \times I$. Note that there exists a section s_0 of $\omega_0\colon E_0 \times_G E_0 \to X = X \times \{0\}$ corresponding to the identity bundle map. Then composing s_0 with with the inclusion into $E \times_G E_0$ we get the following diagram:

$$\begin{tikzcd}
X \times \{0\} \arrow[r, "s_0"] \arrow[d, hook] & E \times_G E_0 \arrow[d, "\omega"] \\
X \times I \arrow[r, equal] \arrow[ur, dashed, "s"] & X \times I
\end{tikzcd}$$

Because ω is a fibration (Remark 1.1.5) we can apply the homotopy lifting property (**??**) to produce a map $s\colon X \times I \to E \times_G E_0$, which is a section of ω. ☐

Proof of Theorem 1.7.10. Let $H\colon X \times I \to B$ be a homotopy between f and g, and consider the following diagram:

$$\begin{tikzcd}
f^*E \arrow[rrr] \arrow[dd, "f^*\pi"'] & & & H^*E \arrow[r, "\tilde{H}"] \arrow[dd, "H^*\pi"] & E \arrow[dd, "\pi"] \\
& & g^*E \arrow[ur] \arrow[dd, "g^*\pi"'] & & \\
X \times \{0\} \arrow[rrr, hook, "i_0"] & & & X \times I \arrow[r, "H"'] \arrow[dr, "\mathrm{pr}_1"'] & B \\
& & X \times \{1\} \arrow[ur, hook, "i_1"'] & & X
\end{tikzcd}$$

By Lemma 1.7.6 we have $f^*\pi \cong i_0^* H^*\pi$. By Lemma 1.7.16 $H^*\pi \cong \mathrm{pr}_1^*(f^*\pi) \cong \mathrm{pr}_1^*(g^*\pi)$, and so $f^*\pi \cong i_0^* H^*\pi \cong i_0^*\mathrm{pr}_1^*(g^*\pi) \cong g^*\pi$. ☐

Exercise 4

Show that a principal G-bundle is trivial if and only if it has a section.

1.8 *Classification of principal G-bundles*

We now move to the most important part of this part of the course, which is the classification theorem for principal G-bundles.

1.8.1 Definition. A principal G-bundle $\pi_G\colon EG \to BG$ is called *universal* if the total space EG is (weakly) contractible. The space $BG \simeq EG/G$ is called the classifying space of G.

The main theorem of this part of the course is the following. We let $\mathcal{P}(X, G)$ denote the set of principal G-bundles over a space X.[16]

[16] The following theorem can also be proved abstractly by Brown representability. But we give a direct proof here.

1.8.2 Theorem. *Let X be a CW-complex, then there is a bijection*

$$\Phi\colon [X, BG] \xrightarrow{\cong} \mathcal{P}(X, G), \quad f \mapsto f^*\pi_G$$

Proof. We first show note that Φ is well-defined by Theorem 1.7.10. Let us first show that Φ is onto. Let $\pi\colon E \to X$ be a principal G-bundle, then we are required to find $f\colon X \to G$ such that $\pi \cong f^*\pi_G$. Equivalently, we are required to find a bundle map $(f, \tilde{f})\colon \pi \to \pi_G$. By Theorem 1.7.14 this is equivalent to finding a section of the bundle $E \times_G EG \to X$ with fiber EG. Because EG is contactible this following from the following lemma.

1.8.3 Lemma. *Let X be a trivial bundle and $\pi\colon E \to X$ a locally trival bundle with fiber G and structure group G with $\pi_i(F) = 0$ for all $i \geq 0$. If $A \subseteq X$ is a subcomplex, then every section of π over A extends to a section defined on all of X. In particular, π has a section. Moreover, any two sections of π are homotopic.*

Proof. Let $s_0\colon A \to E$ be a section of π over A, then we will extend it to a section $s\colon X \to E$ using induction over the cells in $X \setminus A$. In other words, we can assume that $X = A \cup_\phi e^n$ for x^n an n-cell in $X \setminus A$ and attaching map $\phi\colon \partial e^n \to A$. The bundle is trivial over e^n (as e^n is contractible), so we have a commutative diagram

$$\begin{tikzcd}
& \pi^{-1}(e^n) \arrow[r, "\cong", "h"'] \arrow[d, "\pi"] & e^n \times F \arrow[dl, "\mathrm{pr}_1"'] \\
\partial e^n \arrow[ur, "s_0"] \arrow[r, hook] & e^n \arrow[ur, dashed, bend right, "s"'] &
\end{tikzcd}$$

where h is the chart for π over e^n and s is the seciton we wish to define.

The composite $h \circ s_0\colon \partial e^n \to e^n \times F$ is of the form

$$s_0(x) = (x, \tau_0(x)) \in e_n \times F,$$

with $\tau_0\colon \partial e_n \cong S^{n-1} \to F$. Because $\pi_{n-1}F = 0$ by assumption, τ_0 extends to a map $\tau\colon e^n \cong D^n \to F$ which we use to define $s\colon e^n \to e^n \times F$ by $s(x) = (x, \tau(x))$. After composing with h^{-1} we get the desired extension of s_0 over e^n.

To see that the section is unique up to homotopy, assume that we are given another section s'. Consider the bundle $\pi \times \mathrm{id}\colon E \times I \to X \times I$. We can define sections of this bundle over $X \times \{0\}$ and $X \times \{1\}$ using s and s', which together define a section over $X \times \{0, 1\}$. Arguing as in the first part, we can extend this section to a section Σ over $X \times I$. This section is of the form $\Sigma(x, t) = (s_t(x), t)\colon X \times I \to E \times I$, and map s_t provides the desired homotopy between s and s'. $\square$

We now return to the proof of Theorem 1.8.2. It remains to show the injectivity of Φ. That is, if $\pi_0 \cong f^*\pi_G \cong f^*\pi_G \cong \pi_1$, then $f \simeq g$.

Consider the two defining diagrams:

$$\begin{tikzcd} E_0 = f^*EG \arrow[r, "\tilde{f}"] \arrow[d, "\pi_0"'] & EG \arrow[d, "\pi"] \\ X = X \times \{0\} \arrow[r, "f"'] & BG \end{tikzcd} \qquad \begin{tikzcd} E_0 \cong E_1 = g^*EG \arrow[r, "\tilde{g}"] \arrow[d, "\pi_1"'] & EG \arrow[d, "\pi"] \\ X = X \times \{1\} \arrow[r, "g"'] & BG \end{tikzcd}$$

We can combine them to make the diagram:

$$\begin{tikzcd} E_0 \times I \arrow[d, "\pi_0 \times \mathrm{id}"'] & E_0 \times \{0,1\} \arrow[l, hook'] \arrow[r, "\tilde{\alpha}=(\tilde{f},0)\cup(\tilde{g},1)"] \arrow[d, "\pi_0 \times \{0,1\}"] & EG \arrow[d, "\pi"] \\ X \times I & X \times \{0,1\} \arrow[l, hook'] \arrow[r, "\alpha=(f,0)\cup(g,1)"] & BG \end{tikzcd}$$

We will extend the map $(\alpha, \tilde{\alpha})$ to a bundle map $(H, \tilde{H})\colon \pi_0 \times \mathrm{id} \to \pi_G$; then H will give the desired homotopy. Using Theorem 1.7.14 again this corresponds to a section s of the bundle $\omega\colon (E_0 \times I) \times_G EG \to X \times I$. But the map $(\alpha, \tilde{\alpha})$ gives a section s_0 of the bundle $\omega_0\colon (E_0 \times \{0,1\}) \times_G EG \to X \times \{0,1\} \subseteq (E_0 \times I) \times_G EG \to X \times I$. Since EG is contractible, we can use Lemma 1.8.3 to extend the section s_0 to the desired section s. □

1.8.4 Example. We have $\mathcal{P}(S^n, G) \simeq [S^n, BG] \cong \pi_n(BG)$. The long exact sequence in homotopy shows that $\pi_n(BG) \cong \pi_{n-1}(G)$.

1.8.5 Remark. So far we have made no claim about the existence of universal principal G-bundles. Nonetheless, we have the following result of Milnor.

1.8.6 Theorem. *Let G be a locally compact topological group. Then a universal principapl G-bundle exists, and is functorial in the sense that a continuous group homomorphism $f\colon G \to H$ induces a bundle map $(B\mu, E\mu)\colon \pi_G \to \pi_H$. Moreover, the classifying space BG is unique up to homotopy.*

Proof. Let us explain why BG is unique up to homotopy, before commenting on the construction. Suppose $\pi_G\colon EG \to BG$ and $\pi'_G\colon EG' \to BG'$ are universal principal G-bundles. Using the universal properties of π_G and π'_G we can find maps $f\colon BG' \to BG$ and $g\colon BG \to BG'$ such that $\pi'_G \cong f^*\pi_G$ and $\pi_G \cong g^*\pi'_G$. Then,

$$\pi_G \cong g^*\pi'_G \cong g^*f^*\pi_G \cong (f \circ g)^*(\pi_G) \simeq (\mathrm{id}_B)^*\pi_G.$$

By Theorem 1.8.2 we have $f \simeq g \simeq \mathrm{id}_{BG}$. Similarly, we deduce that $g \circ f \simeq \mathrm{id}_{BG'}$. Therefore, $f \simeq g$. □

1.8.7 Remark. There are several different ways to construct the bundle π_G. Here is Milnor's construction. We recall that the join of X and Y is the space[17]

$$X * Y := X \times I \times Y / \sim$$

where $(x, 0, y_1) \sim (x, 0, y_2)$ for all $y_1, y_2 \in Y$ and $(x_1, 1, y) \sim (x_2, 1, y)$ for all $x_1, x_2 \in X$. For example,

$$X * \{y\} = (X \times I)/(X \times \{1\}) \cong CX,$$

[17] Technically, for Milnor's construction, we need to equip the join with coarsest topology which makes $t\colon X * Y \to I$ (given on $X \times I \times Y$ by $(x,t,y) \mapsto t$, constantly 0 on X, and constantly 1 on Y) and the projections $\pi_1\colon t^{-1}(I) \to X$ and $\pi_2\colon t^{-1}(I) \to Y$ continuous.

the cone on X. If $Y = \{y_1, y_2\} = S^0$ is two points, then $X * Y \cong \Sigma X$, the suspension. There is also a reduced version of the join for pointed spaces. For CW-complexes, we have a homotopy equivalence $A * B \simeq \Sigma(A \wedge B)$. For example, $S^n * S^m \simeq S^{n+m-1}$ (in this case, this is actually a homeomorphism).

Now, we let $G^{*(k+1)} := G * \cdots * G$, the join of $k+1$-copies of G. This has a free G-action, given by the diagonal action on the copies of G, and trivial action on I. Let $\mathcal{J}(G) := \operatorname{colim}_k G^{*(k+1)}$. Then, in fact $\mathcal{J}(G)$ has a free G-action and $\mathcal{J}(G) \to \mathcal{J}(G)/G$ is a universal principal G-bundle. The rough idea is that as we join more and more copies of G, the space becomes more and more connective, and in the colimit, (weakly) contractible.

Exercise 5

Show that $B(G \times H) \simeq BG \times BH$ (whenever this makes sense).

1.8.8 Remark. In practice, for us we will construct the universal bundles we need by hand.

1.8.9 Example. We recall from (1.1.18) that we have a principal $O(n)$-bundle

$$O(n) \to V_n(\mathbb{R}^k) \to G_n(\mathbb{R}^k)$$

with $V_n(\mathbb{R}^k)$ $(k-n-1)$-connected. Letting $k \to \infty$, we get the bundle

$$O(n) \to V_n(\mathbb{R}^\infty) \to G_n(\mathbb{R}^\infty)$$

with $V_n(\mathbb{R}^\infty)$ contractible. This is a model for the universal bundle $EO(n) \to BO(n)$, i.e., $BO(n) \simeq G_n(\mathbb{R}^\infty)$.

Now recall that a real vector bundle is a locally trivial bundle with fiber a vector space V of dimension n over $\mathbb{R}$. Using the associated bundle construction and Theorem 1.4.2 we see that there is a bijection

$$\Big\{\text{principal } GL_n(\mathbb{R})\text{-bundles over } X\Big\} \longleftrightarrow \Big\{\text{Rank } n \text{ vector bundles over } X\Big\}$$

By Gram–Schmidt we have $GL_n(\mathbb{R}) \simeq O(n)$, and so we deduce the following from Theorem 1.8.2

$$\mathrm{Vect}_n^{\mathbb{R}}(X) \simeq \mathcal{P}(GL_n(\mathbb{R}), X) \simeq \mathcal{P}(O(n), X) \simeq [x, BO(n)] \simeq [X, G_n(\mathbb{R}^\infty)],$$

and we say that $G_n(\mathbb{R}^\infty)$ is a classifying space for real vector bundles of rank n.

Similarly, $BU(n) \simeq G_n(\mathbb{C}^\infty)$ and this is the classifying space for rank n complex vector bundles.

1.8.10 Example (Classification of real line bundles). Consider the (real) case $n = 1$ in the previous example, so that we are classifying real line bundles. In this case, we have a principal $\mathbb{Z}/2$-bundle $\mathbb{Z}/2 \to S^\infty \to \mathbb{R}P^\infty$, and so $B\mathbb{Z}/2 \simeq \mathbb{R}P^\infty$. But, note that $\mathbb{R}P^\infty \simeq K(\mathbb{Z}/2, 1)$, so **??** gives

$$\mathrm{Vect}_1^{\mathbb{R}}(X) \simeq \mathcal{P}(X, \mathbb{Z}/2) \simeq [X, \mathbb{R}P^\infty] \simeq H^1(X; \mathbb{Z}/2)$$

for any CW-complex X.

Recall that $H^*(\mathbb{R}P^\infty, \mathbb{Z}/2) \cong \mathbb{Z}/2[w]$ for $|w| = 1$. In particular, if π is a real line bundle on X with classifying map $f_\pi \colon X \to \mathbb{R}P^\infty$, we get a well-defined degree one cohomology class

$$w_1(\pi) := f_\pi^\infty(w)$$

called the first Stiefel–Whitney class of π. The bijection $P(X, \mathbb{Z}/2) \simeq H^1(X; \mathbb{Z}/2)$ sends π to $\omega_1(\pi)$ and so real line bundles are completely classified by their first Stiefel–Whitney classes.

1.8.11 Example (Classification of complex line bundles). Consider the complex case $n = 1$ in Example 1.8.9, so that we are classifying complex line bundles. In this case, we have a principal S^1-bundle $S^1 \to S^\infty \to \mathbb{C}P^\infty$, and so $BS^1 \simeq \mathbb{C}P^\infty$. But, note that $\mathbb{C}P^\infty \simeq K(\mathbb{Z}, 2)$, so **??** gives

$$\mathrm{Vect}_1^{\mathbb{C}}(X) \simeq \mathcal{P}(X, BS^1) \simeq [X, \mathbb{C}P^\infty] \simeq H^2(X; \mathbb{Z})$$

for any CW-complex X.

Recall that $H^*(\mathbb{C}P^\infty, \mathbb{Z}) \cong \mathbb{Z}[c]$ for $|c| = 2$. In particular, if π is a complex line bundle on X with classifying map $f_\pi \colon X \to \mathbb{C}P^\infty$, we get a well-defined degree two cohomology class

$$c_1(\pi) := f_\pi^\infty(c)$$

called the first Chern class of π. The bijection $P(X, S^1) \simeq H^2(X; \mathbb{Z})$ sends π to $c_1(\pi)$ and so real line bundles are completely classified by their first Chern classes.

1.8.12 Example. How many (real) vector bundles over $\mathbb{R}P^n$ are there? We have

$$\mathrm{Vect}_1^{\mathbb{R}}(\mathbb{R}^n) \cong H^1(\mathbb{R}P^n; \mathbb{Z}/2) \cong \mathbb{Z}/2,$$

so there are two (up to equivalence). One is the trivial bundle. What is the non-trivial bundle?

Let $x \in S^n$ and $[x] \in \mathbb{R}P^n \simeq S^n/\sim$ the class represented by x. Let $E = \{([x], v) \colon [x] \in \mathbb{R}P^n, v \in [x]\} \subseteq \mathbb{R}P^n \times \mathbb{R}^{n+1}$. Then, we have a bundle γ_1 defined by[18]

$$\gamma_1 \colon E \to \mathbb{R}P^n, \quad ([x], v) \mapsto [x].$$

[18] This bundle is known as the tautological, or canonical, bundle over $\mathbb{R}P^n$.

To see that this is non-trivial note that when $n = 1$ this is exactly the Möbius bundle, which is non-trivial. In general, if γ_1 was trivial, then pull-back along $\mathbb{R}P^1 \to \mathbb{R}P^n$ would also be trivial, but this is the non-trivial Möbius bundle again, a contradiction. So γ_1 must be non-trivial.

1.8.13 Example. Isomorphism classes of principal S^1-bundles over S^2 are given by $[S^2, BS^1] \cong \pi_2(BS^1) \cong \pi_1(S^1) \cong \mathbb{Z} \cong H^2(\mathbb{C}P^\infty; \mathbb{Z})$. The Hopf bundle $H \colon S^1 \to S^3 \to S^2$ is a principal S^1-bundle. In particular, it is given as the pullback along a map $f \colon S^2 \cong \mathbb{C}P^1 \to \mathbb{C}P^\infty$:

$$\begin{tikzcd}
S^3 \arrow[r] \arrow[d, "H"'] & S^\infty \arrow[d, "\pi_{S^1}"] \\
S^2 \cong \mathbb{C}P^1 \arrow[r, "f"'] & \mathbb{C}P^\infty
\end{tikzcd}$$

It turns out that this map is the inclusion $\mathbb{C}P^1 \hookrightarrow \mathbb{C}P^\infty$, and so $H^*(f)\colon H^*(\mathbb{C}P^\infty) \to H^*(\mathbb{C}P^1)$ sends ω to ω. In particular, $c_1(H) \neq 0$, and $c_1(H)$ generates $H^2(\mathbb{C}P^1)$ as a cyclic group.

The following exercise gives another way to compute this Chern class (using **????**)

Exercise 6

Let $\pi\colon E \to X$ be a principal S^1-bundle over a simply-connected space X. Let $a \in H^1(S^1;\mathbb{Z})$ be a generator. Show that

$$c_1(\pi) = d_2(a)$$

where d_2 is the differential on the E_2-page of the Leray–Serre spectral sequence associated to π, i.e., $E_2^{p,q} \cong H^p(X;H^1(S^1)) \implies H^{p+q}(E,\mathbb{Z})$.