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MA3408 - ALGEBRAIC TOPOLOGY II

1

Homotopy theory

1.1 Review of basics on homotopy theory

We begin with a recollection of some facts that have been covered in Algebraic Topology I and Introduction to Topology.

1.1.1 Notation. We let $I = [0, 1]$ denote the unit interval. For a pointed topological space X we will denote the basepoint by x_0 or $*$.

We recall the following definition.

1.1.2 Definition. A homotopy between $f, g: X \rightarrow Y$ is a continuous function $H: X \times I \rightarrow Y$ such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ and $H(x_0, t) = y_0$ for all $t \in I$. We will write $f \simeq g$, or $f \simeq_H g$, if we need to make the choice of homotopy clear.

For a subspace $A \subseteq X$, a relative homotopy is a homotopy with $H(a, t) = f(a) = g(a)$ for all $a \in A, t \in I$.

1.1.3 Remark. Equivalently, we can specify a family of continuous maps $h_t: X \rightarrow Y$ such that $h_0 = f, h_1 = g$ and

$$\begin{aligned} H: X \times I &\rightarrow Y \\ (x, t) &\mapsto h_t(x) \end{aligned}$$

is continuous. We will switch between the two equivalent definitions without comment, using whatever is more convenient.

1.1.4 Proposition. For all spaces X and Y , homotopy is an equivalence relation on the set of maps from X to Y . Furthermore, if we are given $k: A \rightarrow X, \ell: Y \rightarrow B$ and homotopic maps $f \simeq g: X \rightarrow Y$, then $fk \simeq gk: A \rightarrow Y$ and $\ell f \simeq \ell g: X \rightarrow B$.

Proof. Let $f, g: X \rightarrow Y$, then

1. $f \simeq_F f$ via $F(x, t) = f(x)$ for all $x \in X, t \in I$.
2. If $f \simeq_F g$, then $g \simeq_G f$ where $G(x, t) = F(x, 1 - t)$.
3. If $f \simeq_F g$ and $g \simeq_G h$, then $f \simeq_H h$ via

$$H(x, t) = \begin{cases} F(x, 2t) & \text{if } 0 \leq t \leq 1/2 \\ G(x, 2t - 1) & \text{if } 1/2 \leq t \leq 1. \end{cases}$$

For the last part of the proposition let f_t be a homotopy between f and g , then $f_t k$ and ℓf_t give the required homotopy. \square

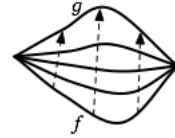


Figure 1.1: A homotopy between f and g .

1.1.5 Definition. For a map $f: X \rightarrow Y$, we let $[f]$ denote the equivalence class containing f . The collection of all homotopy classes of maps from X to Y is denoted $[X, Y]$.¹

¹ If our spaces are based, then these should be homotopy classes of *based* maps.

1.1.6 Remark. Note that if $\alpha = [f] \in [Y, Z]$ and $\beta = [g] \in [X, Y]$, then $\alpha\beta = [f \circ g] \in [X, Z]$, i.e., we can form the category $hTop_*$ whose objects are topological spaces, and whose morphisms are homotopy classes of maps.

1.1.7 Remark. We now very quickly review a number of standard topological constructions.

- Let X be a space and $A \subseteq X$. A map $r: X \rightarrow A$ such that $ri(a) = a$ for all $a \in A$ is called a retraction of X onto A , and A is called a retract of X .
- Let $i: A \hookrightarrow X$ be the inclusion, so that $ri = \text{id}_A$. If $ir \simeq \text{id}_X$, we call this a deformation retraction, and say that A is a deformation retract of X .
- If $f: X \rightarrow Y$, then a section of f is a map $s: Y \rightarrow X$ such that $f \circ s = \text{id}_Y$. We can also ask for a *homotopy* section by requiring only that $f \circ s \simeq \text{id}_Y$.

1.1.8 Definition. A map $f: X \rightarrow Y$ is called null-homotopic if $f \simeq c_y: X \rightarrow Y$ where $c_y: X \rightarrow Y$ is the constant map sending all of X to the point $y \in Y$. A homotopy between f and c_y is called a null-homotopy. A space X is contractible if id_X is null-homotopic.

1.1.9 Definition. Let (X, x_0) be a based topological space and $X \times I$ the cylinder on X . The quotient

$$CX = (X \times I) / (X \times \{1\} \cup \{x_0\} \times I)$$

with the base-point the equivalence class of $(x_0, 1)$ is called the (reduced) cone on X . Note that we have a natural inclusion $X \rightarrow CX$ of based maps given by $x \mapsto [x, 0]$.

1.1.10 Lemma. *The cone CX is contractible.*

Proof. Define $F: CX \times I \rightarrow CX$ by

$$F([x, t], s) = [x, s + (1 - s)t].$$

Note then that we have

$$F([x, t], 0) = [x, t] \quad \text{and} \quad F([x, t], 1) = [x, 1]. \quad \square$$

1.1.11 Lemma. *The following are equivalent:*

- (i) $f: X \rightarrow Y$ is null-homotopic.
- (ii) f can be extended to CX :

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ i \downarrow & \nearrow \exists \tilde{f} & \\ CX & & \end{array}$$

Proof. (i) \implies (ii) : Suppose f is null-homotopic, so $f \simeq_F *$. Then $F(X \times \{1\} \cup \{*\} \times I) = *$, so by the universal property of the quotient, we can find $\tilde{f}: CX \rightarrow Y$ such that $\tilde{f} \circ i = f$.

(ii) \implies (i) : Suppose $\tilde{f} \circ i = f$, then because CX is contractible (Lemma 1.1.10), we have $f = \tilde{f} \circ \text{id}_{CX} \circ i \simeq \tilde{f} \circ (*_{CX}) \circ i \simeq *$, so that f is null-homotopic. \square

1.1.12 Definition. A map $f: X \rightarrow Y$ is a homotopy equivalence if there exists $g: Y \rightarrow X$ such that $fg \simeq \text{id}_Y$ and $gf \simeq \text{id}_X$. We write $X \simeq Y$.

1.1.13 Example. (i) X is contractible if and only if $X \simeq *$.

(ii) If $i: A \hookrightarrow X$, and $r: X \rightarrow A$ is a deformation retract, then i and r are homotopy equivalences, and $A \simeq X$.

1.2 Higher homotopy groups

1.2.1 Notation. We will let $I_n = I^{\times n}, \partial I^n$ be the boundary of I^n , and write $[-, -]$ for homotopy classes of maps (if our spaces are based, these fix the base point).

1.2.2 Definition. For each $n \geq 0$ and X a topological space with $x_0 \in X$, we define

$$\pi_n(X, x_0) = [(I^n, \partial I^n), (X, x_0)].$$

1.2.3 Remark. (i) When $n = 0$, we have $I^0 = \text{pt}$ and $\partial I^0 = \emptyset$, therefore $\pi_0(X)$ is the set of path components of X .

(ii) When $n = 1$, this is a group, but need not be abelian (for example, consider the wedge of two circles).

(iii) Note that $I^n / \partial I^n \simeq S^n$ and $\partial I^n / \partial I^n \simeq s_0$. By the universal property of the quotient map, we see that

$$\pi_n(X, x_0) \cong [(S_n, s_0), (X, x_0)].$$

1.2.4 Definition. A maps of pairs $(X, A) \rightarrow (Y, B)$ is a map $f: X \rightarrow Y$ with $f(A) \subseteq B$, i.e., the diagram:

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

commutes.

1.2.5 Proposition. If $n \geq 1$, then $\pi_n(X, x_0)$ is a group with respect to the operation

$$(f + g)(t_1, \dots, t_n) = \begin{cases} f(2t_1, t_2, \dots, t_n) & 0 \leq t_1 \leq 1/2 \\ g(2t_1 - 1, t_2, \dots, t_n) & 1/2 \leq t_1 \leq 1. \end{cases}$$

Proof. The identity is given by the constant map taking all of I^n to x_0 and the inverse of f is given by

$$-f(t_1, \dots, t_n) = f(1 - t_1, t_2, \dots, t_n). \quad \square$$

1.2.6 Remark. Call the group operation $+_1$. Note that we can also define an operation $+_i$ for $1 \leq i \leq n$ by the same formula on the i -th coordinate.

1.2.7 Theorem. *All of these operations agree, and for $n \geq 2$, these give $\pi_n(X, x_0)$ the structure of an abelian group.*

This is a consequence of the following exercise, known as the Eckmann–Hilton lemma.

Exercise 1: Eckmann–Hilton lemma

Let M be a set and let $*$, \bullet be two binary operations on M , $*$, $\bullet: M \times M \rightarrow M$, both with unit elements. Suppose that

$$(a * b) \bullet (c * d) = (a \bullet c) * (b \bullet d)$$

for all $a, b, c, d \in M$. Show that the units agree, these two operations agree, and that the multiplication is commutative and associative.

1.2.8 Remark. Let us show that

$$(f +_1 g) +_2 (h +_1 i) \simeq (f +_2 h) +_1 (g +_2 i).$$

Indeed, both of these are the following map

$$(t_1, t_2, \dots) \mapsto \begin{cases} f(2t_1, 2t_2, \dots) & [1/2, 0] \times [1/2, 0] \\ g(2t_1 - 2, 2t_2, \dots) & [1/2, 1] \times [0, 1/2] \\ h(2t_1, 2t_2 - 2, \dots) & [0, 1/2] \times [1/2, 1] \\ i(2t_1 - 1, 2t_2 - 2, \dots) & [1/2, 1] \times [1/2, 1]. \end{cases}$$

1.2.9 Remark. Another approach is given by the following visualization: That is, so long as $n \geq 2$, we can shrink the domain of f and g

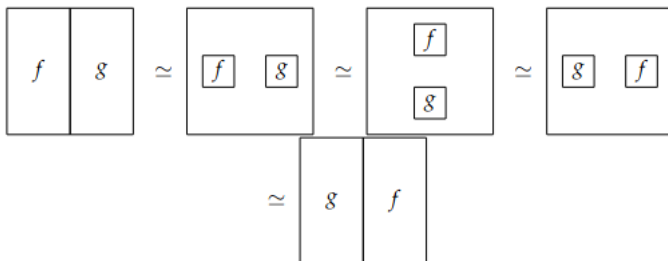


Figure 1.2: $f + g \simeq g + f$.

to smaller cubes (mapping the remaining region to the base point), slide f and g past each other, and then increase the domains back again.

Exercise 2

Let G be a topological group with identity element e , then $\pi_1(G, e)$ is abelian.

Hint: Use Eckmann–Hilton, or note the following: A topological group is a group object in the category of topological spaces. What is a group object in the category of groups?

1.2.10 Proposition. If $n \geq 1$ and X is path connected, then there is an isomorphism $\beta_\gamma : \pi_n(X, x_0) \xrightarrow{\cong} \pi_n(X, x_0)$ given by $\beta_\gamma([f]) = [\gamma \circ f]$ where γ is a path in X from x_1 to x_0 and $\gamma \circ f$ is constructed by first shrinking the domain of f to a smaller cube inside of I^n , and then inserting the path γ radially from x_1 to x_0 on the boundaries of these cubes.

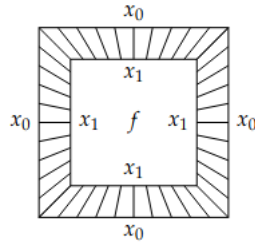
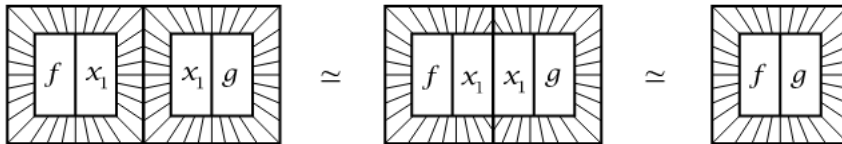


Figure 1.3: β_γ .

Proof. Observe the following:

1. $\gamma \circ (f + g) \simeq \gamma \circ f + \gamma \circ g$, i.e., β_γ is a group homomorphism.
2. $(\gamma \circ \eta) \circ f \simeq \gamma \circ (\eta \circ f)$, for η a path from x_0 to x_1 .
3. $c_{x_0} \circ f \simeq f$, where c_{x_0} denotes the constant path based at x_0 .
4. β_γ is well-defined with respect to homotopies of f or γ .

The only point that is perhaps not clear is (i). For this, we deform f and g to be constant on the right and left halves of I^n , respectively, producing maps we call $f + 0$ and $0 + g$. We then excise a wider symmetric middle slab of $\gamma(f + 0)$ and $\gamma(0 + g)$ until it becomes $\gamma(f + g)$: \square



1.2.11 Remark. Therefore if X is path-connected, different choices of base point x_0 yield isomorphic groups $\pi_n(X, x_0)$, which may then simply be written as $\pi_n(X)$.

1.2.12 Lemma. If $\{X_\alpha\}$ is a collection of path-connected spaces, then $\pi_n(\prod_\alpha X_\alpha) \cong \prod_\alpha \pi_n(X_\alpha)$.

Proof. Note that $\text{Hom}(Y, \prod_{\alpha} X_{\alpha}) \simeq \prod_{\alpha} \text{Hom}(Y, X_{\alpha})$. In particular, a map $S^n \rightarrow \text{Hom}(Y, \prod_{\alpha} X_{\alpha})$ is determined by a collection of maps $S^n \rightarrow X_{\alpha}$. Likewise, a homotopy $S^n \times I \rightarrow \prod_{\alpha} X_{\alpha}$ is determined by a collection of homotopies $S^n \times I \rightarrow X_{\alpha}$. This implies the result. \square

1.2.13 Proposition. *Homotopy groups are functorial: given a map $\phi: X \rightarrow Y$ we get group homomorphisms $\phi_*: \pi_n(X, x_0) \rightarrow \pi_n(Y, \phi(x_0))$ given by $[f] \mapsto [\phi \circ f]$ for all $n \geq 1$.*

Proof. We have the following:

1. ϕ_* is well-defined: if $f \simeq g$ via ψ_t , then $\phi \circ \psi_t$ defines a homotopy between $\phi \circ f$ and $\phi \circ g$.
2. This is a group homomorphism: $\phi \circ (f + g) \simeq \phi \circ f + \phi \circ g$ by the definition of the addition operation. Therefore.

$$\phi_*[f + g] = \phi_*[f] + \phi_*[g].$$

\square

Exercise 3

If $\phi: X \rightarrow Y$ is homotopy equivalence (not necessarily base-point preserving), then $\pi_*: \pi_n(X, x_0) \rightarrow \pi_n(Y, \phi(y_0))$ is an isomorphism.

1.2.14 Remark. We recall the following lifting property: Suppose $p: (\tilde{X}, \tilde{x}_0) \rightarrow (X, x_0)$ is a covering, and there is a map $f: (Y, y_0) \rightarrow (X, x_0)$ with Y path-connected and locally path-connected. Then a lift \tilde{f} exists if and only if $f_*\pi_1(Y, y_0) \subseteq p_*\pi_1(\tilde{X}, \tilde{x}_0)$.

$$\begin{array}{ccc} & (\tilde{X}, \tilde{x}_0) & \\ \tilde{f} \nearrow & \downarrow p & \\ (Y, y_0) & \xrightarrow{f} & (X, x_0) \end{array}$$

1.2.15 Proposition. *If p is a covering, then $p_*: \pi_n(\tilde{X}, \tilde{x}_0) \rightarrow \pi_n(X, x_0)$ is an isomorphism for all $n \geq 2$.*

Proof. Let us first show surjectivity. To that end, suppose we have a map $f: (S^n, s_0) \rightarrow (X, x_0)$ where $n \geq 2$. The assumption on n gives $\pi_1(S^n) = 0$, so $f_*\pi_1(S^n, s_0) \subseteq p_*\pi_1(\tilde{X}, \tilde{x}_0)$ holds. We therefore find a lift in the following:

$$\begin{array}{ccc} & (\tilde{X}, \tilde{x}_0) & \\ \tilde{f} \nearrow & \downarrow p & \\ (S^n, s_0) & \xrightarrow{f} & (X, x_0) \end{array}$$

Then $p_*[\tilde{f}] = [f]$, and p_* is surjective.

To see that p_* is injective, let $[\tilde{f}] \in \ker(p_*)$, i.e., $p_*[\tilde{f}] = [p \circ \tilde{f}] = 0$. Let $f = p \circ \tilde{f}$, then this is homotopic to the constant map $f \simeq c_{x_0}$

via a homotopy $\phi_t: (S^n, s_0) \rightarrow (X, x_0)$ with $\phi_1 = f$ and $\phi_0 = c_{x_0}$. By the same argument as above, the homotopy ϕ_t can be lifted to $\tilde{\phi}_t$. This satisfies $p \circ \tilde{\phi}_1 \simeq \phi_1$ and $p \circ \tilde{\phi}_0 \simeq \phi_0$. By the uniqueness of lifts, we must have $\tilde{\phi}_1 \simeq \tilde{f}$ and $\tilde{\phi}_0 \simeq c_{x_0}$. In other words, $\tilde{\phi}_t$ gives a homotopy between \tilde{f} and c_{x_0} , so that $[\tilde{f}] = 0$, and p_* is injective. \square

1.2.16 Example. S^1 has universal cover $p: \mathbb{R} \rightarrow S^1$, $p(t) = e^{2\pi it}$. Then $\pi_n(S^1) \cong \pi_n(\mathbb{R}) \cong 0$ for $n \geq 2$.

Exercise 4

Find two spaces X, Y with $\pi_n X \cong \pi_n Y$ but $X \not\cong Y$.

Hint: What is the universal cover of $\mathbb{R}P^n$?

1.2.17 Remark (Relative homotopy groups). Suppose we have (X, x_0) and a subspace A containing x_0 . We note that $i_*: \pi_n(A, x_0) \rightarrow \pi_n(X, x_0)$ is not injective in general (example, take S^1 into \mathbb{R}^2). An element in the kernel of i_* is a map $f: (I^n, \partial I^n) \rightarrow (A, x_0)$ such that

$$(I^n, \partial I^n) \xrightarrow{f} (A, x_0) \xrightarrow{i} (X, x_0)$$

is homotopic to c_{x_0} . This means there exists a homotopy

$$H: I^n \times I \rightarrow X$$

such that $H(-, 1) = f$, $H(-, 0) = c_{x_0}$ and $H|_{\partial I^n \times I} = c_{x_0}$.

If we define $J^n = I^n \times \{0\} \cup \partial I^n \times I \subseteq I^n \times I$, then this is a map of triples

$$H: (I^{n+1}, \partial I^{n+1}, J^n) \rightarrow (X, A, x_0).$$

1.2.18 Definition.

$$\pi_n(X, A, x_0) = [(I^n, \partial I^n, J^{n-1}), (X, A, x_0)]$$

1.2.19 Remark. Equivalently,

$$\pi_n(X, A, x_0) = [(D^n, S^{n-1}, s_0), (X, A, x_0)]$$

1.2.20 Proposition. If $n \geq 2$, then $\pi_n(X, A, x_0)$ is a group, and if $n \geq 3$, then it is abelian.

For all $n \geq 2$, a map of pairs $\phi: (X, A, x_0) \rightarrow (Y, B, y_0)$ induces homomorphisms $\phi_*: \pi_n(X, A, x_0) \rightarrow \pi_n(Y, B, y_0)$ for all $n \geq 2$.

Proof. This is similar to the case of $\pi_n(X)$ itself, and the details are left to the reader. \square

1.2.21 Theorem. The relative homotopy groups (X, A, x_0) fit into a long exact sequence

$$\cdots \rightarrow \pi_n(A, x_0) \xrightarrow{i_*} \pi_n(X, x_0) \xrightarrow{j_*} \pi_n(X, A, x_0) \xrightarrow{\partial_n} \pi_{n-1}(A, x_0) \rightarrow \cdots$$

where the map ∂_n is defined by $\partial_n([f]) = [f|_{I^{n-1}}]$.

The proof relies on the following.

1.2.22 Lemma (Compression criterion). *A map $f: (D^n, S^{n-1}, x_0) \rightarrow (X, A, x_0)$ represents 0 in $\pi_n(X, A, x_0)$ if and only if $f \sim g \text{ rel } S^{n-1}$, where g is a map whose image is contained entirely in A .*

Proof. Suppose $[f] = [g]$ with g as in the statement of the lemma. Note that there is a deformation of D^n onto x_0 , and so $[f] = [g] = 0$ in $\pi_n(X, A, x_0)$.

Conversely, suppose that $[f]$ represents 0 in $\pi_n(X, A, x_0)$. This means there exists a homotopy, relative to S^{n-1} , $F: D^n \times I \rightarrow X$ with $F|_{D^n \times \{0\}} = f$, $F|_{D^n \times \{1\}} = c_{x_0}$ and $F|_{S^{n-1} \times I} \subseteq A$. We can restrict F to a family of n -disks in $D^n \times I$ starting with $D^n \times \{0\}$ and ending with the disk $D^n \times \{1\} \cup S^{n-1} \times \{1\}$, all the disks in the family having the same boundary, then we get a homotopy from f to a map in A , stationary on S^{n-1} (said in other words, we can deformation retract $D^n \times [0, 1]$ onto $D^n \times \{1\} \cup S^{n-1} \times I$). \square

We now prove the existence of the long exact sequence.²

² This is the type of proof that is best done by the reader themselves.

Proof of Theorem 1.2.21. Step 1. Let us first show exactness at $\pi_n(X, x_0)$.

We first show $\text{im}(i_*) \subseteq \ker(j_*)$. Note that j_*i_* is induced by the composition $j \circ i$ and that these are both inclusion maps. Therefore, for $[f] \in \pi_n(A, x_0)$ we have $j_*i_*[f] = [j \circ i \circ f]$, but this has image contained in A , and so $j_*i_*[f] = 0$. This shows $\text{im}(i_*) \subseteq \ker(j_*)$.

To see the converse (namely, $\ker(j_*) \subseteq \text{im}(i_*)$) let $[f] \in \ker(j_*)$, i.e. $[j \circ f] = 0$. Note that again j is an inclusion map, and by the compression criteria $f \simeq g'$ relative to S^{n-1} , where g' has image contained in A . Since $x_0 \in S^{n-1}$, the homotopy fixes the basepoint, i.e. $[f] = [g'] \in \pi_n(X, x_0)$. But because g' has image in A , $[g'] \in \pi_n(A, x_0)$ and $i_*[g'] = [i \circ g'] = [f]$, so $[f] \in \text{im}(i_*)$.

Step 2. Let us now show exactness at $\pi_n(X, A, x_0)$.

Note that the composite $\partial \circ j_* = 0$ since the restriction of a map $(I^n, \partial I^n, J^{n-1}) \rightarrow (X, x_0, x_0)$ to I^{n-1} has image x_0 and so represents 0 in $\pi_{n-1}(A, x_0)$. Therefore, $\text{im}(j_*) \subseteq \ker(\partial)$. For the converse, suppose $[f] \in \ker(\partial)$. This means there exists a basepoint preserving homotopy $H: I^{n-1} \times I \rightarrow A$ (relative to ∂I^{n-1}) from $f|_{I^{n-1} \times \{0\}}$ to the constant map where the image of H is contained entirely in A . We can then define another homotopy G , such that $G_0 = f$, $G_t|_{I^{n-1}} = H_t$ and the rest of the image of G_t is $f[I^n]$ union with the image of H_s for $0 \leq s \leq t$. This homotopy maps S^{n-1} into A at all times, so $[f] = [G_1]$. Moreover, G_1 maps the boundary of I^n to x_0 , so $[G_1] \in \pi_n(X, x_0)$. Altogether,

$$j_*[G_1] = [j \circ G_1] = [G_1] = [f]$$

so $\ker(\partial) \subseteq \text{im}(j_*)$.

Step 3: Exactness at $\pi_n(A, x_0)$.

Let $[f] \in \pi_n(X, A, x_0)$ then $i_*\partial \in \pi_{n-1}(X, x_0)$ is the class represented by $f|_{I^{n-1}}$ and this is homotopic relative J^{n-2} to the constant map to x_0 , via f viewed as a homotopy. So this implies $\text{im}(\partial_*) \subseteq \ker(i_*)$. Conversely, let $[f] \in \ker(i_*)$ i.e., $i_*[f] = [i \circ f] = 0$.

Therefore, there exists a homotopy H between f and a constant map through a homotopy that has image in X and preserves x_0 . Since $H_0 = f$ has image in A and H_1 has image $\{x_0\}$, and H_0 takes the boundary to $\{x_0\}$, we see that $[H] \in \pi_n(X, A, x_0)$, and moreover $\partial([H]) \simeq f$. Therefore, $[f] \in \text{im}(\partial)$, and $\text{im}(\partial) = \ker(i_*)$. \square

1.2.23 Definition. A pair (X, A) with basepoint x_0 is said to be n -connected if $\pi_i(X, A) = 0$ for all $i \leq n$.

1.2.24 Lemma. A pair (X, A) is n -connected if and only if $\pi_i(A) \xrightarrow{i_*} \pi_i(X)$ is an isomorphism for $i < n$ and a surjection for $i = n$.

Proof. Use the long exact sequence in homotopy. \square

Exercise 5

Let X be a path-connected space, and CX the cone on X . Show that

$$\pi_n(CX, X, X_0) \cong \pi_{n-1}(X, x_0)$$

for $n \geq 1$.

1.3 Cofibrations and the homotopy extension property

1.3.1 Definition. Let \mathcal{C} be a class of topological spaces. A map $i: A \rightarrow X$ has the homotopy extension property (HEP) if, for every $Y \in \mathcal{C}$, the following extension property has a solution³

³ Here $i_0(x) = (x, 0)$.

$$\begin{array}{ccc} A & \xrightarrow{i_0} & A \times I \\ i \downarrow & & \downarrow i \times \text{id} \\ X & \xrightarrow{i_0} & X \times I \end{array} \quad \begin{array}{c} \searrow H \\ \downarrow \exists \tilde{H} \\ \downarrow f \end{array} \quad \begin{array}{c} \\ \\ Y \end{array}$$

A map $f: A \rightarrow X$ is a cofibration if it has the HEP with respect to all spaces Y .⁴

1.3.2 Remark. Note that we do not ask that \tilde{H} is unique.

1.3.3 Remark. If we are in a 'nice' category of topological spaces (see CREF), which we always assume, then we have an adjunction

$$\text{Hom}(X, \text{Hom}(Y, Z)) \cong \text{Hom}(X \otimes Y, Z)$$

of topological spaces, where $\text{Hom}(Y, Z)$ is given the compact open topology. Writing, $Z^Y := \text{Hom}(Y, Z)$, the homotopy extension property admits a reformulation in the following diagram

$$\begin{array}{ccc} A & \xrightarrow{h} & Y^I \\ i \downarrow & \nearrow \exists \tilde{h} & \downarrow p \\ X & \xrightarrow{f} & Y \end{array}$$

where $p: Y^I \rightarrow Y$ is the evaluation at 0 map. It is often easier to work with this equivalent diagram.

⁴ We will see later that cofibrations are always inclusions, and, if X is Hausdorff, are always closed maps.

Exercise 6

Let (X, A) have the HEP, and assume moreover that $i: A \rightarrow X$ is a retract up to homotopy. Show that A is a retract of X .

1.3.4 Lemma. Let $J = [0, 1]$.

- (i) The inclusion $i_0: X \rightarrow X \times J$ has the homotopy extension property for all Y .
- (ii) The inclusion $i_0: X \rightarrow CX$ has the homotopy extension property for all Y .

Proof. The proof in both cases is very similar; we do the first case in some detail. We are claiming there exists a lift \tilde{H} in the following diagram:

$$\begin{array}{ccc}
 X & \xrightarrow{i_0} & X \times I \\
 i \downarrow & & \downarrow i \times \text{id} \\
 X \times J & \xrightarrow{i_0} & X \times J \times I \\
 & \searrow f & \downarrow \exists \tilde{H} \\
 & & Y
 \end{array}$$

(A curved arrow labeled H goes from $X \times I$ to Y , and a curved arrow labeled f goes from $X \times J$ to Y .)

Geometrically, we will do this in two parts: we will define a map that "stacks" the two intervals on top of each other, i.e., we construct a map $G: X \times J \times I \rightarrow X \times [0, 2]$. We will then do H on one part of the cylinder, and f on the remaining part.

For the first part, let $G: X \times J \times I \rightarrow X \times [0, 2]$ be defined as⁵

$$G(x, t, s) = (x, t(1 + s)).$$

We then define $F: X \times [0, 2] \rightarrow Y$ by

$$F(x, k) = \begin{cases} f(x, k) & 0 \leq k \leq 1 \\ H(x, k/2) & 1 \leq k \leq 2. \end{cases}$$

Putting these together and defining $\tilde{H} := F \circ G$, we see that⁶

$$\tilde{H}((x, t), s) = \begin{cases} f(x, 1 - (1 - t)(1 + s)), & (1 - t)(1 + s) \leq 1 \\ H(x, (1 - t)(1 + s) - 1), & (1 - t)(1 + s) \geq 1. \end{cases}$$

One verifies directly that this gives the required extension. \square

1.3.5 Remark. We recall that given a map $f: X \rightarrow Y$, the mapping cylinder (see Figure 1.4) is the pushout

$$\begin{array}{ccc}
 X & \xrightarrow{i_0} & X \times I \\
 f \downarrow & & \downarrow \\
 Y & \longrightarrow & M_f
 \end{array}$$

In formulas,

$$M_f = ((X \times I) \amalg Y) / ((0, x) \sim f(x), \forall x \in X)$$

⁵ To see what is going on it is worth testing some cases and drawing pictures. For example, when $t = 0$ we have $G(x, 0, s) = (x, 0)$. When $t = 1$ we have $G(x, 1, s) = (x, 1 + s)$. When $s = 0$ we have $G(x, t, 0) = (x, t)$ and when $s = 1$ we have $G(x, t, 1) = (x, 2t)$.

⁶ Again, it is worthwhile to consider some cases. For example, if $t = 0$, then $(1 - t)(1 + s) = (1 + s) \geq 1$ for all s , so $\tilde{H}((x, 0), s) = H(x, s)$. At the other extreme, if $t = 1$, then $(1 - t)(1 + s) = 0 \leq 1$ for all s , so $\tilde{H}((x, 1), s) = f(x, s)$.

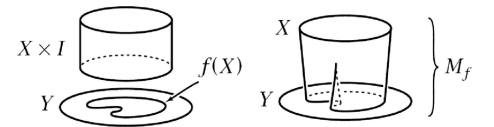


Figure 1.4: The mapping cylinder.

Note that M_f deformation retracts on Y by sliding each point $(x, t) \in M_f$ to the end-point. Note that we have a natural map $j: X \rightarrow M_f$ sending x to $(x, 1)$.

1.3.6 Lemma. *The map $j: X \rightarrow M_f$ has the HEP for all spaces Y .*

Proof. The proof is similar to the previous lemma; one just has to modify the end point by defining

$$\tilde{H}|_{Y \times I}(y, s) = f(y, 0).$$

□

1.3.7 Corollary. *The inclusion $S^{n-1} \rightarrow D^n$ is a cofibration.*

Proof. Simply note that $D^n \simeq CS^{n-1}$.

□

There is a universal test space for cofibrations.

1.3.8 Proposition. *Let $i: A \rightarrow X$, and let M_i be the mapping cylinder. Then $i: A \rightarrow X$ is a cofibration if and only if there exists a map $r: X \times I \rightarrow M_i$ making the diagram*

$$\begin{array}{ccc} A & \xrightarrow{i_0} & A \times I \\ \downarrow i & \nearrow i \times id & \downarrow \\ X & \xrightarrow{i_0} & M_i \end{array} \quad \begin{array}{c} X \times I \\ \nearrow i_0 \\ X \end{array} \quad \begin{array}{c} \exists r \\ \searrow \\ M_i \end{array}$$

commute.

Proof. If i is a cofibration, then the map r exists as a consequence of the HEP.

For the other direction, if r exists, then for any maps $f: X \rightarrow Y$ and $H: A \times I \rightarrow Y$ making the obvious diagram commute, the universal property of the pushout gives us a map $H': M_i \rightarrow Y$. Then let $\tilde{H} = H' \circ r$, and we are done.

□

1.3.9 Corollary. *If $A \subseteq X$, then $i: A \rightarrow X$ is a cofibration if and only if $X \times I$ is a retract of $M_i = X \times \{0\} \cup A \times I$.*

1.3.10 Corollary. *A cofibration $i: A \rightarrow X$ is an injection. If X is Hausdorff, then $i(A)$ is closed in X .*

Proof. Let $J: A \times I \rightarrow M_i$ be the canonical map (arising from the definition of M_i as a pushout). Then, $J(a, 1) = r(i(a), 1)$, and observe that $J|_{A \times \{1\}}$ is the identity, as it is the top of the mapping cylinder. So, $i(a) \neq i(a')$ if $a \neq a'$, i.e., i is injective.

Because $i: A \rightarrow X$ is a cofibration, so is $i(A) \rightarrow X$. Hence $X \times I$ retracts onto $X \times \{0\} \cup i(A) \times I$ (Corollary 1.3.9). For a Hausdorff space, the image of a retract is closed, and so $X \times \{0\} \cup i(A) \times I$ is a closed subspace of $X \times I$. Intersecting with $X \times \{1\}$, we see that $i(A) \times \{1\}$ is closed in $X \times \{1\}$, i.e., $i(A)$ is closed in X .

□

The following (rather pathological) example shows that i is not always a closed map if X is not Hausdorff.

Exercise 7

Let $A = \{a\}$ and $X = \{a, b\}$ with the trivial topology. Show that the inclusion $A \rightarrow X$ is a cofibration whose image is not closed.

1.3.11 Remark. The next goal is to show that CW-complexes (X, A) are always cofibrations. The key is the following exercise.

Exercise 8

- (a) Suppose $\{(X_i, A_i)\}$ are a collection of spaces satisfying the HEP, then so does $(\coprod X_i, \coprod A_i)$.
- (b) Suppose (X, A) satisfies the HEP, and $f: A \rightarrow B$ is a continuous map. Let $Y = X \cup_f B$ be the pushout, then (Y, B) satisfies the HEP.
- (c) Suppose $A = X_0 \subseteq X_1 \subseteq \cdots \subseteq X_n \subseteq X_{n+1} \subseteq \cdots$.
Let $X = \text{colim } X_i$. If each (X_i, X_{i-1}) satisfies the HEP, then so does (X, A) .

1.3.12 Theorem. A relative CW-complex (X, A) satisfies the HEP.

Proof. Using Corollary 1.3.7 and the previous exercise we see that (S^{n-1}, D^n) satisfies the HEP $\implies (\coprod S^{n-1}, \coprod D^n)$ satisfies the HEP. Inductively, (X_{n-1}, A) satisfies the HEP and by the exercise (X, A) satisfies the HEP. \square

1.3.13 Remark. One can also prove this directly by constructing a deformation retract $r: X \times I \rightarrow X \times \{0\} \cup A \times I$.

1.3.14 Remark. One can consider the following question: Suppose that $A \subset X$ with A contractible, then is $X \simeq X/A$? Surprisingly, this is not true in general. Indeed, let $A = S^1 \setminus \{(1, 0)\}$ and consider $A \rightarrow S^1$. Then $S^1/A \cong T$, the $T = \{a, b\}$ the two point space with open sets $\emptyset, \{a\}, \{a, b\}$ (this is the Sierpiński space). One can check that this space is contractible.⁷ The exact condition we need is that $A \rightarrow X$ is a cofibration.

⁷ See <https://math.stackexchange.com/a/264789/64273>.

1.3.15 Definition. A contracting homotopy is a map $H: X \times I \rightarrow X$ such that $H(x, 0) = \text{id}_X$ and $H(x, 1) = c_{x_0}$, the constant map at x_0 .

1.3.16 Proposition. Suppose $A \subseteq X$ and $x_0 \in A$. Suppose there exists a map $H: X \times I \rightarrow X$ such that $H|_{X \times \{0\}} = \text{id}_X$ and $H|_{A \times I}$ has image in A and is a contracting homotopy for A . Then $q: X \rightarrow X/A$ is a homotopy equivalence.

Proof. We need to find $p: X/A \rightarrow X$ such that $q \circ p \simeq \text{id}_{X/A}$ and $p \circ q \simeq \text{id}_X$. The quotient map has a set-theoretic section given by

$$s(\bar{x}) = \begin{cases} x & x \notin A \\ x_0 & x \in A \end{cases}$$

Define $p: X/A \rightarrow X$ by the following diagram

$$\begin{array}{ccccc} X & \xrightarrow{q} & X/A & \xrightarrow{s} & X \\ & & \searrow p & & \downarrow H|_{X \times \{1\}} \\ & & & & X \end{array}$$

Assume for a moment that p is continuous. Then $p \circ q = H|_{X \times \{1\}}$, and so H gives a homotopy between id_X and $p \circ q = H|_{X \times \{1\}}$. Likewise, if we define G by

$$\begin{array}{ccccc} X/A \times I & \xrightarrow{s \times \text{id}} & X \times I & \xrightarrow{H} & X \\ & \searrow G & & & \downarrow q \\ & & & & X/A \end{array}$$

and assume that G is continuous, then

$$G(\bar{x}, 1) = q \circ (H|_{X \times \{1\}} \circ s) = q \circ p,$$

so that G is a homotopy between $\text{id}_{X/A}$ and $q \circ p$. To see that p is continuous, let $U \subset X$ be open, then

$$q^{-1}p^{-1}(U) = (p \circ q)^{-1}(U) = (H|_{X \times \{1\}})^{-1}(U)$$

is open in X by the continuity of $H|_{X \times \{1\}}$, hence $p^{-1}(U)$ is open in X/A by the definition of the quotient topology, and so p is continuous. We leave the proof of continuity of G to the reader. \square

1.3.17 Theorem. Let $A \subseteq X$ be a subspace with A contractible. Suppose that the inclusion $i: A \rightarrow X$ is a cofibration, then $X \rightarrow X/A$ is a homotopy equivalence.

Proof. Let $h: A \rightarrow I \rightarrow A$ be a contracting homotopy. Let $H: A \times I \rightarrow X$ be the composition of h with the inclusion map of A into X , i.e., the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{i_0} & A \times I \\ \downarrow i & & \downarrow i \times \text{id} \\ X & \xrightarrow{i_0} & X \times I \end{array} \quad \begin{array}{c} \searrow H \\ \downarrow \exists \tilde{H} \\ \searrow \text{id}_X \end{array} \quad \begin{array}{c} \\ \\ X \end{array}$$

By the HEP, the dotted map \tilde{H} exists as in the diagram. This map satisfies the conditions of Proposition 1.3.16:

- (i) $\tilde{H}: X \times \{0\} \rightarrow X$ is the identity.
- (ii) $\tilde{H}(A \times I) = H(A \times I) = h(A \times I) \subseteq A$.
- (iii) $\tilde{H}(A \times \{1\}) = x_0$.

Therefore, $q: X \rightarrow X/A$ is a homotopy equivalence, as claimed. \square

Exercise 9: Cofibrations are pushout closed.

Let $i: A \rightarrow X$ be a cofibration, and $g: A \rightarrow B$ any map, then the induced map $B \rightarrow B \cup_g X$ is a fibration.

1.4 Fibrations and the homotopy lifting property

The dual notion of a cofibration is a fibration, where the homotopy extension property is replaced by the homotopy lifting property.

1.4.1 Definition. Let \mathcal{E} be a class of topological spaces. Assume that $p: E \rightarrow B$ is a continuous map, then we say that p has the homotopy lifting property (with respect to \mathcal{E}) if for every $X \in \mathcal{E}$, and map $f: X \rightarrow E$ and every homotopy $H: X \times I \rightarrow B$ that begins with $p \circ f$, we can lift it to a homotopy $\tilde{H}: X \times I \rightarrow E$ that begins with f , i.e., $p \circ \tilde{H} = H$ and $\tilde{H}(x, 0) = f(x)$. In a diagram, we require the lift \tilde{H} in the following:

$$\begin{array}{ccc} X & \xrightarrow{f} & E \\ i_0 \downarrow & \nearrow \exists \tilde{H} & \downarrow p \\ X \times I & \xrightarrow{H} & B \end{array}$$

If \mathcal{E} is the class of all topological spaces, then p is called a (Hurewicz) fibration, while if $\mathcal{E} = \{I^n\}$ (or equivalently, the class of CW-complexes), then p is called a Serre fibration.

1.4.2 Remark. As in Remark 1.3.3, there is an equivalent way to present the homotopy lifting property: we ask for the lift \tilde{h} as shown in the following

$$\begin{array}{ccccc} X & & \xrightarrow{f} & & E \\ & \searrow \exists \tilde{h} & & \searrow ev_0 & \\ & & E^I & \xrightarrow{ev_0} & E \\ & \searrow h & \downarrow p_* & & \downarrow p \\ & & B^I & \xrightarrow{ev_0} & B \end{array}$$

This makes it clear how the homotopy lifting property is dual to the homotopy extension property.

1.4.3 Remark. We can also talk about the homotopy lifting property with respect to a pair (X, A) : namely, a map $p: E \rightarrow B$ has the homotopy lifting property with respect to a pair (X, A) if each homotopy $H: X \times I \rightarrow B$ lifts to a homotopy $\tilde{H}: X \times I \rightarrow E$ which agrees with a given homotopy H_A on $A \times I$. In a diagram, we ask for the lift \tilde{H} in the following:

$$\begin{array}{ccc} X \cup (A \times I) & \xrightarrow{f \cup H_A} & E \\ i_0 \downarrow & \nearrow \exists \tilde{H} & \downarrow p \\ X \times I & \xrightarrow{H} & B \end{array}$$

1.4.4 Theorem. The following are equivalent:

- (i) p is a Serre fibration.
- (ii) p has the homotopy lifting property with respect to all n -discs D^n .
- (iii) p has relative homotopy property with respect to all pairs (D^n, S^{n-1})

(iv) p has the relative homotopy property with respect to all CW-pairs (X, A) .

Proof sketch. (i) \implies (ii) is immediate from the definitions.

(ii) \implies (iii) follows because the pairs $(D^n \times I, D^n \times \{0\})$ and $(D^n \times I, D^n \times \{0\} \cup S^{n-1} \times I)$ are homeomorphic.

(iii) \implies (iv) by induction over the skeleton of X ; one reduces to the case (iii).

(iv) \implies (i) by taking $A = \emptyset$. \square

Exercise 10

Show that the composition of fibrations is a fibration.

1.4.5 Definition. We recall the construction of pullbacks in topological spaces: given maps $p: E \rightarrow B$ and $f: B' \rightarrow B$, we let

$$E' = \{(b', e) \in B' \times E \mid p(e) = f(b')\}.$$

This comes with natural projection maps $f': E' \rightarrow E$ and $p': E' \rightarrow B'$. Then E' is the pull-back in topological spaces, and so we often also denote it by f^*E .

The following is dual to Exercise 9.

1.4.6 Lemma. If $p: E \rightarrow B$ satisfies the HLP with respect to the class \mathcal{E} , then so does $p': E' \rightarrow B'$.

Proof. Consider the following diagram:

$$\begin{array}{ccccc} X & \longrightarrow & E' & \xrightarrow{f'} & E \\ i_0 \downarrow & & p' \downarrow & \lrcorner & \downarrow p \\ X \times I & \longrightarrow & B' & \xrightarrow{f} & B \end{array}$$

Because $p: E \rightarrow B$ satisfies the HLP, there is a lift $\tilde{H}': X \times I \rightarrow E$ of $X \times I \rightarrow B$. Then, by the universal property of the pullback, we get a map $\tilde{H}: X \times I \rightarrow E'$ satisfying the desired properties. \square

1.4.7 Definition. If $p: E \rightarrow B$ is a fibration, then $F := p^{-1}(*)$ is called the fiber, E is called the total space, and B is the base space. We write this as

$$F \rightarrow E \rightarrow B.$$

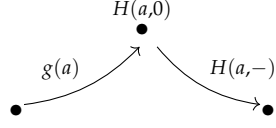
1.4.8 Example. Given a based space X , let

$$PX = \text{Hom}_*(I, X) = \{f: I \rightarrow X \mid f(0) = *\}$$

be the space of paths starting at the base-point. Then $PX \xrightarrow{p_1} X$ is a fibration with fiber ΩX , the loop space in X (i.e., $f(0) = f(1) = *$). To see this, consider our test diagram, where we must show that \tilde{H} exists:

$$\begin{array}{ccc} A & \xrightarrow{g} & PX \\ i_0 \downarrow & \nearrow \exists \tilde{H} & \downarrow p_1 \\ A \times I & \xrightarrow{H} & X \end{array}$$

Note that for each $a \in A$, $g(a)$ is a path in X which ends at $p_1 g(a) = H(a, 0)$. This point is the start of the path $H(a, -)$.



We will define $\tilde{H}(a, s)(t)$ to be a path running along $g(a)$ and then part-way along $H(a, -)$ ending at $H(a, s)$. In symbols,

$$\tilde{H}(a, s)(t) = \begin{cases} g(a)((1+s)t) & 0 \leq t \leq 1/(1+s) \\ H(a, (1+s)t - 1) & 1/(1+s) \leq t \leq 1. \end{cases}$$

Then $\tilde{H}(a, 0) = g(a)$ and $p_1 \tilde{H}(a, s) = \tilde{H}(a, s)(1) = H(a, s)$, as required.

The same argument shows that there is a fibration

$$p_* Y \rightarrow Y^I \xrightarrow{p_1} Y$$

where $p_* Y$ is the space of paths with end-point $*$.

1.4.9 Definition. Given $f: X \rightarrow Y$ the mapping path space P_f (or mapping cocylinder), is the pullback of f along $Y^I \xrightarrow{p_1} Y$, i.e.,

$$\begin{array}{ccc} P_f & \xrightarrow{\quad} & Y^I \\ p' \downarrow & \lrcorner & \downarrow p_1 \\ X & \xrightarrow{\quad f \quad} & Y \end{array}$$

Note that $P_f \simeq X$.

1.4.10 Proposition. The map $p: P_f \rightarrow Y$ given by $p(x, \alpha) = \alpha(1)$ is a fibration.

Proof. This is very similar to Example 1.4.8. Our test diagram is the following:

$$\begin{array}{ccc} A & \xrightarrow{g} & P_f \\ i_0 \downarrow & \nearrow \exists \tilde{H} & \downarrow p \\ A \times I & \xrightarrow{H} & Y \end{array}$$

Note that $g(a) \in P_f \subset X \times Y^I$, so we can write $g(a) = (g_1(a), g_2(a))$. Here $g_1(a)$ maps via f to the starting point of the path $g_2(a)$ and the commutativity of the diagram implies that the endpoint of the path $g_2(a)$ is the starting point of $H(a, -)$. The lift \tilde{H} will have two components. The x component will be constant in s , i.e., $\tilde{H}_1(a, s) = g_1(a)$. Overall, we define

$$\tilde{H}(a, s) = (g_1(a), \tilde{H}_2(a, s)(-)) \in P_f$$

where⁸

$$\tilde{H}_2(a, s)(t) = \begin{cases} g_2(a)((1+s)t) & 0 \leq t \leq 1/(1+s) \\ H(a, (1+s)t - 1) & 1/(1+s) \leq t \leq 1. \end{cases}$$

One check directly that $\tilde{H}(a, s)$ has the required properties. \square

⁸ Compare this to the formula in Example 1.4.8.

As with the homotopy extension property, we have a universal test space. The details (which are dual to Proposition 1.3.8) are left to the reader.

1.4.11 Proposition. *Let $f: E \rightarrow B$ be a continuous map, then f is a fibration if and only if there exists $s: P_f \rightarrow E^I$ making the following diagram commute:*

$$\begin{array}{ccccc}
 P_f & & & \xrightarrow{\pi_E} & E \\
 & \searrow \exists s & & \searrow & \downarrow f \\
 & & E^I & \xrightarrow{ev_0} & E \\
 \pi_{B^I} \swarrow & & \downarrow f_* & & \downarrow f \\
 & & B^I & \xrightarrow{ev_0} & B
 \end{array}$$

where π_{B^I} and π_E are the projection maps coming from the construction of P_f as a pullback.

1.4.12 Remark. One property of cofibrations that does not dualize to fibrations is that cofibrations are inclusions, but fibrations need not be surjective. Indeed, given $p: E \rightarrow B$ a fibration, then the composite

$$E \xrightarrow{p} B \hookrightarrow B \coprod *$$

is also a fibration, but is not surjective.

1.4.13 Remark. We will want to talk about exact sequences where the terms appearing may not have a group structure, but are rather only sets with base-points. Therefore, given a sequence of functions

$$A \xrightarrow{f} B \xrightarrow{g} C$$

of sets with base-points, we say that this is exact at B if $f(A) = g^{-1}(c_0)$ where c_0 is the base-point of C . Note that if A, B, C are groups with base-points the identity elements of the group, then exactness of sets corresponds to exactness of groups.

1.4.14 Theorem. *Let $p: E \rightarrow B$ be a fibration with fiber F and B path-connected. Let Y be any space, then*

$$[Y, F] \xrightarrow{i_*} [Y, E] \xrightarrow{p_*} [Y, B]$$

is exact.

Proof. For one direction, it is clear that $p_*(i_*[g]) = 0$.

Suppose $f \in [Y, E]$ is such that $p_*[f] = [\text{const}]$, i.e., $p \circ f$ is null-homotopic. Let $G: Y \times I \rightarrow B$ be a null-homotopy, and let $H: Y \times I \rightarrow E$ be a solution to the lifting problem indicated in the following diagram, using that p is a fibration:

$$\begin{array}{ccc}
 Y \times \{0\} & \xrightarrow{f} & E \\
 i_0 \downarrow & \nearrow H & \downarrow p \\
 Y \times I & \xrightarrow{G} & B
 \end{array}$$

Note now that $p \circ H(y, 1) = G(y, 1) = b_0$, so that $H(y, 1) \in F := p^{-1}(b_0)$. It follows that $[f] = i_*[H(-, 1)]$. \square

We have an analogous result for cofibration.

1.4.15 Theorem. *Let $i: A \rightarrow X$ be a cofibration, and $q: X \rightarrow X/A$ the quotient map. Let Y be any path-connected space, then the sequence of pointed sets*

$$[X/A, Y] \xrightarrow{q^*} [X, Y] \xrightarrow{i^*} [A, Y]$$

is exact.

Proof. Again, one inclusion is clear: we have $i^*(g^*([g])) = [g \circ q \circ i] = [\text{const}]$.

Now suppose that $f: X \rightarrow Y$ is a map with $f|_A: A \rightarrow Y$ null-homotopic. Let $h: A \times I \rightarrow Y$ be a hull-homotopy, and let $F: X \times I \rightarrow Y$ be the extension as shown in the following diagram:

$$\begin{array}{ccc} A & \xrightarrow{i_0} & A \times I \\ i \downarrow & & \downarrow i \times \text{id} \\ X & \xrightarrow{i_0} & X \times I \end{array} \quad \begin{array}{c} \searrow H \\ \downarrow F \\ \searrow f \end{array} \quad \begin{array}{c} \\ \\ Y \end{array}$$

Let $f' := F(-, 1)$. Then, $f \sim f'$ and $f'(A) = F(A, 1) = y_0$. By the universal property of the quotient, we can find $g: X/A \rightarrow Y$ making the following diagram commute:

$$\begin{array}{ccc} X & \xrightarrow{f'} & Y \\ q \downarrow & \nearrow g' & \\ X/A & & \end{array}$$

Therefore $[f] = [f'] = q^*[g']$. \square

As an extension of Theorem 1.4.14 we have the following.

1.4.16 Theorem. *Given a (Serre) fibration $p: E \rightarrow B$, and base points $b \in B$ and $e \in F := p^{-1}(b)$, then there is an isomorphism $p_*: \pi_n(E, F, e) \xrightarrow{\cong} \pi_n(B, b)$ for all $n \geq 1$. Hence, if B is path-connected, there is a long exact sequence of homotopy groups*

$$\begin{aligned} \cdots \pi_n(F, e) \rightarrow \pi_n(E, e) &\xrightarrow{p_*} \pi_n(B, b) \rightarrow \pi_{n-1}(F, e) \rightarrow \cdots \\ &\cdots \rightarrow \pi_0(E, e) \rightarrow 0. \end{aligned}$$

Proof. We first show that p_* is surjective. Let $[f] \in \pi_n(B, b)$, represented by a map $f: (I^n, \partial I^n) \rightarrow (B, b)$. Note that $I^{n-1} \times \{0\} \subseteq \partial I^n$, so we can form the diagram

$$\begin{array}{ccc} I^{n-1} \times \{0\} & \xrightarrow{*} & E \\ \downarrow & \nearrow \tilde{f} & \downarrow p \\ I^n & \xrightarrow{f} & B \end{array}$$

where the lift \tilde{f} exists because p is a Serre fibration. Because $f(\partial I^n) = b$, we have $\tilde{f}(\partial I^n) \subseteq F$. So \tilde{f} represents an element of $\pi_n(E, F, e)$ with $p_*([\tilde{f}]) = [p \circ \tilde{f}] = [f]$.

To show injectivity, let $\tilde{f}_0, \tilde{f}_1: (I^n, \partial I^n, I^{n-1}) \rightarrow (E, F, e)$ be such that $p_*(\tilde{f}_0) = p_*(\tilde{f}_1)$. Let $H: (I^n \times I, \partial I^n \times I) \rightarrow (B, b)$ be a homotopy from $p\tilde{f}_0$ to $p\tilde{f}_1$. We can find a lift in the following diagram:

$$\begin{array}{ccc} W & \xrightarrow{f} & E \\ \downarrow & \nearrow \tilde{H} & \downarrow p \\ I^n \times I & \xrightarrow{H} & B \end{array}$$

where $W = I^n \times \{0\} \cup I^n \times \{1\} \cup \partial I^n \times I$, and f is \tilde{f}_0 on $I^n \times \{0\}$, \tilde{f}_1 on $I^n \times \{1\}$ and f is constant on $\partial I^n \times I$. The homotopy lifting property gives \tilde{H} defining a homotopy between \tilde{f}_0 and \tilde{f}_1 .

The result then follows (modulo some noise in the low homotopy groups, which can be checked by hand) from Theorem 1.2.21. \square

1.4.17 *Example* (Hopf fibrations). Let $\mathbb{F} = \mathbb{R}, \mathbb{C}$ or \mathbb{H} and fix an integer $d = 1, 2$ or 4 , respectively.

Let

$$\mathbb{F}^{n+1} = \begin{cases} \mathbb{R}^{n+1} & \mathbb{F} = \mathbb{R} \\ \mathbb{C}^{n+1} \cong \mathbb{R}^{2(n+1)} & \mathbb{F} = \mathbb{C} \\ \mathbb{H}^{n+1} \cong \mathbb{R}^{4(n+1)} & \mathbb{F} = \mathbb{H}. \end{cases}$$

In other words, $\mathbb{F}^{n+1} \cong \mathbb{R}^{d(n+1)}$. We define the $d(n+1) - 1$ dimensional sphere inside \mathbb{F}^{n+1} :

$$S^{d(n+1)-1} = \{(u_0, \dots, u_n) \mid u_i \in \mathbb{F}, \sum_{k=0}^n |u_k|^2 = 1\}.$$

We define the \mathbb{F} -projective space by

$$\mathbb{F}P^n := \mathbb{F}^{n+1} \setminus \{0\} / \sim$$

where $(u_0, \dots, u_n) \simeq (v_0, \dots, v_n)$ if and only if there exists $\lambda \in \mathbb{F} \setminus \{0\}$ such that $v_i = \lambda u_i$ for $i = 0, \dots, n$.

Now we have a map $\phi: S^{d(n+1)-1} \rightarrow \mathbb{F}P^n$ that sends (u_0, \dots, u_n) to its equivalence class $[u_0, \dots, u_n]$. Let $F = \phi^{-1}[1, \dots, 0] = \{(\lambda, 0, \dots, 0) \mid \lambda \in \mathbb{F}, |\lambda| = 1\} \cong S^{d-1}$.

We will see later in the course that $S^{d-1} \rightarrow S^{d(n+1)-1} \rightarrow \mathbb{F}P^n$ is a fibration. Explicitly, the fibrations are

$$\begin{aligned} S^0 &\rightarrow S^n \rightarrow \mathbb{R}P^n \\ S^1 &\rightarrow S^{2n+1} \rightarrow \mathbb{C}P^n \\ S^3 &\rightarrow S^{4n+3} \rightarrow \mathbb{H}P^n. \end{aligned}$$

The case $n = 1$ is of interest, as then projective spaces are just spheres, and we obtain the following Hopf fibrations

$$\begin{aligned} S^0 &\rightarrow S^1 \rightarrow S^1 \\ S^1 &\rightarrow S^3 \rightarrow S^2 \\ S^3 &\rightarrow S^7 \rightarrow S^4. \end{aligned}$$

There is also a fibration $S^7 \rightarrow S^{15} \rightarrow S^8$. It is a difficult theorem of Adams that these are the only fibrations between spheres.

1.5 The homotopy extension and lifting property

We recall that given $f: X \rightarrow Y$ we defined the mapping path space P_f in Definition 1.4.9, and that $P_f \rightarrow Y$ is a fibration.

1.5.1 Definition. The homotopy fiber F_f of $f: X \rightarrow Y$ is the fiber of the fibration $P_f \rightarrow Y$. This is well-defined up to homotopy.

The following is an extremely useful definition in homotopy theory; as we will see later, any weak equivalence between CW-complexes is in fact a homotopy equivalence.

1.5.2 Definition. A map $f: (X, x_0) \rightarrow (Y, y_0)$ is a weak equivalence if $f_0: \pi_0(X, x_0) \rightarrow \pi_0(Y, y_0)$ is a bijection and $f_*: \pi_k(X, x_0) \rightarrow \pi_k(Y, y_0)$ is an isomorphism for all $k \geq 1$.

1.5.3 Lemma. If $f: X \rightarrow Y$ is a weak-equivalence, then $\pi_k(F_f) = 0$ for all $k \geq 0$.

Proof. This follows from the long exact sequence of a fibration (Theorem 1.4.16). \square

1.5.4 Remark. We now make a series of remarks about a map $f: X \rightarrow Y$ with homotopy fiber F_f .

(i) A map $\phi: S^{n-1} \rightarrow F_f$ corresponds to a diagram

$$\begin{array}{ccc} S^{n-1} & \xrightarrow{g} & X \\ \downarrow & & \downarrow f \\ C(S^{n-1}) \cong D^n & \xrightarrow{h} & Y \end{array} \quad (1.5.5)$$

where g is the composite $S^{n-1} \xrightarrow{\phi} F_f \rightarrow X$ (use Lemma 1.1.11).

(ii) The boundary map $\pi_n(Y) \rightarrow \pi_{n-1}(F_f)$ in the long exact sequence corresponds to the map sending the class of $\bar{h}: S^n \rightarrow Y$ to the class of $\pi_{n-1}(F_f)$ represented by the diagram

$$\begin{array}{ccc} S^{n-1} & \xrightarrow{c} & X \\ \downarrow & & \downarrow f \\ D^n & \xrightarrow{h} & Y \end{array}$$

where $c = c_{x_0}$ is the constant map, and h is the composite $D^n \rightarrow D^n / S^{n-1} \cong S^n \xrightarrow{\bar{h}} Y$.

(iii) Similarly, the map $\pi_{n-1}(F_f) \rightarrow \pi_{n-1}(X)$ corresponds to sending the diagram (1.5.5) to the class $[g]$.

(iv) In particular, $\pi_{n-1}(F_f) = 0$ is equivalent to completing the

diagram (1.5.5) in the following way:

$$\begin{array}{ccc}
 S^{n-1} & \xrightarrow{g} & X \\
 \downarrow & \searrow & \uparrow G \\
 & C(S^{n-1}) & \\
 & \downarrow & \\
 & C(D^n) & \\
 \uparrow & \nearrow & \downarrow H \\
 D^n & \xrightarrow{h} & Y
 \end{array}$$

f

We can restate the last remark in the following lemma.

1.5.6 Lemma. Suppose $f: X \rightarrow Y$ is a map with homotopy fiber F_f . Then $\pi_{n-1}(F_f) = 0$ if and only if each diagram

$$\begin{array}{ccc}
 S^{n-1} \times \{1\} & \xrightarrow{g} & X \\
 \downarrow & & \downarrow f \\
 S^{n-1} \times I \cup D^n \times \{0\} & \xrightarrow{h} & Y
 \end{array}$$

can be completed to a diagram

$$\begin{array}{ccc}
 S^{n-1} \times \{1\} & \xrightarrow{g} & X \\
 \downarrow & \searrow & \uparrow G \\
 & D^n \times \{1\} & \\
 & \downarrow & \\
 & D^n \times I & \\
 \uparrow & \nearrow & \downarrow H \\
 S^{n-1} \times I \cup D^n \cup \{0\} & \xrightarrow{h} & Y
 \end{array}$$

f

Proof. For any disk we have a homeomorphism $CD^n \cong D^n \times I$ which sends the cone point to the center of $D^n \times \{1\}$, D^n to $S^{n-1} \times I \cup D^n \times \{0\}$, S^{n-1} to $S^{n-1} \times \{1\}$ and CS^{n-1} to $D^n \times \{1\}$. Thus the statement follows from the last part of the remark. \square

This extends to relative CW-complexes.⁹

⁹ The following result is perhaps difficult to remember, but very useful!

1.5.7 Theorem (Homotopy extension and lifting property (HELP)). Let (X, A) be a relative CW-pair and $f: Y \rightarrow Z$ a weak equivalence. Then every diagram

$$\begin{array}{ccc}
 A \times \{1\} & \xrightarrow{g} & Y \\
 \downarrow & & \downarrow f \\
 A \times I \cup X \times \{0\} & \xrightarrow{h} & Z
 \end{array}$$

can be completed to a diagram

$$\begin{array}{ccccc}
 A \times \{1\} & \xrightarrow{g} & & Y & \\
 \downarrow & \searrow & \nearrow G & \downarrow f & \\
 & X \times \{1\} & & & \\
 & \downarrow & & & \\
 & X \times I & \searrow H & & \\
 A \times I \cup X \times \{0\} & \xrightarrow{h} & & Z &
 \end{array}$$

Proof. The proof is by induction over the n -skeleton, with the base case being straightforward. For the inductive step, one reduces to attaching a single cell using the diagram

$$\begin{array}{ccccc}
 X_{n-1} \times \{1\} & \xrightarrow{G_{n-1}} & & Y & \\
 \downarrow & \searrow & \nearrow G_n & \downarrow f & \\
 & X_n \times \{1\} & & & \\
 & \downarrow & & & \\
 & X_n \times I & \searrow H_n & & \\
 X_{n-1} \times I \cup X_n \times \{0\} & \xrightarrow{H_{n-1} \cup h} & & Z &
 \end{array}$$

□

1.5.8 Remark. If (X, A) is a relative CW-complex of dimension n , and $f: Y \rightarrow Z$ is an n -equivalence¹⁰, the same argument goes through to show that the conclusion of HELP also holds in this case.

¹⁰ That is, $f_*: \pi_m(X) \rightarrow \pi_m(Y)$ is an isomorphism for $m < n$ and an epimorphism for $m = n$.

Exercise 11

Show that if $f = \text{id}$ in HELP, then we recover the homotopy extension property.

Our first application of this will be Whitehead's theorem. We start with the following lemma.

1.5.9 Lemma. For any weak equivalence $f: Y \rightarrow Z$ and any CW-complex X , the induced map $f_*: [X, Y] \rightarrow [X, Z]$ is a bijection.

Proof. We first show surjectivity. The pair $X = (X, \emptyset)$ is a relative CW-complex, and so we can apply HELP. Then, for any $h: X \rightarrow Z$

we have a diagram

$$\begin{array}{ccc}
 * & \xrightarrow{\quad} & Y \\
 \downarrow & \searrow & \uparrow G \\
 & X \times \{1\} & \\
 & \downarrow & \\
 & X \times I & \\
 \swarrow & \searrow H & \\
 X \times \{0\} & \xrightarrow{h} & Z
 \end{array}$$

f (vertical arrow from Y to Z)

The homotopy $H: X \times I \rightarrow Z$ satisfies $H_0 = h$ and $H_1 = f \circ G$.

Therefore, $[h] = f_*[G]$.

Now assume that $g_0, g_1 \in [X, Y]$ with $f_*[g_0] = f_*[g_1]$. Let $F: X \times I \rightarrow Z$ be a homotopy between $f \circ g_0$ and $f \circ g_1$. Consider the pair $(X \times I, X \times \partial I)$. This is a relative CW-pair, and HELP gives a diagram

$$\begin{array}{ccc}
 X \times \partial I \times \{1\} & \xrightarrow{g} & Y \\
 \downarrow & \searrow & \uparrow G \\
 & X \times I \times \{1\} & \\
 & \downarrow & \\
 & X \times I \times J & \\
 \swarrow & \searrow H & \\
 X \times \partial I \times J \cup X \times I \times \{0\} & \xrightarrow{h} & Z
 \end{array}$$

f (vertical arrow from Y to Z)

Here $g: X \times \partial I \rightarrow Y$ sends (X, ν) to $g_\nu(x)$ for $\nu = 1, 2$ and $h: X \times \partial I \times J \rightarrow Z$ sends (x, ν, s) to $f \circ g_\nu(x)$. The lift $G: X \times I \rightarrow Y$ gives a homotopy between g_0 and g_1 , i.e., $[g_0] = [g_1]$, and so f_* is injective. \square

1.5.10 Theorem (Whitehead theorem). *If $f: X \rightarrow Y$ is weak-equivalence between CW-complexes, then it is a homotopy equivalence.*

Proof. Suppose $f: X \rightarrow Y$ is a weak equivalence, so $f_*: [Y, X] \xrightarrow{\cong} [Y, Y]$. In other words, there exists a $g: Y \rightarrow X$ such that $f_*[g] = [f \circ g] = [\text{id}_Y]$, i.e., $f \circ g \simeq \text{id}_Y$. Then, $f \circ g \circ f \simeq f$ as well. But, we also have $f_*: [X, X] \xrightarrow{\cong} [X, Y]$, which sends id_X to f and $g \circ f$ to $f \circ g \circ f \simeq f$. Therefore, $\text{id}_X \simeq g \circ f$, and so $X \simeq Y$. \square

1.5.11 Corollary. *If X is a CW-complex with $\pi_i(X) = 0$ for all i , then X is contractible.*

Proof. Apply Whitehead's theorem to the unique map $X \rightarrow *$. \square

1.5.12 Remark. We cannot drop any assumptions from this theorem, as the following examples show:

- (i) We must have a map inducing the weak equivalence; the homotopy groups cannot be abstractly isomorphic, e.g., consider $\mathbb{R}P^2 \times S^3$ and $\mathbb{R}P^3 \times S^2$.

- (ii) The Warsaw circle¹¹ is an example of a space with $\pi_n X = 0$ for all n , but for which X is not contractible.

¹¹ See, for example, <https://wildtopology.com/bestiary/warsaw-circle/>

A

A nice category of topological spaces

A.1 The compact open topology

In this appendix we briefly discuss how to give the set of continuous maps between topological spaces X and Y a topology, such that the product is left adjoint to the Hom functor. To begin, we fix some notation.

A.1.1 Remark. Let X and Y be topological spaces. Let $M(X, Y)$ denote the set of continuous homomorphisms from X to Y . There is an evaluation map

$$e': \text{Hom}_{\text{Sets}}(X, Y) \times X \rightarrow Y$$

given by $e'(f, x) = f(x)$. This restricts to a function

$$e: M(X, Y) \times X \rightarrow Y.$$

A.1.2 Definition. A topology on $M(X, Y)$ is called admissible if e is continuous with respect to this topology.

A.1.3 Remark. It is possible that $M(X, Y)$ has no admissible topologies.

A.1.4 Definition. The compact-open topology on $M(X, Y)$ has as a sub-base the family of sets

$$U^K = \{f \in M(X, Y) \mid f(K) \subseteq U\}$$

where $K \subseteq X$ is compact and U is open in Y .

A.1.5 Proposition. *If X is a locally compact¹ Hausdorff space, the compact-open topology on $M(X, Y)$ is admissible.*

¹ i.e., every point in X has a compact neighborhood

A.1.6 Remark. The compact-open topology is the coarsest admissible topology: for any admissible topology τ we have $\tau_{\text{co}} \subseteq \tau$.

A.1.7 Remark. Suppose we have sets X, Y, Z . Then there is an adjoint equivalence

$$\text{Hom}_{\text{Sets}}(X \times Y, Z) \xrightleftharpoons[\psi]{\phi} \text{Hom}_{\text{Sets}}(X, \text{Hom}_{\text{Sets}}(Y, Z))$$

given by

$$\phi(f)(x)(y) = f(x, y) \quad \text{and} \quad \psi(g)(x, y) = g(x)(y).$$

A.1.8 Proposition. *If X, Y, Z are topological spaces with Y Hausdorff, locally compact, then*

$$\phi: M(X \times Y, Z) \xrightarrow{\cong} M(X, M(Y, Z))$$

is an isomorphism of sets. If X is Hausdorff, then it is a homeomorphism (using the compact-open topology).