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MA8403 - Equivariant homotopy theory

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Preface

This are the courses notes for MA8403 - Equivariant homotopy theory, held during the Autumn semester 2023 at NTNU. The notes are mainly based on two excellent sets of lectures notes, one by Guillou [2], and one by Blumberg [1]. The notes will be continually updated during the semester.

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Chapter 1

Equivariance in algebra

Group actions in algebra

We recall that if X is an object in a category $\mathcal{C}$, then the set of endomorphisms $\mathrm{End}(X)$ forms a monoid (that is, a set equipped with an associative binary operation and an identity element) under composition. The set of automorphisms of X (that is, those endomorphisms that are invertible) form a group. Moreover, we have

$$\mathrm{Aut}(X) = \mathrm{End}(X) \cap \mathrm{Iso}(\mathcal{C})$$

Note that any group is a monoid, simply by forgetting the existence of inverses.

Definition 1.1. An action of a group G on an object $X \in \mathcal{C}$ is a monoid homomorphism $a\colon G \to \mathrm{End}(X)$, or equivalently a group homomorphism $a\colon G \to \mathrm{Aut}(X)$ (note that a monoid homomorphism between groups is a group homomorphism).

Remark 1.2. Unwinding the definition, this means that we have:

a. For each $g \in G$, there is a morphism $a(g)\colon G \to G$.

b. a preserves composition, i.e., $a(g \cdot h) = a(h) \cdot a(h)$.

c. a preserves identifies, so $a(e) = \mathrm{id}_X$.

Example 1.1

Let $\mathcal{C}$ be the category of sets and functions. Then, $a\colon G \to \mathrm{End}(X) = \left\{ f\colon X \to X \right\}$ correspond to a function $\overline{a}\colon G \times X \to X$. The conditions above mean that the diagrams

$$\begin{tikzcd} G \times G \times X \arrow[r, "m \times \mathrm{id}"] \arrow[d, "\mathrm{id} \times \overline{a}"'] & G \times X \arrow[d, "\overline{a}"] \\ G \times X \arrow[r, "\overline{a}"'] & X \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \{*\} \times X \arrow[r, "e \times \mathrm{id}"] \arrow[d, "\cong"'] & G \times X \arrow[d, "\overline{a}"] \\ X \arrow[r, equal] & X \end{tikzcd}$$

commute, where $m\colon G \times G \to G$ denotes the group multiplication. Equivalently, in symbols, we have

$$\overline{a}(h, \overline{a}(h, x)) = \overline{a}(gh, x)$$

and

$$\overline{a}(e, x) = x.$$

Remark 1.3. Let BG denote the category with one object $*$ and with $\mathrm{Hom}(*, *) = G$. Then an action of G in the category $\mathcal{C}$ is the same as a functor $\rho\colon BG \to \mathcal{C}$. The object X in the previous definition is the object $\rho(*) \in \mathcal{C}$.

Remark 1.4. Let $\mathcal{C} = \mathrm{Mod}_R$ for a commutative ring R. An action of G on $M \in \mathrm{Mod}_R$ is a monoid homomorphism

$$a\colon G \to \mathrm{Hom}_R(M, M).$$

We recall that $\mathrm{Hom}_R(M, M)$ actually has the structure of an R-algebra

Definition 1.5. The (R-linear) group ring on R is the R-algebra $R[G]$ whose:

(a) underlying R-module is the free R-module with basis on the underlying set of G.

(b) whose multiplication is given on basis elements by the group operation.

Example 1.2

Let $R = \mathbb{Z}$ and $G = C_2 = \langle \sigma \rangle$. An element of $\mathbb{Z}[C_2]$ is of the form $a + b\sigma$ where $a, b \in \mathbb{Z}$. Multiplication is given by

$$(a_1 \cdot 1 + b_1\sigma) \cdot (a_2 \cdot 1 + b_2\sigma) = (a_1a_2 + b_1b_2) \cdot 1 + (a_1b_2 + b_1a_2)\sigma.$$

This is the same thing as the polynomial ring $\mathbb{Z}[\sigma]/(\sigma^2 - 1)$.

Remark 1.6. Categorically, the group ring construction is left adjoint to the functor that takes an R-algebra to its group of units, i.e., there is an adjudication

$$R[-]\colon \mathrm{Grp} \leftrightarrows \mathrm{Mod}_R\colon (-)^\times$$

Returning to group actions, we have the following:

Proposition 1.7. *Let R be a commutative ring, and G a finite group. The following data on an R-module M are equivalent:*

a. *A monoid homomorphism $G \to \mathrm{End}_R(M)$.*

b. *A group homomorphism $G \to \mathrm{Aut}_R(M)$.*

c. *A homomorphism of R-algebras $R[G] \to \mathrm{Hom}_R(M, M)$.*

d. *An $R[G]$-module structure on M whose underlying R-module structure is M.*

Definition 1.8. A representation of G over R is an $R[G]$-module.

Example 1.3

If $R = k$ is a field, then the underlying R-module is a k-vector space V. If $\dim_k(V) = n$, then $\mathrm{Aut}_k(V) = GL_n(k)$, and a k-representation is the same thing as a group homomorphism $G \to GL_n(k)$.

Definition 1.9. The $R[G]$-module $R[G]$ is known as the regular representation. More generally, if X is a finite G-set, then the free R-module $R[X]$ inherits the structure of a $R[G]$-module (the case of $R[G]$ itself corresponds to the finite G-set G, considered as an $R[G]$-module over itself). Representations obtained this way are known as permutation representations.

Example 1.4

Taking X to be the trivial G-set, we obtain the (one-dimensional) trivial representation. This is simply the $R[G]$-module R, where G acts trivially.

Definition 1.10. Let $G = C_2 = \langle \tau \rangle$, and suppose that $-1 \neq 1 \in R$. Then the sign representation of G is the one-dimensional representation where τ acts as -1 (if $-1 = 1$ in R this still makes sense, but is just the trivial representation). Note that this is an example of a representation that is not a permutation representation.

Example 1.5

Let $C_n = \langle \sigma \rangle$ be the cyclic group of order n. Let us calculate all complex 1-dimensional representations of C_n, i.e., homomorphisms $\rho \colon C_n \to \mathbb{C}$. Note that if we define $\rho(\sigma) = c$, then $\rho(\sigma^n) = c^n = 1$, so that c must be an n-th root of unity. There are precisely n-of these (take $\zeta_n = e^{2\pi/ni}$), and so there are precisely n-representations. For example, when $G = C_4$, the four representations correspond to sending σ to either $1, i, -1$ or $-i$. Note that we can also consider these as 2-dimensional *real* representations.

Notation 1.11. We let $\rho = \rho_G$ denote the regular representation of G, and the trivial n-dimensional representation by $\mathbf{n} = R^{\oplus n}$.

Definition 1.12. A subrepresentation is a submodule.

Example 1.6

The regular representation always has a one-dimensional trivial representation, generated by the sum $\sum_{g \in G} g$.

Definition 1.13. A representation V is irreducible if the only subrepresentations of V are 0 and V.

Theorem 1.14 (Maschke)**.** *Suppose that k is a field of characteristic not dividing $|G|$. Then every representation splits as a direct sum of irreducible representations.*

Proof. We prove the following: if $V \subseteq W$ is a subrepresentation, then there exists $U \subseteq W$ such that $U \oplus V \simeq W$.

To see this, let $\pi\colon W \to W$ be any k-linear projection of W onto V. This map need not be G-equivariant, but we can make it so by 'averaging'. That is, we define a new map $\phi\colon W \to W$ by

$$\phi(\mathbf{w}) = \frac{1}{|G|} \sum_{g \in G} g \cdot \pi(g^{-1} \cdot \mathbf{w}).$$

Moreover the map is G-equivariant: for $h \in G$ we have

$$\begin{aligned} \phi(h \cdot \mathbf{w}) &= \frac{1}{|G|} \sum_{g \in G} g \cdot \pi(g^{-1} \cdot h \cdot \mathbf{w}) \\ &= \frac{1}{|G|} \sum_{u \in G} u \cdot h \cdot \pi(u^{-1} \cdot \mathbf{w}) \\ &= h \cdot \phi(\mathbf{w}), \end{aligned}$$

where $u = gh^{-1}$, and so ϕ is $k[G]$-linear. Furthermore, the map ϕ is the identity on V By the splitting lemma, $W = V \oplus \ker(\phi)$. □

Remark 1.15. We used the assumption on k to ensure that we could divide by $|G|$. Without that assumption, the theorem is false. Indeed, let $G = C_2$, $R = \mathbb{F}_2$ and consider the representation defined by $\rho(\tau) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. This is not irreducible, but does not split as a direct sum of indecomposable representations.

Corollary 1.16. *Suppose that k is a field of characteristic not dividing $|G|$. If V is an irreducible representation, then V is isomorphism to a subgroup of $k[G]$ (slogan: all irreducibles are submodules of the regular representation).*

Proof. Let $\mathbf{v} \in V$ be non-trivial. Then the homomorphism $\phi\colon k[G] \to V$ given by sending 1 to $\mathbf{v}$ must be surjective, because V is irreducible. Let $U = \ker(\phi)$, then apply Maschke's theorem. □

Example 1.7

Let $G = C_2 = \langle \tau \rangle$, and k a field of characteristic not equal to 2. We have the trivial representation $\mathbf{1}$ and the sign representation $\mathbf{1}_{\text{sgn}}$. Then, $\mathbf{1}$ is generated by the sum $1 + \tau$, while $\mathbf{1}_{\text{sgn}}$ is generated by $1 - \tau$, and we deduce that

$$\rho_{C_2} = \mathbf{1} \oplus \mathbf{1}_{\text{sgn}}.$$

The representation ring

Let k be a field, and suppose that V and W are G-representations, then the k-linear tensor sum $V \oplus W$ can be given the structure of a $k[G]$-module, by taking the diagonal G-action. If we think of a representation in terms of a homomorphism $\rho\colon G \to GL_n(k)$, then this direct sum corresponds to the 'block sum'

$$G \to GL_n(k) \times GL_m(k) \to GL_{n+m}(k)$$

Similarly, we can define a tensor product of representations using the 'Kronecker tensor product' (or matrix direct product). Equivalently, this is the k-linear tensor product $V \otimes W$ with the g-action defined on simple tensors by

$$g \cdot (\mathbf{v} \otimes \mathbf{w}) = g \cdot \mathbf{v} \otimes g\mathbf{w}.$$

We leave it for the reader to verify the following straightforward computations:

(a) $\mathbf{1} \otimes V \cong V \cong V \otimes \mathbf{1}$.

(b) $\mathbf{n} \otimes V \cong V^{\oplus n} \cong V \otimes \mathbf{n}$.

Example 1.8

Let us compute the tensor product $\mathbf{1}_{\text{sgn}} \otimes \mathbf{1}_{\text{sgn}}$. The underlying vector space is simply $k \otimes k \cong k$, while τ acts as $\tau \cdot (1 \otimes 1) = (\tau \cdot 1) \otimes (\tau \cdot 1) = -1 \otimes -1 = 1 \otimes 1$. So the tensor product $\mathbf{1}_{\text{sgn}} \otimes \mathbf{1}_{\text{sgn}} = \mathbf{1}$.

Example 1.9

Take $G = C_3$ and $k = \mathbb{R}$. We have a two-dimensional representation λ_3 corresponding to rotation by an angle of $2\pi/3$. What is $\lambda_3 \otimes \lambda_3$? This is a 4-dimensional representation, and by working out all irreducible representations must be either $\mathbf{4}, \mathbf{2} \oplus \lambda_3$ or $\lambda_3 \oplus \lambda_3$. If you know a little bit of character theory, you can see that it must be $\mathbf{2} \oplus \lambda_3$: we have

$$\chi_{\lambda_3 \otimes \lambda_3}(1) = 4, \quad \chi_{\lambda_3 \otimes \lambda_3}(\tau) = 1$$

$$\begin{aligned} \chi_{\mathbf{4}}(1) &= 4, & \chi_{\mathbf{4}}(\tau) &= 1 \\ \chi_{\mathbf{2} \oplus \lambda_3}(1) &= 4, & \chi_{\mathbf{2} \oplus \lambda_3}(\tau) &= 1 \\ \chi_{\lambda_3 \oplus \lambda_3}(1) &= 4, & \chi_{\lambda_3 \oplus \lambda_3}(\tau) &= -2 \end{aligned}$$

Remark 1.17. By passing to isomorphism classes of representations, the set of finite dimensional representations has the structure of a semiring. Using the Grothendieck construction, we can produce a commutative ring.

Definition 1.18. For a finite group G the real representation ring $RO(G)$ is the Grothendieck group of the above semi-ring. Explicitly,

$$RO(G) \coloneqq \mathbb{Z}\left\{\begin{matrix}\text{isomorphism classes of finite-}\\ \text{dimensional real } G\text{-representations}\end{matrix}\right\} / \langle [V \oplus W] - [V] - [W] \rangle.$$

Remark 1.19. As an abelian group, $RO(G)$ is a direct sum of copies of $\mathbb{Z}$, with rank equal to the number of isomorphism classes of irreducible representations.

Remark 1.20. We can make the same definition for other fields, for example when $k = \mathbb{C}$ we get the complex representation ring $R(G)$.

Example 1.10

We have $RO(C_2) \cong \mathbb{Z}\{\mathbf{1}\} \oplus \mathbb{Z}\{\mathbf{1}_{\mathrm{sgn}}\}$. The ring structure is determined by Example (1.8): we have $RO(C_2) \cong \mathbb{Z}[\sigma]/(\sigma^2 - 1)$. The same is true for the complex representation ring. Note that this is the same as $\mathbb{Z}[C_2]$. In fact, the complex representation ring of a finite abelian group is always (non-canonically) isomorphic to the group ring: it is the group ring of the character group.

Example 1.11

When $G = C_3$ we have that

$$RO(C_3) = \mathbb{Z}\{\mathbf{1}\} \oplus \mathbb{Z}\{\lambda_3\}.$$

By Example (1.9) we have $[\lambda_3]^2 = 2 + [\lambda_3]$ and we see that

$$RO(C_3) \cong \mathbb{Z}[\lambda]/(\lambda^2 - \lambda - 2).$$

On the other hand, the complex representation ring is given by

$$R(C_3) \cong \mathbb{Z}[\zeta]/(\zeta^3 - 1).$$

By tensoring a real representation with $\mathbb{C}$, there is a map

$$RO(C_3) \to R(C_3)$$

given by $\lambda \mapsto \zeta + \zeta^2$.

Definition 1.21. Let $\phi \colon H \to G$ be a morphism of groups, then the pullback $\phi^*(V)$ of a G-representation is the $k[H]$-module induced by restriction of scalars along $k[H] \to k[G]$. Equivalently, it is the representation given by the composite $H \to G \xrightarrow{a} \mathrm{End}(V)$.

Remark 1.22. We have

$$\phi^*(V \oplus W) = \phi^*(V) \oplus \phi^*(W) \quad \text{and } \phi^*(V \otimes W) \cong \phi^*(V) \otimes \phi^*(W).$$

Therefore we deduce:

Corollary 1.23. *A group homomorphism* $\phi\colon H \to G$ *induces a ring homomorphism* $\phi^*\colon RO(G) \to RO(H)$.

Example 1.12

Consider the group homomorphism $\phi\colon G \to G/G \cong e$. Then we have $\mathbf{n} = \phi^*(\mathbf{k}^k)$.

Definition 1.24. An *injective* group homomorphism $\iota\colon H \hookrightarrow G$ gives rise to a restriction functor for representations, which we denote by Res^G_H.

Example 1.13

Let $G = C_4 = \langle r \rangle$ and $H = C_2 \subseteq C_4$ the subgroup generated by r^2, and take $k = \mathbb{R}$. Pulling back the sign representation of C_2 along the quotient $C_4 \twoheadrightarrow C_2$ gives rise to the sign representation σ. This is one of three irreducible C_4 real representations: we have the trivial representation $\mathbf{1}$ and the rotation representation λ_4. The inclusion $H \hookrightarrow G$ gives a morphism

$$RO(C_4) = \mathbb{Z}\{\mathbf{1}\} \oplus \mathbb{Z}\{\sigma\} \oplus \mathbb{Z}\{\lambda_4\} \to \mathbb{Z}\{\mathbf{1}\} \oplus \mathbb{Z}\{\mathbf{1}_{\mathrm{sgn}}\} = RO(C_2).$$

This map is determined by

$$\mathbf{1} \mapsto \mathbf{1}, \quad \sigma \mapsto \mathbf{1}, \quad \lambda_4 \mapsto 2 \cdot \mathbf{1}_{\mathrm{sgn}},$$

where the image of σ is determined from its definition as the pull-back, and the image of λ_4 comes from the fact that r acts as multiplication by $\pi/4$ and so r^2 acts as multiplication by -1, and so restricts to a 2-dimensional sign representation $\mathbf{1}_{\mathrm{sgn}} \oplus \mathbf{1}_{\mathrm{sgn}}$. We can conclude that $\lambda_4^2 \mapsto (2 \cdot \mathbf{1}_{\mathrm{sgn}})^2 = \mathbf{4}$, so that λ_4^2 must be either $\mathbf{4}, \mathbf{3} \oplus \sigma, \mathbf{2} \oplus 2\sigma, \mathbf{1} \oplus 3\sigma$ or 4σ.

There is also another functor, which will turn out to be adjoint to restriction.

Definition 1.25. Given $H \leq G$, and a H-representation V, we define the induced representation Ind^G_H is be the tensor product $k[G] \otimes_{k[H]} V$.

Remark 1.26. Induction plays well with direct sums: we have $\mathrm{Ind}^G_H(V \oplus W) \cong \mathrm{Ind}^G_H(V) \oplus \mathrm{Ind}^G_H(W)$. But a dimension check shows that it does not commute with tensor products. Therefore, we get an induced map of abelian groups, but *not* of commutative rings

$$\mathrm{Ind}^G_H\colon RO(H) \to RO(G)$$

Directly from the definition we deduce the following:

Lemma 1.27. *If $K \leq H \leq G$ then $\mathrm{Ind}_H^G \mathrm{Ind}_K^H V$ for any K-representation V.*

Example 1.14

The regular representation $\rho_G = k[G] \cong k[G] \otimes_{k[G]} k \simeq \mathrm{Ind}_e^G(\mathbf{1})$. More generally, we have $\mathrm{Ind}_H^G \rho_H \cong \rho_G$.

Example 1.15

Let $C_2 \subseteq C_4$ and $k = \mathbb{R}$. What is $\mathrm{Ind}_{C_2}^{C_4}(\mathbf{1})$? We have

$$\mathrm{Ind}_{C_2}^{C_4}(\mathbf{1}) = \mathbb{R}[C_4] \otimes_{\mathbb{R}[C_2]} \mathbb{R} \cong \mathbb{R}[C_4/C_2] \cong \phi^*(\rho_{C_2})$$

for $\phi\colon C_4 \to C_4/C_2 \cong C_2$ the quotient map.[a] This means that

$$\mathrm{Ind}_{C_2}^{C_4}(\mathbf{1}) = \mathbf{1} \oplus \sigma$$

To work out $\mathrm{Ind}_{C_2}^{C_4}(\mathbf{1}_{\mathrm{sgn}})$ we have that

$$\begin{aligned}\rho_{C_4} \cong \mathrm{Ind}_{C_2}^{C_4}(\rho_{C_2})) = \mathrm{Ind}_{C_2}^{C_4}(\mathbf{1} \oplus \mathbf{1}_{\mathrm{sgn}}) &\cong \mathrm{Ind}_{C_2}^{C_4}(\mathbf{1}) \oplus \mathrm{Ind}_{C_2}^{C_4}(\mathbf{1}_{\mathrm{sgn}}) \\ &\cong 1 \oplus \sigma \oplus \mathrm{Ind}_{C_2}^{C_4}(\mathbf{1}_{\mathrm{sgn}}).\end{aligned}$$

By $\rho_{C_4} \cong \mathbf{1} \oplus \sigma \oplus \lambda_4,$. To see this, once can note that

$$\mathbb{R}[\mathbb{Z}/4] \cong \mathbb{R}[X]/(X^4 - 1) \cong \mathbb{R}[X]/\prod_{d|4} \Phi_d \cong \bigoplus_{d|4} \mathbb{R}[X]/\Phi_d$$

so that

$$\mathbb{R}[\mathbb{Z}/4] \cong \mathbb{R}[X]/(X-1) \oplus \mathbb{R}[X]/(X+1) \oplus \mathbb{R}[X]/(X^2-1).$$

Hence, $\mathrm{Ind}_{C_2}^{C_4}(\mathbf{1}_{\mathrm{sgn}}) \cong \lambda_4$.

We deduce that the map

$$\mathrm{Ind}_{C_2}^{C_4}\colon RO(C_2) \to RO(C_4)$$

is determined by

$$\mathbf{1} \mapsto \mathbf{1} \oplus \sigma \quad \text{and} \quad \mathbf{1}_{\mathrm{sgn}} \mapsto \lambda_4.$$

[a] This always works: For $H \leq G$ a normal subgroup we have $\mathrm{Ind}_H^G(\mathbf{1}) = \phi^*(\rho_{G/H})$.

In general, if we have commutative rings R and S and a ring map $f\colon R \to S$, then we can define induction and restriction between R-modules and S-modules and induction is left adjoint to restriction. As a special case, we have:

Lemma 1.28. *If $H \leq G$, then induction is left adjoint to restriction.*

Proposition 1.29 (The projection formula)**.** *Let $H \leq G$, then there is a natural equivalence*

$$\mathrm{Ind}_H^G(\mathrm{Res}_H^G(V) \otimes W) \xrightarrow{\sim} V \otimes \mathrm{Ind}_H^G(W)$$

for $V \in RO(G)$ and $W \in RO(G)$

Proof. We first construct the map: by adjunction, such a map is equivalent to a H-equivariant map

$$\mathrm{Res}_H^G(V) \otimes W \to \mathrm{Res}_H^G(V \otimes \mathrm{Ind}_H^G(W)) \cong \mathrm{Res}_H^G(V) \otimes \mathrm{Res}_H^G \mathrm{Ind}_H^G(W).$$

This map is given as $\mathrm{id} \otimes \eta$ where $\eta\colon W \to \mathrm{Res}_H^G \mathrm{Ind}_H^G(W)$ is the unit of the induction/restriction adjunction. To check this is an equivalence, it suffices to check on underlying vector spaces, which then just boils down to the isomorphism

$$\bigoplus_{G/H}(V \otimes W) \cong V \otimes (\bigoplus_{G/H} W).$$

$\square$

Example 1.16

Let us finish our calculation of λ_4^2 in $RO(C_4)$. We have just seen that $\mathrm{Ind}_{C_2}^{C_4}(\mathbf{1}_{\mathrm{sgn}}) \cong \lambda_4$, and hence

$$\begin{aligned}
\lambda_4 \otimes \lambda_4 \cong \lambda_4 \otimes (\mathrm{Ind}_{C_2}^{C_4}(\mathbf{1}_{\mathrm{sgn}})) &\cong \mathrm{Ind}_{C_2}^{C_4}(\mathrm{Res}_{C_2}^{C_4}(\lambda_4) \otimes \mathbf{1}_{\mathrm{sgn}}) \\
&\cong \mathrm{Ind}_{C_2}^{C_4}(2 \cdot \mathbf{1}_{\mathrm{sgn}} \otimes \mathbf{1}_{\mathrm{sgn}}) \\
&\cong \mathrm{Ind}_{C_2}^{C_4}(\mathbf{2}) \\
&\cong \mathbf{2} \oplus 2\sigma
\end{aligned}$$

For a a ring map $R \to S$, restriction also has a right adjoint, given by coinduction, denoted Coind_R^S and defined by $M \mapsto \mathrm{Hom}_S(R, M)$. A special fact about representation theory is that these adjoint are equal.

Proposition 1.30. *There is a natural equivalence of functors $\mathrm{Ind}_H^G \simeq \mathrm{Coind}_H^G$.*

Proof. If M is an R-module, we use the notation $M^* \cong \mathrm{Hom}_R(M, R)$ for the linear dual. In the case $R = k[G]$, then the natural k-linear isomorphisms

$$\mathrm{Hom}_k(M, N) \cong M^* \otimes_k N \quad \text{ and } M^{**} \cong M$$

are actually $k[G]$-module isomorphisms. To prove the proposition, we note that

$$k[G] \cong k[G]^* = \mathrm{Hom}_k(k[G], k)$$

so that

$$\mathrm{Ind}_H^G(M) = k[G] \otimes_{k[H]} M \cong k[G]^* \otimes_{k[H]} M \cong \mathrm{Hom}_{k[H]}(k[G], M) = \mathrm{Coind}_H^G(M)$$

naturally in M. $\square$

Remark 1.31. More generally, if $f\colon R \to S$ is a morphism of rings, then induction and coinduction agree if and only if S is finitely-generated and projective over R, and there is an isomorphism of (S, R)-bimodules

$$S \to \mathrm{Hom}_R(S, R).$$

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