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MA8403 - Equivariant homotopy theory

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Preface

This are the courses notes for MA8403 - Equivariant homotopy theory, held during the Autumn semester 2023 at NTNU. The notes are mainly based on two excellent sets of lectures notes, one by Guillou [2], and one by Blumberg [1]. The notes will be continually updated during the semester.

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Chapter 1

Equivariance in algebra

Group actions in algebra

We recall that if X is an object in a category \mathcal{C} , then the set of endomorphisms $\text{End}(X)$ forms a monoid (that is, a set equipped with an associative binary operation and an identity element) under composition. The set of automorphisms of X (that is, those endomorphisms that are invertible) form a group. Moreover, we have

$$\text{Aut}(X) = \text{End}(X) \cap \text{Iso}(\mathcal{C})$$

Note that any group is a monoid, simply by forgetting the existence of inverses.

Definition 1.1. An action of a group G on an object $X \in \mathcal{C}$ is a monoid homomorphism $a: G \rightarrow \text{End}(X)$, or equivalently a group homomorphism $a: G \rightarrow \text{Aut}(X)$ (note that a monoid homomorphism between groups is a group homomorphism).

Remark 1.2. Unwinding the definition, this means that we have:

- a. For each $g \in G$, there is a morphism $a(g): G \rightarrow G$.
- b. a preserves composition, i.e., $a(g \cdot h) = a(h) \cdot a(g)$.
- c. a preserves identities, so $a(e) = \text{id}_X$.

Let \mathcal{C} be the category of sets and functions. Then, $a: G \rightarrow \text{End}(X) = \{f: X \rightarrow X\}$ correspond to a function $\bar{a}: G \times X \rightarrow X$. The conditions above mean that the diagrams

$$\begin{array}{ccc} G \times G \times X & \xrightarrow{m \times \text{id}} & G \times X \\ \text{id} \times \bar{a} \downarrow & & \downarrow \bar{a} \\ G \times X & \xrightarrow{\bar{a}} & X \end{array} \quad \text{and} \quad \begin{array}{ccc} \{*\} \times X & \xrightarrow{e \times \text{id}} & G \times X \\ \cong \downarrow & & \downarrow \bar{a} \\ X & \xlongequal{\quad} & X \end{array}$$

commute, where $m: G \times G \rightarrow G$ denotes the group multiplication. Equivalently, in symbols, we have

$$\bar{a}(h, \bar{a}(g, x)) = \bar{a}(gh, x)$$

and

$$\bar{a}(e, x) = x.$$

Remark 1.3. Let BG denote the category with one object $*$ and with $\text{Hom}(*, *) = G$. Then an action of G in the category \mathcal{C} is the same as a functor $\rho: BG \rightarrow \mathcal{C}$. The object X in the previous definition is the object $\rho(*) \in \mathcal{C}$.

Remark 1.4. Let $\mathcal{C} = \text{Mod}_R$ for a commutative ring R . An action of G on $M \in \text{Mod}_R$ is a monoid homomorphism

$$a: G \rightarrow \text{Hom}_R(M, M).$$

We recall that $\text{Hom}_R(M, M)$ actually has the structure of an R -algebra

Definition 1.5. The $(R$ -linear) group ring on R is the R -algebra $R[G]$ whose:

- (a) underlying R -module is the free R -module with basis on the underlying set of G .
- (b) whose multiplication is given on basis elements by the group operation.

Let $R = \mathbb{Z}$ and $G = C_2 = \langle \sigma \rangle$. An element of $\mathbb{Z}[C_2]$ is of the form $a + b\sigma$ where $a, b \in \mathbb{Z}$. Multiplication is given by

$$(a_1 \cdot 1 + b_1\sigma) \cdot (a_2 \cdot 1 + b_2\sigma) = (a_1a_2 + b_1b_2) \cdot 1 + (a_1b_2 + b_1a_2)\sigma.$$

This is the same thing as the polynomial ring $\mathbb{Z}[\sigma]/(\sigma^2 - 1)$.

Remark 1.6. Categorically, the group ring construction is left adjoint to the functor that takes an R -algebra to its group of units, i.e., there is an adjunction

$$R[-]: \text{Grp} \rightleftarrows \text{Mod}_R: (-)^\times$$

Returning to group actions, we have the following:

Proposition 1.7. *Let R be a commutative ring, and G a finite group. The following data on an R -module M are equivalent:*

- a. A monoid homomorphism $G \rightarrow \text{End}_R(M)$.
- b. A group homomorphism $G \rightarrow \text{Aut}_R(M)$.
- c. A homomorphism of R -algebras $R[G] \rightarrow \text{Hom}_R(M, M)$.
- d. An $R[G]$ -module structure on M whose underlying R -module structure is M .

Definition 1.8. A representation of G over R is an $R[G]$ -module.

If $R = k$ is a field, then the underlying R -module is a k -vector space V . If $\dim_k(V) = n$, then $\text{Aut}_k(V) = GL_n(k)$, and a k -representation is the same thing as a group homomorphism $G \rightarrow GL_n(k)$.

Definition 1.9. The $R[G]$ -module $R[G]$ is known as the regular representation. More generally, if X is a finite G -set, then the free R -module $R[X]$ inherits the structure of a $R[G]$ -module (the case of $R[G]$ itself corresponds to the finite G -set G , considered as an $R[G]$ -module over itself). Representations obtained this way are known as permutation representations.

Taking X to be the trivial G -set, we obtain the (one-dimensional) trivial representation. This is simply the $R[G]$ -module R , where G acts trivially.

Definition 1.10. Let $G = C_2 = \langle \tau \rangle$, and suppose that $-1 \neq 1 \in R$. Then the sign representation of G is the one-dimensional representation where τ acts as -1 (if $-1 = 1$ in R this still makes sense, but is just the trivial representation). Note that this is an example of a representation that is not a permutation representation.

Let $C_n = \langle \sigma \rangle$ be the cyclic group of order n . Let us calculate all complex 1-dimensional representations of C_n , i.e., homomorphisms $\rho: C_n \rightarrow \mathbb{C}$. Note that if we define $\rho(\sigma) = c$, then $\rho(\sigma^n) = c^n = 1$, so that c must be an n -th root of unity. There are precisely n of these (take $\chi_n = e^{2\pi i/n}$), and so there are precisely n representations. For example, when $G = C_4$, the four representations correspond to sending σ to either $1, i, -1$ or $-i$. Note that we can also consider these as 2-dimensional real representations.

Notation 1.11. We let $\rho = \rho_G$ denote the regular representation of G , and the trivial n -dimensional representation by $\mathbf{n} = R^{\oplus n}$.

Definition 1.12. A subrepresentation is a submodule.

The regular representation always has a one-dimensional trivial representation, generated by the sum $\sum_{g \in G} g$.

Definition 1.13. A representation V is irreducible if the only subrepresentations of V are 0 and V .

Theorem 1.14 (Maschke). *Suppose that k is a field of characteristic not dividing $|G|$. Then every representation splits as a direct sum of irreducible representations.*

Proof. We prove the following: if $V \subseteq W$ is a subrepresentation, then there exists $U \subseteq W$ such that $U \oplus V \simeq W$.

To see this, let $\pi: W \rightarrow W$ be any k -linear projection of W onto V . This map need not be G -equivariant, but we can make it so by ‘averaging’. That is,

we define a new map $\phi: W \rightarrow W$ by

$$\phi(\mathbf{w}) = \frac{1}{|G|} \sum_{g \in G} g \cdot \pi(g^{-1} \cdot \mathbf{w}).$$

Moreover the map is G -equivariant: for $h \in G$ we have

$$\begin{aligned} \phi(h \cdot \mathbf{w}) &= \frac{1}{|G|} \sum_{g \in G} g \cdot \pi(g^{-1} \cdot h \cdot \mathbf{w}) \\ &= \frac{1}{|G|} \sum_{u \in G} u \cdot h \cdot \pi(u^{-1} \cdot \mathbf{w}) \\ &= h \cdot \phi(\mathbf{w}), \end{aligned}$$

where $u = gh^{-1}$, and so ϕ is $k[G]$ -linear. Furthermore, the map ϕ is the identity on V . By the splitting lemma, $W = V \oplus \ker(\phi)$. \square

Remark 1.15. We used the assumption on k to ensure that we could divide by $|G|$. Without that assumption, the theorem is false. Indeed, let $G = C_2$, $R = \mathbb{F}_2$ and consider the representation defined by $\rho(\tau) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. This is not irreducible, but does not split as a direct sum of indecomposable representations.

Corollary 1.16. *Suppose that k is a field of characteristic not dividing $|G|$. If V is an irreducible representation, then V is isomorphic to a subgroup of $k[G]$ (slogan: all irreducibles are submodules of the regular representation).*

Proof. Let $\mathbf{v} \in V$ be non-trivial. Then the homomorphism $\phi: k[G] \rightarrow V$ given by sending 1 to \mathbf{v} must be surjective, because V is irreducible. Let $U = \ker(\phi)$, then apply Maschke's theorem. \square

Let $G = C_2 = \langle \tau \rangle$, and k a field of characteristic not equal to 2. We have the trivial representation $\mathbf{1}$ and the sign representation $\mathbf{1}_{\text{sgn}}$. Then, $\mathbf{1}$ is generated by the sum $1 + \tau$, while $\mathbf{1}_{\text{sgn}}$ is generated by $1 - \tau$, and we deduce that

$$\rho_{C_2} = \mathbf{1} \oplus \mathbf{1}_{\text{sgn}}.$$

Bibliography

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