Drew Heard

MA8403 - Equivariant homotopy theory

August 25, 2023

Preface

This are the courses notes for MA8403 - Equivariant homotopy theory, held during the Autumn semester 2023 at NTNU. The notes are mainly based on two excellent sets of lectures notes, one by Guillou [2], and one by Blumberg [1]. The notes will be continually updated during the semester.

Contents

<div align="center">

Chapter 1

</div>

<div align="center">

Equivariance in algebra

</div>

Group actions in algebra

We recall that if X is an object in a category $\mathcal{C}$, then the set of endomorphisms $\mathrm{End}(X)$ forms a monoid (that is, a set equipped with an associative binary operation and an identity element) under composition. The set of automorphisms of X (that is, those endomorphisms that are invertible) form a group. Moreover, we have

$$\mathrm{Aut}(X) = \mathrm{End}(X) \cap \mathrm{Iso}(\mathcal{C})$$

Note that any group is a monoid, simply by forgetting the existence of inverses.

Definition 1.1. An action of a group G on an object $X \in \mathcal{C}$ is a monoid homomorphism $a\colon G \to \mathrm{End}(X)$, or equivalently a group homomorphism $a\colon G \to \mathrm{Aut}(X)$ (note that a monoid homomorphism between groups is a group homomorphism).

Remark 1.2. Unwinding the definition, this means that we have:

    a. For each $g \in G$, there is a morphism $a(g)\colon X \to X$.

    b. a preserves composition, i.e., $a(g \cdot h) = a(g) \cdot a(h)$.

    c. a preserves identifies, so $a(e) = \mathrm{id}_X$.

Example 1.1

Let $\mathcal{C}$ be the category of sets and functions. Then, $a\colon G \to \mathrm{End}(X) = \big\{ f\colon X \to X \big\}$ correspond to a function $\bar{a}\colon G \times X \to X$. The conditions above mean that the diagrams

$$
\begin{array}{ccc}
G \times G \times X & \xrightarrow{\ m \times \mathrm{id}\ } & G \times X \\
{\scriptstyle \mathrm{id} \times \bar{a}} \downarrow & & \downarrow {\scriptstyle \bar{a}} \\
G \times X & \xrightarrow[\ \bar{a}\ ]{} & X
\end{array}
\qquad \text{and} \qquad
\begin{array}{ccc}
\{*\} \times X & \xrightarrow{\ e \times \mathrm{id}\ } & G \times X \\
{\scriptstyle \cong} \downarrow & & \downarrow {\scriptstyle \bar{a}} \\
X & =\!\!=\!\!=\!\!= & X
\end{array}
$$

commute, where $m\colon G \times G \to G$ denotes the group multiplication. Equivalently, in symbols, we have

$$\bar{a}(g, \bar{a}(h, x)) = \bar{a}(gh, x)$$

and

$$\bar{a}(e, x) = x.$$

Remark 1.3. Let BG denote the category with one object $*$ and with $\mathrm{Hom}(*, *) = G$. Then an action of G in the category $\mathcal{C}$ is the same as a functor $\rho \colon BG \to \mathcal{C}$. The object X in the previous definition is the object $\rho(*) \in \mathcal{C}$.

Remark 1.4. Let $\mathcal{C} = \mathrm{Mod}_R$ for a commutative ring R. An action of G on $M \in \mathrm{Mod}_R$ is a monoid homomorphism

$$a \colon G \to \mathrm{Hom}_R(M, M).$$

We recall that $\mathrm{Hom}_R(M, M)$ actually has the structure of an R-algebra

Definition 1.5. The (R-linear) group ring on R is the R-algebra $R[G]$ whose:

(a) underlying R-module is the free R-module with basis on the underlying set of G.

(b) whose multiplication is given on basis elements by the group operation.

> **Example 1.2**
>
> Let $R = \mathbb{Z}$ and $G = C_2 = \langle \sigma \rangle$. An element of $\mathbb{Z}[C_2]$ is of the form $a + b\sigma$ where $a, b \in \mathbb{Z}$. Multiplication is given by
>
> $$(a_1 \cdot 1 + b_1 \sigma) \cdot (a_2 \cdot 1 + b_2 \sigma) = (a_1 a_2 + b_1 b_2) \cdot 1 + (a_1 b_2 + b_1 a_2) \sigma.$$
>
> This is the same thing as the polynomial ring $\mathbb{Z}[\sigma]/(\sigma^2 - 1)$.

Remark 1.6. Categorically, the group ring construction is left adjoint to the functor that takes an R-algebra to its group of units, i.e., there is an adjudication

$$R[-] \colon \mathrm{Grp} \leftrightarrows \mathrm{Alg}_R \colon (-)^\times$$

Returning to group actions, we have the following:

Proposition 1.7. *Let R be a commutative ring, and G a finite group. The following data on an R-module M are equivalent:*

a. *A monoid homomorphism $G \to \mathrm{End}_R(M)$.*

b. *A group homomorphism $G \to \mathrm{Aut}_R(M)$.*

c. *A homomorphism of R-algebras $R[G] \to \mathrm{Hom}_R(M, M)$.*

d. *An $R[G]$-module structure on M whose underlying R-module structure is M.*

Definition 1.8. A representation of G over R is an $R[G]$-module.

> **Example 1.3**
>
> If $R = k$ is a field, then the underlying R-module is a k-vector space V. If $\dim_k(V) = n$, then $\mathrm{Aut}_k(V) = GL_n(k)$, and a k-representation is the same thing as a group homomorphism $G \to GL_n(k)$.

Definition 1.9. The $R[G]$-module $R[G]$ is known as the regular representation. More generally, if X is a finite G-set, then the free R-module $R[X]$ inherits the structure of a $R[G]$-module (the case of $R[G]$ itself corresponds to the finite G-set G, considered as an $R[G]$-module over itself). Representations obtained this way are known as permutation representations.

> **Example 1.4**
>
> Taking X to be the trivial G-set, we obtain the (one-dimensional) trivial representation. This is simply the $R[G]$-module R, where G acts trivially.

Definition 1.10. Let $G = C_2 = \langle \tau \rangle$, and suppose that $-1 \neq 1 \in R$. Then the sign representation of G is the one-dimensional representation where τ acts as -1 (if $-1 = 1$ in R this still makes sense, but is just the trivial representation). Note that this is an example of a representation that is not a permutation representation.

> **Example 1.5**
>
> Let $C_n = \langle \sigma \rangle$ be the cyclic group of order n. Let us calculate all complex 1-dimensional representations of C_n, i.e., homomorphisms $\rho \colon C_n \to \mathbb{C}$. Note that if we define $\rho(\sigma) = c$, then $\rho(\sigma^n) = c^n = 1$, so that c must be an n-th root of unity. There are precisely n-of these (take $\zeta_n = e^{2\pi/ni}$), and so there are precisely n-representations. For example, when $G = C_4$, the four representations correspond to sending σ to either $1, i, -1$ or $-i$. Note that we can also consider these as 2-dimensional *real* representations.

Notation 1.11. We let $\rho = \rho_G$ denote the regular representation of G, and the trivial n-dimensional representation by $\mathbf{n} = R^{\oplus n}$.

Definition 1.12. A subrepresentation is a submodule.

> **Example 1.6**
>
> The regular representation always has a one-dimensional trivial representation, generated by the sum $\sum_{g \in G} g$.

Definition 1.13. A representation V is irreducible if the only subrepresentations of V are 0 and V.

Theorem 1.14 (Maschke). *Suppose that k is a field of characteristic not dividing $|G|$. Then every representation splits as a direct sum of irreducible representations.*

Proof. We prove the following: if $V \subseteq W$ is a subrepresentation, then there exists $U \subseteq W$ such that $U \oplus V \simeq W$.

To see this, let $\pi \colon W \to W$ be any k-linear projection of W onto V. This map need not be G-equivariant, but we can make it so by 'averaging'. That is, we define a new map $\phi \colon W \to W$ by

$$\phi(\mathbf{w}) = \frac{1}{|G|} \sum_{g \in G} g \cdot \pi(g^{-1} \cdot \mathbf{w}).$$

Moreover the map is G-equivariant: for $h \in G$ we have

$$\phi(h \cdot \mathbf{w}) = \frac{1}{|G|} \sum_{g \in G} g \cdot \pi(g^{-1} \cdot h \cdot \mathbf{w})$$

$$= \frac{1}{|G|} \sum_{u \in G} u \cdot h \cdot \pi(u^{-1} \cdot \mathbf{w})$$

$$= h \cdot \phi(\mathbf{w}),$$

where $u = gh^{-1}$, and so ϕ is $k[G]$-linear. Furthermore, the map ϕ is the identity on V By the splitting lemma, $W = V \oplus \ker(\phi)$. $\qquad\square$

Remark 1.15. We used the assumption on k to ensure that we could divide by $|G|$. Without that assumption, the theorem is false. Indeed, let $G = C_2$, $R = \mathbb{F}_2$ and consider the representation defined by $\rho(\tau) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. This is not irreducible, but does not split as a direct sum of irreducible representations.

Corollary 1.16. *Suppose that k is a field of characteristic not dividing $|G|$. If V is an irreducible representation, then V is isomorphism to a submodule of $k[G]$ (slogan: all irreducibles are submodules of the regular representation).*

Proof. Let $\mathbf{v} \in V$ be non-trivial. Then the homomorphism $\phi \colon k[G] \to V$ given by sending 1 to $\mathbf{v}$ must be surjective, because V is irreducible. Let $U = \ker(\phi)$, then apply Maschke's theorem. $\qquad\square$

> **Example 1.7**
>
> Let $G = C_2 = \langle \tau \rangle$, and k a field of characteristic not equal to 2. We have the trivial representation $\mathbf{1}$ and the sign representation $\mathbf{1}_{\mathrm{sgn}}$. Then, $\mathbf{1}$ is generated by the sum $1 + \tau$, while $\mathbf{1}_{\mathrm{sgn}}$ is generated by $1 - \tau$, and we deduce that
> $$\rho_{C_2} = \mathbf{1} \oplus \mathbf{1}_{\mathrm{sgn}}.$$

The representation ring

Let k be a field, and suppose that V and W are G-representations, then the k-linear tensor sum $V \oplus W$ can be given the structure of a $k[G]$-module, by taking the diagonal G-action. If we think of a representation in terms of a homomorphism $\rho \colon G \to GL_n(k)$, then this direct sum corresponds to the 'block sum'

$$G \to GL_n(k) \times GL_m(k) \to GL_{n+m}(k)$$

Similarly, we can define a tensor product of representations using the 'Kronecker tensor product' (or matrix direct product). Equivalently, this is the k-linear tensor product $V \otimes W$ with the g-action defined on simple tensors by

$$g \cdot (\mathbf{v} \otimes \mathbf{w}) = g \cdot \mathbf{v} \otimes g\mathbf{w}.$$

We leave it for the reader to verify the following straightforward computations:

(a) $\mathbf{1} \otimes V \cong V \cong V \otimes \mathbf{1}$.

(b) $\mathbf{n} \otimes V \cong V^{\oplus n} \cong V \otimes \mathbf{n}$.

Example 1.8

Let us compute the tensor product $\mathbf{1}_{\mathrm{sgn}} \otimes \mathbf{1}_{\mathrm{sgn}}$. The underlying vector space is simply $k \otimes k \cong k$, while τ acts as $\tau \cdot (1 \otimes 1) = (\tau \cdot 1) \otimes (\tau \cdot 1) = -1 \otimes -1 = 1 \otimes 1$. So the tensor product $\mathbf{1}_{\mathrm{sgn}} \otimes \mathbf{1}_{\mathrm{sgn}} = \mathbf{1}$.

Example 1.9

Take $G = C_3$ and $k = \mathbb{R}$. We have a two-dimensional representation λ_3 corresponding to rotation by an angle of $2\pi/3$. What is $\lambda_3 \otimes \lambda_3$? This is a 4-dimensional representation, and by working out all irreducible representations must be either $\mathbf{4}, \mathbf{2} \oplus \lambda_3$ or $\lambda_3 \oplus \lambda_3$. If you know a little bit of character theory, you can see that it must be $\mathbf{2} \oplus \lambda_3$: we have

$$\chi_{\lambda_3 \otimes \lambda_3}(1) = 4, \quad \chi_{\lambda_3 \otimes \lambda_3}(\tau) = 1$$

$$\chi_{\mathbf{4}}(1) = 4, \quad \chi_{\mathbf{4}}(\tau) = 4$$

$$\chi_{\mathbf{2} \oplus \lambda_3}(1) = 4, \quad \chi_{\mathbf{2} \oplus \lambda_3}(\tau) = 1$$

$$\chi_{\lambda_3 \oplus \lambda_3}(1) = 4, \quad \chi_{\lambda_3 \oplus \lambda_3}(\tau) = -2$$

Remark 1.17. By passing to isomorphism classes of representations, the set of finite dimensional representations has the structure of a semiring. Using the Grothendieck construction, we can produce a commutative ring.

Definition 1.18. For a finite group G the real representation ring $RO(G)$ is the Grothendieck group of the above semi-ring. Explicitly,

$$RO(G) := \mathbb{Z}\left\{\begin{matrix}\text{isomorphism classes of finite-}\\ \text{dimensional real } G\text{-representations}\end{matrix}\right\} / \langle [V \oplus W] - [V] - [W] \rangle.$$

Remark 1.19. As an abelian group, $RO(G)$ is a direct sum of copies of $\mathbb{Z}$, with rank equal to the number of isomorphism classes of irreducible representations.

Remark 1.20. We can make the same definition for other fields, for example when $k = \mathbb{C}$ we get the complex representation ring $R(G)$.

Example 1.10

We have $RO(C_2) \cong \mathbb{Z}\{\mathbf{1}\} \oplus \mathbb{Z}\{\mathbf{1}_{\text{sgn}}\}$. The ring structure is determined by Example (1.8): we have $RO(C_2) \cong \mathbb{Z}[\sigma]/(\sigma^2 - 1)$. The same is true for the complex representation ring. Note that this is the same as $\mathbb{Z}[C_2]$. In fact, the complex representation ring of a finite abelian group is always (non-canonically) isomorphic to the group ring: it is the group ring of the character group.

Example 1.11

When $G = C_3$ we have that

$$RO(C_3) = \mathbb{Z}\{\mathbf{1}\} \oplus \mathbb{Z}\{\lambda_3\}.$$

By Example (1.9) we have $[\lambda_3]^2 = 2 + [\lambda_3]$ and we see that

$$RO(C_3) \cong \mathbb{Z}[\lambda]/(\lambda^2 - \lambda - 2).$$

On the other hand, the complex representation ring is given by

$$R(C_3) \cong \mathbb{Z}[\zeta]/(\zeta^3 - 1).$$

By tensoring a real representation with $\mathbb{C}$, there is a map

$$RO(C_3) \to R(C_3)$$

given by $\lambda \mapsto \zeta + \zeta^2$.

Definition 1.21. Let $\phi \colon H \to G$ be a morphism of groups, then the pullback $\phi^*(V)$ of a G-representation is the $k[H]$-module induced by restriction of scalars along $k[H] \to k[G]$. Equivalently, it is the representation given by the composite $H \to G \xrightarrow{a} \text{End}(V)$.

Remark 1.22. We have

$$\phi^*(V \oplus W) = \phi^*(V) \oplus \phi^*(W) \quad \text{and } \phi^*(V \otimes W) \cong \phi^*(V) \otimes \phi^*(W).$$

Therefore we deduce:

Corollary 1.23. *A group homomorphism $\phi\colon H \to G$ induces a ring homomorphism $\phi^*\colon RO(G) \to RO(H)$.*

Example 1.12

Consider the group homomorphism $\phi\colon G \to G/G \cong e$. Then we have $\mathbf{n} = \phi^*(\mathbf{k}^n)$.

Definition 1.24. An *injective* group homomorphism $\iota\colon H \hookrightarrow G$ gives rise to a restriction functor for representations, which we denote by Res_H^G.

Example 1.13

Let $G = C_4 = \langle r \rangle$ and $H = C_2 \subseteq C_4$ the subgroup generated by r^2, and take $k = \mathbb{R}$. Pulling back the sign representation of C_2 along the quotient $C_4 \twoheadrightarrow C_2$ gives rise to the sign representation σ. This is one of three irreducible C_4 real representations: we have the trivial representation $\mathbf{1}$ and the rotation representation λ_4. The inclusion $H \hookrightarrow G$ gives a morphism

$$RO(C_4) = \mathbb{Z}\{\mathbf{1}\} \oplus \mathbb{Z}\{\sigma\} \oplus \mathbb{Z}\{\lambda_4\} \to \mathbb{Z}\{\mathbf{1}\} \oplus \mathbb{Z}\{\mathbf{1}_{\mathrm{sgn}}\} = RO(C_2).$$

This map is determined by

$$\mathbf{1} \mapsto \mathbf{1}, \quad \sigma \mapsto \mathbf{1}, \quad \lambda_4 \mapsto 2 \cdot \mathbf{1}_{\mathrm{sgn}},$$

where the image of σ is determined from its definition as the pull-back, and the image of λ_4 comes from the fact that r acts as multiplication by $\pi/4$ and so r^2 acts as multiplication by -1, and so restricts to a 2-dimensional sign representation $\mathbf{1}_{\mathrm{sgn}} \oplus \mathbf{1}_{\mathrm{sgn}}$. We can conclude that $\lambda_4^2 \mapsto (2 \cdot \mathbf{1}_{\mathrm{sgn}})^2 = \mathbf{4}$, so that λ_4^2 must be either $\mathbf{4}, \mathbf{3} \oplus \sigma, \mathbf{2} \oplus 2\sigma, \mathbf{1} \oplus 3\sigma$ or 4σ.

There is also another functor, which will turn out to be adjoint to restriction.

Definition 1.25. Given $H \leq G$, and a H-representation V, we define the induced representation Ind_H^G is be the tensor product $k[G] \otimes_{k[H]} V$.

Remark 1.26. Induction plays well with direct sums: we have $\operatorname{Ind}_H^G(V \oplus W) \cong \operatorname{Ind}_H^G(V) \oplus \operatorname{Ind}_H^G(W)$. But a dimension check shows that it does not commute with tensor products. Therefore, we get an induced map of abelian groups, but *not* of commutative rings

$$\operatorname{Ind}_H^G\colon RO(H) \to RO(G)$$

Directly from the definition we deduce the following:

Lemma 1.27. *If $K \leq H \leq G$ then $\mathrm{Ind}_H^G \mathrm{Ind}_K^H V \simeq \mathrm{Ind}_K^G V$ for any K-representation V.*

Example 1.14

The regular representation $\rho_G = k[G] \cong k[G] \otimes_k k \simeq \mathrm{Ind}_e^G(\mathbf{1})$. More generally, we have $\mathrm{Ind}_H^G \rho_H \cong \rho_G$.

Example 1.15

Let $C_2 \subseteq C_4$ and $k = \mathbb{R}$. What is $\mathrm{Ind}_{C_2}^{C_4}(\mathbf{1})$? We have

$$\mathrm{Ind}_{C_2}^{C_4}(\mathbf{1}) = \mathbb{R}[C_4] \otimes_{\mathbb{R}[C_2]} \mathbb{R} \cong \mathbb{R}[C_4/C_2] \cong \phi^*(\rho_{C_2})$$

for $\phi \colon C_4 \to C_4/C_2 \cong C_2$ the quotient map.[a] This means that

$$\mathrm{Ind}_{C_2}^{C_4}(\mathbf{1}) = \mathbf{1} \oplus \sigma$$

To work out $\mathrm{Ind}_{C_2}^{C_4}(\mathbf{1}_{\mathrm{sgn}})$ we have that

$$\rho_{C_4} \cong \mathrm{Ind}_{C_2}^{C_4}(\rho_{C_2})) = \mathrm{Ind}_{C_2}^{C_4}(\mathbf{1} \oplus \mathbf{1}_{\mathrm{sgn}}) \cong \mathrm{Ind}_{C_2}^{C_4}(\mathbf{1}) \oplus \mathrm{Ind}_{C_2}^{C_4}(\mathbf{1}_{\mathrm{sgn}})$$
$$\cong \mathbf{1} \oplus \sigma \oplus \mathrm{Ind}_{C_2}^{C_4}(\mathbf{1}_{\mathrm{sgn}}).$$

But $\rho_{C_4} \cong \mathbf{1} \oplus \sigma \oplus \lambda_4,$. To see this, one can note that

$$\mathbb{R}[\mathbb{Z}/4] \cong \mathbb{R}[X]/(X^4 - 1) \cong \mathbb{R}[X]/\prod_{d|4} \Phi_d \cong \bigoplus_{d|4} \mathbb{R}[X]/\Phi_d$$

so that

$$\mathbb{R}[\mathbb{Z}/4] \cong \mathbb{R}[X]/(X - 1) \oplus \mathbb{R}[X]/(X + 1) \oplus \mathbb{R}[X]/(X^2 + 1).$$

Hence, $\mathrm{Ind}_{C_2}^{C_4}(\mathbf{1}_{\mathrm{sgn}}) \cong \lambda_4$.
    We deduce that the map

$$\mathrm{Ind}_{C_2}^{C_4} \colon RO(C_2) \to RO(C_4)$$

is determined by

$$\mathbf{1} \mapsto \mathbf{1} \oplus \sigma \quad \text{and} \quad \mathbf{1}_{\mathrm{sgn}} \mapsto \lambda_4.$$

---

[a]This always works: For $H \leq G$ a normal subgroup we have $\mathrm{Ind}_H^G(\mathbf{1}) = \phi^*(\rho_{G/H})$.

In general, if we have commutative rings R and S and a ring map $f \colon R \to S$, then we can define induction and restriction between R-modules and S-modules and induction is left adjoint to restriction. As a special case, we have:

Lemma 1.28. *If $H \leq G$, then induction is left adjoint to restriction.*

Proposition 1.29 (The projection formula)**.** *Let $H \leq G$, then there is a natural equivalence*

$$\text{Ind}_H^G(\text{Res}_H^G(V) \otimes W) \xrightarrow{\sim} V \otimes \text{Ind}_H^G(W)$$

for $V \in RO(G)$ and $W \in RO(G)$

Proof. We first construct the map: by adjunction, such a map is equivalent to a H-equivariant map

$$\text{Res}_H^G(V) \otimes W \to \text{Res}_H^G(V \otimes \text{Ind}_H^G(W)) \cong \text{Res}_H^G(V) \otimes \text{Res}_H^G \text{Ind}_H^G(W).$$

This map is given as $\text{id} \otimes \eta$ where $\eta \colon W \to \text{Res}_H^G \text{Ind}_H^G(W)$ is the unit of the induction/restriction adjunction. To check this is an equivalence, it suffices to check on underlying vector spaces, which then just boils down to the isomorphism

$$\bigoplus_{G/H}(V \otimes W) \cong V \otimes (\bigoplus_{G/H} W).$$

$\square$

Example 1.16

Let us finish our calculation of λ_4^2 in $RO(C_4)$. We have just seen that $\text{Ind}_{C_2}^{C_4}(\mathbf{1}_{\text{sgn}}) \cong \lambda_4$, and hence

$$\begin{aligned}
\lambda_4 \otimes \lambda_4 &\cong \lambda_4 \otimes (\text{Ind}_{C_2}^{C_4}(\mathbf{1}_{\text{sgn}})) \cong \text{Ind}_{C_2}^{C_4}(\text{Res}_{C_2}^{C_4}(\lambda_4) \otimes \mathbf{1}_{\text{sgn}}) \\
&\cong \text{Ind}_{C_2}^{C_4}(2 \cdot \mathbf{1}_{\text{sgn}} \otimes \mathbf{1}_{\text{sgn}}) \\
&\cong \text{Ind}_{C_2}^{C_4}(\mathbf{2}) \\
&\cong \mathbf{2} \oplus 2\sigma
\end{aligned}$$

For a a ring map $R \to S$, restriction also has a right adjoint, given by coinduction, denoted Coind_R^S and defined by $M \mapsto \text{Hom}_S(R, M)$. A special fact about representation theory is that these adjoint are equal.

Proposition 1.30. *There is a natural equivalence of functors $\text{Ind}_H^G \simeq \text{Coind}_H^G$.*

Proof. If M is an R-module, we use the notation $M^* \cong \text{Hom}_R(M, R)$ for the linear dual. In the case $R = k[G]$, then the natural k-linear isomorphisms

$$\text{Hom}_k(M, N) \cong M^* \otimes_k N \quad \text{and } M^{**} \cong M$$

are actually $k[G]$-module isomorphisms. To prove the proposition, we note that

$$k[G] \cong k[G]^* = \text{Hom}_k(k[G], k)$$

so that

$$\operatorname{Ind}_H^G(M) = k[G] \otimes_{k[H]} M \cong k[G]^* \otimes_{k[H]} M \cong \operatorname{Hom}_{k[H]}(k[G], M) = \operatorname{Coind}_H^G(M)$$

naturally in M. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ $\square$

Remark 1.31. More generally, if $f\colon R \to S$ is a morphism of rings, then induction and coinduction agree if and only if S is finitely-generated and projective over R, and there is an isomorphism of (S, R)-bimodules

$$S \to \operatorname{Hom}_R(S, R).$$

The double-coset formula

Let us return for a moment to $RO(C_4)$. We have constructed maps

$$RO(C_2) \to RO(C_4) \to RO(C_2)$$

which send

$$\mathbf{1} \mapsto \mathbf{1} \oplus \sigma \mapsto \mathbf{2}$$

and

$$\mathbf{1}_{\mathrm{sgn}} \mapsto \lambda_4 \mapsto 2 \cdot \mathbf{1}_{\mathrm{sgn}}$$

so that the composite map $RO(C_2) \to RO(C_2)$ is multiplication by 2. This is actually a completely general phenomena.

Definition 1.32. Let $H, K \leq G$ be subgroups, then a double coset HgK is the set

$$HgK = \left\{ x \in G \,\middle|\, x = hgk \text{ for some } h \in H, k \in K \right\}.$$

Theorem 1.33 (Double coset formula)**.** *For subgroups $H, K, \leq G$ and a H-representation K we have a decomposition of K-representations*

$$\operatorname{Res}_K^G \operatorname{Ind}_H^G(V) = \sum_{HgK \in H\backslash G/K} \operatorname{Ind}_{H^{g-1} \cap K}^K c_g^* \operatorname{Res}_{H \cap K^g}^H(V)$$

where $c_g \colon H \cap K^g \xrightarrow{\sim} H^{g^{-1}} \cap K$ is the conjugation by g homomorphism.

Corollary 1.34. *Suppose that G is abelian, and $H = K$, then the composite*

$$RO(H) \xrightarrow{\operatorname{Ind}_H^G} RO(G) \xrightarrow{\operatorname{Res}_H^G} RO(H)$$

is given by multiplication by the index $|G/H|$ of H inside of G.

Indeed, in this case $H\backslash G/H = G/H$.

Remark 1.35. The restriction that G is abelian is really necessary. See [2, Example 1.1.51].

<center>Chapter 2</center>

<center>| Mackey functors</center>

The definition of a Mackey functor

We can axiomatize the structure we have on seen on the representation ring in an algebraic object called a *Mackey functor*. As we will see later, these play the same role in equivariant stable homotopy that abelian groups play in ordinary stable homotopy.

Definition 2.1. A Mackey functor $\underline{M}$ for a finite group G consists of the following data:

(a) An abelian group $\underline{M}(H)$ for each $H \leq G$.

(b) A restriction map $R_H^G \colon \underline{M}(H) \to \underline{M}(K)$ for each $K \leq H$.

(c) A transfer map $I_K^H \colon \underline{M}(K) \to \underline{M}(H)$ for each $K \leq H$.

(d) A conjugation homomorphism $c_g \colon \underline{M}(H) \to \underline{M}(H^g)$ for each $g \in G$.

subject to the following rules:

(i) R_H^H and I_H^H are the identity for each $H \leq K$. Moreover, for each $h \in H$, c_h is the identity on $\underline{M}(H)$.

(ii) If $L \leq K \leq H$, then $R_L^K \circ R_K^H \simeq R_L^H$ and $I_H^K \circ I_L^K \simeq I_L^H$.

(iii) $c_g \circ c_h \simeq c_{gh}$ for all $g, h \in G$.

(iv) $R_{K^g}^{H^g} c_g = c_g R_K^H$ and $I_{K^g}^{H^g} c_g = c_g I_K^H$.

(v) The double coset formula holds:
$$R_L^H \circ I_K^H = \sum_{KhL \in K \backslash H / L} I_{K^h \cap L}^L R_{K^h \cap L}^{K^h} c_h$$

for all $L, K \leq H \leq G$.

Example 2.1

In the previous section we have shown that there is a Mackey functor $\underline{RO}(G)$ with $\underline{RO}(G)(H) \cong RO(H)$.

<center>11</center>

Remark 2.2. There are numerous different ways to record the data of a Mackey functor, and depending on precisely what you want to do, some may be better than others. For example, although it's maybe not so hard to define a category of Mackey functors from this perspective, its formal properties might be a bit hard to see (for example, is it an abelian category? Is it symmetric monoidal?), while from other perspectives this becomes much clearer.

Remark 2.3. Let $\underline{M}$ be a Mackey functor. Note that if g normalizes H so that $H^g = H$, then c_g maps $\underline{M}$ to itself, i.e. we have an action of $N_G(H)$ on $\underline{M}(H)$. Moreover, the normal subgroup $H \le N_G(H)$ acts trivially, so we get an action of $W_G(H) = N_G(H)/H$ on $\underline{M}(H)$. For example, in the case $H = e$, we get an action of $N_G(e)/e = G$ on $\underline{M}(e)$. It is customary to write Mackey functors via a *Lewis diagram*: for example, when $G = C_p$ this is a diagram of the form:

$$\underline{M}(C_p)$$
$$R \Big\downarrow \Big\uparrow I$$
$$\underline{M}(e)$$
$$\circlearrowleft$$
$$C_p$$

> ### Example 2.2
>
> Given an $\mathbb{Z}[G]$-module M we can produce a Mackey functor $\underline{M}$ defined by
> $$\underline{M}(H) = M^H.$$
> Here restriction is defined by inclusion of fixed points for a larger subgroup, which the transfer map is defined by
> $$I_K^H \colon M^K \to M_H, \quad I_K^H(m) = \sum_{hK \in H/K} h \cdot M$$
> This does not depend on coset representatives, since m is assumed to be fixed by K. Finally, conjugation c_g is multiplication by g.
>   This gives a functor $FP \colon \mathrm{Mod}_{\mathbb{Z}[G]} \to \mathrm{Mack}_G$

Lemma 2.4. *The functor FP is right adjoint to the evaluation functor that takes a Mackey functor $\underline{M}$ to the $\mathbb{Z}[G]$-module $\underline{M}(e)$.*

Proof. Given a morphism $\alpha \colon \underline{M}(e) \to V$ of $\mathbb{Z}[G]$-modules we show that there is a unique morphism $\underline{\alpha} \colon \underline{M} \to FP(V)$ of Mackey functors (the converse association is easy, and is just given by evaluation at e). This is defined by setting

$$\underline{\alpha}(H) \colon \underline{M}(H) \to V^H$$

to be the composite

$$\underline{M}(H) \xrightarrow{R_e^H} \underline{M}(e) \xrightarrow{\alpha} V^H.$$

Note that this does indeed land in V^H as h acts trivially on $\underline{M}(H)$ and commutes with α, so that $c_h \alpha R_e^H = \alpha R_e^H$. It follows that we get a commutative diagram

$$
\begin{array}{ccc}
\underline{M}(H) & \xrightarrow{\underline{\alpha}(H)} & V^H \\
{\scriptstyle R_e^H}\downarrow & & \downarrow \\
\underline{M}(e) & \xrightarrow{\alpha} & V
\end{array}
$$

We must show that

$$\underline{\alpha}(H) I_K^H = I_K^H \underline{\alpha}(K)$$

and

$$\underline{\alpha}(H) R_K^H = R_K^H \underline{\alpha}(K).$$

We show the first, and leave the second as an exercise. We have[1]

$$\underline{\alpha}(H) I_K^H = \alpha R_e^H I_K^H = \alpha \sum_{hK \in H/K} c_h R_e^K = \sum_{hK \in H/K} h{\cdot}\alpha R_e^K = I_K^H \alpha R_e^K = I_K^H \underline{\alpha}(K).$$

The second equation is similar, but simpler, and we leave it for the reader. $\square$

Remark 2.5. Evaluation also has a left adjoint, the 'fixed quotient' Mackey functor, defined by $\underline{M}(H) = M_H$, the largest quotient of M on which H acts trivially.

Remark 2.6. A special case of the fixed-point Mackey functor comes from taking an abelian group M considered with trivial G-action. This gives the *constant Mackey functor* $\underline{M}$ with $\underline{M}(H) = H$, restriction and conjugation the identity, and transfer from K to H given by multiplication by the index of K in H.

Example 2.3

There is a C_2-Mackey functor described by the Lewis diagram

$$
\begin{array}{c}
\mathbb{Z} \\
{\scriptstyle \Delta}\Big(\ \Big){\scriptstyle \nabla} \\
\mathbb{Z} \oplus \mathbb{Z} \\
\circlearrowright \\
\text{swap}
\end{array}
$$

This is an example of a fixed-point Mackey functor applied to the free module $\mathbb{Z}[C_2]$.

---

[1]Note that $H\backslash G/K = H\backslash G$ if $K = e$.

The Burnside ring

Another example of a Mackey functor comes from the Burnside ring. This is very important in both the theory of Mackey functors and in equivariant homotopy theory, as it plays the role of the unit $\mathbb{Z}$.

Definition 2.7. The Burnside ring of a finite group G is the Grothendieck group of the category of finite G-sets under coproduct. More explicitly,

$$A(G) := \mathbb{Z}\{\text{isomorphism classes of finite } G\text{-set}\}/\langle[X \amalg Y] - [X] - [Y]\rangle.$$

Remark 2.8. Because every finite G-set decomposes into a coproduct of orbits and the isomorphism type of an orbit G/H only depends on the conjugacy class of H, we have an additive decomposition

$$A(G) \cong \bigoplus_{\text{Conj}(G)} \mathbb{Z}.$$

Suppose $H \leq G$ then there are maps $A(G) \to A(H)$ induced by restriction of G-sets, and $A(H) \to A(K)$, induced by taking an H-set X to the G-set $G \times_H X$. There is also a conjugation map $c_g \colon A(H) \to A(H^g)$ defined by taking a H-set X defined by $H \to \text{Aut}(X)$ to the H^g-set defined by $H^g \xrightarrow{c_g^{-1}} H \to \text{Aut}(X)$.

Definition 2.9. The Burnside Mackey functor $\mathbb{A}_G$ is the Mackey functor with $\mathbb{A}_G(H) = A(H)$, and structure maps as in the previous remark.

Remark 2.10. Sending a G-set X to the associated permutation representation, we get a ring morphism $A(G) \to R(G)$, which extends to a morphism of Mackey functors.

Example 2.4

Let $G = C_p$, then there are two orbits, namely C_p/C_p and C_p/e, so we have
$$A(C_p) \cong \mathbb{Z}\{C_p/C_p, C_p\}$$
while clearly
$$A(e) \cong \mathbb{Z}\{e\}.$$
The Mackey functor looks as follows

$$\mathbb{Z}\{C_p/C_p, C_p/e\}$$
$$R \Big\downarrow \Big\uparrow I$$
$$\mathbb{Z}\{e\}$$
$$\circlearrowleft$$

where the maps are determined by $C_p/C_p \mapsto e$, $C_p/e \mapsto p$ (essentially, counting the number of points in the set), while the transfer sends e to

> C_p/e (it takes p copies of the singleton, with action that permutes the copies). The Weyl group action is trivial.

Induction, restriction and inflation for Mackey functors

We have seen that we can defined (co)induction and restriction for Mackey functors. A similar construction exists for Mackey functors. The restriction functor is particularly simple.

Definition 2.11. Given a G-Mackey functor $\underline{M}$ and a subgroup $H \leq G$ the restricted Mackey functor H-Mackey functor $\mathrm{Res}_H^G \underline{M}$ is the Mackey functor with $\underline{M}(K) = \underline{M}(K)$ for $K \leq H$.

Remark 2.12. Although simple, there is one small subtlety to be aware of: when we restrict the Mackey functor we have less conjugation maps available: the Weyl group is smaller. For example, restriction to the trivial group gives a functor to $\mathrm{Mack}_e \simeq \mathrm{Mod}_\mathbb{Z}$, i.e., we have to Weyl group action anymore.

Remark 2.13. Similarly to representations, this functor has a left and right adjoint. From our definition of a Mackey functor, this is a little annoying to define, but really this comes from similar functors on the category of finite G-sets. Indeed, we have indexed our Mackey functors on subgroup $H \leq G$, but we could equally well have done this on the G-set G/H. Since any finite G-set decomposes as a disjoint union of orbits, we can define $\underline{M}(X)$ for X a finite G-set to be the direct sum of $\underline{M}(G/H)$ for the orbits appearing in a decomposition of X.

In fact, one can define a Mackey functor as a bifunctor $\underline{M} = (\underline{M}^*, \underline{M}_*)$ from finite G-sets to abelian groups satisfying some axioms: here $\underline{M}^*$ and $\underline{M}_*$ are contravariant and covariant functors that agree on objects.

Definition 2.14. Let $\underline{M}$ be a H-Mackey functor, then the induction from H to G, $\mathrm{Ind}_H^G(\underline{M})$ is the G-Mackey functor defined on a finite G-set X by

$$\mathrm{Ind}_H^G(\underline{M})(X) = \underline{M}(\mathrm{Res}_H^G X).$$

Remark 2.15. One can give an explicit description of this Mackey functor as

$$\mathrm{Ind}_H^G \underline{M}(K) = \bigoplus_{HgK \in H\backslash G/K} \underline{M}(H \cap K^g)$$

It is rather tedious to check the axioms for this, but see [4, Section 2] if you want the details.

Remark 2.16. Similarly, restriction is equivalently defined by

$$\mathrm{Res}_H^G(\underline{M})(X) = \underline{M}(\mathrm{Ind}_H^G X).$$

Remark 2.17. The following result is slightly surprising since induction is the left, but not right, adjoint of restriction on the category of finite G-sets.

Proposition 2.18. *Induction for Mackey functors is both the left and right adjoint of restriction.*

Proof. Let

$$\epsilon\colon (G \times_H X \downarrow_H^G) \to X \quad \text{and} \quad \eta\colon Y \to (G \times_H Y) \downarrow_H^G$$

by the counit and unit for the adjunction on the category of finite G-sets. Here ϵ maps the class of (g, x) to gx and η maps y to the class of $(1, y)$.

To show that induction is left adjoint, we define the counit and unit on Mackey functors

$$p(N)\colon \operatorname{Ind}_H^G \operatorname{Res}_H^G N \to N, \quad N_*(\epsilon)$$

and

$$q(M)\colon M \to \operatorname{Res}_H^G \operatorname{Ind}_H^G M$$

by specifying their value on a G-set X and a H-set Y by

$$p(N) = N_*(\epsilon) \quad \text{and} \quad q(M) = M_*(\eta).$$

That this defines an adjunction then follows from the fact that ϵ and η are the counit and unit of an adjunction.

To show induction is right adjoint is very similar, except we use the contravariant structure M^* instead of the covariant structure. $\qquad \square$

Example 2.5

What is $\operatorname{Ind}_e^{C_p}(\mathbb{Z})$? Note that on G-sets we have that $\operatorname{Res}_e^G(C_p/C_p)$ is a point, while $\operatorname{Res}_e^G(C_p/e)$ is p points. We deduce that

$$\operatorname{Ind}_e^{C_p} \mathbb{Z} = \Delta \bigg(\!\!\begin{array}{c} \mathbb{Z} \\ \downarrow \uparrow \\ \mathbb{Z}[C_p] \end{array}\!\!\bigg) \nabla$$

$$\overset{\curvearrowright}{\underset{W}{\bigcup}}$$

where the Weyl group acts by cyclic permutations. When $p = 2$, we have seen this Mackey functor in Example (2.3).

If you want a fun exercise, you can try and compute $\operatorname{Ind}_{C_2}^{C_4} \underline{M}$ for $\underline{M} \in \operatorname{Mack}_{C_2}$.

There is one final functor that we would like to introduce for Mackey functors, and that is inflation.

Definition 2.19. Let $N \trianglelefteq G$ be a normal subgroup. We define the inflation functor $\operatorname{Mack}_{G/N} \to \operatorname{Mack}_G$ by

$$\operatorname{Inf}_{G/N}^G M(H) = \begin{cases} M(H/N) & \text{if } N \leq H \\ 0 & \text{else.} \end{cases}$$

Restriction, transfer, and conjugation are defined in the obvious way.

> **Example 2.6**
>
> For any abelian group M (seen as an object of Mack_e), and group G, there is an inflated Mackey functor $\text{Inf}_{G/G}^G M$ whose value is M at G and is zero elsewhere.

> **Example 2.7**
>
> Let $G = C_{p^2}$ and $N = C_p$, and suppose $M \in \text{Mack}_{C_p^2/C_p} \simeq \text{Mack}_{C_p}$. Then $N := \text{Inf}_{C_p}^{C_p^2}$ is the Mackey functor with $N(C_{p^2}) = \underline{M}(C_p), N(C_p) = \underline{M}(e)$ and $N(e) = 0$.

Inflation has both a left and a right adjoint.

Definition 2.20. Let $M \in \text{Mack}_G$ and $N \trianglelefteq G$, then we define G/N-Mackey functors M^+ and M^- by

$$M^+(G/K) = M(G)/ \sum_{J \leq K, J \ngeq N} I_J^K M(J)$$

and

$$M^-(G/K) = \bigcap_{J \leq K, J \ngeq N} \ker(R_J^K)$$

Restrictions, transfer, and conjugation come from those for L. These define functors

$$^+, \, ^- \colon \text{Mack}_{G/N} \to \text{Mack}_G .$$

Proposition 2.21. $^+$ *is left adjoint to* $\text{Inf}_{G/N}^G$ *and* $^-$ *is right adjoint to* $\text{Inf}_{G/N}^G$.

Proof. Let L be a Mackey functor for G and M a Mackey functor for G/N. Given a morphism $L \to \text{Inf}_{G/N}^G M$, we note that this is necessarily zero on $L(J)$ with $J \ngeq N$, and hence must vanish on $\sum_{J \leq K, J \ngeq N} I_J^K M(J)$ for each subgroup K. We therefore get an induced morphism $L^+ \to M$. Conversely, given $\beta \colon L^+ \to M$ we can construct a morphism $L \to \text{Inf}_{G/N}^G M$ by defining it to be 0 on $L(J)$ when $J \ngeq N$, and defining it to be the composite

$$L(K) \to L(K)/ \sum_{J \leq K, J \ngeq N} I_J^K L(J) = L^+(K/N) \xrightarrow{\beta} M(K/N) = \text{Inf}_{G/N}^G M(K)$$

when $K \geq N$. One can check that this defines a map of Mackey functors, and that this defines an adjunction. The proof for $^-$ is similar. $\qquad\square$

Remark 2.22. From the perspective of G-sets, inflation arises in the following way: a functor $M \colon \text{Fin}_{G/N} \to \text{Mod}_\mathbb{Z}$ gives rise to a functor $\text{Fin}_G \to \text{Mod}_\mathbb{Z}$ by composition with the fixed-point functor $\text{Fin}_G \to \text{Fin}_{G/N}$.

The category of G-spaces

Definition 3.1. A G-spaces is a topological space X with a continuous action $G \times X \to X$ (this is the special case of Definition 1.1 in the case $\mathcal{C} = \mathrm{Top}$). We let $G\,\mathrm{Top}$ denote the category of G-spectra, where the morphisms are morphisms of topological spaces that commute with the G-action. By taking based spaces, we can define $G\,\mathrm{Top}_*$, the category of *based G-spaces*.

Example 3.1

We can always give a topological space the trivial action. However, there are of course many more examples. For example, the n-sphere S^n has a natural C_2-action, giving by sending a point to its antipode. This is the same as saying that the non-trivial element of C_2-acts by -1. This action is *free*, in the sense it has no fixed-points.

Example 3.2

Here is an important example, and is the reason why representations show up in equivariant homotopy theory. Let V be a real representation of G of dimension n. We let S^V denote the one-point compactification of V. This is (non-equivariantly) an n-sphere S^n. This has a natural G-action where the new point ∞ is fixed by G. In fact, S^V is a based topological space, with base point ∞. If $V = \mathbb{R}^n$, this is S^n with the trivial action, so our representation spheres generalize the usual notion of a sphere.

For a non-trivial example of a representation sphere, we can consider $G = C_2$ and the sign representation $\mathbf{1}_{\mathrm{sgn}}$. In this case, we can draw the sphere $S^{\mathbf{1}_{\mathrm{sgn}}}$ (usually denoted S^σ or $S^{1,1}$)

This is simply the circle with the C_2-action given by reflection across

---

the equator. The two marked points are the only points fixed by the C_2-action.

We can also consider the regular representation ρ. This is a two dimensional representation, and so the underlying space is a 2-sphere. We can think of it as the one point complexification of $\mathbb{C}$, also known as $\mathbb{C}P^1$, equipped with the complex conjugation action. Since $\mathbb{R} \subseteq \mathbb{C}$ is fixed by conjugation, we see that $\mathbb{R}P^1 \subseteq \mathbb{C}P^1$ is fixed by this action.
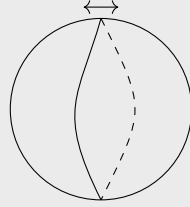
The C_2-action is again given by reflection across an equator.

Notation 3.2. Let X and Y be based spaces, then we let $\mathrm{Map}_G(X, Y) \subseteq \mathrm{Map}(X, Y)$ denote the space of G-equivariant maps $X \to Y$, with the subspace topology. Moreover, we can write define a G-space $G\mathrm{Map}(X, Y)$ to have underlying space $\mathrm{Map}(X, Y)$ with G-action given by

$$(g \cdot f)(x) := g^{-1}f(gx).$$

The two are related: We have $G\mathrm{Map}(X, Y)^G = \mathrm{Map}_G(X, Y)$ as topological spaces, essentially by definition.

Remark 3.3. Given a group homomorphism $H \to G$ we get a pullback functor $\phi^* \colon G\,\mathrm{Top} \to H\,\mathrm{Top}$ is the obvious way. For an injective homomorphism, this gives us restriction functors $\mathrm{Res}_H^G \colon G\,\mathrm{Top} \to H\,\mathrm{Top}$. This has a left and right adjoint, as we now explain.

Definition 3.4. Let X be a H-space, then we define a G-space $\mathrm{Ind}_H^G X = G \times_H X$ as the quotient of $G \times X$ by the relation $(g \cdot h, x) \sim (g, h \cdot x)$ for $g \in G, h \in H$.

Example 3.3

If $H = e$ is the trivial group, then $\mathrm{Ind}_e^G X \cong G \times X$ is a free G-space.

Example 3.4

We have $\mathrm{Ind}_H^G(*) \cong G/H$. More generally, if X is a space with trivial H-action, we have $\mathrm{Ind}_H^G(X) \cong X \times G/H$, where G acts on G/H, and trivially on X.

In analogy with Mackey functors, we have the following:

Proposition 3.5. *The functor* $\mathrm{Ind}_H^G \colon H\,\mathrm{Top} \to G\,\mathrm{Top}$ *is left adjoint to* $\mathrm{Res}_H^G \colon G\,\mathrm{Top} \to H\,\mathrm{Top}$.

Remark 3.6. There is a based version, which sends X to $G_+ \wedge_H X$, the quotient of $G_+ \wedge X$ by the same relation as before.

Remark 3.7. Similarly, there is a functor $\mathrm{Coind}_H^G \colon H\,\mathrm{Top} \to G\,\mathrm{Top}$, which sends X to $\mathrm{Map}_H(G, X)$, the space of H-equivariant maps from G to X, with G-action given by $(gf) \cdot g' = f(g \cdot g')$. This functor is right adjoint to restriction.

  In contrast to what we saw for Mackey functors, the left and right adjoint do not agree for spaces (Exercise: try and find an example).

Remark 3.8. In fact, given $f \colon H \to G$ a group homomorphism, the pullback functor $f^* \colon G\,\mathrm{Top} \to H\,\mathrm{Top}$ always has a left and right adjoint. Taking H to be the trivial group we can get functors called fixed points and orbits, which we now explicitly define.

Definition 3.9. For $X \in G\,\mathrm{Top}$, the fixed points X^G is the subspace

$$X^G = \left\{\, x \in X \mid g \cdot x = x \text{ for all } g \in G \,\right\} \subseteq X.$$

This defines a functor $(-)^G \colon G\,\mathrm{Top} \to \mathrm{Top}$, which is right adjoint to pullback along $G \to e$.

  More generally, we can define X^H as the composite

$$G\,\mathrm{Top} \xrightarrow{\ \mathrm{Res}_H^G\ } H\,\mathrm{Top} \xrightarrow{\ (-)^H\ } \mathrm{Top}\,.$$

> **Example 3.5**
>
> Returning to the representation spheres considered in Example (3.2) we have $(S^\sigma)^{C_2} = S^0$, while $(S^\rho)^{C_2} = S^1$.

Definition 3.10. For $X \in G\,\mathrm{Top}$, the orbit of a point $x \in X$ is the set

$$G \cdot x = \left\{\, g \cdot x \mid g \in G \,\right\}$$

We can define an equivalence relation on X by saying that $x \sim y$ if and only if there exists a $g \in G$ such that $g \cdot x = y$. The quotient by this equivalence relation is the orbit space X/G of the action. This is the left adjoint of pullback along $G \to e$.

Remark 3.11. In summary: we have

$$\mathrm{Map}_G(Y, X) \simeq \mathrm{Map}(Y, X^G)$$

and

$$\mathrm{Map}_G(X, Y) \cong \mathrm{Map}(X/G, Y)$$

where X is a G-space and Y has the trivial G-action.

There is an obvious notion of homotopy in equivariant homotopy.

Definition 3.12. Two equivariant maps $f, g\colon X \to Y$ between G-spaces are said to be G-homotopic if there is an equivariant map $h\colon I \times X \to Y$ such that $h|_{\{0\}\times X} \simeq f$ and $h|_{\{1\}\times X} \simeq g$. Here we give I the trivial G-action.

> ### Example 3.6
>
> Let $* \mathrel{\substack{\infty \\ \xrightarrow{\hspace{1.2em}} \\ 0}} S^\sigma$ be the inclusion of the two fixed-points of S^σ. Because S^∞ is path-connected, these are non-equivariantly homotopic. But they are not equivariantly homotopic, since the space of fixed points is not path-connected.

We can now define G-homotopy groups, G-homotopy equivalences, etc. Note that by adjointness, the notion of equivariant homotopy will just be computing $\pi_n(X^G)$. Instead, it is better to remember the fixed points of all subgroups $H \leq G$. To fix notation, let $\pi_n^H(X) := \pi_n(X^H)$.

Definition 3.13. A morphism $f\colon X \to Y$ of G-spaces is a weak G-homotopy equivalence if the induced maps $\pi_n^H(f)\colon \pi_n^H(X) \to \pi_n^H(Y)$ are isomorphisms for all $H \leq G$.

Remark 3.14. By playing around with adjunctions we have $\pi_n^H(Y) = [G/H \times S^n, X]$, and so this leads naturally to the notion of G-CW-complex.

Definition 3.15. A G-CW-complex is a sequential colimit of spaces X_n where X_{n+1} is a pushout (in $G\,\mathrm{Top}$)

$$
\begin{array}{ccc}
\coprod G/H \times S^n & \longrightarrow & X_n \\
\downarrow & & \downarrow \\
\coprod G/H \times D^{n+1} & \longrightarrow & X_{n+1}
\end{array}
$$

and H ranges over all the subgroups of G. There is also a relative version where we allow $X_{-1} = A$, for some G-space A.

> ### Example 3.7
>
> We consider again the representation sphere S^σ. Here we start with the C_2-fixed 0-skeleton S^0. Here we attach a single free 1-cell $(C_2/e) \times D_1$ where one (free) endpoint is attached to 0 and the other to the point ∞.

The G-Whitehead theorem

Our next goal is to prove the equivariant Whitehead theorem.

Theorem 3.16 (Equivariant Whitehead theorem)**.** *A weak G-homotopy equivalence between G-CW complexes is a G-homotopy equivalence.*

There are two ways to prove this theorem, both generalizing the two non-equivariant ways. The first is an inductive argument, and can be found in [2, Theorem 2.1.31]. The second is to prove a more general theorem, the equivariant homotopy extension and lifting property (HELP). The main subtly when extending the proof from the non-equivariant case is to define a suitable notion of the connectivity of a map.

Definition 3.17. Let $\theta\colon \mathrm{Sub}(G)/G \to \left\{\, x \in \mathbb{Z} \,\middle|\, x \geq -1 \,\right\}$. A map $f\colon X \to Y$ of G-spaces is θ-connected if f^H is $\theta(H)$-connected for all $H \leq G$. A G-CW complex is θ-dimensional if all cells of orbit type G/H have non-equivariant dimension at most $\theta(H)$.

With these definitions in place, we now have an equivariant version of the HELP theorem.[1]

Proposition 3.18. *Let (A, X) be a θ-dimensional relative G-CW complex, and let $e\colon Y \to Z$ be a θ-connected G-map between G-CW complexes. Given $g\colon A \to Y$, $h\colon A \times I \to Z$, and $f\colon X \to Z$ such that $eg = hi_1$ and $f|_A = hi_0$, there exist maps $\widetilde{g}\colon X \to Y$ and $\widetilde{h}\colon X \times I \to Z$ that make the following diagram commute:*


Sketch of proof. The desired maps are constructed by induction on the skeleton of X, and then cell by cell, so we can assume that $X = G/H \times D^{n+1}$ and $A = G/H \times S^n$. By playing around with adjunctions, this reduces to the ordinary non-equivariant HELP theorem with $e\colon Y^H \to Z^H$, $X = D^n$ and $A = S^{n-1}$. $\qquad\square$

Remark 3.19. An equivalent way of phrasing it: under the assumption of the theorem, any diagram

$$
\begin{array}{ccc}
A \times \{1\} & \xrightarrow{\ g\ } & Y \\
\downarrow & & \downarrow{\scriptstyle e} \\
A \times I \cup X \times \{0\} & \xrightarrow[h \cup f]{} & Z
\end{array}
$$

can be completed to the diagram

$$
\begin{array}{ccc}
A \times \{1\} & \xrightarrow{\;\;g\;\;} & Y \\
& X \times \{1\} & \\
& X \times I & \\
A \times I \cup X \times \{0\} & \xrightarrow{\;h \cup f\;} & Z
\end{array}
$$
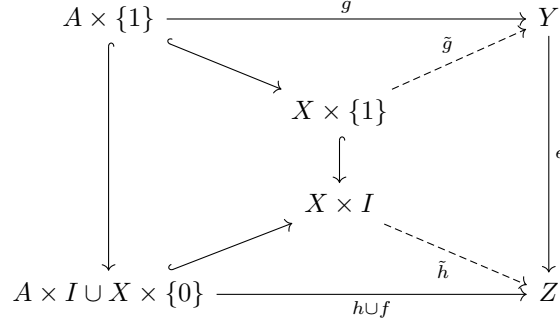
Theorem 3.20. *Let X be a G-CW complex and let $e\colon Y \to Z$ be a θ-connected map between G-CW complexes, then*

$$
e_*\colon [X, Y]_G \to [X, Z]_G
$$

is a bijection if $\dim(X) \le \theta$ and surjective if $\dim X = \theta$. In particular, if e is a weak equivalence and X is any G-CW complexes, then e_* is a bijection.

Proof. Taking $A = \emptyset$ gives a map $\tilde{g}\colon X \to Y$ and a homotopy $\tilde{h}\colon X \times I \to Z$ satisfying $h(-,0) = f$ and $h(-,1) = e \circ \tilde{g}$, hence $[f] = [e \circ \tilde{g}] \in [X, Z]_G$ and therefore e_* is surjective. Now assume we are given two maps $g_0, g_1\colon X \to Y$ and an equivalence $[e \circ g_0] = [e \circ g_1]$. In particular, there exists a G-homotopy $H\colon X \times [0,1] \to Z$.

We want to apply HELP to the pair $(X \times [0,1], X \times \{0,1\})$. We define $g\colon X \times \{0,1\} \to Y$ to be e_0 at time 0, and e_1 at time 1. Similarly, we define $h\colon X \times \{0,1\} \times I \to Z$ to be $H(x,s)$. The HELP theorem then gives us a G-homotopy $\tilde{g}$ between e_0 and e_1. $\qquad\square$

As a corollary, we obtain an equivariant version of Whitehead's theorem.

Corollary 3.21. *Any θ-connected G-map $e\colon Y \to Z$ between G-CW complexes of dimension less than θ is a G-homotopy equivalence. In particular, if e is a weak equivalence, then it is a G-homotopy equivalence.*

Proof. A map $f\colon Z \to Y$ such that $e_*[f] = \mathrm{id}$ is a homotopy inverse to e. $\qquad\square$

Remark 3.22. Using the equivariant version of HELP, one can also prove equivariant versions of cellular approximation and G-CW approximation.

Elmendorf's theorem

An abstract way to 'do' homotopy theory in a category is to put a model structure on it. In case you haven't seen these before, we will briefly state the definitions:

Definition 3.23. A weak factorization system (WFS) on a category $\mathcal{C}$ is a pair $(\mathcal{L}, \mathcal{R})$ of classes of morphisms of $\mathcal{C}$ such that:

- Every morphism $f \colon X \to Y$ of $\mathcal{C}$ may be factored as the composition of a morphism in $\mathcal{L}$ followed by one in $\mathcal{R}$:

$$f \colon X \xrightarrow{\in \mathcal{L}} Z \xrightarrow{\in \mathcal{R}} Y.$$

- The classes are closed under having the lifting property against each other:

  - $\mathcal{L}$ is precisely the class of morphisms having the left lifting property against every morphism in $\mathcal{R}$.
  - $\mathcal{R}$ is precisely the class of morphisms having the right lifting property against every morphism in $\mathcal{L}$.

$$\begin{array}{ccc} & \xrightarrow{e \in \mathcal{L}} & \\ u \downarrow & \overset{m}{\cdots\cdots} & \downarrow v \\ & \xrightarrow[w \in \mathcal{R}]{} & \end{array}$$

Definition 3.24 (Model Structure). A model structure on a category $\mathcal{C}$ is a choice of three distinguished classes of morphisms: cofibrations $\mathcal{C}$, fibrations $\mathcal{F}$, and weak equivalences $\mathcal{W}$, satisfying the following conditions:

- $\mathcal{W}$ contains all isomorphisms and is closed under two-out-of-three: given a composable pair of morphisms f and g, if two out of the three morphisms f, g, $g \circ f$ are in $\mathcal{W}$, then so is the third.

- $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ and $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ are two weak factorization systems on $\mathcal{C}$. The morphisms $\mathcal{F} \cap \mathcal{W}$ are called acyclic fibrations while those in $\mathcal{C} \cap \mathcal{W}$ are called acyclic cofibrations.

When a category $\mathcal{C}$ is a complete and cocomplete category with a model structure, we call it a model category.

Notation 3.25. Some more terminology: An object is cofibrant if the unique morphism $\emptyset \to X$ from the initial object is a cofibration, and is fibrant if the unique morphism $X \to *$ is a fibration.[2]

The homotopy category $\mathrm{Ho}(\mathcal{C})$ of a model category is the universal way of inverting the weak equivalences in $\mathcal{C}$: $\mathrm{Ho}(\mathcal{C}) \coloneqq \mathcal{C}[\mathcal{W}^{-1}]$.

Remark 3.26. This definition of homotopy category does not depend on the choice of fibrations and cofibrations. It only depends on the underlying category with weak equivalences. However, the model structure makes the homotopy category easier to handle. In fact, with a model structure, the homotopy category is equivalent to the category whose objects are those which are both fibrant and cofibrant, and morphisms are the equivalence classes of morphism under left homotopy. This definition of homotopy category avoids the set theory technical issues one may meet with while doing localization.

---

[2]If $\mathcal{C}$ is a model category, then it has an initial object, the colimit of the empty diagram, and a terminal object, the limit of the empty diagram.

Remark 3.27. The axioms for a model category are over-determined. Indeed, one can show that a map is a cofibration (a trivial cofibration) if and only if it has the left lifting property with respect to all trivial fibrations (fibrations). Dually, a map is a fibration (a trivial fibration) if and only if it has the right lifting property with respect to all trivial cofibrations (cofibrations). In particular, we only need to specify the weak equivalences and (co)fibrations, and then the remaining class is formally determined.

> ### Example 3.8
>
> The category Top of topological spaces has a model structure with fibrations as Serre fibrations, equivalences are the weak homotopy equivalences, and cofibrations are the retracts of relative cell complexes. We denote this by $\mathrm{Top}_{\mathrm{Quillen}}$.
>
> The category Set of sets has exactly 9 model structures on it!

Definition 3.28. Suppose that $\mathcal{C}$ and $\mathcal{D}$ are model categories. We say that a pair $F\colon \mathcal{C} \leftrightarrows \mathcal{D}\colon U$ of adjoint functors with F the left adjoint is a Quillen adjunction if the following equivalent conditions are satisfied:

- F preserves cofibrations and acyclic cofibrations;

- U preserves fibrations and acyclic fibrations;

- F preserves cofibrations and U preserves fibrations;

- F preserves acyclic cofibrations and U preserves acyclic fibrations.

Definition 3.29. Let $\mathcal{C}$ and $\mathcal{D}$ be model categories equipped with a Quillen adjunction $F\colon \mathcal{C} \leftrightarrows \mathcal{D}\colon U$. Then we say that $\mathcal{C}$ and $\mathcal{D}$ are Quillen equivalent if the derived adjunction $F\colon \mathrm{Ho}(\mathcal{C}) \leftrightarrows \mathrm{Ho}(\mathcal{D})\colon U$ is an equivalence of categories. This is equivalent to the following condition:

- For any cofibrant $X \in \mathcal{C}$ and fibrant $Y \in \mathcal{D}$, $FX \to Y$ is a weak equivalence if and only if the adjoint $X \to UY$ is an equivalence.

> ### Example 3.9
>
> There is a model structure (the Kan model structure) on simplicial sets, where a morphism $f\colon X \to Y$ is
>
> - cofibration if it is a monomorphism (i.e., a level-wise injection).
>
> - weak equivalence if its geometric realization is a weak equivalence in Top.
>
> Then there is a Quillen equivalence
>
> $$|-|\colon \mathrm{sSet}_{\mathrm{Kan}} \leftrightarrows \mathrm{Top}_{\mathrm{Quillen}}\colon S(-)$$

given by geometric realization and singular functors.

Proposition 3.30. *There is a model structure on $G\,\mathrm{Top}$ where a map $f\colon X \to Y$ is a*

- *fibration if $f^H\colon X^H \to Y^H$ is a fibration for all $H \leq G$.*

- *weak equivalence if $f^H\colon X^H \to Y^H$ is a weak equivalence for all $H \leq G$.*

Elmendorf's theorem relates this to the category of presheaves on the orbit category.

Definition 3.31. The orbit category of a finite group $\mathcal{O}_G$ is the category whose objects are the orbits G/H for $H \leq G$ a subgroup, and whose morphisms are the G-equivariant continuous functions. Equivalently, this is the full subcategory of $G\,\mathrm{Top}$ on the objects G/H.

Proposition 3.32. *There is a model structure on $\mathrm{Fun}(\mathcal{O}_G^{op}, \mathrm{Top})$ where the weak equivalences and fibrations are taken objectwise.*

Lemma 3.33. *There is an adjunction*

$$\theta\colon \mathrm{Fun}(\mathcal{O}_G^{op}, \mathrm{Top}) \leftrightarrows G\,\mathrm{Top}\colon \psi$$

where θ is evaluation at G/e and $\psi(X)(G/H) = X^H$.

Sketch of proof. We show this by constructing a unit and a counit. We first note that $\theta \circ \psi(X) = X^e$, and this comes with a G-action (by functoriality) that agrees with the original G-action. We can then take the counit $\epsilon\colon \theta \circ \psi \to \mathrm{id}$ to be the identity.

For the unit, note that for any $T\colon \mathcal{O}_G^{op} \to \mathrm{Top}$ and subgroups $K \leq H \leq G$, the map $T(G/H) \to T(G/K)$ factors through the fixed points $T(G/K)^H$. We then have

$$(\psi \circ \theta(T))(G/H) = (\psi T(G/e))(G/H) = T(G/e)^H,$$

and the factorization of the map $T(G/H) \to T(G/e)$ through $T(G/e)^H$ defines η. It is then straightforward to check the unit and counit identities. $\qquad\square$

Remark 3.34. Since the counit is an equivalence, the right adjoint is fully-faithful.

Theorem 3.35 (Elmendorf's theorem)**.** *The adjunction of Lemma 3.33 is a Quillen equivalence with respect to the model structures of Propositions 3.30 and 3.32.*

Sketch of proof. We first show that the adjunction is Quillen. But this is straightforward: the functor ψ preserves fibrations and weak equivalences essentially by definition.

Now in general, the unit of the adjunction of Lemma 3.33 is not an equivalence. But we claim that it is for any $\mathcal{X}$ that is cofibrant. Indeed, one can explicitly check that the cofibrant objects in the model structure on $\mathrm{Fun}(\mathcal{O}_G^{op}, \mathrm{Top})$ are retracts of the cellular objects, and these are generated under pushouts along inclusions and directed colimits by

$$\left\{ \mathrm{Map}_G(-, G/H) \times X \,\middle|\, H \subseteq G, X \text{ is a cell in } \mathrm{Top} \right\}$$

Now in general the fixed point functors $(-)^H$ do not preserve colimits, but one can show that they do preserve retracts, pushouts and directed colimits. One can then directly check the claim from the following sequence of equivalences:

$$\mathcal{X}(G/K) = \mathrm{Map}_G(G/K, G/H) \times Y = (G/H)^K \times Y = (G/H \times Y)^K = \mathcal{X}(G/e)^K$$

Now suppose we are given a morphism $f \colon \theta(\mathcal{X}) \to \mathcal{Y}$ in $G\,\mathrm{Top}$ (note that in fact every object in $G\,\mathrm{Top}$ is fibrant). We can factor the H-fixed points of the adjoint $g^H \colon \mathcal{X}(G/H) \to \psi(\mathcal{Y})^H = \mathcal{Y}^H$ as

$$\mathcal{X}(G/H) \xrightarrow[\cong]{\eta_{G/H}} \mathcal{X}(G/e)^H \xrightarrow{f^H} Y^H.$$

By the 2 out of 3 axiom, we see that g^H is an equivalence if and only if f^H is an equivalence. By the definition of the model structures, we deduce that g is an equivalence if and only if f is an equivalence, as required. $\qquad\square$

Remark 3.36. Elmendorf's original theorem was only a statement about homotopy categories. More specifically, one defines a homotopy inverse to ψ by using the bar construction.

Remark 3.37. Here is an application. Let $\mathcal{F}$ be a family of subgroups (i.e., a collection closed under taking subgroups and conjugations). Then we can construct a G-space determined by the property that

$$E\mathcal{F}^K = \begin{cases} * & \text{if } K \in \mathcal{F} \\ \emptyset & \text{if } K \notin \mathcal{F}. \end{cases}$$

Indeed, simply define $E\mathcal{F}'$ to be the presheaf with the analogous property, and define $E\mathcal{F}$ to be the image under the Quillen equivalence.

Forms of cohomology

There are several different flavors of cohomology for equivariant homotopy. We begin with Borel (co)homology, before moving on to the more powerful Bredon (co)homology.

Borel cohomology

The simplest, and perhaps most common, form of equivariant cohomology is known as Borel equivariant cohomology.

The following can always be constructed, for example using the Milnor construction.

Definition 4.1. For any group G, we let EG denote a G-CW complex whose underlying space is contractible, and such that the action of G-is free. We let the orbit space EG/G be denoted by BG. This is known as the classifying space of the group G.

> **Example 4.1**
>
> Let $G = C_2$ then a model for EC_2 is S^∞ with the antipodal action. Here we have $S^\infty/C_2 \simeq \mathbb{R}P^\infty$, and so $BC_2 \simeq \mathbb{R}P^\infty$.
>
> More generally, taking the limit of the inclusion of the unit sphere $S^{2n-1} \subseteq \mathbb{C}^n$, we can think of S^∞ as the unit sphere inside of $\mathbb{C}^\infty$. We can think of C_n as acting on each complex coordinate as multiplication by an n-th root of unity. This gives a free action of C_n on S^∞, and so $EC_n \simeq S^\infty$. We deduce that $BC_n \simeq S^\infty/C_n$. This is known as an infinite-dimensional lens space.

Remark 4.2. It is a good exercise to check that $EG \times EH \simeq E(G \times H)$. In particular, we deduce that $B(G \times H) \simeq BG \times BH$.

Proposition 4.3. *The space BG is a $K(G,1)$, i.e., $\pi_1(BG) \simeq G$ and all other homotopy groups are trivial.*

Proof. This follows from the fact that $G \to EG \to BG$ is a fibration, and the long exact sequence in homotopy. $\square$

Definition 4.4. Give a G-space X, the *Borel construction* on X is the orbit space of the diagonal G-action on $EG \times X$. This is usually denoted $EG \times_G X$.

The Borel equivariant homology and cohomology are defined by

$$H_*^{Borel}(X) := H_*(EG \times_G X) \quad \text{and} \quad H_{Borel}^*(X) := H_*(EG \times_G X).$$

Remark 4.5. When compared with Bredon cohomology, this is much easier to compute. On the other hand, it can't tell the difference between EG and a point. More generally, we have the following:

Definition 4.6. A map $f: X \to Y$ of G-spaces is an underlying equivalence if it induces an equivalence on underlying spaces.

The following follows essentially by definition (check on fixed points!):

Proposition 4.7. *The functor $EG \times (-): G\operatorname{Top} \to G\operatorname{Top}$ takes underlying equivalences to G-equivalences. Therefore, Borel cohomology takes underlying equivalences to cohomology isomorphisms.*

Remark 4.8. Taking X to be a point, we see that

$$H_*^{Borel}(*) = H_*(BG) \quad \text{and } H_{Borel}^*(*) = H^*(BG).$$

It is either a theorem, or a definition, that this is the group (co)homology of the group G.

Bredon cohomology

There is a more refined version of equivariant cohomology known as Bredon cohomology. We need the following definition:

Definition 4.9. Let G be a finite group, then a *coefficient system* is a presheaf $X \in \operatorname{Fun}(\mathcal{O}_G^{op}, \operatorname{Ab})$.

Note that Elemendorf's theorem says that for any coefficient system we can produce an Eilenberg–MacLane G-space.

> ### Example 4.2
>
> Any Mackey functor $\underline{M}$ gives rise to a coefficient system by forgetting the data of the transfer map. Strictly speaking, given our definition of Mackey functors, this is not entirely obvious: one needs to note that any G-map $\phi: G/H \twoheadrightarrow G/K$ can be factored (not uniquely) as a composite $G/H \twoheadrightarrow G/K^\gamma \cong G/K$ where H is contained in the conjugate K^γ. Then the map $M(\phi): M(G/K) \to M(G/H)$ is defined by to
>
> $$\underline{M}(K) \xrightarrow{c_\gamma} \underline{M}(K^\gamma) \xrightarrow{R_H^{K^\gamma}} \underline{M}(H).$$
>
> One can check that the composite does not depend on the choice of factorization.

Example 4.3

For $n \geq 2$ the equivariant homotopy groups $\pi_n(X^H)$ for $H \leq G$ assemble into a coefficient system denoted $\underline{\pi}_n(X)$. Indeed, the functor $G/H \mapsto X^H$ defines a coefficient system valued in spaces, and then we simply apply π_n to this. This clearly applies in other contexts, for example the assignment $\underline{H}_n(X; A) := H_n(X^K; A)$ determines a coefficient system.

Example 4.4

Let $G = C_2$ and $X = S^\sigma$. Then $X^{C_2} \simeq S^0$ and $X^1 \simeq S^1$. It follows that

$$\underline{H}_0(S^\sigma; \mathbb{Z}) \cong \quad \begin{array}{c} \mathbb{Z}^2 \\ \downarrow {\scriptstyle \nabla} \\ \mathbb{Z} \end{array} \qquad \underline{H}_1(S^\sigma; \mathbb{Z}) \cong \quad \begin{array}{c} 0 \\ \\ \mathbb{Z}_{\mathrm{sgn}} \end{array}$$

Here we have written $\mathbb{Z}_{\mathrm{sgn}}$ for $\underline{H}_1(S^\sigma)(C_2/e)$ since the nontrivial element of C_2 acts as an orientation-reversing map on S^σ.

Example 4.5

Here is an important example: for a G-CW complex X, the functor that sends G/K to $C_n^{cell}(X^K)$ defines a coefficient system denoted $\underline{C_n}(X)$. This relies on the observation that the H-fixed points of a G-CW complex inhert a CW-structure.

Bibliography

[1] Andrew Blumberg, *Lecture Notes on Equivariant Stable Homotopy Theory*, Spring 2017, Available at https://web.ma.utexas.edu/users/a.debray/lecture_notes/m392c_EHT_notes.pdf. iii

[2] Bertrand Guillou, *Class notes - Equivariant Homotopy and Cohomology, Math 751*, Fall 2020, Available at https://www.ms.uky.edu/~guillou/F20/751Notes.pdf. iii, 10, 22

[3] J. P. May, *A concise course in algebraic topology*, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL, 1999. MR 1702278 22

[4] Hiroki Sasaki, *Green correspondence and transfer theorems of Wielandt type for G-functors*, J. Algebra **79** (1982), no. 1, 98–120. MR 679973 15