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MA8403 - Equivariant homotopy theory

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Preface

This are the courses notes for MA8403 - Equivariant homotopy theory, held during the Autumn semester 2023 at NTNU. The notes are mainly based on two excellent sets of lectures notes, one by Guillou [3], and one by Blumberg [2]. The notes will be continually updated during the semester.

Contents

Chapter 1

Equivariance in algebra

Group actions in algebra

We recall that if X is an object in a category $\mathcal{C}$, then the set of endomorphisms $\mathrm{End}(X)$ forms a monoid (that is, a set equipped with an associative binary operation and an identity element) under composition. The set of automorphisms of X (that is, those endomorphisms that are invertible) form a group. Moreover, we have

$$\mathrm{Aut}(X) = \mathrm{End}(X) \cap \mathrm{Iso}(\mathcal{C})$$

Note that any group is a monoid, simply by forgetting the existence of inverses.

Definition 1.1. An action of a group G on an object $X \in \mathcal{C}$ is a monoid homomorphism $a\colon G \to \mathrm{End}(X)$, or equivalently a group homomorphism $a\colon G \to \mathrm{Aut}(X)$ (note that a monoid homomorphism between groups is a group homomorphism).

Remark 1.2. Unwinding the definition, this means that we have:

a. For each $g \in G$, there is a morphism $a(g)\colon X \to X$.

b. a preserves composition, i.e., $a(g \cdot h) = a(g) \cdot a(h)$.

c. a preserves identifies, so $a(e) = \mathrm{id}_X$.

Example 1.1

Let $\mathcal{C}$ be the category of sets and functions. Then, $a\colon G \to \mathrm{End}(X) = \left\{ f\colon X \to X \right\}$ correspond to a function $\overline{a}\colon G \times X \to X$. The conditions above mean that the diagrams

$$\begin{tikzcd} G \times G \times X \arrow[r, "m\times\mathrm{id}"] \arrow[d, "\mathrm{id}\times\overline{a}"'] & G \times X \arrow[d, "\overline{a}"] \\ G \times X \arrow[r, "\overline{a}"'] & X \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \{*\} \times X \arrow[r, "e\times\mathrm{id}"] \arrow[d, "\cong"'] & G \times X \arrow[d, "\overline{a}"] \\ X \arrow[r, equal] & X \end{tikzcd}$$

commute, where $m\colon G\times G \to G$ denotes the group multiplication. Equivalently, in symbols, we have

$$\overline{a}(g, \overline{a}(h, x)) = \overline{a}(gh, x)$$

and

$$\overline{a}(e, x) = x.$$

Remark 1.3. Let BG denote the category with one object $*$ and with $\mathrm{Hom}(*,*) = G$. Then an action of G in the category $\mathcal{C}$ is the same as a functor $\rho\colon BG \to \mathcal{C}$. The object X in the previous definition is the object $\rho(*) \in \mathcal{C}$.

Remark 1.4. Let $\mathcal{C} = \mathrm{Mod}_R$ for a commutative ring R. An action of G on $M \in \mathrm{Mod}_R$ is a monoid homomorphism

$$a\colon G \to \mathrm{Hom}_R(M, M).$$

We recall that $\mathrm{Hom}_R(M, M)$ actually has the structure of an R-algebra

Definition 1.5. The (R-linear) group ring on R is the R-algebra $R[G]$ whose:

(a) underlying R-module is the free R-module with basis on the underlying set of G.

(b) whose multiplication is given on basis elements by the group operation.

Example 1.2

Let $R = \mathbb{Z}$ and $G = C_2 = \langle\sigma\rangle$. An element of $\mathbb{Z}[C_2]$ is of the form $a + b\sigma$ where $a, b \in \mathbb{Z}$. Multiplication is given by

$$(a_1 \cdot 1 + b_1\sigma) \cdot (a_2 \cdot 1 + b_2\sigma) = (a_1a_2 + b_1b_2) \cdot 1 + (a_1b_2 + b_1a_2)\sigma.$$

This is the same thing as the polynomial ring $\mathbb{Z}[\sigma]/(\sigma^2 - 1)$.

Remark 1.6. Categorically, the group ring construction is left adjoint to the functor that takes an R-algebra to its group of units, i.e., there is an adjudication

$$R[-]\colon \mathrm{Grp} \leftrightarrows \mathrm{Alg}_R \colon (-)^\times$$

Returning to group actions, we have the following:

Proposition 1.7. *Let R be a commutative ring, and G a finite group. The following data on an R-module M are equivalent:*

a. *A monoid homomorphism $G \to \mathrm{End}_R(M)$.*

b. *A group homomorphism $G \to \mathrm{Aut}_R(M)$.*

c. *A homomorphism of R-algebras $R[G] \to \mathrm{Hom}_R(M, M)$.*

d. *An $R[G]$-module structure on M whose underlying R-module structure is M.*

Definition 1.8. A representation of G over R is an $R[G]$-module.

Example 1.3

If $R = k$ is a field, then the underlying R-module is a k-vector space V. If $\dim_k(V) = n$, then $\mathrm{Aut}_k(V) = GL_n(k)$, and a k-representation is the same thing as a group homomorphism $G \to GL_n(k)$.

Definition 1.9. The $R[G]$-module $R[G]$ is known as the regular representation. More generally, if X is a finite G-set, then the free R-module $R[X]$ inherits the structure of a $R[G]$-module (the case of $R[G]$ itself corresponds to the finite G-set G, considered as an $R[G]$-module over itself). Representations obtained this way are known as permutation representations.

Example 1.4

Taking X to be the trivial G-set, we obtain the (one-dimensional) trivial representation. This is simply the $R[G]$-module R, where G acts trivially.

Definition 1.10. Let $G = C_2 = \langle \tau \rangle$, and suppose that $-1 \neq 1 \in R$. Then the sign representation of G is the one-dimensional representation where τ acts as -1 (if $-1 = 1$ in R this still makes sense, but is just the trivial representation). Note that this is an example of a representation that is not a permutation representation.

Example 1.5

Let $C_n = \langle \sigma \rangle$ be the cyclic group of order n. Let us calculate all complex 1-dimensional representations of C_n, i.e., homomorphisms $\rho \colon C_n \to \mathbb{C}$. Note that if we define $\rho(\sigma) = c$, then $\rho(\sigma^n) = c^n = 1$, so that c must be an n-th root of unity. There are precisely n-of these (take $\zeta_n = e^{2\pi/ni}$), and so there are precisely n-representations. For example, when $G = C_4$, the four representations correspond to sending σ to either $1, i, -1$ or $-i$. Note that we can also consider these as 2-dimensional *real* representations.

Notation 1.11. We let $\rho = \rho_G$ denote the regular representation of G, and the trivial n-dimensional representation by $\mathbf{n} = R^{\oplus n}$.

Definition 1.12. A subrepresentation is a submodule.

Example 1.6

The regular representation always has a one-dimensional trivial representation, generated by the sum $\sum_{g \in G} g$.

Definition 1.13. A representation V is irreducible if the only subrepresentations of V are 0 and V.

Theorem 1.14 (Maschke)**.** *Suppose that k is a field of characteristic not dividing $|G|$. Then every representation splits as a direct sum of irreducible representations.*

Proof. We prove the following: if $V \subseteq W$ is a subrepresentation, then there exists $U \subseteq W$ such that $U \oplus V \simeq W$.

To see this, let $\pi \colon W \to W$ be any k-linear projection of W onto V. This map need not be G-equivariant, but we can make it so by 'averaging'. That is, we define a new map $\phi \colon W \to W$ by

$$\phi(\mathbf{w}) = \frac{1}{|G|} \sum_{g \in G} g \cdot \pi(g^{-1} \cdot \mathbf{w}).$$

Moreover the map is G-equivariant: for $h \in G$ we have

$$\begin{aligned} \phi(h \cdot \mathbf{w}) &= \frac{1}{|G|} \sum_{g \in G} g \cdot \pi(g^{-1} \cdot h \cdot \mathbf{w}) \\ &= \frac{1}{|G|} \sum_{u \in G} u \cdot h \cdot \pi(u^{-1} \cdot \mathbf{w}) \\ &= h \cdot \phi(\mathbf{w}), \end{aligned}$$

where $u = gh^{-1}$, and so ϕ is $k[G]$-linear. Furthermore, the map ϕ is the identity on V By the splitting lemma, $W = V \oplus \ker(\phi)$. $\square$

Remark 1.15. We used the assumption on k to ensure that we could divide by $|G|$. Without that assumption, the theorem is false. Indeed, let $G = C_2$, $R = \mathbb{F}_2$ and consider the representation defined by $\rho(\tau) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. This is not irreducible, but does not split as a direct sum of irreducible representations.

Corollary 1.16. *Suppose that k is a field of characteristic not dividing $|G|$. If V is an irreducible representation, then V is isomorphism to a submodule of $k[G]$ (slogan: all irreducibles are submodules of the regular representation).*

Proof. Let $\mathbf{v} \in V$ be non-trivial. Then the homomorphism $\phi \colon k[G] \to V$ given by sending 1 to $\mathbf{v}$ must be surjective, because V is irreducible. Let $U = \ker(\phi)$, then apply Maschke's theorem. $\square$

Example 1.7

Let $G = C_2 = \langle \tau \rangle$, and k a field of characteristic not equal to 2. We have the trivial representation $\mathbf{1}$ and the sign representation $\mathbf{1}_{\mathrm{sgn}}$. Then, $\mathbf{1}$ is generated by the sum $1 + \tau$, while $\mathbf{1}_{\mathrm{sgn}}$ is generated by $1 - \tau$, and we deduce that

$$\rho_{C_2} = \mathbf{1} \oplus \mathbf{1}_{\mathrm{sgn}}.$$

The representation ring

Let k be a field, and suppose that V and W are G-representations, then the k-linear tensor sum $V \oplus W$ can be given the structure of a $k[G]$-module, by taking the diagonal G-action. If we think of a representation in terms of a homomorphism $\rho\colon G \to GL_n(k)$, then this direct sum corresponds to the 'block sum'

$$G \to GL_n(k) \times GL_m(k) \to GL_{n+m}(k)$$

Similarly, we can define a tensor product of representations using the 'Kronecker tensor product' (or matrix direct product). Equivalently, this is the k-linear tensor product $V \otimes W$ with the g-action defined on simple tensors by

$$g \cdot (\mathbf{v} \otimes \mathbf{w}) = g \cdot \mathbf{v} \otimes g\mathbf{w}.$$

We leave it for the reader to verify the following straightforward computations:

(a) $\mathbf{1} \otimes V \cong V \cong V \otimes \mathbf{1}$.

(b) $\mathbf{n} \otimes V \cong V^{\oplus n} \cong V \otimes \mathbf{n}$.

Example 1.8

Let us compute the tensor product $\mathbf{1}_{\mathrm{sgn}} \otimes \mathbf{1}_{\mathrm{sgn}}$. The underlying vector space is simply $k \otimes k \cong k$, while τ acts as $\tau \cdot (1 \otimes 1) = (\tau \cdot 1) \otimes (\tau \cdot 1) = -1 \otimes -1 = 1 \otimes 1$. So the tensor product $\mathbf{1}_{\mathrm{sgn}} \otimes \mathbf{1}_{\mathrm{sgn}} = \mathbf{1}$.

Example 1.9

Take $G = C_3$ and $k = \mathbb{R}$. We have a two-dimensional representation λ_3 corresponding to rotation by an angle of $2\pi/3$. What is $\lambda_3 \otimes \lambda_3$? This is a 4-dimensional representation, and by working out all irreducible representations must be either $\mathbf{4}, \mathbf{2} \oplus \lambda_3$ or $\lambda_3 \oplus \lambda_3$. If you know a little bit of character theory, you can see that it must be $\mathbf{2} \oplus \lambda_3$: we have

$$\begin{aligned}
\chi_{\lambda_3 \otimes \lambda_3}(1) &= 4, \quad \chi_{\lambda_3 \otimes \lambda_3}(\tau) = 1 \\
\chi_{\mathbf{4}}(1) &= 4, \quad \chi_{\mathbf{4}}(\tau) = 4 \\
\chi_{\mathbf{2} \oplus \lambda_3}(1) &= 4, \quad \chi_{\mathbf{2} \oplus \lambda_3}(\tau) = 1 \\
\chi_{\lambda_3 \oplus \lambda_3}(1) &= 4, \quad \chi_{\lambda_3 \oplus \lambda_3}(\tau) = -2
\end{aligned}$$

Remark 1.17. By passing to isomorphism classes of representations, the set of finite dimensional representations has the structure of a semiring. Using the Grothendieck construction, we can produce a commutative ring.

Definition 1.18. For a finite group G the real representation ring $RO(G)$ is the Grothendieck group of the above semi-ring. Explicitly,

$$RO(G) := \mathbb{Z}\begin{Bmatrix}\text{isomorphism classes of finite-}\\ \text{dimensional real } G\text{-representations}\end{Bmatrix} / \langle [V \oplus W] - [V] - [W] \rangle.$$

Remark 1.19. As an abelian group, $RO(G)$ is a direct sum of copies of $\mathbb{Z}$, with rank equal to the number of isomorphism classes of irreducible representations.

Remark 1.20. We can make the same definition for other fields, for example when $k = \mathbb{C}$ we get the complex representation ring $R(G)$.

Example 1.10

We have $RO(C_2) \cong \mathbb{Z}\{\mathbf{1}\} \oplus \mathbb{Z}\{\mathbf{1}_{\text{sgn}}\}$. The ring structure is determined by Example (1.8): we have $RO(C_2) \cong \mathbb{Z}[\sigma]/(\sigma^2 - 1)$. The same is true for the complex representation ring. Note that this is the same as $\mathbb{Z}[C_2]$. In fact, the complex representation ring of a finite abelian group is always (non-canonically) isomorphic to the group ring: it is the group ring of the character group.

Example 1.11

When $G = C_3$ we have that

$$RO(C_3) = \mathbb{Z}\{\mathbf{1}\} \oplus \mathbb{Z}\{\lambda_3\}.$$

By Example (1.9) we have $[\lambda_3]^2 = 2 + [\lambda_3]$ and we see that

$$RO(C_3) \cong \mathbb{Z}[\lambda]/(\lambda^2 - \lambda - 2).$$

On the other hand, the complex representation ring is given by

$$R(C_3) \cong \mathbb{Z}[\zeta]/(\zeta^3 - 1).$$

By tensoring a real representation with $\mathbb{C}$, there is a map

$$RO(C_3) \to R(C_3)$$

given by $\lambda \mapsto \zeta + \zeta^2$.

Definition 1.21. Let $\phi \colon H \to G$ be a morphism of groups, then the pullback $\phi^*(V)$ of a G-representation is the $k[H]$-module induced by restriction of scalars along $k[H] \to k[G]$. Equivalently, it is the representation given by the composite $H \to G \xrightarrow{a} \mathrm{End}(V)$.

Remark 1.22. We have

$$\phi^*(V \oplus W) = \phi^*(V) \oplus \phi^*(W) \quad \text{and } \phi^*(V \otimes W) \cong \phi^*(V) \otimes \phi^*(W).$$

Therefore we deduce:

Corollary 1.23. *A group homomorphism $\phi\colon H \to G$ induces a ring homomorphism $\phi^*\colon RO(G) \to RO(H)$.*

Example 1.12

Consider the group homomorphism $\phi\colon G \to G/G \cong e$. Then we have $\mathbf{n} = \phi^*(\mathbf{k}^n)$.

Definition 1.24. An *injective* group homomorphism $\iota\colon H \hookrightarrow G$ gives rise to a restriction functor for representations, which we denote by Res_H^G.

Example 1.13

Let $G = C_4 = \langle r \rangle$ and $H = C_2 \subseteq C_4$ the subgroup generated by r^2, and take $k = \mathbb{R}$. Pulling back the sign representation of C_2 along the quotient $C_4 \twoheadrightarrow C_2$ gives rise to the sign representation σ. This is one of three irreducible C_4 real representations: we have the trivial representation $\mathbf{1}$ and the rotation representation λ_4. The inclusion $H \hookrightarrow G$ gives a morphism

$$RO(C_4) = \mathbb{Z}\{\mathbf{1}\} \oplus \mathbb{Z}\{\sigma\} \oplus \mathbb{Z}\{\lambda_4\} \to \mathbb{Z}\{\mathbf{1}\} \oplus \mathbb{Z}\{\mathbf{1}_{\mathrm{sgn}}\} = RO(C_2).$$

This map is determined by

$$\mathbf{1} \mapsto \mathbf{1}, \quad \sigma \mapsto \mathbf{1}, \quad \lambda_4 \mapsto 2 \cdot \mathbf{1}_{\mathrm{sgn}},$$

where the image of σ is determined from its definition as the pull-back, and the image of λ_4 comes from the fact that r acts as multiplication by $\pi/4$ and so r^2 acts as multiplication by -1, and so restricts to a 2-dimensional sign representation $\mathbf{1}_{\mathrm{sgn}} \oplus \mathbf{1}_{\mathrm{sgn}}$. We can conclude that $\lambda_4^2 \mapsto (2 \cdot \mathbf{1}_{\mathrm{sgn}})^2 = \mathbf{4}$, so that λ_4^2 must be either $\mathbf{4}, \mathbf{3} \oplus \sigma, \mathbf{2} \oplus 2\sigma, \mathbf{1} \oplus 3\sigma$ or 4σ.

There is also another functor, which will turn out to be adjoint to restriction.

Definition 1.25. Given $H \leq G$, and a H-representation V, we define the induced representation Ind_H^G is be the tensor product $k[G] \otimes_{k[H]} V$.

Remark 1.26. Induction plays well with direct sums: we have $\mathrm{Ind}_H^G(V \oplus W) \cong \mathrm{Ind}_H^G(V) \oplus \mathrm{Ind}_H^G(W)$. But a dimension check shows that it does not commute with tensor products. Therefore, we get an induced map of abelian groups, but *not* of commutative rings

$$\mathrm{Ind}_H^G\colon RO(H) \to RO(G)$$

Directly from the definition we deduce the following:

Lemma 1.27. *If $K \leq H \leq G$ then $\operatorname{Ind}_H^G \operatorname{Ind}_K^H V \simeq \operatorname{Ind}_K^G V$ for any K-representation V.*

Example 1.14

The regular representation $\rho_G = k[G] \cong k[G] \otimes_k k \simeq \operatorname{Ind}_e^G(\mathbf{1})$. More generally, we have $\operatorname{Ind}_H^G \rho_H \cong \rho_G$.

Example 1.15

Let $C_2 \subseteq C_4$ and $k = \mathbb{R}$. What is $\operatorname{Ind}_{C_2}^{C_4}(\mathbf{1})$? We have

$$\operatorname{Ind}_{C_2}^{C_4}(\mathbf{1}) = \mathbb{R}[C_4] \otimes_{\mathbb{R}[C_2]} \mathbb{R} \cong \mathbb{R}[C_4/C_2] \cong \phi^*(\rho_{C_2})$$

for $\phi\colon C_4 \to C_4/C_2 \cong C_2$ the quotient map.[a] This means that

$$\operatorname{Ind}_{C_2}^{C_4}(\mathbf{1}) = \mathbf{1} \oplus \sigma$$

To work out $\operatorname{Ind}_{C_2}^{C_4}(\mathbf{1}_{\mathrm{sgn}})$ we have that

$$\begin{aligned}\rho_{C_4} \cong \operatorname{Ind}_{C_2}^{C_4}(\rho_{C_2})) = \operatorname{Ind}_{C_2}^{C_4}(\mathbf{1} \oplus \mathbf{1}_{\mathrm{sgn}}) &\cong \operatorname{Ind}_{C_2}^{C_4}(\mathbf{1}) \oplus \operatorname{Ind}_{C_2}^{C_4}(\mathbf{1}_{\mathrm{sgn}}) \\ &\cong 1 \oplus \sigma \oplus \operatorname{Ind}_{C_2}^{C_4}(\mathbf{1}_{\mathrm{sgn}}).\end{aligned}$$

But $\rho_{C_4} \cong \mathbf{1} \oplus \sigma \oplus \lambda_4$,. To see this, one can note that

$$\mathbb{R}[\mathbb{Z}/4] \cong \mathbb{R}[X]/(X^4 - 1) \cong \mathbb{R}[X]/\prod_{d|4} \Phi_d \cong \bigoplus_{d|4} \mathbb{R}[X]/\Phi_d$$

so that

$$\mathbb{R}[\mathbb{Z}/4] \cong \mathbb{R}[X]/(X-1) \oplus \mathbb{R}[X]/(X+1) \oplus \mathbb{R}[X]/(X^2+1).$$

Hence, $\operatorname{Ind}_{C_2}^{C_4}(\mathbf{1}_{\mathrm{sgn}}) \cong \lambda_4$.

We deduce that the map

$$\operatorname{Ind}_{C_2}^{C_4}\colon RO(C_2) \to RO(C_4)$$

is determined by

$$\mathbf{1} \mapsto \mathbf{1} \oplus \sigma \quad \text{and} \quad \mathbf{1}_{\mathrm{sgn}} \mapsto \lambda_4.$$

[a] This always works: For $H \leq G$ a normal subgroup we have $\operatorname{Ind}_H^G(\mathbf{1}) = \phi^*(\rho_{G/H})$.

In general, if we have commutative rings R and S and a ring map $f\colon R \to S$, then we can define induction and restriction between R-modules and S-modules and induction is left adjoint to restriction. As a special case, we have:

Lemma 1.28. *If $H \leq G$, then induction is left adjoint to restriction.*

Proposition 1.29 (The projection formula)**.** *Let $H \leq G$, then there is a natural equivalence*

$$\mathrm{Ind}_H^G(\mathrm{Res}_H^G(V) \otimes W) \xrightarrow{\sim} V \otimes \mathrm{Ind}_H^G(W)$$

for $V \in RO(G)$ and $W \in RO(G)$

Proof. We first construct the map: by adjunction, such a map is equivalent to a H-equivariant map

$$\mathrm{Res}_H^G(V) \otimes W \to \mathrm{Res}_H^G(V \otimes \mathrm{Ind}_H^G(W)) \cong \mathrm{Res}_H^G(V) \otimes \mathrm{Res}_H^G \mathrm{Ind}_H^G(W).$$

This map is given as $\mathrm{id} \otimes \eta$ where $\eta\colon W \to \mathrm{Res}_H^G \mathrm{Ind}_H^G(W)$ is the unit of the induction/restriction adjunction. To check this is an equivalence, it suffices to check on underlying vector spaces, which then just boils down to the isomorphism

$$\bigoplus_{G/H}(V \otimes W) \cong V \otimes (\bigoplus_{G/H} W).$$

□

Example 1.16

Let us finish our calculation of λ_4^2 in $RO(C_4)$. We have just seen that $\mathrm{Ind}_{C_2}^{C_4}(\mathbf{1}_{\mathrm{sgn}}) \cong \lambda_4$, and hence

$$\begin{aligned}
\lambda_4 \otimes \lambda_4 \cong \lambda_4 \otimes (\mathrm{Ind}_{C_2}^{C_4}(\mathbf{1}_{\mathrm{sgn}})) &\cong \mathrm{Ind}_{C_2}^{C_4}(\mathrm{Res}_{C_2}^{C_4}(\lambda_4) \otimes \mathbf{1}_{\mathrm{sgn}}) \\
&\cong \mathrm{Ind}_{C_2}^{C_4}(2 \cdot \mathbf{1}_{\mathrm{sgn}} \otimes \mathbf{1}_{\mathrm{sgn}}) \\
&\cong \mathrm{Ind}_{C_2}^{C_4}(\mathbf{2}) \\
&\cong \mathbf{2} \oplus 2\sigma
\end{aligned}$$

For a a ring map $R \to S$, restriction also has a right adjoint, given by coinduction, denoted Coind_R^S and defined by $M \mapsto \mathrm{Hom}_S(R, M)$. A special fact about representation theory is that these adjoint are equal.

Proposition 1.30. *There is a natural equivalence of functors $\mathrm{Ind}_H^G \simeq \mathrm{Coind}_H^G$.*

Proof. If M is an R-module, we use the notation $M^* \cong \mathrm{Hom}_R(M, R)$ for the linear dual. In the case $R = k[G]$, then the natural k-linear isomorphisms

$$\mathrm{Hom}_k(M, N) \cong M^* \otimes_k N \quad \text{ and } M^{**} \cong M$$

are actually $k[G]$-module isomorphisms. To prove the proposition, we note that

$$k[G] \cong k[G]^* = \mathrm{Hom}_k(k[G], k)$$

so that

$$\operatorname{Ind}_H^G(M) = k[G] \otimes_{k[H]} M \cong k[G]^* \otimes_{k[H]} M \cong \operatorname{Hom}_{k[H]}(k[G], M) = \operatorname{Coind}_H^G(M)$$

naturally in M. $\square$

Remark 1.31. More generally, if $f\colon R \to S$ is a morphism of rings, then induction and coinduction agree if and only if S is finitely-generated and projective over R, and there is an isomorphism of (S, R)-bimodules

$$S \to \operatorname{Hom}_R(S, R).$$

The double-coset formula

Let us return for a moment to $RO(C_4)$. We have constructed maps

$$RO(C_2) \to RO(C_4) \to RO(C_2)$$

which send

$$\mathbf{1} \mapsto \mathbf{1} \oplus \sigma \mapsto \mathbf{2}$$

and

$$\mathbf{1}_{\mathrm{sgn}} \mapsto \lambda_4 \mapsto 2 \cdot \mathbf{1}_{\mathrm{sgn}}$$

so that the composite map $RO(C_2) \to RO(C_2)$ is multiplication by 2. This is actually a completely general phenomena.

Definition 1.32. Let $H, K \leq G$ be subgroups, then a double coset HgK is the set

$$HgK = \left\{ x \in G \,\middle|\, x = hgk \text{ for some } h \in H, k \in K \right\}.$$

Theorem 1.33 (Double coset formula). *For subgroups $H, K, \leq G$ and a H-representation K we have a decomposition of K-representations*

$$\operatorname{Res}_K^G \operatorname{Ind}_H^G(V) = \sum_{HgK \in H\backslash G/K} \operatorname{Ind}_{H^{g^{-1}} \cap K}^K c_g^* \operatorname{Res}_{H \cap K^g}^H(V)$$

where $c_g\colon H \cap K^g \xrightarrow{\sim} H^{g^{-1}} \cap K$ is the conjugation by g homomorphism.

Corollary 1.34. *Suppose that G is abelian, and $H = K$, then the composite*

$$RO(H) \xrightarrow{\operatorname{Ind}_H^G} RO(G) \xrightarrow{\operatorname{Res}_H^G} RO(H)$$

is given by multiplication by the index $|G/H|$ of H inside of G.

Indeed, in this case $H\backslash G/H = G/H$.

Remark 1.35. The restriction that G is abelian is really necessary. See [3, Example 1.1.51].

Chapter 2

Mackey functors

The definition of a Mackey functor

We can axiomatize the structure we have on seen on the representation ring in an algebraic object called a *Mackey functor*. As we will see later, these play the same role in equivariant stable homotopy that abelian groups play in ordinary stable homotopy.

Definition 2.1. A Mackey functor $\underline{M}$ for a finite group G consists of the following data:

(a) An abelian group $\underline{M}(H)$ for each $H \leq G$.

(b) A restriction map $R^G_H \colon \underline{M}(H) \to \underline{M}(K)$ for each $K \leq H$.

(c) A transfer map $I^H_K \colon \underline{M}(K) \to \underline{M}(H)$ for each $K \leq H$.

(d) A conjugation homomorphism $c_g \colon \underline{M}(H) \to \underline{M}(H^g)$ for each $g \in G$.

subject to the following rules:

(i) R^H_H and I^H_H are the identity for each $H \leq K$. Moreover, for each $h \in H$, c_h is the identity on $\underline{M}(H)$.

(ii) If $L \leq K \leq H$, then $R^K_L \circ R^H_K \simeq R^H_L$ and $I^K_H \circ I^K_L \simeq I^H_L$.

(iii) $c_g \circ c_h \simeq c_{gh}$ for all $g, h \in G$.

(iv) $R^{H^g}_{K^g} c_g = c_g R^H_K$ and $I^{H_g}_{K^g} c_g = c_g I^H_K$.

(v) The double coset formula holds:

$$R^H_L \circ I^H_K = \sum_{KhL \in K \backslash H/L} I^L_{K^h \cap L} R^{K^h}_{K^h \cap L} c_h$$

for all $L, K \leq H \leq G$.

Example 2.1

In the previous section we have shown that there is a Mackey functor $\underline{RO}(G)$ with $\underline{RO}(G)(H) \cong RO(H)$.

Remark 2.2. There are numerous different ways to record the data of a Mackey functor, and depending on precisely what you want to do, some may be better than others. For example, although it's maybe not so hard to define a category of Mackey functors from this perspective, its formal properties might be a bit hard to see (for example, is it an abelian category? Is it symmetric monoidal?), while from other perspectives this becomes much clearer.

Remark 2.3. Let $\underline{M}$ be a Mackey functor. Note that if g normalizes H so that $H^g = H$, then c_g maps $\underline{M}$ to itself, i.e. we have an action of $N_G(H)$ on $\underline{M}(H)$. Moreover, the normal subgroup $H \le N_G(H)$ acts trivially, so we get an action of $W_G(H) = N_G(H)/H$ on $\underline{M}(H)$. For example, in the case $H = e$, we get an action of $N_G(e)/e = G$ on $\underline{M}(e)$. It is customary to write Mackey functors via a *Lewis diagram*: for example, when $G = C_p$ this is a diagram of the form:

$$\begin{tikzcd} \underline{M}(C_p) \arrow[d, bend right, "R"'] \\ \underline{M}(e) \arrow[u, bend right, "I"'] \arrow[loop below, "C_p"] \end{tikzcd}$$

Example 2.2

Given an $\mathbb{Z}[G]$-module M we can produce a Mackey functor $\underline{M}$ defined by

$$\underline{M}(H) = M^H.$$

Here restriction is defined by inclusion of fixed points for a larger subgroup, which the transfer map is defined by

$$I_K^H \colon M^K \to M^H, \quad I_K^H(m) = \sum_{hK \in H/K} h \cdot m$$

This does not depend on coset representatives, since m is assumed to be fixed by K. Finally, conjugation c_g is multiplication by g.

This gives a functor $FP\colon \mathrm{Mod}_{\mathbb{Z}[G]} \to \mathrm{Mack}_G$

Lemma 2.4. *The functor FP is right adjoint to the evaluation functor that takes a Mackey functor $\underline{M}$ to the $\mathbb{Z}[G]$-module $\underline{M}(e)$.*

Proof. Given a morphism $\alpha\colon \underline{M}(e) \to V$ of $\mathbb{Z}[G]$-modules we show that there is a unique morphism $\underline{\alpha}\colon \underline{M} \to FP(V)$ of Mackey functors (the converse association is easy, and is just given by evaluation at e). This is defined by setting

$$\underline{\alpha}(H)\colon \underline{M}(H) \to V^H$$

to be the composite

$$\underline{M}(H) \xrightarrow{R_e^H} \underline{M}(e) \xrightarrow{\alpha} V^H.$$

Note that this does indeed land in V^H as h acts trivially on $\underline{M}(H)$ and commutes with α, so that $c_h \alpha R_e^H = \alpha R_e^H$. It follows that we get a commutative diagram

$$\begin{tikzcd} \underline{M}(H) \arrow[r, "\alpha(H)"] \arrow[d, "R_e^H"'] & V^H \arrow[d, hook] \\ \underline{M}(e) \arrow[r, "\alpha"'] & V \end{tikzcd}$$

We must show that

$$\underline{\alpha}(H) I_K^H = I_K^H \underline{\alpha}(K)$$

and

$$\underline{\alpha}(H) R_K^H = R_K^H \underline{\alpha}(K).$$

We show the first, and leave the second as an exercise. We have[1]

$$\underline{\alpha}(H) I_K^H = \alpha R_e^H I_K^H = \alpha \sum_{hK \in H/K} c_h R_e^K = \sum_{hK \in H/K} h \cdot \alpha R_e^K = I_K^H \alpha R_e^K = I_K^H \underline{\alpha}(K).$$

The second equation is similar, but simpler, and we leave it for the reader. □

Remark 2.5. Evaluation also has a left adjoint, the 'fixed quotient' Mackey functor, defined by $\underline{M}(H) = M_H$, the largest quotient of M on which H acts trivially.

Remark 2.6. A special case of the fixed-point Mackey functor comes from taking an abelian group M considered with trivial G-action. This gives the *constant Mackey functor* $\underline{M}$ with $\underline{M}(H) = H$, restriction and conjugation the identity, and transfer from K to H given by multiplication by the index of K in H.

Example 2.3

There is a C_2-Mackey functor described by the Lewis diagram

$$\begin{tikzcd} \mathbb{Z} \arrow[d, bend right, "\Delta"'] \\ \mathbb{Z} \oplus \mathbb{Z} \arrow[u, bend right, "\nabla"'] \arrow[loop below, "\text{swap}"] \end{tikzcd}$$

This is an example of a fixed-point Mackey functor applied to the free module $\mathbb{Z}[C_2]$.

[1] Note that $H \backslash G / K = H \backslash G$ if $K = e$.

The Burnside ring

Another example of a Mackey functor comes from the Burnside ring. This is very important in both the theory of Mackey functors and in equivariant homotopy theory, as it plays the role of the unit $\mathbb{Z}$.

Definition 2.7. The Burnside ring of a finite group G is the Grothendieck group of the category of finite G-sets under coproduct. More explicitly,

$$A(G) \coloneqq \mathbb{Z}\,\{\text{isomorphism classes of finite } G\text{-set}\}\,/\langle [X \amalg Y] - [X] - [Y] \rangle.$$

Remark 2.8. Because every finite G-set decomposes into a coproduct of orbits and the isomorphism type of an orbit G/H only depends on the conjugacy class of H, we have an additive decomposition

$$A(G) \cong \bigoplus_{\mathrm{Conj}(G)} \mathbb{Z}.$$

Suppose $H \leq G$ then there are maps $A(G) \to A(H)$ induced by restriction of G-sets, and $A(H) \to A(K)$, induced by taking an H-set X to the G-set $G \times_H X$. There is also a conjugation map $c_g \colon A(H) \to A(H^g)$ defined by taking a H-set X defined by $H \to \mathrm{Aut}(X)$ to the H^g-set defined by $H^g \xrightarrow{c_g^{-1}} H \to \mathrm{Aut}(X)$.

Definition 2.9. The Burnside Mackey functor $\mathbb{A}_G$ is the Mackey functor with $\mathbb{A}_G(H) = A(H)$, and structure maps as in the previous remark.

Remark 2.10. Sending a G-set X to the associated permutation representation, we get a ring morphism $A(G) \to RO(G)$, which extends to a morphism of Mackey functors $\mathbb{A}_G \to \underline{RO}(G)$.

Example 2.4

Let $G = C_p$, then there are two orbits, namely C_p/C_p and C_p/e, so we have

$$A(C_p) \cong \mathbb{Z}\{C_p/C_p, C_p/e\}$$

while clearly

$$A(e) \cong \mathbb{Z}\{e\}.$$

The Mackey functor looks as follows

$$\begin{tikzcd}
\mathbb{Z}\{C_p/C_p, C_p/e\} \arrow[d, bend right, "R"'] \\
\mathbb{Z}\{e\} \arrow[u, bend right, "I"'] \arrow[loop below]
\end{tikzcd}$$

where the maps are determined by $C_p/C_p \mapsto e$, $C_p/e \mapsto p$ (essentially, counting the number of points in the set), while the transfer sends e to

C_p/e (it takes p copies of the singleton, with action that permutes the copies). The Weyl group action is trivial.

Induction, restriction and inflation for Mackey functors

We have seen that we can define (co)induction and restriction for Mackey functors. A similar construction exists for Mackey functors. The restriction functor is particularly simple.

Definition 2.11. Given a G-Mackey functor $\underline{M}$ and a subgroup $H \leq G$ the restricted Mackey functor H-Mackey functor $\mathrm{Res}_H^G \underline{M}$ is the Mackey functor with $\underline{M}(K) = \underline{M}(K)$ for $K \leq H$.

Remark 2.12. Although simple, there is one small subtlety to be aware of: when we restrict the Mackey functor we have less conjugation maps available: the Weyl group is smaller. For example, restriction to the trivial group gives a functor to $\mathrm{Mack}_e \simeq \mathrm{Mod}_{\mathbb{Z}}$, i.e., we have no Weyl group action anymore.

Remark 2.13. Similarly to representations, this functor has a left and right adjoint. From our definition of a Mackey functor, this is a little annoying to define, but really this comes from similar functors on the category of finite G-sets. Indeed, we have indexed our Mackey functors on subgroups $H \leq G$, but we could equally well have done this on G-sets G/H. Since any finite G-set decomposes as a disjoint union of orbits, we can define $\underline{M}(X)$ for X a finite G-set to be the direct sum of $\underline{M}(G/H)$ for the orbits appearing in a decomposition of X.

In fact, one can define a Mackey functor as a bifunctor $\underline{M} = (\underline{M}^*, \underline{M}_*)$ from finite G-sets to abelian groups satisfying some axioms: here $\underline{M}^*$ and $\underline{M}_*$ are contravariant and covariant functors that agree on objects.

Definition 2.14. Let $\underline{M}$ be a H-Mackey functor, then the induction from H to G, $\mathrm{Ind}_H^G(\underline{M})$ is the G-Mackey functor defined on a finite G-set X by

$$\mathrm{Ind}_H^G(\underline{M})(X) = \underline{M}(\mathrm{Res}_H^G X).$$

Remark 2.15. One can give an explicit description of this Mackey functor as

$$\mathrm{Ind}_H^G \underline{M}(K) = \bigoplus_{HgK \in H\backslash G/K} \underline{M}(H \cap K^g)$$

It is rather tedious to check the axioms for this, but see [5, Section 2] if you want the details.

Remark 2.16. Similarly, restriction is equivalently defined by

$$\mathrm{Res}_H^G(\underline{M})(X) = \underline{M}(\mathrm{Ind}_H^G X).$$

Remark 2.17. The following result is slightly surprising since induction is the left, but not right, adjoint of restriction on the category of finite G-sets.

Proposition 2.18. *Induction for Mackey functors is both the left and right adjoint of restriction.*

Proof. Let

$$\epsilon\colon (G\times_H X\downarrow^G_H)\to X \quad \text{and} \quad \eta\colon Y\to (G\times_H Y)\downarrow^G_H$$

by the counit and unit for the adjunction on the category of finite G-sets. Here ϵ maps the class of (g,x) to gx and η maps y to the class of $(1,y)$.

To show that induction is left adjoint, we define the counit and unit on Mackey functors

$$p(N)\colon\ \mathrm{Ind}_H^G \mathrm{Res}_H^G N\to N$$

and

$$q(M)\colon M\to \mathrm{Res}_H^G \mathrm{Ind}_H^G M$$

by specifying their value on a G-set X and a H-set Y by

$$p(N)=N_*(\epsilon) \quad \text{and } q(M)=M_*(\eta).$$

That this defines an adjunction then follows from the fact that ϵ and η are the counit and unit of an adjunction.

To show induction is right adjoint is very similar, except we use the contravariant structure M^* instead of the covariant structure. □

Example 2.5

What is $\mathrm{Ind}_e^{C_p}(\mathbb{Z})$? Note that on G-sets we have that $\mathrm{Res}_e^{C_p}(C_p/C_p)$ is a point, while $\mathrm{Res}_e^{C_p}(C_p/e)$ is p points. We deduce that

$$\mathrm{Ind}_e^{C_p}\mathbb{Z}= \begin{tikzcd} \mathbb{Z} \arrow[d, bend right, "\Delta"'] \\ \mathbb{Z}[C_p] \arrow[u, bend right, "\nabla"'] \arrow[loop below, "W"] \end{tikzcd}$$

where the Weyl group acts by cyclic permutations. When $p=2$, we have seen this Mackey functor in Example (2.3).

If you want a fun exercise, you can try and compute $\mathrm{Ind}_{C_2}^{C_4}\underline{M}$ for $\underline{M}\in\mathrm{Mack}_{C_2}$.

There is one final functor that we would like to introduce for Mackey functors, and that is inflation.

Definition 2.19. Let $N\trianglelefteq G$ be a normal subgroup. We define the inflation functor $\mathrm{Mack}_{G/N}\to\mathrm{Mack}_G$ by

$$\mathrm{Inf}_{G/N}^G M(H)=\begin{cases} M(H/N) & \text{if } N\le H\\ 0 & \text{else.}\end{cases}$$

Restriction, transfer, and conjugation are defined in the obvious way.

Example 2.6

For any abelian group M (seen as an object of Mack_e), and group G, there is an inflated Mackey functor $\mathrm{Inf}_{G/G}^{G} M$ whose value is M at G and is zero elsewhere.

Example 2.7

Let $G = C_{p^2}$ and $N = C_p$, and suppose $M \in \mathrm{Mack}_{C_p^2/C_p} \simeq \mathrm{Mack}_{C_p}$. Then $N := \mathrm{Inf}_{C_p}^{C_p^2}$ is the Mackey functor with $N(C_{p^2}) = \underline{M}(C_p), N(C_p) = \underline{M}(e)$ and $N(e) = 0$.

Inflation has both a left and a right adjoint.

Definition 2.20. Let $M \in \mathrm{Mack}_G$ and $N \trianglelefteq G$, then we define G/N-Mackey functors M^+ and M^- by

$$M^+(K/N) = M(K)/\sum_{J\leq K, J\not\geq N} I_J^K M(J)$$

and

$$M^-(K/N) = \bigcap_{J\leq K, J\not\geq N} \ker(R_J^K)$$

Restrictions, transfer, and conjugation come from those for M. These define functors

$$^+, ^-\colon \mathrm{Mack}_{G/N} \to \mathrm{Mack}_G\,.$$

Proposition 2.21. *$^+$ is left adjoint to $\mathrm{Inf}_{G/N}^{G}$ and $^-$ is right adjoint to $\mathrm{Inf}_{G/N}^{G}$.*

Proof. Let L be a Mackey functor for G and M a Mackey functor for G/N. Given a morphism $L \to \mathrm{Inf}_{G/N}^{G} M$, we note that this is necessarily zero on $L(J)$ with $J \not\geq N$, and hence must vanish on $\sum_{J\leq K, J\not\geq N} I_J^K M(J)$ for each subgroup K. We therefore get an induced morphism $L^+ \to M$. Conversely, given $\beta\colon L^+ \to M$ we can construct a morphism $L \to \mathrm{Inf}_{G/N}^{G} M$ by defining it to be 0 on $L(K)$ when $K \not\geq N$, and defining it to be the composite

$$L(K) \to L(K)/\sum_{J\leq K, J\not\geq N} I_J^K L(J) = L^+(K/N) \xrightarrow{\beta} M(K/N) = \mathrm{Inf}_{G/N}^{G} M(K)$$

when $K \geq N$. One can check that this defines a map of Mackey functors, and that this defines an adjunction. The proof for $^-$ is similar. □

Remark 2.22. From the perspective of G-sets, inflation arises in the following way: a functor $M\colon \mathrm{Fin}_{G/N} \to \mathrm{Mod}_{\mathbb{Z}}$ gives rise to a functor $\mathrm{Fin}_G \to \mathrm{Mod}_{\mathbb{Z}}$ by composition with the fixed-point functor $\mathrm{Fin}_G \to \mathrm{Fin}_{G/N}$.

Chapter 3

The category of G-spaces

Definition 3.1. A G-spaces is a topological space X with a continuous action $G \times X \to X$ (this is the special case of Definition 1.1 in the case $\mathcal{C} = \mathrm{Top}$). We let $G\,\mathrm{Top}$ denote the category of G-spaces, where the morphisms are morphisms of topological spaces that commute with the G-action. By taking based spaces, we can define $G\,\mathrm{Top}_*$, the category of *based* G-spaces.

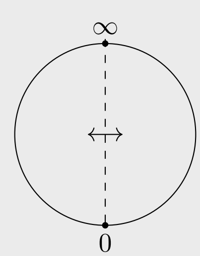
Example 3.1

We can always give a topological space the trivial action. However, there are of course many more examples. For example, the n-sphere S^n has a natural C_2-action, giving by sending a point to its antipode. This is the same as saying that the non-trivial element of C_2-acts by -1. This action is *free*, in the sense it has no fixed-points.

Example 3.2

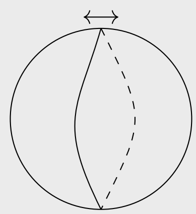
Here is an important example, and is the reason why representations show up in equivariant homotopy theory. Let V be a real representation of G of dimension n. We let S^V denote the one-point compactification of V. This is (non-equivariantly) an n-sphere S^n. This has a natural G-action where the new point ∞ is fixed by G. In fact, S^V is a based topological space, with base point ∞. If $V = \mathbb{R}^n$, this is S^n with the trivial action, so our representation spheres generalize the usual notion of a sphere.

For a non-trivial example of a representation sphere, we can consider $G = C_2$ and the sign representation $\mathbf{1}_{\mathrm{sgn}}$. In this case, we can draw the sphere $S^{\mathbf{1}_{\mathrm{sgn}}}$ (usually denoted S^σ or $S^{1,1}$)



This is simply the circle with the C_2-action given by reflection across the equator. The two marked points are the only points fixed by the C_2-action.

We can also consider the regular representation ρ. This is a two dimensional representation, and so the underlying space is a 2-sphere. We can think of it as the one point complexification of $\mathbb{C}$, also known as $\mathbb{C}P^1$, equipped with the complex conjugation action. Since $\mathbb{R} \subseteq \mathbb{C}$ is fixed by conjugation, we see that $\mathbb{R}P^1 \subseteq \mathbb{C}P^1$ is fixed by this action.



The C_2-action is again given by reflection across an equator.

Notation 3.2. Let X and Y be based spaces, then we let $\mathrm{Map}_G(X, Y) \subseteq \mathrm{Map}(X, Y)$ denote the space of G-equivariant maps $X \to Y$, with the subspace topology. Moreover, we can write define a G-space $G\,\mathrm{Map}(X, Y)$ to have underlying space $\mathrm{Map}(X, Y)$ with G-action given by

$$(g \cdot f)(x) \coloneqq g^{-1} f(gx).$$

The two are related: We have $G\,\mathrm{Map}(X, Y)^G = \mathrm{Map}_G(X, Y)$ as topological spaces, essentially by definition.

Remark 3.3. Given a group homomorphism $H \to G$ we get a pullback functor $\phi^* \colon G\,\mathrm{Top} \to H\,\mathrm{Top}$ is the obvious way. For an injective homomorphism, this gives us restriction functors $\mathrm{Res}^G_H \colon G\,\mathrm{Top} \to H\,\mathrm{Top}$. This has a left and right adjoint, as we now explain.

Definition 3.4. Let X be a H-space, then we define a G-space $\mathrm{Ind}^G_H X = G \times_H X$ as the quotient of $G \times X$ by the relation $(g \cdot h, x) \sim (g, h \cdot x)$ for $g \in G, h \in H$.

Example 3.3

If $H = e$ is the trivial group, then $\mathrm{Ind}^G_e X \cong G \times X$ is a free G-space.

Example 3.4

We have $\mathrm{Ind}^G_H(*) \cong G/H$. More generally, if X is a space with trivial H-action, we have $\mathrm{Ind}^G_H(X) \cong X \times G/H$, where G acts on G/H, and trivially on X.

In analogy with Mackey functors, we have the following:

Proposition 3.5. *The functor* $\mathrm{Ind}_H^G\colon H\,\mathrm{Top} \to G\,\mathrm{Top}$ *is left adjoint to* $\mathrm{Res}_H^G\colon G\,\mathrm{Top} \to H\,\mathrm{Top}$.

Remark 3.6. There is a based version, which sends X to $G_+ \wedge_H X$, the quotient of $G_+ \wedge X$ by the same relation as before.

Remark 3.7. Similarly, there is a functor $\mathrm{Coind}_H^G\colon H\,\mathrm{Top} \to G\,\mathrm{Top}$, which sends X to $\mathrm{Map}_H(G, X)$, the space of H-equivariant maps from G to X, with G-action given by $(gf)\cdot g' = f(g\cdot g')$. This functor is right adjoint to restriction.

In contrast to what we saw for Mackey functors, the left and right adjoint do not agree for spaces (we will see an example on the exercise sheet).

Remark 3.8. In fact, given $f\colon H \to G$ a group homomorphism, the pullback functor $f^*\colon G\,\mathrm{Top} \to H\,\mathrm{Top}$ always has a left and right adjoint. Taking G to be the trivial group we can get functors called fixed points and orbits, which we now explicitly define.

Definition 3.9. For $X \in G\,\mathrm{Top}$, the fixed points X^G is the subspace

$$X^G = \left\{\, x \in X \,\middle|\, g\cdot x = x \text{ for all } g \in G \,\right\} \subseteq X.$$

This defines a functor $(-)^G\colon G\,\mathrm{Top} \to \mathrm{Top}$, which is right adjoint to pullback along $G \to e$.

More generally, we can define X^H as the composite

$$G\,\mathrm{Top} \xrightarrow{\mathrm{Res}_H^G} H\,\mathrm{Top} \xrightarrow{(-)^H} \mathrm{Top}\,.$$

Example 3.5

Returning to the representation spheres considered in Example (3.2) we have $(S^\sigma)^{C_2} = S^0$, while $(S^\rho)^{C_2} = S^1$.

Definition 3.10. For $X \in G\,\mathrm{Top}$, the orbit of a point $x \in X$ is the set

$$G\cdot x = \left\{\, g\cdot x \,\middle|\, g \in G \,\right\}$$

We can define an equivalence relation on X by saying that $x \sim y$ if and only if there exists a $g \in G$ such that $g\cdot x = y$. The quotient by this equivalence relation is the orbit space X/G of the action. This is the left adjoint of pullback along $G \to e$.

Remark 3.11. In summary: we have

$$\mathrm{Map}_G(Y, X) \simeq \mathrm{Map}(Y, X^G)$$

and

$$\mathrm{Map}_G(X, Y) \cong \mathrm{Map}(X/G, Y)$$

where X is a G-space and Y has the trivial G-action.

There is an obvious notion of homotopy in equivariant homotopy.

Definition 3.12. Two equivariant maps $f, g\colon X \to Y$ between G-spaces are said to be G-homotopic if there is an equivariant map $h\colon I \times X \to Y$ such that $h|_{\{0\}\times X} \simeq f$ and $h|_{\{1\}\times X} \simeq g$. Here we give I the trivial G-action.

Example 3.6

Let $* \underset{0}{\overset{\infty}{\rightrightarrows}} S^\sigma$ be the inclusion of the two fixed-points of S^σ. Because S^σ is path-connected, these are non-equivariantly homotopic. But they are not equivariantly homotopic, since the space of fixed points is not path-connected.

We can now define G-homotopy groups, G-homotopy equivalences, etc. Note that by adjointness, the notion of equivariant homotopy will just be computing $\pi_n(X^G)$. Instead, it is better to remember the fixed points of all subgroups $H \leq G$. To fix notation, let $\pi_n^H(X) \coloneqq \pi_n(X^H)$.

Definition 3.13. A morphism $f\colon X \to Y$ of G-spaces is a weak G-homotopy equivalence if the induced maps $\pi_n^H(f)\colon \pi_n^H(X) \to \pi_n^H(Y)$ are isomorphisms for all $H \leq G$.

Remark 3.14. By playing around with adjunctions we have $\pi_n^H(Y) = [G/H \times S^n, X]$, and so this leads naturally to the notion of G-CW-complex.

Definition 3.15. A G-CW-complex is a sequential colimit of spaces X_n where X_{n+1} is a pushout (in $G\,\mathrm{Top}$)

$$\begin{tikzcd} \coprod G/H \times S^n \arrow[r] \arrow[d] & X_n \arrow[d] \\ \coprod G/H \times D^{n+1} \arrow[r] \arrow[ur, phantom, "\lrcorner", very near end] & X_{n+1} \end{tikzcd}$$

and H ranges over all the subgroups of G. There is also a relative version where we allow $X_{-1} = A$, for some G-space A.

Example 3.7

We consider again the representation sphere S^σ. Here we start with the C_2-fixed 0-skeleton S^0. Here we attach a single free 1-cell $(C_2/e) \times D^1$ where one (free) endpoint is attached to 0 and the other to the point ∞.

The G-Whitehead theorem

Our next goal is to prove the equivariant Whitehead theorem.

Theorem 3.16 (Equivariant Whitehead theorem)**.** *A weak G-homotopy equivalence between G-CW complexes is a G-homotopy equivalence.*

There are two ways to prove this theorem, both generalizing the two non-equivariant ways. The first is an inductive argument, and can be found in [3, Theorem 2.1.31]. The second is to prove a more general theorem, the equivariant homotopy extension and lifting property (HELP). The main subtly when extending the proof from the non-equivariant case is to define a suitable notion of the connectivity of a map.

Definition 3.17. Let $\theta\colon \mathrm{Sub}(G)/G \to \{\, x \in \mathbb{Z} \,|\, x \geq -1 \,\}$. A map $f\colon X \to Y$ of G-spaces is θ-connected if f^H is $\theta(H)$-connected for all $H \leq G$. A G-CW complex is θ-dimensional if all cells of orbit type G/H have non-equivariant dimension at most $\theta(H)$.

With these definitions in place, we now have an equivariant version of the HELP theorem.[1]

Proposition 3.18. *Let (A, X) be a θ-dimensional relative G-CW complex, and let $e\colon Y \to Z$ be a θ-connected G-map between G-CW complexes. Given $g\colon A \to Y$, $h\colon A \times I \to Z$, and $f\colon X \to Z$ such that $eg = hi_1$ and $f|_A = hi_0$, there exist maps $\widetilde{g}\colon X \to Y$ and $\widetilde{h}\colon X \times I \to Z$ that make the following diagram commute:*

$$\begin{tikzcd}
A \arrow[rr, "i_0"] \arrow[dd] & & A \times I \arrow[dl, "h"'] \arrow[dd] & & A \arrow[ll, "i_1"'] \arrow[dl, "g"] \arrow[dd] \\
& Z & & Y \arrow[ll, "e"'] & \\
X \arrow[ur, "f"] \arrow[rr, "i_0"'] & & X \times I \arrow[ul, dashed, "\tilde{h}"] & & X \arrow[ll, "i_1"] \arrow[ul, dashed, "\tilde{g}"']
\end{tikzcd}$$

Sketch of proof. The desired maps are constructed by induction on the skeleton of X, and then cell by cell, so we can assume that $X = G/H \times D^{n+1}$ and $A = G/H \times S^n$. By playing around with adjunctions, this reduces to the ordinary non-equivariant HELP theorem with $e\colon Y^H \to Z^H$, $X = D^n$ and $A = S^{n-1}$. ☐

Remark 3.19. An equivalent way of phrasing it: under the assumption of the theorem, any diagram

$$\begin{tikzcd}
A \times \{1\} \arrow[r, "g"] \arrow[d, hook] & Y \arrow[d, "e"] \\
A \times I \cup X \times \{0\} \arrow[r, "h \cup f"'] & Z
\end{tikzcd}$$

[1] For the details in the non-equivariant case, see [4, Chapter 10.3].

can be completed to the diagram

$$\begin{tikzcd}
A \times \{1\} \arrow[rr, "g"] \arrow[dr, hook] \arrow[ddd, hook] & & Y \arrow[ddd, "e"] \\
& X \times \{1\} \arrow[ur, dashed, "\tilde{g}"] \arrow[d, hook] & \\
& X \times I \arrow[dr, dashed, "\tilde{h}"] & \\
A \times I \cup X \times \{0\} \arrow[ur, hook] \arrow[rr, "h \cup f"'] & & Z
\end{tikzcd}$$

Theorem 3.20. *Let X be a G-CW complex and let $e\colon Y \to Z$ be a θ-connected map between G-CW complexes, then*

$$e_*\colon [X, Y]_G \to [X, Z]_G$$

is a bijection if $\dim(X) \leq \theta$ and surjective if $\dim X = \theta$. In particular, if e is a weak equivalence and X is any G-CW complexes, then e_* is a bijection.

Proof. Taking $A = \emptyset$ gives a map $\tilde{g}\colon X \to Y$ and a homotopy $\tilde{h}\colon X \times I \to Z$ satisfying $h(-, 0) = f$ and $h(-, 1) = e \circ \tilde{g}$, hence $[f] = [e \circ \tilde{g}] \in [X, Z]_G$ and therefore e_* is surjective. Now assume we are given two maps $g_0, g_1\colon X \to Y$ and an equivalence $[e \circ g_0] = [e \circ g_1]$. In particular, there exists a G-homotopy $H\colon X \times [0, 1] \to Z$.

We want to apply HELP to the pair $(X \times [0, 1], X \times \{0, 1\})$. We define $g\colon X \times \{0, 1\} \to Y$ to be g_0 at time 0, and g_1 at time 1. Similarly, we define $h\colon X \times \{0, 1\} \times I \to Z$ to be a constant homotopy. The HELP theorem then gives us a G-homotopy $\tilde{g}$ between g_0 and g_1. □

As a corollary, we obtain an equivariant version of Whitehead's theorem.

Corollary 3.21. *Any θ-connected G-map $e\colon Y \to Z$ between G-CW complexes of dimension less than θ is a G-homotopy equivalence. In particular, if e is a weak equivalence, then it is a G-homotopy equivalence.*

Proof. A map $f\colon Z \to Y$ such that $e_*[f] = \mathrm{id}$ is a homotopy inverse to e. □

Remark 3.22. Using the equivariant version of HELP, one can also prove equivariant versions of cellular approximation and G-CW approximation.

Elmendorf's theorem

An abstract way to 'do' homotopy theory in a category is to put a model structure on it. In case you haven't seen these before, we will briefly state the definitions:

Definition 3.23. A weak factorization system (WFS) on a category $\mathcal{C}$ is a pair $(\mathcal{L}, \mathcal{R})$ of classes of morphisms of $\mathcal{C}$ such that:

- Every morphism $f\colon X \to Y$ of $\mathcal{C}$ may be factored as the composition of a morphism in $\mathcal{L}$ followed by one in $\mathcal{R}$:

$$f\colon X \xrightarrow{\in\mathcal{L}} Z \xrightarrow{\in\mathcal{R}} Y.$$

- The classes are closed under having the lifting property against each other:
  - $\mathcal{L}$ is precisely the class of morphisms having the left lifting property against every morphism in $\mathcal{R}$.
  - $\mathcal{R}$ is precisely the class of morphisms having the right lifting property against every morphism in $\mathcal{L}$.

$$\begin{tikzcd} {} \arrow[r, "e\in\mathcal{L}"] \arrow[d, "u"'] & {} \arrow[d, "v"] \arrow[dl, dotted, "m"'] \\ {} & {} \arrow[l, "w\in\mathcal{R}"] \end{tikzcd}$$

Definition 3.24 (Model Structure). A model structure on a category $\mathcal{C}$ is a choice of three distinguished classes of morphisms: cofibrations $\mathcal{C}$, fibrations $\mathcal{F}$, and weak equivalences $\mathcal{W}$, satisfying the following conditions:

- $\mathcal{W}$ contains all isomorphisms and is closed under two-out-of-three: given a composable pair of morphisms f and g, if two out of the three morphisms f, g, $g \circ f$ are in $\mathcal{W}$, then so is the third.

- $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ and $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ are two weak factorization systems on $\mathcal{C}$. The morphisms $\mathcal{F} \cap \mathcal{W}$ are called acyclic fibrations while those in $\mathcal{C} \cap \mathcal{W}$ are called acyclic cofibrations.

When a category $\mathcal{C}$ is a complete and cocomplete category with a model structure, we call it a model category.

Notation 3.25. Some more terminology: An object is cofibrant if the unique morphism $\emptyset \to X$ from the initial object is a cofibration, and is fibrant if the unique morphism $X \to *$ is a fibration.[2]

The homotopy category $\mathrm{Ho}(\mathcal{C})$ of a model category is the universal way of inverting the weak equivalences in $\mathcal{C}$: $\mathrm{Ho}(\mathcal{C}) \coloneqq \mathcal{C}[\mathcal{W}^{-1}]$.

Remark 3.26. This definition of homotopy category does not depend on the choice of fibrations and cofibrations. It only depends on the underlying category with weak equivalences. However, the model structure makes the homotopy category easier to handle. In fact, with a model structure, the homotopy category is equivalent to the category whose objects are those which are both fibrant and cofibrant, and morphisms are the equivalence classes of morphism under left homotopy. This definition of homotopy category avoids the set theory technical issues one may meet with while doing localization.

[2] If $\mathcal{C}$ is a model category, then it has an initial object, the colimit of the empty diagram, and a terminal object, the limit of the empty diagram.

Remark 3.27. The axioms for a model category are over-determined. Indeed, one can show that a map is a cofibration (a trivial cofibration) if and only if it has the left lifting property with respect to all trivial fibrations (fibrations). Dually, a map is a fibration (a trivial fibration) if and only if it has the right lifting property with respect to all trivial cofibrations (cofibrations). In particular, we only need to specify the weak equivalences and (co)fibrations, and then the remaining class is formally determined.

Example 3.8

The category Top of topological spaces has a model structure with fibrations as Serre fibrations, equivalences are the weak homotopy equivalences, and cofibrations are the retracts of relative cell complexes. We denote this by $\mathrm{Top}_{\mathrm{Quillen}}$.

The category Set of sets has exactly 9 model structures on it!

Definition 3.28. Suppose that $\mathcal{C}$ and $\mathcal{D}$ are model categories. We say that a pair $F\colon \mathcal{C} \leftrightarrows \mathcal{D}\colon U$ of adjoint functors with F the left adjoint is a Quillen adjunction if the following equivalent conditions are satisfied:

- F preserves cofibrations and acyclic cofibrations;
- U preserves fibrations and acyclic fibrations;
- F preserves cofibrations and U preserves fibrations;
- F preserves acyclic cofibrations and U preserves acyclic fibrations.

Definition 3.29. Let $\mathcal{C}$ and $\mathcal{D}$ be model categories equipped with a Quillen adjunction $F\colon \mathcal{C} \leftrightarrows \mathcal{D}\colon U$. Then we say that $\mathcal{C}$ and $\mathcal{D}$ are Quillen equivalent if the derived adjunction $F\colon\ \mathrm{Ho}(\mathcal{C}) \leftrightarrows \mathrm{Ho}(\mathcal{D})\colon U$ is an equivalence of categories. This is equivalent to the following condition:

- For any cofibrant $X \in \mathcal{C}$ and fibrant $Y \in \mathcal{D}$, $FX \to Y$ is a weak equivalence if and only if the adjoint $X \to UY$ is an equivalence.

Example 3.9

There is a model structure (the Kan model structure) on simplicial sets, where a morphism $f\colon X \to Y$ is

- cofibration if it is a monomorphism (i.e., a level-wise injection).
- weak equivalence if its geometric realization is a weak equivalence in Top.

Then there is a Quillen equivalence

$$|-|\colon\ \mathrm{sSet}_{\mathrm{Kan}} \leftrightarrows \mathrm{Top}_{\mathrm{Quillen}}\colon S(-)$$

given by geometric realization and singular functors.

Proposition 3.30. *There is a model structure on* $G\,\mathrm{Top}$ *where a map* $f\colon X \to Y$ *is a*

- *fibration if* $f^H\colon X^H \to Y^H$ *is a fibration for all* $H \leq G$*.*
- *weak equivalence if* $f^H\colon X^H \to Y^H$ *is a weak equivalence for all* $H \leq G$*.*

Elmendorf's theorem relates this to the category of presheaves on the orbit category.

Definition 3.31. The orbit category of a finite group $\mathcal{O}_G$ is the category whose objects are the orbits G/H for $H \leq G$ a subgroup, and whose morphisms are the G-equivariant continuous functions. Equivalently, this is the full subcategory of $G\,\mathrm{Top}$ on the objects G/H.

Remark 3.32. It is useful to note that $\mathrm{Hom}_{\mathcal{O}_G}(G/H, G/K)$ is non-zero if H is subconjugate in G to K. Moreover, the automorphism group of G/H is exactly the Weyl group, $W_G(H)$.

Example 3.10

When $G = C_2$, the orbit category is the following category:

$$\begin{tikzcd} C_2/e \arrow[loop above, "C_2"] \arrow[d] \\ C_2/C_2 \end{tikzcd}$$

Proposition 3.33. *There is a model structure on* $\mathrm{Fun}(\mathcal{O}_G^{op}, \mathrm{Top})$ *where the weak equivalences and fibrations are taken objectwise.*

Lemma 3.34. *There is an adjunction*

$$\theta\colon \mathrm{Fun}(\mathcal{O}_G^{op}, \mathrm{Top}) \leftrightarrows G\,\mathrm{Top}\colon \psi$$

where θ *is evaluation at* G/e *and* $\psi(X)(G/H) = X^H$*.*

Sketch of proof. We show this by constructing a unit and a counit. We first note that $\theta \circ \psi(X) = X^e$, and this comes with a G-action (by functoriality) that agrees with the original G-action. We can then take the counit $\epsilon\colon \theta \circ \psi \to \mathrm{id}$ to be the identity.

For the unit, note that for any $T\colon \mathcal{O}_G^{op} \to \mathrm{Top}$ and subgroups $K \leq H \leq G$, the map $T(G/H) \to T(G/K)$ factors through the fixed points $T(G/K)^H$. We then have

$$(\psi \circ \theta(T))(G/H) = (\psi T(G/e))(G/H) = T(G/e)^H,$$

and the factorization of the map $T(G/H) \to T(G/e)$ through $T(G/e)^H$ defines η. It is then straightforward to check the unit and counit identities. □

Remark 3.35. Since the counit is an equivalence, the right adjoint is fully-faithful.

Theorem 3.36 (Elmendorf's theorem). *The adjunction of Lemma 3.34 is a Quillen equivalence with respect to the model structures of Propositions 3.30 and 3.33.*

Sketch of proof. We first show that the adjunction is Quillen. But this is straightforward: the functor ψ preserves fibrations and weak equivalences essentially by definition.

Now in general, the unit of the adjunction of Lemma 3.34 is not an equivalence. But we claim that it is for any $\mathcal{X}$ that is cofibrant. Indeed, one can explicitly check that the cofibrant objects in the model structure on $\mathrm{Fun}(\mathcal{O}_G^{op}, \mathrm{Top})$ are retracts of the cellular objects, and these are generated under pushouts along inclusions and directed colimits by

$$\left\{ \mathrm{Map}_G(-, G/H) \times X \;\middle|\; H \subseteq G, X \text{ is a cell in } \mathrm{Top} \right\}$$

Now in general the fixed point functors $(-)^H$ do not preserve colimits, but one can show that they do preserve retracts, pushouts and directed colimits. One can then directly check the claim from the following sequence of equivalences:

$$\mathcal{X}(G/K) = \mathrm{Map}_G(G/K, G/H) \times Y = (G/H)^K \times Y = (G/H \times Y)^K = \mathcal{X}(G/e)^K$$

Now suppose we are given a morphism $f\colon \theta(\mathcal{X}) \to \mathcal{Y}$ in $G\,\mathrm{Top}$ (note that in fact every object in $G\,\mathrm{Top}$ is fibrant). We can factor the H-fixed points of the adjoint $g^H\colon \mathcal{X}(G/H) \to \psi(\mathcal{Y})^H = \mathcal{Y}^H$ as

$$\mathcal{X}(G/H) \xrightarrow[\cong]{\eta_{G/H}} \mathcal{X}(G/e)^H \xrightarrow{f^H} Y^H.$$

By the 2 out of 3 axiom, we see that g^H is an equivalence if and only if f^H is an equivalence. By the definition of the model structures, we deduce that g is an equivalence if and only if f is an equivalence, as required. □

Remark 3.37. Elmendorf's original theorem was only a statement about homotopy categories. More specifically, one defines a homotopy inverse to ψ by using the bar construction.

Remark 3.38. Here is an application. Let $\mathcal{F}$ be a family of subgroups (i.e., a collection closed under taking subgroups and conjugations). Then we can construct a G-space determined by the property that

$$E\mathcal{F}^K = \begin{cases} * & \text{if } K \in \mathcal{F} \\ \emptyset & \text{if } K \notin \mathcal{F}. \end{cases}$$

Indeed, simply define $E\mathcal{F}'$ to be the presheaf with the analogous property, and define $E\mathcal{F}$ to be the image under the Quillen equivalence.

Chapter 4

Forms of equivariant cohomology

There are several different flavors of cohomology for equivariant homotopy. We begin with Borel (co)homology, before moving on to the more powerful Bredon (co)homology.

Borel cohomology

The simplest, and perhaps most common, form of equivariant cohomology is known as Borel equivariant cohomology.

The following can always be constructed, for example using the Milnor construction.

Definition 4.1. For any group G, we let EG denote a G-CW complex whose underlying space is contractible, and such that the action of G-is free. We let the orbit space EG/G be denoted by BG. This is known as the classifying space of the group G.

Example 4.1

Let $G = C_2$ then a model for EC_2 is S^∞ with the antipodal action. Here we have $S^\infty/C_2 \simeq \mathbb{R}P^\infty$, and so $BC_2 \simeq \mathbb{R}P^\infty$.

More generally, taking the limit of the inclusion of the unit sphere $S^{2n-1} \subseteq \mathbb{C}^n$, we can think of S^∞ as the unit sphere inside of $\mathbb{C}^\infty$. We can think of C_n as acting on each complex coordinate as multiplication by an n-th root of unity. This gives a free action of C_n on S^∞, and so $EC_n \simeq S^\infty$. We deduce that $BC_n \simeq S^\infty/C_n$. This is known as an infinite-dimensional lens space.

Remark 4.2. It is a good exercise to check that $EG \times EH \simeq E(G \times H)$. In particular, we deduce that $B(G \times H) \simeq BG \times BH$.

Proposition 4.3. *The space BG is a $K(G, 1)$, i.e., $\pi_1(BG) \simeq G$ and all other homotopy groups are trivial.*

Proof. This follows from the fact that $G \to EG \to BG$ is a fibration, and the long exact sequence in homotopy. □

Definition 4.4. Give a G-space X, the *Borel construction* on X is the orbit space of the diagonal G-action on $EG \times X$. This is usually denoted $EG \times_G X$.

The Borel equivariant homology and cohomology are defined by

$$H_*^{Borel}(X) \coloneqq H_*(EG \times_G X) \quad \text{and} \quad H^*_{Borel}(X) \coloneqq H_*(EG \times_G X).$$

Remark 4.5. When compared with Bredon cohomology, this is much easier to compute. On the other hand, it can't tell the difference between EG and a point. More generally, we have the following:

Definition 4.6. A map $f\colon X \to Y$ of G-spaces is an underlying equivalence if it induces an equivalence on underlying spaces.

The following follows essentially by definition (check on fixed points!):

Proposition 4.7. *The functor $EG \times (-)\colon G\,\mathrm{Top} \to G\,\mathrm{Top}$ takes underlying equivalences to G-equivalences. Therefore, Borel cohomology takes underlying equivalences to cohomology isomorphisms.*

Remark 4.8. Taking X to be a point, we see that

$$H_*^{Borel}(*) = H_*(BG) \quad \text{and } H^*_{Borel}(*) = H^*(BG).$$

It is either a theorem, or a definition, that this is the group (co)homology of the group G.

Bredon cohomology

There is a more refined version of equivariant cohomology known as Bredon cohomology. We need the following definition:

Definition 4.9. Let G be a finite group, then a *coefficient system* is a presheaf $X \in \mathrm{Fun}(\mathcal{O}_G^{op}, \mathrm{Ab})$.

Note that Elemendorf's theorem says that for any coefficient system we can produce an Eilenberg–MacLane G-space.

Example 4.2

Any Mackey functor $\underline{M}$ gives rise to a coefficient system by forgetting the data of the transfer map. Strictly speaking, given our definition of Mackey functors, this is not entirely obvious: one needs to note that any G-map $\phi\colon G/H \twoheadrightarrow G/K$ can be factored (not uniquely) as a composite $G/H \twoheadrightarrow G/K^\gamma \cong G/K$ where H is contained in the conjugate K^γ. Then the map $M(\phi)\colon M(G/K) \to M(G/H)$ is defined by to

$$\underline{M}(K) \xrightarrow{c_\gamma} \underline{M}(K^\gamma) \xrightarrow{R_H^{K^\gamma}} \underline{M}(H).$$

One can check that the composite does not depend on the choice of factorization.

Example 4.3

For $n \geq 2$ the equivariant homotopy groups $\pi_n(X^H)$ for $H \leq G$ assemble into a coefficient system denoted $\underline{\pi}_n(X)$. Indeed, the functor $G/H \mapsto X^H$ defines a coefficient system valued in (based) spaces, and then we simply apply π_n to this. This clearly applies in other contexts, for example the assignment $\underline{H}_n(X;A)(G/H) \coloneqq H_n(X^K;A)$ determines a coefficient system.

Example 4.4

Let $G = C_2$ and $X = S^\sigma$. Then $X^{C_2} \simeq S^0$ and $X^1 \simeq S^1$. It follows that

$$\underline{H}_0(S^\sigma;\mathbb{Z}) \cong \begin{array}{c} \mathbb{Z}^2 \\ \big\downarrow \nabla \\ \mathbb{Z} \end{array} \qquad \underline{H}_1(S^\sigma;\mathbb{Z}) \cong \begin{array}{c} 0 \\ \\ \mathbb{Z}_{\mathrm{sgn}} \end{array}$$

Here we have written $\mathbb{Z}_{\mathrm{sgn}}$ for $\underline{H}_1(S^\sigma)(C_2/e)$ since the nontrivial element of C_2 acts as an orientation-reversing map on S^σ.

Example 4.5

Here is an important example: for a G-CW complex X, the functor that sends G/K to $C_n^{cell}(X^K)$ defines a coefficient system denoted $\underline{C_n}(X)$. This relies on the observation that the H-fixed points of a G-CW complex inherit a CW-structure.

Example 4.6

We constructed a cell structure on S^σ in Example (3.7). On fixed points this gives S^0, with two 0-cells and no higher cells, while on the underlying space it gives the cell structure for S^1 with two 0-cells and two 1-cells. Then:

$$\underline{C}_0(S^\sigma;\mathbb{Z}) \cong \begin{array}{c} \mathbb{Z}^2 \\ \big\downarrow \mathrm{id} \\ \mathbb{Z}^2 \end{array} \qquad \underline{C}_1(S^\sigma;\mathbb{Z}) \cong \begin{array}{c} 0 \\ \\ \mathbb{Z}[C_2] \end{array}$$

Here we write $\mathbb{Z}[C_2]$ for $\underline{C}_1(S^\sigma)(C_2/e)$ since the C_2-action exchanges the two 1-cells in $(S^\sigma)^e$.

Definition 4.10. A map of G-coefficient systems $C \to D$ is a natural transformation. We will write $\mathrm{Hom}_{\mathrm{coeff}}(C, D)$ for the set (actually abelian group) of maps between C and D.

Remark 4.11. If we fix a subgroup K and allow n to vary in Example (4.5), then $\underline{C}_*(X) = C_*^{cell}(X^K)$ is a chain complex. The differentials commute with restriction and conjugation in the obvious manner, and so we have maps

$$d_n \colon \underline{C}_n(X) \to \underline{C}_{n-1}(X)$$

of coefficient systems, which make $\underline{C}_*(X)$ into a chain complex.

Definition 4.12 (Bredon cohomology). Let X be a G-space, and M a G-coefficient system. $\mathrm{Hom}_{coeff}(\underline{C}_*(X), M)$ is a cochain complex of abelian groups, which we write $C^*_{\mathrm{coeff}}(X; M)$, and the *Bredon cohomology* of X with coefficients in M is

$$H^n_G(X; M) \coloneqq H^n(C^*_{\mathrm{coeff}}(X; M)).$$

In the case of $M = \underline{\mathbb{Z}}$, this turns out to be very simple.

Lemma 4.13. *Let C be a G-coefficient system, then*

$$\mathrm{Hom}_{\mathrm{coeff}}(C, \underline{\mathbb{Z}}) \cong \mathrm{Hom}_{AbGrps}(C(G/e)/G, \mathbb{Z}).$$

Proof. Exercise. □

Example 4.7

In Example (4.6) we computed the cellular complex for S^σ, up to determining the boundary maps. There is only one map left to determine, and that is the map

$$\mathbb{Z}[C_2] \cong \underline{C}_1(S^\sigma)(C_2/e) \to \underline{C}_0(S^\sigma)(C_2/e) \cong \mathbb{Z} \oplus \mathbb{Z}.$$

Recall that the differential of a 1-cell is just the signed sum of its two boundary 0-cells:

$$d(e) = \phi(1) - \phi(0)$$

for $\phi \colon S^0 \to X^0$ the attached map. In other words, $\underline{C}_*(S^\sigma)$ is

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & \mathbb{Z} \oplus \mathbb{Z} \arrow[d, "\mathrm{id}"] \\ \mathbb{Z}[C_2] \arrow[r, "\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}"'] & \mathbb{Z} \oplus \mathbb{Z} \end{tikzcd}$$

Now we want to apply $\mathrm{Hom}_{coeff}(-, \underline{\mathbb{Z}})$. By Lemma 4.13 this is

$$\mathbb{Z} \oplus \mathbb{Z} \xrightarrow{(1\ -1)} \mathbb{Z}$$

Hence,

$$H^i_{C_2}(S^\sigma; \underline{\mathbb{Z}}) \cong \begin{cases} \mathbb{Z} & i = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Example 4.8

Let us do the case $S = S^{2\sigma}$. This is the 2-sphere with action given by rotation by a half-turn. This has a C_2-CW structure given by the following:

a. There are two-zero cells corresponding to the two-fixed points, the north and south poles.

b. There is a single free one-cell corresponding to the boundary of the hemispheres.

c. There is a single 2-cell.

See Figure 4.1. We can then produce the cellular chain complex:

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \mathbb{Z} \oplus \mathbb{Z} \arrow[d, "\mathrm{id}"] \\ \mathbb{Z}[C_2] \arrow[r, "1-g"'] & \mathbb{Z}[C_2] \arrow[r, "\left(\begin{smallmatrix} 1 & -1 \\ 1 & -1 \end{smallmatrix}\right)"'] & \mathbb{Z} \oplus \mathbb{Z} \end{tikzcd}$$

where $C_2 = \langle g \rangle$. Applying $\mathrm{Hom}_{\mathrm{coeff}}(-, \mathbb{Z})$ we get the chain complex

$$\mathbb{Z} \oplus \mathbb{Z} \xrightarrow{(1\ -1)} \mathbb{Z} \xrightarrow{0} \mathbb{Z}$$

and so

$$H^i_{C_2}(S^{2\sigma}; \underline{\mathbb{Z}}) \cong \begin{cases} \mathbb{Z} & i = 0, 2 \\ 0 & \text{otherwise.} \end{cases}$$

Both of these could have been computed using the following general result.

Lemma 4.14. *For any G-space X we have*

$$H^i_G(X; \underline{\mathbb{Z}}) \cong H^i(X/G; \mathbb{Z}).$$

Indeed, in our first computation the orbit space is contractible, while in the second example, it is equivalent to S^2 (we have attached a two-cell to a contractible 1-skeleton). This lemma is an easy consequence of Lemma 4.13, but will also follow from an axiomatic treatment of equivariant cohomology theories.

There is also a way to get fixed points instead of orbits.

Lemma 4.15. *For any G-space X we have $H^n_G(X; \mathrm{Inf}^G_e(\mathbb{Z})) \cong H^n(X^G; \mathbb{Z})$.*

Proof. Exercise. $\square$

Finally, we can also get the ordinary homology of the underlying non-equivariant space of X:

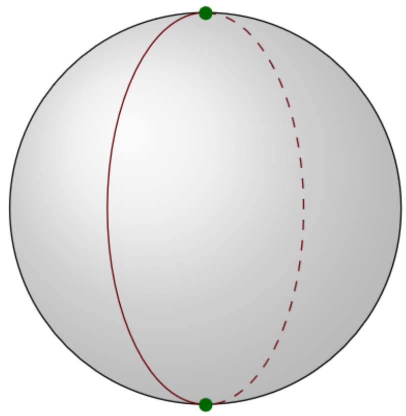


Figure 4.1: C_2-CW structure on $S^{2\sigma}$, the 2-sphere with a C_2-action by rotation through 180°. The two green dots are the two 0-cells $C_2/C_2 \times D^0$; the red circle is the single 1-cell $C2/e \times D^1$, and the gray hemispheres are the single 2-cell $C_2/e \times D^2$.

.

Lemma 4.16. *For any G-space X, we have $H^n(X; \mathrm{Ind}_e^G(\mathbb{Z})) \cong H^n(X; \mathbb{Z})$.*

Proof. This follows from the following claim: there is an equivalence

$$\mathrm{Hom}_{\mathrm{coeff}}(C, \mathrm{Ind}_e^G(\mathbb{Z})) \cong \mathrm{Hom}_{AbGrp}(C(e), \mathbb{Z}),$$

which is essentially a consequence of Proposition 2.18: induction is right adjoint to the forgetful functor. □

Remark 4.17. There was nothing special about $\mathbb{Z}$ in these three results; we could have used any abelian group A. Later we will consider the case $A = \mathbb{F}_p$.

Bredon homology

To define Bredon homology with need to work with functors $\mathcal{O}_G \to \mathrm{Ab}$ rather than $\mathcal{O}_G^{op} \to \mathrm{Ab}$. We call these $\mathcal{O}_G$-modules. With M such an $\mathcal{O}_G$-module we define:

Definition 4.18. The Bredon homology of X with coefficients in M is

$$H_n^G(X; M) := H_n(\underline{C}_n(X) \otimes_{\mathcal{O}_G} M).$$

Here $\underline{C}_n(X) \otimes_{\mathcal{O}_G} M$ denotes the coend

$$\underline{C}_n(X) \otimes_{\mathcal{O}_G} M := \int^{G/H \in \mathcal{O}_G} \underline{C}_n(X)(G/H) \otimes M(G/H).$$

If you haven't seen coends before, they can be given as a coequalizer: If $S\colon \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{C}$ is a functor, then the coend of S is equivalent to the coequalizer

$$\coprod_{c' \to c} S(c', c) \rightrightarrows \coprod_{c \in \mathcal{C}} S(c, c) \longrightarrow \int^{c \in \mathcal{C}} S(c, c)$$

To ground your intuition, thinking of a ring as an Ab-enriched category with one object, then a left R-module B is an additive functor $R \to Ab$ sending the one object of R to the abelian group B and sending each arrow $r \in R$ to the scalar multiplication $r_*\colon b \mapsto rb$. A right R-module is an additive functor $R^{op} \to Ab$. In particular, $R \mapsto A \otimes B$ is a bifunctor $R^{op} \times R \to Ab$. Then,

$$\int^R A \otimes B = A \otimes_R B$$

is the usual tensor product over R.

Remark 4.19. One gets an $\mathcal{O}_G$-module from a Mackey functor by forgetting the *restriction* maps.

One has analogs of the results about Bredon cohomology:

Lemma 4.20. *There are isomorphisms*

(i) $H_n^G(X;\underline{\mathbb{Z}}) \cong H_n(X/G;\mathbb{Z})$,

(ii) $H_n^G(X;\mathrm{Inf}_e^G(\mathbb{Z})) \cong H_n(X^G;\mathbb{Z})$, *and*

(iii) $H_n^G(X;\mathrm{Ind}_e^G(\mathbb{Z})) \cong H_n(X;\mathbb{Z})$.

Remark 4.21. Let us relate Bredon (co)homology with Borel cohomology. We have

$$H^*_{Borel}(X;\mathbb{Z}) \coloneqq H^*(EG \times_G X;\mathbb{Z}) \cong H^*(EG \times X;\underline{\mathbb{Z}})$$

and

$$H_*^{Borel}(X;\mathbb{Z}) \coloneqq H_*(EG \times_G X;\mathbb{Z}) \cong H_*(EG \times X;\underline{\mathbb{Z}})$$

So Borel (co)homology is just a special case of Bredon (co)homology. Note then that the projection map $EG \times X \to X$ induces a map

$$\phi\colon H^*(X;\underline{\mathbb{Z}}) \to H^*_{Borel}(X;\mathbb{Z})$$

Lemma 4.22. *Suppose that X is a free G-CW complex, i.e., it can be built from free cells $G_+ \times D^n$, then ϕ is an isomorphism.*

Proof. For such spaces the projection map $EG \times X \to X$ is a G-equivariant weak equivalence. Since X was assumed to be a G-CW complex, the Whitehead theorem (Theorem 3.16) implies that the projection is a G-equivariant homotopy equivalence. □

Application - Smith theory and the Conner conjecture

We will use Bredon homology to prove the following theorem.

Theorem 4.23 (Smith)**.** *Let G be a finite p-group and X be a finite G-CW complex such that the underlying topological space of X is an $\mathbb{F}_p$-cohomology sphere, i.e., $H^*(X;\mathbb{F}_p) \cong H^*(S^n;\mathbb{F}_p)$ for some n. Then X^G is either empty or an $\mathbb{F}_p$-cohomology sphere (of equal or smaller dimension).*

Proof. We claim that we can easily reduce to the case $G = \mathbb{Z}/p$. Indeed, if $H \subseteq G$ is a normal subgroup, then $X^G \cong (X^H)^{(G/H)}$, and so we can induct on the order of the group using the Sylow theorems. We have already seen that there exist coefficient systems M and N such that

$$H^*_G(X; M) \cong H^*(X; \mathbb{F}_p) \quad \text{and} \quad H^*_G(X; M) \cong H^*(X^G; \mathbb{F}_p).$$

namely $M = \mathrm{Ind}_e^G(\mathbb{F}_p)$ and $N = \mathrm{Inf}_e^G(\mathbb{F}_p)$. We claim that there also exists a coefficient system L such that $H^*_{C_p}(X; L) \cong \widetilde{H}^*((X/X^{C_p})/C_p; \mathbb{F}_p)$. Indeed, we leave it as an exercise to show that $L := \ker(\underline{\mathbb{F}}_p \to \mathrm{Inf}_e^{C_p}(\mathbb{F}_p))$ works.

We now fix $p = 2$ for simplicity. We have a short exact sequence

$$\mathbb{F}_2 \xrightarrow{\Delta} \mathbb{F}_2[C_2] \xrightarrow{\nabla} \mathbb{F}_2$$

which extends to a short exact sequence of coefficient systems

$$0 \to L \to M \to I \oplus L \to 0$$

where I is the inflation from e to C_2 of $\mathbb{F}_2$. This induces a long exact sequence in cohomology, which we can rewrite (using the definitions of the coefficient systems) as

$$\cdots \to \widetilde{H}^n((X/X^{C_2})/C_2; \mathbb{F}_2) \to H^n(X; \mathbb{F}_2) \to \widetilde{H}^n((X/X^{C_2})/C_2; \mathbb{F}_2) \oplus H^n(X^{C_2}; \mathbb{F}_2)$$
$$\to \widetilde{H}^{n+1}((X/X^{C_2})/C_2; \mathbb{F}_2) \to \cdots$$

Let

$$a_n = \dim \widetilde{H}^n((X/X^{C_2})/C_2; \mathbb{F}_2), \quad b_n = \dim H^n(X; \mathbb{F}_2), \quad c_n = H^n(X^{C_2}; \mathbb{F}_2).$$

Then exactness of the long exact sequence (at the direct sum) implies that

$$a_n + c_n \leq b_n + a_{n+1}.$$

Note that

$$a_0 + c_0 + c_1 \leq b_0 + a_1 + c_1 \leq b_0 + b_1 + a_2$$

or more generally that

$$a_0 + c_0 + c_1 + \ldots c_n \leq b_0 + b_1 + \cdots + b_n + a_{n+1}$$

for any n. Taking n large enough, we deduce that $\sum c_i \leq \sum b_i$. But by assumption $\sum b_i = 2$. It follows that $\sum c_i \leq 2$. If we assume that X^{C_2} is non-zero, then we must have $\sum c_i > 0$, so it remains to eliminate the possibility that $\sum c_i = 1$. But applying the Euler characteristic[1] we deduce that

$$\chi(X) = \chi(X^{C_2}) + 2\chi((X/X^{C_2})/C_2) \equiv \chi(X^{C_2}) \mod (2).$$

This finishes the proof. $\square$

[1]Recall that this is the alternating sum of the ranks of the cohomology

Theorem 4.24 (Conner conjecture, proved by Oliver)**.** *Let G be a finite group, and let X have the homotopy type of a finite dimensional G-CW complex. Then,*

$$\widetilde{H}^*(X;A) = 0 \implies \widetilde{H}^*(X/G;A) = 0.$$

Remark 4.25*.* The finiteness assumption is neccessary. Note that the free G-space EG is contractible, but $EG/G = BG$ has non trivial cohomology unless $G = e$.

Proof. We will prove this for $G = C_p$ - again, once can reduce to this case. We will still stick with $p = 2$. First, we assume that X^{C_2} has a fixed point - if not, we can replace it with ΣX.

By Smith theory, and the assumption that X is acyclic, we have

$$\sum \dim(H^i(X^{C_2};\mathbb{F}_2)) \le \sum \dim(H^i(X;\mathbb{F}_2)) = 1.$$

and

$$\chi(X) \equiv \chi(X^{C_2}) = 1 \mod (2).$$

This is only possible if $\widetilde{H}^*(X^{C_2};\mathbb{F}_2) = 0$. Likewise, the inequalities,

$$a_q + c_q \le b_q + a_{q+1}$$

□

Axioms for equivariant homology and cohomology

We know that (co)homology can be characterized on CW complexes by the Eilenberg–Steenrod axioms. Are there similar axioms for equivariant (co)homology theories? The answer is yes. This takes place in the category of pairs of G-CW complexes, i.e., the category with objects (X,A) for X a G-CW complex and A a sub-complex, and morphisms $(X,A) \to (Y,B)$ morphisms $f\colon X \to Y$ such that $f(A) \subseteq B$.

Definition 4.26*.* A (ordinary) homology theory on G-CW complexes is a sequence of functors $h_n(X,A)$ on pairs of G-CW complexes and natural transformations $\delta\colon h_n(X,A) \to h_{n-1}(A,\emptyset)$ satisfying the following axioms:

(1) (Homotopy invariance) If $f \simeq g$, then $f_* = g_*$.

(2) (Long exact sequence) If we let $h_n(X) \coloneqq h_n(X,\emptyset)$ then there is an exact sequence

$$\cdots \to h_n(A) \to h_n(X) \to h_n(X,A) \xrightarrow{\delta} h_{n-1}(A) \to \cdots$$

(3) (Excision) If X is the union of subcomplexes A and B, then the inclusion $(A, A\cap B) \hookrightarrow (X,B)$ induces an isomorphism

$$h_n(A, A\cap B) \cong h_n(X,B)$$

(4) (Additivity) If (X, A) is a disjoint union of pairs (X_i, A_i), then the inclusions $(X_i, A_i) \to (X, A)$ induce an isomorphism

$$\bigoplus_i h_n(X_i, A) \cong h_n(X, A).$$

(5) (Dimension) $h_n(G/H) = 0$ if $n \neq 0$.

For cohomology theories, reverse the directions of the arrows.

Theorem 4.27. *Bredon (co)homology is an equivariant (co)cohomology theory, where we set* $\underline{C}_*(X, A) = \widetilde{\underline{C}}_*(X/A)$.

Proof. Homotopy invariant, excision and additivity and excision are relatively straightforward. Let us discuss the dimension axiom and the long exact sequence. Recall that $\underline{C}_*(X)$ is a coefficient system whose value at (G/K) is the free abelian group $C_*^{cell}(X^K)$. We can do homological algebra in the category of coefficient systems, and we claim that $\underline{C}_*(X)$ is a projective object here. We will not prove this, but it is a consequence of a more general result:

Lemma 4.28. *If S is a finite G-set, then the coefficient system F_S with*

$$F_S(G/H) = F(S^H)$$

the free abelian group on the set S^H, is a projective object in the category of coefficient systems.

This lemma implies that $\mathrm{Hom}(\underline{C}_*(X), -)$ preserves short exact sequences of coefficient systems, and it follows that we get the desired long exact sequence. A Yoneda lemma argument shows that

$$H^0_G(G/H; M) := H^0(\mathrm{Hom}_{\mathrm{coeff}}(\underline{C}_*(G/H), M)) \cong M(G/H)$$

while

$$H^i_G(G/H; M) \cong 0$$

for $i > 0$. □

Remark 4.29. In fact, a uniqueness theorem similar to the usual theorem for non-equivariant cohomology shows that Bredon cohomology with coefficients in M is the only equivariant cohomology theory with those coefficients.

Remark 4.30. There is a spectral sequence

$$\mathrm{Ext}^p_{coeff}(h_q(X), M) \implies H^{p+q}_G(X; M),$$

where $h_q(X)(G/H) = H_q(X^H)$.

Remark 4.31 ($RO(G)$-grading)*.* So far we have indexed our cohomology theories on the integers $\mathbb{Z}$. However, we have already seen that we have more notions of a sphere in equivariant homotopy. So, can we (and should be) grade on the group $RO(G)$? There are some subtle issues here.[2] The point is that $RO(G)$ consists of *equivalence* classes, and we really want to the remember the isomorphisms between isomorphic representations. The usual fix is to rather fix an choice of preferred representatives. For example, this is easy to do when we talk about $G = \mathbb{Z}/2$ when the representation is simple. We will tend to ignore this remark from now, but please be aware of it.

In any case, we end up with suspension and loop functors for a representation V, and the following axiom:

Axiom 4.32 (Equivariant suspension axiom)**.** *For each $\alpha \in RO(G)$ and actual representation V, there is a natural isomorphism*

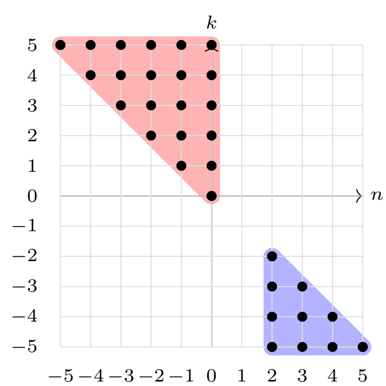
$$\Sigma^V : \widetilde{H}^\alpha_G(X) \xrightarrow{\sim} \widetilde{H}^{\alpha+V}_G(\Sigma^V X) = \widetilde{H}^{\alpha+V}_G(S^V \wedge X).$$

In general, even determining $\widetilde{H}^\alpha_G(G/K_+)$ for $\alpha \notin \mathbb{Z}$ is very complicated.

Remark 4.33*.* One might wonder if Bredon cohomology with coefficients in M extends to an $RO(G)$-graded cohomology theory in general. This turns out to not be true. In fact, it is a theorem of Lewis, May, and McClure that it does so if and only M is the underlying coefficient system of a Mackey functor.

We will do one single computation, that of the Bredon cohomology of a point with $G = \mathbb{Z}/2$ and coefficients in the constant coefficient system $\underline{\mathbb{F}_2}$. We recall that $RO(C_2) \cong \mathbb{Z}\{\mathbf{1}\} \oplus \mathbb{Z}\{\sigma\}$, so our groups will be bigraded. The result is best described pictorially:

Theorem 4.34. *The bigraded groups $H^{n+k\sigma}_{C_2}(*, \underline{\mathbb{F}_2})$ are as follows, where each dot represents a copy of $\mathbb{F}_2$:*



[2] For amusement, see [1, Section 6].

Proof. We will start with the blue region. These are the groups $H^{n-k\sigma}(*,\underline{\mathbb{F}_2})$ for $k>0$, which, using the suspension axiom becomes,

$$H^{n-k\sigma}_{C_2}(*,\underline{\mathbb{F}_2}) \cong \widetilde{H}^{n-k\sigma}_{C_2}(S^0,\underline{\mathbb{F}_2}) \cong \widetilde{H}^{n}_{C_2}(S^{k\sigma},\underline{\mathbb{F}_2}) \cong \widetilde{H}^{n}(S^{k\sigma}/C_2,\mathbb{F}_2)$$

by the $\mathbb{F}_2$-analog of Lemma 4.14. It is an exercise on the worksheet to show that $S^{k\sigma}/C_2 \simeq \Sigma\mathbb{RP}^{k-1}$. These groups are determined by

$$\widetilde{H}^n(\Sigma\mathbb{RP}^{k-1};\mathbb{F}_2) \cong \begin{cases} \mathbb{F}_2, & n=2,3,\dots,k \\ 0 & \text{else.} \end{cases}$$

So, for $k>0$, we have

$$H^{n-k\sigma}(*;\underline{\mathbb{F}_2}) \cong \begin{cases} \mathbb{F}_2 & 2\le n\le k \\ 0 & \text{else} \end{cases}$$

These determines the blue region of the diagram (this is often referred to as the 'negative' cone).

Now we turn to the harder part, which is the computation of $H^{n+k\sigma}_{C_2}(*;\underline{\mathbb{F}_2})$ for $k\ge 0$. Here we have

$$H^{n+k\sigma}_{C_2}(*;\underline{\mathbb{F}_2}) \cong \widetilde{H}^{n+k\sigma}_{C_2}(S^0;\underline{\mathbb{F}_2}) \cong \widetilde{H}^{C_2}_{-n-k\sigma}(S^0;\underline{\mathbb{F}_2})$$

where the last equivalence uses an equivariant version of Poincaré duality. This is in fact the motivation for $RO(G)$-grading - if we want to have Poincaré duality, we are forced into grading over representations. Taking this on faith, we must therefore compute the groups $\widetilde{H}^{C_2}_{-n}(S^{k\sigma},\underline{\mathbb{F}_2})$ Since we work with homology, the $\mathcal{O}_G$-module $\underline{\mathbb{F}_2}$ here refers to the $\mathcal{O}_G$-module whose transfer maps are all 0. This $\mathcal{O}_G$-module splits as $\mathrm{Inf}_e^{C_2}\mathbb{F}_2 = N$ and L, the $\mathcal{O}_G$-module that is $\mathbb{F}_2$ at the bottom, and 0 at the top. So

$$\widetilde{H}^{C_2}_{-n}(S^{k\sigma};\underline{\mathbb{F}_2}) \cong \widetilde{H}^{C_2}_{-n}(S^{k\sigma};N\oplus L) \cong \widetilde{H}^{C_2}_{-n}(S^{k\sigma};N)\oplus \widetilde{H}^{C_2}_{-n}(S^{k\sigma};L)$$

By Lemma 4.20 and the homology analog of the argument given in the proof of Smith theory, we have

$$\widetilde{H}^{C_2}_{-n}(S^{k\sigma};\underline{\mathbb{F}_2}) \cong \widetilde{H}_{-n}(S^0;\mathbb{F}_2)\oplus \widetilde{H}_{-n}(((S^{k\sigma})/S^0)/C_2;\mathbb{F}_2)$$

The first term gives us an $\mathbb{F}_2$ when $n=0$, and gives the dots on the positive y-axis in the figure. In order to compute this, we note that there is a cofiber sequence

$$S(k\sigma)_+ \to D(k\sigma)_+ \to S^{k\sigma},$$

where $S(V)$ and $D(V)$ denote the unit sphere and unit disk, respectively, inside an orthogonal representation. Now $S(k\sigma)$ is actually S^{k-1}_a, the antipodal sphere of dimension $k-1$. Using the $D(k\sigma)$ is equivariantly contractible, passing to quotients gives us the sequence

$$\mathbb{RP}^{k-1}_+ \to S^0 \to S^{k\sigma}/C_2 \to \Sigma\mathbb{RP}^{k-1}_+$$

In the end, we deduce that

$$\widetilde{H}^{C_2}_{-n}(S^{k\sigma};\underline{\mathbb{F}_2}) \cong \widetilde{H}_{-n}(S^0;\mathbb{F}_2) \oplus \widetilde{H}_{-n}(\Sigma\mathbb{RP}^{k-1}_+;\mathbb{F}_2)$$

The first term contributes an $\mathbb{F}_2$ when $n = 0$, while the second does so whenever $n \in [1, k]$. This gives the red region in the diagram. $\square$

Chapter 5

From Gspaces to G-spectra

From spaces to spectra

We begin with some motivation for the category of spectra.

Remark 5.1. We recall that for a space X the Freudenthal suspension theorem says that the maps

$$\pi_n(X) \to \pi_{n+1}(\Sigma X) \to \pi_{n+2}(\Sigma^2 X)$$

eventually stabilize (after finitely many steps). We call these the *stable* homotopy groups of X. One can wonder: is there an object whose n-th homotopy group is the n-th stable homotopy groups of X. This is what spectra are designed to do.

Definition 5.2. A spectrum E is a sequence of spaces $E = \{E_n\}$ and maps $\sigma_n \colon \Sigma E_n \to E_{n+1}$. The r-th homotopy group of E is defined to be

$$\pi_r(E) \coloneqq \lim_{n\to\infty} \pi_{n+r}(E_n),$$

where the colimit is taken over the diagram where the maps are

$$\pi_{n+r}(E_n) \xrightarrow{\Sigma_n} \pi_{n+r+1}(\Sigma E_n) \to \pi_{n+r+1}(E_{n+1}).$$

A CW-spectrum additionally has the property that each E_n is a CW-complex, and the maps $\Sigma E_n \to E_{n+1}$ exhibit ΣE_n as a subcomplex of E_{n+1}.

Remark 5.3. We note that stable homotopy groups are in general defined for all $r \in \mathbb{Z}$, where we must start the colimit for $n + r \geq 0$. For example, $\pi_{-1}(E)$ is the colimit of

$$\pi_0(E_1) \to \pi_1(E_2) \to \cdots$$

Example 5.1

For a based space X, the suspension spectrum $\Sigma^\infty X$ is the spectrum consisting of spaces $\{\Sigma^n X\}$ with maps $\sigma_n \colon \Sigma^{n+1} X \to \Sigma^{n+1} X$ the identity map. For example, $\Sigma^\infty S^0$ is what people refer to as the *sphere spectrum* and is often denoted by $\mathbb{S}$. The n-th homotopy group of $\Sigma^\infty X$ is exactly the n-th stable homotopy group of X.

Another reason to care about these is that they represent cohomology theories.

Theorem 5.4. *(Adams' Brown representability theorem). Every cohomology (resp. homology) theory on CW-complexes is represented (resp. corepresented) by a spectrum.*

What is a morphism of spectra? There is an obvious guess:

Definition 5.5. A *strict morphism* of spectra $f\colon E \to F$ of degree r is a sequence of maps $f_n\colon E_n \to F_{n-r}$ such that the diagram

$$\begin{tikzcd} \Sigma E_n \arrow[r, "\Sigma f_n"] \arrow[d] & \Sigma F_{n-r} \arrow[d] \\ E_{n+1} \arrow[r, "f_{n+1}"'] & F_{n+1-r} \end{tikzcd}$$

strictly commutes.

This turns out to not be the correct thing. For example, we would like the Hopf map $\eta\colon S^3 \to S^2$ to give a morphism $\mathbb{S} \to \mathbb{S}$ of degree 1, but this map does not desuspend to a map $S^2 \to S^1$ or a map $S^1 \to S^0$. There is a fix for this.

Definition 5.6. Let E be a CW spectrum. A subspectrum of E is a sequence of subcomplexes $F_n \subseteq E_n$ so that ΣF_n is a subcomplex of F_{n+1}. We say that F is cofinal if every cell of E contains an element which is a cell of some F_n. This is equivalent to the following: for each n and each finite subcomplex $K \subseteq E_n$ there is a d such that $\Sigma^d K \subseteq F_{n+d}$.

Definition 5.7. A morphism between CW spectra $f\colon X \to Y$ is defined to be a strict map $X' \to Y$ for some cofinal subspectrum X' of X. Two maps $X \to Y$ between CW-spectra are considered to be the same if if they take the same values on a common cofinal subspectrum. Since the intersection of two cofinal subspectra is a cofinal subspectrum, this amounts to saying that replacing the cofinal subspectrum on which a spectrum map is defined by a smaller cofinal subspectrum is regarded as giving the same map. We need to check this is well defined, and that we can compose maps, but this can be done.

One can then go on to define a notion of homotopy by defining $X \times I$ to be the spectrum with $(X \times I)_n \simeq X_n \times I$ (the reduced product, i.e., the basepoint cross I is collapsed to a point). This leads to the stable homotopy category Ho(Sp). Here are two useful facts about this category.

Theorem 5.8 (Whitehead theorem)**.** *A morphism $f\colon X \to Y$ between CW-complexs that induces an isomorphism on homotopy groups is a homotopy equivalence.*

Theorem 5.9. *The map $[X,Y] \to [\Sigma X, \Sigma Y]$ is a bijection, and Σ is an auto-equivalence of our category.*

Example 5.2

Let $X_n = K(G,n)$ for an abelian group G. Then we can define a spectrum HG with $(HG)_n \simeq K(G,n)$ and the map $\sigma_n \colon \Sigma K(G,n) \to K(G,n+1)$ the adjoint to the equivalence $K(G,n) \xrightarrow{\simeq} \Omega K(G,n+1)$. We know that

$$H^n(X;G) \cong [X, K(G,n)]$$

and

$$H^n(X;G) \cong [\Sigma^\infty X, HG]_{-n}$$

One could like to get read of the 'homotopy' part of Ho(Sp) and build a stable model category, or ∞-category, whose homotopy category is the homotopy category considered above. It took a long time to find the right technical tools to do this, but there are now several different approaches. Note that what we have really done is inverted the suspension functor, and made its inverse the loops functor. With a bit of machinery, one can even make this into a definition:

$$\mathrm{Sp} \coloneqq \varprojlim(\mathrm{Top}_* \xrightarrow{\Omega} \mathrm{Top}_* \xrightarrow{\Omega} \cdots)$$

Here we take the colimit in the category of categories. Equivalently, this is the colimit in the category of presentable categories

$$\mathrm{Sp} \coloneqq \mathrm{Top}_*[S^{-1}] \simeq \varinjlim(\mathrm{Top}_* \xrightarrow{-\wedge S^1} \mathrm{Top}_* \xrightarrow{\wedge S^1} \cdots)$$

Remark 5.10. It is useful to recall by analogy that in commutative algebra we have

$$R[x^{-1}] \cong \varinjlim(R \xrightarrow{\cdot x} R \xrightarrow{\cdot x} R \cdots$$

G-spaces and G-spectra

We have seen that we have G-spaces, and so we can try and stabilize and talk about G-spectra. Our first guess might be to directly take the definition of spectra and put a G-everywhere.

Definition 5.11. A G-equivariant S^1 spectrum E is a sequence of G-spaces $E = \{E_n\}$ and equivariant maps $\sigma_n \colon \Sigma E_n \to E_{n+1}$.

This defines for us the so-called category of 'naive' G-spectra. By a variant of Elmendorf's theorem, we have the following:

Theorem 5.12. *The category of naive G-spectra is equivalent to the category* $\mathrm{Fun}(\mathcal{O}_G^{op}, \mathrm{Sp})$.

Although these are not ultimately not what we will be interested in, they are still strong enough to represent $\mathbb{Z}$-graded cohomology theories.

Theorem 5.13. *Every cohomology (resp. homology) theory on G-CW complexes is represented (resp. corepresented) by a naive G-spectrum.*

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