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MA8403 - Equivariant homotopy theory

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Preface

This are the courses notes for MA8403 - Equivariant homotopy theory, held during the Autumn semester 2023 at NTNU. The notes are mainly based on two excellent sets of lectures notes, one by Guillou [2], and one by Blumberg [1]. The notes will be continually updated during the semester.

Contents

Chapter 1

Equivariance in algebra

Group actions in algebra

We recall that if X is an object in a category $\mathcal{C}$, then the set of endomorphisms $\mathrm{End}(X)$ forms a monoid (that is, a set equipped with an associative binary operation and an identity element) under composition. The set of automorphisms of X (that is, those endomorphisms that are invertible) form a group. Moreover, we have

$$\mathrm{Aut}(X) = \mathrm{End}(X) \cap \mathrm{Iso}(\mathcal{C})$$

Note that any group is a monoid, simply by forgetting the existence of inverses.

Definition 1.1. An action of a group G on an object $X \in \mathcal{C}$ is a monoid homomorphism $a\colon G \to \mathrm{End}(X)$, or equivalently a group homomorphism $a\colon G \to \mathrm{Aut}(X)$ (note that a monoid homomorphism between groups is a group homomorphism).

Remark 1.2. Unwinding the definition, this means that we have:

a. For each $g \in G$, there is a morphism $a(g)\colon G \to G$.

b. a preserves composition, i.e., $a(g \cdot h) = a(h) \cdot a(h)$.

c. a preserves identifies, so $a(e) = \mathrm{id}_X$.

Example 1.1

Let $\mathcal{C}$ be the category of sets and functions. Then, $a\colon G \to \mathrm{End}(X) = \{ f\colon X \to X \}$ correspond to a function $\overline{a}\colon G \times X \to X$. The conditions above mean that the diagrams

$$\begin{tikzcd} G \times G \times X \arrow[r, "m \times \mathrm{id}"] \arrow[d, "\mathrm{id} \times \overline{a}"'] & G \times X \arrow[d, "\overline{a}"] \\ G \times X \arrow[r, "\overline{a}"'] & X \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \{*\} \times X \arrow[r, "e \times \mathrm{id}"] \arrow[d, "\cong"'] & G \times X \arrow[d, "\overline{a}"] \\ X \arrow[r, equal] & X \end{tikzcd}$$

commute, where $m\colon G \times G \to G$ denotes the group multiplication. Equivalently, in symbols, we have

$$\overline{a}(h, \overline{a}(h, x)) = \overline{a}(gh, x)$$

and

$$\overline{a}(e, x) = x.$$

Remark 1.3. Let BG denote the category with one object $*$ and with $\mathrm{Hom}(*,*) = G$. Then an action of G in the category $\mathcal{C}$ is the same as a functor $\rho\colon BG \to \mathcal{C}$. The object X in the previous definition is the object $\rho(*) \in \mathcal{C}$.

Remark 1.4. Let $\mathcal{C} = \mathrm{Mod}_R$ for a commutative ring R. An action of G on $M \in \mathrm{Mod}_R$ is a monoid homomorphism

$$a\colon G \to \mathrm{Hom}_R(M, M).$$

We recall that $\mathrm{Hom}_R(M, M)$ actually has the structure of an R-algebra

Definition 1.5. The (R-linear) group ring on R is the R-algebra $R[G]$ whose:

(a) underlying R-module is the free R-module with basis on the underlying set of G.

(b) whose multiplication is given on basis elements by the group operation.

Example 1.2

Let $R = \mathbb{Z}$ and $G = C_2 = \langle \sigma \rangle$. An element of $\mathbb{Z}[C_2]$ is of the form $a + b\sigma$ where $a, b \in \mathbb{Z}$. Multiplication is given by

$$(a_1 \cdot 1 + b_1\sigma) \cdot (a_2 \cdot 1 + b_2\sigma) = (a_1a_2 + b_1b_2) \cdot 1 + (a_1b_2 + b_1a_2)\sigma.$$

This is the same thing as the polynomial ring $\mathbb{Z}[\sigma]/(\sigma^2 - 1)$.

Remark 1.6. Categorically, the group ring construction is left adjoint to the functor that takes an R-algebra to its group of units, i.e., there is an adjudication

$$R[-]\colon \mathrm{Grp} \leftrightarrows \mathrm{Mod}_R\colon (-)^\times$$

Returning to group actions, we have the following:

Proposition 1.7. *Let R be a commutative ring, and G a finite group. The following data on an R-module M are equivalent:*

a. *A monoid homomorphism $G \to \mathrm{End}_R(M)$.*

b. *A group homomorphism $G \to \mathrm{Aut}_R(M)$.*

c. *A homomorphism of R-algebras $R[G] \to \mathrm{Hom}_R(M, M)$.*

d. *An $R[G]$-module structure on M whose underlying R-module structure is M.*

Definition 1.8. A representation of G over R is an $R[G]$-module.

Example 1.3

If $R = k$ is a field, then the underlying R-module is a k-vector space V. If $\dim_k(V) = n$, then $\mathrm{Aut}_k(V) = GL_n(k)$, and a k-representation is the same thing as a group homomorphism $G \to GL_n(k)$.

Definition 1.9. The $R[G]$-module $R[G]$ is known as the regular representation. More generally, if X is a finite G-set, then the free R-module $R[X]$ inherits the structure of a $R[G]$-module (the case of $R[G]$ itself corresponds to the finite G-set G, considered as an $R[G]$-module over itself). Representations obtained this way are known as permutation representations.

Example 1.4

Taking X to be the trivial G-set, we obtain the (one-dimensional) trivial representation. This is simply the $R[G]$-module R, where G acts trivially.

Definition 1.10. Let $G = C_2 = \langle \tau \rangle$, and suppose that $-1 \neq 1 \in R$. Then the sign representation of G is the one-dimensional representation where τ acts as -1 (if $-1 = 1$ in R this still makes sense, but is just the trivial representation). Note that this is an example of a representation that is not a permutation representation.

Example 1.5

Let $C_n = \langle \sigma \rangle$ be the cyclic group of order n. Let us calculate all complex 1-dimensional representations of C_n, i.e., homomorphisms $\rho \colon C_n \to \mathbb{C}$. Note that if we define $\rho(\sigma) = c$, then $\rho(\sigma^n) = c^n = 1$, so that c must be an n-th root of unity. There are precisely n-of these (take $\zeta_n = e^{2\pi/ni}$), and so there are precisely n-representations. For example, when $G = C_4$, the four representations correspond to sending σ to either $1, i, -1$ or $-i$. Note that we can also consider these as 2-dimensional *real* representations.

Notation 1.11. We let $\rho = \rho_G$ denote the regular representation of G, and the trivial n-dimensional representation by $\mathbf{n} = R^{\oplus n}$.

Definition 1.12. A subrepresentation is a submodule.

Example 1.6

The regular representation always has a one-dimensional trivial representation, generated by the sum $\sum_{g \in G} g$.

Definition 1.13. A representation V is irreducible if the only subrepresentations of V are 0 and V.

Theorem 1.14 (Maschke)**.** *Suppose that k is a field of characteristic not dividing $|G|$. Then every representation splits as a direct sum of irreducible representations.*

Proof. We prove the following: if $V \subseteq W$ is a subrepresentation, then there exists $U \subseteq W$ such that $U \oplus V \simeq W$.

To see this, let $\pi\colon W \to W$ be any k-linear projection of W onto V. This map need not be G-equivariant, but we can make it so by 'averaging'. That is, we define a new map $\phi\colon W \to W$ by

$$\phi(\mathbf{w}) = \frac{1}{|G|} \sum_{g \in G} g \cdot \pi(g^{-1} \cdot \mathbf{w}).$$

Moreover the map is G-equivariant: for $h \in G$ we have

$$\begin{aligned}\phi(h \cdot \mathbf{w}) &= \frac{1}{|G|} \sum_{g \in G} g \cdot \pi(g^{-1} \cdot h \cdot \mathbf{w}) \\ &= \frac{1}{|G|} \sum_{u \in G} u \cdot h \cdot \pi(u^{-1} \cdot \mathbf{w}) \\ &= h \cdot \phi(\mathbf{w}),\end{aligned}$$

where $u = gh^{-1}$, and so ϕ is $k[G]$-linear. Furthermore, the map ϕ is the identity on V By the splitting lemma, $W = V \oplus \ker(\phi)$. $\square$

Remark 1.15. We used the assumption on k to ensure that we could divide by $|G|$. Without that assumption, the theorem is false. Indeed, let $G = C_2$, $R = \mathbb{F}_2$ and consider the representation defined by $\rho(\tau) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. This is not irreducible, but does not split as a direct sum of indecomposable representations.

Corollary 1.16. *Suppose that k is a field of characteristic not dividing $|G|$. If V is an irreducible representation, then V is isomorphism to a subgroup of $k[G]$ (slogan: all irreducibles are submodules of the regular representation).*

Proof. Let $\mathbf{v} \in V$ be non-trivial. Then the homomorphism $\phi\colon k[G] \to V$ given by sending 1 to $\mathbf{v}$ must be surjective, because V is irreducible. Let $U = \ker(\phi)$, then apply Maschke's theorem. $\square$

Example 1.7

Let $G = C_2 = \langle \tau \rangle$, and k a field of characteristic not equal to 2. We have the trivial representation $\mathbf{1}$ and the sign representation $\mathbf{1}_{\mathrm{sgn}}$. Then, $\mathbf{1}$ is generated by the sum $1 + \tau$, while $\mathbf{1}_{\mathrm{sgn}}$ is generated by $1 - \tau$, and we deduce that

$$\rho_{C_2} = \mathbf{1} \oplus \mathbf{1}_{\mathrm{sgn}}.$$

The representation ring

Let k be a field, and suppose that V and W are G-representations, then the k-linear tensor sum $V \oplus W$ can be given the structure of a $k[G]$-module, by taking the diagonal G-action. If we think of a representation in terms of a homomorphism $\rho\colon G \to GL_n(k)$, then this direct sum corresponds to the 'block sum'

$$G \to GL_n(k) \times GL_m(k) \to GL_{n+m}(k)$$

Similarly, we can define a tensor product of representations using the 'Kronecker tensor product' (or matrix direct product). Equivalently, this is the k-linear tensor product $V \otimes W$ with the g-action defined on simple tensors by

$$g \cdot (\mathbf{v} \otimes \mathbf{w}) = g \cdot \mathbf{v} \otimes g\mathbf{w}.$$

We leave it for the reader to verify the following straightforward computations:

(a) $\mathbf{1} \otimes V \cong V \cong V \otimes \mathbf{1}$.

(b) $\mathbf{n} \otimes V \cong V^{\oplus n} \cong V \otimes \mathbf{n}$.

Example 1.8

Let us compute the tensor product $\mathbf{1}_{\mathrm{sgn}} \otimes \mathbf{1}_{\mathrm{sgn}}$. The underlying vector space is simply $k \otimes k \cong k$, while τ acts as $\tau \cdot (1 \otimes 1) = (\tau \cdot 1) \otimes (\tau \cdot 1) = -1 \otimes -1 = 1 \otimes 1$. So the tensor product $\mathbf{1}_{\mathrm{sgn}} \otimes \mathbf{1}_{\mathrm{sgn}} = \mathbf{1}$.

Example 1.9

Take $G = C_3$ and $k = \mathbb{R}$. We have a two-dimensional representation λ_3 corresponding to rotation by an angle of $2\pi/3$. What is $\lambda_3 \otimes \lambda_3$? This is a 4-dimensional representation, and by working out all irreducible representations must be either $\mathbf{4}, \mathbf{2} \oplus \lambda_3$ or $\lambda_3 \oplus \lambda_3$. If you know a little bit of character theory, you can see that it must be $\mathbf{2} \oplus \lambda_3$: we have

$$\chi_{\lambda_3 \otimes \lambda_3}(1) = 4, \quad \chi_{\lambda_3 \otimes \lambda_3}(\tau) = 1$$

$$\begin{aligned} \chi_{\mathbf{4}}(1) &= 4, & \chi_{\mathbf{4}}(\tau) &= 1 \\ \chi_{\mathbf{2} \oplus \lambda_3}(1) &= 4, & \chi_{\mathbf{2} \oplus \lambda_3}(\tau) &= 1 \\ \chi_{\lambda_3 \oplus \lambda_3}(1) &= 4, & \chi_{\lambda_3 \oplus \lambda_3}(\tau) &= -2 \end{aligned}$$

Remark 1.17. By passing to isomorphism classes of representations, the set of finite dimensional representations has the structure of a semiring. Using the Grothendieck construction, we can produce a commutative ring.

Definition 1.18. For a finite group G the real representation ring $RO(G)$ is the Grothendieck group of the above semi-ring. Explicitly,

$$RO(G) := \mathbb{Z}\left\{\begin{matrix}\text{isomorphism classes of finite-}\\ \text{dimensional real } G\text{-representations}\end{matrix}\right\} / \langle [V \oplus W] - [V] - [W] \rangle.$$

Remark 1.19. As an abelian group, $RO(G)$ is a direct sum of copies of $\mathbb{Z}$, with rank equal to the number of isomorphism classes of irreducible representations.

Remark 1.20. We can make the same definition for other fields, for example when $k = \mathbb{C}$ we get the complex representation ring $R(G)$.

Example 1.10

We have $RO(C_2) \cong \mathbb{Z}\{\mathbf{1}\} \oplus \mathbb{Z}\{\mathbf{1}_{\text{sgn}}\}$. The ring structure is determined by Example (1.8): we have $RO(C_2) \cong \mathbb{Z}[\sigma]/(\sigma^2 - 1)$. The same is true for the complex representation ring. Note that this is the same as $\mathbb{Z}[C_2]$. In fact, the complex representation ring of a finite abelian group is always (non-canonically) isomorphic to the group ring: it is the group ring of the character group.

Example 1.11

When $G = C_3$ we have that

$$RO(C_3) = \mathbb{Z}\{\mathbf{1}\} \oplus \mathbb{Z}\{\lambda_3\}.$$

By Example (1.9) we have $[\lambda_3]^2 = 2 + [\lambda_3]$ and we see that

$$RO(C_3) \cong \mathbb{Z}[\lambda]/(\lambda^2 - \lambda - 2).$$

On the other hand, the complex representation ring is given by

$$R(C_3) \cong \mathbb{Z}[\zeta]/(\zeta^3 - 1).$$

By tensoring a real representation with $\mathbb{C}$, there is a map

$$RO(C_3) \to R(C_3)$$

given by $\lambda \mapsto \zeta + \zeta^2$.

Definition 1.21. Let $\phi\colon H \to G$ be a morphism of groups, then the pullback $\phi^*(V)$ of a G-representation is the $k[H]$-module induced by restriction of scalars along $k[H] \to k[G]$. Equivalently, it is the representation given by the composite $H \to G \xrightarrow{a} \mathrm{End}(V)$.

Remark 1.22. We have

$$\phi^*(V \oplus W) = \phi^*(V) \oplus \phi^*(W) \quad \text{and } \phi^*(V \otimes W) \cong \phi^*(V) \otimes \phi^*(W).$$

Therefore we deduce:

Corollary 1.23. *A group homomorphism $\phi\colon H \to G$ induces a ring homomorphism $\phi^*\colon RO(G) \to RO(H)$.*

Example 1.12

Consider the group homomorphism $\phi\colon G \to G/G \cong e$. Then we have $\mathbf{n} = \phi^*(\mathbf{k}^k)$.

Definition 1.24. An *injective* group homomorphism $\iota\colon H \hookrightarrow G$ gives rise to a restriction functor for representations, which we denote by Res^G_H.

Example 1.13

Let $G = C_4 = \langle r \rangle$ and $H = C_2 \subseteq C_4$ the subgroup generated by r^2, and take $k = \mathbb{R}$. Pulling back the sign representation of C_2 along the quotient $C_4 \twoheadrightarrow C_2$ gives rise to the sign representation σ. This is one of three irreducible C_4 real representations: we have the trivial representation $\mathbf{1}$ and the rotation representation λ_4. The inclusion $H \hookrightarrow G$ gives a morphism

$$RO(C_4) = \mathbb{Z}\{\mathbf{1}\} \oplus \mathbb{Z}\{\sigma\} \oplus \mathbb{Z}\{\lambda_4\} \to \mathbb{Z}\{\mathbf{1}\} \oplus \mathbb{Z}\{\mathbf{1}_{\mathrm{sgn}}\} = RO(C_2).$$

This map is determined by

$$\mathbf{1} \mapsto \mathbf{1}, \quad \sigma \mapsto \mathbf{1}, \quad \lambda_4 \mapsto 2 \cdot \mathbf{1}_{\mathrm{sgn}},$$

where the image of σ is determined from its definition as the pull-back, and the image of λ_4 comes from the fact that r acts as multiplication by $\pi/4$ and so r^2 acts as multiplication by -1, and so restricts to a 2-dimensional sign representation $\mathbf{1}_{\mathrm{sgn}} \oplus \mathbf{1}_{\mathrm{sgn}}$. We can conclude that $\lambda_4^2 \mapsto (2 \cdot \mathbf{1}_{\mathrm{sgn}})^2 = \mathbf{4}$, so that λ_4^2 must be either $\mathbf{4}, \mathbf{3} \oplus \sigma, \mathbf{2} \oplus 2\sigma, \mathbf{1} \oplus 3\sigma$ or 4σ.

There is also another functor, which will turn out to be adjoint to restriction.

Definition 1.25. Given $H \leq G$, and a H-representation V, we define the induced representation Ind^G_H is be the tensor product $k[G] \otimes_{k[H]} V$.

Remark 1.26. Induction plays well with direct sums: we have $\mathrm{Ind}^G_H(V \oplus W) \cong \mathrm{Ind}^G_H(V) \oplus \mathrm{Ind}^G_H(W)$. But a dimension check shows that it does not commute with tensor products. Therefore, we get an induced map of abelian groups, but *not* of commutative rings

$$\mathrm{Ind}^G_H\colon RO(H) \to RO(G)$$

Directly from the definition we deduce the following:

Lemma 1.27. *If $K \leq H \leq G$ then $\operatorname{Ind}_H^G \operatorname{Ind}_K^H V$ for any K-representation V.*

Example 1.14

The regular representation $\rho_G = k[G] \cong k[G] \otimes_{k[G]} k \simeq \operatorname{Ind}_e^G(\mathbf{1})$. More generally, we have $\operatorname{Ind}_H^G \rho_H \cong \rho_G$.

Example 1.15

Let $C_2 \subseteq C_4$ and $k = \mathbb{R}$. What is $\operatorname{Ind}_{C_2}^{C_4}(\mathbf{1})$? We have

$$\operatorname{Ind}_{C_2}^{C_4}(\mathbf{1}) = \mathbb{R}[C_4] \otimes_{\mathbb{R}[C_2]} \mathbb{R} \cong \mathbb{R}[C_4/C_2] \cong \phi^*(\rho_{C_2})$$

for $\phi\colon C_4 \to C_4/C_2 \cong C_2$ the quotient map.[a] This means that

$$\operatorname{Ind}_{C_2}^{C_4}(\mathbf{1}) = \mathbf{1} \oplus \sigma$$

To work out $\operatorname{Ind}_{C_2}^{C_4}(\mathbf{1}_{\mathrm{sgn}})$ we have that

$$\begin{aligned} \rho_{C_4} \cong \operatorname{Ind}_{C_2}^{C_4}(\rho_{C_2})) = \operatorname{Ind}_{C_2}^{C_4}(\mathbf{1} \oplus \mathbf{1}_{\mathrm{sgn}}) &\cong \operatorname{Ind}_{C_2}^{C_4}(\mathbf{1}) \oplus \operatorname{Ind}_{C_2}^{C_4}(\mathbf{1}_{\mathrm{sgn}}) \\ &\cong 1 \oplus \sigma \oplus \operatorname{Ind}_{C_2}^{C_4}(\mathbf{1}_{\mathrm{sgn}}). \end{aligned}$$

By $\rho_{C_4} \cong \mathbf{1} \oplus \sigma \oplus \lambda_4,$. To see this, once can note that

$$\mathbb{R}[\mathbb{Z}/4] \cong \mathbb{R}[X]/(X^4 - 1) \cong \mathbb{R}[X]/\prod_{d|4} \Phi_d \cong \bigoplus_{d|4} \mathbb{R}[X]/\Phi_d$$

so that

$$\mathbb{R}[\mathbb{Z}/4] \cong \mathbb{R}[X]/(X-1) \oplus \mathbb{R}[X]/(X+1) \oplus \mathbb{R}[X]/(X^2-1).$$

Hence, $\operatorname{Ind}_{C_2}^{C_4}(\mathbf{1}_{\mathrm{sgn}}) \cong \lambda_4$.

We deduce that the map

$$\operatorname{Ind}_{C_2}^{C_4}\colon RO(C_2) \to RO(C_4)$$

is determined by

$$\mathbf{1} \mapsto \mathbf{1} \oplus \sigma \quad \text{and} \quad \mathbf{1}_{\mathrm{sgn}} \mapsto \lambda_4.$$

[a] This always works: For $H \leq G$ a normal subgroup we have $\operatorname{Ind}_H^G(\mathbf{1}) = \phi^*(\rho_{G/H})$.

In general, if we have commutative rings R and S and a ring map $f\colon R \to S$, then we can define induction and restriction between R-modules and S-modules and induction is left adjoint to restriction. As a special case, we have:

Lemma 1.28. *If $H \leq G$, then induction is left adjoint to restriction.*

Proposition 1.29 (The projection formula)**.** *Let $H \leq G$, then there is a natural equivalence*

$$\mathrm{Ind}_H^G(\mathrm{Res}_H^G(V) \otimes W) \xrightarrow{\sim} V \otimes \mathrm{Ind}_H^G(W)$$

for $V \in RO(G)$ and $W \in RO(G)$

Proof. We first construct the map: by adjunction, such a map is equivalent to a H-equivariant map

$$\mathrm{Res}_H^G(V) \otimes W \to \mathrm{Res}_H^G(V \otimes \mathrm{Ind}_H^G(W)) \cong \mathrm{Res}_H^G(V) \otimes \mathrm{Res}_H^G \mathrm{Ind}_H^G(W).$$

This map is given as $\mathrm{id} \otimes \eta$ where $\eta \colon W \to \mathrm{Res}_H^G \mathrm{Ind}_H^G(W)$ is the unit of the induction/restriction adjunction. To check this is an equivalence, it suffices to check on underlying vector spaces, which then just boils down to the isomorphism

$$\bigoplus_{G/H}(V \otimes W) \cong V \otimes (\bigoplus_{G/H} W).$$

$\square$

Example 1.16

Let us finish our calculation of λ_4^2 in $RO(C_4)$. We have just seen that $\mathrm{Ind}_{C_2}^{C_4}(\mathbf{1}_{\mathrm{sgn}}) \cong \lambda_4$, and hence

$$\begin{aligned}\lambda_4 \otimes \lambda_4 \cong \lambda_4 \otimes (\mathrm{Ind}_{C_2}^{C_4}(\mathbf{1}_{\mathrm{sgn}})) &\cong \mathrm{Ind}_{C_2}^{C_4}(\mathrm{Res}_{C_2}^{C_4}(\lambda_4) \otimes \mathbf{1}_{\mathrm{sgn}}) \\ &\cong \mathrm{Ind}_{C_2}^{C_4}(2 \cdot \mathbf{1}_{\mathrm{sgn}} \otimes \mathbf{1}_{\mathrm{sgn}}) \\ &\cong \mathrm{Ind}_{C_2}^{C_4}(\mathbf{2}) \\ &\cong \mathbf{2} \oplus 2\sigma\end{aligned}$$

For a a ring map $R \to S$, restriction also has a right adjoint, given by coinduction, denoted Coind_R^S and defined by $M \mapsto \mathrm{Hom}_S(R, M)$. A special fact about representation theory is that these adjoint are equal.

Proposition 1.30. *There is a natural equivalence of functors* $\mathrm{Ind}_H^G \simeq \mathrm{Coind}_H^G$.

Proof. If M is an R-module, we use the notation $M^* \cong \mathrm{Hom}_R(M, R)$ for the linear dual. In the case $R = k[G]$, then the natural k-linear isomorphisms

$$\mathrm{Hom}_k(M, N) \cong M^* \otimes_k N \quad \text{ and } M^{**} \cong M$$

are actually $k[G]$-module isomorphisms. To prove the proposition, we note that

$$k[G] \cong k[G]^* = \mathrm{Hom}_k(k[G], k)$$

so that

$$\mathrm{Ind}_H^G(M) = k[G]\otimes_{k[H]} M \cong k[G]^* \otimes_{k[H]} M \cong \mathrm{Hom}_{k[H]}(k[G], M) = \mathrm{Coind}_H^G(M)$$

naturally in M. □

Remark 1.31. More generally, if $f\colon R \to S$ is a morphism of rings, then induction and coinduction agree if and only if S is finitely-generated and projective over R, and there is an isomorphism of (S, R)-bimodules

$$S \to \mathrm{Hom}_R(S, R).$$

The double-coset formula

Let us return to a moment to $RO(C_4)$. We have constructed maps

$$RO(C_2) \to RO(C_4) \to RO(C_2)$$

which send

$$\mathbf{1} \mapsto \mathbf{1} \oplus \sigma \mapsto \mathbf{2}$$

and

$$\mathbf{1}_{\mathrm{sgn}} \mapsto \lambda_4 \mapsto 2 \cdot \mathbf{1}_{\mathrm{sgn}}$$

so that the composite map $RO(C_2) \to RO(C_2)$ is multiplication by 2. This is actually a complete general phenomena.

Definition 1.32. Let $H, K \le G$ be subgroups, then a double coset HgK is the set

$$HgK = \left\{ x \in G \,\middle|\, x = hgk \text{ for some } h \in H, k \in K \right\}.$$

Theorem 1.33 (Double coset formula)**.** *For subgroups $H, K, \le G$ and a H-representation K we have a decomposition of K-representations*

$$\mathrm{Res}_K^G \mathrm{Ind}_H^G(V) = \sum_{HgK \in H\backslash G/K} \mathrm{Ind}_{H^{g-1}\cap K}^K c_g^* \mathrm{Res}_{H\cap K^g}^H(V)$$

where $c_g\colon H \cap K^g \xrightarrow{\sim} H^{g^{-1}} \cap K$ is the conjugation by g homomorphism.

Corollary 1.34. *Suppose that G is abelian, and $H = K$, then the composite*

$$RO(H) \xrightarrow{\mathrm{Ind}_H^G} RO(G) \xrightarrow{\mathrm{Res}_H^G} RO(H)$$

is given by multiplication by the index $|G/H|$ of H inside of G.

Indeed, in this case $H\backslash G/H = G/H$.

Remark 1.35. The restriction that G is abelian is really necessary. See [2, Example 1.1.51].

Chapter 2

Mackey functors

The definition of a Mackey functor

We can axiomatize the structure we have on seen on the representation ring in an algebraic object called a *Mackey functor*. As we will see later, these play the same role in equivariant stable homotopy that abelian groups play in ordinary stable homotopy.

Definition 2.1. A Mackey functor $\underline{M}$ for a finite group G consists of the following data:

(a) An abelian group $\underline{M}(H)$ for each $H \leq G$.

(b) A restriction map $R^G_H \colon \underline{M}(H) \to \underline{M}(K)$ for each $K \leq H$.

(c) A transfer map $I^H_K \colon \underline{M}(K) \to \underline{M}(H)$ for each $K \leq H$.

(d) A conjugation homomorphism $c_g \colon \underline{M}(H) \to \underline{M}(H^g)$ for each $g \in G$.

subject to the following rules:

(i) R^H_H and I^H_H are the identity for each $H \leq K$. Moreover, for each $h \in H$, c_h is the identity on $\underline{M}(H)$.

(ii) If $L \leq K \leq H$, then $R^K_L \circ R^H_K \simeq R^H_L$ and $I^K_H \circ I^K_L \simeq I^H_L$.

(iii) $c_g \circ c_h \simeq c_{gh}$ for all $g, h \in G$.

(iv) $R^{H^g}_{K^g} c_g = c_g R^H_K$ and $I^{H_g}_{K^g} c_g = c_g I^H_K$.

(v) The double coset formula holds:

$$R^H_L \circ I^H_K = \sum_{KhL \in K \backslash H/L} I^L_{K^h \cap L} R^{K^h}_{K^h \cap L} c_h$$

for all $L, K \leq H \leq G$.

Example 2.1

In the previous section we have shown that there is a Mackey functor $\underline{RO}(G)$ with $\underline{RO}(G)(H) \cong RO(H)$.

Remark 2.2. There are numerous different ways to record the data of a Mackey functor, and depending on precisely what you want to do, some may be better than others. For example, although it's maybe not so hard to define a category of Mackey functors from this perspective, its formal properties might be a bit hard to see (for example, is it an abelian category? Is it symmetric monoidal?), while from other perspectives this becomes much clearer.

Remark 2.3. Let $\underline{M}$ be a Mackey functor. Note that if g normalizes H so that $H^g = H$, then c_g maps $\underline{M}$ to itself, i.e. we have an action of $N_G(H)$ on $\underline{M}(H)$. Moreover, the normal subgroup $H \leq N_G(H)$ acts trivially, so we get an action of $W_G(H) = N_G(H)/H$ on $\underline{M}(H)$. For example, in the case $H = e$, we get an action of $N_G(e)/e = G$ on $\underline{M}(e)$. It is customary to write Mackey functors via a *Lewis diagram*: for example, when $G = C_p$ this is a diagram of the form:

$$\begin{tikzcd} \underline{M}(C_p) \arrow[d, bend right, "R"'] \\ \underline{M}(e) \arrow[u, bend right, "I"'] \arrow[loop below, "C_p"] \end{tikzcd}$$

Example 2.2

Given an $\mathbb{Z}[G]$-module M we can produce a Mackey functor $\underline{M}$ defined by

$$\underline{M}(H) = M^H.$$

Here restriction is defined by inclusion of fixed points for a larger subgroup, which the transfer map is defined by

$$I_K^H \colon M^K \to M_H, \quad I_K^H(m) = \sum_{hK \in H/K} h \cdot M$$

This does not depend on coset representatives, since m is assumed to be fixed by K. Finally, conjugation c_g is multiplication by g.

This gives a functor $FP \colon \mathrm{Mod}_{\mathbb{Z}[G]} \to \mathrm{Mack}_G$

Lemma 2.4. *The functor FP is right adjoint to the evaluation functor that takes a Mackey functor $\underline{M}$ to the $\mathbb{Z}[G]$-module $\underline{M}(e)$.*

Proof. Given a morphism $\alpha \colon \underline{M}(e) \to V$ of $\mathbb{Z}[G]$-modules we show that there is a unique morphism $\underline{\alpha} \colon \underline{M} \to FP(V)$ of Mackey functors (the converse association is easy, and is just given by evaluation at e). This is defined by setting

$$\underline{\alpha}(H) \colon \underline{M}(H) \to V^H$$

to be the composite

$$\underline{M}(H) \xrightarrow{R^H_e} \underline{M}(e) \xrightarrow{\alpha} V^H.$$

Note that this does indeed land in V^H as h acts trivially on $\underline{M}(H)$ and commutes with α, so that $c_h \alpha R^H_e = \alpha R^H_e$. It follows that we get a commutative diagram

$$\begin{tikzcd}
\underline{M}(H) \arrow[r, "\underline{\alpha}(H)"] \arrow[d, "R^H_e"'] & V^H \arrow[d, hook] \\
\underline{M}(e) \arrow[r, "\alpha"'] & V
\end{tikzcd}$$

We must show that

$$\underline{\alpha}(H) I^H_K = I^H_K \underline{\alpha}(K)$$

and

$$\underline{\alpha}(H) R^H_K = R^H_K \underline{\alpha}(K).$$

We show the first, and leave the second as an exercise. We have[1]

$$\underline{\alpha}(H) I^H_K = \alpha R^H_e I^H_K = \alpha \sum_{hK \in H/K} c_h R^K_e = \sum_{hK \in H/K} h \cdot \alpha R^K_e = I^H_K \alpha R^K_e = I^H_K \underline{\alpha}(K).$$

The second equation is similar, but simpler, and we leave it for the reader. □

Remark 2.5. Evaluation also has a left adjoint, the 'fixed quotient' Mackey functor, defined by $\underline{M}(H) = M_H$, the largest quotient of M on which H acts trivially.

Remark 2.6. A special case of the fixed-point Mackey functor comes from taking an abelian group M considered with trivial G-action. This gives the *constant Mackey functor* $\underline{M}$ with $\underline{M}(H) = H$, restriction and conjugation the identity, and transfer from K to H given by multiplication by the index of K in H.

Example 2.3

There is a C_2-Mackey functor described by the Lewis diagram

$$\begin{tikzcd}
\mathbb{Z} \arrow[d, bend right, "\Delta"'] \\
\mathbb{Z} \oplus \mathbb{Z} \arrow[u, bend right, "\nabla"'] \arrow[loop below, "\text{swap}"]
\end{tikzcd}$$

This is an example of a fixed-point Mackey functor applied to the free module $\mathbb{Z}[C_2]$.

[1] Note that $H \backslash G / K = H \backslash G$ if $K = e$.

The Burnside ring

Another example of a Mackey functor comes from the Burnside ring. This is very important in both the theory of Mackey functors and in equivariant homotopy theory, as it plays the role of the unit $\mathbb{Z}$.

Definition 2.7. The Burnside ring of a finite group G is the Grothendieck group of the category of finite G-sets under coproduct. More explicitly,

$$A(G) \coloneqq \mathbb{Z}\left\{\text{isomorphism classes of finite } G\text{-set}\right\}/\langle [X \amalg Y] - [X] - [Y] \rangle.$$

Remark 2.8. Because every finite G-set decomposes into a coproduct of orbits and the isomorphism type of an orbit G/H only depends on the conjugacy class of H, we have an additive decomposition

$$A(G) \cong \bigoplus_{\mathrm{Conj}(G)} \mathbb{Z}.$$

Suppose $H \leq G$ then there are maps $A(G) \to A(H)$ induced by restriction of G-sets, and $A(H) \to A(K)$, induced by taking an H-set X to the G-set $G \times_H X$. There is also a conjugation map $c_g \colon A(H) \to A(H^g)$ defined by taking a H-set X defined by $H \to \mathrm{Aut}(X)$ to the H^g-set defined by $H^g \xrightarrow{c_g^{-1}} H \to \mathrm{Aut}(X)$.

Definition 2.9. The Burnside Mackey functor $\mathbb{A}_G$ is the Mackey functor with $\mathbb{A}_G(H) = A(H)$, and structure maps as in the previous remark.

Remark 2.10. Sending a G-set X to the associated permutation representation, we get a ring morphism $A(G) \to R(G)$, which extends to a morphism of Mackey functors.

Example 2.4

Let $G = C_2$, then there are two orbits, namely C_2/C_2 and C_2/e, so we have

$$A(C_2) \cong \mathbb{Z}\{C_2/e, C_2/C_2\}$$

while clearly

$$A(e) \cong \mathbb{Z}\{e\}.$$

The Mackey functor looks as follows

$$\begin{tikzcd}
\mathbb{Z}\{C_2/e, C_2/C_2\} \arrow[d, bend right] \\
\mathbb{Z}e \arrow[u, bend right] \arrow[loop below]
\end{tikzcd}$$

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