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MA8403 - Equivariant homotopy theory

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Preface

This are the courses notes for MA8403 - Equivariant homotopy theory, held during the Autumn semester 2023 at NTNU. The notes are mainly based on two excellent sets of lectures notes, one by Guillou [2], and one by Blumberg [1]. The notes will be continually updated during the semester.

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Chapter 1

Equivariance in algebra

Group actions in algebra

We recall that if X is an object in a category $\mathcal{C}$, then the set of endomorphisms $\mathrm{End}(X)$ forms a monoid (that is, a set equipped with an associative binary operation and an identity element) under composition. The set of automorphisms of X (that is, those endomorphisms that are invertible) form a group. Moreover, we have

$$\mathrm{Aut}(X) = \mathrm{End}(X) \cap \mathrm{Iso}(\mathcal{C})$$

Note that any group is a monoid, simply by forgetting the existence of inverses.

Definition 1.1. An action of a group G on an object $X \in \mathcal{C}$ is a monoid homomorphism $a\colon G \to \mathrm{End}(X)$, or equivalently a group homomorphism $a\colon G \to \mathrm{Aut}(X)$ (note that a monoid homomorphism between groups is a group homomorphism).

Remark 1.2. Unwinding the definition, this means that we have:

a. For each $g \in G$, there is a morphism $a(g)\colon G \to G$.

b. a preserves composition, i.e., $a(g \cdot h) = a(h) \cdot a(h)$.

c. a preserves identifies, so $a(e) = \mathrm{id}_X$.

Example 1.1

Let $\mathcal{C}$ be the category of sets and functions. Then, $a\colon G \to \mathrm{End}(X) = \left\{ f\colon X \to X \right\}$ correspond to a function $\overline{a}\colon G \times X \to X$. The conditions above mean that the diagrams

$$\begin{tikzcd} G \times G \times X \arrow[r, "m \times \mathrm{id}"] \arrow[d, "\mathrm{id} \times \overline{a}"'] & G \times X \arrow[d, "\overline{a}"] \\ G \times X \arrow[r, "\overline{a}"'] & X \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \{*\} \times X \arrow[r, "e \times \mathrm{id}"] \arrow[d, "\cong"'] & G \times X \arrow[d, "\overline{a}"] \\ X \arrow[r, equal] & X \end{tikzcd}$$

commute, where $m\colon G \times G \to G$ denotes the group multiplication. Equivalently, in symbols, we have

$$\overline{a}(h, \overline{a}(h, x)) = \overline{a}(gh, x)$$

and

$$\overline{a}(e, x) = x.$$

Remark 1.3. Let BG denote the category with one object $*$ and with $\mathrm{Hom}(*,*) = G$. Then an action of G in the category $\mathcal{C}$ is the same as a functor $\rho\colon BG \to \mathcal{C}$. The object X in the previous definition is the object $\rho(*) \in \mathcal{C}$.

Remark 1.4. Let $\mathcal{C} = \mathrm{Mod}_R$ for a commutative ring R. An action of G on $M \in \mathrm{Mod}_R$ is a monoid homomorphism

$$a\colon G \to \mathrm{Hom}_R(M, M).$$

We recall that $\mathrm{Hom}_R(M, M)$ actually has the structure of an R-algebra

Definition 1.5. The (R-linear) group ring on R is the R-algebra $R[G]$ whose:

(a) underlying R-module is the free R-module with basis on the underlying set of G.

(b) whose multiplication is given on basis elements by the group operation.

Example 1.2

Let $R = \mathbb{Z}$ and $G = C_2 = \langle \sigma \rangle$. An element of $\mathbb{Z}[C_2]$ is of the form $a + b\sigma$ where $a, b \in \mathbb{Z}$. Multiplication is given by

$$(a_1 \cdot 1 + b_1\sigma) \cdot (a_2 \cdot 1 + b_2\sigma) = (a_1a_2 + b_1b_2) \cdot 1 + (a_1b_2 + b_1a_2)\sigma.$$

This is the same thing as the polynomial ring $\mathbb{Z}[\sigma]/(\sigma^2 - 1)$.

Remark 1.6. Categorically, the group ring construction is left adjoint to the functor that takes an R-algebra to its group of units, i.e., there is an adjudication

$$R[-]\colon \mathrm{Grp} \leftrightarrows \mathrm{Mod}_R \colon (-)^\times$$

Returning to group actions, we have the following:

Proposition 1.7. *Let R be a commutative ring, and G a finite group. The following data on an R-module M are equivalent:*

a. *A monoid homomorphism $G \to \mathrm{End}_R(M)$.*

b. *A group homomorphism $G \to \mathrm{Aut}_R(M)$.*

c. *A homomorphism of R-algebras $R[G] \to \mathrm{Hom}_R(M, M)$.*

d. *An $R[G]$-module structure on M whose underlying R-module structure is M.*

Definition 1.8. A representation of G over R is an $R[G]$-module.

Example 1.3

If $R = k$ is a field, then the underlying R-module is a k-vector space V. If $\dim_k(V) = n$, then $\mathrm{Aut}_k(V) = GL_n(k)$, and a k-representation is the same thing as a group homomorphism $G \to GL_n(k)$.

Definition 1.9. The $R[G]$-module $R[G]$ is known as the regular representation. More generally, if X is a finite G-set, then the free R-module $R[X]$ inherits the structure of a $R[G]$-module (the case of $R[G]$ itself corresponds to the finite G-set G, considered as an $R[G]$-module over itself). Representations obtained this way are known as permutation representations.

Example 1.4

Taking X to be the trivial G-set, we obtain the (one-dimensional) trivial representation. This is simply the $R[G]$-module R, where G acts trivially.

Definition 1.10. Let $G = C_2 = \langle \tau \rangle$, and suppose that $-1 \neq 1 \in R$. Then the sign representation of G is the one-dimensional representation where τ acts as -1 (if $-1 = 1$ in R this still makes sense, but is just the trivial representation). Note that this is an example of a representation that is not a permutation representation.

Example 1.5

Let $C_n = \langle \sigma \rangle$ be the cyclic group of order n. Let us calculate all complex 1-dimensional representations of C_n, i.e., homomorphisms $\rho\colon C_n \to \mathbb{C}$. Note that if we define $\rho(\sigma) = c$, then $\rho(\sigma^n) = c^n = 1$, so that c must be an n-th root of unity. There are precisely n-of these (take $\zeta_n = e^{2\pi/ni}$), and so there are precisely n-representations. For example, when $G = C_4$, the four representations correspond to sending σ to either $1, i, -1$ or $-i$. Note that we can also consider these as 2-dimensional *real* representations.

Notation 1.11. We let $\rho = \rho_G$ denote the regular representation of G, and the trivial n-dimensional representation by $\mathbf{n} = R^{\oplus n}$.

Definition 1.12. A subrepresentation is a submodule.

Example 1.6

The regular representation always has a one-dimensional trivial representation, generated by the sum $\sum_{g \in G} g$.

Definition 1.13. A representation V is irreducible if the only subrepresentations of V are 0 and V.

Theorem 1.14 (Maschke)**.** *Suppose that k is a field of characteristic not dividing $|G|$. Then every representation splits as a direct sum of irreducible representations.*

Proof. We prove the following: if $V \subseteq W$ is a subrepresentation, then there exists $U \subseteq W$ such that $U \oplus V \simeq W$.

To see this, let $\pi\colon W \to W$ be any k-linear projection of W onto V. This map need not be G-equivariant, but we can make it so by 'averaging'. That is, we define a new map $\phi\colon W \to W$ by

$$\phi(\mathbf{w}) = \frac{1}{|G|} \sum_{g \in G} g \cdot \pi(g^{-1} \cdot \mathbf{w}).$$

Moreover the map is G-equivariant: for $h \in G$ we have

$$\begin{aligned} \phi(h \cdot \mathbf{w}) &= \frac{1}{|G|} \sum_{g \in G} g \cdot \pi(g^{-1} \cdot h \cdot \mathbf{w}) \\ &= \frac{1}{|G|} \sum_{u \in G} u \cdot h \cdot \pi(u^{-1} \cdot \mathbf{w}) \\ &= h \cdot \phi(\mathbf{w}), \end{aligned}$$

where $u = gh^{-1}$, and so ϕ is $k[G]$-linear. Furthermore, the map ϕ is the identity on V By the splitting lemma, $W = V \oplus \ker(\phi)$. $\square$

Remark 1.15. We used the assumption on k to ensure that we could divide by $|G|$. Without that assumption, the theorem is false. Indeed, let $G = C_2$, $R = \mathbb{F}_2$ and consider the representation defined by $\rho(\tau) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. This is not irreducible, but does not split as a direct sum of indecomposable representations.

Corollary 1.16. *Suppose that k is a field of characteristic not dividing $|G|$. If V is an irreducible representation, then V is isomorphism to a subgroup of $k[G]$ (slogan: all irreducibles are submodules of the regular representation).*

Proof. Let $\mathbf{v} \in V$ be non-trivial. Then the homomorphism $\phi\colon k[G] \to V$ given by sending 1 to $\mathbf{v}$ must be surjective, because V is irreducible. Let $U = \ker(\phi)$, then apply Maschke's theorem. $\square$

Example 1.7

Let $G = C_2 = \langle \tau \rangle$, and k a field of characteristic not equal to 2. We have the trivial representation $\mathbf{1}$ and the sign representation $\mathbf{1}_{\mathrm{sgn}}$. Then, $\mathbf{1}$ is generated by the sum $1 + \tau$, while $\mathbf{1}_{\mathrm{sgn}}$ is generated by $1 - \tau$, and we deduce that

$$\rho_{C_2} = \mathbf{1} \oplus \mathbf{1}_{\mathrm{sgn}}.$$

The representation ring

Let k be a field, and suppose that V and W are G-representations, then the k-linear tensor sum $V \oplus W$ can be given the structure of a $k[G]$-module, by taking the diagonal G-action. If we think of a representation in terms of a homomorphism $\rho\colon G \to GL_n(k)$, then this direct sum corresponds to the 'block sum'

$$G \to GL_n(k) \times GL_m(k) \to GL_{n+m}(k)$$

Similarly, we can define a tensor product of representations using the 'Kronecker tensor product' (or matrix direct product). Equivalently, this is the k-linear tensor product $V \otimes W$ with the g-action defined on simple tensors by

$$g \cdot (\mathbf{v} \otimes \mathbf{w}) = g \cdot \mathbf{v} \otimes g\mathbf{w}.$$

We leave it for the reader to verify the following straightforward computations:

(a) $\mathbf{1} \otimes V \cong V \cong V \otimes \mathbf{1}$.

(b) $\mathbf{n} \otimes V \cong V^{\oplus n} \cong V \otimes \mathbf{n}$.

Example 1.8

Let us compute the tensor product $\mathbf{1}_{\text{sgn}} \otimes \mathbf{1}_{\text{sgn}}$. The underlying vector space is simply $k \otimes k \cong k$, while τ acts as $\tau \cdot (1 \otimes 1) = (\tau \cdot 1) \otimes (\tau \cdot 1) = -1 \otimes -1 = 1 \otimes 1$. So the tensor product $\mathbf{1}_{\text{sgn}} \otimes \mathbf{1}_{\text{sgn}} = \mathbf{1}$.

Example 1.9

Take $G = C_3$ and $k = \mathbb{R}$. We have a two-dimensional representation λ_3 corresponding to rotation by an angle of $2\pi/3$. What is $\lambda_3 \otimes \lambda_3$? This is a 4-dimensional representation, and by working out all irreducible representations must be either $\mathbf{4}, \mathbf{2} \oplus \lambda_3$ or $\lambda_3 \oplus \lambda_3$. If you know a little bit of character theory, you can see that it must be $\mathbf{2} \oplus \lambda_3$: we have

$$\chi_{\lambda_3 \otimes \lambda_3}(1) = 4, \quad \chi_{\lambda_3 \otimes \lambda_3}(\tau) = 1$$

$$\begin{aligned} \chi_{\mathbf{4}}(1) &= 4, & \chi_{\mathbf{4}}(\tau) &= 1 \\ \chi_{\mathbf{2} \oplus \lambda_3}(1) &= 4, & \chi_{\mathbf{2} \oplus \lambda_3}(\tau) &= 1 \\ \chi_{\lambda_3 \oplus \lambda_3}(1) &= 4, & \chi_{\lambda_3 \oplus \lambda_3}(\tau) &= -2 \end{aligned}$$

Remark 1.17. By passing to isomorphism classes of representations, the set of finite dimensional representations has the structure of a semiring. Using the Grothendieck construction, we can produce a commutative ring.

Definition 1.18. For a finite group G the real representation ring $RO(G)$ is the Grothendieck group of the above semi-ring. Explicitly,

$$RO(G) := \mathbb{Z}\left\{\begin{array}{l}\text{isomorphism classes of finite-}\\ \text{dimensional real } G\text{-representations}\end{array}\right\} / \langle [V \oplus W] - [V] - [W] \rangle.$$

Remark 1.19. As an abelian group, $RO(G)$ is a direct sum of copies of $\mathbb{Z}$, with rank equal to the number of isomorphism classes of irreducible representations.

Remark 1.20. We can make the same definition for other fields, for example when $k = \mathbb{C}$ we get the complex representation ring $R(G)$.

Example 1.10

We have $RO(C_2) \cong \mathbb{Z}\{\mathbf{1}\} \oplus \mathbb{Z}\{\mathbf{1}_{\mathrm{sgn}}\}$. The ring structure is determined by Example (1.8): we have $RO(C_2) \cong \mathbb{Z}[\sigma]/(\sigma^2 - 1)$. The same is true for the complex representation ring. Note that this is the same as $\mathbb{Z}[C_2]$. In fact, the complex representation ring of a finite abelian group is always (non-canonically) isomorphic to the group ring: it is the group ring of the character group.

Example 1.11

When $G = C_3$ we have that

$$RO(C_3) = \mathbb{Z}\{\mathbf{1}\} \oplus \mathbb{Z}\{\lambda_3\}.$$

By Example (1.9) we have $[\lambda_3]^2 = 2 + [\lambda_3]$ and we see that

$$RO(C_3) \cong \mathbb{Z}[\lambda]/(\lambda^2 - \lambda - 2).$$

On the other hand, the complex representation ring is given by

$$R(C_3) \cong \mathbb{Z}[\zeta]/(\zeta^3 - 1).$$

By tensoring a real representation with $\mathbb{C}$, there is a map

$$RO(C_3) \to R(C_3)$$

given by $\lambda \mapsto \zeta + \zeta^2$.

Bibliography

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