Stabilizing a Double Inverted Pendulum System

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Abstract

In this paper, we outline our solution to the double inverted pendulum problem, wherein a double pendulum is inverted and attached to a cart. The objective is to balance the double pendulum above the cart by only exerting force on the cart. Our approach utilized the Linear Quadratic Regulator feedback controller to exert optimal control on the cart and stabilize the system. We were successful in creating a robust control system that balances the inverted double pendulum indefinitely regardless of initial conditions on cart position or velocity, pendulum joint position or velocity, or pendulum tip position or velocity.

1 Background

The problem investigated in this report is the upright stabilization of an inverted double pendulum. The objective of the problem is to hold the double pendulum above a cart by moving the cart with the only control available being force exerted on the cart to move it left or right. Given cart position and velocity, joint position and angular velocity, and tip position and angular velocity, the system should choose the force applied to the cart in order to optimize the total time that the double pendulum is stabilized above the cart.

The inverted double pendulum is a well-known problem in optimal control and reinforcement learning. Much research has been done on this problem due to the simplicity of the system ideas and its easy accessibility. Thus, many different approaches have been taken towards the problem including linear quadratic regulators, the state-dependent Ricatti equation, neural network optimal control, a stabilization fuzzy controller, harmonic balance identification, and bifurcation theory. While each of these methods has shown some level of success, we will attempt to derive a solution to the problem from scratch using what we've learned in this course.

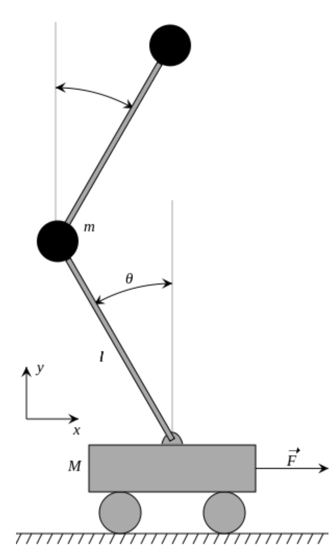
2 Mathematical Representation

Our goal is to keep the pendulum up as long as possible. Thus we seek a solution that will maximize the potential energy of the system. Figure 1 illustrates our system, although we will change some variable names and make additions to clarify our problem.

We define our variables as follows:

- C_p - the position of the cart on the x-axis
- l_1 - the length of the first pendulum
- l_2 - the length of the second pendulum
- J_θ - the off vertical angle of the first pendulum, which we call the joint angle
- T_θ - the off vertical angle of the second pendulum, which we call the tip angle
- g - the force due to gravity

Figure 1: Double Inverted Pendulum System[1]



- m_1 - the mass of the joint
- m_2 - the mass of the tip
- M - the mass of the cart
- t - time
- F - the force applied to the cart.

We make one simplifying assumption of the system. Namely, $l_1 = l_2 = 1$.

The functional that we will be observing is based on kinetic and potential energy. We wish to maximize $U - T$ or minimize $T - U$ where T is kinetic energy an U is potential energy. Defining L to be our Lagrangian, we get the functional

$$J(y) = \int_{t_0}^{t_f} L dt \tag{1}$$

where

$$L = T - U \tag{2}$$

Now we must define T and U based on the variables we know and the control we have as well as external forces.

First, we observe that the position of the cart, joint, and tip are given by $(C_p, 0)$, $(C_p - \sin J_\theta, \cos J_\theta)$, and $(C_p - \sin J_\theta - \sin T_\theta, \cos J_\theta + \cos T_\theta)$. Next, we break up the kinetic and potential energy into three different parts, being the first pendulum, the second pendulum, and the cart. Then the total potential and kinetic energies will be the sum of these three parts. The total potential and kinetic energies are given by

$$U = m_1 g \cos J_\theta + m_2 g(\cos J_\theta + \cos T_\theta) \tag{3}$$

and

$$\begin{aligned} T &= \frac{M}{2} v_c^2 + \frac{m_1}{2} v_1^2 + + \frac{m_2}{2} v_2^2 \\ &= \frac{M}{2} \dot{C_p}^2 + \frac{m_1}{2}((\dot{C}_p - \dot{J}_\theta \cos J_\theta)^2 - \dot{J}_\theta \sin^2 J_\theta) \\ &+ \frac{m_2}{2}((\dot{C}_p - \dot{J}_\theta \cos J_\theta - \dot{T}_\theta \cos T_\theta)^2 + (-\dot{J}_\theta \sin^2 J_\theta - \dot{T}_\theta \sin^2 T_\theta)^2). \end{aligned} \tag{4}$$

3 Solution

We now use the model developed above to solve the system with a Linear Quadratic Regulator. To start, we apply the Euler-Lagrange Equations to get the following equations:

$$\begin{aligned} F &= \frac{\partial L}{\partial C_p} - \frac{d}{dt} \frac{\partial L}{\partial \dot{C}_p} \\ 0 &= \frac{\partial L}{\partial J_\theta} - \frac{d}{dt} \frac{\partial L}{\partial \dot{J}_\theta} \\ 0 &= \frac{\partial L}{\partial T_\theta} - \frac{d}{dt} \frac{\partial L}{\partial \dot{T}_\theta}. \end{aligned} \tag{5}$$

Since we want to stabilize the double pendulum vertically, we will use small angle approximation to linearize our system around $(J_\theta, \dot{J}_\theta) = (0, 0)$ and $(T_\theta, \dot{T}_\theta) = (0, 0)$. This simplifies our very messy Euler-Lagrange equations considerably to get:

$$
\begin{aligned}
F &= (M_c + m_1 + m_1)\ddot{C}_p - (m_1 + m_2)\ddot{J}_\theta - m_2\ddot{T}_\theta \\
0 &= (m_1 + m_2)(\ddot{J}_\theta - \ddot{C}_p) + m_2\ddot{T}_\theta - (m_1 + m_2)gJ_\theta \\
0 &= \ddot{J}_\theta + \ddot{T}_\theta - gT_\theta - \ddot{C}_p.
\end{aligned}
\tag{6}
$$

Solving for $\ddot{C}_p$, $\ddot{J}_\theta$, and $\ddot{T}_\theta$ gives our equations of motion:

$$
\begin{aligned}
\ddot{C}_p &= \frac{F + g(m_1 + m_2)J_\theta(t)}{M} \\
\ddot{J}_\theta &= \frac{Fm_1 - Mgm_2T_\theta(t) + g(Mm_1 + Mm_2 + m_1(m_1 + m_2))J_\theta(t)}{Mm_1} \\
\ddot{T}_\theta &= \frac{g(m_1 + m_2)(T_\theta(t) - J_\theta(t))}{m_1}.
\end{aligned}
\tag{7}
$$

In order to use a Linear Quadratic Regulator, we wish to write our system in the form $\dot{x}(t) = Ax(t) + Bu(t)$, where

$$
\dot{x}(t) = \begin{bmatrix} \dot{C}_p \\ \ddot{C}_p \\ \dot{J}_\theta \\ \ddot{J}_\theta \\ \dot{T}_\theta \\ \ddot{T}_\theta \end{bmatrix}, \ x(t) = \begin{bmatrix} C_p \\ \dot{C}_p \\ J_\theta \\ \dot{J}_\theta \\ T_\theta \\ \dot{T}_\theta \end{bmatrix}, \text{ and } u(t) = F.
$$

This produces

$$
A = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{g(m_1+m_2)}{M} & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & \frac{g(Mm_1+Mm_2+m_1^2+m_1m_2}{Mm_1} & 0 & \dfrac{-gm_2}{m_1} & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & \frac{-g(m_1+m_2)}{m_1} & 0 & \frac{g(m_1+m_2)}{m_1} & 0
\end{bmatrix} \text{ and } B = \begin{bmatrix} 0 \\ \dfrac{1}{M} \\ 0 \\ \dfrac{1}{M} \\ 0 \\ 0 \end{bmatrix}.
$$

We can solve for Q and R with the equation $L = x^T(t)Qx(t) + u^T(t)Ru(t)$. However, the Lagrangian is very difficult to solve for analytically. As it turns out, as long as Q and R are positive semidefinite and positive definite, the cost function has a unique minimum that can be obtained by solving the Algebraic Riccati Equation. We simply choose Q to contain appropriate weights on the diagonal to penalize the state variables and R a weight to penalize the control. We choose

$$
Q = \begin{bmatrix}
5000 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 100 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 100 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix} \text{ and } R = \begin{bmatrix} 1 \end{bmatrix}.
$$

With everything in the system defined, we employ the `solve_continuous_are()` function from SciPy's linear algebra module to get P such that

$$
PA + A^TP + Q - PBR^{-1}B^TP = 0. \tag{8}
$$

From here, P can be used to solve for the optimal feedback gain. The feedback law for the Algebraic Riccati Equation is given by

$$u = -R^{-1}B^TPx. \tag{9}$$

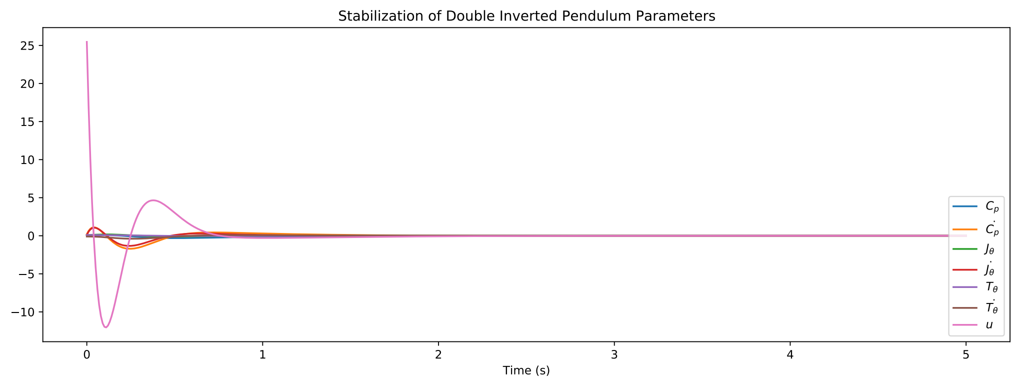
We use a numerical root solver to find x at each time step such that

$$(A - BR^{-1}B^TP)x = 0 \tag{10}$$

with the given initial conditions. Then our force at each time step is given by equation 9 with the calculated values of x. This is our final solution.

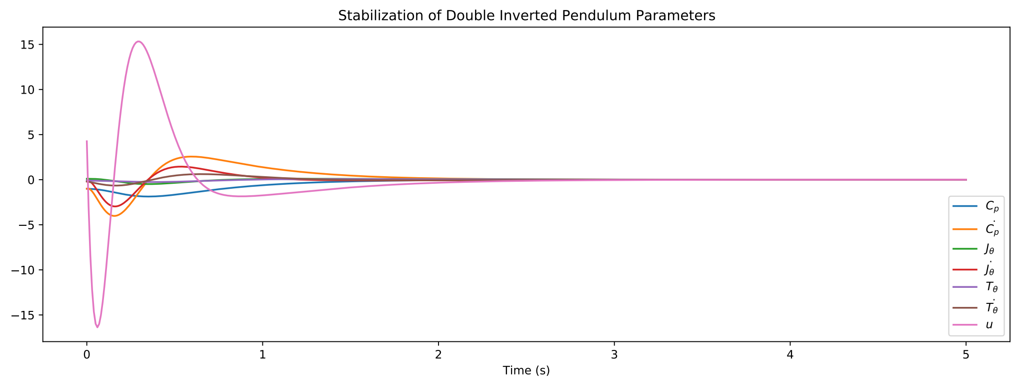
4 Interpretation

Starting with any feasible initial conditions for our system (cart position, cart velocity, joint angle, joint angular velocity, tip angle, and tip angular velocity), we can solve the associated Algebraic Riccati Equation to get a solution to our infinite horizon problem. The solution optimizes potential energy for the system. Thus, keeping the pendulum above the cart will always be the goal of the force we control. If the system starts with a relatively vertical position, but a cart that has a velocity in some direction, the force we need to exert will be to counteract the movement of the cart and return underneath the pendulum.



The solutions assume that there is no random outside noise being added to the system, thus we can reach a perfect balance of the pendulum in which the parts are practically stationary.

Different initial conditions will guarantee different optimal solutions. Given a cart that starts with the pendulum below it, the cart will need to move in such a manner that the pendulum swings up, thus the force will be sharp in one direction, then the other to move the cart quickly in opposing directions to swing the pendulum up, then the system will mirror all other solutions and keep the pendulum balanced directly above the cart, thus maximizing potential energy.



Overall, we were impressed with how robust this LQR solution was. It was able to respond to any initial conditions we threw at it and quickly stabilize the system.

References

[1] Tristeng, "Double inverted pendulum." https://github.com/tristeng/control/tree/master/res/img, January 2019.

5 Appendix

All code for this project can be accessed in a Jupyter Notebook by contacting the authors.