Differential Equations - Project 1

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1 Introduction

In this problem, we are analyzing the relationship between evicted households and renting households. Due to the effects of COVID-19 in 2020, rental assistance was enacted by The Centers for Disease Control and Prevention to help protect residential renters. However the eviction moratorium on residential evictions is no longer in effect. Thus those that are still feeling the negative economic effects of COVID-19 are left with no rental assistance. The goal of this problem is to analyze, visualize, and understand this relationship between evicted households and renting households. Through different factors we can model the behavior between the relationship of renting and eviction. Allowing us to predict which factors negatively and positively affect the relationship between renting and evictions.

2 Methods

In this problem, we are assuming the only forces acting on the flow between rented households and evicted households are time, with the eviction and renting rates being constants. For this problem, we use two models; the first model is a simple linear model that assumes the following:
a constant fraction of the renting group transitions to the evicted group regardless of the vacancy rate, and a constant fraction of the evicted group transitions to the renting group.
The second model we use is an improved mode with the following assumptions:
The magnitude of the flow rate from the renting group to the evicted group increases as the number of vacancies decrease.
The magnitude of the flow rate from the evicted group to the renting group decreases as the vacancies decrease.

For the purposes of this problem, we are utilizing data found from the Eviction Lab for the city of Waterbury, Connecticut. The data found gives us a non-homeowner households (N) value of 59,245, a rental rate (α) of 6.1%, an eviction rate (β) of 2.03%, and a starting value $R(0)$ of 56,283 renters. We also know that the starting amount of evicted persons $E(0)$, based on the data given is 2,962.

Based on the relationship between rented households and evicted households, we can establish that

$$N = E + R \tag{1}$$

As a result, we can say that:

$$E = N - R \tag{2}$$

And

$$R = N - E \tag{3}$$

Using this data we can formulate the following:

$$\frac{dR}{dt} = \beta(E) - \alpha(R) \tag{4}$$

$$\frac{dE}{dt} = \alpha(R) - \beta(E) \tag{5}$$

Utilizing these relationships, we can rewrite our differential equations to be:

$$\frac{dR}{dt} = \beta(N - R) - \alpha(R) \tag{6}$$

And

$$\frac{dE}{dt} = \alpha(N - E) - \beta(E) \tag{7}$$

Solving these differential equations using the Separation of Variables method:

$$\frac{dR}{dt} = \beta(N - R) - \alpha(R) \tag{8}$$

Distribute the β

$$\frac{dR}{dt} = \beta N - \beta R - \alpha R \tag{9}$$

Factor the $-R$ out

$$\frac{dR}{dt} = \beta N - R(\beta + \alpha) \tag{10}$$

Subtract the βN from both sides

$$\frac{dR}{dt} - \beta N = -R(\beta + \alpha) \tag{11}$$

Divide the $-R$ on both sides

$$\frac{dR}{dt} - \frac{\beta N}{-R} = \beta + \alpha \tag{12}$$

Multiply both sides by dt

$$dR - \frac{\beta N}{-R} = (\beta + \alpha)dt \tag{13}$$

Take out the multiple of βN and begin integrating both sides

$$\beta N \int -\frac{1}{-r} dt = \int (\beta + \alpha)dt \tag{14}$$

The result is:

$$\beta N \; ln|R| = \beta t + \alpha t + C1 \tag{15}$$

Where C1 is an arbitrary constant to be determined later. Divide both sides by βN

$$ln|R| = \frac{\beta t + \alpha t + C1}{\beta N} \tag{16}$$

We finally arrive at R(t) after exponentiating both sides of the equation:

$$R(t) = e^{\frac{\beta t + \alpha t + C1}{\beta N}} \tag{17}$$

Employing a similar method to solve for E(t):

$$\frac{dE}{dt} = \alpha R - \beta E \tag{18}$$

We know that $R = N - E$, so applying that here gives us

$$\frac{dE}{dt} = \alpha(N - E) - \beta E \tag{19}$$

We can distribute the α

$$\frac{dE}{dt} = \alpha N - \alpha E - \beta E \tag{20}$$

We can then factor out the $-E$

$$\frac{dE}{dt} = \alpha N - E(\alpha + \beta) \tag{21}$$

We can then subtract the αN from both sides

$$\frac{dE}{dt} - \alpha N = -E(\alpha + \beta) \tag{22}$$

And then divide the $-E$ on both sides while multiplying by dt

$$\frac{dE}{dt} - \frac{\alpha N}{-E} = (\alpha + \beta)dt \tag{23}$$

Pull out the αN and begin the process of integrating both sides

$$\alpha N \int -\frac{1}{-E} dE = \int (\alpha + \beta) dt \tag{24}$$

This leads us to:

$$\alpha N \; ln|E| = \alpha t + \beta t + C2 \tag{25}$$

Where C2 is an arbitrary constant to be determined later, divide both sides by αN

$$ln|E| = \frac{\alpha t + \beta t + C2}{\alpha N} \tag{26}$$

Exponentiate both sides to remove the $ln|E|$

$$E(t) = e^{\frac{\alpha t + \beta t + C2}{\alpha N}} \tag{27}$$

Taking these, we can apply the formula of $\beta = \frac{1}{3}\alpha$ to solve for the equilibrium solutions of R(t) and E(t)

$$\frac{dR}{dt} = 0, 0 = \frac{1}{3}\alpha(N - R) - \alpha R \tag{28}$$

Which leads to:

$$R = \frac{N}{4} \tag{29}$$

For E(t), applying a similar formula:

$$\frac{dE}{dt} = 0, 0 = \alpha N - E(\alpha + \frac{1}{3}\alpha) \tag{30}$$

This leads to:

$$E = \frac{3N}{4} \tag{31}$$

For our second, more improved model, we operated under the following assumptions: As R gets larger and closer to N, the magnitude of the flow rate from the renting group to the evicted group should increase. And, as R gets larger and closer to N, the magnitude of the flow rate from the evicted group to the renting group should decrease.

In other words, as R increases, the flow of E to R should decrease while the flow from R to E should increase.

Using these assumptions led us to an altered flow rate from the renting group to the evicted group of

$$\alpha \frac{R^2}{N} \tag{32}$$

and for the evicted group to the renting group, we used:

$$-\beta \frac{E^2}{N} \tag{33}$$

Which led us to the new differential equations of:

$$\frac{dR}{dt} = \beta\frac{E^2}{N} - \alpha\frac{R^2}{N} \tag{34}$$

And

$$\frac{dE}{dt} = \alpha\frac{R^2}{N} - \beta\frac{E^2}{N} \tag{35}$$

Due to the fact that solving these equations is currently beyond our technical ability with differential equations, we turned to technology to solve the differential equations behind the improved model. The results of this will be covered in the Results Section.

3 Results

First we will consider the case where 95% of non-homeowner households are renters. Therefore as mentioned previously in methods, a rental rate (α) of 6.1%, an eviction rate (β) of 2.03%, and non-homeowner households (N) value of 59,245,Thus, $R(0)$ = 56,283 renters, and $E(0)$ = 2,962 renters. Based on the previously mentioned relationship between rented households and evicted households

$$R = N - E \tag{36}$$

and

$$E = N - R \tag{37}$$

Transforming our linear model into

$$\frac{dR}{dt} = \beta(N - R) - \alpha(R) \tag{38}$$

and

$$\frac{dE}{dt} = \alpha(N - E) - \beta(E) \tag{39}$$

Through the method of integrating factors and steps mentioned in methods we can solve for these two equations.

$$R(t) = e^{\frac{\beta t + \alpha t + C1}{\beta N}}$$

(40)

and

$$E(t) = e^{\frac{\alpha t + \beta t + C2}{\alpha N}}$$

(41)

Implementing our initial conditions $R(0)$ = 56,283 renters and $E(0)$ = 2,962 renters.
Renting house holds become

$$56,283 = e^{\frac{C1}{\beta N}}$$

(42)

and Evicted house holds become

$$2,962 = e^{\frac{C2}{\alpha N}} \tag{43}$$

Implementing our initial condition of (N) = 59,245, (α) of 6.1%, and (β) of 2.03% we can solve for the constants.
For Renting households

$$56,283 = e^{\frac{C1}{(2.03\%)(59,245)}} \tag{44}$$

$$C1 = 13155 \tag{45}$$

Thus

$$R(t) = e^{\frac{\beta t + \alpha t + 13155}{\beta N}}$$

(46)

For number of evicted households the same logic follows.

$$2,962 = e^{\frac{C2}{(6.1\%)(59,245)}} \tag{47}$$

$$C2 = 28,889 \tag{48}$$

thus

$$E(t) = e^{\frac{\alpha t + \beta t + 28,889}{\alpha N}}$$

(49)

Solving the initial value problem

$$\frac{dR}{dt} = \beta(E) - \alpha(R) \tag{50}$$

$R(0)$ = 56,283 renters.

$$\frac{dE}{dt} = \alpha(R) - \beta(E) \tag{51}$$

$E(0)$ = 2,962 renters
Graphically the IVP looks like

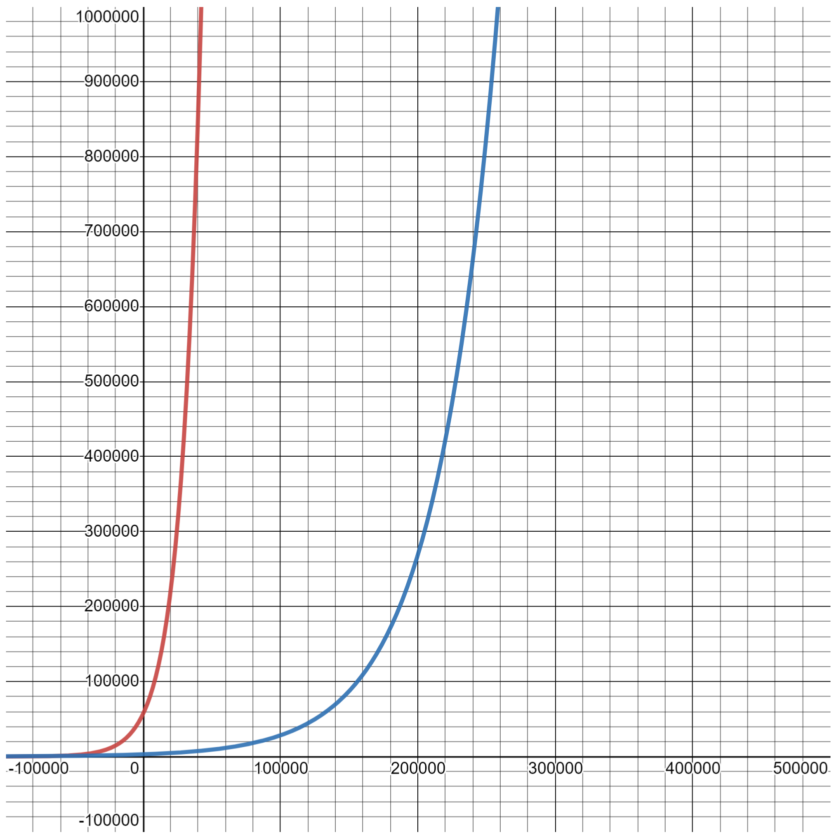

Figure 1: IVP, Red is R(t). Blue is E(t)

This model predicts that both both renting and evicted households will grow exponentially. The limit of both R(t) and E(t) are ∞. We can also see that there will never be an equal number of renting and evicted non-homeowner households.

Secondly we will consider the case where the flow rate from the renting group to the evicted group is

$$\alpha \frac{R^2}{N} \tag{52}$$

and the flow rate from the evicted group to renting group is

$$-\beta \frac{E^2}{N} \tag{53}$$

Thus our new non linear differential equations become

$$\frac{dR}{dt} = \beta \frac{E^2}{N} - \alpha \frac{R^2}{N} \tag{54}$$

And

$$\frac{dE}{dt} = \alpha\frac{R^2}{N} - \beta\frac{E^2}{N} \tag{55}$$

From the relationship between N, R, and E, established in methods we can rewrite these equations as

$$\frac{dR}{dt} = \beta\frac{(N-R)^2}{N} - \alpha\frac{R^2}{N} \tag{56}$$

$$\frac{dE}{dt} = \alpha\frac{(N-E)^2}{N} - \beta\frac{E^2}{N} \tag{57}$$

Due to these differential equations being non linear solving for R(t) and E(t) analytically have no algebraic solution. However we can employ the use of directional fields and equilibrium solutions to find an approximation.
As mentioned in the methods, our equilibrium equations look like

$$R^* = \frac{N(\beta - \sqrt{\beta\alpha})}{\beta - \alpha} \tag{58}$$

And

$$E^* = \frac{N(\alpha - \sqrt{\alpha\beta})}{\alpha - \beta} \tag{59}$$

To find the equilibrium solutions, we begin by substituting in the initial conditions (N) = 59,245, (α) of 6.1%, and (β) of 2.03%, we can solve for R^* and E^*

$$R^* = \frac{59245((2.03\%) - \sqrt{(2.03\%)(6.1\%)}}{(2.03\%) - (6.1\%)} \tag{60}$$

$$E^* = \frac{59245((6.1\%) - \sqrt{(6.1\%)(2.03\%)})}{(6.1\%) - (2.03\%)} \tag{61}$$

Thus $R^* = 21674$ and $E^* = 37571$
We can graphically see these equilibrium solutions with directional Fields.

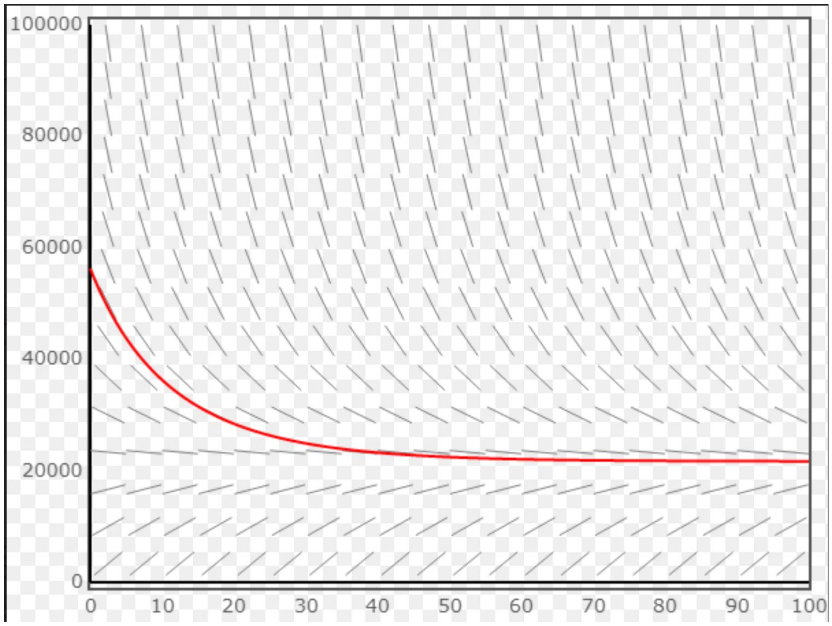

Figure 2: R^* with R(0) = 56283

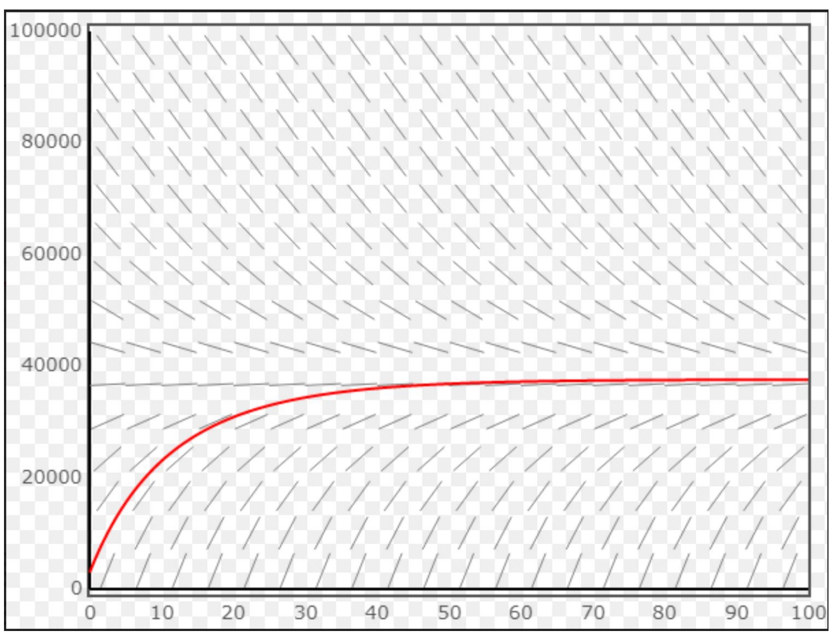

Figure 3: E^* with E(0) = 2962

The model predicts that the long term percentages of non-homeowner households who are renting will reach an equilibrium value of 21674 as seen in Figure 2.
The model also predicts that the long term percentages of non-homeowner households who are evicted will reach an equilibrium value of 37571 as seen in Figure 3. There will never be an equal amount of renting and evicted households, however $R^* = 21674$ and $E^* = 37571$ create a stable system.

$$R^* + E^* = 59245 \tag{62}$$

The value of 59245 is our N value.
Now we will investigate the case when we change the initial percentage of non-homeowner households that are renters to 50%.

$$R = (50\%)(59,245) \tag{63}$$

$$E = (50\%)(59,245) \tag{64}$$

$R = 29{,}622.5$ and $E = 29622.5$.
We will keep (α) of 6.1%, and (β) of 2.03% the same. This is graphically what the new model looks like

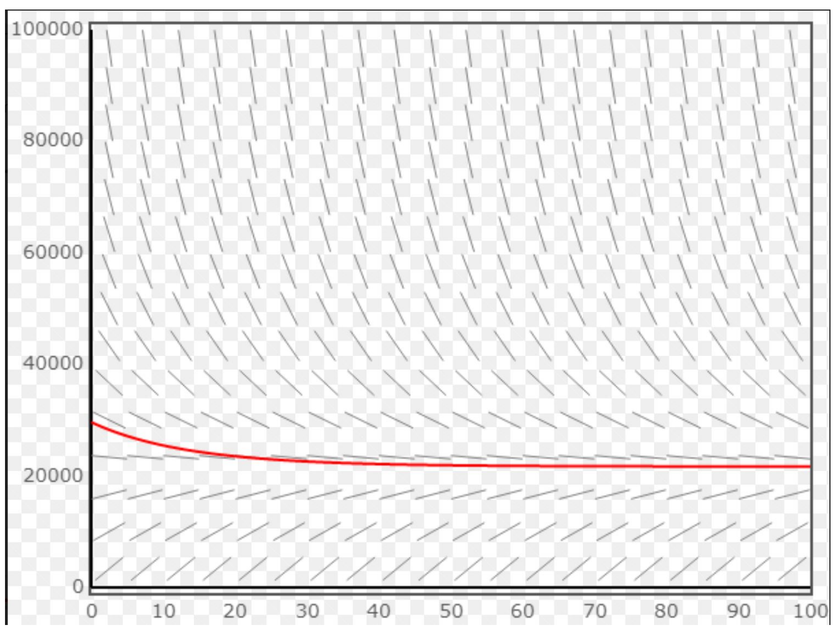

Figure 4: R^* with R(0) = 29622.5

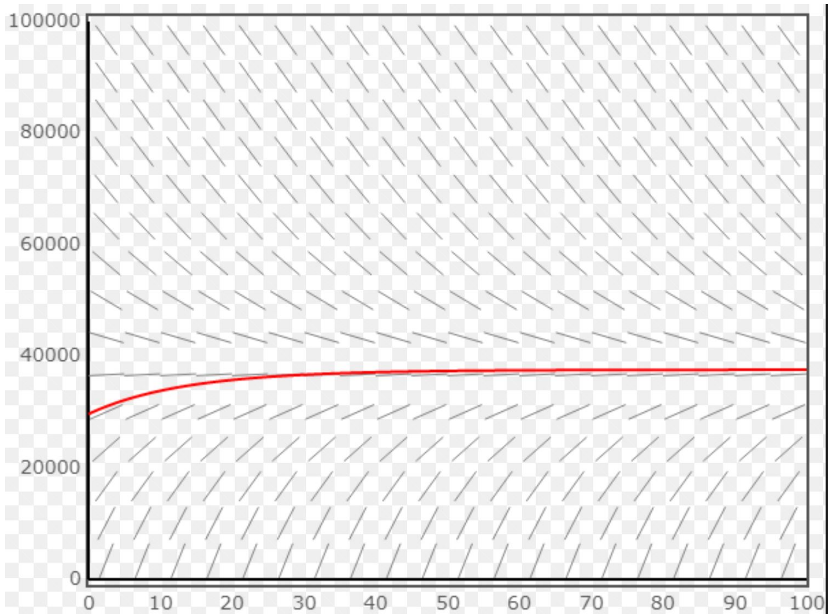

Figure 5: E^* with E(0) = 29622.5

In this new model the long term percentage of renting households and evicted house holds does not change. Changing the initial percentage of households that are renters and households that are evicted only change the rate at which renters and evicted hit the equilibrium.
Now its time to change our (β) the renting rate. In this model out $R(0)$ will be the same at which we started which was 56,283. The same goes for $E(0)$ which is 2,962. (α) will remain the same at 6.1% however (β) will become $\frac{1}{2}$ (α) instead of $\frac{1}{3}$ (α). This gives (β) a new value of 3.05% instead of 2.03%.
This is what our new model looks like

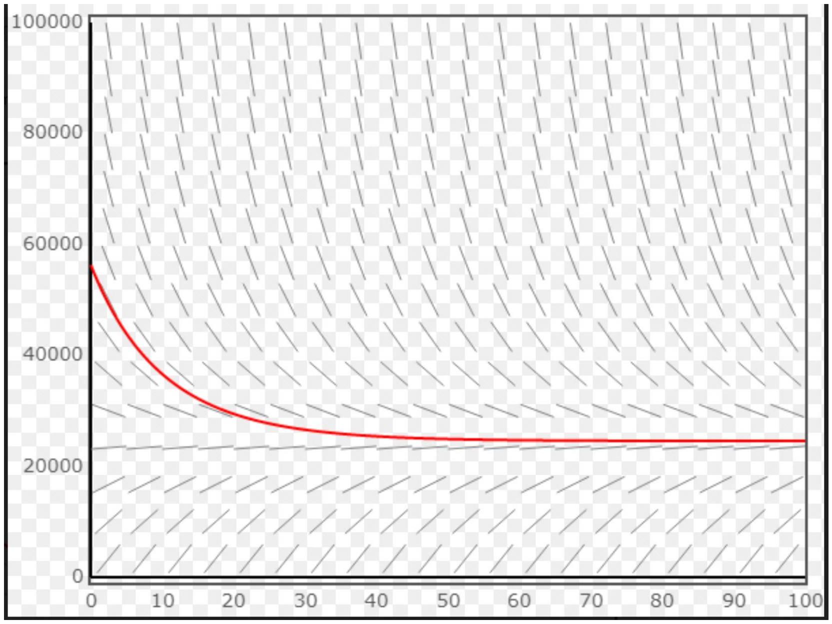

Figure 6: R^* with R(0) = 56,283 and (β) =3.05%

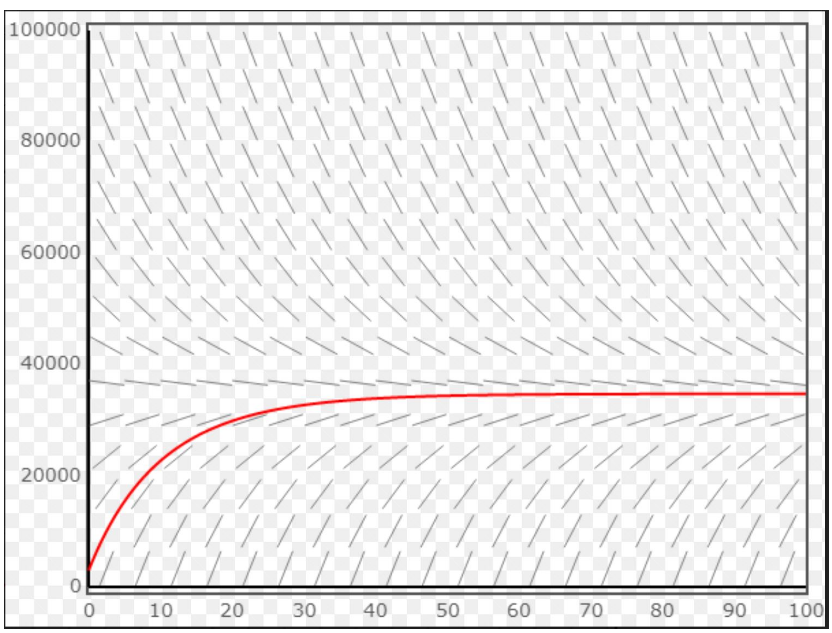

Figure 7: E^* with E(0) = 2,962 and (β) =3.05%

Changing the (β) value did have an effect on our model and did change the long term percentages of renting and evicted households. Although the change doesn't seem significant on the graph we can look at the rate of change of our line segments at the equilibrium points. In Figures 3 and 5 the line segments for E^* appear horizontal telling us the rate of change is near 0. However in Figure 7 E^* line segments appear to have a slight negative slope. Reading the slope field like this allows me to mathematically induce that E^* in Figure 7 must be less than E^* in Figures 3 and 5. We can use the same thought process to mathematically induce that R^* in Figure 6 is greater than R^* in Figures 2 and 4.

4 Discussion

Overall, we used differential equations to model the eviction and rental rate of households based on data during and after the moratorium policy enacted during the COVID-19 pandemic. We discovered that, in the initial model, the amount of renters and evicted persons would grow without bound in the long-run. Given the fact that $N = 59245$ which $R+E$ must equal, this model doesn't really work. We also found that there will never be an equal number of renting and evicted non-homeowner households. In the improved model, we found that there is at first an exchange of rentals and evicted households, but as time goes on, they reach a limiting value, as mentioned earlier.

In conclusion, eviction trends seem to stabilize as time goes on, and in the city of Waterbury, Connecticut, the eviction rate at this time is high enough that the number of evicted households will eventually surpass the number of rented households. However, it is important to note that it is likely that current events are impacting this exercise in modeling, as it has had significant impact on all aspects of the world. As such, the model may not necessarily show accurate results as it is based on data from a unique event that can not be considered normal. That being said, this model does adequately show the effects that COVID-19 and the moratorium on evictions has had on housing in Waterbury.