Differential Equations - Project 2

Abdur-Rahman Khwaja and Andrew Kotarski

21 March 2022

1 Introduction

In this paper we will be examining and modeling the oscillation of springs through ordinary differential equations. The case specifically where we have a spring and attach a mass to the end of the spring. We then displace the weighted spring from its initial equilibrium position. If we let go of the mass, then the spring begins to oscillate up and down creating harmonic motion. At any time there are three forces acting on the mass. The force of the spring $-kx$, the restive force $-cx$' and the external force $F(t)$. Using Newtons second law and Hooke's law we can describe the relationship between these variables and the motion of the spring. Hooke's law states the force needed to displace a spring by extending or compressing it is proportional to the displacement. Newtons second law can be represented as a pair of differential equations. One that states that the rate of change of the position of an object is its velocity, and the acceleration of an object is the net force acting on the object by its mass. Therefore the position of the mass attached to the spring must satisfy the differential equation

$$mx'' + cx' + kx = F(t) \tag{1}$$

were $x(t)$ is position, $F(t)$ is the external force, c is the restive force, k is the spring force, and m is the mass of the weight.

By changing the restrictive force motion can be un-damped $c = 0$. Under-dampened $c^2 < 4mk$, and over-damped $c^2 > 4mk$.

We can manipulate the external force to create free motion. Free motion is when there is no external force acting on the mass $F(t) = 0$. The case when $F \neq 0$ creates a periodic external force acting on the mass

By manipulating these factors we will be able to mathematically and visually describe the motion of the mass attached to a spring at any given time

2 Methods

For the purposes of all the differential equations, the object attached to the end of the spring will have a mass of 1 kilogram $m = 1kg$, and the spring constant will be 4N/m $k = 4N/m$

Problem 1

In this case, we will create simple harmonic movement. We will create a system free of external force $F = 0$, and the system will be un-dampened $c = 0$. Thus the differential equation becomes

$$mx'' + kx = 0 \tag{2}$$

Implementing variable conditions, and rewriting in terms of the characteristic equation

$$r^2 + 4 = 0 \tag{3}$$

solving for the roots yeilds

$$r = \pm 2i \tag{4}$$

These are classified as complex roots, thus we can write them in the General Complex form.

$$x(t) = e^{\alpha t}[A\cos(\beta t) + B\sin(\beta t)] \tag{5}$$

Thus the general equation of the position x with respect to time is

$$x(t) = A\cos(2t) + B\sin(2t) \tag{6}$$

Problem 2

Case 1

In case 1 we will observe a under-damped system where $c = 2 \cdot N/s$, that is free from external force $F(t) = 0$, while mass and spring constant remain the same $m = 1kg$ and $k = 4N/m$. Thus the differential equation becomes

$$mx'' + 2x' + 4 = 0 \tag{7}$$

Implementing variable conditions and rewriting in terms of the characteristic equation

$$r^2 + 2r + 4 = 0 \tag{8}$$

Solving for the roots yeilds

$$r = -1 \pm i\sqrt{3} \tag{9}$$

These are classified as complex roots, thus we can write them in the General Complex form.

$$x(t) = e^{\alpha t}[A\cos(\beta t) + B\sin(\beta t)] \tag{10}$$

equation

Thus the position x of the object at time t.

$$x(t) = e^{-t}[A\cos(\sqrt{3}t) + B\sin(\sqrt{3}t)] \tag{11}$$

Case 2

In case 2 we will observe an over-damped system $c = 5 \cdot N/s$, that is free from external force $F(t) = 0$, while mass and spring constant remain the same $m = 1kg$ and $k = 4N/m$. Thus the differential equation becomes

$$mx'' + 5x' + 4 = 0 \tag{12}$$

Implementing variable conditions and rewriting in terms of the characteristic equation

$$r^2 + 5r + 4 = 0 \tag{13}$$

Solving for the roots yields

$$r_1 = -4, r_2 = -1 \tag{14}$$

equation These are distinct and real roots and can be written in terms of the general solution

$$x(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t} \tag{15}$$

Thus the position of object x and time t is

$$x(t) = C_1 e^{-4t} + C_2 e^{-1t} \tag{16}$$

Problem 3

In Problem 3, we began with the assumption that this is an undamped system $c = 0$ that is acted on by a periodic external force. It is also given that this system is currently at rest and will start at the equilibrium position, which means

$$x(0) = 0, x'(0) = 0 \tag{17}$$

Case 1

For case 1, our first external force is:

$$F(t) = 0.5\cos(2.2t) \tag{18}$$

This makes our differential equation:

$$x'' + 4x = 0.5\cos(2.2t) \tag{19}$$

We can rewrite in terms of our characteristic equation as:

$$r^2 + 4 = 0.5\cos(2.2t) \tag{20}$$

.

In this case, we will need to use the Method of Undetermined Coefficients, which requires us to solve a X_h, a homogeneous equation and a X_p, a particular solution. We will begin by solving our homogeneous equation, and then move to our particular solution.

$$r^2 + 4 = 0 \tag{21}$$

$$r^2 = -4 \tag{22}$$

$$r = \sqrt{-4} \tag{23}$$

$$r = \pm 2i \tag{24}$$

We can then rewrite in terms of our general solution for differential equations of this form:

$$e^0[A\cos(2t) + B\sin(2t)] \tag{25}$$

We will now solve for our particular solution. We start by making a guess for ourX_p based on F(t). Given that F(t) =

$$0.5\cos(2.2t) \tag{26}$$

We can say that our initial guess will be

$$x = [C\sin(2.2t) + D\cos(2.2)] \tag{27}$$

This is based on the form that if F(t) is

$$\cos(xt) \tag{28}$$

Then the guess will be:

$$A\sin(xt) + B\cos(xt) \tag{29}$$

Checking our guess with Y_h shows no duplication, so we can substitute this value of x into our non-homogeneous ODE to solve for coefficients C and D. To start, we need to solve for the second derivative of x.

$$x' = [2.2C\cos(2.2t) - 2.2D\sin(2.2t)] \tag{30}$$

$$x'' = [-4.84C\sin(2.2t) - 4.84D\cos(2.2t)] \tag{31}$$

Substitute into our non-homogeneous ODE and solve for coefficients:

$$(-4.84C\sin(2.2t) - 4.84D\cos(2.2t)) + 4(C\sin(2.2t) + D\cos(2.2t)) = 0.5\cos(2.2t) \tag{32}$$

This simplifies to:

$$(-4.84C\sin(2.2t) - 4.84D\cos(2.2t)) + (4C\sin(2.2t) + 4D\cos(2.2t)) = 0.5\cos(2.2t) \tag{33}$$

We can further simplify:

$$(-0.84C\sin(2.2t) - 0.84D\cos(2.2t)) = 0.5\cos(2.2t) \tag{34}$$

We can cancel the first term, as sine is not present in $F(t)$

$$-0.84D\cos(2.2t) = 0.5\cos(2.2t) \tag{35}$$

since our cosine function is the same factor on both sides, we can divide both sides by it leaving us with:

$$-0.84D = 0.5 \tag{36}$$

This gives us

$$D = -1.68 \tag{37}$$

This means X_p is equal to

$$-1.68\cos(2.2t) \tag{38}$$

We will now write our general solution and solve for the remaining coefficients in the Results section

$$x(t) = (A\cos(2t) + B\sin(2t)) - (1.68\cos(2.2t)) \tag{39}$$

Case 2

For case 2 of our undamped system, we use a periodic external force of

$$F(t) = 6\cos(2t) \tag{40}$$

This means that our differential equation will be:

$$x'' + 4x = 6\cos(2t) \tag{41}$$

We chose to use technology for this differential equation, as solving it will be fairly difficult. Giving us the following for our general solution:

$$x(t) = -\frac{3}{2}(t\ \cos(2t) - \sin(t)\cos(t)) \tag{42}$$

3 Results

Problem 1
(a)
The position x of the object at time t is

$$x(t) = A\cos(2t) + B\sin(2t) \tag{43}$$

We are given initial conditions that the mass is stretched 0.1 meters beyond its equilibrium position with an initial velocity of 0.4 m/s.

$$x(0) = 0.1m \tag{44}$$

$$x'(0) = 0.4\, m/s \tag{45}$$

We can determine C_1 and C_2 by using the initial conditions

$$0.1 = A\cos(0) + B\sin(0) \tag{46}$$

Thus

$$A = 0.1 \tag{47}$$

The derivative of x(t) is

$$x'(t) = -2A\sin(2t) + B\cos(2t) \tag{48}$$

Substituting in initial conditions

$$.4 = -2A\sin(0) + 2B\cos(0) \tag{49}$$

Thus

$$B = 0.2 \tag{50}$$

Then we can write the position of the object in implicit form.

$$x(t) = 0.1\cos(2t) + 0.2\sin(2t) \tag{51}$$

(b)

Circular frequency and Period of the free vibration:
Circular frequency can be calculated by the following equation

$$\omega_0 = \sqrt{\frac{k}{m}} \tag{52}$$

Substituting in known variables yields

$$\omega_0 = 2 \tag{53}$$

The equation to calculate period is

$$T = \frac{2\pi}{\omega_0} \tag{54}$$

Substitute in known values yields

$$T = \pi \tag{55}$$

(c)

Determine the amplitude A and phase angle α of the vibrations:
Amplitude can be calculated using the following equation

$$A = \sqrt{C_1^2 + C_2^2} \tag{56}$$

Substituting in the values for C_1 and C_2 found in part a

$$A = \frac{\sqrt{5}}{10} \tag{57}$$

Phase angle can be calculated by the following equation

$$\tan(\alpha) = \frac{C_2}{C_1} \tag{58}$$

Substituting in the values for C_1 and C_2 found in part a

$$\alpha = \tan^{-1}(2) \tag{59}$$

(d)
Solution in phase-amplitude form:
The formula for phase amplitude is

$$x(t) = A\cos(\omega_0 t - \alpha) \tag{60}$$

Substituting in known values

$$x(t) = 0.1\cos(2t - \tan^{-1}(2)) \tag{61}$$

(e)
Graph $x(t)$:

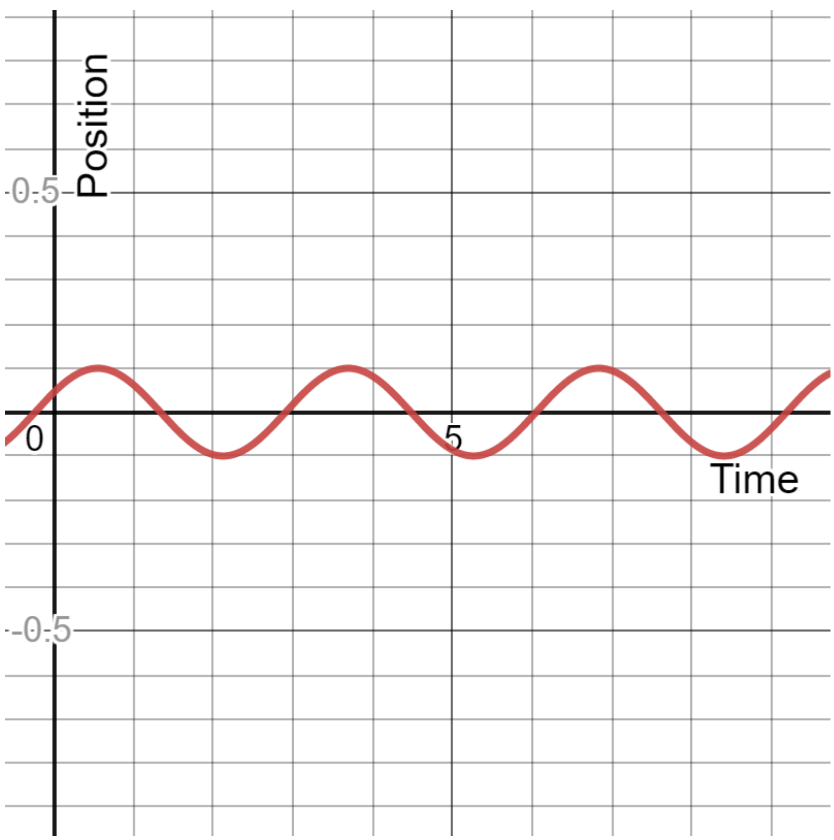

Figure 1: Graph of $x(t) = 0.1\cos(2t - \tan^{-1}(2))$

(d)
The motion of the spring is simple harmonic motion. Since there are no external

forces acting on the object as well as no damping effect, the sinusoidal wave oscillates between $-0.1m < y < 0.1m$ for an infinite amount of time.

Problem 2, Case 1

(a) The position x of the object at time t is

$$x(t) = e^{-t}[A\cos(\sqrt{3}t) + B\sin(\sqrt{3}t)] \tag{62}$$

We are given initial conditions that the mass is stretched 0.1 meters beyond its equilibrium position with an initial velocity of 0.4 m/s.

$$x(0) = 0.1m \tag{63}$$

$$x'(0) = 0.4\, m/s \tag{64}$$

We can determine C_1 and C_2 by using the initial conditions

$$0.1 = e^0[A\cos(0) + B\sin(0)] \tag{65}$$

Thus

$$0.1 = A \tag{66}$$

The second derivative of $x(t)$ is

$$x'(t) = -e^{-t}[A\cos(\sqrt{3}t) + B\sin(\sqrt{3}t)] + e^{-t}[-\sqrt{3}A\sin(\sqrt{3}t) + \sqrt{3}B\cos(\sqrt{3}t)] \tag{67}$$

Substituting in initial conditions

$$0.4 = -e^0[A\cos(0) + B\sin(0)] + e^0[-\sqrt{3}A\sin(0) + \sqrt{3}B\cos(0)] \tag{68}$$

$$0.4 = A + \sqrt{3}B \tag{69}$$

Substituting in the value for A found above

$$B = \frac{\sqrt{3}}{100} \tag{70}$$

Then we can write the position of the object in implicit form.

$$x(t) = e^{-t}[0.1\cos(\sqrt{3}t) + \frac{\sqrt{3}}{100}\sin(\sqrt{3}t)] \tag{71}$$

(b)
Graph $x(t)$:

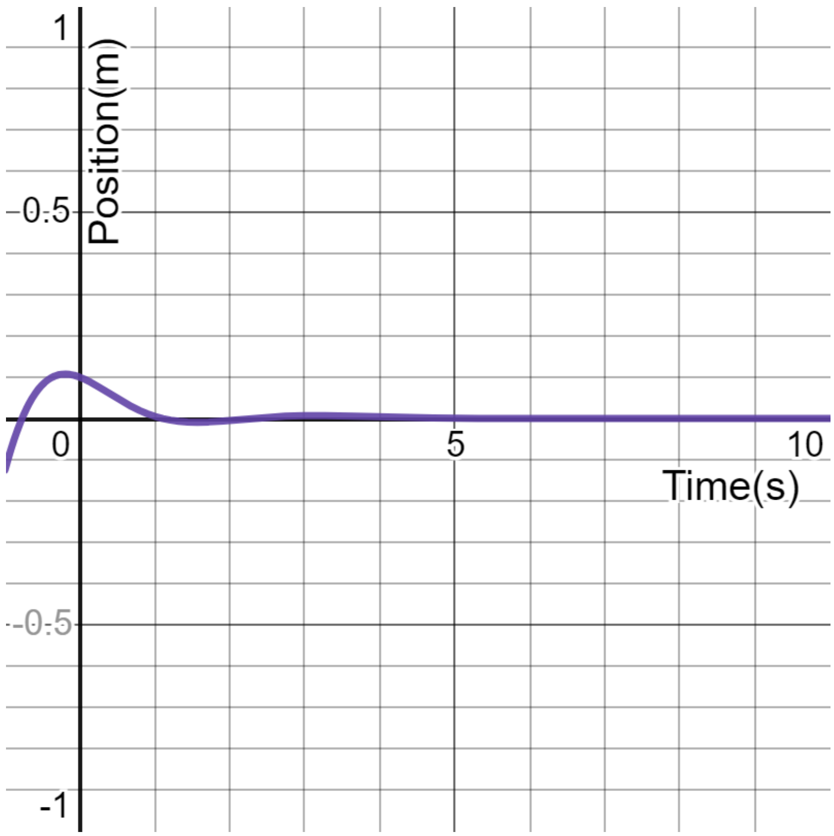

Figure 2: The graph of $x(t) = e^{-t}[.1\cos(\sqrt{3}t) + \frac{\sqrt{3}}{100}\sin(\sqrt{3}t)]$

(c)

Amplitudes as $t \to \infty$:

We can calculate the amplitudes as $t \to \infty$ by taking the limit of

$$\lim_{t \to +\infty} e^{-t}[0.1\cos(\sqrt{3}t) + \frac{\sqrt{3}}{100}\sin(\sqrt{3}t)] \tag{72}$$

We can reduce the equation to

$$\lim_{t \to +\infty} e^{-t} \tag{73}$$

$$\lim_{t \to +\infty} x(t) = 0 \tag{74}$$

Thus the amplitude converges to 0 meters as time goes on.

(d)

The motion of the object quickly loses velocity as time increases. This a result of having an under-damped spring. This model of the object is better than

Figure 1 created in Question 1 because Figure 2 converges to 0 m. Figure 1 oscillates forever.

Problem 2, Case 2

(a)

The position of object x at time t is

$$x(t) = C_1 e^{-4t} + C_2 e^{-1t} \tag{75}$$

We are given initial conditions that the mass is stretched 0.1 meters beyond its equilibrium position with an initial velocity of 0.4 m/s.

$$x(0) = 0.1m \tag{76}$$

$$x'(0) = 0.4\, m/s \tag{77}$$

We can determine C_1 and C_2 by using the initial conditions

$$x(0) = C_1 e^0 + C_2 e^0 \tag{78}$$

$$0.1 = C_1 + C_2 \tag{79}$$

The second derivative of $x(t)$ is

$$x'(t) = -4C_1 e^{-4t} - C_2 e^{-t} \tag{80}$$

Substituting in intial conditions

$$0.4 = -4C_1 - C_2 \tag{81}$$

Solving the two coeffecient equations yeilds

$$C_1 = 0.1 \tag{82}$$

$$C_2 = 0 \tag{83}$$

Thus we can write the implicit solution as

$$x(t) = 0.1e^{-4t} \tag{84}$$

(b)

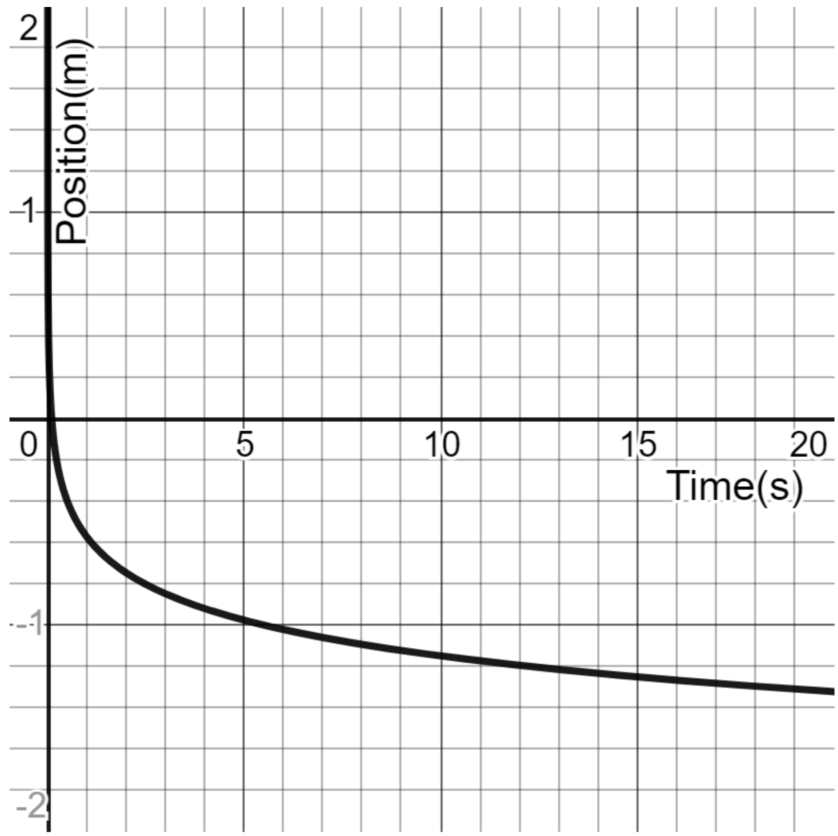

Figure 3: The graph of $x(t) = 0.1e^{-4t}$

(c)

Amplitudes as $t \to \infty$:
We can calculate the amplitudes as $t \to \infty$ by taking the limit of

$$\lim_{t \to +\infty} 0.1e^{-4t} \tag{85}$$

We can reduce the equation to

$$\lim_{t \to +\infty} e^{-4t} \tag{86}$$

$$\lim_{t \to +\infty} x(t) = 0 \tag{87}$$

Thus the amplitude converges to 0 meters as time goes on.
(d)
The motion of the over-damped spring seems to be stretched out as it reaches its equilibrium. This is more realistic than the motion found in Figure 1, because the amplitude stops as time approaches ∞.

Problem 3
Case 1
Beginning with our general solution of:

$$x(t) = (A\cos(2t) + B\sin(2t)) - (1.68\cos(2.2t)) \tag{88}$$

And initial conditions of:

$$x(0) = 0, x'(0) = 0 \tag{89}$$

We can substitute our first initial condition in:

$$0 = (A\cos(0) + B\sin(0)) - (1.68\cos(0)) \tag{90}$$

Simplifying:

$$0 = A(1) + B(0) - 1.68(1) \tag{91}$$

This leaves us with

$$0 = A - 1.68 \tag{92}$$

So,

$$A = 1.68 \tag{93}$$

Solve for the derivative of x(t):

$$x'(t) = (-2A\sin(2t) + 2B\cos(2t)) - (3.696\sin(2.2t) \tag{94}$$

Substitute the second initial condition:

$$0 = (-2A\sin(0) + 2B\cos(0)) - (3.696\sin(0)) \tag{95}$$

Simplifying:

$$0 = 0 + 2B - 0 \tag{96}$$

So,

$$B = 0 \tag{97}$$

This means

$$x(t) = 1.68\cos(2t) - 1.68\cos(2.2t) \tag{98}$$

Which can be simplified into the form:

$$x(t) = 1.68(\cos(2t) - \cos(2.2t)) \tag{99}$$

Where 2 is the natural circular frequency of the mass and 2.2 is the circular frequency of the external force. While this equation is nice, we can rewrite it as a product of two sine functions using the following trigonometric identity:

$$\cos(\alpha t - \beta t) - \cos(\alpha t + \beta t) = 2\sin(\alpha t)\sin(\beta t) \tag{100}$$

where $\alpha = \frac{1}{2}(2.2 + 2)$ and $\beta = \frac{1}{2}(2.2 - 2)$.

This results in

$$\alpha = \frac{1}{2}(2.2 + 2) = 2.1 \tag{101}$$

and

$$\beta = \frac{1}{2}(2.2 - 2) = 0.1 \tag{102}$$

So, we can rewrite $x(t)$ as

$$1.68(2\sin(2.1t)\sin(0.1t)) \tag{103}$$

Which can be simplified to

$$3.36(\sin(2.1t)\sin(0.1t)) \tag{104}$$

Graphically we can find that the maximum amplitude of the oscillations in this equation is 3.36 meters.

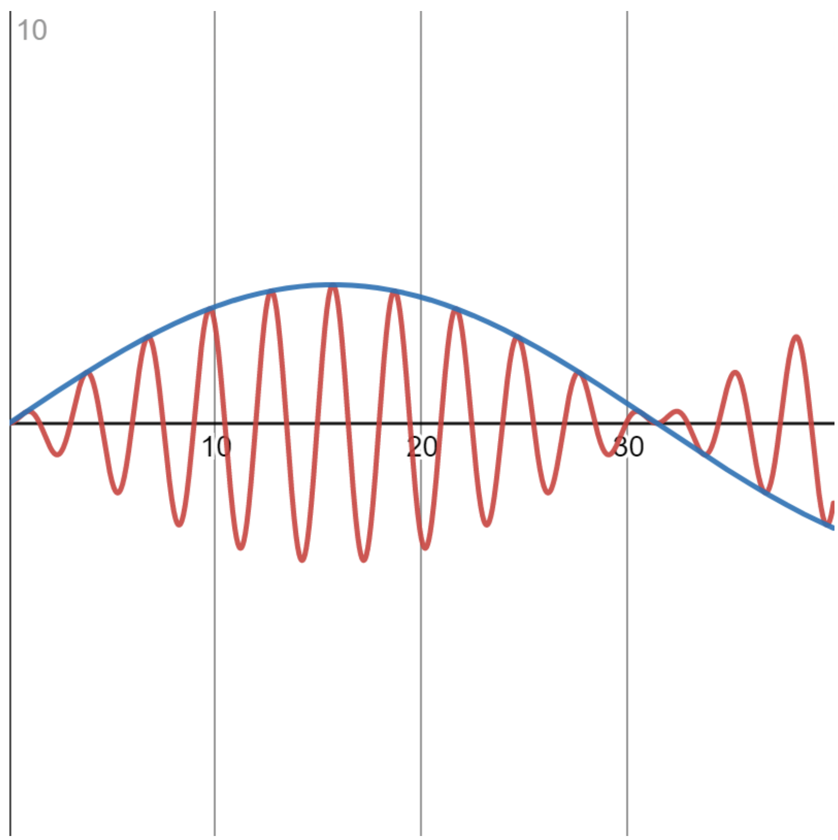

Figure 4: The graph of Problem 3 Case 1. x-axis is time in seconds, y axis is the position of the spring in meters. The red line is x(t) while the blue line is the function 3.36 sin(0.1t)

Case 2

As was mentioned before, we utilized technology to solve the differential equation for x(t), arriving at

$$x(t) = -\frac{3}{2}(t\ \cos(2t) - \sin(t)\cos(t)) \tag{105}$$

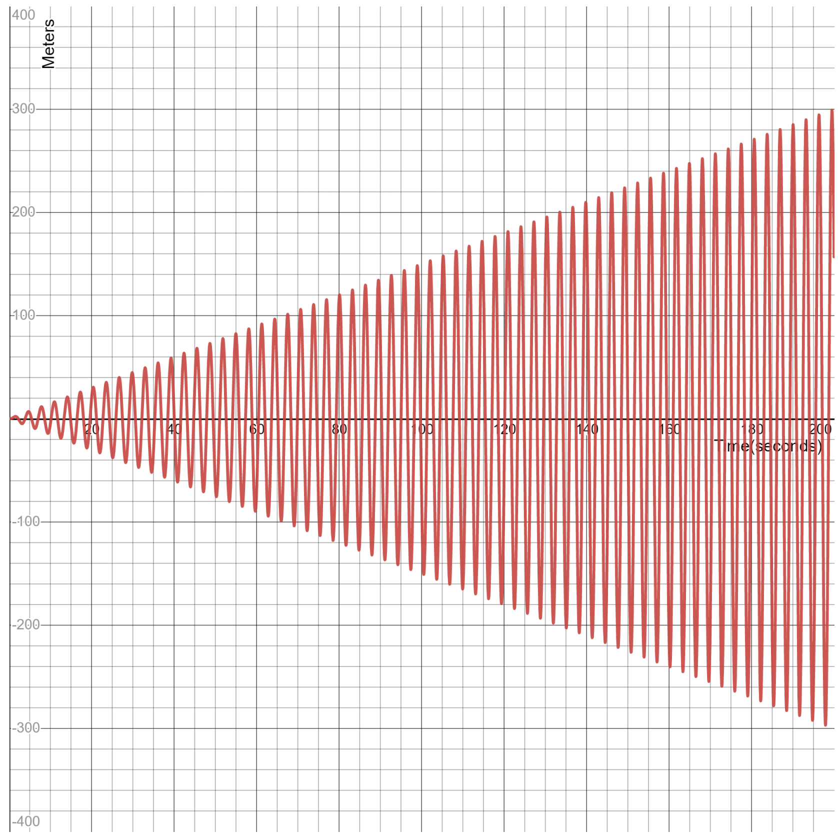

Figure 5: The graph of Problem 3 Case 2. x-axis is time in seconds, y axis is the position of the spring in meters

We also found that as $t \to \infty$, the vibrations constantly increased with no bound. However, this is not realistic because the displacement of the spring increases with no bound, and this is not valid within the context of this mathematical model.

4 Discussion

In this paper, we examined and modeled the oscillation of a spring with a mass attached to the end of it. We then displaced the spring and manipulated several of the variables that influence how the spring would oscillate. Understanding how springs oscillate through Hooke's Law has many applications in the world, and modeling springs with these variables allow us to better understand and then apply Hooke's Law to the world. Hooke's Law has already seen a great deal of usage within the world, it has been utilized greatly in the field of engineering, but has also been used in seismology and molecular mechanics.