Logistic Equation: Continuous, Discrete, Chaos

Andrew Kotarsi

May 2022

1 Introduction

What is a logistic equation

The logistic growth model is a first order linear differential equation that is used to model population growth. The simplest version of the logistic growth equation is the Malthusian growth model which was published by Thomas Malthus in 1798. Thomas Malthus was a British economist and philosopher who noted that populations have a natural tendency to grow exponentially when resources are abundant. Malthus modeled population as a function of time by taking the rate of change of a population with respect to time proportional to the population itself. The model shows that the rate at which a population increases will increase as population increases.

$$\frac{dN}{dt} = rN(t) \tag{1}$$

$N(t)$ is the Population and r is the intrinsic growth rate.
(The intrinsic growth rate of a population is the reproduction rate less the death rate)
The idea that populations will increase exponentially was the general assumption that ecologists and philosophers accepted during the 1700's. While Malthus created a model for the accepted idea of exponential growth, in *An Essay on the Principle of Population*, Malthus believed that real populations are limited by an environments resources. Malthus was criticized for this idea, however the Malthusian growth model was accepted by economists and philosophers.
It wasn't until 1838 that Malthus's idea on populations being bounded by environmental resources was mathematically modeled. After reading Malthus's essay Pierre Francois Verhulst, created a new model in which population is limited by environmental resources. The Verhulst model essentially dampens the Malthusian growth model as the function approaches a limiting factor. As the population $N(t)$ approaches a ceiling function k the rate of growth approaches 0.

$$\frac{dN}{dt} = rN(t)(1 - \frac{N(t)}{k}) \tag{2}$$

When our population $N(t)$ is small, the term $(1 - \frac{N(t)}{k})$ approaches 1, and the rate of change of the function becomes exponential. However when the population $N(t)$ approaches the limiting factor k the term $(1-\frac{N(t)}{k})$ approaches 0, and the rate of change of population decreases.

2 Method

Malthusian Growth Model
Given

$$\frac{dN}{dt} = rN(t) \tag{3}$$

This is a first order linear equation. Using the method of separation of variables we can solve this differential equation for $N(t)$

$$\frac{dN}{N} = rdt \tag{4}$$

$$\int \frac{1}{N}(dN) = \int (r)dt \tag{5}$$

Anti-differentiating the equation

$$ln(N) = rt + c \tag{6}$$

Solving for $N(t)$

$$N = e^{rt} \times e^{c} \tag{7}$$

Since c is just a constant, e^c is a new constant. We will rename this new constant C. Thus

$$N(t) = Ce^{rt} \tag{8}$$

This is the general solution to the Malthusian Growth Model. As mentioned in the introduction, the solution to Malthusian Growth Model is exponential growth.

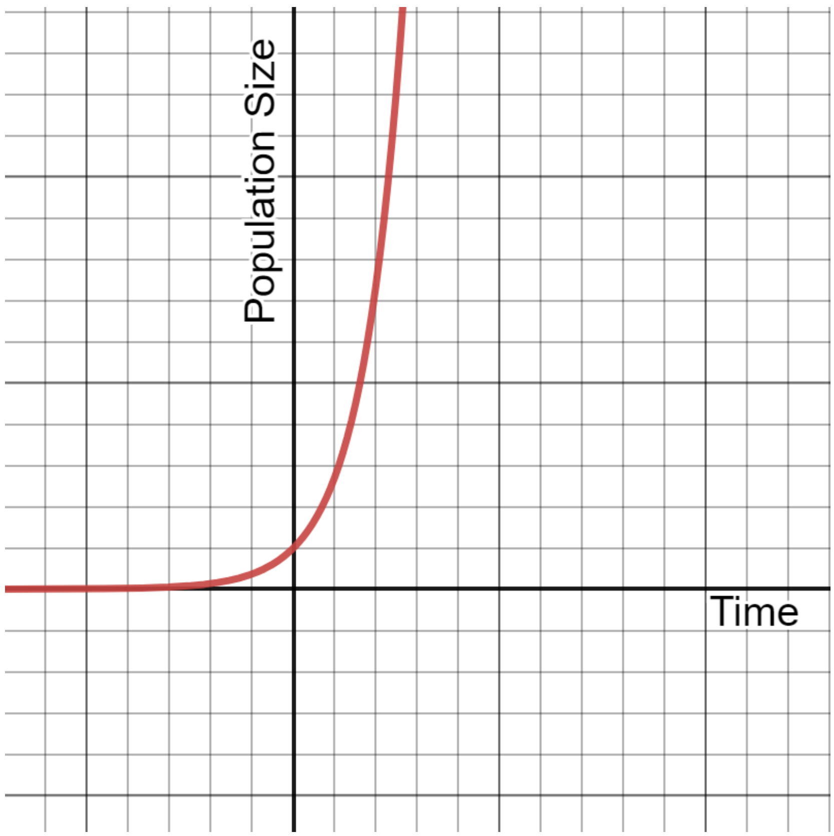

Figure 1: Solution to Malthusian Growth Model

Verhulst Growth Model

Given

$$\frac{dN}{dt} = rN(1 - \frac{N}{k}) \tag{9}$$

The Verhulst Growth Model is a first order differential equation. We can solve for $N(t)$ by using separation of variables.

$$\frac{1}{N(1-\frac{N}{k})}dN = rdt \tag{10}$$

Anti-differentiate both sides.

$$\int \frac{1}{N(1-\frac{N}{k})}dN = \int (r)dt \tag{11}$$

The right hand side of the equation becomes

$$\int \frac{1}{N(1-\frac{N}{k})}dN = rt + c \tag{12}$$

To solve the left hand side of the equation, first we need to eliminate the fractions. This is achieved by multiplying the numerator and denominator by k.

$$\int \frac{k}{kN(1-\frac{N}{k})} dN \tag{13}$$

Distributing the k in the denominator

$$\int \frac{k}{N(k-N)} dN \tag{14}$$

Now we have a complex rational function. Using partial fraction decomposition, we can assign each factor its own fraction, and set equal to the original expression.

$$\frac{A}{N} + \frac{B}{k-N} = \frac{k}{N(k-N)} \tag{15}$$

Multiply both sides of the equation by the denominator to cancel it.

$$(N(k-N))\frac{A}{N} + \frac{B}{k-N} = \frac{k}{N(k-N)}(N(k-N)) \tag{16}$$

Distributing the term and switching left hand side and ride hand side gives

$$k = A(k-N) + Bk \tag{17}$$

Let $N = 0$, then

$$k = A(k) \tag{18}$$

Thus

$$A = 1 \tag{19}$$

Let $N = k$, then

$$k = Bk \tag{20}$$

Thus

$$B = 1 \tag{21}$$

Thus the integral becomes

$$\int \frac{1}{N} + \frac{1}{k-N} dN = rt + c \tag{22}$$

Anti-differentiating the left hand side,

$$ln|N| - ln|k-N| = rt + c \tag{23}$$

Multiply both sides by -1,

$$-ln|N| + ln|k-N| = -rt - c \tag{24}$$

By properties of logarithms we can write the left hand side as a fraction.

$$ln|\frac{k-N}{N}| = -rt - c \tag{25}$$

Raise both sides to the power of e to cancel out the ln

$$\frac{k-N}{N} = Ce^{-rt} \tag{26}$$

Simply the left hand side fraction

$$\frac{k}{N} - 1 = Ce^{-rt} \tag{27}$$

Add the 1 to the right hand side:

$$\frac{k}{N} = Ce^{-rt} + 1 \tag{28}$$

Multiply both side by N and divide both side by Ce^{-rt}

$$N = \frac{k}{1 + Ce^{-rt}} \tag{29}$$

This is the general solution to the Verhulst model. The general solution graphed is the Sigmoid function.

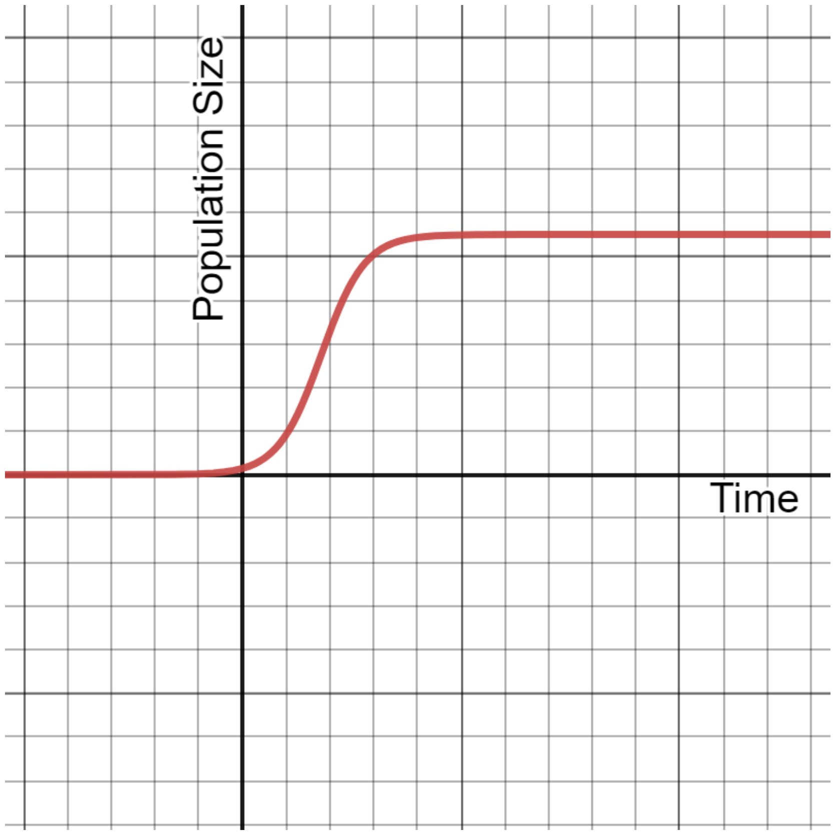

Figure 2: Solution to Verhulst Growth Model

Visually we can see that initially when the population size is low the growth it takes exponential growth. As mentioned in the introduction when our population $N(t)$ is small, the term $(1 - \frac{N(t)}{k})$ approaches 1, and the rate of change of the function becomes exponential.
Visually we can also see that when the population $N(t)$ approaches the limiting factor k the term $(1 - \frac{N(t)}{k})$ approaches 0, and the rate of change of population decreases.

3 Results

Malthusian growth model
Consider for the Malthusian growth model that we are given an initial population of 2 rabbits. $N(0) = 2$. First we need to solve for our constant C

$$2 = Ce^{r(0)} \tag{30}$$

Thus

$$C = 2 \tag{31}$$

Therefore our explicit solution is

$$N(t) = 2e^{rt} \tag{32}$$

Consider that we observe 3 groups of rabbits each with different intrinsic growth rates, but starting populations of 2. As mentioned in the introduction, the intrinsic growth rate of a population is the reproduction rate less the death rate. Consider $r = 10\%$, $r = 20\%$, and $r = 50\%$.

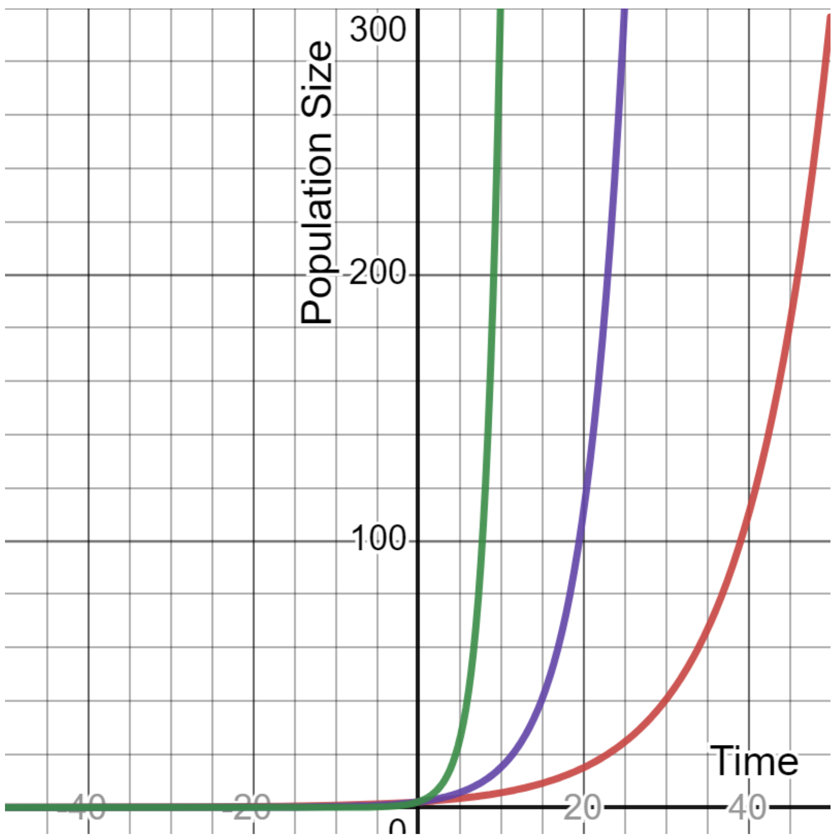

Figure 3: Green: $r = 50\%$, Purple: $r = 20\%$, Red: $r = 10\%$

The higher the the value of the intrinsic growth rate the faster the population reproduces over time. In context, a group of rabbits that has a high birth rate and low deaths will have a higher population over time than a group that has a low birth rate and high death rate. Observe $r = 50\%$ compared to $r = 10\%$.
Consider two more groups of rabbits, both have an intrinsic growth rate of

$r = 20\%$ but group 1 has an initial population of 50 and group 2 has an initial population 10.

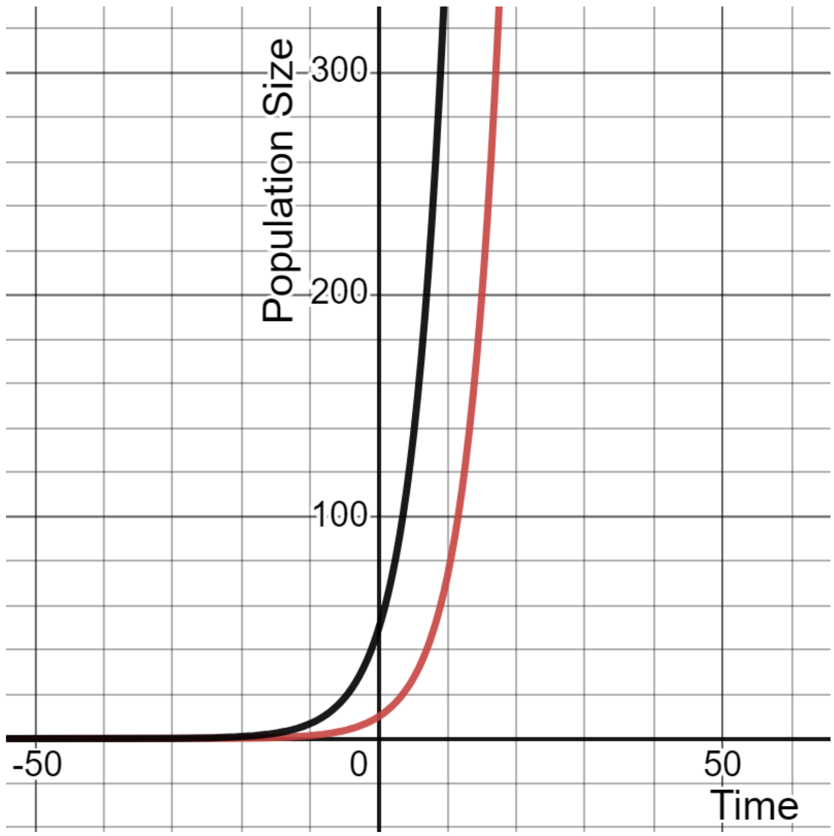

Figure 4: Black: $C = 50$, Red $C = 20$

As observed in solving for the explicit solution, changing the initial population will result in the x-intercept of the growth-rate being at that initial population.

Verhulst growth model

Given the general solution

$$N = \frac{k}{1 + Ce^{-rt}} \tag{33}$$

Lets solve for the explicit solution at $t = 0$. Then Population size at $t = 0$ becomes

$$N(0) = N_0 \tag{34}$$

From equation 26 in methods,

$$\frac{k - N}{N} = Ce^{-rt} \tag{35}$$

At $t = 0$,

$$\frac{k - N_0}{N_0} = Ce^{-r(0)} \tag{36}$$

Thus the constant C is a ratio of the limiting factor of the population minus the initial population divided by the initial population.

$$\frac{k - N_0}{N_0} = C \tag{37}$$

Consider at $t = 0$, our initial population of 3 groups of rabbits is N_0 is 10. Consider, the three intrinsic growth rates of $r = 10\%$, $r = 20\%$, and $r = 50\%$. Consider the environment can only sustain 100 rabbits.

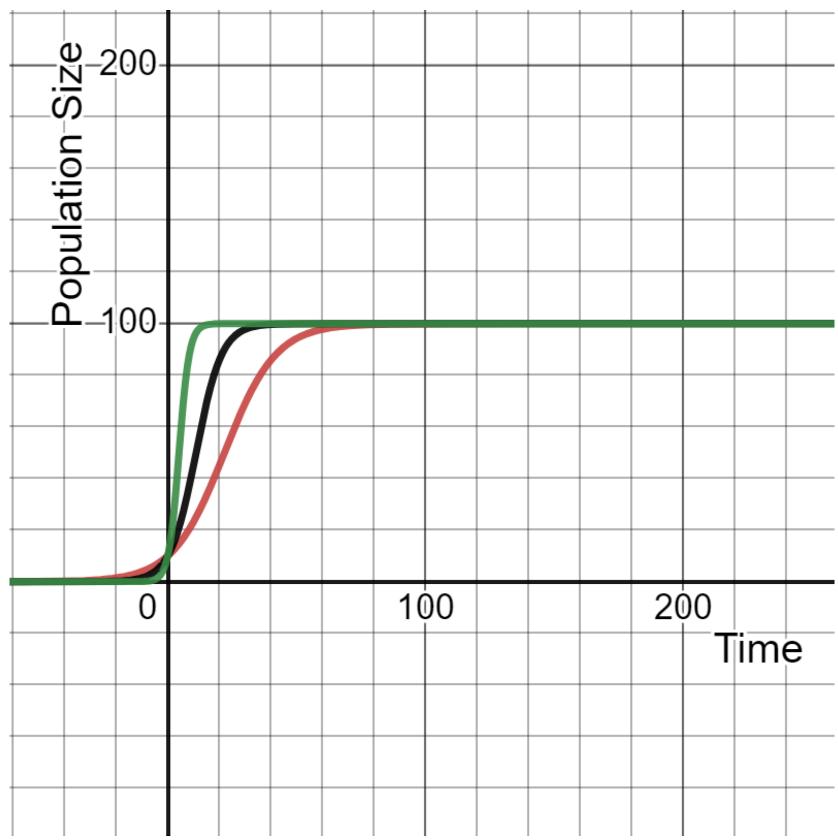

Figure 5: Green: $r = 50\%$, Black: $r = 20\%$, Red: $r = 10\%$

Each group of Rabbits can not surpass the limiting size of the population which is 100 rabbits. Higher intrinsic rates hit the ceiling function faster than lower intrinsic rates, which makes sense in context. The populations that have a higher intrinsic growth reach a higher population faster.

Consider group 1 that has an initial population of 10 and a limiting factor of 100. Consider group 2 that has a initial population of 10 and a limiting factor of 200. Both have an intrinsic growth rate of $r = 10\%$.

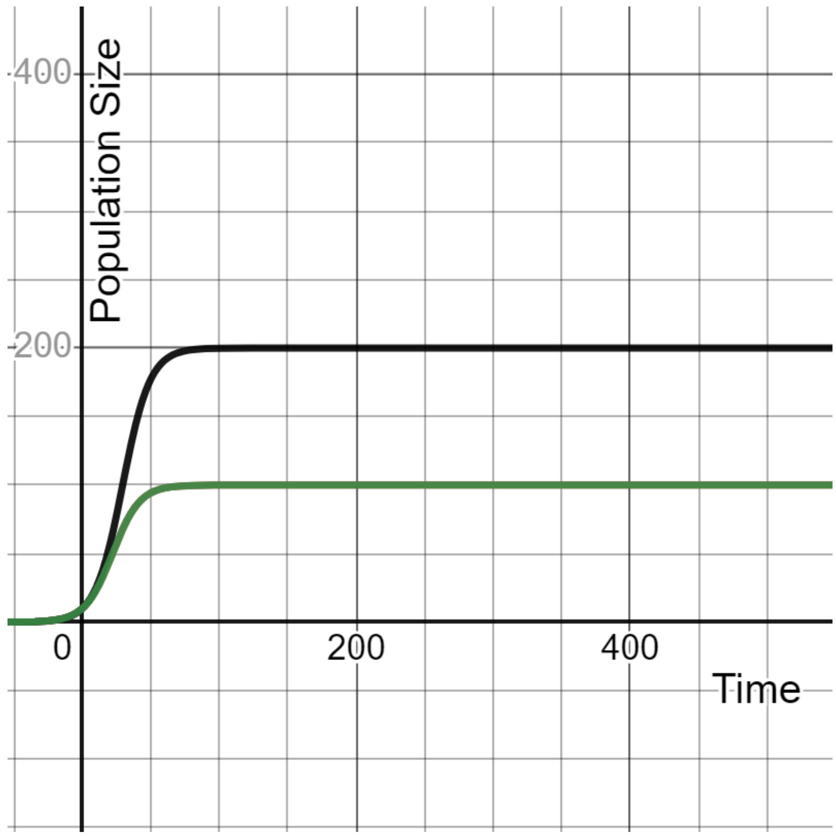

Figure 6: Group 1: Green, Group 2: Black

As Group 1 approaches the limiting factor the rate of change starts to decrease, but since group 2 has a higher limiting factor, the rate of change of group 2 continues to have exponential growth until it approaches its limiting factor.
Consider group 1 of rabbits with an initial population of 50, an intrinsic growth rate of $r = 20\%$ and a carrying capacity of 350 rabbits. Consider group 2 of rabbits with an initial population of 100, an intrinsic growth rate of $r = 15\%$ and a carrying capacity of 400 rabbits.

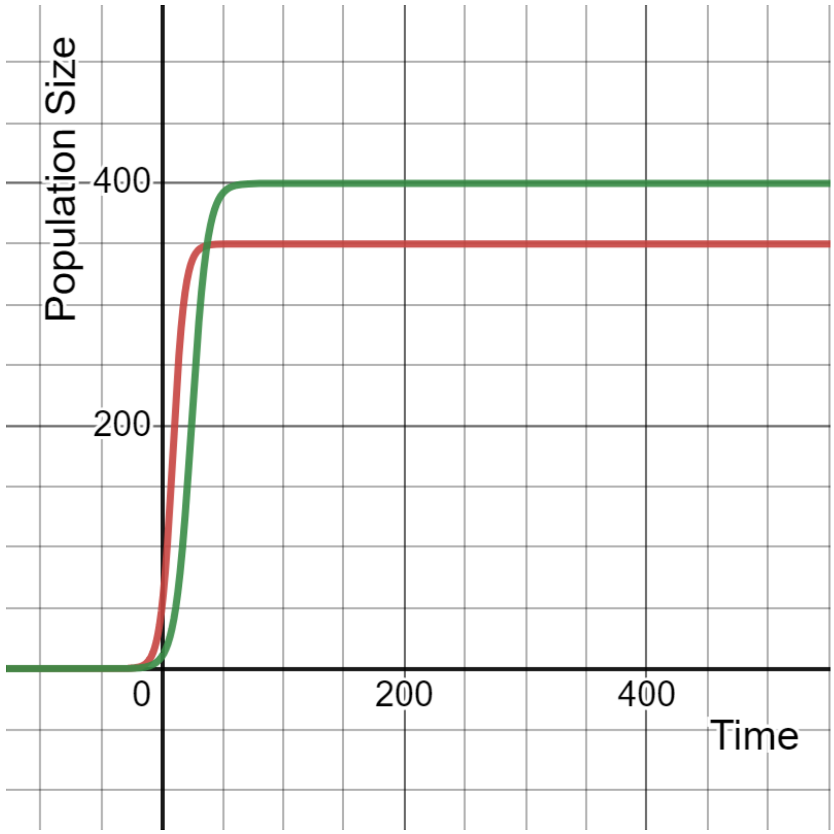

Figure 7: Group 1: Red, Group 2: Green

Since group 1 has a higher intrinsic growth rate, lower limiting factor, and it approaches its limiting factor faster. Group 2 has a higher rate of change since it has a higher limiting factor.
Consider a group of rabbits with a limiting factor of 400 and an intrinsic growth rate of $r = 2\%$.

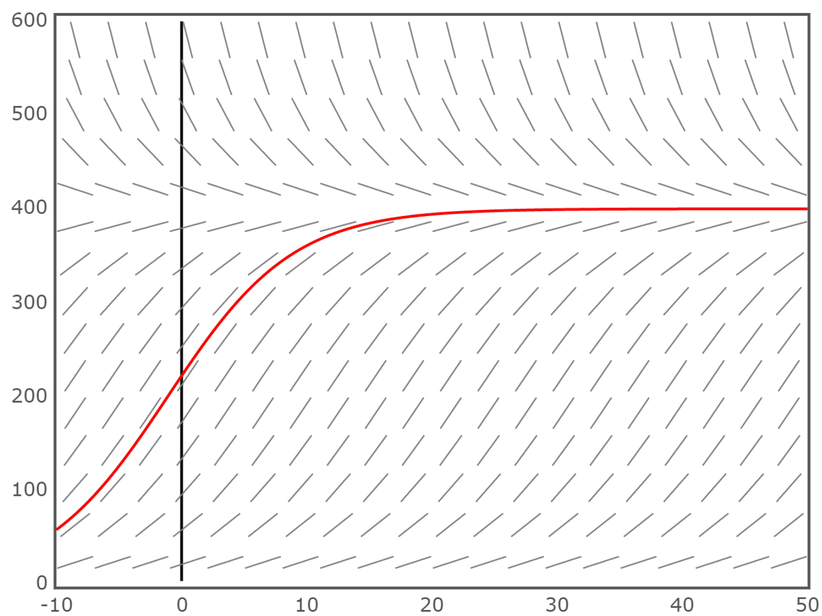

Figure 8: Directional Field

The directional field shows the limiting factor of the population is the equilibrium of the directional field. The directional field shows all possible solutions to the Verhulst model with the given conditions of a growth rate of 2% and limiting factor of 400 rabbits.

4 Discrete Method

Another way to think about the Verhulst model is the discrete logistic equation. Discrete mathematics is a separate branch of mathematics that only includes objects that have distinct separate values. Discrete mathematics lives in the integer world where values are finite. Whereas calculus, such as the Malthusian and Verhulst Models require real numbers and vary between points smoothly. The discrete logistic equation for population growth over time is

$$N_{t+1} = rN_t(1 - N_t) \tag{38}$$

N_{t+1} is the population next year, and N_t is the population of the current year, and r is the intrinsic growth rate. The population N_t ranges between 0 and 1 and signifies the proportion to the maximum population. To get the population of each subsequent year, iterative the function.

$$N_1 = rN_0(1 - N_0) \tag{39}$$

$$N_2 = rN_1(1 - N_1) \tag{40}$$

$$N_3 = rN_2(1 - N_2) \tag{41}$$

Continuing the iteration we can calculate the population for every subsequent year.

Varying Parameters

By varying the r parameter, we can observe different types behavior of that our populations take on.

When $0 < r < 1$ the population will always go extinct, independent of the initial population. Let $r = .70$ and let the proportion to the maximum population start at $N_0 = .087$. Iterating the function yields:

$$x_n = 0.087, 0.056, 0.037, 0.025, 0.017, 0.012, 0.008, 0.006, 0.004, 0.003, 0.002, 0.001, 0.001, 0.001, 0.000, 0.000, 0.000$$

Figure 9: Discrete Logistic iteration

Graphing the iteration:

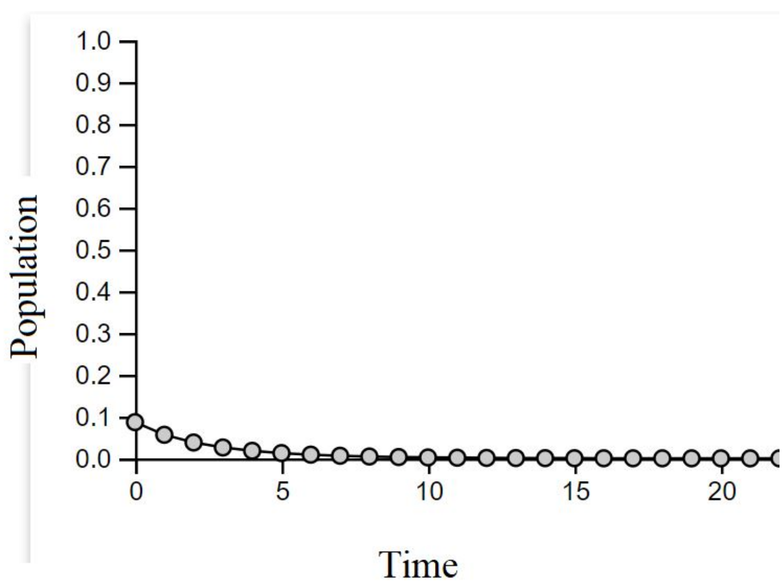

Figure 10: Iteration Graphed

The population proportion reaches an equilibrium 0 because each time we iterate the function, since $r < 1$ our death rate is greater than the birth rate each generation.

When $1 < r < 2$ the population will approach the proportion

$$N_t = \frac{r-1}{r} \tag{42}$$

regardless of initial population proportion. Suppose that we have an initial proportion to the maximum $N_0 = .836$ and an intrinsic growth rate of $r = 1.507$. Then our population should reach an equilibrium of

$$N_t = \frac{1.507 - 1}{1.507} \tag{43}$$

$$N_t = .336 \tag{44}$$

Iterating the discrete logistic function:

$$x_n = 0.837, 0.206, 0.247, 0.280, 0.304, 0.319, 0.327, 0.332, 0.334, 0.335, 0.336, 0.336, 0.336, 0.336, 0.336, 0.336, 0.336$$

Figure 11: Discrete Logistic iteration

Graphing the iteration

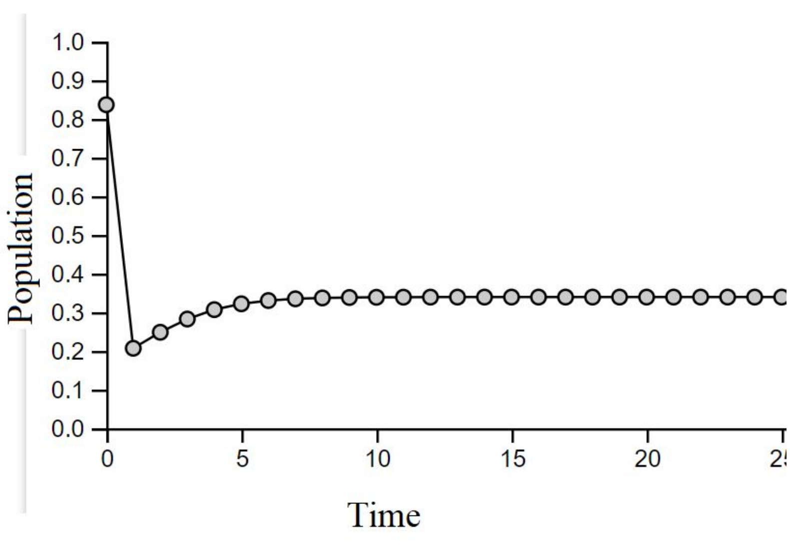

Figure 12: Iteration Graphed

Our population reaches an equilibrium of $\frac{r-1}{r}$, which matches what we see in real populations. Populations usually stay the same as long as birth and death rates are equal.
When $2 < r < 3$ the population will approach the value $\frac{r-1}{r}$, however it will fluctuate around that value, independent of starting population. For example take a starting population proportion of $N_0 = .770$ and an intrinsic growth rate of $r = 2.847$. Then our population portion should reach an equilibrium value of

$$N_t = \frac{2.847 - 1}{2.847} \tag{45}$$

$$N_t = .649 \tag{46}$$

however it should oscillate around that value until it converges. We can see this oscillation and convergence in our plot:

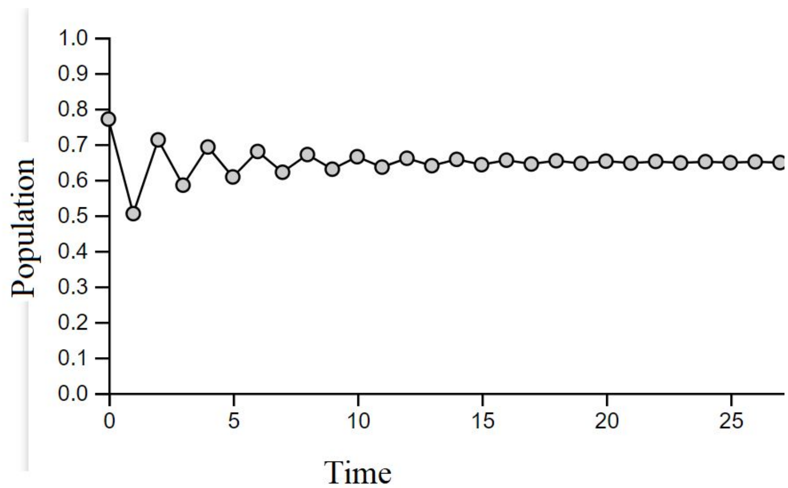

Figure 13: Population Value Oscillating till Convergence

However when r takes on specific values greater than 3, the populations permanently oscillate between two values, then four values, then 8 values, etc. This is called the period doubling cascade. When $3 < r < 3.44949$ there is permanent oscillation between 2 values.

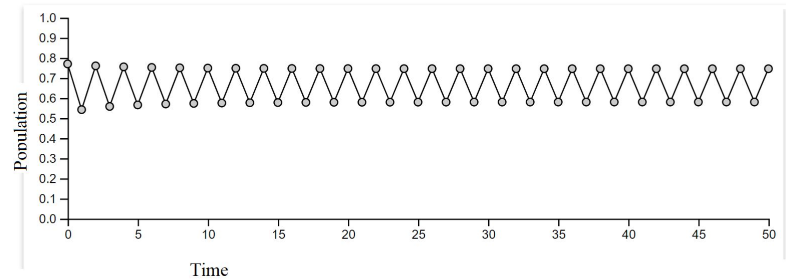

Figure 14: oscillation of period 2 $r = 3.063$

When $3.44949 < r < 3.54409$ there is permanent oscillation between 4 values.

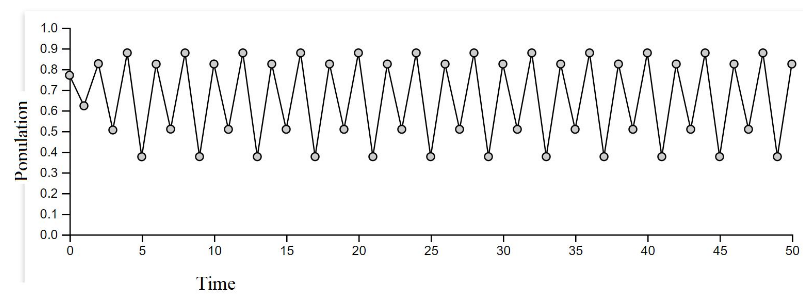

Figure 15: oscillation of period 4 when $r = 3.513$

Lets create a new graph with the intrinsic growth rate r on the x axis and the equilibrium value of the population proportion on the y axis. By doing this we can visualize the oscillations between equilibrium populations. When $r < 3$ we know that the population will reach an equilibrium. This can be observed in the new graph.

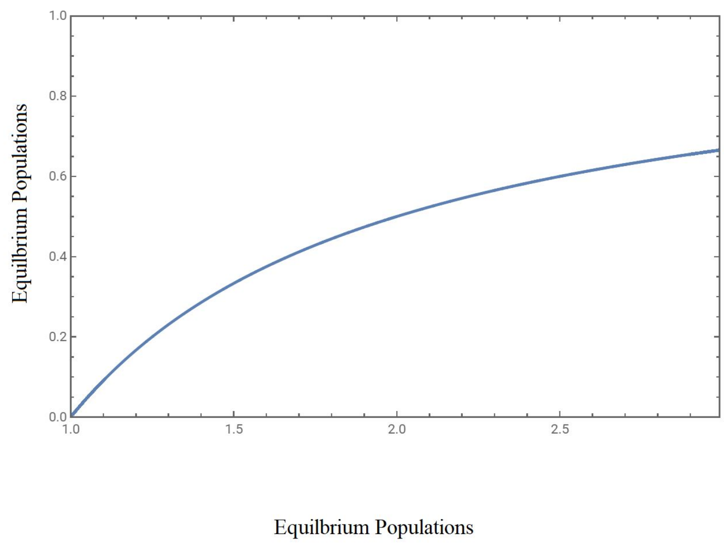

Figure 16: $r < 3$ equilibrium population points

When When $3 < r < 3.44949$ we know that r will oscillate between two values. As r increases the the distance between the higher population and lower population will increase. This can be visualized

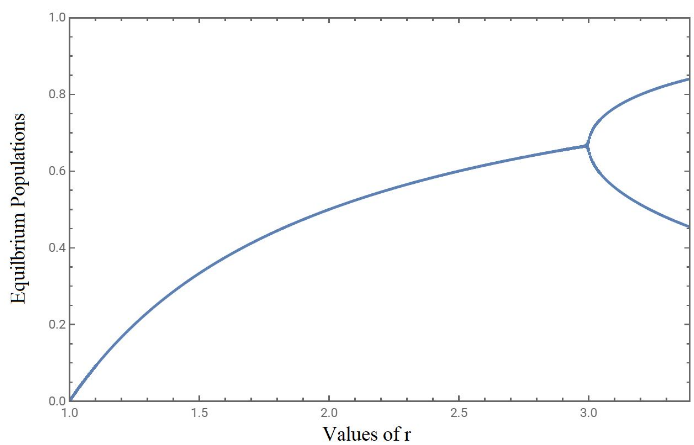

Figure 17: oscillation of period 2 when $3 < r < 3.44949$

When $3.44949 < r < 3.54409$, we will observe oscillation of period 4.

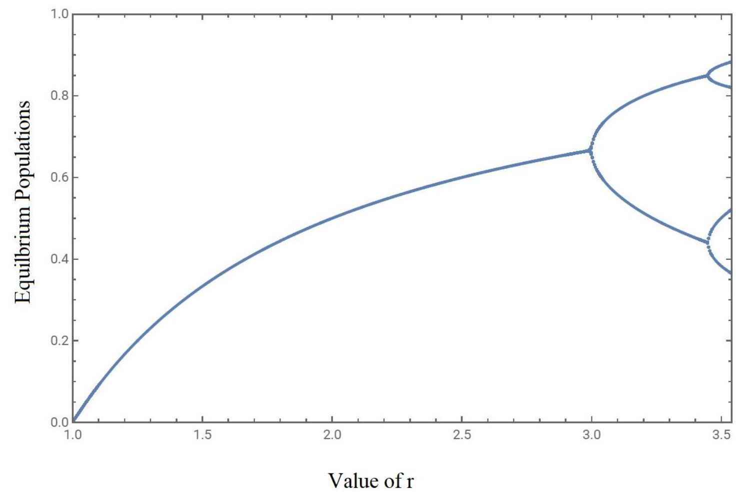

Figure 18: oscillation of period 4 when $3.44949 < r < 3.54409$

These are known as period doubling bifurcations. As the value of r keeps increasing the period doubling bifurcations come between faster intervals. For $3.54409 < r < 3.57$ we see bifurcation of periods 8, 16, 32, and 64.

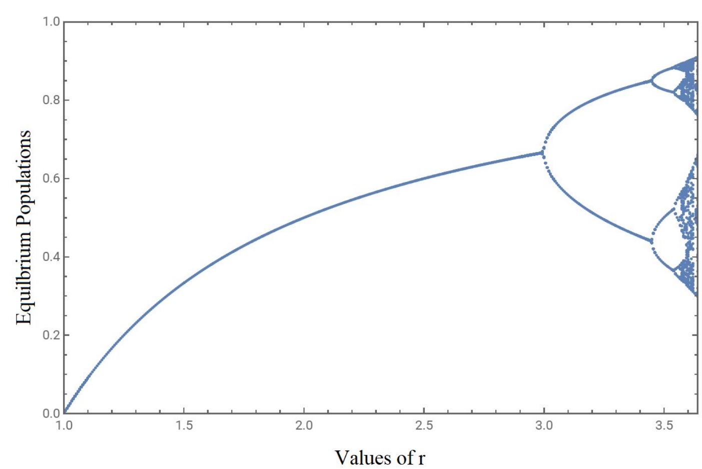

Figure 19: bifurcations of 8,16,32,64 for $r < 3.57$

Then when $r \geq 3.57$ we are introduced to chaos. The population never settles on a fixed value. We can observe this in our iteration graph, and in our bifurcation graph.

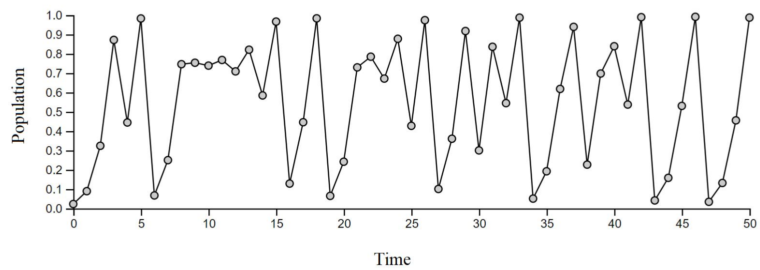

Figure 20: Population Points for when $r \geq 3.57$

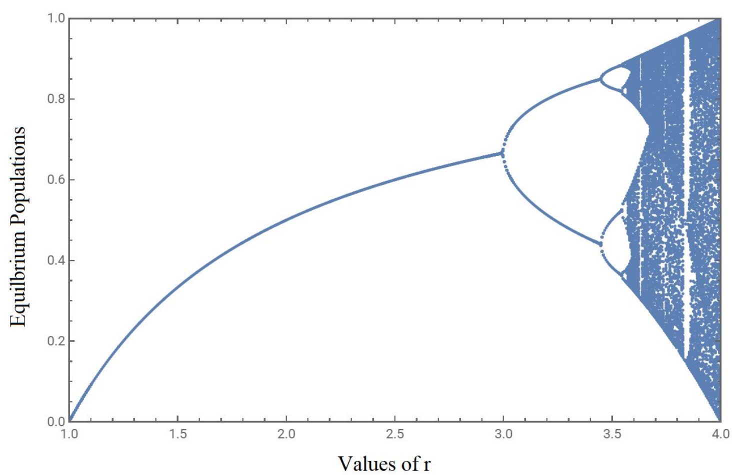

Figure 21: Population values for when $r \geq 3.57$

In figure 21, we can see the chaos of the discrete logistic equation unfold. Depending on what value of r you select you will be able to find a period of oscillation of any quantity.

5 Discussion

Through this paper we explored both continuous and discrete logistic equations. We introduced two continuous models, the Malthusian and Verhulst model. The Malthusian model describes the rate of change of a population with respect to time is proportional to the population itself. We then solved this ODE and saw that exponential growth satisfies the Malthusian equation. We explored how changing the initial conditions and proportionality constant affected our solution. We then explored the Verhulst model which dampens the exponential growth of the Malthusian model to more accurately describe population growth. We explored how the dampening term works, and solved the ODE. We saw that the sigmoid function satisfies this equation. We then explored how changing the proportionality constant, limiting factor, and initial population affected our solution. Next we explored the discrete model of the logistic equation. We explained how the iteration function worked and looked at how changing the intrinsic growth rate affected the yearly population. If $r < 1$ population goes to 0. If $1 < r < 2$ our population converges to the value $\frac{r-1}{r}$. If $2 < r < 3$ the population oscillates around the value $\frac{r-1}{r}$ until it finally converges. We then explored values of r that introduce permanent oscillations of periods, 2, 4, 8, 16, 32, and 64. Finally when $r \geq 3.57$ the model created chaos. We visualized each value of r with a bifurcation diagram. The overall goal of this project was to help explain, visualize, and understand how different models can affect our understanding of the same problem.

6 Sources

https://www.khanacademy.org/math/ap-calculus-bc/bc-differential-equations-new/bc-7-9/v/modeling-population-with-differential-equations
https://en.wikipedia.org/wiki/Malthusian$_{g}rowth_{m}odel : \quad : text = A$
$https : //www.complexity - explorables.org/flongs/logistic/$
$https : //www.youtube.com/watch?v = ovJcsL7vyrkt = 407s$