1 Introduction

Despite that planar geometry plays a less important role in higher level of math, it is intrinsically beautiful and has prepossessed a lot of famous mathematicians. Recorded in the Euclid's *Elements*, planar geometry was a protagonist of the math community dating back to Ancient Greece. Even if nowadays there're less ongoing research in planar geometry, the subject still fascinates math people and occasionally appears in the Putnam contest. In this handout, we introduce the Euclid's Postulates and present several problems in planar geometry. Some of them are less in the spirit of Euclid, being based on algebraic or combinatorial considerations. Here "imagination is more important than knowledge" (A. Einstein).

2 Resources

The problems above and below are shamelessly ripped from the books *Putnam and Beyond* by Andreescu and Gelca and *Problem Solving Strategies* by Arthur Engel. We also stole some problems from the Putnam competition. Many other problem solving books have induction sections as well. These problems are arranged roughly in order of difficulty, but difficulty is relative to each person so take it with a grain of salt.

3 Euclid's Postulates

- A straight line segment can be drawn joining any two points.
- Any straight line segment can be extended indefinitely in a straight line.
- Given any straight lines segment, a circle can be drawn having the segment as radius and one endpoint as center.
- All right angles are congruent.
- If two lines are drawn which intersect a third in such a way that the sum of the inner angles on one side is less than two Right Angles, then the two lines inevitably must intersect each other on that side if extended far enough. This postulate is equivalent to what is known as the Parallel Postulate.

4 Easy

Exercise 1. Prove the Pythagorean Theorem. How do you tell whether a triangle is acute, right, or obtuse?

Exercise 2. Prove that the midpoints of the sides of a quadrilateral form a parallelogram.

Exercise 3. In triangle ABC, $AB = 13$, $BC = 14$, and $CA = 15$. Distinct points D, E, and F lie on segments BC, CA, and DE, respectively, such that $AD \perp BC$, $DE \perp AC$, and $AF \perp BF$. The length of segment DF can be written as mn , where m and n are relatively prime positive integers. What is $m + n$?

Exercise 4. A straight line cuts the asymptotes of a hyperbola in points A and B and the curve in points P and Q. Prove that $AP = BQ$.

Exercise 5 (2010 B2). Given A, B, and C are noncollinear points in the plane with integer coordinates such that the distances AB, AC, and BC are integers, what is the smallest possible value of AB?

Exercise 6. In the figure shown below, circle B is tangent to circle A at X, circle C is tangent to circle A at Y , and circles B and C are tangent to each other. If $AB = 6, AC = 5$, and $BC = 9$, what is AX?

1.png

5 Medium

Exercise 7. Let $ABCD$ be a convex quadrilateral, and define P_1, P_2, P_3, P_4, P_5, and P_6 to be the midpoints of line segments AB, BC, CD, DA, AC, and BD respectively. Prove that lines P_1P_3, P_2P_4, and P_5P_6 all intersect in a single point.

Exercise 8 (2012 A1). Let $d_1, d_2, \ldots, d_{12}$ be real numbers in the open interval $(1, 12)$. Show that there exist distinct indices i, j, k such that d_i,d_j, d_k are the side lengths of an acute triangle.

Exercise 9 (1998 B2). Given a point (a, b) with $0 < b < a$, determine the minimum perimeter of a triangle with one vertex at (a, b), one on the x-axis, and one on the line $y = x$. You may assume that a triangle of minimum perimeter exists.

Exercise 10 (2015 A1). Let A and B be points on the same branch of the hyperbola $xy = 1$. Suppose that P is a point lying between A and B on this hyperbola, such that the area of the triangle APB is as large as possible. Show that the region bounded by the hyperbola and the chord AP has the same area as the region bounded by the hyperbola and the chord PB.

Exercise 11 (1992 CMO). A convex quadrilateral $ABCD$ is inscribed in a circle with center O. The diagonals AC, BD of $ABCD$ meet at P. Circumcircles of ABP and CDP meet at P and Q (O, P, Q are pairwise distinct). Show that the angle of OQP is $90°$.

6 Hard

Exercise 12. Acute-angled triangle ABC is inscribed into circle Ω. Lines tangent to Ω at B and C intersect at P. Points D and E are on AB and AC such that PD and PE are perpendicular to AB and AC respectively. Prove that the orthocenter of triangle ADE is the midpoint of BC.

Exercise 13 (2017 B5). A line in the plane of a triangle T is called an equalizer if it divides T into two regions having equal area and equal perimeter. Find positive integers $a > b > c$, with a as small as possible, such that there exists a triangle with side lengths a, b, c that has exactly two equalizers.

Exercise 14 (2018 A6). Suppose that A, B, C, and D are distinct points, no three of which lie on a line, in the Euclidean plane. Show that if the squares of the lengths of the line segments AB, AC, AD, BC, BD, and CD are rational numbers, then the quotient

$$\frac{\text{area}(\Delta ABC)}{\text{area}(\Delta ABD)}$$

is a rational number.

7 Hint

- Various proofs
- Use coordinates
- Draw the picture
- Write down the standard form of a hyperbola
- First guess the answer, and then prove that no smaller values are possible
- The intersection points are important
- Make use of the result of Exercise 2
- What's the property for an acute triangle
- Use reflections
- Shoelace's formula, and then AM-GM
- Connect AO, AQ, DO, and DQ

8 Appendix

Here are important things in middle/high school Olympiads competitions. Some of the results might be used to solve planar geometry problems in the Putnam contest.

- Properties of parallelism, orthogonality, similar triangles, and cyclic quadrilateral
- The Law of Sines, The Law of Cosines, Heron's Formula
- Intersecting Chords Theorem, Tangent Secant Theorem
- Pythagorean Theorem
- Geometric Mean Theorem
- Stewart's Theorem
- Menelaus's Theorem, Ceva's Theorem
- Ptolemy's Theorem, Simson's Theorem
- The incenter, circumcenter, centroid, orthocenter, and escenter of a triangle
- Radical axis of two circles
- Euler's Theorem, Fermat Point, Nine-point circle
- Coordinate geometry: line, circle, parabola, ellipse, hyperbola, and second definitions