Markov Categories and Entropy

Paolo Perrone, IEEE Transactions on Information Theory, 2023

Utku Boduroğlu, Drew McNeely, and Nico Wittrock

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April 11, 2024

What Markov categories are <u>not</u>

Definition

A symmetric monoidal category with

objects: commutative comonoids $(X, \mathrm{del}_X, \mathrm{copy}_X)$

morphisms: morphisms of comonoids

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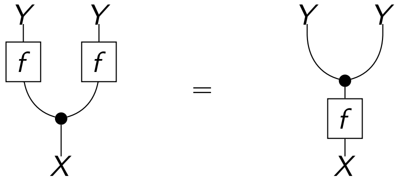

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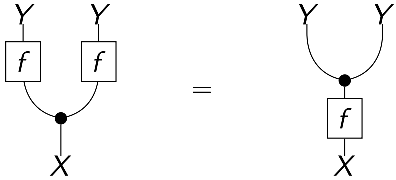


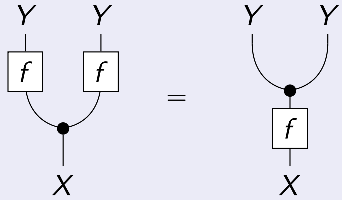
run f twice
with same input

run f once
and copy output

Determinism in Markov categories

Definition

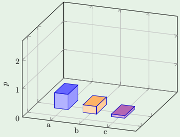
$f : X \to Y$ is deterministic if



Determinism in FinStoch

Recall: $p : I \to A := \{a, b, c\}$ is a distribution $\{p(a), p(b), p(c)\}$.

Example



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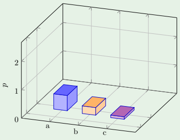

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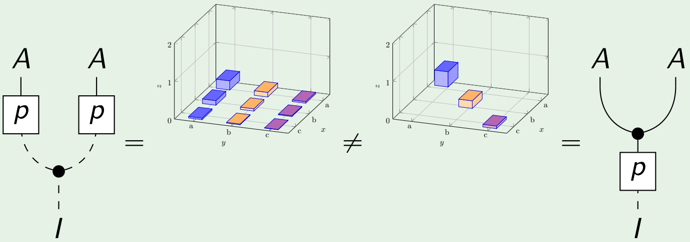

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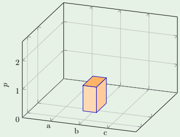

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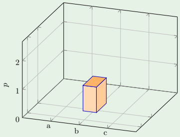

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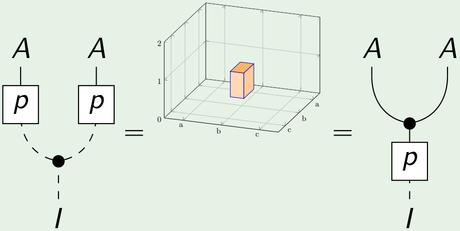
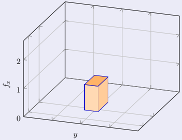


Determinism in FinStoch

Recall: $f : X \to Y$ has distributions f_x on Y.

Proof.

f is deterministic $\Leftrightarrow$ all f_x of form 

$\Leftrightarrow$ f corresponds to Set-function

Determinism and Products

Theorem

Let C *be a Markov category.*

The deterministic morphisms form Markov subcategory $\mathsf{C}_{det} \subset \mathsf{C}$.

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Example

$$\mathsf{FinStoch} \supset \mathsf{FinStoch}_{\mathsf{det}} \cong \mathsf{FinSet}$$

Products and Determinism

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Let C be cartesian monoidal (i.e. $\otimes$ is product functor).

C is Markov.

All morphisms are deterministic.

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Theorem

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Example (Markov categories without uncertainty)

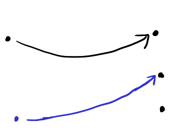
Set, FinSet, Meas, BorelMeas, ...

Probability Monads

…introduce uncertainty to cartesian Markov categories

Probability Monads

Example (Powerset Monad)

Set is deterministic 

Probability Monads

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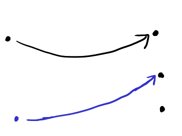
Set is deterministic 

$$\rightsquigarrow$$

(slightly adjusted) powerset monad $2^{\circ} : \mathsf{Set} \to \mathsf{Set}$

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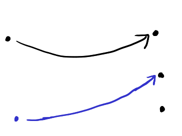
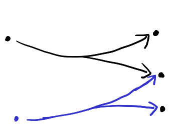
Set is deterministic 

$\rightsquigarrow$

(slightly adjusted) powerset monad $2^{\circ} : \mathsf{Set} \to \mathsf{Set}$

$\rightsquigarrow$

Kleisli category $\mathsf{Set}_{2^{\circ}}$ describes <u>possibility</u> 

Probability Monads

Theorem

For a cartesian monoidal category C *with monad* $T : \mathsf{C} \to \mathsf{C}$ *which is*

- *compatible with* I ($\rightsquigarrow$ *powerset monad was adjusted*)
- *compatible with* $\otimes$

its Kleisli category C_T *is Markov.*

Probability Monads

Example

$$\begin{tikzcd}
\mathsf{Set} \arrow[d, rightsquigarrow, "2^{\circ}"] & \mathsf{FinSet} \arrow[d, rightsquigarrow, "2^{\circ}"] & \mathsf{Meas} \arrow[d, rightsquigarrow, "\text{Giry monad}"] & \mathsf{BorelMeas} \arrow[d, rightsquigarrow, "\text{Giry monad}"] \\
\mathsf{Poss} & \mathsf{FinPoss} & \mathsf{Stoch} & \mathsf{BorelStoch}
\end{tikzcd}$$

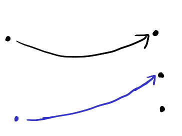
Probability Monads

Probability Monads

Uncertainty captured by morphisms!

Probability Monads

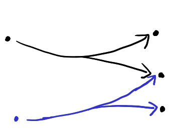
Example (Powerset Monad)



Set is cartesian monoidal $\Rightarrow$ morphisms are deterministic

$$\text{Powerset monad } 2^{\bullet} : \mathsf{Set} \to \mathsf{Set},$$
$$X \mapsto 2^X := \{M \subseteq X\}$$

Kleisli category $\mathsf{Set}_{2^\bullet}$ has morphisms:



$$f : X \to 2^Y \quad \text{i.e. } f(x) \subseteq Y$$

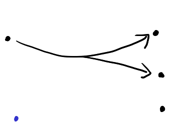
As Pablo stated: describes possibility

Probability Monads

Example (Powerset Monad)

Problem: Kleisli category $\mathrm{Set}_{2^\bullet}$ has morphism:

$$f : X \to 2^Y \quad \text{with} \quad f(x_i) = \varnothing$$



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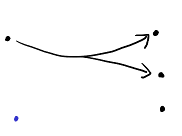

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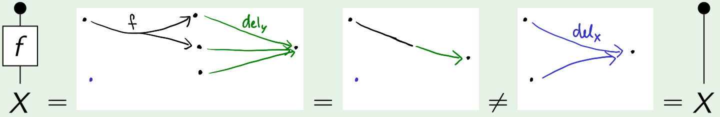

$\Rightarrow$ $\mathrm{Set}_{2^\bullet}$ is <u>not</u> Markov!

$$\rightsquigarrow \text{adjust monad} \quad T : \mathrm{Set} \to \mathrm{Set},$$
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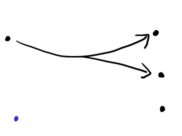

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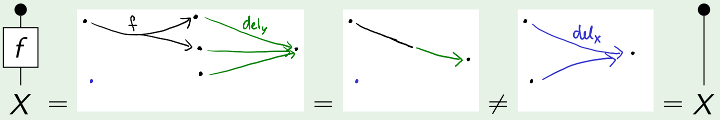

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$\Rightarrow \mathrm{Set}_T =: \mathrm{Poss}$ is Markov. With uncertainty.

Outline

This is just me fooling around with Beamer to get a feel for it. None of the contents should be taken seriously at this point.

1. Introduction to Markov Categories
2. Useful ways to construct Markov categories
   - Randomness in Markov categories
   - Introducing Uncertainty by Probability Monads
3. Properties and additional axioms
4. Bonus Slides
   - Powerset monad
5. Markov Categories
   - Axioms
   - Examples
6. Enriched Categories
7. Divergences
   - Entropy

Definition of Markov Category

Definition

A Markov category is a semi-Cartesian category in which every object is a commutative comonoid.

Example Markov category

Okay if we create an example, it absolutely needs to involve the cloud cover forecast during the total solar eclipse.

Enriched Categories

So yeah an enriched category is like a category with more structure on the morphisms. Instead of having hom-sets, ya got yourself some hom-objects.

Divergence

Aight, so a divergence is like a metric for probability distributions...
Except it ain't a metric, cause it ain't <u>sym</u>metric.

Entropy

Entropy is a measure of how "random" a probability distribution is, given through the divergence between both sides of the defining equation for deterministic kernels. Neat!!