CSE 5361 Homework 4 Spring 2023

due 02/16/2023

Problem 4.1.1. Use the Lagrange interpolation process to obtain a polynomial of least degree that assumes the values:

x	0	2	3	4
y	7	11	28	63

Problem 4.1.4. Verify that the polynomials

$$p(x) = 5x^3 - 27x^2 + 45x - 21$$
$$q(x) = x^4 - 5x^3 + 8x^2 - 5x + 3$$

interpolate the data

x	1	2	3	4
y	2	1	6	47

and explain why this does not violate the uniqueness part of the theorem on existence of polynomial interpolation.

Problem 4.1.7. Compute the following divided-difference tables, and use them to obtain polynomials of degree 3 that interpolate the function values indicated:

b .

x	$f[\,]$	$f[\,,\,]$	$f[\,,\,,\,]$	$f[\,,\,,\,,\,]$
-1	2			
1	-4		2	
3	6			
		2		
5	10			

Write the final polynomials in a form most efficient for computing.

Problem 4.1.15. It is suspected that the table

x	-2	-1	0	1	2	3
y	1	4	11	16	13	-4

comes from a cubic polynomial. How can this be tested? Explain.

Problem 4.2.10. Let the function $f(x) = \ln(x)$ be approximated by an interpolation polynomial of degree 9 with 10 nodes uniformly distributed on the interval $[1, 2]$. What bound can be placed on the error?

Computer Problem 4.1.3. Write a simple program that interpolate e^x by a polynomial of degree 10 on $[0, 2]$ and then compares the polynomial to $f(x) = e^x$ at 100 points.

Computer Problem 4.1.9. Write a procedure for carrying out inverse interpolation to solve equations of the form $f(x) = 0$. Test it out on the following data:

x	1.4	1.25
y	3.7	3.9

4.1.1)

Problem 4.1.1. Use the Lagrange interpolation process to obtain a polynomial of least degree that assumes the values:

x	0	2	3	4
y	7	11	28	63

For n points we need a polynomial of at most degree n-1.

$$l_0 = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} = \frac{(x-2)(x-3)(x-4)}{(0-2)(0-3)(0-4)} = \frac{(x-2)(x-3)(x-4)}{-24}$$

$$l_1 = \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} = \frac{(x)(x-3)(x-4)}{(2-0)(2-3)(2-4)} = \frac{(x)(x-3)(x-4)}{4}$$

$$l_2 = \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} = \frac{(x)(x-2)(x-4)}{(3-0)(3-2)(3-4)} = \frac{(x)(x-2)(x-4)}{-3}$$

$$l_3 = \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} = \frac{(x)(x-2)(x-3)}{(4-0)(4-2)(4-3)} = \frac{(x)(x-2)(x-3)}{8}$$

$$p(x) = \frac{(x-2)(x-3)(x-4)}{-24}(7) + \frac{(x)(x-3)(x-4)}{4}(11)$$

$$+ \frac{(x)(x-2)(x-4)}{-3}(28) + \frac{(x)(x-2)(x-3)}{8}(63)$$

4.1.4)

Problem 4.1.4. Verify that the polynomials

$$p(x) = 5x^3 - 27x^2 + 45x - 21$$
$$q(x) = x^4 - 5x^3 + 8x^2 - 5x + 3$$

interpolate the data

x	1	2	3	4
y	2	1	6	47

and explain why this does not violate the uniqueness part of the theorem on existence of polynomial interpolation.

* partially evaluated polynomials using Julia

$p(x)$:

$$p(1) = 5(1)^3 - 27(1)^2 + 45(1) - 21 = 2$$
$$p(2) = 5(2)^3 - 27(2)^2 + 45(2) - 21 = 1$$
$$p(3) = 5(3)^3 - 27(3)^2 + 45(3) - 21 = 6$$
$$p(4) = 5(4)^3 - 27(4)^2 + 45(4) - 21 = 47$$

$\forall (x_i, y_i)$ $p(x_i) = y_i$ so, p interpolates the data

$q(x)$:

$$q(1) = (1)^4 - 5(1)^3 + 8(1)^2 - 5(1) + 3 = 2$$
$$q(2) = (2)^4 - 5(2)^3 + 8(2)^2 - 5(2) + 3 = 1$$
$$q(3) = (3)^4 - 5(3)^3 + 8(3)^2 - 5(3) + 3 = 6$$
$$q(4) = (4)^4 - 5(4)^3 + 8(4)^2 - 5(4) + 3 = 47$$

$\forall (x_i, y_i)$ $q(x_i) = y_i$ so, q interpolates the data

The theorem on the existence of interpolating polynomials only states there to be a unique polynomial of at most degree $n-1$ that interpolates n data points.

So, it doesn't make any claims about the uniqueness of these polynomials with a degree $\geq n$.

4.1.7)

Problem 4.1.7. Compute the following divided-difference tables, and use them to obtain polynomials of degree 3 that interpolate the function values indicated:

b .

x	$f[\,]$	$f[\,,\,]$	$f[\,,\,,\,]$	$f[\,,\,,\,,\,]$
-1	2			
1	-4		2	
3	6			
		2		
5	10			

Write the final polynomials in a form most efficient for computing.

	x	f[]	f[,]	f[,,]	f[,,,]
0	-1	2			
			-3		
1	1	-4	5	2	-7/12
				-3/2	
2	3	6			
			2		
3	5	10			

$$f[x_i, x_{i+1}, \dots, x_j] = \frac{f[x_{i+1}, x_{i+2}, \dots, x_j] - f[x_i, \dots, x_{j-1}]}{x_j - x_i}$$

$$f[0,1] = \frac{f[1] - f[0]}{x_1 - x_0} = \frac{-4-2}{1-(-1)} = \frac{-6}{2} = -3$$

$$f[1,2] = \frac{f[2] - f[1]}{x_2 - x_1} = \frac{6-(-4)}{3-1} = \frac{10}{2} = 5$$

$$f[1,2,3] = \frac{f[2,3] - f[1,2]}{x_3 - x_1} = \frac{2-5}{5-3} = \frac{-3}{2}$$

$$f[0,1,2,3] = \frac{f[1,2,3] - f[0,1,2]}{x_3 - x_0} = \frac{-3/2 - 2}{5-(-1)} = \frac{-\frac{7}{2}}{6}$$

$$= \frac{-7}{12}$$

$$p(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + a_3(x-x_0)(x-x_1)(x-x_2)$$

$a_0 = 2$

$a_1 = -3$

$a_2 = 2$

$a_3 = -7/12$

$$p(x) = 2 - 3(x+1) + 2(x+1)(x-1) - \frac{7}{12}(x+1)(x-1)(x-3)$$

4.1.15)

Problem 4.1.15. It is suspected that the table

x	-2	-1	0	1	2	3
y	1	4	11	16	13	-4

comes from a cubic polynomial. How can this be tested? Explain.

If interpolating 6 datapoints still results in a degree 3 polynomial then we know this data came from a cubic polynomial.

Using my Julia implementation of the divided differences method I obtain

$a_0 = 1$

$a_1 = 3$

$a_2 = 2$

$a_3 = -1$

$a_4 = 0$

$a_5 = 0$

$$y = p(x) = 1 + 3(x+2) + 2(x+2)(x+1) - (x+2)(x+1)x$$

Since $p(x)$ is a cubic, we conclude the data came from a cubic polynomial.

4.2.10)

Problem 4.2.10. Let the function $f(x) = \ln(x)$ be approximated by an interpolation polynomial of degree 9 with 10 nodes uniformly distributed on the interval $[1, 2]$. What bound can be placed on the error?

$$|f(x) - p(x)| \leq \frac{1}{4(n+1)} M h^{n+1}$$

$$h = \frac{b-a}{n}$$

$$|f(x) - p(x)| \leq \frac{1}{4(10+1)} M\left(\frac{2-1}{10}\right)^{11} \leq \frac{M}{44(10)^{11}}$$

$$|f^{(n+1)}(x)| \leq M$$

$$\frac{3628800}{x^{11}} \leq M$$

M can be largest when $x = 1$

$$\frac{3628800}{(1)^{11}} \leq M$$

$$3628800 \leq M$$

$$|f(x) - p(x)| \leq \frac{M}{44(10)^{11}} \leq \frac{3628800}{44(10)^{11}} \leq 8.247 \times 10^{-7}$$