Stats100A **Summer 2024**

Week 1: (Ungraded) Challenge Problems 1 - Solutions

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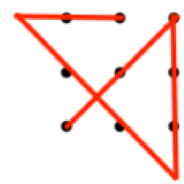
1.1 Solutions

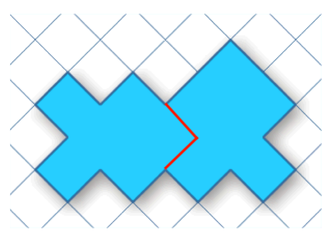
Problem 1 (Thinking Outside the Box): Given the 3×3 grid below, place your pencil at any point. Then, without lifting your pencil you must draw four straight lines to cross out all the 9 dots.

Solution: You can draw lines beyond the implicit boundaries of the grid, hence "thinking outside of the box". See the solution below:

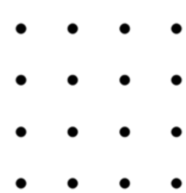


Problem 2 (One Cut - Two Shapes): Given the shape below and using a single curve to cut the shape (the curve can be angled or have as many bends as you wish), cut the shape into two identically shaped pieces.

Solution: There are many solutions to this problem. Here is one such solution:

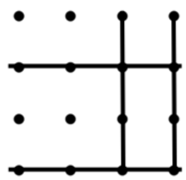


Problem 3 (Dots and Rectangles): Given a 4×4 grid of dots, how many rectangles can be formed by connecting 4 of the dots, such that the sides of rectangle are parallel to the sides of the grid?



One possible approach to this problem is to break it down into cases of rectangles of dimensions 1×1, 1×2, 2×1, ..., 4×4. And then count how many rectangles there are for each case and total them up for you final answer.

While this works it's also time consuming. The astute of you may have made a clever observation. Suppose I placed two horizontal lines on the grid, and then two vertical lines on the grid, then the intersection of these lines will form a rectangle. See below for an example of this:



Thus to determine the total number of rectangles, we need to find the number of ways to place two horizontal lines and two vertical lines on the grid. Notice that there are 4 choose 2 ways to place the horizontal lines, denoted $\binom{4}{2} = 6$. Likewise, there are 6 ways to place the two vertical lines. To reiterate there are 6 ways to place the horizontal lines and then 6 ways to place the vertical lines, for a total of $6 \times 6 = 36$ ways to form a rectangle on the grid.

Problem 4 (A Biased Coin): I give you a coin that is biased in favor of heads. What can you do in order to make it into a fair coin?

Solution: Let's consider two coin flips and look at the possible states $\{HH, HT, TH, TT\}$. Since the coin is biased in favor of heads, thus HH is most likely and TT is least likely. Also we consider the coin flips to be independent so TH and HT are just as likely. So here is one proposal to make the bias coin fair:

Flip the coin twice. If you get HH or TT disregard the result and flip twice again. Otherwise if you get HT relabel this as H and if you get TH relabel this as T. Since HT is just as likely as TH, this ensure that our relabeling trick assigns equal probabilities to H and T, which makes the coin fair again.

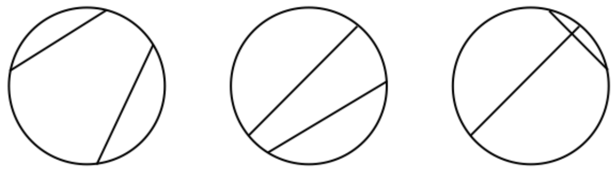
Problem 5 (Half Your Height): At what age do you expect the average person to be half of their (full-grown) height. Without looking up a statistic on this, can you come up with a plausible way to deduce this answer?

> *Remark:* There isn't really a "correct" way to do this. This problem is more to test if you can make a reasonable conclusion based on your basic prior knowledge about the world.

Solution: It's commonly stated that you are fully grown around the age of 18. Let's just say the average height of the typical adult is 6 feet. Well if you were half this height at the age of 9 then you would be 3 feet at the age of 9. This seems off, 3 is more associated with a toddler, say around the age of $9/2 = 4.5 \approx 4$ years old. Now I stated 6 feet was the average height, but realistically that's probably too tall and the average height is maybe closer to 5 feet 6 inches. So 4 years old might still be an estimate, so I would argue you're about half your height around an age between 3 and 4.

> The real life statistics is more around the range of 2 to 3.

Problem 6 (A Circle and Two Chords): I have a unit circle and draw two chords at random. What is the probability that the chords intersect? (Third circle below depicts two chords intersecting).



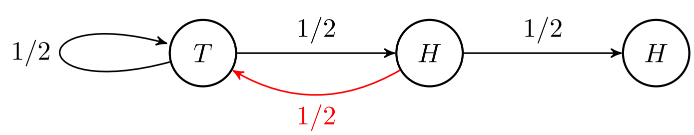
Solution: Let's place 4 points on the circle in an arbitrary position. How many ways can you draw the pairs of chords given these 4 points? You should see that there are only 3 ways to do this, one of which depicts two chords intersecting. Thus there is a $\frac{1}{3}$ probability of the chords intersecting. Since the choice of the 4 points was arbitrary, this argument applies for any arrangements of 4 points in general position and thus for the distribution of drawing two chords on the circle. Therefore the probability of two chords intersecting is $\frac{1}{3}$.

Problem 7 (HH vs. HT): I can choose to flip a fair coin until I see two heads in a row, denoted HH. Or I can choose to flip the coin until I see a heads followed by a tail, i.e. HT. On average should it take more flips to see HH or more flips until you see HT? Or should we expect roughly the same number of flips?

Solution: This problem is counterintuitive. The answer is that it take more flips on average to see HH than it does HT. To get an intuitive understanding consider the following argument:

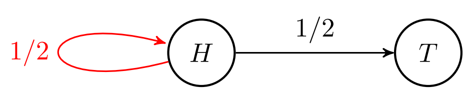
Case HH:

Suppose we just flipped a H (center circle in figure below). Now if we flip a coin and fail to get H (i.e. T), then at best we are 2 flips away from obtaining HH.



Case HT:

Suppose we just flipped a H (left circle in figure below). Now if we flip a coin and fail to get T (i.e. H), then at best we are 1 flips away from obtaining HT.



So it's more likely that you see HT before you see HH and therefore we expect to see HT in fewer flips than HH.