Stats100A **Summer 2024**

Week 4: (Ungraded) Challenge Problems 4 - Solutions

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4.1 Solutions

Problem 1 (Repulsive Probability): The event A is said to be repelled by the event B if $P(A|B) < P(A)$, and it is said to be attracted to B if $P(A|B) > P(A)$. Show that if B attracts A, then A attracts B, and B^c repels A. If A attracts B, and B attracts C, does A attract C?

Solution: Suppose A attracts B, then by definition we have that $P(B|A) > P(B)$. Also recall that $P(B|A) = \frac{P(A \cap B)}{P(A)}$. So we have that $\frac{P(A \cap B)}{P(A)} > P(B)$, and multiplying both sides of the inequality by $\frac{P(A)}{P(B)}$, we have $\frac{P(A \cap B)}{P(B)} > P(A)$. Therefore $P(A|B) > P(A)$, which proves that A attracts B. To show that B^c repels A, notice that by the total law of probability that $P(A) = P(A|B^c)P(B^c) + P(A|B)P(B)$, and since B attracts A, we have the inequality:

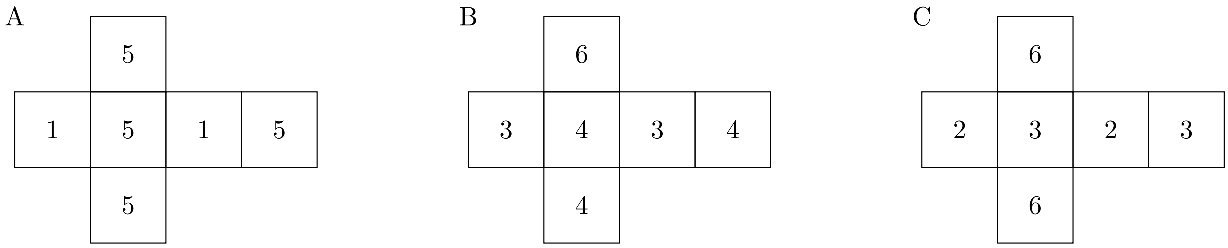
$$
\begin{aligned}
P(A) &> P(A|B^c)P(B^c) + P(A)P(B) \\
P(A)(1 - P(B)) &> P(A|B^c)P(B^c) \\
P(A)P(B^c) &> P(A|B^c)P(B^c) \\
P(A) &> P(A|B^c)
\end{aligned}
$$

Therefore A is repelled by B^c.
Lastly, the transitive property doesn't hold. Suppose $A \cap C = \emptyset$, then $P(A|C) = 0 \leq P(A)$.

Problem 2 (Around The Quarter): A quarter is glued to a tabletop. You then place another quarter tangent to the tabletop quarter and begin rotating it around the tabletop quarter. You keep rotating it until your quarter reaches the point it started at. How many times will your quarter rotate?

Solution: The coin rotates twice.Once relative to the stationary quarter, and once more with respect to it's own revolution.

Problem 3 (Rigged Dice): I have dice, A, B, C with face values depicted below:



Show that $P(A > B) = P(B > C)$. Is it the case that $P(A > C) = P(A > B) = P(B > C)$?

Solution: Again, we can solve this problem via the law of total probability. Consider the following:

$$\begin{aligned} P(A > B) &= P(A > B|A = 5)P(A = 5) + P(A > B|A = 1)P(A = 1) \\ &= \left(\frac{5}{6}\right)\left(\frac{4}{6}\right) + (0)\left(\frac{2}{6}\right) \\ &= \frac{20}{36} \end{aligned}$$

$$\begin{aligned} P(B > C) &= P(B > C|B = 3)P(B = 3) + P(B > C|B = 4)P(B = 4) + P(B > C|B = 6)P(B = 6) \\ &= \left(\frac{2}{6}\right)\left(\frac{2}{6}\right) + \left(\frac{4}{6}\right)\left(\frac{3}{6}\right) + \left(\frac{4}{6}\right)\left(\frac{1}{6}\right) \\ &= \frac{20}{36} \\ P(A > C) &= P(A > C|A = 5)P(A = 5) + P(A > C|A = 1)P(A = 1) \\ &= \left(\frac{4}{6}\right)\left(\frac{4}{6}\right) + (0)\left(\frac{2}{6}\right) \\ &= \frac{16}{36} \end{aligned}$$

Thus $P(A > B) = P(B > C)$ however $P(A > C) \neq P(A > B)$ and $P(A > C) \neq P(B > C)$.

Problem 4 (Coin Game I): Player A has 100 fair coins and player B has 101 fair coins. Both of them toss their respective coins simultaneously and count the number of heads they each respectively received. Whoever has more heads is declared the winner winner. What's the probability that player B beats player A.

Solution: Say we compared 100 coins of players B to player A's 100 coins. Since all these coins are fair, the expectation of number of heads for each player is the same for their respective 100 coins. However, player B has an additional coin, which can be considered a tie-breaker coin, and so with a $\frac{1}{2}$ probability the expected number of heads for B is 1 plus the expected number of heads for A. In other words, there is a $\frac{1}{2}$ player B wins.

Problem 5 (Coin Game II): Players A and B are playing a game where they take turns flipping a biased coin, with probability p of landing heads (and winning). Player A starts the game, and then the players pass the coin back and forth until one person flips heads and wins.

What is the probability that A wins?

Solution: Denote $P(A)$ denote the probability that A wins. There is p chance that player A wins on the first flip. If the first flip is tails (a $1-p$ chance), then the probability of winning is equal to the probability that B gets a tails ($1-p$ chance) and A wins (which is still a $P(A)$ chance by independence). This can also be expressed by the law of total probability:

$$P(A) = P(A|H)P(H) + P(A|T)P(T)$$
$$P(A) = P(A|H)P(H) + (P(T)P(A))P(T)$$
$$P(A) = P(H) + (P(T)P(A))P(T)$$
$$P(A) = p + (1-p)^2 P(A)$$
$$P(A)(1-(1-p)^2) = p$$
$$P(A)(1-1+2p-p^2) = p$$
$$P(A)(2p-p^2) = p$$
$$P(A)(2-p)p = p$$
$$P(A) = \frac{1}{2-p}$$