The Probability Current for the Landau Problem

Drew Silcock

Wednesday 26th November, 2014

1 The Straight/Landau Gauge

The Landau problem is trying to find the wavefunction for a uniform everywhere magnetic field:

$$\mathbf{B} = B\hat{\mathbf{z}} \tag{1}$$

We will only consider the ground state solutions here. In order to find a solution for the wavefunction Ψ, we must first choose a gauge. For the straight gauge, $\mathbf{A} = Bx\hat{\mathbf{y}}$, the ground state solution is:

$$\Psi_{s,0}(x,t) = \mathcal{N} e^{-\frac{1}{2}\frac{M\Omega}{\hbar}x^2} e^{-i\frac{1}{2}\hbar\Omega t} \tag{2}$$

Where $\mathcal{N}$ is a normalisation factor, M is the mass of the particle, $\Omega \stackrel{\text{def}}{=} \frac{qB}{M} \in \mathbb{R}$ is the angular frequency.

Writing Ψ as $\Psi = \sqrt{\rho}e^{i\theta}$ where ρ is the probability density and θ is the phase of the wavefunction, we find:

$$\rho = \mathcal{N}^2 e^{-\frac{M\Omega}{\hbar}x^2} \tag{3}$$

$$\text{and } \theta = -\frac{1}{2}\hbar\Omega t \tag{4}$$

Thus, the probability current for the straight gauge is:

$$\begin{aligned}
\mathbf{J}_s &= \frac{\hbar\rho}{M}\left(\nabla\theta - \frac{q}{\hbar}\mathbf{A}\right) && (5)\\
&= \frac{\hbar}{M}\mathcal{N}^2 e^{-\frac{M\Omega}{\hbar}x^2}\left(\nabla\left\{-\frac{1}{2}\hbar\Omega t\right\} - \frac{q}{\hbar}Bx\hat{\mathbf{y}}\right) && (6)\\
&= -\frac{qB}{M}\mathcal{N}^2 e^{-\frac{M\Omega}{\hbar}x^2} x\hat{\mathbf{y}} \text{ as } \nabla\left\{-\frac{1}{2}\hbar\Omega t\right\} = 0 && (7)\\
&= -\mathcal{N}^2\Omega e^{-\frac{M\Omega}{\hbar}x^2} x\hat{\mathbf{y}} && (8)
\end{aligned}$$

2 The Circular Gauge

If we take the circular gauge, $\mathbf{A} = \frac{1}{2}B\left(x\hat{\mathbf{y}} - y\hat{\mathbf{x}}\right)$, then we find the ground state wavefunction is:

$$\Psi_{c,0}(x,y,t) = \mathcal{N} f(x,y) e^{-\frac{1}{4}\frac{M\Omega}{\hbar}\left(x^2+y^2\right)} e^{-i\frac{1}{2}\hbar\Omega t} \tag{9}$$

where $f(x,y)$ is an complex analytic function of x and y and $\mathcal{N}$ is a different, generic normalisation.

If we write this analytic function $f(x,y)$ as $f(x,y) = Fe^{i\phi}$, then the associated probability current is:

$$\begin{aligned}
\mathbf{J}_c(F,\phi) &= \frac{\hbar\rho}{M}\left(\nabla\theta - \frac{q}{\hbar}\mathbf{A}\right) && (10)\\
&= \frac{\hbar F^2}{M}\mathcal{N}^2 e^{-\frac{1}{2}\frac{M\Omega}{\hbar}\left(x^2+y^2\right)}\left(\nabla\phi - \frac{q}{\hbar}\frac{1}{2}B\left(x\hat{\mathbf{y}} - y\hat{\mathbf{x}}\right)\right) && (11)\\
&= \frac{\hbar F^2}{M}\mathcal{N}^2 e^{-\frac{1}{2}\frac{M\Omega}{\hbar}\left(x^2+y^2\right)}\left(\nabla\phi - \frac{M\Omega}{2\hbar}\left(x\hat{\mathbf{y}} - y\hat{\mathbf{x}}\right)\right) && (12)
\end{aligned}$$

3 Equivalence of straight and circular

We can choose this analytic function $f(x,y)$ such that the probability current for the circular gauge reduces to the probability current for the straight gauge. If we choose:

$$
\begin{aligned}
f_s(x,y) &= e^{-\frac{1}{4}\frac{M\Omega}{\hbar}(x+iy)^2} && (13)\\
&= e^{-\frac{1}{4}\frac{M\Omega}{\hbar}\left(x^2-y^2\right)} e^{-i\frac{1}{2}\frac{M\Omega}{\hbar}xy} && (14)\\
\Rightarrow F_s &= e^{-\frac{1}{4}\frac{M\Omega}{\hbar}\left(x^2-y^2\right)} && (15)\\
\text{and } \phi_s &= -\frac{1}{2}\frac{M\Omega}{\hbar}xy && (16)
\end{aligned}
$$

Then $\mathbf{J}_c$ becomes:

$$
\begin{aligned}
\mathbf{J}_c(F_s,\phi_s) &= \frac{\hbar F_s^2}{M}\mathcal{N}^2 e^{-\frac{1}{2}\frac{M\Omega}{\hbar}\left(x^2+y^2\right)}\left(\nabla\phi_s - \frac{M\Omega}{2\hbar}\left(x\hat{\mathbf{y}} - y\hat{\mathbf{x}}\right)\right) && (17)\\
&= \frac{\hbar e^{-\frac{1}{2}\frac{M\Omega}{\hbar}\left(x^2-y^2\right)}}{M}\mathcal{N}^2 e^{-\frac{1}{2}\frac{M\Omega}{\hbar}\left(x^2+y^2\right)}\left(\nabla\left\{-\frac{1}{2}\frac{M\Omega}{\hbar}xy\right\} - \frac{M\Omega}{2\hbar}\left(x\hat{\mathbf{y}} - y\hat{\mathbf{x}}\right)\right) && (18)\\
&= -\frac{\Omega}{2}\mathcal{N}^2 e^{-\frac{1}{2}\frac{M\Omega}{\hbar}x^2}\left(\nabla\left\{xy\right\} + x\hat{\mathbf{y}} - y\hat{\mathbf{x}}\right) && (19)\\
&= -\frac{\Omega}{2}\mathcal{N}^2 e^{-\frac{1}{2}\frac{M\Omega}{\hbar}x^2}\left(x\hat{\mathbf{y}} + y\hat{\mathbf{x}} + x\hat{\mathbf{y}} - y\hat{\mathbf{x}}\right) && (20)\\
&= -\mathcal{N}^2\Omega e^{-\frac{1}{2}\frac{M\Omega}{\hbar}x^2}x\hat{\mathbf{y}} && (21)\\
&= \mathbf{J}_s && (22)
\end{aligned}
$$