Math Sec 1.4

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Exercise. 1.20
Proof: We must prove properties (v) through (viii)
Let $(\mathbf{x} + W) \in V/W$, $(\mathbf{y} + W) \in V/W$, $a, b \in \mathbb{F}$
(v):

$$\begin{aligned} a \boxdot [(\mathbf{x} + W) \boxplus (\mathbf{y} + W)] &= a \boxdot [(\mathbf{x} + \mathbf{y}) + W] \\ &= (a\mathbf{x} + a\mathbf{y} + W) \\ &= a \boxdot (\mathbf{x} + W) \boxplus a \boxdot (\mathbf{y} + W) \end{aligned}$$

(vi):

$$\begin{aligned} (a + b) \boxplus (\mathbf{x} + W) &= (a + b)\mathbf{x} + W \\ &= (a\mathbf{x} + b\mathbf{x} + W) \\ &= a \boxplus (\mathbf{x} + W) \boxplus b \boxdot (\mathbf{x} + W) \end{aligned}$$

(vii):

$$\begin{aligned} 1 \boxdot (\mathbf{x} + W) &= (1\mathbf{x}) + W \\ &= (\mathbf{x} + W) \end{aligned}$$

(viii):

$$\begin{aligned} (ab) \boxdot (\mathbf{x} + W) &= a \boxdot (b\mathbf{x} + W) \\ &= (ab\mathbf{x}) + W \\ &= (ba\mathbf{x} + W) \\ &= b \boxdot (a\mathbf{x} + W) \\ &= (ba) \boxdot (\mathbf{x} + W) \end{aligned}$$

Exercise. 1.21

$$\begin{aligned} (a \boxdot (\mathbf{x} + W)) \boxplus (b \boxdot (\mathbf{y} + W)) &= (a\mathbf{x} + W) \boxplus (b\mathbf{y} + W) \\ &= (a\mathbf{x} + b\mathbf{y}) + W \end{aligned}$$

Exercise. 1.22
Proof: By definition 1.4.1, $\mathbf{x} - \mathbf{y}$ is obviously in V because both $\mathbf{x}$, and $\mathbf{y}$ are in V. Thus they are all in the same coset, and equivalent to eachother, and that coset is the only element of the quotient space V/V.

Exercise. 1.23
Let ϕ:V/$\{\mathbf{0}\}$ $\to$ V be defined such that for each $(\mathbf{x} + V) \in V/\{\mathbf{0}\}$ $\phi((\mathbf{x} + V)) = \mathbf{x}$. Note that

$$\begin{aligned}\phi((\mathbf{x} + \{\mathbf{0}\}) \boxplus (\mathbf{y} + \{\mathbf{0}\})) &= \phi((\mathbf{x} + \mathbf{y}) + \{\mathbf{0}\}) \\ &= (\mathbf{x} + \mathbf{y}) \\ &= \phi((\mathbf{x} + \{\mathbf{0}\}) + \phi((\mathbf{y} + \{\mathbf{0}\})\end{aligned}$$

And for scaler multiplication,

$$\begin{aligned}\phi(c \boxdot (\mathbf{x} + \{\mathbf{0}\}) &= \phi((c\mathbf{x}) + \{\mathbf{0}\}) \\ &= c\mathbf{x} \\ &= c\phi(\mathbf{x} + \{\mathbf{0}\})\end{aligned}$$

Exercise. 1.24
Let $\psi : V/W \to \mathbb{F}[y]$ be defined such that

$$\psi(x^i + W) = \begin{cases} 0 & \text{if i is odd} \\ x^{i/2} & \text{if i is even} \end{cases}$$

Note, for $p, q \in \mathbb{F}[x]$, and $c \in \mathbb{F}$

$$\begin{aligned}\psi(p + W \boxplus q + W) &= \psi((p + q) + W) \\ &= p + q \\ &= \psi(p + W) + \psi(q + W)\end{aligned}$$

And,

$$\begin{aligned}\psi(c \boxdot (p + W)) &= \psi((cp) + W) \\ &= cp \\ &= c\psi(p + W)\end{aligned}$$

To show surjectivity, choose $f(x) \in \mathbb{F}[x]$. Note that

$$\psi(f'(x^2) + W) = f(x)$$

To show injectivity, choose $f(x) + W,\ f'(x) + W \in V/W$ s.t. $\psi(f(x) + W) = \psi(f'(x) + W)$. Note,

$$\begin{aligned}\psi(f(x) + W) - \psi(f'(x) + W) &= \psi(f(x) - f'(x) + W)) \\ &= 0 \\ &= \psi(0 + W)\end{aligned}$$

Thus, $f(x),\ f'(x) \in W$, which implies that the cosets are equal.