Math Sec 2.6

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Exercise. 2.32
Note,

$$E1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3.5 & 0 & 1 \end{bmatrix} \quad E3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

$$U = E_3E_2E_1A = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0.5 & -2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & 6 \\ 4 & 5 & 6 \\ 7 & 8 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 6 \\ 0 & -3 & 6 \\ 0 & 0 & -9 \end{bmatrix}$$

By multiplying, we find

$$A = (E_3E_2E_1)^{-1}U = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ 3.5 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & 6 \\ 0 & -3 & 6 \\ 0 & 0 & -9 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 6 \\ 4 & 5 & 6 \\ 7 & 8 & 0 \end{bmatrix}$$

Exercise. 2.33
(i)

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$U = \left(\begin{array}{ccc|c} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\mathscr{N}(A) = span \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Thus,

$$\mathbf{x} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

(ii)

$$A = \left(\begin{array}{ccc|c} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{array}\right)$$

Null space is obviously empty, and we have

$$\mathbf{x} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

(iii)

$$A = \left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 4 & 5 & 4 & 1 \\ 7 & 7 & 7 & 0 \end{array}\right)$$

$$U = \left(\begin{array}{ccc|c} 1 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\mathcal{N}(A) = span \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Thus,

$$\mathbf{x} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

(iv)

$$A = \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 1 & 2 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$U = \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\mathcal{N}(A) = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

Thus,

$$\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

(v)

$$A = \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 4 & 25 & 6 & 1 \\ 7 & 8 & 9 & 0 \end{array}\right)$$

$$U = \left(\begin{array}{ccc|c} 1 & 0 & 0 & -.765 \\ 0 & 1 & 0 & .025 \\ 0 & 0 & 1 & .5708333 \end{array}\right)$$

Null space empty and solution is

$$\mathbf{x} = \begin{bmatrix} -.7625 \\ .025 \\ .57083 \end{bmatrix}$$

Exercise. 2.34 **(i)**
Given that

$$\mathbf{e_j} = \begin{bmatrix} e_{j1} \\ e_{j2} \\ \vdots \\ e_{jn} \end{bmatrix} \mathbf{e_i} = \begin{bmatrix} e_{i1} \\ e_{i2} \\ \vdots \\ e_{in} \end{bmatrix}$$

are elements of the standard basis, we know that the product $\mathbf{e}_i\mathbf{e}_j^T$ will yield an nxn matrix, E, where $E_{1,1} = \mathbf{e_{j1}}\mathbf{e_{i1}}$, $E_{1,2} = \mathbf{e_{j2}}\mathbf{e_{i1}}$, and $E_{m,k} = \mathbf{e_{jk}}\mathbf{e_{im}}$. Note every element of $\mathbf{e}$ is zero, except for $\mathbf{e_{jj}}$, and $\mathbf{e_{ii}}$, where they equal 1. Thus, the only element of E That is not zero, is $E_{ij} = 1$.

(ii)

$$
\begin{aligned}
(I - a\mathbf{u}\mathbf{v}^T)^{-1} &= \left(I - \frac{a\mathbf{u}\mathbf{v}^T}{a\mathbf{v}^T\mathbf{u} - 1}\right) \\
(I - a\mathbf{u}\mathbf{v}^T)^{-1}(I - a\mathbf{u}\mathbf{v}^T) &= \left(I - \frac{a\mathbf{u}\mathbf{v}^T}{a\mathbf{v}^T\mathbf{u} - 1}\right)(I - a\mathbf{u}\mathbf{v}^T) \\
I &= II - a\mathbf{u}\mathbf{v}^T - I\frac{a\mathbf{u}\mathbf{v}^T}{a\mathbf{v}^T\mathbf{u} - 1} + \frac{a\mathbf{u}\mathbf{v}^T}{a\mathbf{v}^T\mathbf{u} - 1}(a\mathbf{u}\mathbf{v}^T) \\
I &= I - a\mathbf{u}\mathbf{v}^T - \frac{a\mathbf{u}\mathbf{v}^T}{a\mathbf{v}^T\mathbf{u} - 1} + \frac{a\mathbf{u}\mathbf{v}^T a\mathbf{u}\mathbf{v}^T}{a\mathbf{v}^T\mathbf{u} - 1} \\
0 &= \frac{-a\mathbf{u}\mathbf{v}^T(a\mathbf{v}^T\mathbf{u} - 1) - a\mathbf{u}\mathbf{v}^T + a\mathbf{u}\mathbf{v}^T a\mathbf{u}\mathbf{v}^T}{a\mathbf{v}^T\mathbf{u} - 1} \\
0 &= \frac{a\mathbf{u}\mathbf{v}^T - a\mathbf{u}\mathbf{v}^T a\mathbf{v}^T\mathbf{u} - a\mathbf{u}\mathbf{v}^T + a\mathbf{u}\mathbf{v}^T a\mathbf{u}\mathbf{v}^T}{a\mathbf{v}^T\mathbf{u} - 1} \\
0 &= \frac{a\mathbf{u}\mathbf{v}^T a\mathbf{u}\mathbf{v}^T - a\mathbf{u}\mathbf{v}^T a\mathbf{v}^T\mathbf{u}}{a\mathbf{v}^T\mathbf{u} - 1} \\
0 &= \frac{a^2(\mathbf{u}\mathbf{v}^T\mathbf{u}\mathbf{v}^T - \mathbf{u}\mathbf{v}^T\mathbf{v}^T\mathbf{u})}{a\mathbf{v}^T\mathbf{u} - 1} \\
0 &= \frac{a^2(\mathbf{v}^T\mathbf{u})(\mathbf{u}\mathbf{v}^T - \mathbf{u}\mathbf{v}^T)}{a\mathbf{v}^T\mathbf{u} - 1} \\
0 &= \frac{a^2(\mathbf{v}^T\mathbf{u})(0)}{a\mathbf{v}^T\mathbf{u} - 1} \\
0 &= \frac{0}{a\mathbf{v}^T\mathbf{u} - 1}
\end{aligned}
$$

Exercise. 2.35
Suppose A is a matrix not in RREF. Then one of the following is not true:

- It is in REF

- Leading Coefficent of every row is equal to one
- Leading Coefficent of each row is only non-zero entry in the column

Suppose (i) is not true, we can use elementary matrix III to get zeros in each column as needed. If there are any zero rows, use matrix I to switch the rows accordingly. Thus A can be made sure to be in REF.
Suppose (ii) is not true, to get leading coefficents of every row equal to 1, use matrix II to take a scaler combonation of the inverse of the leading coefficent of the row in question. Thus A can be made sure to have ones in each leading coefficent.
To finish, suppose (iii) is not true. We can use any number of matrix III's to cancel all previous elements above the leading coefficents (which are all equal to 1). Thus A

can be assured to be in REF, with ones as leading coefficents, and no elements above the leading coefficents, therefore in RREF.

Exercise. 2.36

By definition of row equivalence between A and B, we know that for elementary matrices $(E_1, E_2, \ldots, E_k)$ it is true that $A = E_1E_2 \ldots E_kB$ and then also that $B = E_k^{-1}E_{k-1}^{-1} \ldots E_1^{-1}A$. We know that every elementary matrix is invertible by proposition 2.6.2. Showing that the inverse elementary matrices perform changes on A that create B and since A and B are row equivalent all of the inverse elemntary matrices are also elemetary matrices.

To show Reflexive:
A is row equivalent to IA and thus $A \equiv A$ since I is an elementary matrix.

To show symmetric:
Suppose $A \equiv B$. Thus there exists some $L = E_kE_{k-1} \ldots E_1$, such that $A = LB$, and note that $L^{-1} = E_1^{-1} \ldots E_{k-1}^{-1}E_1^{-1}$ and $L^{-1}A = B$. Thus, $B \equiv A$
To show Transativity:
Suppose $A \equiv B$ and $B \equiv C$. Thus, there exists an L_1, L_2 such that L_1, L_2 are combinations of Elementary matracies similar to above. $A = L_1B$ and $B = L_2C$. Thus $L_1^{-1}A = L_2C$, and $A = L_1L_2C$ and $A \equiv C$ Thus it is an equivalence relation

Exercise. 2.37

The matrix representing the transformation from problem 2.17 is:

$$L = \begin{bmatrix} \alpha & 1 & 0 \\ 0 & \alpha & 2 \\ 0 & 0 & \alpha \end{bmatrix}$$

In this case $\alpha = 1$ giving the derivitive operator as:

$$L = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

So we see that L^{-1} is given by:

$$L = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

These are unique because $\mathscr{N}(L) = \mathbf{0}$ and $\mathscr{N}(L^{-1}) = \mathbf{0}$.

Exercise. 2.38
We can express the basis for this space as:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & x^2 & 0 \\ 0 & 0 & 0 & x^3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

We express the transformation of each of the elements of the basis as:

$$\begin{aligned}
(x-1)0 &= 0 \\
(x-1)1 &= x-1 \\
(x-1)2x &= 2x^2-2x \\
(x-1)3x^2 &= 3x^3-3x^2
\end{aligned}$$

Giving the transformation matrix:

$$A = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 2 & -3 \\ 0 & 0 & 0 & 3 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

From this matrix we can determine that the bases for the kernel and range:

$$\mathscr{N}(A) = \left\{ \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\} \quad \mathscr{R}(A) = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$