Sec 3.2
Math 320

Rex McArthur

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Exercise. 3.7

Consider

$$||\text{proj}x(v)||^2 = \sum_{i=1}^{m} \left| \frac{\langle x_i, v \rangle}{||x_i||} \right|^2 \leq ||v||^2$$

$$\implies \sum_{i=1}^{m} \frac{|\langle x_i, v \rangle|^2}{||x_i||^2} \leq ||v||^2$$

$$\implies \sum_{i=1}^{m} |\langle x_i, v \rangle|^2 \leq ||v||^2 ||x_i||^2$$

$$\implies \sum_{i=1}^{m} |\langle x_i, v \rangle| \leq ||v|| ||x_i||$$

$$\implies |\langle x_i, v \rangle| \leq ||v|| ||x_i||$$

Which is the desired result.

Exercise. 3.8
(i) We show that the set is normal by showing that for $s_i \in S$ $\|s_i\| = 1$, and that it is orthogonal by showing that $\langle s_i, s_j \rangle = 0$ $\forall s_i, s_j \in S$ where $i \neq j$.

$$\langle cos(x), cos(x) \rangle = \frac{1}{\pi} \int_{-pi}^{\pi} cos^2(x) dx = \frac{1}{\pi} [x + sin(x)cos(x)]_0^{\pi} = 1$$

$$\langle sin(x), sin(x) \rangle = \frac{1}{\pi} \int_{-pi}^{\pi} sin^2(x) dx = \frac{1}{\pi} [x - sin(x)cos(x)]_0^{\pi} = 1$$

implying normality. The same result follows for the other two elements of the set as the only discrepancy between them and these elements of the set is the period. As for orthogonality

$$\langle cos(x), sin(x)\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} cos(x)sin(x)dx = (\frac{1}{\pi})(\frac{-1}{2})cos^2(x)_{-\pi}^{\pi} = 0$$

$$\begin{aligned}
\langle cos(x), cos(2x)\rangle &= \frac{1}{\pi}\int_{-\pi}^{\pi} cos(x)cos(2x)dx = \frac{1}{\pi}\int_{-\pi}^{\pi} cos(x)[cos^2(x) - sin^2(x)]dx \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} cos^3(x)dx - \int_{-\pi}^{\pi} sin^2(x)cos(x)dx = \frac{1}{\pi}(0) = 0 \\
\langle cos(x), sin(2x)\rangle &= \frac{1}{\pi}\int_{-\pi}^{\pi} cos(x)sin(2x) = \frac{1}{\pi}[\frac{-2}{3}cos^3(x)]_{-\pi}^{\pi} = 0 \\
\langle sin(x), cos(2x)\rangle &= \frac{1}{\pi}\int_{-\pi}^{\pi} cos^2(x)sin(x)dx - \int_{-\pi}^{\pi} sin^3(x)dx = -\frac{1}{3\pi}cos^3(x)_{-\pi}^{\pi} = 0 \\
\langle sin(x), sin(2x)\rangle &= \frac{1}{\pi}\int_{-\pi}^{\pi} sin(x)sin(2x)dx = \frac{2}{3\pi}sin^3(x)_{-\pi}^{\pi} = 0
\end{aligned}$$

Again, the result follows for the other two elements of the set because the other two elements are only a change in period from $\langle cos(x), sin(x)\rangle$ (ii)

$$\|t\| = \langle t, t\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} t^2dt = \frac{1}{3\pi}[t^3]_{-\pi}^{\pi} = \frac{2\pi^2}{3}$$

(iii)

$$\begin{aligned}
proj_X(cos(3t)) &= \langle cos(t), cos(3t)\rangle cos(t) + \langle sin(t), cos(3t)\rangle sin(t) + \\
&\quad \langle (cos(2t), cos(3t)\rangle cos(2t) + \langle sin(2t), cos(3t)\rangle sin(2t) \\
&= \frac{1}{\pi}\Big(\int_{-\pi}^{\pi} cos(t)cos(3t)dt\ cos(t) + \int_{-\pi}^{\pi} sin(t)(cos(3t)dt\ sin(t) + \\
&\quad \int_{-\pi}^{\pi} cos(2t)cos(3t)dt\ cos(2t) + \int_{-\pi}^{\pi} sin(2t)cos(3t)dt\ sin(2t)\Big) \\
&= \frac{1}{\pi}\big(0 \cdot cos(t) + \frac{-3\pi}{2}sin(t) + 0 + 4\pi sin(2t)\big) = \frac{-3}{2}sin(t) + 4sin(2t)
\end{aligned}$$

This is a linear combination of the starting equations.

(iv)

$$\begin{aligned}
proj_x(t) &= \langle cos(t), t\rangle cos(t) + \langle sin(t), t\rangle sin(t) + \langle (cos(2t), t\rangle cos(2t) + \langle sin(2t), t\rangle sin(2t) \\
&= \frac{1}{\pi}\Big(\int_{-\pi}^{\pi} cos(t)t\ dt\ cos(t) + \int_{-\pi}^{\pi} sin(t)t\ dt\ sin(t) + \\
&\quad \int_{-\pi}^{\pi} cos(2t)t\ dt\ cos(2t) + \int_{-\pi}^{\pi} sin(2t)t\ dt\ sin(2t)\Big) \\
&= \frac{1}{\pi}(2\pi sin(t) - \pi sin(2t)) = 2sin(t) - sin(2t)
\end{aligned}$$

This is a linear combination of the starting equations.

Exercise. 3.9

We know that an orthonormal transformation preserves vector length. Here, as a counterexample, let $x, y \in V$ be two unit vectors that are orthogonal to X. In this case, $\text{proj}X(y) = 0 \neq 1 \quad \text{proj}X(x) = 0 \neq 1$ and we have the desired result.

Exercise. 3.10

If we rotate two vectors x, y by θ, we preserve the angle between two vectors since we rotate them both by θ. Furthermore, we preserve vector length since we are only rotating them and not performing other operations on them. Therefore, this is an orthonormal transformation.

Exercise. 3.11
(i)
By properties of orthogonality,

$$\|Qx\|_2 = \langle Qx, Qx\rangle = (Qx)^H Qx = x^H Q^H Qx = x^H Ix = x^H x = \langle x, x\rangle = \|x\|_2$$

(ii)
By (v)

$$(Q_1Q_2)(Q_1Q_2)^H = Q_1Q_2Q_2^HQ_1^H = Q_1Q_1^{-1} = I$$

which implies orthogonality. (iii)
Because Q is orthonormal, when we take the Hermetian of the inverse this results in the original Q allowing us to perform the following proof:

$$\langle Q^{-1}, Q^{-1}\rangle = Q^{-1}(Q^{-1})^H = Q^{-1}Q = I$$

(iv)
$\Rightarrow$ By (iii), if Q is orthonormal, the resulting statement is true.

$$Q^H Q = Q^{-1}Q = QQ^{-1} = I$$

$\Leftarrow$ If $Q^HQ = QQ^H = Q^{-1}Q = QQ^{-1} = I$ implying orthonormality yielding that $\|Q\| = 1$ and is orthonormal.

(v)
Because we know that if Q is orthonormal then either the rows or the columns must be orthonormal. Suppose that the rows are orthonormal. Because we know that Q^T is also orthonormal then the columns of Q are orthonormal.

(vi)
We know that $Q^{-1} = Q \implies Q^{-1}Q = I \implies \det(Q^{-1}Q) = \det(I) = \det(Q^{-1})\det(Q) = \det(I) \implies \det(Q^H)\det(Q) = \det(I) \implies \det(Q)\det(Q) = 1 \implies \det(Q) = \pm 1 \implies |\det(Q)| = 1$

The converse is not true. Take the matrix A as an example where

$$\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{bmatrix}$$

Whose determinant is 1 but whose columns are not orthonormal.