Sec 3.2
Math 320

Rex McArthur

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Exercise. 3.7

$$||\text{proj}x(v)||^2 = \sum_{i=1}^{m} \left| \frac{\langle x_i, v\rangle}{||x_i||} \right|^2 \leq ||v||^2$$

$$\implies \sum_{i=1}^{m} \frac{|\langle x_i, v\rangle|^2}{||x_i||^2} \leq ||v||^2$$

$$\implies \sum_{i=1}^{m} |\langle x_i, v\rangle|^2 \leq ||v||^2||x_i||^2$$

$$\implies \sum_{i=1}^{m} |\langle x_i, v\rangle| \leq ||v||||x_i||$$

$$\implies |\langle x_i, v\rangle| \leq ||v||||x_i||$$

Exercise. 3.8
(i) You can show that is normal, but showing the length of each is one, and the inner product of each combo = 0. Note,

$$\langle cos(x), cos(x)\rangle = \frac{1}{\pi}\int_{-pi}^{\pi} cos^2(x)dx = \frac{1}{\pi}[x + sin(x)cos(x)]_0^{\pi} = 1$$

$$\langle sin(x), sin(x)\rangle = \frac{1}{\pi}\int_{-pi}^{\pi} sin^2(x)dx = \frac{1}{\pi}[x - sin(x)cos(x)]_0^{\pi} = 1$$

Thus, they are each normal.
The other two follow immediatly from above, because the only difference is the period.

To show orthogonality,

$$
\begin{aligned}
\langle cos(x), sin(x)\rangle &= \frac{1}{\pi}\int_{-\pi}^{\pi} cos(x)sin(x)dx = (\frac{1}{\pi})(\frac{-1}{2})cos^2(x)_{-\pi}^{\pi} = 0 \\
\langle cos(x), cos(2x)\rangle &= \frac{1}{\pi}\int_{-\pi}^{\pi} cos(x)cos(2x)dx = \frac{1}{\pi}\int_{-\pi}^{\pi} cos(x)[cos^2(x) - sin^2(x)]dx \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} cos^3(x)dx - \int_{-\pi}^{\pi} sin^2(x)cos(x)dx = \frac{1}{\pi}(0) = 0 \\
\langle cos(x), sin(2x)\rangle &= \frac{1}{\pi}\int_{-\pi}^{\pi} cos(x)sin(2x) = \frac{1}{\pi}[\frac{-2}{3}cos^3(x)]_{-\pi}^{\pi} = 0 \\
\langle sin(x), cos(2x)\rangle &= \frac{1}{\pi}\int_{-\pi}^{\pi} cos^2(x)sin(x)dx - \int_{-\pi}^{\pi} sin^3(x)dx = -\frac{1}{3\pi}cos^3(x)_{-\pi}^{\pi} = 0 \\
\langle sin(x), sin(2x)\rangle &= \frac{1}{\pi}\int_{-\pi}^{\pi} sin(x)sin(2x)dx = \frac{2}{3\pi}sin^3(x)_{-\pi}^{\pi} = 0
\end{aligned}
$$

Again, the result for the last two follow from these, because the only difference is a change in the period.

(ii)

$$
\|t\| = \langle t, t\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} t^2 dt = \frac{1}{3\pi}[t^3]_{-\pi}^{\pi} = \frac{2\pi^2}{3}
$$

(iii)

$$
\begin{aligned}
\text{proj}_X(cos(3t)) &= \langle cos(t), cos(3t)\rangle cos(t) + \langle sin(t), cos(3t)\rangle sin(t) + \\
&\quad \langle (cos(2t), cos(3t)\rangle cos(2t) + \langle sin(2t), cos(3t)\rangle sin(2t) \\
&= \frac{1}{\pi}\bigg(\int_{-\pi}^{\pi} cos(t)cos(3t)dt\ cos(t) + \int_{-\pi}^{\pi} sin(t)(cos(3t)dt\ sin(t) + \\
&\quad \int_{-\pi}^{\pi} cos(2t)cos(3t)dt\ cos(2t) + \int_{-\pi}^{\pi} sin(2t)cos(3t)dt\ sin(2t)\bigg) \\
&= \frac{1}{\pi}\big(0 \cdot cos(t) + \frac{-3\pi}{2}sin(t) + 0 + 4\pi sin(2t)\big) = \frac{-3}{2}sin(t) + 4sin(2t)
\end{aligned}
$$

This is a linear combination of the starting equations.

(iv)

$$
\begin{aligned}
\mathrm{proj}_x(t) &= \langle cos(t), t\rangle cos(t) + \langle sin(t), t\rangle sin(t) + \langle (cos(2t), t\rangle cos(2t) + \langle sin(2t), t\rangle sin(2t) \\
&= \frac{1}{\pi}\Bigg(\int_{-\pi}^{\pi} cos(t)t\ dt\ cos(t) + \int_{-\pi}^{\pi} sin(t)t\ dt\ sin(t) + \\
&\qquad \int_{-\pi}^{\pi} cos(2t)t\ dt\ cos(2t) + \int_{-\pi}^{\pi} sin(2t)t\ dt\ sin(2t)\Bigg) \\
&= \frac{1}{\pi}(2\pi sin(t) - \pi sin(2t)) = 2sin(t) - sin(2t)
\end{aligned}
$$

This is a linear combination of the starting equations.

Exercise. 3.9

Consider the counterexample of two unit vectors $x, y \in V$ that are orthogonal to X. In this case, $\mathrm{proj}X(y) = 0 \neq 1 \quad \mathrm{proj}X(x) = 0 \neq 1$ and we have the desired result, because orthonormal transformations perserves vector length.

Exercise. 3.10

If we rotate two vectors x, y by θ, we preserve the angle between two vectors since we rotate them both by θ. Also, length is unaffected by rotation. and not performing other operations on them. Therefore, this is an orthonormal transformation.

Exercise. 3.11
(i)
By properties of orthogonality,

$$\|Qx\|_2 = \langle Qx, Qx\rangle = (Qx)^H Qx = x^H Q^H Qx = x^H I x = x^H x = \langle x, x\rangle = \|x\|_2$$

(ii)
By (v)

$$(Q_1Q_2)(Q_1Q_2)^H = Q_1Q_2Q_2^H Q_1^H = Q_1Q_1^{-1} = I$$

which implies orthogonality. (iii)
Because Q is orthonormal, when we take the Hermetian of the inverse this results in the original Q allowing us to perform the following proof:

$$\langle Q^{-1}, Q^{-1}\rangle = Q^{-1}(Q^{-1})^H = Q^{-1}Q = I$$

(iv)
$\Rightarrow$ By (iii), if Q is orthonormal, the resulting statement is true.

$$Q^H Q = Q^{-1}Q = QQ^{-1} = I$$

$\Leftarrow$ If $Q^H Q = QQ^H = Q^{-1}Q = QQ^{-1} = I$ implying orthonormality yielding that $\|Q\| = 1$ and is orthonormal.

(v)
Because we know that if Q is orthonormal then either the rows or the columns must be orthonormal. Suppose that the rows are orthonormal. Because we know that Q^T is also orthonormal then the columns of Q are orthonormal.

(vi)
We know that $Q^{-1} = Q \implies Q^{-1}Q = I \implies \det(Q^{-1}Q) = \det(I) = \det(Q^{-1})\det(Q) = \det(I) \implies \det(Q^H)\det(Q) = \det(I) \implies \det(Q)\det(Q) = 1 \implies \det(Q) = \pm 1 \implies |\det(Q)| = 1$

The converse is not true. Take the matrix A as an example where

$$\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{bmatrix}$$

Whose determinant is 1 but whose columns are not orthonormal.