Math Sec 1.7

Rex McArthur
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Exercise. 1.37
Proof: We proceed by induction. For n=1, there are 1 permutations possible, obviously. 1 element can only be arranged in 1 unique way.
Assume for n-1 elements, there are $(n-1)!$ permutations. If you add one element to the set, you have n times as many permutations, one for each spot to put the last (n^{th}) element. Thus there are $n(n-1)! = n!$ permutations.

Exercise. 1.38

i. 6!

ii. $5! \cdot 2$

iii. $4! \cdot 3!$

iv. $2(3!)^2$ because you have two groups, with three people with three places to sit.

Exercise. 1.39
$C(4,2)^2 = 6^2$ 6 ways, for each pair.
$C(13,2) = 78$ Ways to pick the different ranks of pairs
$4 \cdot C(11,1) = 44$ Picking the rank of the last card, and the suit
Thus, $6^2 \cdot 11 \cdot 4 = 123,552$

Exercise. 1.40
Given they chose 5 balls, and you have to pick the right 'powerball', the number of ways to match 3 of the 5 are $C(5,3) = 10$, and the other two have to come from the other 54, so $C(54,2) = 1,431$. Thus, 14,310 total combanations win \$100.
We were given that the total number of combanations was 175,223,510. Thus, the probability is $\frac{14{,}310}{175{,}223{,}510} \approx .000081667$

Exercise. 1.41

(i):

$$\begin{aligned}
S_n &= \sum_{k=1}^{n} \binom{n}{k} k \\
&= \sum_{k=1}^{n} \binom{n}{(n-k)} (n-k) \text{ By symmetry of Binomial Thm.} \\
&= n\sum_{k=1}^{n} \binom{n}{(n-k)} - \sum_{k=1}^{n} \binom{n}{(n-k)} k \\
&= n\sum_{k=0}^{n} \binom{n}{k+1} - \sum_{k=1}^{n} \binom{n}{k} k \\
S_n &= n2^n - S_n \\
2S_n &= n2^n \\
S_n &= n2^{n-1}
\end{aligned}$$

(ii):

$$\begin{aligned}
S_n &= \sum_{k=1}^{n} \binom{n}{k} k^2 \\
&= \sum_{k=1}^{n} \frac{n!k^2}{k!(n-k)!} \\
&= \sum_{k=1}^{n} \frac{n!k}{(k-1)!(n-k)!} \\
&= \sum_{k=1}^{n} \frac{n!}{(k-1)!(n-k)!} k \\
&= \sum_{k=1}^{n} \binom{n-1}{k-1} k \\
&= \sum_{k=1}^{n} \binom{n}{k} - \binom{n-1}{k} k \\
&= \sum_{k=1}^{n} \binom{n}{k} k - \sum_{k=1}^{n} \binom{n-1}{k} k \\
&= n2^{n-1} - \sum_{k=1}^{n} \frac{(n-1)!}{k!(n-1-k)!} k \frac{n}{(n-k)} \cdot \frac{(n-k)}{n}
\end{aligned}$$

$$\begin{aligned}
2S_n &= n2^{n-1} + n^2 \sum_{k=1}^{n} \binom{n}{k} k \\
2S_n &= n2^{n-1} + n^2 2^{n-1} \\
S_n &= \frac{n2^{n-1}(n+1)}{2} \\
&= n(n+1)2^{n-1}
\end{aligned}$$

Exercise. 1.42
Note, we cean re-index using $r = k + j$ and $j = r - k$, in order to preserve powers.

$$(1+x)^n(1+x)^m = \sum_{k=0}^{n} \binom{n}{k} x^k \cdot \sum_{j=0}^{m} \binom{m}{j} x^j$$

$$(1+x)^{n+m} = \sum_{k=0}^{n} \binom{n}{k} \sum_{j=0}^{m} \binom{m}{j} x^{j+k} = \sum_{r=0}^{n+m} \binom{m+n}{r} x^r \quad \to \sum_{r=0}^{m+n} \sum_{k=0}^{r} \binom{n}{k} \binom{m}{r-k} x^r$$

Because monomials are linearly independent, it is sufficent for coefficents to be equal.

$$\to \sum_{k=0}^{r} \binom{n}{k} \binom{m}{r-k} = \binom{m+n}{r}$$