Math Sec 1.4

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Exercise. 2.1

(i): Note,

$$\begin{aligned} L(a(x_1, y_1)) + L(b(x_2, y_2)) &= a(x_1, y_1) + b(x_2, y_2) \\ &= aL(x_1, y_1) + bL(x_1, y_2) \end{aligned}$$

Thus, it is a linear transformation. Note that,

$$\begin{aligned} \mathscr{N} &= \{\mathbf{0}\} \\ \mathscr{R} &= \mathbb{R}^2 \end{aligned}$$

(ii): Note,

$$\begin{aligned} L(a(x_1, y_1)) + L(b(x_2, y_2)) &= a(x_1, 0) + b(x_2, 0) \\ &= aL(x_1, y_1) + bL(x_1, y_2) \end{aligned}$$

Thus, it is a linear transformation. Note that,

$$\begin{aligned} \mathscr{N} &= \{(0, y) |\ y \in \mathbb{R}\} \\ \mathscr{R} &= \{(x, 0) |\ x \in \mathbb{R}\} \end{aligned}$$

(iii): Note, $L(\mathbf{0}) \neq \mathbf{0}$, Thus, it is not a linear transformation.
(iv): Note,

$$\begin{aligned} L(a(x_1, y_1)) + L(b(x_2, y_2)) &= (a^2x_1^2, a^2y_1^2) + (b^2x_2^2, b^2y_2^2) \\ &= a^2(x_1^2, y_1^2) + b^2(x_2^2, y_2^2) \\ &= a^2L(x_1, y_1) + b^2L(x_2, y_2) \\ &\neq aL(x_1, y_1) + bL(x_2, y_2) \end{aligned}$$

Thus, it is NOT a linear mapping.

Exercise. 2.2
Let $p(x),\ q(x) \in \mathbb{F}_2$

(i): Not a linear transformation because,

$$\begin{aligned} L(a(p(x))) + L(b(q(x))) &= x^2 + x^2 \\ &\neq aL(p(x)) + bL(q(x)) \end{aligned}$$

(ii): Note that $xp(x) \in \mathbb{F}[x]_4 \forall p(x) \in \mathbb{F}[x]_2$

$$\begin{aligned} L(a(p(x))) + L(b(q(x))) &= axp(x) + bxq(x) \\ &= aL(p(x)) + bL(q(x)) \end{aligned}$$

Thus, it is a linear transformation.

(iii): Note that $x^4 + p(x) \in \mathbb{F}[x]_4 \forall p(x) \in \mathbb{F}[x]_2$

$$\begin{aligned} L(a(p(x))) + L(b(p'(x))) &= ax^4 + ap(x) + bx^4 + bq(x) \\ &= a(x^4 + p(x)) + b(x^4 + q(x)) \\ &= aL(p(x)) + bL(q(x)) \end{aligned}$$

(iv): Note that $(4x^2 - 3x)p'(x) \in \mathbb{F}[x]_4 \forall p(x) \in \mathbb{F}[x]_2$

$$\begin{aligned} L(a(p(x))) + L(b(q(x))) &= (4x^2 - 3x)ap'(x) + (4x^2 - 3x)bq'(x) \\ &= a((4x^2 - 3x)p'(x)) + b((4x^2 - 3x)q'(x)) \\ &= aL(p(x)) + bL(q(x)) \end{aligned}$$

Thus, it is a linear transformation.

Exercise. 2.3
Let $f(x),\ g(x) \in C^1([0,1];\mathbb{F})$. Note, $\forall f(x),\ f(x) + f'(x)$ is continious because both $f(x)$ and $f'(x)$ are continious.

$$\begin{aligned} L(a(f(x))) + L(b(g(x))) &= af(x) + af'(x) + bg(x) + bg'(x) \\ &= a(f(x) + f'(x)) + b(g(x) + g'(X)) \\ &= aL(f(x)) + bL(g(x)) \end{aligned}$$

To verify that $L(f) = g$,

$$\begin{aligned} L(f) &= e^{-x}\int_0^x g(t)e^t dt + Ce^{-x} + (-e^{-x}\int_0^x g(t)e^t dt) + e^{-x}g(x)e^x - Ce^{-x} \\ &= g(x) + e^{-x} - e^{-x} \\ &= g(x) \end{aligned}$$

Exercise. 2.4
Let $L, K, M \in \mathscr{L}(V, W)$, thus L,K both map from V to W. Let $\mathbf{v} \in V$, and $a, b \in \mathbb{F}$.

(i):
By properties of linear maps,

$$(L+K)(\mathbf{v}) = L(\mathbf{v}) + K(\mathbf{v}) = K(\mathbf{v}) + L(\mathbf{v}) = (K+L)(\mathbf{v})$$

(ii):
By properties of linear maps,

$$(L+K)(\mathbf{v}) + M(\mathbf{v}) = (L(\mathbf{v}) + K(\mathbf{v})) + M(\mathbf{v}) = L(\mathbf{v}) + (K(\mathbf{v})) + M(\mathbf{v})) = L + (K+M)(\mathbf{v})$$

(iii):
Note, the linear map $M(\mathbf{v}) = \mathbf{0}$ is a linear map that satisfies the additive identity.

(iv):
Because L is a vector space, let $L'(\mathbf{v}) = -\mathbf{v}$. This is obviously a linear transformation, and works as the additive inverse.

(v):
By properties of Linear transformations for L, K

$$a(L+K)(\mathbf{v}) = a(L(\mathbf{v}) + K(\mathbf{v})) = aL(\mathbf{v}) + aK(\mathbf{v}) = a(K(\mathbf{v}) + L(\mathbf{v})) = a(K+L)(\mathbf{v})$$

(vi):

$$(a+b)L(\mathbf{v}) = aL(\mathbf{v}) + bL(\mathbf{v})) = bL(\mathbf{v}) + aL(\mathbf{v}) = (b+a)L(\mathbf{v})$$

(vii):
There exists an element of W such that, Note $1L(\mathbf{v}) = 1 * \mathbf{w} = \mathbf{w} = L(\mathbf{v})$

(viii):
By properties of vector spaces, there are elements in W such that,

$$(ab)L(\mathbf{v}) = ab(\mathbf{w}) = a(b\mathbf{w}) = a(bL(\mathbf{v})$$

Exercise. 2.5
We proceed by induction. For n=1, we have V_1, V_2 and $L_1 : V_1 \to V_2$. Obviously, $(L_1)^{-1} = L_1^{-1}$.
Suppose that $(L_{n-1}L_{n-1} \dots L_1)^{-1} = L_1 - 1 L_2^{-1} \dots L_{n-1}^{-1}$. For $\{V_i\}_{i=1}^{n+1}$, and $\{L_i\}_{i=1}^{n}$, we have

$$(L_n L_{n-1} \dots L_1)^{-1} = (L_n(L_{n-1} \dots L_1))^{-}1$$

By remark 2.1.20, we can switch the order

$$= ((L_{n-1} \dots L_n)^{-1} L_n^{-1})$$

and by inductive hypothesis

$$= L_1^{-1} \dots L_n^{-1}$$

Exercise. 2.6
To show $\mathscr{N}(KL) = L^{-1}\mathscr{N}(K) = \{\mathbf{v}|L(\mathbf{v}) \in \mathscr{N}(K)\}$, we note by definiton:

$$\mathscr{N}(KL) = \{\mathbf{v} \in V|KL(\mathbf{v}) = \mathbf{0}\}$$

$$\mathscr{N}(K) = \{\mathbf{w} \in W|K(\mathbf{w}) = \mathbf{0}\}$$

We also know that $L^{-1} : W \to V$ is a bijective map, because the two spaces are isomorphic. Let $\mathbf{v} \in \mathscr{N}(KL)$. Thus $KL(\mathbf{v}) = \mathbf{0}$, and $KL(\mathbf{v}) \in W$. Thus, $\mathbf{v} \in L^{-1}KL(\mathbf{v}) \in V$.
To show the other direction, let $\mathbf{v} \in L^{-1}\mathscr{N}(K)$. Because L inverse is bijective, there exists $\mathbf{v} \in V$, for every $\mathbf{w} \in W$ that is in the nullspace of K, and $L^{-1}\mathscr{N}(K) = \{v \in V|\mathbf{v} = L^{-1}(\mathscr{N}(K)\}$, and thus $\mathbf{v} \in \mathscr{N}(KL)$.

To show $\mathscr{R}(KL) \cong \mathscr{R}(K)$, we note by Definition:

$$\mathscr{R}(KL) = \{\mathbf{u} \in U|\exists\mathbf{v} \in V \text{Where } KL(\mathbf{v}) = \mathbf{u}\}$$

$$\mathscr{R}(K) = \{\mathbf{u} \in U|\exists\mathbf{w} \in W \text{Where } K(\mathbf{w}) = \mathbf{u}\}$$

Let $\mathbf{u} \in \mathscr{R}(KL)$. Thus, $\exists\mathbf{v} \in V$, where $KL(\mathbf{v}) = \mathbf{w}$. Note $L(\mathbf{v}) \in W$, and $K(L(\mathbf{v})) = \mathbf{u}$. Thus, $\mathbf{u} \in \mathscr{R}(K)$.
To show the other direction, let $\mathbf{u} \in \mathscr{R}(K)$.Thus $\exists\mathbf{w} \in W$, where $K(\mathbf{w}) = \mathbf{u}$. Because $L \cong W$, $\exists\mathbf{v} \in V$ s.t. $L(\mathbf{v}) = \mathbf{w}$, and $KL(\mathbf{v}) = \mathbf{u}$. Thus $\mathbf{u} \in \mathscr{R}(KL)$.
Thus, $\mathscr{R}(KL) = \mathscr{R}(K)$.

Exercise. 2.7
(i): Let $\mathbf{x} \in V$, and $\mathbf{x} \in \mathscr{N}(L^k)$. Thus, $L^k\mathbf{x} = \mathbf{0}$. If follows that $L(L^k\mathbf{x}) = L(\mathbf{0}) = \mathbf{0}$. Thus, $\mathbf{x} \in \mathscr{N}(L^{k+1})$

(ii):
Let $\mathbf{w} \in \mathscr{R}(L^{k+1})$. Thus, there exists $\mathbf{v} \in V$ s.t. $L^{k+1}(\mathbf{v}) = L(L(\mathbf{v}))$. Thus, there exists $\mathbf{v}' \in V$ s.t. $L(\mathbf{v}) = \mathbf{v}'$. Thus $L^k(\mathbf{v}') = \mathbf{w}$ and $\mathbf{w} \in \mathscr{R}(L^k)$.
$\mathscr{R}(L^{k+1}) \subset \mathscr{R}(L^k)$