Math Sec 3.1

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Exercise. 3.31
$\Leftarrow$ Suppose that $a^p = b^q$. Then $(a^p/p + b^q/q) = a^p(1/p + 1/q) = a^p$ and $ab = (a^p)^{1/p}(b^q)^{1/q} = (a^p)^{1/p}(a^p)^{1/q} = a^p$, showing that $ab = a^p/p + b^q/q$.

$\Rightarrow$ Suppose that $ab = (1/p)a^p + (1/q)b^q$ Upon dividing by ab and using the fact that $a^p/b = (a^p/b^q)^{1/q}$ and $b^q/a = (b^q/a^p)^{1/p}$, we see that

$$\frac{1}{p}(\frac{a^p}{b^q})^{1/q} + \frac{1}{q}(\frac{b^q}{a^p})^{1/p} = 1$$

Let $x = a^p/b^q$. Multiplying by $x^{1/p}$, we obtain

$$\frac{1}{p}x^{1/p+1/q} + \frac{1}{q} = 1$$

$$\rightarrow \frac{1}{p}x + \frac{1}{q} = 1$$

This implies that $x = a^p/b^q = 1$, so $a^p = b^q$, as desired.

Exercise. 3.32
Suppose $\epsilon \geq 1$. Thus, by Young's inequality,

$$\begin{aligned} ab &\leq \frac{a^2}{2} + \frac{b^2}{2} \\ &\leq \frac{\epsilon^2}{\epsilon}(\frac{a^2}{2} + \frac{b^2}{2}) \\ &\leq \frac{a^2 + \epsilon^2 b^2}{2\epsilon} \\ ab &\leq \frac{a^2}{2\epsilon} + \frac{\epsilon b^2}{2} \end{aligned}$$

Now, suppose $\epsilon < 1$. Thus, by Young's inequality,

$$\begin{aligned} ab &\leq \frac{a^2}{2} + \frac{b^2}{2} \\ &\leq \frac{1}{\epsilon}(\frac{a^2 + b^2}{2}) \\ &\leq \frac{a^2}{2\epsilon} + \frac{b^2}{2\epsilon} \\ ab &\leq \frac{a^2}{2\epsilon} + \frac{\epsilon b^2}{2} \end{aligned}$$

In all cases, $ab \leq \frac{a^2}{2\epsilon} + \frac{\epsilon b^2}{2}$

Exercise. 3.33
We note first that if $a = b$,

$$a^\theta b^{1-\theta} = a^\theta a^{1-\theta} = a = \theta a + (1-\theta)a = \theta a + (1-\theta)b$$

Now suppose $a \neq b$

$$\begin{aligned} a^\theta b^{1-\theta} &\leq \theta a + (1-\theta)b \\ \ln(a^\theta b^{1-\theta}) &\leq \ln(\theta a + (1-\theta)b) \end{aligned}$$

Note that, because the natural log is convex, $\theta \ln(a) < \ln(\theta a)$. Thus,

$$\begin{aligned} \ln(a^\theta b^{1-\theta}) &= \theta \ln(a) + (1-\theta)\ln(b) \\ &< \ln(\theta a) + \ln((1-\theta)b) \\ &< \ln(\theta a + (1-\theta)b) \end{aligned}$$

Thus, equality holds if and only if $a = b$.

Exercise. 3.34
Letting $\theta = \frac{1}{2}$, we get :

$$\begin{aligned} &a^{\frac{1}{2}} b^{\frac{1}{2}} \leq \frac{1}{2}(a+b) \\ &(ab)^{\frac{1}{2}} \leq \frac{1}{2}(a+b) \\ &(\text{area})^{\frac{1}{2}} \leq \frac{1}{4}\text{Perimeter} \\ &P \geq 4\sqrt{A} \end{aligned}$$

The minimum lies at $P = 4\sqrt{A}$ which holds only for

$$2(a+b) = 4\sqrt{ab} \Rightarrow 4(a+b)^2 = 16(ab) \Rightarrow 4(a-b)^2 = 0 \Rightarrow a = b$$

Exercise. 3.35
For arbitrary dimensions, we may extend the Arithmetic Geometric Mean inequality to say,

$$(x_1 \cdot ... \cdot x_n)^{\frac{1}{n}} \leq \frac{x_1 + \cdots + x_n}{n}$$

Where equality holds if and only if each x_i is equal to all the others.
The n-dim. cube must have n verticies, and 2^{n-1} edges, because each vertex is connected to n edges, so the total length of all vertecies is going to be $2^{n-1}(x_1 + x_2 + \cdots + x_n)$, where each x is a length of a vertex. The volume is simply going to be $2^n(x_1 \cdot \cdots \cdot x_n)^{\frac{1}{n}}$, Thus, the inequality gives us

$$\begin{aligned} 2^n(x_1 \cdot ... \cdot x_n)^{\frac{1}{n}} &\leq 2^{n-1}\sum_{i=1}^{n} x_i \\ (x_1 \cdot ... \cdot x_n)^{\frac{1}{n}} &\leq \frac{1}{n}\sum_{i=1}^{n} x_i \\ \text{Area}^{\frac{1}{n}} &\leq \frac{1}{n}\sum_{i=1}^{n} x_i \end{aligned}$$

The minimum value is found here when these are equal.
Note, if $x_i = y$ for all i, then we know

$$y^n = \frac{1}{n}\sum_{i=1}^{n} x_i = \frac{1}{n}(n(y^n)) = y^n$$

Otherwise, we know that these won't be equal, by Exercise 33.

Exercise. 3.36