Math Sec 1.6

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Math 320

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Exercise. 1.19
(i) Proof: Given an integer $a = \sum_{k=0}^{n-1} a_k 10^k = a_0 10^0 + \cdots + a_{n-1}10^{n-1}$, Note that

$$\begin{aligned} a &= \sum a_k 10^k \\ &= a_0 + \cdots + a_{n-1}10^{n-1} \\ &= (9a_1 + 99a_2 + \cdots + (10^{n-1} - 1)a_{n-1}) + (a_0 + a_1 + \cdots + a_{n-1} \\ &= 3(3a_1 + 33a_2 + \cdots + (10^{n-1} - 1)/3a_{n-1}) + (a_0 + \cdots + a_{n-1}) \\ &= 3(3a_1 + 33a_2 + \cdots + (10^{n-1} - 1)/3a_{n-1}) + \sum_{k=0}^{n-1} a_k \end{aligned}$$

Note that $3|a$ if and only if $3|\sum_{k=0}^{n-1} a_k$, by definition of divisibility, because it is apparent that 3 divides the first term.
(ii) Proof: Given an integer $a = \sum_{k=0}^{n-1} a_k 10^k = a_0 10^0 + \cdots + a_{n-1}10^{n-1}$, Note that

$$\begin{aligned} a &= \sum a_k 10^k \\ &= a_0 + \cdots + a_{n-1}10^{n-1} \\ &= (9a_1 + 99a_2 + \cdots + (10^{n-1} - 1)a_{n-1}) + (a_0 + a_1 + \cdots + a_{n-1}) \\ &= 9(1a_1 + 11a_2 + \cdots + (10^{n-1} - 1)/3a_{n-1}) + (a_0 + \cdots + a_{n-1}) \\ &= 9(1a_1 + 11a_2 + \cdots + (10^{n-1} - 1)/3a_{n-1}) + \sum_{k=0}^{n-1} a_k \end{aligned}$$

Note that $9|a$ if and only if $9|\sum_{k=0}^{n-1} a_k$, by definition of divisibility, because it is apparent that 9 divides the first term.
(iii) Proof: Given an integer $a = \sum_{k=0}^{n-1} a_k 10^k = a_0 10^0 + \cdots + a_{n-1}10^{n-1}$, Note that

$$
\begin{aligned}
a &= \sum_{k=0}^{n-1} a_k 10^k \\
&= a_0 10^0 + a_1 + 10^1 + \cdots + a_{n-1} 10^{n-1} \\
&= (11a_1 + 99a_2 + 1001a_3 + \cdots + (10^{n-1} - 1) + a_{n-1}) + (a_0 - a_1 + a_2 - \cdots + (-1)^{n-1} a_{n-1}) \\
&\text{(Where the coefficients of } a_i^{th} \text{ term given by } 10^i - (-1)^{i-1}) \\
&= 11(a_1 + 9a_2 + 91a_3 + 909a_4 \cdots + ((10^{n-1} - 1)/11) a_{n-1}) + (a_0 - a_1 + a_2 \cdots + (-1)^{n-1} a_{n-1})
\end{aligned}
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Note that a is divisible by 11 if and only if the second expression is divisible by 11, by definition of divisiblity, since the entire first part is divisible by 11.

Exercise. 1.20