

Math Sec 1.4

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Exercise. 1.26

Proof: Use substitution $j = k - 5$

$$\sum_{k=5}^n (k-5)^2 = \sum_{j=1}^{n-5} j^2 = \frac{(2n+1)(n+1)n}{6}$$

Exercise. 1.27

Use substitution $i = k - 4$, $k = i + 4$

$$\sum_{i=0}^n \sum_{j=-3}^{k-8} (k-4) = \sum_{i=0}^n \sum_{j=-3}^{i-4} i$$

Note that the second sum has $i - 4 - (-3) + 1 = i$ iterations. Thus,

$$= \sum_{i=0}^n (i^2) = \frac{n(n+1)(n+2)}{6}$$

Exercise. 1.28

Using the substitution $i = k - 3$ and $k = i + 3$, we have

$$\begin{aligned}
\sum_{j=-3}^{n-3} \sum_{k=j+3}^{n+3} k - 3 &= \sum_{j=-3}^{n-3} \sum_{k=j}^n i \\
&= \sum_{j=-3}^{n-3} \left(\sum_{i=0}^n i - \sum_{i=0}^{j-1} i \right) \\
&= \sum_{j=-3}^{n-3} \left(\frac{n(n+1)}{2} - \frac{j(j-1)}{2} \right)
\end{aligned}$$

By FTOFC

$$= \frac{1}{2} \sum_{j=-3}^{n-3} (n)(n+1) - \sum_{j=0}^{n-3} j^2 - \sum_{j=0}^{n-3} j - ((-3)^2 + (-2)^2 + (-1)^2) + (-3 + -2 + -1)$$

By subtracting the finite part, below 0

$$= \frac{1}{2} \left[n(n+1)^2 - \frac{(n-3)(n-2)(2n-5)}{6} + \frac{(n-3)(n-2)}{2} - 20 \right]$$

Exercise. 1.29

$$\begin{aligned}
\text{If } f(x) = k^4, \text{ Then } (\Delta f)(x) &= (k+1)^4 - k^4 \\
&= (k^2 + 2k + 1)^2 - k^4 \\
(\Delta f)(k) &= 4k^3 + 6k^2 + 4k + 1
\end{aligned}$$

It follows that,

$$\begin{aligned}
\sum_{k=1}^{b-1} (4k^3 + 6k^2 + 4k + 1) &= \sum_{k=1}^{b-1} (f\Delta) \\
&= f(b) - f(a) = b^4 - a^4 \\
b^4 - a^4 &= \sum_{k=1}^{b-1} (4k^3 + 6k^2 + 4k + 1) = 4 \sum_{k=1}^{b-1} (k^3) + 6 \sum_{k=1}^{b-1} (k^2) + 4 \sum_{k=1}^{b-1} (k) + (b - a)
\end{aligned}$$

Note, that $(b^4 - a^4) - (b - a) = b^4 - b - (a^4 - a) = b(b^3 - 1) - a(a^3 - 1)$. Thus,

$$b(b^3 - 1) - a(a^3 - 1) = 4 \sum_{k=1}^{b-1} (k^3) + 6 \sum_{k=1}^{b-1} (k^2) + 4 \sum_{k=1}^{b-1} (k)$$

Use $a = 1$, $b = n + 1$

$$\begin{aligned}
 (n+1)((n+1)^3 - 1) &= 4 \sum_{k=1}^{b-1} (k^3) + 6 \sum_{k=1}^{b-1} (k^2) + 4 \sum_{k=1}^{b-1} (k) \\
 n^4 + 4n^3 + 6n^2 + 3n &= 4 \sum_{k=1}^{b-1} (k^3) + (2n+1)(n+1)(n) + 2n(n+1) \\
 &= 4 \sum_{k=1}^{b-1} (k^3) + 2n^3 + 5n^2 + 3n
 \end{aligned}$$

Thus,

$$\begin{aligned}
 4 \sum_{k=1}^{b-1} (k^3) &= n^4 + 2n^3 + n^2 = (n^2 + n)^2 = (n(n+1))^2 \\
 \sum_{k=1}^{b-1} (k^3) &= \frac{1}{4} (n(n+1))^2 = \left(\frac{n(n+1)}{2} \right)^2
 \end{aligned}$$

Exercise. 1.30

If $f(i) = \frac{-1}{i}$,

$$\begin{aligned}
 (\Delta f)(i) &= \left(\frac{-1}{(i+1)} \right) + \frac{1}{i} \\
 &= \frac{-i}{i(i+1)} + \frac{i+1}{i(i+1)} \\
 &= \frac{1}{i(i+1)}
 \end{aligned}$$

Thus, by the Fundamental Theorem of Finite Calculus,

$$\begin{aligned}
 \sum_{i=1}^n \frac{1}{i(i+1)} &= \sum_{i=1}^n (\Delta f) = \frac{-1}{n+1} - \frac{-1}{1} \\
 &= 1 - \frac{1}{n+1}
 \end{aligned}$$

Exercise. 1.31

Proof:

(i)

$$\rightarrow \sum_{k=0}^n \sum_{j=k}^n j$$

$$\begin{aligned}
\sum_{k=0}^n \sum_{j=k}^n j &= \sum_{k=0}^n \left(\sum_{j=0}^n j - \sum_{j=0}^{k-1} j \right) \\
&= \sum_{k=0}^n \left(\frac{n(n+1)}{2} - \frac{(k-1)k}{2} \right) \\
&= \frac{1}{2} - \sum_{k=0}^n k^2 + \sum_{k=0}^n k \\
&= \frac{1}{2} \left(n(n+1)^2 - \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} \right) \\
&= \frac{1}{2} \left(n^3 + 2n^2 + n - \left(\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6} \right) + \frac{n^2}{2} + \frac{n}{2} \right) \\
&= \frac{n^3}{3} + \frac{3n^2}{2} + \frac{2n}{3}
\end{aligned}$$

$$\leftarrow \sum_{j=k}^n \sum_{k=0}^n j$$

$$\sum_{j=k}^n \sum_{k=0}^n j = \sum_{j=0}^n \sum_{k=0}^n j$$

Because there are $j+1$ steps in the second sum,

$$\begin{aligned}
&= \sum_{j=0}^n j^2 + \sum_{j=0}^n j \\
&= \sum_{j=1}^n j^2 + \sum_{j=1}^n j \\
&= \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} \\
&= \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6} + \frac{n^2}{2} + \frac{n}{2} \\
&= \frac{n^3}{3} + \frac{3n^2}{2} + \frac{2n}{3}
\end{aligned}$$

Thus, they are equivalent.

(ii):

$$\rightarrow \sum_{k=0}^n \sum_{j=0}^k j$$

$$\begin{aligned} \sum_{k=0}^n \sum_{j=0}^k j &= \sum_{k=0}^n \frac{(k+1)k}{2} \\ &= \frac{1}{2} \left(\sum_{k=0}^n k^2 + \sum_{k=0}^n k \right) \\ &= \frac{1}{2} \left(\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} \right) \\ &= \frac{n(n+1)(n+2)}{6} \end{aligned}$$

$$\leftarrow \sum_{j=0}^k \sum_{k=0}^n j$$

$$\begin{aligned} \sum_{j=0}^k \sum_{k=0}^n j &= \sum_{j=0}^n \sum_{k=j}^n j \\ &= \sum_{j=0}^n j(n-j+1) \\ &= \sum_{j=0}^n jn - j^2 + j \\ &= \frac{n^2(n+1)}{2} - \frac{(2n+1)(n+1)(n)}{6} + \frac{n(n+1)}{2} \\ &= n(n+1) \left(\frac{n}{2} - \frac{(2n+1)}{6} + \frac{1}{2} \right) \\ &= n(n+1) \left(\frac{(n+2)}{6} \frac{n+2}{6} \right) \\ &= \frac{n(n+1)(n+2)}{6} \end{aligned}$$

Exercise. 1.32

By taking the derivative of both sides, we get

$$\sum_{t=0}^n \beta^{t-1} = \frac{(\beta-1)(N+1)\beta^N - (\beta^{N+1} - 1)}{(\beta-1)^2}$$

Now taking the limit as N approaches infinity, All β^N terms will go to zero.

$$\lim_{N \rightarrow \infty} \sum_{t=0}^N t\beta^{t-1} = \frac{1}{(\beta-1)^2}$$

This implies that,

$$\sum_{t=0}^N (t\beta^{t-1})/\beta = \frac{1}{(\beta-1)^2}$$

Which implies,

$$\sum_{t=0}^N t\beta^{t-1} = \frac{\beta}{(1-\beta)^2}$$