Sec 3.3
Math 320

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Exercise. 3.14
(i)

$$f_X(\alpha) = \begin{cases} \frac{1}{2} & \text{if } \alpha = a, b, c \\ \frac{1}{6} & \text{if } \alpha = d \\ \frac{1}{3} & \text{if } \alpha = e, f \\ 0 & \text{elsewhere} \end{cases}$$

(ii)

$$\frac{1}{2} + \frac{1}{6} \cdot 2 + \frac{1}{3} \cdot 3.5 = 2$$

(iii)

$$\text{var}(x) = (1-2)^2 \cdot \frac{1}{2} + (0)^2 \cdot \frac{1}{6} + (1.5)^2 \cdot \frac{1}{3} = \frac{5}{4}$$

Exercise. 3.15
Let X be a discrete set, and have probability distribution $p(x = i) = g_X$ Suppose h is a continuous function s.t. $h : \mathbb{R} \to \mathbb{R}$, and $\exists \epsilon > 0$ between each point in X where there are countable number of points. Thus, because h is continuous, $\exists \epsilon' > 0$ between each point. Let $A = \{\epsilon'\}_{i \in I}$. Choosing the $\inf(A)$ will yield an ϵ', such that for every point p, $\exists E(p, \frac{\epsilon'}{2})$, such that the neighborhood E contains no other points in X. Thus, $h(X)$ is a random variable, and by LOTUS,

$$E[h(X)] = \sum_{i \in I} h(i) p(x = i) = \sum_{i \in I} h(i) g_X(i)$$

Exercise. 3.16

$$\begin{aligned} \text{var}[X] &= E((X - \mu)^2) = E(X^2 - 2X\mu + \mu^2) \\ &= E(X^2) - 2\mu E(X) + E(\mu^2) \\ &= E(X^2) - \mu^2 \end{aligned}$$

Exercise. 3.17

$$\begin{aligned}
\operatorname{var}(\alpha x+\beta y) &= E(\alpha x+\beta y)^2-(E(\alpha x+\beta y))^2\\
&= E(\alpha^2x^2+\alpha x\beta y+\alpha\beta xy+\beta^2y^2)-(\alpha E(x)+\beta E(Y))^2\\
&= \alpha^2E(x^2)+2\alpha\beta E(xy)+\beta^2E(y^2)-\alpha E(x)^2-2\alpha\beta E(x)E(y)-\beta^2E(y)^2\\
&= \alpha^2(E(x^2)-E(x)^2)+2\alpha\beta(E(xy)-E(x)E(y))+\beta^2(E(y^2)-E(y)^2)\\
&= \alpha^2\operatorname{var}[x]+2\alpha\beta(E(xy)-E(x)E(y))+\beta^2\operatorname{var}[y]
\end{aligned}$$

As desired, and if x, y are independent, we know that $E(xy)-E(x)E(y)=0$, and thus

$$\operatorname{var}(\alpha x+\beta y)=\alpha^2\operatorname{var}[x]+\beta^2\operatorname{var}[y]$$

Exercise. 3.18
Note, by LOTUS $E\big(\frac{1}{X+1}\big)$ is a random variable.

$$\begin{aligned}
E(\frac{1}{X+1}) &= \sum_{i=0}^{n}\frac{1}{i+1}\binom{(}{n}, i)p^i(1-p)^{n-i}\\
&= \sum_{i=0}^{n}\frac{n!}{(i+1)!(n-i)!}p^i(1-p)^{n-i}\\
&= \sum_{i=1}^{n+1}\frac{n!}{i!(n-i+1)!}p^{i-1}(1-p)^{n-i+1}\\
&= \frac{1}{p(n+1)}\sum_{i=1}^{n+1}\frac{(n+1)!}{i!(n+1-i)!}p^i(1-p)^{n+1-i}\\
&= \frac{1}{p(n+1)}\sum_{i=1}^{n+1}\binom{(}{n}+1, i)p^i(1-p)^{n+1-i}\\
&= \frac{1}{p(n+1)}\Big(\sum_{i=0}^{n+1}\binom{(}{n}+1, i)p^i(1-p)^{n+1-i}-(1-p)^{n+1}\Big)\\
&= \frac{1}{p(n+1)}\cdot(1-(1-p)^{n+1})\\
&= \frac{1-(1-p)^{n+1}}{p(n+1)}
\end{aligned}$$

Exercise. 3.19

$$
\begin{aligned}
E(X) &= \sum_{x \in X} x \cdot P(X = x) \\
&= \sum_{i \in I} x \sum x \in X \frac{xP(X = x) \bigcap B_i) P(B_i)}{P(B_i)} \\
&= \sum_{i} \sum_{x} x P(X = x | B_i) P(B_i) \\
&= \sum_{i} E(X | B_i) P(B_i)
\end{aligned}
$$