Math 344
Sec 2.8

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Exercise. 2.44
Note, by Theorem 2.8.1, $\det(A) = \det(A^T)$ Thus,

$$\det(V_n) = \det \begin{bmatrix} 1 & 1 & 1 \cdots 1 \\ x_0 & x_1 & x_2 \cdots x_n \\ x_0^2 & x_1^2 & x_2^2 \cdots x_n^2 \\ \cdots & \cdots & \cdots\cdots\cdots \\ x_0^n & x_1^n & x_2^n \cdots x_n^n \end{bmatrix} \implies \det \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & x_1 - x_0 & x_2 - x_0 & \cdots & x_n - x_0 \\ 0 & x_1^2 - x_1x_0 & x_2^2 - x_2x_0 & \cdots & x_n^2 - x_nx_0 \\ 0 & x_1^3 - x_1^2x_0 & x_2^3 - x_2^2x_0 & \cdots & x_n^3 - x_n^2x_0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & x_1^n - x_1^{n-1}x_0 & x_2^n - x_2^{n-1}x_0 & \cdots & x_n^3 - x_n^2x_0 \end{bmatrix}$$

$$\implies 1 \cdot \det \begin{bmatrix} x_1 - x_0 & x_2 - x_0 & \cdots & x_n - x_0 & \\ x_1^2 - x_1x_0 & x_2^2 - x_2x_0 & \cdots & x_n^2 - x_nx_0 & \\ x_1^3 - x_1^2x_0 & x_2^3 - x_2^2x_0 & \cdots & x_n^3 - x_n^2x_0 & \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ x_1^n - x_1^{n-1}x_0 & x_2^n - x_2^{n-1}x_0 & \cdots & x_n^3 - x_n^2x_0 & \end{bmatrix}$$

$$\implies 1 \cdot \det \begin{bmatrix} x_1 - x_0 & x_1^2 - x_1x_0 & x_1^3 - x_1^2x_0 & \cdots & x_1^n - x_1^{n-1}x_0 \\ x_2 - x_0 & x_2^2 - x_2x_0 & x_2^3 - x_2^2x_0 & \cdots & x_2^n - x_2^{n-1}x_0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ x_n - x_0 & x_n^2 - x_nx_0 & x_n^3 - x_n^2x_0 & \cdots & x_n^3 - x_n^2x_0 \end{bmatrix}$$

$$\implies 1(x_1 - x_0)(x_2 - x_0)\cdots(x_n - x_0)\det \begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^n \\ 1 & x_2 & x_2^2 & \cdots & x_2^n \\ 1 & x_3 & x_3^2 & \cdots & x_3^n \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & x_3 & x_3^2 & \cdots & x_3^n \end{bmatrix}$$

$$\implies 1(x_1 - x_0)(x_2 - x_0)\cdots(x_n - x_0)\det \begin{bmatrix} 1 & 1 & 1 \cdots 1 \\ x_1 & x_2 & x_2 \cdots x_n \\ x_1^2 & x_2^2 & x_2^2 \cdots x_n^2 \\ \cdots & \cdots & \cdots\cdots\cdots \\ x_1^n & x_2^n & x_2^n \cdots x_n^n \end{bmatrix}$$

$$\implies (x_1-x_0)(x_2-x_0)\cdots(x_n-x_0)\det\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & x_2-x_1 & x_3-x_1 & \cdots & x_n-x_1 \\ 0 & x_2^2-x_2x_1 & x_3^2-x_3x_1 & \cdots & x_n^2-x_nx_1 \\ 0 & x_2^3-x_2^2x_1 & x_3^3-x_3^2x_1 & \cdots & x_n^3-x_n^2x_1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & x_2^n-x_2^{n-1}x_1 & x_3^n-x_3^{n-1}x_1 & \cdots & x_n^3-x_n^2x_1 \end{bmatrix}$$

$$\implies 1(x_1-x_0)(x_2-x_0)\cdots(x_n-x_0)\det\begin{bmatrix} x_2-x_1 & x_3-x_1 & \cdots & x_n-x_1 \\ x_2^2-x_2x_1 & x_3^2-x_3x_1 & \cdots & x_n^2-x_nx_1 \\ x_2^3-x_2^2x_1 & x_3^3-x_3^2x_1 & \cdots & x_n^3-x_n^2x_1 \\ \cdots & \cdots & \cdots & \cdots \\ x_2^n-x_2^{n-1}x_1 & x_3^n-x_3^{n-1}x_1 & \cdots & x_n^3-x_n^2x_1 \end{bmatrix}$$

$$\implies 1(x_1-x_0)(x_2-x_0)\cdots(x_n-x_0)\det\begin{bmatrix} x_2-x_1 & x_2^2-x_2x_1 & x_2^3-x_2^2x_1 & \cdots & x_2^n-x_2^{n-1}x_1 \\ x_3-x_1 & x_3^2-x_3x_1 & x_3^3-x_3^2x_1 & \cdots & x_3^n-x_3^{n-1}x_1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ x_n-x_1 & x_n^2-x_nx_1 & x_n^3-x_n^2x_1 & \cdots & x_n^3-x_n^2x_1 \end{bmatrix}$$

$$\implies 1(x_1-x_0)(x_2-x_0)\cdots(x_n-x_0)(x_2-x_1)(x_3-x_1)\cdots(x_n-x_1)\det\begin{bmatrix} 1 & x_2 & x_2^2 & \cdots & x_2^n \\ 1 & x_3 & x_3^2 & \cdots & x_3^n \\ 1 & x_4 & x_4^2 & \cdots & x_4^n \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{bmatrix}$$

By going backwards, we can continue to operate the same way on the matrix, and we will end up with all ones on the diagonal, and the product of all (x_j-x_i) where $i < j$, thus,

$$\det(V_n) = \Pi_{i<j}(x_j - x_i)$$

Exercise. 2.45 Let

$$A = \begin{bmatrix} x_{11} & x_{12} & x_{13} & \cdots & x_{1n} \\ x_{21} & x_{22} & x_{23} & \cdots & x_{2n} \\ x_{31} & x_{32} & x_{33} & \cdots & x_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ x_{n1} & x_{n2} & x_{n3} & \cdots & x_{nn} \end{bmatrix}$$

$$\implies \det(A) = \det\begin{bmatrix} x_{11} & x_{12} & x_{13} & \cdots & x_{1n} \\ x_{21} & x_{22} & x_{23} & \cdots & x_{2n} \\ x_{31} & x_{32} & x_{33} & \cdots & x_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ x_{n1} & x_{n2} & x_{n3} & \cdots & x_{nn} \end{bmatrix}$$

Now we also have that

$$\alpha A = \begin{bmatrix} \alpha x_{11} & \alpha x_{12} & \alpha x_{13} & \cdots & \alpha x_{1n} \\ \alpha x_{21} & \alpha x_{22} & \alpha x_{23} & \cdots & \alpha x_{2n} \\ \alpha x_{31} & \alpha x_{32} & \alpha x_{33} & \cdots & \alpha x_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \alpha x_{n1} & \alpha x_{n2} & \alpha x_{n3} & \cdots & \alpha x_{nn} \end{bmatrix}$$

$$\implies \det(\alpha A) = \implies \det \begin{bmatrix} \alpha x_{11} & \alpha x_{12} & \alpha x_{13} & \cdots & \alpha x_{1n} \\ \alpha x_{21} & \alpha x_{22} & \alpha x_{23} & \cdots & \alpha x_{2n} \\ \alpha x_{31} & \alpha x_{32} & \alpha x_{33} & \cdots & \alpha x_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \alpha x_{n1} & \alpha x_{n2} & \alpha x_{n3} & \cdots & \alpha x_{nn} \end{bmatrix}$$

$$\implies \alpha\det \begin{bmatrix} x_{11} & x_{12} & x_{13} & \cdots & x_{1n} \\ \alpha x_{21} & \alpha x_{22} & \alpha x_{23} & \cdots & \alpha x_{2n} \\ \alpha x_{31} & \alpha x_{32} & \alpha x_{33} & \cdots & \alpha x_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \alpha x_{n1} & \alpha x_{n2} & \alpha x_{n3} & \cdots & \alpha x_{nn} \end{bmatrix}$$

$$\implies \alpha^2\det \begin{bmatrix} x_{11} & x_{12} & x_{13} & \cdots & x_{1n} \\ x_{21} & x_{22} & x_{23} & \cdots & x_{2n} \\ \alpha x_{31} & \alpha x_{32} & \alpha x_{33} & \cdots & \alpha x_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \alpha x_{n1} & \alpha x_{n2} & \alpha x_{n3} & \cdots & \alpha x_{nn} \end{bmatrix}$$

Recursively, we have that

$$\implies \det(\alpha A) = \alpha^n\det \begin{bmatrix} x_{11} & x_{12} & x_{13} & \cdots & x_{1n} \\ x_{21} & x_{22} & x_{23} & \cdots & x_{2n} \\ x_{31} & x_{32} & x_{33} & \cdots & x_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ x_{n1} & x_{n2} & x_{n3} & \cdots & x_{nn} \end{bmatrix} = \alpha^n\det(A)$$

Exercise. 2.46 The matrix B can be row reduced to some uper triangular matrix, so that det(B) = $\Pi_{i=1}^n B_{ii}\beta_i$, where β_i is the necessary row operations to reduce the matrix. Similarly, D can be reduced so that det(D) = $\Pi_{i=1}^n D_{ii}\delta_i$.
Thus we have that

$$A = \begin{bmatrix} \beta_1 B_{11} & \beta_1 B_{12} & \dots & \beta_1 B_{1n} & C_{11} & & \dots & C_{1n} & \\ 0 & \beta_2 B_{22} & \dots & \beta_1 B_{2n} & C_{21} & & \dots & C_{2n} & \\ 0 & 0 & \dots & \beta_1 B_{2n} & C_{n1} & & \dots & C_{nn} & \\ \vdots & & & \vdots & & & & & \\ O & & & \beta_n B_{nn} & C_{n1} & & \dots & C_{nn} & \\ 0 & \dots & & & \delta_1 D_{11} & \delta_1 D_{12} & & \dots & \delta_1 D_{1n} \\ 0 & \dots & & & 0 & \delta_1 D_{12} & & \dots & \delta_1 D_{1n} \\ 0 & \dots & & & \vdots & & & & \vdots \\ 0 & \dots & & & 0 & 0 & & \dots & \delta_1 D_{1n} \end{bmatrix}$$

$$\begin{aligned} \det(A) &= (\beta_1 B_{11})\dots(\beta_n B_{nn})(\delta_1 D_{11})\dots(\delta_n D_{nn}) \\ &= \Pi_{i=1}^n \beta_i B_{ii} \Pi_{i=1}^n \delta_i D_{ii} \\ &= \det(B)\det(D) \end{aligned}$$

Exercise. 2.47 Knowing that

$$\begin{bmatrix} I & \mathbf{0} \\ -\mathbf{y^H} & 1 \end{bmatrix} \begin{bmatrix} I-\mathbf{xy^H} & \mathbf{x} \\ \mathbf{0^H} & 1 \end{bmatrix} \begin{bmatrix} I & \mathbf{0} \\ \mathbf{y^H} & 1 \end{bmatrix} = \begin{bmatrix} I & \mathbf{x} \\ \mathbf{0^H} & 1-\mathbf{y^H x} \end{bmatrix}$$

$$\implies \det\left(\begin{bmatrix} I & \mathbf{0} \\ -\mathbf{y^H} & 1 \end{bmatrix} \begin{bmatrix} I-\mathbf{xy^H} & \mathbf{x} \\ \mathbf{0^H} & 1 \end{bmatrix} \begin{bmatrix} I & \mathbf{0} \\ \mathbf{y^H} & 1 \end{bmatrix} \right) = \det\left(\begin{bmatrix} I & \mathbf{x} \\ \mathbf{0^H} & 1-\mathbf{y^H x} \end{bmatrix} \right)$$

By theorem 2.8.7,

$$\det\left(\begin{bmatrix} I & \mathbf{0} \\ -\mathbf{y^H} & 1 \end{bmatrix} \right) \det\left(\begin{bmatrix} I-\mathbf{xy^H} & \mathbf{x} \\ \mathbf{0^H} & 1 \end{bmatrix} \right) \det\left(\begin{bmatrix} I & \mathbf{0} \\ \mathbf{y^H} & 1 \end{bmatrix} \right) = \det\left(\begin{bmatrix} I & \mathbf{x} \\ \mathbf{0^H} & 1-\mathbf{y^H x} \end{bmatrix} \right)$$

Because the first and third matrices are triangular, we know that their determinants are equal to the product of the diagonals, or 1.

$$\det\left(\begin{bmatrix} I & \mathbf{x} \\ \mathbf{0^H} & 1-\mathbf{y^H x} \end{bmatrix} \right) = 1 \cdot 1 \cdot \dots \cdot (1-\mathbf{y^H x}) = (1-\mathbf{y^H x})$$

This has a determinant of $(1-\mathbf{y^H x})$, and by the previous exercise,

$$\det\left(\begin{bmatrix} I-\mathbf{xy^H} & \mathbf{x} \\ \mathbf{0^H} & 1 \end{bmatrix} \right) = \det(I-\mathbf{xy^H})\det(1) = \det(I-\mathbf{xy^H})$$

$$\det(I-\mathbf{xy^H}) = (1-\mathbf{y^H x})$$

Exercise. 2.48

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 4 \\ 5 & 6 & 7 \end{bmatrix}$$

We have that $A^{-1} = \frac{\text{adj}(A)}{\det(A)}$.

$$Adj(A) = \begin{bmatrix} -10 & 4 & 2 \\ 13 & -8 & -1 \\ -4 & 4 & 0 \end{bmatrix}$$

$\det(A) = 4$, Thus,

$$A^{-1} = \frac{1}{4} \begin{bmatrix} -10 & 4 & 2 \\ 13 & -8 & -1 \\ -4 & 4 & 0 \end{bmatrix}$$

Exercise. 2.50

(i)

By contrapositive: We must show that $\mathscr{C}$ is linearly dependent. Thus, one of the rows can be written as a combonation of the others, and

$$W(\mathbf{x}) = \det \begin{bmatrix} y_1 & y_2 & \dots & y_n \\ \vdots & & & \vdots \\ y_1^{n-1} & y_2^{n-1} & \dots & y_n^{n-1} \\ 0 & 0 & \dots & 0 \end{bmatrix}$$

Thus, we know that since we can co-factor expand along the last row, det $= 0$ and $W(x) = 0 \quad \forall\, \mathbf{x}$ (ii)

$$
\begin{aligned}
W(x) &= \det \begin{bmatrix} e^{\alpha x} & xe^{\alpha x} & x^2e^{\alpha x} \\ \alpha e^{\alpha x} & e^{\alpha x} + \alpha x e^{\alpha x} & 2xe^{\alpha x} + \alpha x^2 e^{\alpha x} \\ \alpha^2 e^{\alpha x} & 2\alpha e^{\alpha x} + \alpha^2 x e^{\alpha x} & 2e^{\alpha x} + 4xe^{\alpha x} + \alpha^2 x^2 e^{\alpha x} \end{bmatrix} \\
&= \det \begin{bmatrix} e^{\alpha x} & xe^{\alpha x} & x^2e^{\alpha x} \\ 0 & e^{\alpha x} & 2xe^{\alpha x} \\ 0 & 2\alpha e^{\alpha x} & 2e^{\alpha x} + 4xe^{\alpha x} \end{bmatrix} \\
&= \det \begin{bmatrix} e^{\alpha x} & xe^{\alpha x} & x^2e^{\alpha x} \\ 0 & e^{\alpha x} & 2xe^{\alpha x} \\ 0 & 0 & 2e^{\alpha x} \end{bmatrix} \\
&= 2e^{3(\alpha x)} \neq 0 \quad \forall\, \mathbf{x}
\end{aligned}
$$