Math Sec 1.4

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Exercise. 1.26
Proof: Use substiution $j = k - 5$

$$\sum_{k=5}^{n}(k-5)^2 = \sum_{j-1}^{n-5} j^2 = \frac{(2n+1)(n+1)n}{6}$$

Exercise. 1.27

Use substitution $i = k - 4$, $k = i + 4$

$$\sum_{i=0}^{n}\sum_{j=-3}^{k-8}(k-4) = \sum_{i=0}^{n}\sum_{j=-3}^{i-4} i$$

Note that the second sum has $i - 4 - (-3) + 1 = i$ iterations. Thus,

$$= \sum_{i=0}^{n}(i^2) = \frac{n(n+1)(n+2)}{6}$$

Exercise. 1.28

Using the substitution $i = k - 3$ and $k = i + 3$, we have

$$
\begin{aligned}
\sum_{j=-3}^{n-3}\sum_{k=j+3}^{n+3} k-3 &= \sum_{j=-3}^{n-3}\sum_{k=j}^{n} i \\
&= \sum_{j=-3}^{n-3}\Big(\sum_{i=0}^{n} i - \sum_{i=0}^{j-1} i\Big) \\
&= \sum_{j=-3}^{n-3}\Big(\frac{n(n+1}{2} - \frac{j(j-1)}{2}\Big) \\
&\text{By FTOFC} \\
&= \frac{1}{2}\sum_{j=-3}^{n-3}(n)(n+1) - \sum_{j=0}^{n-3} j^2 - \sum_{j=0}^{n-3} j - ((-3)^2+(-2)^2+(-1)^2) + (-3+-2+-1) \\
&\text{By subtracting the finite part, below 0} \\
&= \frac{1}{2}\Big[n(n+1)^2 - \frac{(n-3)(n-2)(2n-5)}{6} + \frac{(n-3)(n-2)}{2} - 20\Big]
\end{aligned}
$$

Exercise. 1.29

If $f(x) = k^4$, Then $(\Delta f)(x) = (k+1)^4 - k^4$

$$
\begin{aligned}
&= (k^2+2k+1)^2 - k^4 \\
(\Delta f)(k) &= 4k^3 + 6x^2 + 4x + 1
\end{aligned}
$$

It follows that,

$$
\begin{aligned}
&\sum_{k=1}^{b-1}(4k^3+6x^2+4x+1) = \sum_{k=1}^{b-1}(f\Delta) \\
&= f(b) - f(a) = b^4 - a^4 \\
&b^4 - a^4 = \sum_{k=1}^{b-1}(4k^3+6k^2+4k+1) = 4\sum_{k=1}^{b-1}(k^3) + 6\sum_{k=1}^{b-1}(k^2) + 4\sum_{k=1}^{b-1}(k) + (b-a)
\end{aligned}
$$

Note, that $(b^4-a^4)-(b-a) = b^4 - b - (a^4 - a) = b(b^3-1) - a(a^3-1)$. Thus,

$$
b(b^3-1) - a(a^3-1) = 4\sum_{k=1}^{b-1}(k^3) + 6\sum_{k=1}^{b-1}(k^2) + 4\sum_{k=1}^{b-1}(k)
$$

Use $a = 1,\ b = n + 1$

$$
\begin{aligned}
(n+1)((n+1)^3 - 1) &= 4\sum_{k=1}^{b-1}(k^3) + 6\sum_{k=1}^{b-1}(k^2) + 4\sum_{k=1}^{b-1}(k) \\
n^4 + 4n^3 + 6n^2 + 3n &= 4\sum_{k=1}^{b-1}(k^3) + (2n+1)(n+1)(n) + 2n(n+1) \\
&= 4\sum_{k=1}^{b-1}(k^3) + 2n^3 + 5n^2 + 3n
\end{aligned}
$$

Thus,

$$
\begin{aligned}
4\sum_{k=1}^{b-1}(k^3) &= n^4 + 2n^3 + n^2 = (n^2+n)^2 = (n(n+1))^2 \\
\sum_{k=1}^{b-1}(k^3) &= \frac{1}{4}(n(n+1))^2 = \Big(\frac{n(n+1)}{2}\Big)^2
\end{aligned}
$$

Exercise. 1.30
If $f(i) = \frac{-1}{i}$,

$$
\begin{aligned}
(\Delta f)(i) &= \Big(\frac{-1}{(i+1)}\Big) + \frac{1}{i} \\
&= \frac{-i}{i(i+1)} + \frac{i+1}{i(i+1)} \\
&= \frac{1}{i(i+1)}
\end{aligned}
$$

Thus, by the Fundamental Theorem of Finite Calculus,

$$
\begin{aligned}
\sum_{i=1}^{n}\frac{1}{i(i+1)} = \sum_{i=1}^{n}(\Delta f) &= \frac{-1}{n+1} - \frac{-1}{1} \\
&= 1 - \frac{1}{n+1}
\end{aligned}
$$

Exercise. 1.31
Proof:

(i)

$\rightarrow \sum_{k=0}^{n} \sum_{j=k}^{n} j$

$$\begin{aligned}
\sum_{k=0}^{n} \sum_{j=k}^{n} j &= \sum_{k=0}^{n} \Big(\sum_{j=0}^{n} j - \sum_{j=0}^{k-1} j \Big) \\
&= \sum_{k=0}^{n} \Big(\frac{n(n+1)}{2} - \frac{(k-1)k}{2} \Big) \\
&= \frac{1}{2} - \sum_{k=0}^{n} k^2 + \sum_{k=0}^{n} k \\
&= \frac{1}{2} \Big(n(n+1)^2 - \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} \Big) \\
&= \frac{1}{2} \Big(n^3 + 2n^2 + n - (\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}) + \frac{n^2}{2} + \frac{n}{2} \Big) \\
&= \frac{n^3}{3} + \frac{3n^2}{2} + \frac{2n}{3}
\end{aligned}$$

$\leftarrow \sum_{j=k}^{n} \sum_{k=0}^{n} j$

$$\sum_{j=k}^{n} \sum_{k=0}^{n} j = \sum_{j=0}^{n} \sum_{k=0}^{n} j$$

Because there are $j+1$ steps in the second sum,

$$\begin{aligned}
&= \sum_{j=0}^{n} j^2 + \sum_{j=0}^{n} j \\
&= \sum_{j=1}^{n} j^2 + \sum_{j=1}^{n} j \\
&= \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} \\
&= \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6} + \frac{n^2}{2} + \frac{n}{2} \\
&= \frac{n^3}{3} + \frac{3n^2}{2} + \frac{2n}{3}
\end{aligned}$$

Thus, they are equivalent.

(ii):

$\rightarrow \sum_{k=0}^{n} \sum_{j=0}^{k} j$

$$\begin{aligned}
\sum_{k=0}^{n} \sum_{j=0}^{k} j &= \sum_{k=0}^{n} \frac{(k+1)k}{2} \\
&= \frac{1}{2}\left(\sum_{k=0}^{n} k^2 + \sum_{k=0}^{n} k\right) \\
&= \frac{1}{2}\left(\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}\right) \\
&= \frac{n(n+1)(n+2)}{6}
\end{aligned}$$

$\leftarrow \sum_{j=0}^{k} \sum_{k=0}^{n} j$

$$\begin{aligned}
\sum_{j=0}^{k} \sum_{k=0}^{n} j = \sum_{j=0}^{n} \sum_{k=j}^{n} j \\
&= \sum_{j=0}^{n} j(n-j+1) \\
&= \sum_{j=0}^{n} jn - j^2 + j \\
&= \frac{n^2(n+1)}{2} - \frac{(2n+1)(n+1)(n)}{6} \frac{n(n+1)}{2} \\
&= n(n+1)\left(\frac{n}{2} - \frac{(2n+1)}{6} + \frac{1}{2}\right) \\
&= n(n+1)\left(\frac{(n+2)}{6} \frac{n+2}{6}\right) \\
&= \frac{n(n+1)(n+2)}{6}
\end{aligned}$$

Exercise. 1.32

By taking the derivative of both sides, we get

$$\sum_{t=0}^{n} \beta^{t-1} = \frac{(\beta-1)(N+1)\beta^N - (\beta^{N+1}-1)}{(\beta-1)^2}$$

Now taking the limit as N approaches infinity, All β^N terms will go to zero.

$$lim_{N\rightarrow\infty} \sum_{t=0}^{N} t\beta^{t-1} = \frac{1}{(\beta-1)^2}$$

This implies that,

$$\sum_{t=0}^{N} (t\beta^{t-1})/\beta = \frac{1}{(\beta-1)^2}$$

Which implies,

$$\sum_{t=0}^{N} t\beta^{t-1} = \frac{\beta}{(1-\beta)^2} =$$