Math Sec 3.1

Rex McArthur
Math 344

October 5, 2015

Exercise. 3.1
(i)

$$\begin{aligned}
\langle \mathbf{x}, \mathbf{y} \rangle &= \frac{1}{2}(\langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{x}, \mathbf{y} \rangle) \\
&= \frac{1}{2}(\frac{1}{2}(\langle \mathbf{x}, \mathbf{x} \rangle + 2\langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{y}, \mathbf{y} \rangle - (\langle \mathbf{x}, \mathbf{x} \rangle - 2\langle \mathbf{x}, \mathbf{y} \rangle - \langle \mathbf{y}, \mathbf{y} \rangle))) \\
&= \frac{1}{4}(\langle \mathbf{x}+\mathbf{y}, \mathbf{x}+\mathbf{y} \rangle - \langle \mathbf{x}-\mathbf{y}, \mathbf{x}-\mathbf{y} \rangle) \\
&= \frac{1}{4}(||x+y||)^2 - (||x-y||^2)
\end{aligned}$$

(ii)

$$\begin{aligned}
\|x\|^2 + \|y\|^2 &= \frac{1}{2}(\langle \mathbf{x}, \mathbf{x} \rangle + \langle \mathbf{x}, \mathbf{y} \rangle) \\
&= \frac{1}{2}(\langle \mathbf{x}, \mathbf{x} \rangle + \langle \mathbf{y}, \mathbf{y} \rangle\langle \mathbf{x}, \mathbf{x} \rangle + \langle \mathbf{y}, \mathbf{y} \rangle) \\
&= \frac{1}{2}(\frac{1}{2}(\langle \mathbf{x}, \mathbf{x} \rangle + 2\langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{y}, \mathbf{y} \rangle - (\langle \mathbf{x}, \mathbf{y} \rangle - \langle \mathbf{y}, \mathbf{y} \rangle))) \\
&= \frac{1}{4}(\langle \mathbf{x}+\mathbf{y}, \mathbf{x}+\mathbf{y} \rangle - \langle \mathbf{x}-\mathbf{y}, \mathbf{x}-\mathbf{y} \rangle) \\
&= \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2)
\end{aligned}$$

Exercise. 3.2

$$\begin{aligned}
\langle \mathbf{x}, \mathbf{y} \rangle &= \frac{1}{2}(\langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{y}, \mathbf{x} \rangle - \langle \mathbf{y}, \mathbf{x} \rangle + \langle \mathbf{x}, \mathbf{y} \rangle) \\
&= \frac{1}{2}(\langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{x}, \mathbf{y} \rangle - i^2\langle \mathbf{x}, \mathbf{y} \rangle + i^2\langle \mathbf{y}, \mathbf{x} \rangle) \\
&= \frac{1}{2}(\frac{1}{2}(\|x+y\|^2 - \|x-y\|^2) + \frac{1}{2}(i\langle \mathbf{x}, \mathbf{iy} \rangle - i\langle \mathbf{x}, \mathbf{iy} \rangle \\
&= \frac{1}{4}(\|x+y\|^2 - \|x-y\| + i\|x-iy\| - i\|x+iy\|^2)
\end{aligned}$$

Exercise. 3.3

$$
\begin{aligned}
cos(\theta) &= \frac{\langle x, x^5\rangle}{\|x\|\|x^5\|} \\
\langle x, x^5\rangle &= \int_0^1 x^6 dx = \frac{1}{7}x^7|_0^1 = \frac{1}{7} \\
\|x\| = \sqrt{\langle x, x\rangle} &= \int_0^1 x^2 dx = \frac{1}{3}x^3|_0^1 = \frac{1}{\sqrt{3}} \\
\|x^5\| = \sqrt{\langle x, x\rangle} &= \int_0^1 x^{10} dx = \frac{1}{11}x^{11}|_0^1 = \frac{1}{\sqrt{11}} \\
\theta = cos^{-1}(\frac{\sqrt{33}}{7}) &\approx .60824
\end{aligned}
$$

ii)

$$
\begin{aligned}
cos(\theta) &= \frac{\langle x^2, x^4\rangle}{\|x^2\|\|x^4\|} \\
\langle x, x^4\rangle &= \int_0^1 x^5 dx = \frac{1}{7}x^7|_0^1 = \frac{1}{7} \\
\|x^2\| = \sqrt{\langle x, x\rangle} &= \int_0^1 x^4 dx = \frac{1}{5}x^5|_0^1 = \frac{1}{\sqrt{5}} \\
\|x^4\| = \sqrt{\langle x, x\rangle} &= \int_0^1 x^8 dx = \frac{1}{9}x^9|_0^1 = \frac{1}{\sqrt{9}} \\
\theta = cos^{-1}(\frac{\sqrt{45}}{7}) &\approx .289
\end{aligned}
$$

Exercise. 3.4
Suppose $\|T\mathbf{x}\| = a\|\mathbf{x}\|$, Thus,

$$
\begin{aligned}
\frac{\langle \mathbf{Tx}, \mathbf{Ty}\rangle}{\|x\|\|y\|} &= \frac{\frac{1}{4}(||Tx + Ty||)^2 - (||Tx - Ty||^2)}{\|Tx\|\|Ty\|} \\
&= \frac{\frac{1}{4}a^2\|x + y\|^2 - \|x - y\|^2}{a^2\|x\|\|y\|} \\
&= \frac{\langle \mathbf{x}, \mathbf{y}\rangle}{\|x\|\|y\|}
\end{aligned}
$$

Now suppose, $\frac{\langle \mathbf{Tx}, \mathbf{Ty}\rangle}{\|Tx\|\|Ty\|} = \frac{\langle \mathbf{x}, \mathbf{y}\rangle}{\|x\|\|y\|}$. Thus, $\langle \mathbf{Tx}, \mathbf{Ty}\rangle = \langle \mathbf{x}, \mathbf{y}\rangle = 0$ iff $\mathbf{x}, \mathbf{y}$ are orthogonal, thus it preserves angle. Note that $\mathbf{y} = \mathrm{proj}_x\mathbf{y} + \mathbf{r}$,

$$
\frac{\langle \mathbf{Tx}, \mathbf{Ty}\rangle}{\|T\mathbf{x}\|\|T\mathbf{y}\|} = \frac{\langle \mathbf{Tx}, \mathbf{T}(\alpha\mathbf{x} + \mathbf{r})\rangle}{\|Tx\|\|Ty\|}
$$

Thus,

$$\begin{aligned}
\frac{\langle \mathbf{Tx}, \mathbf{T}(\alpha\mathbf{x}+\mathbf{r})\rangle}{\|Tx\|\|Ty\|} &= \frac{\langle \mathbf{x}, \alpha\mathbf{x}+\mathbf{r}\rangle}{\|x\|\|y\|} \\
\frac{\alpha\|Tx\|^2}{\|Tx\|\|Ty\|} &= \frac{\alpha\|x\|^2}{\|x\|\|y\|} \\
\frac{\alpha\|Tx\|}{\|Ty\|} &= \frac{\alpha\|x\|}{\|y\|} \\
\frac{\alpha\|Tx\|}{\|x\|} &= \frac{\alpha\|Ty\|}{\|y\|}
\end{aligned}$$

Which is equal to some constant for both, and thus,

$$(\|T\mathbf{x}\| = a\|\mathbf{x}\|)$$

and,

$$(\|T\mathbf{y}\| = a\|\mathbf{y}\|)$$

Exercise. 3.5

$$f = e^x \qquad g = x - 1$$

$$\begin{aligned}
\text{proj}_u(f) &= \frac{\langle g, f\rangle}{\langle g, g\rangle} \cdot g \\
\langle g, f\rangle &= \langle e^x, x-1\rangle = \int_0^1 xe^x - e^x dx = xe^x - 2e^x|_0^1 = (e-2e) + (2) = -e+2 \\
\langle g, g\rangle &= \int_0^1 (x-1)^2 = \int_0^1 x^2 - 2x + 1 dx = \frac{1}{3}x^3 - x^2 + x|_0^1 = \frac{1}{3}
\end{aligned}$$

Thus, we have that projection e^x onto $(x-1)$ is

$$\frac{-e+2}{\frac{1}{3}} \cdot (x-1) = (-3e+6)(x-1)$$

Exercise. 3.6
Note, $0 \leq \|x - \lambda y\|^2$ and let $\lambda = \frac{\langle x,y\rangle}{\langle y,y\rangle}$.

$$\|x - \lambda y\|^2 = \langle x - \lambda y, x - \lambda y\rangle$$

Note that $x - \lambda y$ is orthogonal to y and λy, because

$$\langle \mathbf{x} - \frac{\langle \mathbf{x}, \mathbf{y}\rangle}{\|y\|^2}, \frac{-\langle \mathbf{x}, \mathbf{y}\rangle y}{\|y\|^2}\rangle = \langle \mathbf{x} - \text{proj}_y\mathbf{x}, -\text{proj}_y\mathbf{x}\rangle$$

Which leads to y being orthogonal to both, because it is a projection onto it. Thus, $\langle x - \lambda y, -\lambda y\rangle = 0$ and $\langle -\lambda y, x - \lambda y\rangle = 0$. and by Bessel's inequality we get

$$\begin{aligned}
0 \leq \|x - \lambda y\|^2 &= \langle x - \lambda y, x - \lambda y\rangle \\
&= \langle x, x\rangle - \langle \lambda y, x\rangle \\
&= \|x^2\| - \frac{\langle x, y\rangle^2}{\langle y, y\rangle^2} \\
&= \|x^2\| - \frac{\langle x, y\rangle^2}{\|y\|^2} \\
0 \leq \|x^2\| - \frac{\langle x, y\rangle^2}{\|y\|^2} \\
&\Rightarrow \frac{\langle x, y\rangle^2}{\|y\|^2} \leq \|x^2\| \\
&\Rightarrow \frac{\langle x, y\rangle}{\|y\|} \leq \|x\| \\
&\Rightarrow \langle x, y\rangle \leq \|x\| \ \|y\|
\end{aligned}$$