Algorithms for solving for SS and Time Path of OG model with heterogeneous static firms

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The model assumed for these solutions is one without taxes or bequests. Endogenous labor. J ability types. M firms. I consumption goods. T periods from initial state to the steady state.

1 Steady State

1. Guess $\bar{r}$ and $\bar{w}$

2. Determine output prices, $\bar{p}_m$ using zero profit condition and firm FOCS together. These imply:

$$\bar{p}_m = \left[(1-\gamma_m)\left(\frac{\bar{w}}{\bar{A}_m}\right)^{1-\epsilon_m} + \gamma_m\left(\frac{(\bar{r}+\delta_m)}{\bar{A}_m}\right)^{1-\epsilon_m}\right]^{\frac{1}{1-\epsilon_m}} \tag{1}$$

3. Determine consumption goods prices using the fixed coefficient matrix Π mapping output goods to consumption goods: $\bar{p}^c_i = \sum_{m=1}^{M} \pi_{i,m}\bar{p}_m$

4. Determine the price of the composite consumption good: $\tilde{p} = \prod_{i=1}^{I}\left(\frac{\bar{p}^c_i}{\alpha_i}\right)^{\alpha_i}$

5. With prices $\bar{r}, \bar{w}, \bar{p}^c_i, \tilde{p}$, we can solve the HH problem. We can solve the problem for each type j separately using a root-finding algorithm to solve for the $2\times S$ unknowns from $2\times S$ equations: the HH FOCs for the choices of savings and labor supply in each year of life. This yields $\bar{b}_{j,s}$ and $\bar{n}_{j,s}$.

6. Using the HH's budget constraint and the vector of prices, we can solve for composite consumption, $\tilde{c}_{j,s}$.

7. We can solve for the total demand for each consumption good using the necessary conditions from the HH's subutility function. These imply (where $\bar{c}_i$ is the minimum expenditure on good i):

$$\bar{c}_{i,j,s} = \frac{\alpha_i \tilde{p}_s \tilde{c}_{j,s}}{\bar{p}^c_i} + \bar{c}_i, \tag{2}$$

8. Summing over J and S we can get total demand for consumption of good i: $\bar{C}_i = \sum_{j=1}^{J}\sum_{s=1}^{S} \bar{c}_{i,j,s}$

9. With $\bar{C}_i$ and prices, we can solve for the demand for output from each industry using the resource constraint, the fixed coefficient matrices Π and Ξ (where Ξ is the fixed coefficient input-output matrix - mapping production goods to capital goods), and the demand for investment.

   - Investment demand for industry m in the SS is given by: $\bar{I}_m = \delta_m \bar{K}_m$
   - Demand for industry m output from consumption is given by: $\bar{X}^c_m = \sum_{i=1}^{I} \pi_{i,m}\bar{C}_i$
   - Demand for industry m output from investment demand is given by: $\bar{X}^i_m = \sum_{j=1}^{M} \xi_{j,m}\bar{I}_j = \sum_{j=1}^{M} \xi_{j,m}\delta_j \bar{K}_j$
   - And we can use the firm's FOC for capital, labor, and the production function together, to write the demand for capital as a function of output and factor prices:

$$\bar{K}_m = \frac{\bar{X}_m}{\bar{A}_m}\left[\gamma_m^{\frac{1}{\epsilon_m}} + (1-\gamma_m)^{\frac{1}{\epsilon_m}}\left(\frac{\bar{r}+\delta_m}{\bar{w}}\right)^{\epsilon_m - 1}\left(\frac{1-\gamma_m}{\gamma_m}\right)^{\frac{\epsilon_m-1}{\epsilon_m}}\right]^{\frac{\epsilon_m}{1-\epsilon_m}} \tag{3}$$

- Resource constraint for industry m:

$$
\begin{aligned}
\bar{X}_m &= \bar{X}^c_m + \bar{X}^i_m \\
&= \sum_{i=1}^{I} \pi_{i,m}\bar{C}_i + \sum_{j=1}^{M} \xi_{j,m}\bar{I}_j \\
&= \sum_{i=1}^{I} \pi_{i,m}\bar{C}_i + \sum_{j=1}^{M} \xi_{j,m}\delta_j\bar{K}_j \\
&= \sum_{i=1}^{I} \pi_{i,m}\bar{C}_i + \sum_{j=1}^{M} \xi_{j,m}\delta_j\frac{\bar{X}_j}{\bar{A}_j}\left[\gamma_j^{\frac{1}{\epsilon_j}} + (1-\gamma_j)^{\frac{1}{\epsilon_j}}\left(\frac{\bar{r}+\delta_j}{\bar{w}}\right)^{\epsilon_j-1}\left(\frac{1-\gamma_j}{\gamma_j}\right)^{\frac{\epsilon_j-1}{\epsilon_j}}\right]^{\frac{\epsilon_j}{1-\epsilon_j}}
\end{aligned}
\tag{4}
$$

- The above is a system of M equations and M unknowns: a root finder can be used to solve for $\bar{X}_m$

10. With $\bar{X}_m$ and prices, we can solve for the demand for capital from each industry. We'll use the capital market clearing condition ($\sum_{m=1}^{M}\bar{K}_m = \sum_{j=1}^{J}\sum_{s=1}^{S}\bar{b}_{j,s}$) and the firm's FOC for capital together to find these demands. Denoting the supply of capital as $\bar{K}^s = \sum_{j=1}^{J}\sum_{s=1}^{S}\bar{b}_{j,s}$, we have:

$$
\bar{K}_m = \bar{K}^s - \sum_{k\neq m}\gamma_k\bar{X}_k\left(\frac{\bar{p}_m\left(\frac{\gamma_m\bar{X}_m}{\bar{K}_m}\right)\bar{A}_m^{\frac{\epsilon_m-1}{\epsilon_m}} - \delta_m + \delta_k}{\bar{p}_k}\bar{A}_k^{\frac{1-\epsilon_k}{\epsilon_k}}\right)^{-\epsilon_k}
\tag{5}
$$

This is a system of M nonlinear equations that can be solved for $\bar{K}_m$ using a root finding algorithm.

11. With $\bar{X}_m$ and prices, we can solve for the demand for labor from each industry. We'll use the labor market clearing condition ($\sum_{m=1}^{M}\bar{EL}_m = \sum_{j=1}^{J}\sum_{s=1}^{S}e_{j,s}\bar{n}_{j,s}$) and the firm's FOC for labor together to find these demands. Denoting the supply of labor as $\bar{L}^s = \sum_{j=1}^{J}\sum_{s=1}^{S}e_{j,s}\bar{n}_{j,s}$, we have:

$$\bar{EL}_m = \bar{L}^s - \sum_{k \neq m} (1-\gamma_k)\bar{X}_k \left(\frac{\bar{p}_m \left(\frac{(1-\gamma_m)\bar{X}_m}{\bar{EL}_m}\right) \bar{A}_m^{\frac{\epsilon_m - 1}{\epsilon_m}} - \delta_m + \delta_k}{\bar{p}_k} \bar{A}_k^{\frac{1-\epsilon_k}{\epsilon_k}} \right)^{-\epsilon_k} \tag{6}$$

This is a system of M nonlinear equations that can be solved for $\bar{EL}_m$ using a root finding algorithm.

12. Use the demands for capital and labor found above in the FOC for a particular firm m (can be any firm), this will imply and interest rate and wage rate. Call these r_{new} and w_{new}, respectively. We have:

$$r_{new} = \bar{p}_m \left(\frac{\gamma_m \bar{X}_m}{\bar{K}_m}\right) \bar{A}_m^{\frac{\epsilon_m - 1}{\epsilon_m}} - \delta_m, \forall\ m \tag{7}$$

$$w_{new} = \bar{p}_m \left(\frac{(1-\gamma_m)\bar{X}_m}{\bar{EL}_m}\right) \bar{A}_m^{\frac{\epsilon_m - 1}{\epsilon_m}}, \forall\ m \tag{8}$$

13. If r_{new} and w_{new} are equal to $\bar{r}$ and $\bar{w}$ guessed, then stop. Else, update the guesses over $\bar{r}$ and $\bar{w}$ and repeat. One can use r_{new} and w_{new} to inform the new guess.

2 Time Path

1. Guess the time paths for factor prices: r_t and w_t

2. Determine time path for output prices, $p_{m,t}$ using zero profit condition and firm FOCS together. These imply:

$$p_{m,t} = \left[(1-\gamma_m) \left(\frac{w_t}{A_{m,t}}\right)^{1-\epsilon_m} + \gamma_m \left(\frac{(r_t + \delta_m)}{A_{m,t}}\right)^{1-\epsilon_m} \right]^{\frac{1}{1-\epsilon_m}} \tag{9}$$

3. Determine the time path for consumption goods prices using the fixed coefficient matrix Π mapping output goods to consumption goods: $p^c_{i,t} = \sum_{m=1}^{M} \pi_{i,m} p_{m,t}$

4. Determine time path for the price of the composite consumption good: $\tilde{p}_t = \prod_{i=1}^{I}\left(\frac{p^c_{i,t}}{\alpha_i}\right)^{\alpha_i}$

5. With the time paths of prices $r_t, w_t, p^c_{i,t}, \tilde{p}_t$, we can solve the HH problem. I think we can solve the path for each type j and cohort separately using a root-finding algorithm to solve for the $2 \times S$ unknowns from $2 \times S$ equations: the HH FOCs for the choices of savings and labor supply. This yields, $b_{j,s,t}$ and $n_{j,s,t}$.

6. Using the HH's budget constraint and the vector of prices, we can solve for composite consumption, $\tilde{c}_{j,s,t}$.

7. We can solve for the total demand for each consumption good using the necessary conditions from the HH's subutility function. These imply (where $\bar{c}_i$ is the minimum expenditure on good i):

$$c_{i,j,s,t} = \frac{\alpha_i \tilde{p}_{s,t}\tilde{c}_{j,s,t}}{p_{i,t}} + \bar{c}_i, \tag{10}$$

8. Summing over J and S we can get time path for the demand for consumption of good i: $C_{i,t} = \sum_{j=1}^{J}\sum_{s=1}^{S} c_{i,j,s,t}$

9. With $C_{i,t}$ and prices, we can solve for the demand for output from each industry using the resource constraint, the fixed coefficient matrices Π and Ξ (where Ξ is the fixed coefficient input-output matrix - mapping production goods to capital goods), and the demand for investment.

   - Investment demand for industry m in period t is given by: $I_{m,t} = K_{m,t+1} - (1-\delta_m)K_{m,t}$
   - Demand for industry m output in period t from consumption is given by: $X^c_{m,t} = \sum_{i=1}^{I}\pi_{i,m}C_{i,t}$
   - Demand for industry m output in period t from investment demand is given by: $X^i_{m,t} = \sum_{j=1}^{M}\xi_{j,m}I_{j,t} = \sum_{j=1}^{M}\xi_{j,m}\left(K_{j,t+1} - (1\delta_m)K_{j,t}\right)$

- And we can use the firm's FOC for capital, labor, and the production function together, to write the demand for capital in each period as a function of output and factor prices:

$$K_{m,t} = \frac{X_{m,t}}{A_{m,t}} \left[\gamma_m^{\frac{1}{\epsilon_m}} + (1-\gamma_m)^{\frac{1}{\epsilon_m}} \left(\frac{r_t + \delta_m}{w_t} \right)^{\epsilon_m - 1} \left(\frac{1-\gamma_m}{\gamma_m} \right)^{\frac{\epsilon_m - 1}{\epsilon_m}} \right]^{\frac{\epsilon_m}{1-\epsilon_m}} \tag{11}$$

- Resource constraint for industry m in year t:

$$\begin{aligned} X_{m,t} &= X^c_{m,t} + X^i_{m,t} \\ &= \sum_{i=1}^{I} \pi_{i,m} C_{i,t} + \sum_{j=1}^{M} \xi_{j,m} I_{j,t} \\ &= \sum_{i=1}^{I} \pi_{i,m} C_{i,t} + \sum_{j=1}^{M} \xi_{j,m} \left(K_{j,t+1} - (1\delta_j) K_{j,t} \right) \\ &= \sum_{i=1}^{I} \pi_{i,m} C_{i,t} ... \\ &\quad + \sum_{j=1}^{M} \xi_{j,m} \left[\left(\frac{X_{j,t+1}}{A_{j,t+1}} \left[\gamma_j^{\frac{1}{\epsilon_j}} + (1-\gamma_j)^{\frac{1}{\epsilon_j}} \left(\frac{r_{t+1} + \delta_j}{w_{t+1}} \right)^{\epsilon_j - 1} \left(\frac{1-\gamma_j}{\gamma_j} \right)^{\frac{\epsilon_j - 1}{\epsilon_j}} \right]^{\frac{\epsilon_j}{1-\epsilon_j}} \right) ... \right. \\ &\quad \left. -(1-\delta_j) \left(\frac{X_{j,t}}{A_{j,t}} \left[\gamma_j^{\frac{1}{\epsilon_j}} + (1-\gamma_j)^{\frac{1}{\epsilon_j}} \left(\frac{r_t + \delta_j}{w_t} \right)^{\epsilon_j - 1} \left(\frac{1-\gamma_j}{\gamma_j} \right)^{\frac{\epsilon_j - 1}{\epsilon_j}} \right]^{\frac{\epsilon_j}{1-\epsilon_j}} \right) \right] \end{aligned} \tag{12}$$

- The above is a system of $M \times T$ equations and $M \times T$ unknowns: a root finder can be used to solve for $X_{m,t}$
- Alternatively, one can leverage the recursive nature of the problem. In particular, since we know the steady state solution, we have $K_{m,T}$ for all m. Thus we can find $I_{m,T-1} = K_{m,T} - (1\delta_m)K_{m,T-1}$. Then using the equations above, we can solve a system of M equations and M unknowns to find $X_{m,T-1}$. We then repeat this for each period from $T-1$ back to the

initial period. This yields the time path of output demand, $X_{m,t}$ without having to solve a very large system of equations at once.

10. With the time path for output demand, $X_{m,t}$, and prices, we can solve for the demand for capital from each industry in each period t. We'll use the capital market clearing condition ($\sum_{m=1}^{M} K_{m,t} = \sum_{j=1}^{J}\sum_{s=1}^{S} b_{j,s,t}$) and the firm's FOC for capital together to find these demands. Denoting the supply of capital as $K_t^s = \sum_{j=1}^{J}\sum_{s=1}^{S} b_{j,s,t}$, we have:

$$K_{m,t} = K^{s,t} - \sum_{k \neq m} \gamma_k X_{k,t} \left(\frac{p_{m,t}\left(\frac{\gamma_m X_{m,t}}{K_{m,t}}\right) A_{m,t}^{\frac{\epsilon_m - 1}{\epsilon_m}} - \delta_m + \delta_k}{p_{k,t}} A_{k,t}^{\frac{1-\epsilon_k}{\epsilon_k}} \right)^{-\epsilon_k} \tag{13}$$

For each year, t, this is a system of M nonlinear equations that can be solved for $K_{m,t}$ using a root finding algorithm.

11. With the time path for output demand, $X_{m,t}$ and prices, we can solve for the demand for labor from each industry. We'll use the labor market clearing condition ($\sum_{m=1}^{M} EL_{m,t} = \sum_{j=1}^{J}\sum_{s=1}^{S} e_{j,s}n_{j,s,t}$) and the firm's FOC for labor together to find these demands. Denoting the supply of labor as $L^{s,t} = \sum_{j=1}^{J}\sum_{s=1}^{S} e_{j,s}n_{j,s,t}$, we have:

$$EL_{m,t} = L^{s,t} - \sum_{k \neq m} (1-\gamma_k) X_{k,t} \left(\frac{p_{m,t}\left(\frac{(1-\gamma_m) X_{m,t}}{EL_{m,t}}\right) A_{m,t}^{\frac{\epsilon_m - 1}{\epsilon_m}} - \delta_m + \delta_k}{p_{k,t}} A_{k,t}^{\frac{1-\epsilon_k}{\epsilon_k}} \right)^{-\epsilon_k} \tag{14}$$

For each year, t, this is a system of M nonlinear equations that can be solved for $EL_{m,t}$ using a root finding algorithm.

12. Use the demands for capital and labor found above in the FOC for a particular firm, this will imply and interest rate and wage rate in each period, t. Call these $r_{new,t}$ and $w_{new,t}$, respectively. We have:

$$r_{new,t} = p_m \left(\frac{\gamma_m X_{m,t}}{K_{m,t}} \right) A_{m,t}^{\frac{\epsilon_m - 1}{\epsilon_m}} - \delta_m, \forall\ m, t \tag{15}$$

$$w_{new} = p_{m,t} \left(\frac{(1-\gamma_m) X_{m,t}}{EL_{m,t}} \right) A_{m,t}^{\frac{\epsilon_m - 1}{\epsilon_m}}, \forall\ m, t \tag{16}$$

13. If $r_{new,t}$ and $w_{new,t}$ are equal the time paths r_t and w_t guessed, then stop. Else, update the guesses over r_t and w_t and repeat. One can use $r_{new,t}$ and $w_{new,t}$ to inform the new guess.