**Freescale Semiconductor**
APPLICATION NOTE

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# Trigonometry Approximations

**Mark Pedley and Michael Stanley**

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## 1 Introduction

This application note documents mathematical approximations to inverse trigonometric functions used in the *Freescale Sensor Fusion Library* and contained in the file *approximations.c*.

The approximations are more efficient than standard C floating point library when implemented on an integer microcontroller but are still highly accurate. The results are also computed directly in degrees rather than radians saving the additional floating point multiplication required to convert from radians to degrees.

The mathematics underlying these approximations first transforms inverse sine and inverse cosine calls to an inverse tangent with modified argument. The argument of the inverse tangent is then transformed again allowing a call to a Pade[3,2] rational approximation to the inverse tangent in the limited range 0 to 15°.

The benchmarks in the following table were measured on the Freescale FRDM-KL25Z board which uses a 32-bit ARM M0+ integer core running at 48 MHz (48 million clock ticks per second). A floating addition or multiply, by comparison, typically take just 120 to 150 clock ticks when emulated on an integer 32 bit core. Inverse sine, cosine and tangent calls are extremely expensive.

| C99 library function | approximations.c |
|---|---|
| `float asinf(float x)` 4000–6000 clock ticks | `float fasin_deg(float x)` 3000–4000 clock ticks |
| `float acosf(float x)` 4000–6000 clock ticks | `float facos_deg(float x)` 3000–4000 clock ticks |
| `float atanf(float x)` 3800–4800 clock ticks | `float fatan_deg(float x)` 1900–3500 clock ticks |

## 1.1 Software Functions

The following is a list of Freescale Sensor Fusion Library software functions found in the file *approximations.c*.

**Table 1. Sensor Fusion software functions**

| Functions | Description | Section |
|---|---|---|
| `float fasin_deg (float x);` | Inverse sine function (deg) range –90° to 90°. Worst case error is 10.29x10$^{-6}$ deg. | 2.1 |
| `float facos_deg (float x);` | Inverse cosine function (deg) range 0° to 180°. Worst case error is 14.67x10$^{-6}$ deg. | 2.2 |
| `float fatan_deg (float x);` | Inverse arctangent (deg) range –90° to 90°. Worst case error is 9.84x10$^{-6}$ deg. | 2.3 |
| `float fatan2_deg (float y, float x);` | Inverse arctangent (deg) range –180° to 180°. Worst case error is 9.84x10$^{-6}$ deg. | 2.3 |
| `float fatan_15deg (float x);` | Inverse arctangent (deg) in range –15° to +15° only | 2.4 |

# 2 Mathematics

## 2.1 Approximation to Inverse Sine (–90° to +90°)

The function `fasin_deg` computes the inverse sine of $x$ as the inverse tangent of the new argument $\frac{x}{\sqrt{1-x^2}}$. The overhead of the square root and division is still less than the overhead of the standard C inverse sine function.

Putting $x = sin\theta$ into the definition of the tangent gives:

$$tan\theta = \frac{sin\theta}{cos\theta} = \frac{x}{\sqrt{1-x^2}} \Rightarrow \theta = sin^{-1}x = tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right) \quad \textbf{(1)}$$

## 2.2 Approximation to Inverse Cosine (0° to +180°)

The function `facos_deg` allows the inverse cosine of $x$ to be determined as the inverse tangent of the new argument $\frac{\sqrt{1-x^2}}{x}$. Because the inverse tangent returns an angle in the range –90° to +90°, 180° is added if the argument is negative to give the inverse cosine in the standard range 0° to 180°.

Putting $x = cos\theta$ into the definition of the tangent gives:

$$tan\theta = \frac{sin\theta}{cos\theta} = \frac{\sqrt{1-x^2}}{x} \Rightarrow \theta = cos^{-1}x = tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right) \quad \textbf{(2)}$$

## 2.3 Approximation to Inverse Tangent (–90° to +90°)

The inverse tangent for angles in the range –90° to +90° is computed in function `fatan_deg` which successively maps its argument to the inverse tangent of an angle in the range from 0° to +15°.

Negative arguments are mapped to positive arguments using the identity:

$$tan^{-1}(-x) = -tan^{-1}(x) \quad \textbf{(3)}$$

An argument $x$ greater than 1 (implying an angle above +45°) is mapped to argument less than 1 (implying an angle below +45°) using the identity:

$$tan\left(\frac{\pi}{2} - \theta\right) = \frac{sin\left(\frac{\pi}{2} - \theta\right)}{cos\left(\frac{\pi}{2} - \theta\right)} = \frac{cos\theta}{sin\theta} = \frac{1}{tan\theta} \quad \textbf{(4)}$$

The new argument is then compared with $tan(15°)$. If the angle is above 15° (in the range 15° to 45°) then it is mapped to the range –15° to 15° using the identity:

$$tan(\theta + \phi) = \frac{tan\theta + tan\phi}{1 - tan\theta tan\phi} \quad \textbf{(5)}$$

Substituting $\phi = \frac{-\pi}{6}$ (equal to -30°) gives:

$$tan\left(\theta - \frac{\pi}{6}\right) = \frac{tan\theta - tan\left(\frac{\pi}{6}\right)}{1 + tan\theta tan\left(\frac{\pi}{6}\right)} = \frac{tan\theta - \left(\frac{1}{\sqrt{3}}\right)}{1 + tan\theta\left(\frac{1}{\sqrt{3}}\right)} = \frac{\sqrt{3}tan\theta - 1}{tan\theta + \sqrt{3}} \quad \textbf{(6)}$$

With the substitution $x = tan\theta$:

$$tan\left(\theta-\frac{\pi}{6}\right)=\frac{x\sqrt{3}-1}{x+\sqrt{3}}\Rightarrow\theta=\left(\frac{\pi}{6}\right)+tan^{-1}\left(\frac{x\sqrt{3}-1}{x+\sqrt{3}}\right) \quad \textbf{(7)}$$

## 2.4 Pade[3, 2] Approximation to Inverse Tangent (0° to +15°)

The standard Pade[3,2] rational approximation to the inverse tangent expanded about $x = 0$ is:

$$tan^{-1}(x)\approx\frac{x+\left(\frac{4}{15}\right)x^3}{1+\left(\frac{3}{5}\right)x^2}=\frac{x\left\{1+\left(\frac{4}{15}\right)x^2\right\}}{1+\left(\frac{3}{5}\right)x^2}\ rad \quad \textbf{(8)}$$

$$=\frac{x\left\{\left(\frac{180}{\pi}\right)+\left(\frac{180}{\pi}\right)\left(\frac{4}{15}\right)x^2\right\}}{1+\left(\frac{3}{5}\right)x^2}\ deg=\frac{x\left\{\left(\frac{180}{\pi}\right)\left(\frac{5}{3}\right)+\left(\frac{180}{\pi}\right)\left(\frac{4}{9}\right)x^2\right\}}{\left(\frac{5}{3}\right)+x^2}\ deg \quad \textbf{(9)}$$

$$=\frac{x(95.492965855+25.464790894x^2)}{1.666666666+x^2}\ deg \quad \textbf{(10)}$$

This rational expression is used in function `fatan_15deg` with slightly modified Pade parameters selected to minimize the maximum error in the range 0° to +15°.

$$tan^{-1}(x)\approx\frac{x(96.644395816+25.086941612x^2)}{1.6867633134+x^2}\ deg \quad \textbf{(11)}$$

# 3 Revision history

**Table 2. Revision history**

| Rev. No. | Date | Description |
|---|---|---|
| 1 | 9/2015 | Initial release |

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