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APPLICATION NOTE

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Sensor Fusion Kalman Filters

Mark Pedley and Michael Stanley

Contents

1 Introduction

1.1 Summary

This application note documents the mathematics used by the two Kalman filters which implement the fusion of i) accelerometer, magnetometer and gyroscope data and ii) accelerometer and gyroscope data. These two algorithms are labeled "Gaming Handset" and "Gyro Stabilized Compass" in the Freescale Sensor Fusion Toolbox.

The Freescale Application Note (AN5018) *Basic Kalman Filter Theory*" provides an introduction to the mathematics of standard Kalman filters. The sensor fusion Kalman filters are a variant termed i) 'complementary', in that orientation estimates are provided independently by the gyroscope and by the accelerometer and magnetometer and ii) 'indirect' in that the Kalman filter tracks error signals rather than the underlying process.

The sensor fusion Kalman filters have been designed to be operate reliably with low power in an environment where interference from acceleration may vary by 60dB (from a few mg at rest to several thousand mg during aggressive handset motion) and where interference from stray magnetic disturbance fields may also vary by 60dB (from a few μT in normal conditions to 1000μT or more when a magnet is brought close). The quality of the estimated gravity and geomagnetic vectors estimated by the accelerometer and magnetometer sensors used to stabilize the gyroscope sensor therefore varies dramatically. This document is therefore organized around solving this problem and using the available sensor data in an optimal manner.

Section 2 describes a simple method for estimating the noise components affecting i) the accelerometer estimate of the gravity vector and ii) the magnetometer estimate of the geomagnetic vector in terms of deviations from the 1g sphere and geomagnetic sphere respectively[1]. Acceleration is modeled as noise on the accelerometer estimate of the gravity vector and magnetic disturbance is modeled as noise on the magnetometer estimate of the geomagnetic vector.

Given these noise estimates, section 3 derives the least squares orientation solution which optimally weights the orientation fit to the estimated gravity and geomagnetic vectors. This ensures that the orientation is i) still stabilized relative to the geomagnetic vector using the magnetometer when the accelerometer data is meaningless as a result of aggressive motion and ii) still stabilized in tilt relative to the gravity vector using the accelerometer when the magnetometer data is meaningless as a result of a magnet introducing a strong magnetic disturbance.

Section 4 derives simple (and therefore low computational power) quaternion expressions for the quaternion errors i) between the gyroscope and accelerometer estimates of the gravity vector and ii) between the gyroscope and magnetometer estimates of the geomagnetic vector.

Sections 5 documents the model of the gyroscope sensor and its zero rate offset error and drift. The zero rate offset is the primary source of errors in the gyroscope sensor estimate of orientation. Unlike the accelerometer, the gyroscope is highly insensitive to acceleration and, unlike the magnetometer, is completely insensitive to magnetic fields.

Section 6 discusses the estimation of the geomagnetic inclination angle. This angle is a required input to the least squares weighted orientation fit algorithm of section 3.

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[1] Here we refer to the fact that if you rotate an accelerometer or magnetometer in space, and then plot the readings taken during those rotations, you should ideally obtain a sphere. This results from the fact that regardless of orientation, you are always measuring the same vector. From the sensor's perspective, the readings map onto a sphere of radius equal to the magnitude of the vector in question.

Section 7 derives the Kalman filter which stabilizes the two error quaternions derived in section 4 according to the relative confidence in the gyroscope, accelerometer and magnetometer orientation estimates.

Section 8 derives a simplified variant of the Kalman of section 7 designed for use in systems with accelerometer and gyroscope sensors only. Here only the gravity error quaternion is updated according to the relative confidence in the accelerometer and gyroscope estimates of the gravity vector.

1.2 Terminology

Symbol	Definition
Superscript -	Denotes an *a priori* estimate made without use of the current measurement.
Superscript +	Denotes an *a posteriori* estimate made with use of the current measurement.
Subscript ε	Denotes an error component.
Subscript k	Refers to iteration k.
Subscript G	Denotes an accelerometer measurement or estimate.
Subscript M	Denotes a magnetometer measurement or estimate.
Subscript Y	Denotes a gyroscope measurement or estimate.
Superscript G	Denotes that the measurement or estimate is in the global (earth) frame.
Superscript S	Denotes that the measurement or estimate is in the sensor frame.
${}^{G}\boldsymbol{a}_k, {}^{S}\boldsymbol{a}_k$	Acceleration in the global and sensor frames (units *g*) at iteration k
$\boldsymbol{A}_k$	Linear prediction or state matrix for the error process $\boldsymbol{x}_{\varepsilon,k}$ at iteration k
$\boldsymbol{b}_k$	Gyroscope zero rate offset vector (units deg/s) at iteration k
$\boldsymbol{b}_{\varepsilon,k}$	Gyroscope zero rate offset error vector (units deg/s) at iteration k
B	The local geomagnetic field strength (units µT)
${}^{S}\boldsymbol{B}_{c,k}$	Calibrated magnetometer measurement in the sensor frame (units µT^2) at iteration k
$\boldsymbol{C}_k$	The measurement matrix relating the measured process $\boldsymbol{z}_{\varepsilon,k}$ to the underlying error process $\boldsymbol{x}_{\varepsilon,k}$ at iteration k $$\boldsymbol{z}_{\varepsilon,k} = \boldsymbol{C}_k \boldsymbol{x}_{\varepsilon,k} + \boldsymbol{v}_k$$
${}^{G}\boldsymbol{d}_k, {}^{S}\boldsymbol{d}_k$	The magnetic disturbance in global and sensor frames (units µT) at iteration k
$E[]$	Expectation operator.
$E(\boldsymbol{R}), E(q)$	Orientation fit error as a function of rotation matrix $\boldsymbol{R}$ and quaternion q
g	Magnitude of the gravity vector equal to 1g in all measurement frames
${}^{G}\boldsymbol{g}_k, {}^{S}\boldsymbol{g}_k$	The gravity vector in the global and sensor frames (units *g*) at iteration k
${}^{S}\boldsymbol{G}_k$	Accelerometer measurement in the sensor frame (*g*) at iteration k

Symbol	Definition
$\boldsymbol{I}_n$	n by n identity matrix
$\boldsymbol{K}_k$	The Kalman filter gain matrix at iteration k
${}^G\boldsymbol{m}_k, {}^S\boldsymbol{m}_k$	The geomagnetic vector in the global and sensor frames (units µT) at iteration k
$\hat{\boldsymbol{n}}$	Normalized rotation axis
$\boldsymbol{O}_n$	n by n zero matrix
$\boldsymbol{P}_k^-$	*A priori* covariance matrix.
$\boldsymbol{P}_k^+$	*A posteriori* covariance matrix
$q = \{q_0, q_1, q_2, q_3\}$	Orientation quaternion transforming from global to sensor frame as a result of coordinate system rotation
q_k	Orientation quaternion transforming from global to sensor frame at iteration k
$\hat{q}_k^-$	*A priori* estimate of the orientation quaternion q_k
$\hat{q}_k^+$	*A posteriori* estimate of the orientation quaternion q_k
$q_{zg\varepsilon,k}, \boldsymbol{q}_{zg\varepsilon,k}$	Measured rotation quaternion between the accelerometer and magnetometer (6DOF) estimate and *a priori* gravity vector estimate from the gyroscope. $q_{zg\varepsilon,k}$ is the full quaternion and $\boldsymbol{q}_{zg\varepsilon,k}$ is the vector component.
$q_{zm\varepsilon,k}, \boldsymbol{q}_{zm\varepsilon,k}$	Measured rotation quaternion between the accelerometer and magnetometer (6DOF) estimate and *a priori* geomagnetic vector estimate from the gyroscope. $q_{zm\varepsilon,k}$ is the full quaternion and $\boldsymbol{q}_{zm\varepsilon,k}$ is the vector component.
$q_{g\varepsilon,k}, \boldsymbol{q}_{g\varepsilon,k}$	The orientation tilt error quaternion relative to the true gravity vector modeled in the Kalman filter. $q_{g\varepsilon,k}$ is the full quaternion and $\boldsymbol{q}_{g\varepsilon,k}$ is the vector component.
$q_{m\varepsilon,k}, \boldsymbol{q}_{m\varepsilon,k}$	The orientation tilt error quaternion relative to the true geomagnetic vector modeled in the Kalman filter. $q_{m\varepsilon,k}$ is the full quaternion and $\boldsymbol{q}_{m\varepsilon,k}$ is the vector component.
$Q_{a,k}$	Acceleration covariance (units g^2) at iteration k $$Q_{a,k} = E[\lvert {}^S\boldsymbol{a}_k \rvert^2]$$
$Q_{d,k}$	Magnetic disturbance covariance (units µT^2) at iteration k $$Q_{d,k} = E[\lvert {}^S\boldsymbol{d}_k \rvert^2]$$
$Q_{vB,k}$	Magnetometer sensor noise covariance (units µT^2) at iteration k $$Q_{vB,k} = E\left[\left\lvert \boldsymbol{v}_{B,k} \right\rvert^2\right]$$
$Q_{vG,k}$	Accelerometer sensor noise covariance (units g^2) at iteration k $$Q_{vG,k} = E\left[\left\lvert \boldsymbol{v}_{G,k} \right\rvert^2\right]$$

Symbol	Definition
$Q_{wb,k}$	Covariance of gyroscope zero rate offset random walk (units deg^2/sec^2) at iteration k $$Q_{w\delta,k} = E\left[{w_{b,k}}^2\right]$$
$Q_{w\delta,k}$	Covariance of geomagnetic inclination angle random walk (units deg^2) at iteration k $$Q_{w\delta,k} = E\left[{w_{\delta,k}}^2\right]$$
$\boldsymbol{Q}_{w,k}$	Covariance matrix of noise process $\boldsymbol{w}_k$ in the underlying process $\boldsymbol{x}_{\varepsilon,k}$ $$\boldsymbol{Q}_{w,k} = cov\{\boldsymbol{w}_k, \boldsymbol{w}_k\} = E[\boldsymbol{w}_k {\boldsymbol{w}_k}^T]$$
$\boldsymbol{Q}_{v,k}$	Covariance matrix of measurement noise process $\boldsymbol{v}_k$ in the measured process $\boldsymbol{z}$ $$\boldsymbol{Q}_{v,k} = cov\{\boldsymbol{v}_k, \boldsymbol{v}_k\} = E[\boldsymbol{v}_k {\boldsymbol{v}_k}^T]$$
$\boldsymbol{r}_k$	*k-th* fixed reference vector in the global frame
$\boldsymbol{R}$	Rotation (aka orientation) matrix
$\boldsymbol{R}_k$	Orientation matrix transforming from global to sensor frame at iteration k
$\widehat{\boldsymbol{R}}_k^-$	*A priori* estimate of the orientation matrix $\boldsymbol{R}_k$ at iteration k
$\widehat{\boldsymbol{R}}_k^+$	*A posteriori* estimate of the orientation matrix $\boldsymbol{R}_k$ at iteration k
$\boldsymbol{s}_k$	*k-th* measured vector in the sensor frame
$tr(\boldsymbol{A})$	Trace of matrix $\boldsymbol{A}$
$\boldsymbol{u}, \boldsymbol{v}$	Arbitrary vectors.
$\boldsymbol{v}_k$	Additive noise in the Kalman filter measured error process $\boldsymbol{z}_{\varepsilon,k}$: $$\boldsymbol{z}_{\varepsilon,k} = \boldsymbol{C}_k \boldsymbol{x}_{\varepsilon,k} + \boldsymbol{v}_k$$
$\boldsymbol{v}_{B,k}$	Magnetometer sensor additive noise (units µT) at iteration k
$\boldsymbol{v}_{G,k}$	Accelerometer sensor additive noise (units *g*) at iteration k
$\boldsymbol{v}_{Y,k}$	Gyroscope sensor additive noise (units deg/s) at iteration k
$\boldsymbol{w}_k$	Additive noise in error of underlying Kalman filter process $\boldsymbol{x}_{\varepsilon,k}$: $$\boldsymbol{x}_{\varepsilon,k} = \boldsymbol{A}_k \boldsymbol{x}_{\varepsilon,k-1} + \boldsymbol{w}_k$$
$\boldsymbol{w}_{b,k}$	Driving noise for gyroscope offset random walk (units deg/s) at iteration k
$w_{\delta,k}$	Driving noise for geomagnetic inclination angle random walk (units deg) at iteration k
$\boldsymbol{x}_{\varepsilon,k}$	The underlying Kalman filter error process at iteration k
$\widehat{\boldsymbol{x}}_{\varepsilon,k}^-$	The *a priori* estimate of the Kalman filter error process $\boldsymbol{x}_{\varepsilon,k}$
$\widehat{\boldsymbol{x}}_{\varepsilon,k}^+$	The *a posteriori* estimate of the Kalman filter error process $\boldsymbol{x}_{\varepsilon,k}$
${}^S\boldsymbol{Y}_k$	Gyroscope measurement (units deg/s) at iteration k
$\boldsymbol{z}_{\varepsilon,k}$	The measurement error vector

Symbol	Definition
α	Used as scaling constant in covariance matrices converting units of deg/s to radians $$\alpha = \left(\frac{\pi\delta t}{180}\right)$$
α_k	Weighting applied to measurement of the *k-th* reference vector in orientation fit.
α_g	Weighting applied to gravity reference vector
α_m	Weighting applied to geomagnetic reference vector
δ	Geomagnetic inclination angle (deg)
δ_k	Geomagnetic inclination angle (deg) at iteration k
$\delta_{\varepsilon,k}$	Geomagnetic inclination angle error (deg) at iteration k
$\delta_{z\varepsilon,k}$	Error between the Kalman filter estimate of inclination angle δ and the accelerometer plus magnetometer estimate.
δt	Sampling interval of the Kalman filter (units s)
η	Rotation angle (degrees or radians)
$\boldsymbol{\kappa}$	4x4 symmetric matrix whose eigenvectors give the optimum orientation quaternion.
λ	Eigenvalue and Lagrange Multiplier
$\boldsymbol{\omega}_k$	True angular velocity (deg/s)
$\boldsymbol{\omega}_k^-$	The *a priori* estimate of the angular velocity $\boldsymbol{\omega}_k$ (deg/s)

1.3 Software Functions

Table 1. Sensor Fusion Library software functions

Kalman Filter Initialization Functions (fusion.c)	Section

`void fInit_6DOF_GY_KALMAN`
`(struct SV_6DOF_GY_KALMAN *pthisSV, struct AccelSensor *pthisAccel);`
 	
`void fInit_9DOF_GBY_KALMAN`
`(struct SV_9DOF_GBY_KALMAN *pthisSV, struct AccelSensor *pthisAccel, struct MagSensor *pthisMag, struct MagCalibration *pthisMagCal);`
	7, 8

Kalman Filter Execution Functions (fusion.c)	**Section**

`void fRun_6DOF_GY_KALMAN`
`(struct SV_6DOF_GY_KALMAN *pthisSV, struct AccelSensor *pthisAccel, struct GyroSensor *pthisGyro);`
 	
`void fRun_9DOF_GBY_KALMAN`
`(struct SV_9DOF_GBY_KALMAN *pthisSV, struct AccelSensor *pthisAccel,`
	7, 8

Kalman Filter Initialization Functions (fusion.c)	Section
`struct MagSensor *pthisMag, struct GyroSensor *pthisGyro, struct MagCalibration *pthisMagCal);`	
Least Squares Accelerometer and Magnetometer Orientation (orientation.c)	**Section**

`void fLeastSquareseCompassNED (struct fquaternion *pfq, float fB, float fDelta, float fsinDelta, float fcosDelta, float *pfDelta6DOF, float fBc[], float fGs[], float *pfQvBQd, float *pfQvGQa)`
 	
`void fLeastSquareseCompassAndroid(struct fquaternion *pfq, float fB, float fDelta, float fsinDelta, float fcosDelta, float *pfDelta6DOF, float fBc[], float fGs[], float *pfQvBQd, float *pfQvGQa)`
 	
`void fLeastSquareseCompassWin8 (struct fquaternion *pfq, float fB, float fDelta, float fsinDelta, float fcosDelta, float *pfDelta6DOF, float fBc[], float fGs[], float *pfQvBQd, float *pfQvGQa)`
	3

Quaternion Algebra (orientation.c)	**Section**
`void fveqconjgquq (struct fquaternion *pfq, float fu[], float fv[])`	4
Compile Time Constants (fusion.h)	**Section**
FQVBQD_MIN_9DOF_GBY_KALMAN	2
FQVY_9DOF_GBY_KALMAN FQVY_6DOF_GY_KALMAN	5
FQWB_9DOF_GBY_KALMAN FQWB_6DOF_GY_KALMAN	5
FQWDLT_9DOF_GBY_KALMAN	5
FGYRO_OFFSET_MIN_9DOF_GBY_KALMAN FGYRO_OFFSET_MIN_9DOF_GBY_KALMAN	7
FDELTA_MIN_9DOF_GBY_KALMAN FDELTA_MAX_9DOF_GBY_KALMAN	7

2 Estimating Acceleration and Magnetic Disturbance

Compared to versions 4.22 and earlier versions of the library, version 5.00 has somewhat simplified the Kalman filters by treating linear acceleration and magnetic interference as “noise”, rather than tracking them as explicit variables in the filter. Somewhat counter intuitively, this appears to improve dynamic response of the filters.

2.1 Introduction

This section describes a simple technique for estimating the interfering “noise” affecting the accelerometer estimate of the gravity vector and the magnetometer estimate of the geomagnetic vector.

The accelerometer sensor at rest has high frequency measurement noise of approximately 3mg and an orientation dependent error of order 30mg or so resulting from various sources including i) non-linearity in its signal chain response ii) imperfectly calibrated offset iii) imperfectly calibrated gain and iv) uncorrected cross-axis interference. In addition, the accelerometer will also experience physical acceleration up to 8000mg in gaming applications with orientation changes at 2000dps meaning that the acceleration can change by 16000mg in 0.1s. The accuracy of the accelerometer estimate of the gravity vector therefore varies dramatically depending on the dynamics by a factor of 60dB.

The magnetometer sensor at rest has high frequency measurement noise of one or two uT and an orientation dependent error of comparable size resulting from an imperfect estimate of the hard and soft iron interference resulting from magnetic sources on the PCB. Magnetic disturbance, defined as resulting from sources in the global frame unlike hard and soft iron interference which are fixed in the PCB frame, can vary from zero to clipping the magnetometer sensor at 1000uT or more when a magnet is brought close. The magnetometer provides the estimate of the geomagnetic vector to stabilize the gyroscope orientation estimate and the accuracy of this estimate also varies dramatically by a similar factor of 60dB.

There are two approaches to estimating the acceleration and magnetic disturbance:

a) the acceleration and magnetic disturbance vectors are estimated in the Kalman filter on a sample by sample basis in order to allow the estimated acceleration and magnetic disturbance vectors to be subtracted from the accelerometer and magnetometer measurements

b) the acceleration and magnetic disturbance magnitudes are estimated independently of the Kalman filter and used in the Kalman filter to determine the weighting of the gyroscope orientation estimate versus the accelerometer and magnetometer estimate.

The first approach is computationally expensive since the Kalman filter is more complicated and must execute at a rate sufficient to accurately track the linear acceleration and magnetic disturbance vectors without aliasing. In conditions of highly aggressive gaming motion, the Kalman filter needs to run at several hundred Hz which dramatically increases both the microcontroller cost and the power dissipation.

The second approach is computationally cheap since it permits a relatively simple Kalman filter which can run at rates of just 25Hz with the associated cost and power savings. The Kalman filter will, for example, compute its orientation estimate predominantly from the gyroscope and magnetometer sensors when a high accelerometer noise covariance is present from acceleration. Gyroscope integration errors will accumulate slowly over time in the orientation estimate but given that the orientation is, by definition, changing rapidly under these conditions, the error is academic. Once the acceleration stops, the Kalman filter rapidly corrects the orientation error between the gyroscope and accelerometer estimates of the gravity vector.

Similarly, the Kalman filter will compute its orientation estimate predominantly from the gyroscope and accelerometer sensors when a magnet is brought close by giving an accurate tilt estimate from vertical and a compass heading angle determined from the gyroscope. Once the magnetic disturbance is removed, the deviation between the gyroscope and magnetometer estimates of compass heading is rapidly corrected.

The approach taken in this document is the second approach where acceleration and magnetic disturbance are treated as noise terms whose magnitude only is needed in the noise covariance matrices.

The measurement from a perfectly calibrated accelerometer with no sensor noise and experiencing no acceleration will have magnitude 1g and lie exactly on the 1g sphere. In the presence of sensor noise, imperfect calibration and acceleration, the measurement will no longer have magnitude 1g and no longer lie on the 1g sphere. The deviation of the accelerometer measurement from the 1g sphere therefore provides a simple estimate of noise.

Similarly, the measurement from a perfectly calibrated magnetometer with no sensor noise and experiencing no magnetic disturbance will have magnitude equal to the geomagnetic field strength and lie exactly on the geomagnetic sphere. In the presence of sensor noise, imperfect calibration and magnetic disturbance, the measurement will no longer lie on the geomagnetic sphere and the deviation of the measurement from the geomagnetic sphere therefore provides an estimate of magnetometer noise.

These approximations can be inaccurate for a single measurement where the vector sum of the noise terms cancels out and results in measurements lying on or close to the 1g gravity or geomagnetic spheres respectively. A simple example for the accelerometer would be a 2g acceleration downwards resulting in a net accelerometer measurement of 1g and a zero noise estimate rather than the true 2g acceleration noise value. Statistically, however the approximations used are valid over large numbers of measurements.

2.2 Accelerometer Sensor Model and Noise Covariance

The accelerometer measurement ${}^S\boldsymbol{G}_k$ combines acceleration ${}^S\boldsymbol{a}_k$, the gravitational component ${}^S\boldsymbol{g}_k$ and a noise term $\boldsymbol{v}_{G,k}$.

The Aerospace (NED) and Windows 8 are 'gravity positive' standards whereas Android is an 'acceleration positive' standard. The accelerometer sensor models for these three cases are:

$$ {}^S\boldsymbol{G}_k = -{}^S\boldsymbol{a}_k + {}^S\boldsymbol{g}_k - \boldsymbol{v}_{G,k} \text{ (Aerospace, Windows 8)} \qquad \textbf{(1)} $$

$$ {}^S\boldsymbol{G}_k = {}^S\boldsymbol{a}_k - {}^S\boldsymbol{g}_k + \boldsymbol{v}_{G,k}, \text{ (Android)} \qquad \textbf{(2)} $$

The accelerometer sensor noise component $\boldsymbol{v}_{G,k}$ comprises the high frequency, nearly white, noise which is approximately uncorrelated from sample to sample plus the low frequency noise resulting from imperfect sensor gain, offset and cross-axis, calibration. The bandwidth of the low frequency is proportional to the bandwidth of the acceleration and PCB orientation dynamics and is approximately DC if the PCB is stationary at a fixed orientation.

For all three coordinate systems:

$$ |{}^S\boldsymbol{G}_k|^2 - g^2 = \left|{}^S\boldsymbol{a}_k - {}^S\boldsymbol{g}_k + \boldsymbol{v}_{G,k}\right|^2 - g^2 \qquad \textbf{(3)} $$

Taking the expectation value over samples k gives:

$$ E[|{}^S\boldsymbol{G}_k|^2 - g^2] = E\left[\left|{}^S\boldsymbol{a}_k - {}^S\boldsymbol{g}_k + \boldsymbol{v}_{G,k}\right|^2\right] - g^2 \qquad \textbf{(4)} $$

$$= E\left[\left({}^S\boldsymbol{a}_k - {}^S\boldsymbol{g}_k + \boldsymbol{v}_{G,k}\right).\left({}^S\boldsymbol{a}_k - {}^S\boldsymbol{g}_k + \boldsymbol{v}_{G,k}\right)\right] - g^2 \tag{5}$$

The vectors ${}^S\boldsymbol{a}_k$, $\boldsymbol{v}_{G,k}$ and ${}^S\boldsymbol{g}_k$ can be assumed to be statistically uncorrelated over time giving the simplification:

$$E[|{}^S\boldsymbol{G}_k|^2 - g^2] = E[{}^S\boldsymbol{a}_k.{}^S\boldsymbol{a}_k] + E[{}^S\boldsymbol{g}_k.{}^S\boldsymbol{g}_k] + E\left[\boldsymbol{v}_{G,k}.\boldsymbol{v}_{G,k}\right] - g^2 = E[|{}^S\boldsymbol{a}_k|^2] + E\left[\left|\boldsymbol{v}_{G,k}\right|^2\right] \tag{6}$$

The acceleration covariance $Q_{a,k}$ is defined as:

$$Q_{a,k} = E[|{}^S\boldsymbol{a}_k|^2] \tag{7}$$

The accelerometer sensor noise covariance $Q_{vG,k}$ (including high frequency sensor noise and low frequency orientation-dependent noise from calibration errors) is defined as:

$$Q_{vG,k} = E\left[\left|\boldsymbol{v}_{G,k}\right|^2\right] \tag{8}$$

Substituting back shows that the deviation of the squared accelerometer measurement from the 1g sphere is therefore a reasonable estimate of the sum of the acceleration covariance and the accelerometer sensor noise covariance. This is the model for the noise covariance affecting the quality of the accelerometer estimate of the gravity vector.

$$E[|{}^S\boldsymbol{G}_k|^2 - g^2] = Q_{vG,k} + Q_{a,k} \tag{9}$$

Magnetometer Sensor Model and Noise Covariance

The calibrated magnetometer measurement ${}^S\boldsymbol{B}_{c,k}$ is modeled as the sum of the geomagnetic component ${}^S\boldsymbol{m}_k$, any magnetic disturbance ${}^S\boldsymbol{d}_k$ and sensor noise $\boldsymbol{v}_{B,k}$:

$${}^S\boldsymbol{B}_{c,k} = {}^S\boldsymbol{m}_k + {}^S\boldsymbol{d}_k + \boldsymbol{v}_{B,k} \tag{10}$$

The magnetic calibration algorithms remove hard and soft iron magnetic distortion effects which are constant in the sensor frame leaving the calibrated measurement ${}^S\boldsymbol{B}_{c,k}$. The magnetic disturbance ${}^S\boldsymbol{d}_k$ is defined any magnetic interference which does not rotate with the sensor frame and not, therefore, included in the hard and soft iron calibration.

With perfect magnetic calibration and in the absence of any magnetic disturbance and sensor noise, the magnetometer measurement lies on the geomagnetic sphere and has magnitude equal to the geomagnetic field strength B. In practice the measurement will not lie on the geomagnetic sphere as a consequence of i) magnetometer sensor noise ii) imperfect hard and soft iron calibration and iii) the presence of magnetic disturbance in the environment.

Using the same arguments and algebra used for the accelerometer, the deviation of the squared modulus of the magnetometer measurement from the geomagnetic sphere is given by:

$$\left|{}^S\boldsymbol{B}_{c,k}\right|^2 - B^2 = \left|{}^S\boldsymbol{m}_k + {}^S\boldsymbol{d}_k + \boldsymbol{v}_{B,k}\right|^2 - B^2 \tag{11}$$

Since the vectors ${}^S\boldsymbol{d}_k$, $\boldsymbol{v}_{B,k}$ and ${}^S\boldsymbol{m}_k$ are uncorrelated over time:

$$E\left[\left|{}^{S}\boldsymbol{B}_{c,k}\right|^{2}-B^{2}\right]=E[{}^{S}\boldsymbol{m}_{k}.{}^{S}\boldsymbol{m}_{k}]+E[{}^{S}\boldsymbol{d}_{k}.{}^{S}\boldsymbol{d}_{k}]+E\left[\boldsymbol{v}_{B,k}.\boldsymbol{v}_{B,k}\right]-B^{2}=E[|{}^{S}\boldsymbol{d}_{k}|^{2}]+E\left[\left|\boldsymbol{v}_{B,k}\right|^{2}\right] \quad \textbf{(12)}$$

The magnetic disturbance covariance $Q_{d,k}$ is defined as:

$$Q_{d,k}=E[|{}^{S}\boldsymbol{d}_{k}|^{2}] \quad \textbf{(13)}$$

The magnetometer noise covariance $Q_{vB,k}$ (including high frequency sensor noise and low frequency orientation-dependent noise from calibration errors) is defined as:

$$Q_{vB,k}=E\left[\left|\boldsymbol{v}_{B,k}\right|^{2}\right] \quad \textbf{(14)}$$

Substituting back shows that the deviation of the squared magnetometer measurement from the geomagnetic sphere is therefore a reasonable estimate of the sum of the magnetic disturbance covariance and the magnetometer noise covariance. This is the model for the noise covariance affecting the quality of the magnetometer estimate of the geomagnetic vector.

$$E\left[\left|{}^{S}\boldsymbol{B}_{c,k}\right|^{2}-B^{2}\right]=Q_{vB,k}+Q_{d,k} \quad \textbf{(15)}$$

2.3 Compile Time Constants

The sensor fusion software applies a lower bound to the estimate in equation (15). This is defined in the compile time constants found in file fusion.h for the Kalman filters of section 7.

```
#define FQVBQD_MIN_9DOF_GBY_KALMAN 7E0F
```

3 Least Squares Accelerometer and Magnetometer Orientation Matrix

3.1 Introduction

This section contains the mathematics to compute the least squares fit to the orientation matrix using gravity and geomagnetic vector estimates from the accelerometer and magnetometer sensors respectively weighted according to the confidence in those estimates derived from the noise covariances as defined in section 2.

For the case where the PCB is at a fixed orientation with no acceleration and no magnetic disturbance then the gravity vector and geomagnetic will both be given approximately equal weighting.

During gaming motions, where accelerations exceed 1g and can reach 8g or more, the gravity vector estimate is meaningless and the geomagnetic vector estimate is given the higher weighting. The resulting orientation remains stabilized relative to the geomagnetic vector.

In the presence of magnetic disturbance, such as that caused by a magnet being brought up to the PCB while stationary on a desk, the geomagnetic vector estimate is unreliable but the gravity vector remains reliable. A high weighting is then given to the gravity vector estimate resulting in an orientation correctly stabilized in tilt relative to the vertical gravity vector.

The derivation in this section solves the general problem with N reference orientation vectors and then restricts this to $N=2$ for the case of interest with gravity and geomagnetic reference vectors only. The geomagnetic inclination angle δ is assumed known in this section and the mathematics used to estimate δ described in sections 6 and 7.

3.2 Matrix Trace Lemma

The trace of a matrix product is unchanged if the order of the two matrices is commuted.

$$tr(\boldsymbol{AB}) = tr(\boldsymbol{BA}) \tag{16}$$

Proof: If the matrix $\boldsymbol{A}$ has dimensions MxN and a matrix $\boldsymbol{B}$ has dimensions NxM, then $\boldsymbol{C} = \boldsymbol{AB}$ has dimensions MxM and $\boldsymbol{D} = \boldsymbol{BA}$ has dimensions NxN.
The element C_{ij} of matrix $\boldsymbol{C} = \boldsymbol{AB}$ has value:

$$C_{ij} = \sum_{k=0}^{N-1} A_{ik} B_{kj} \Rightarrow tr(\boldsymbol{C}) = tr(\boldsymbol{AB}) = \sum_{i=0}^{M-1} C_{ii} = \sum_{i=0}^{M-1} \sum_{k=0}^{N-1} A_{ik} B_{ki} \tag{17}$$

The element D_{ij} of matrix $\boldsymbol{D} = \boldsymbol{BA}$ has value:

$$D_{ij} = \sum_{k=0}^{M-1} B_{ik} A_{kj} \Rightarrow tr(\boldsymbol{D}) = tr(\boldsymbol{BA}) = \sum_{i=0}^{N-1} D_{ii} = \sum_{i=0}^{N-1} \sum_{k=0}^{M-1} B_{ik} A_{ki} \tag{18}$$

Swapping the dummy summation variables gives:

$$tr(\boldsymbol{D}) = tr(\boldsymbol{BA}) = \sum_{k=0}^{N-1} \sum_{i=0}^{M-1} B_{ki} A_{ik} = \sum_{i=0}^{M-1} \sum_{k=0}^{N-1} A_{ik} B_{ki} = tr(\boldsymbol{AB}) = tr(\boldsymbol{C}) \tag{19}$$

An obvious extension of this result is that the trace operation is invariant under forwards and backwards cyclic permutations of matrix products:

$$tr(\boldsymbol{A}_1 \boldsymbol{A}_2 \boldsymbol{A}_3 \dots \boldsymbol{A}_N) = tr(\boldsymbol{A}_N \boldsymbol{A}_1 \boldsymbol{A}_2 \dots \boldsymbol{A}_{N-1}) = tr(\boldsymbol{A}_2 \dots \boldsymbol{A}_{N-1} \boldsymbol{A}_N \boldsymbol{A}_1) \tag{20}$$

A special case is that for any invertible matrix $\boldsymbol{B}$:

$$tr(\boldsymbol{B}^{-1}\boldsymbol{AB}) = tr(\boldsymbol{ABB}^{-1}) = tr(\boldsymbol{A}) \tag{21}$$

3.3 Definition of the Error Function

The error function $E(\boldsymbol{R})$ is defined as the weighted sum of the squared vector differences between N sensor frame measurements $\boldsymbol{s}_i$ and rotated global frame reference vectors $\boldsymbol{r}_k$. The rotation matrix $\boldsymbol{R}$ follows the Freescale convention of transforming vectors from the global (reference) frame to the sensor frame as a result of a coordinate system rotation.

$$E(\boldsymbol{R}) = \left(\frac{1}{2}\right)\sum_{k=0}^{N-1} \alpha_k \, |\boldsymbol{s}_k - \boldsymbol{R}\boldsymbol{r}_k|^2 = \left(\frac{1}{2}\right)\sum_{k=0}^{N-1} \alpha_k \, tr\{(\boldsymbol{s}_k - \boldsymbol{R}\boldsymbol{r}_k)(\boldsymbol{s}_k - \boldsymbol{R}\boldsymbol{r}_k)^T\} \quad \textbf{(22)}$$

$$= \left(\frac{1}{2}\right)\sum_{k=0}^{N-1} \alpha_k \, tr\{\boldsymbol{s}_k(\boldsymbol{s}_k - \boldsymbol{R}\boldsymbol{r}_k)^T - \boldsymbol{R}\boldsymbol{r}_k(\boldsymbol{s}_k - \boldsymbol{R}\boldsymbol{r}_k)^T\} \quad \textbf{(23)}$$

$$= \left(\frac{1}{2}\right)\sum_{k=0}^{N-1} \alpha_k \, \{tr(\boldsymbol{s}_k{\boldsymbol{s}_k}^T) - tr(\boldsymbol{s}_k{\boldsymbol{r}_k}^T\boldsymbol{R}^T) - tr(\boldsymbol{R}\boldsymbol{r}_k{\boldsymbol{s}_k}^T) + tr(\boldsymbol{R}\boldsymbol{r}_k{\boldsymbol{r}_k}^T\boldsymbol{R}^T)\} \quad \textbf{(24)}$$

The weights α_i determine the confidence of the individual measurements and later will be defined in terms of the noise covariances of the accelerometer and magnetometer measurements. The factor of $\left(\frac{1}{2}\right)$ has no effect on the solution and is for mathematical convenience.

Since $tr(\boldsymbol{A}) = tr(\boldsymbol{A}^T)$, the second and third terms of equation (24) are equal:

$$tr(\boldsymbol{R}\boldsymbol{r}_k{\boldsymbol{s}_k}^T) = tr(\boldsymbol{s}_k{\boldsymbol{r}_k}^T\boldsymbol{R}^T) \quad \textbf{(25)}$$

So far no assumption has been made about the form of the unknown matrix $\boldsymbol{R}$. Restricting $\boldsymbol{R}$ to be a rotation matrix for which $\boldsymbol{R}^T = \boldsymbol{R}^{-1}$ simplifies the final term of equation (24) to:

$$tr(\boldsymbol{R}\boldsymbol{r}_k{\boldsymbol{r}_k}^T\boldsymbol{R}^T) = tr(\boldsymbol{R}^T\boldsymbol{R}\boldsymbol{r}_k{\boldsymbol{r}_k}^T) = tr(\boldsymbol{R}^{-1}\boldsymbol{R}\boldsymbol{r}_k{\boldsymbol{r}_k}^T) = tr(\boldsymbol{r}_k{\boldsymbol{r}_k}^T) \quad \textbf{(26)}$$

Substituting back into the error function gives:

$$E(\boldsymbol{R}) = \left(\frac{1}{2}\right)\sum_{k=0}^{N-1} tr(\alpha_k\boldsymbol{s}_k{\boldsymbol{s}_k}^T) + \left(\frac{1}{2}\right)\sum_{k=0}^{N-1} tr(\alpha_k\boldsymbol{r}_k{\boldsymbol{r}_k}^T) - \sum_{k=0}^{N-1} tr(\alpha_k\boldsymbol{s}_k{\boldsymbol{r}_k}^T\boldsymbol{R}^T) \quad \textbf{(27)}$$

$$= \left(\frac{1}{2}\right)\sum_{k=0}^{N-1} tr(\alpha_k\boldsymbol{s}_k{\boldsymbol{s}_k}^T) + \left(\frac{1}{2}\right)\sum_{k=0}^{N-1} tr(\alpha_k\boldsymbol{r}_k{\boldsymbol{r}_k}^T) - tr\left\{\left(\sum_{k=0}^{N-1}(\alpha_k\boldsymbol{s}_k{\boldsymbol{r}_k}^T)\right)\boldsymbol{R}^T\right\} \quad \textbf{(28)}$$

The matrix $\boldsymbol{B}$ is defined in terms of the weighted reference and measured vectors as:

$$\boldsymbol{B} = \sum_{k=0}^{N-1} \alpha_k \boldsymbol{s}_k \boldsymbol{r}_k{}^T \tag{29}$$

The error function $E(\boldsymbol{R})$ then equals:

$$E(\boldsymbol{R}) = \left(\frac{1}{2}\right) \sum_{k=0}^{N-1} \alpha_k tr(\boldsymbol{s}_k \boldsymbol{s}_k{}^T) + \left(\frac{1}{2}\right) \sum_{k=0}^{N-1} \alpha_k tr(\boldsymbol{r}_k \boldsymbol{r}_k{}^T) - tr(\boldsymbol{B}\boldsymbol{R}^T) \tag{30}$$

For orientation applications it can be assumed that both the measured sensor frame vectors $\boldsymbol{s}_k$ and the reference vectors $\boldsymbol{r}_k$ have unit norm:

$$tr(\boldsymbol{s}_k \boldsymbol{s}_k{}^T) = |\boldsymbol{s}_k|^2 = 1 \tag{31}$$

$$tr(\boldsymbol{r}_k \boldsymbol{r}_k{}^T) = |\boldsymbol{r}_k|^2 = 1 \tag{32}$$

In addition, the relative weightings α_k sum to unity giving:

$$E(\boldsymbol{R}) = \left(\frac{1}{2}\right) \sum_{k=0}^{N-1} \alpha_k + \left(\frac{1}{2}\right) \sum_{k=0}^{N-1} \alpha_k - tr(\boldsymbol{B}\boldsymbol{R}^T) = 1 - tr(\boldsymbol{B}\boldsymbol{R}^T) = 1 - tr(\boldsymbol{R}\boldsymbol{B}^T) \tag{33}$$

The error function $E(\boldsymbol{R})$ is then minimized by maximizing the final term $tr(\boldsymbol{B}\boldsymbol{R}^T)$ with respect to $\boldsymbol{R}$ to give the optimum orientation matrix $\boldsymbol{R}$.

3.4 Solution by Eigen-Decomposition

The optimum rotation matrix $\boldsymbol{R}$ can be written in terms of its equivalent quaternion q using the standard relationship between an orientation matrix and orientation quaternion:

$$\boldsymbol{R} = \begin{pmatrix} R_{xx} & R_{xy} & R_{xz} \\ R_{yx} & R_{yy} & R_{yz} \\ R_{zx} & R_{zy} & R_{zz} \end{pmatrix} = \begin{pmatrix} q_0{}^2 + q_1{}^2 - q_2{}^2 - q_3{}^2 & 2q_1q_2 + 2q_0q_3 & 2q_1q_3 - 2q_0q_2 \\ 2q_1q_2 - 2q_0q_3 & q_0{}^2 - q_1{}^2 + q_2{}^2 - q_3{}^2 & 2q_2q_3 + 2q_0q_1 \\ 2q_1q_3 + 2q_0q_2 & 2q_2q_3 - 2q_0q_1 & q_0{}^2 - q_1{}^2 - q_2{}^2 + q_3{}^2 \end{pmatrix} \tag{34}$$

The error function can then be written as $E(q)$ in terms of the four components of the optimum orientation quaternion q as:

$$E(q) = 1 - tr(\boldsymbol{R}\boldsymbol{B}^T) \tag{35}$$

$$= 1 - tr\left\{\begin{pmatrix} {q_0}^2+{q_1}^2-{q_2}^2-{q_3}^2 & 2q_1q_2+2q_0q_3 & 2q_1q_3-2q_0q_2 \\ 2q_1q_2-2q_0q_3 & {q_0}^2-{q_1}^2+{q_2}^2-{q_3}^2 & 2q_2q_3+2q_0q_1 \\ 2q_1q_3+2q_0q_2 & 2q_2q_3-2q_0q_1 & {q_0}^2-{q_1}^2-{q_2}^2+{q_3}^2 \end{pmatrix} \sum_{k=0}^{N-1} \alpha_k \begin{pmatrix} s_{k,x}r_{k,x} & s_{k,y}r_{k,x} & s_{k,z}r_{k,x} \\ s_{k,x}r_{k,y} & s_{k,y}r_{k,y} & s_{k,z}r_{k,y} \\ s_{k,x}r_{k,z} & s_{k,y}r_{k,z} & s_{k,z}r_{k,z} \end{pmatrix}\right\} \tag{36}$$

$$\begin{aligned} = 1 - \sum_{k=0}^{N-1} &\{\alpha_k({q_0}^2+{q_1}^2-{q_2}^2-{q_3}^2)b_{k,x}r_{k,x} + \alpha_k({q_0}^2+{q_2}^2-{q_1}^2-{q_3}^2)s_{k,y}r_{k,y} \\ &+ \alpha_k({q_0}^2+{q_3}^2-{q_1}^2-{q_2}^2)s_{k,z}r_{k,z}\} \\ &- \sum_{k=0}^{N-1}\{2\alpha_k(q_1q_2-q_0q_3)s_{k,y}r_{k,x} + 2\alpha_k(q_1q_2+q_0q_3)s_{k,x}r_{k,y} + 2\alpha_k(q_1q_3-q_0q_2)s_{k,x}r_{k,z} \\ &+ 2\alpha_k(q_1q_3+q_0q_2)s_{k,z}r_{k,x} + 2\alpha_k(q_2q_3-q_0q_1)s_{k,z}r_{k,y} + 2\alpha_k(q_2q_3+q_0q_1)s_{k,y}r_{k,z}\} \end{aligned} \tag{37}$$

It is mathematically convenient to define the 4 component solution vector $\boldsymbol{q}$ containing the four components of the optimum rotation quaternion q:

$$\boldsymbol{q} = \begin{pmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix} \tag{38}$$

$\boldsymbol{q}$ as defined here is not the three component vector quaternion $\{q_1, q_2, q_3\}$ but contains all four quaternion components.

The error function $E(q)$ is, by inspection, a quadratic function of the components of the optimum quaternion and can therefore be written in terms of a 4x4 matrix $\boldsymbol{\kappa}$ as:

$$E(q) = 1 - \boldsymbol{q}^T\boldsymbol{\kappa}\boldsymbol{q} \tag{39}$$

Since $E(q)$ is a scalar, it is easily proved that $\boldsymbol{\kappa}$ must be symmetric:

$$E(q) = E(q)^T \Rightarrow 1 - \boldsymbol{q}^T\boldsymbol{\kappa}\boldsymbol{q} = (1 - \boldsymbol{q}^T\boldsymbol{\kappa}\boldsymbol{q})^T \Rightarrow \boldsymbol{q}^T(\boldsymbol{\kappa} - \boldsymbol{\kappa}^T)\boldsymbol{q} = 0 \Rightarrow \boldsymbol{\kappa} = \boldsymbol{\kappa}^T \tag{40}$$

By direct evaluation:

$$\boldsymbol{\kappa} = \sum_{k=0}^{N-1} \alpha_k \begin{pmatrix} s_{k,x}r_{k,x}+s_{k,y}r_{k,y}+s_{k,z}r_{k,z} & s_{k,y}r_{k,z}-s_{k,z}r_{k,y} & s_{k,z}r_{k,x}-s_{k,x}r_{k,z} & s_{k,x}r_{k,y}-s_{k,y}r_{k,x} \\ s_{k,y}r_{k,z}-s_{k,z}r_{k,y} & s_{k,x}r_{k,x}-s_{k,y}r_{k,y}-s_{k,z}r_{k,z} & s_{k,x}r_{k,y}+s_{k,y}r_{k,x} & s_{k,x}r_{k,z}+s_{k,z}r_{k,x} \\ s_{k,z}r_{k,x}-s_{k,x}r_{k,z} & s_{k,y}r_{k,x}+s_{k,x}r_{k,y} & -s_{k,x}r_{k,x}+s_{k,y}r_{k,y}-s_{k,z}r_{k,z} & s_{k,y}r_{k,z}+s_{k,z}r_{k,y} \\ s_{k,x}r_{k,y}-s_{k,y}r_{k,x} & s_{k,z}r_{k,x}+s_{k,x}r_{k,z} & s_{k,z}r_{k,y}+s_{k,y}r_{k,z} & -s_{k,x}r_{k,x}-s_{k,y}r_{k,y}+s_{k,z}r_{k,z} \end{pmatrix} \tag{41}$$

Since q is a rotation quaternion it is constrained to have unit norm and therefore $1 - \boldsymbol{q}^T\boldsymbol{q} = 0$. The method of Lagrange Multipliers can be applied to find the stationary point of $E(q)$ subject to this constraint by looking for the stationary point of the modified error function $E'(q)$ defined with Lagrange Multiplier λ:

$$E'(q) = 1 - \boldsymbol{q}^T \boldsymbol{\kappa} \boldsymbol{q} + \lambda(1 - \boldsymbol{q}^T \boldsymbol{q}) \tag{42}$$

Applying the stationary constraint $E'(q + \delta q) = E'(q)$ gives:

$$(\boldsymbol{q} + \delta\boldsymbol{q})^T \boldsymbol{\kappa} (\boldsymbol{q} + \delta\boldsymbol{q}) + \lambda\big(1 - (\boldsymbol{q} + \delta\boldsymbol{q})^T (\boldsymbol{q} + \delta\boldsymbol{q})\big) = \boldsymbol{q}^T \boldsymbol{\kappa} \boldsymbol{q} + \lambda(1 - \boldsymbol{q}^T \boldsymbol{q}) \tag{43}$$

$$\Rightarrow \delta\boldsymbol{q}^T \boldsymbol{\kappa} \boldsymbol{q} + \boldsymbol{q}^T \boldsymbol{\kappa} \delta\boldsymbol{q} - \lambda(\boldsymbol{q}^T \delta\boldsymbol{q} + \delta\boldsymbol{q}^T \boldsymbol{q}) = 0 \tag{44}$$

$$\Rightarrow \delta\boldsymbol{q}^T \boldsymbol{\kappa} \boldsymbol{q} = \lambda \delta\boldsymbol{q}^T \boldsymbol{q} \Rightarrow \delta\boldsymbol{q}^T (\boldsymbol{\kappa} \boldsymbol{q} - \lambda \boldsymbol{q}) = 0 \; for \; all \; \delta\boldsymbol{q} \tag{45}$$

$$\Rightarrow \boldsymbol{\kappa} \boldsymbol{q} = \lambda \boldsymbol{q} \tag{46}$$

The solution vector $\boldsymbol{q}$ containing the four component of the optimum quaternion q is therefore one of the four eigenvectors of $\boldsymbol{\kappa}$.

Since $\boldsymbol{\kappa}$ is symmetric, it has real eigenvalues and real eigenvectors. Eigenvectors associated with different eigenvalues are orthogonal and if eigenvalues are repeated then linear combinations of eigenvectors can still be chosen which are orthogonal.

Substituting gives the error function $E(q)$ as a function of the eigenvalue λ:

$$E(q) = 1 - \boldsymbol{q}^T \boldsymbol{\kappa} \boldsymbol{q} = 1 - \lambda \boldsymbol{q}^T \boldsymbol{q} = 1 - \lambda \tag{47}$$

The optimum rotation quaternion which *minimizes* $E(q)$ is therefore the four component eigenvector associated with the *largest* eigenvalue of $\boldsymbol{\kappa}$.

3.5 $\boldsymbol{\kappa}$ Matrix For Gravity and Geomagnetic Vectors

For $N = 2$ with α_g representing the weighting of the measured and reference gravity vectors $\boldsymbol{r}_g$ and $\boldsymbol{s}_g$ and α_m representing the weighting of the measured and reference geomagnetic vectors $\boldsymbol{r}_m$ and $\boldsymbol{s}_m$, equation (41) simplifies to:

$$\boldsymbol{\kappa} = \alpha_g \begin{pmatrix} s_{g,x}r_{g,x} + s_{g,y}r_{g,y} + s_{g,z}r_{g,z} & s_{g,y}r_{g,z} - s_{g,z}r_{g,y} & s_{g,z}r_{g,x} - s_{g,x}r_{g,z} & s_{g,x}r_{g,y} - s_{g,y}r_{g,x} \\ s_{g,y}r_{g,z} - s_{g,z}r_{g,y} & s_{g,x}r_{g,x} - s_{g,y}r_{g,y} - s_{g,z}r_{g,z} & s_{g,x}r_{g,y} + s_{g,y}r_{g,x} & s_{g,x}r_{g,z} + s_{g,z}r_{g,x} \\ s_{g,z}r_{g,x} - s_{g,x}r_{g,z} & s_{g,y}r_{g,x} + s_{g,x}r_{g,y} & -s_{g,x}r_{g,x} + s_{g,y}r_{g,y} - s_{g,z}r_{g,z} & s_{g,y}r_{g,z} + s_{g,z}r_{g,y} \\ s_{g,x}r_{g,y} - s_{g,y}r_{g,x} & s_{g,z}r_{g,x} + s_{g,x}r_{g,z} & s_{g,z}r_{g,y} + s_{g,y}r_{g,z} & -s_{g,x}r_{g,x} - s_{g,y}r_{g,y} + s_{g,z}r_{g,z} \end{pmatrix} + \alpha_m \begin{pmatrix} s_{m,x}r_{m,x} + s_{m,y}r_{m,y} + s_{m,z}r_{m,z} & s_{m,y}r_{m,z} - s_{m,z}r_{m,y} & s_{m,z}r_{m,x} - s_{m,x}r_{m,z} & s_{m,x}r_{m,y} - s_{m,y}r_{m,x} \\ s_{m,y}r_{m,z} - s_{m,z}r_{m,y} & s_{m,x}r_{m,x} - s_{m,y}r_{m,y} - s_{m,z}r_{m,z} & s_{m,x}r_{m,y} + s_{m,y}r_{m,x} & s_{m,x}r_{m,z} + s_{m,z}r_{m,x} \\ s_{m,z}r_{m,x} - s_{m,x}r_{m,z} & s_{m,y}r_{m,x} + s_{m,x}r_{m,y} & -s_{m,x}r_{m,x} + s_{m,y}r_{m,y} - s_{m,z}r_{m,z} & s_{m,y}r_{m,z} + s_{m,z}r_{m,y} \\ s_{m,x}r_{m,y} - s_{m,y}r_{m,x} & s_{m,z}r_{m,x} + s_{m,x}r_{m,z} & s_{m,z}r_{m,y} + s_{m,y}r_{m,z} & -s_{m,x}r_{m,x} - s_{m,y}r_{m,y} + s_{m,z}r_{m,z} \end{pmatrix} \quad (48)$$

The following sub-sections evaluate the $\boldsymbol{\kappa}$ matrix for the three coordinate systems supported.

3.5.1 Aerospace Coordinate System (NED)

The reference vector $\boldsymbol{r}_g$ is the downwards pointing gravity vector with magnitude 1g aligned with the z axis:

$$\boldsymbol{r}_g = {}^G\boldsymbol{g}_k = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad (49)$$

The measured vector $\boldsymbol{s}_g$ is the gravity vector rotated into the sensor frame and computed from the normalized accelerometer measurement:

$$\boldsymbol{s}_g = {}^S\boldsymbol{g}_k = \frac{{}^S\boldsymbol{G}_k}{|{}^S\boldsymbol{G}_k|} \quad (50)$$

The reference vector $\boldsymbol{r}_m$ points northwards (x axis) and downwards (z axis) from horizontal by the inclination angle δ:

$$\boldsymbol{r}_m = {}^G\boldsymbol{m}_k = \begin{pmatrix} cos\delta \\ 0 \\ sin\delta \end{pmatrix} \quad (51)$$

The measured vector $\boldsymbol{s}_{mg}$ is the geomagnetic vector rotated into the sensor frame and computed from the normalized calibrated magnetometer measurement:

$$\boldsymbol{s}_m = {}^S\boldsymbol{m}_k = \frac{{}^S\boldsymbol{B}_{c,k}}{|{}^S\boldsymbol{B}_{c,k}|} \quad (52)$$

The $\boldsymbol{\kappa}$ matrix evaluates to:

$$\boldsymbol{\kappa}_{NED} = \alpha_g \begin{pmatrix} s_{g,z} & s_{g,y} & -s_{g,x} & 0 \\ s_{g,y} & -s_{g,z} & 0 & s_{g,x} \\ -s_{g,x} & 0 & -s_{g,z} & s_{g,y} \\ 0 & s_{g,x} & s_{g,y} & s_{g,z} \end{pmatrix} + \alpha_m \begin{pmatrix} s_{m,x}cos\delta + s_{m,z}sin\delta & s_{m,y}sin\delta & s_{m,z}cos\delta - s_{m,x}sin\delta & -s_{m,y}cos\delta \\ s_{m,y}sin\delta & s_{m,x}cos\delta - s_{m,z}sin\delta & s_{m,y}cos\delta & s_{m,x}sin\delta + s_{m,z}cos\delta \\ s_{m,z}cos\delta - s_{m,x}sin\delta & s_{m,y}cos\delta & -s_{m,x}cos\delta - s_{m,z}sin\delta & s_{m,y}sin\delta \\ -s_{m,y}cos\delta & s_{m,z}cos\delta + s_{m,x}sin\delta & s_{m,y}sin\delta & -s_{m,x}cos\delta + s_{m,z}sin\delta \end{pmatrix} \quad (53)$$

3.5.2 Android Coordinate System (ENU)

The reference vector $\boldsymbol{r}_g$ is the downwards pointing gravity vector with magnitude 1g aligned in the opposite direction to the upwards pointing *z* axis:

$$\boldsymbol{r}_g = {}^G\boldsymbol{g}_k = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \tag{54}$$

The measured vector $\boldsymbol{s}_g$ is the gravity vector rotated into the sensor frame and computed from the normalized negated accelerometer measurement. The negation occurs because the Android coordinate system is acceleration positive and gravity negative.

$$\boldsymbol{s}_g = {}^S\boldsymbol{g}_k = \frac{-{}^S\boldsymbol{G}_k}{|{}^S\boldsymbol{G}_k|} \tag{55}$$

The reference vector $\boldsymbol{r}_m$ points northwards (*y* axis) and downwards (negative *z* axis) from horizontal by the inclination angle δ:

$$\boldsymbol{r}_m = {}^G\boldsymbol{m}_k = \begin{pmatrix} 0 \\ cos\delta \\ -sin\delta \end{pmatrix} \tag{56}$$

The measured vector $\boldsymbol{s}_{mg}$ is the geomagnetic vector rotated into the sensor frame and computed from the normalized calibrated magnetometer measurement:

$$\boldsymbol{s}_m = {}^S\boldsymbol{m}_k = \frac{{}^S\boldsymbol{B}_{c,k}}{\left|{}^S\boldsymbol{B}_{c,k}\right|} \tag{57}$$

The $\boldsymbol{\kappa}$ matrix evaluates to:

$$\boldsymbol{\kappa} = \alpha_g \begin{pmatrix} -s_{g,z} & -s_{g,y} & s_{g,x} & 0 \\ -s_{g,y} & s_{g,z} & 0 & -s_{g,x} \\ s_{g,x} & 0 & s_{g,z} & -s_{g,y} \\ 0 & -s_{g,x} & -s_{g,y} & -s_{g,z} \end{pmatrix} + \\ \alpha_m \begin{pmatrix} s_{m,y}cos\delta - s_{m,z}sin\delta & -s_{m,y}sin\delta - s_{m,z}cos\delta & s_{m,x}sin\delta & s_{m,x}cos\delta \\ -s_{m,y}sin\delta - s_{m,z}cos\delta & -s_{m,y}cos\delta + s_{m,z}sin\delta & s_{m,x}cos\delta & -s_{m,x}sin\delta \\ s_{m,x}sin\delta & s_{m,x}cos\delta & s_{my}cos\delta + s_{m,z}sin\delta & -s_{m,y}sin\delta + s_{m,z}cos\delta \\ s_{m,x}cos\delta & -s_{m,x}sin\delta & -s_{m,y}sin\delta + s_{m,z}cos\delta & -s_{m,y}cos\delta - s_{m,z}sin\delta \end{pmatrix} \tag{58}$$

3.5.3 Windows 8 Coordinate System (ENU)

The reference vector $\boldsymbol{r}_g$ is the downwards pointing gravity vector with magnitude 1g aligned in the opposite direction to the upwards pointing *z* axis:

$$\boldsymbol{r}_g = {}^G\boldsymbol{g}_k = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \tag{59}$$

The measured vector $\boldsymbol{s}_g$ is the gravity vector rotated into the sensor frame and computed from the normalized accelerometer measurement:

$$\boldsymbol{s}_g = {}^S\boldsymbol{g}_k = \frac{{}^S\boldsymbol{G}_k}{|{}^S\boldsymbol{G}_k|} \tag{60}$$

The reference vector $\boldsymbol{r}_m$ points northwards (*y* axis) and downwards (negative *z* axis) from horizontal by the inclination angle δ:

$$\boldsymbol{r}_m = {}^G\boldsymbol{m}_k = \begin{pmatrix} 0 \\ cos\delta \\ -sin\delta \end{pmatrix} \tag{61}$$

The measured vector $\boldsymbol{s}_{mg}$ is the geomagnetic vector rotated into the sensor frame and computed from the normalized calibrated magnetometer measurement:

$$\boldsymbol{s}_m = {}^S\boldsymbol{m}_k = \frac{{}^S\boldsymbol{B}_{c,k}}{|{}^S\boldsymbol{B}_{c,k}|} \tag{62}$$

The $\boldsymbol{\kappa}$ matrix evaluates to:

$$\boldsymbol{K} = \alpha_g \begin{pmatrix} -s_{g,z} & -s_{g,y} & s_{g,x} & 0 \\ -b_{g,y} & s_{g,z} & 0 & -s_{g,x} \\ s_{g,x} & 0 & s_{g,z} & -s_{g,y} \\ 0 & -s_{g,x} & -s_{g,y} & -s_{g,z} \end{pmatrix} + \\ \alpha_m \begin{pmatrix} s_{m,y}cos\delta - s_{m,z}sin\delta & -s_{m,y}sin\delta - s_{m,z}cos\delta & s_{m,x}sin\delta & s_{m,x}cos\delta \\ -s_{m,y}sin\delta - s_{m,z}cos\delta & -s_{m,y}cos\delta + s_{m,z}sin\delta & s_{m,x}cos\delta & -s_{m,x}sin\delta \\ s_{m,x}sin\delta & s_{m,x}cos\delta & s_{m,y}cos\delta + s_{m,z}sin\delta & -s_{m,y}sin\delta + s_{m,z}cos\delta \\ s_{m,x}cos\delta & -s_{m,x}sin\delta & s_{m,z}cos\delta - s_{m,y}sin\delta & -s_{m,y}cos\delta - s_{m,z}sin\delta \end{pmatrix} \tag{63}$$

3.6 Relative Weightings

The weightings α_g and α_m applied to the accelerometer and magnetometer estimates of the gravity and geomagnetic vectors must be inversely proportional to the noise covariances defined in equations (9) and (15) and sum to unity giving:

$$\alpha_g = \frac{\left(\frac{Q_{d,k} + Q_{vB,k}}{B^2}\right)}{\left(Q_{a,k} + Q_{vG,k}\right) + \left(\frac{Q_{d,k} + Q_{vB,k}}{B^2}\right)} = \frac{\left|\frac{\left|{}^S\boldsymbol{B}_{c,k}\right|^2}{B^2} - 1\right|}{\left|\left|{}^S\boldsymbol{G}_k\right|^2 - 1\right| + \left|\frac{\left|{}^S\boldsymbol{B}_{c,k}\right|^2}{B^2} - 1\right|} \tag{64}$$

$$\alpha_m = \frac{\left(Q_{a,k} + Q_{vG,k}\right)}{\left(Q_{a,k} + Q_{vG,k}\right) + \left(\frac{Q_{d,k} + Q_{vB,k}}{B^2}\right)} = \frac{\left|\left|{}^S\boldsymbol{G}_k\right|^2 - 1\right|}{\left|\left|{}^S\boldsymbol{G}_k\right|^2 - 1\right| + \left|\frac{\left|{}^S\boldsymbol{B}_{c,k}\right|^2}{B^2} - 1\right|} \tag{65}$$

In the limit of increasingly high accelerometer noise, the weightings asymptote to $\alpha_g = 0$ and $\alpha_m = 1$ (full weighting to the magnetometer measurement) and, in the limit of increasingly high magnetometer noise, asymptote to $\alpha_g = 1$ and $\alpha_m = 0$ (full weighting to the accelerometer measurement).

4 Gravity and Geomagnetic Tilt Error Quaternions

4.1 Introduction

The orientation algorithm documented in the previous section gives the optimum (in a least squares sense) orientation fit to the instantaneous accelerometer and magnetometer measurements weighted proportionally to the confidence (which is inversely proportionally to the acceleration and magnetic disturbance noise) in the two readings. This orientation is termed the six degree of freedom or 6DOF acceleration plus magnetometer orientation estimate since it is computed from a total of six sensor channels.

The 6DOF estimate is used in the Kalman filter to stabilize the 3DOF gyroscope orientation estimate which is unaffected by acceleration or magnetic disturbance but which does drift as a result of an imperfectly estimated gyroscope zero rotation rate offset. The combined Kalman filter orientation estimate is termed the nine degree of freedom or 9DOF orientation estimate since it is computed from a total of nine sensor channels. Sensor fusion can therefore be viewed as attempting to produce an orientation estimate which combines the excellent high frequency response of the 3DOF gyroscope estimate with the low drift characteristics of the 6DOF accelerometer plus magnetometer estimate.

Section 4.2 first derives the mathematical expression for the rotation quaternion that rotates one arbitrary vector onto another with equal magnitude.

In sections 4.3 and 4.4, two pairs of vectors are discussed: i) the gravity vectors computed from the 3DOF gyroscope and 6DOF accelerometer plus magnetometer orientation estimates and ii) the geomagnetic vectors computed from the 3DOF gyroscope and 6DOF accelerometer plus magnetometer orientation estimates. The gravity and geomagnetic rotation quaternions computed between these two pairs of vectors form the measurement error quaternions used in the Kalman filter.

The Kalman filter attempts to drive the two measurement error quaternions to zero taking proper account of the confidence in the measurements:

- If a low value of accelerometer noise (including acceleration as discussed in section 2) is present then the 9DOF Kalman filter estimate of tilt from vertical is rapidly updated to the 6DOF tilt angle from vertical. If a high value of accelerometer noise is detected then the gravity tilt estimate from integrating the gyroscope is used with little or no change.
- Similarly, if a low value of magnetometer noise (including magnetic disturbance as discussed in section 2) is detected then the 9DOF Kalman filter estimate of the tilt angle away from the geomagnetic vector is rapidly updated to the 6DOF tilt angle away from the geomagnetic vector. If a high value of magnetic disturbance is detected then the geomagnetic tilt estimate from integrating the gyroscope is used with little or no change.

Since the gravity and geomagnetic vectors are not aligned, except at the north and south geomagnetic poles, the continuous correction of the two tilt angles relative to these two distinct vectors gives a complete estimate of orientation.

4.2 Rotation Quaternion Between Two Vectors

This section therefore derives the expression for the rotation quaternion q required to rotate vector $\boldsymbol{u}$ onto vector $\boldsymbol{v}$ such that:

$$\boldsymbol{v} = q^{*}\boldsymbol{u}q \quad \textbf{(66)}$$

The rotation quaternion q is unchanged if both $\boldsymbol{u}$ and $\boldsymbol{v}$ are negated:

$$(-\boldsymbol{v}) = q^*(-\boldsymbol{u})q \quad \textbf{(67)}$$

Obviously the norm of both vectors must be the same since rotation does not change a vector's magnitude. The rotation sign is defined as the Freescale standard which uses a rotation of the coordinate frame rather a rotation of the vector within the coordinate frame.

The angle η between the two vectors can be determined from the scalar product $\boldsymbol{u}.\boldsymbol{v}$:

$$cos\eta = 2cos^2\left(\frac{\eta}{2}\right) - 1 = \frac{\boldsymbol{u}.\boldsymbol{v}}{|\boldsymbol{u}||\boldsymbol{v}|} \quad \textbf{(68)}$$

Rearranging gives the scalar component q_0 of the quaternion q:

$$q_0 = cos\left(\frac{\eta}{2}\right) = \sqrt{\frac{1}{2} + \frac{\boldsymbol{u}.\boldsymbol{v}}{2|\boldsymbol{u}||\boldsymbol{v}|}} = \sqrt{\frac{|\boldsymbol{u}||\boldsymbol{v}| + \boldsymbol{u}.\boldsymbol{v}}{2|\boldsymbol{u}||\boldsymbol{v}|}} \quad \textbf{(69)}$$

The rotation axis $\hat{\boldsymbol{n}}$ is calculated from the vector product $\boldsymbol{u} \times \boldsymbol{v}$:

$$\hat{\boldsymbol{n}}sin\eta = 2\hat{\boldsymbol{n}}sin\left(\frac{\eta}{2}\right)cos\left(\frac{\eta}{2}\right) = \frac{-\boldsymbol{u} \times \boldsymbol{v}}{|\boldsymbol{u}||\boldsymbol{v}|} \quad \textbf{(70)}$$

The minus sign in equation (70) derives from calculating the rotation axis for a coordinate system rotation by η rather than the axis required to rotate the vector $\boldsymbol{u}$ onto $\boldsymbol{v}$ in a fixed coordinate system.

Substitution gives the vector component $\boldsymbol{q} = \{q_1, q_2, q_3\}$ of the quaternion q:

$$\boldsymbol{q} = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = \hat{\boldsymbol{n}}sin\left(\frac{\eta}{2}\right) = \frac{-\boldsymbol{u} \times \boldsymbol{v}}{2|\boldsymbol{u}||\boldsymbol{v}|\sqrt{\frac{|\boldsymbol{u}||\boldsymbol{v}| + \boldsymbol{u}.\boldsymbol{v}}{2|\boldsymbol{u}||\boldsymbol{v}|}}} \quad \textbf{(71)}$$

The required rotation quaternion q is then:

$$q = cos\left(\frac{\eta}{2}\right) + \hat{\boldsymbol{n}}sin\left(\frac{\eta}{2}\right) = \sqrt{\frac{|\boldsymbol{u}||\boldsymbol{v}| + \boldsymbol{u}.\boldsymbol{v}}{2|\boldsymbol{u}||\boldsymbol{v}|}} - \frac{\boldsymbol{u} \times \boldsymbol{v}}{2|\boldsymbol{u}||\boldsymbol{v}|\sqrt{\frac{|\boldsymbol{u}||\boldsymbol{v}| + \boldsymbol{u}.\boldsymbol{v}}{2|\boldsymbol{u}||\boldsymbol{v}|}}} = \frac{|\boldsymbol{u}||\boldsymbol{v}| + \boldsymbol{u}.\boldsymbol{v} - \boldsymbol{u} \times \boldsymbol{v}}{\sqrt{2|\boldsymbol{u}||\boldsymbol{v}|(|\boldsymbol{u}||\boldsymbol{v}| + \boldsymbol{u}.\boldsymbol{v})}} \quad \textbf{(72)}$$

Equation (72) correctly gives zero scalar quaternion component q_0 for the 180° rotation case where $|\boldsymbol{u}||\boldsymbol{v}| + \boldsymbol{u}.\boldsymbol{v} = 0$.

$$q_0 = cos\left(\frac{\eta}{2}\right) = \sqrt{\frac{|\boldsymbol{u}||\boldsymbol{v}| + \boldsymbol{u}.\boldsymbol{v}}{2|\boldsymbol{u}||\boldsymbol{v}|}} = 0 \tag{73}$$

The vector quaternion component $\boldsymbol{q}$ is, however, undefined for the 180° rotation case. Mathematically, $\boldsymbol{u} \times \boldsymbol{v} = \boldsymbol{0}$ and $|\boldsymbol{u}||\boldsymbol{v}| + \boldsymbol{u}.\boldsymbol{v} = 0$ resulting in zero numerator and denominator. Physically, there are an infinite number of rotation axes orthogonal to both $\boldsymbol{u}$ and $\boldsymbol{v}$ when $\boldsymbol{u}$ and $\boldsymbol{v}$ are anti-parallel.

By inspection, one solution (of the infinite number available) to the vector component $\boldsymbol{q}$ for the 180° rotation case valid for all cases except $u_x = u_y = u_z$ is:

$$\boldsymbol{q} = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = \frac{1}{\sqrt{\left(u_y - u_z\right)^2 + (u_z - u_x)^2 + \left(u_x - u_y\right)^2}} \begin{pmatrix} u_y - u_z \\ u_z - u_x \\ u_x - u_y \end{pmatrix} \tag{74}$$

For the special case $u_x = u_y = u_z$ and 180° rotation, the solution is:

$$\boldsymbol{q} = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \tag{75}$$

For the case when the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ have unit norm, equation (72) can be simplified to:

$$q = cos\left(\frac{\eta}{2}\right) + \hat{\boldsymbol{n}} sin\left(\frac{\eta}{2}\right) = \frac{1 + \boldsymbol{u}.\boldsymbol{v} - \boldsymbol{u} \times \boldsymbol{v}}{\sqrt{2(1 + \boldsymbol{u}.\boldsymbol{v})}} = \frac{1}{\sqrt{2}}\left(\sqrt{1 + \boldsymbol{u}.\boldsymbol{v}} - \frac{\boldsymbol{u} \times \boldsymbol{v}}{\sqrt{(1 + \boldsymbol{u}.\boldsymbol{v})}}\right) \; for \; |\boldsymbol{u}| = |\boldsymbol{v}| = 1 \tag{76}$$

Equation (76) is implemented in function `fveqconjgquq` in file *orientation.c*.

4.3 Gravity Tilt Error Quaternion Measurement

The gravity vector ${}^G\boldsymbol{g}_k$ in the global reference frame is defined in section 3.5 for the three coordinate systems. The gravity vector ${}^S\boldsymbol{g}_k$ in the sensor frame for any orientation matrix $\boldsymbol{R}$ can be computed by multiplying ${}^G\boldsymbol{g}_k$ by the orientation matrix $\boldsymbol{R}$ to transform from the global to sensor frame:

$${}^S\boldsymbol{g}_k = \boldsymbol{R}\,{}^G\boldsymbol{g}_k \tag{77}$$

The rotated gravity vector ${}^S\boldsymbol{g}_k$ in equation (77) can easily be determined from the z axis column vector of the orientation matrix $\boldsymbol{R}$:

$$^{S}\boldsymbol{g}_k = \begin{pmatrix} R_{xx} & R_{xy} & R_{xz} \\ R_{yx} & R_{yy} & R_{yz} \\ R_{zx} & R_{zy} & R_{zz} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} R_{xz} \\ R_{yz} \\ R_{zz} \end{pmatrix} for\ Aerospace/NED \tag{78}$$

$$^{S}\boldsymbol{g}_k = \begin{pmatrix} R_{xx} & R_{xy} & R_{xz} \\ R_{yx} & R_{yy} & R_{yz} \\ R_{zx} & R_{zy} & R_{zz} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} = -\begin{pmatrix} R_{xz} \\ R_{yz} \\ R_{zz} \end{pmatrix} = for\ Android, Windows\ 8 \tag{79}$$

The derivation in section 4.2 can now be applied to determine the quaternion $q_{zg\varepsilon,k}$ which rotates the 6DOF estimate of the gravity vector onto the Kalman filter estimate of the gravity vector. At the start of iteration k, the best orientation estimate is the *a priori* estimate obtained by applying the gyroscope sensor measurement to the *a posteriori* quaternion $\hat{q}_k^-$ and orientation matrix $\hat{\boldsymbol{R}}_k^-$ estimates from the previous iteration. This is discussed in detail in sections 7 and 8 which document the Kalman filters.

With the vector definitions of $\boldsymbol{u}_{g,k}$ and $\boldsymbol{v}_{g,k}$ as the 6DOF and *a priori* estimates of the gravity vector at iteration k:

$$\boldsymbol{u}_{g,k} = \begin{pmatrix} R_{k,xz}^{6DOF} \\ R_{k,yz}^{6DOF} \\ R_{k,zz}^{6DOF} \end{pmatrix}, \boldsymbol{v}_{g,k} = \begin{pmatrix} \hat{R}_{k,xz}^{-} \\ \hat{R}_{k,yz}^{-} \\ \hat{R}_{k,zz}^{-} \end{pmatrix} for\ Aerospace/NED \tag{80}$$

$$\boldsymbol{u}_{g,k} = -\begin{pmatrix} R_{k,xz}^{6DOF} \\ R_{k,yz}^{6DOF} \\ R_{k,zz}^{6DOF} \end{pmatrix}, \boldsymbol{v}_{g,k} = -\begin{pmatrix} \hat{R}_{k,xz}^{-} \\ \hat{R}_{k,yz}^{-} \\ \hat{R}_{k,zz}^{-} \end{pmatrix} for\ Android, Windows\ 8 \tag{81}$$

then substituting into equation (76) gives the measured error quaternion $q_{zg\varepsilon,k}$ between the two gravity vector estimates with the sign convention that $q_{zg\varepsilon,k}$ rotates the 6DOF gravity vector estimate onto the *a priori* gravity vector estimate:

$$q_{zg\varepsilon,k} = \frac{1}{\sqrt{2}}\left(\sqrt{1 + \boldsymbol{u}_{g,k}.\boldsymbol{v}_{g,k}} - \frac{\boldsymbol{u}_{g,k} \times \boldsymbol{v}_{g,k}}{\sqrt{(1 + \boldsymbol{u}_{g,k}.\boldsymbol{v}_{g,k})}}\right) \tag{82}$$

The vector component $\boldsymbol{q}_{zg\varepsilon,k}$ of $q_{zg\varepsilon,k}$ comprises the first three of the seven rows of the Kalman filter measurement vector $\boldsymbol{z}_{\varepsilon,k}$ documented in sections 7 and 8.

4.4 Geomagnetic Tilt Error Quaternion Measurement

The geomagnetic vector $^{G}\boldsymbol{m}_k$ in the global reference frame is also defined in section 3.5 for the three coordinate systems. The geomagnetic vector $^{S}\boldsymbol{m}_k$ in the sensor frame for any orientation matrix $\boldsymbol{R}$ can be computed by multiplying by the orientation matrix:

$$^{S}\boldsymbol{m}_k = \boldsymbol{R}^{G}\boldsymbol{m}_k \tag{83}$$

Evaluating equation (83) for the three coordinate systems gives:

$$^{S}\boldsymbol{m}_k = \begin{pmatrix} R_{xx} & R_{xy} & R_{xz} \\ R_{yx} & R_{yy} & R_{yz} \\ R_{zx} & R_{zy} & R_{zz} \end{pmatrix} B \begin{pmatrix} cos\delta \\ 0 \\ sin\delta \end{pmatrix} = B \begin{pmatrix} R_{xx}cos\delta + R_{xz}sin\delta \\ R_{yx}cos\delta + R_{yz}sin\delta \\ R_{zx}cos\delta + R_{zz}sin\delta \end{pmatrix} for\ Aerospace/NED \tag{84}$$

$$^{S}\boldsymbol{m}_k = \begin{pmatrix} R_{xx} & R_{xy} & R_{xz} \\ R_{yx} & R_{yy} & R_{yz} \\ R_{zx} & R_{zy} & R_{zz} \end{pmatrix} B \begin{pmatrix} 0 \\ cos\delta \\ -sin\delta \end{pmatrix} = B \begin{pmatrix} R_{xy}cos\delta - R_{xz}sin\delta \\ R_{yy}cos\delta - R_{yz}sin\delta \\ R_{zy}cos\delta - R_{zz}sin\delta \end{pmatrix} for\ Android, Windows\ 8 \tag{85}$$

With the vector definitions of $\boldsymbol{u}_{m,k}$ and $\boldsymbol{v}_{m,k}$ as the normalized 6DOF and *a priori* estimates of the geomagnetic vector at filter iteration k:

$$\boldsymbol{u}_{m,k} = \begin{pmatrix} R_{k,xx}^{6DOF}cos\delta + R_{k,xz}^{6DOF}sin\delta \\ R_{k,yx}^{6DOF}cos\delta + R_{k,yz}^{6DOF}sin\delta \\ R_{k,zx}^{6DOF}cos\delta + R_{k,zz}^{6DOF}sin\delta \end{pmatrix}, \boldsymbol{v}_{m,k} = \begin{pmatrix} \hat{R}_{k,xx}^{-}cos\delta + \hat{R}_{k,xz}^{-}sin\delta \\ \hat{R}_{k,yx}^{-}cos\delta + \hat{R}_{k,yz}^{-}sin\delta \\ \hat{R}_{k,zx}^{-}cos\delta + \hat{R}_{k,zz}^{-}sin\delta \end{pmatrix} for\ Aerospace/NED \tag{86}$$

$$\boldsymbol{u}_{m,k} = \begin{pmatrix} R_{k,xy}^{6DOF}cos\delta - R_{k,xz}^{6DOF}sin\delta \\ R_{k,yy}^{6DOF}cos\delta - R_{k,yz}^{6DOF}sin\delta \\ R_{k,zy}^{6DOF}cos\delta - R_{k,zz}^{6DOF}sin\delta \end{pmatrix}, \boldsymbol{v}_{m,k} = \begin{pmatrix} \hat{R}_{k,xy}^{-}cos\delta - \hat{R}_{k,xz}^{-}sin\delta \\ \hat{R}_{k,yy}^{-}cos\delta - \hat{R}_{k,yz}^{-}sin\delta \\ \hat{R}_{k,zy}^{-}cos\delta - \hat{R}_{k,zz}^{-}sin\delta \end{pmatrix} for\ Android, Windows\ 8 \tag{87}$$

then substituting into equation (76) gives the measurement error quaternion $q_{zm\varepsilon,k}$ between the two geomagnetic vector estimates. The vector component $\boldsymbol{q}_{zm\varepsilon,k}$ of $q_{zm\varepsilon,k}$ comprises the second three of the seven rows of the Kalman filter measurement vector $\boldsymbol{z}_{\varepsilon,k}$ documented in section 7.

$$q_{zm\varepsilon,k} = \frac{1}{\sqrt{2}}\left(\sqrt{1 + \boldsymbol{u}_{m,k}.\boldsymbol{v}_{m,k}} - \frac{\boldsymbol{u}_{m,k} \times \boldsymbol{v}_{m,k}}{\sqrt{(1 + \boldsymbol{u}_{m,k}.\boldsymbol{v}_{m,k})}}\right) \tag{88}$$

5 Gyroscope Sensor Model

5.1 Zero Rate Offset Model

The model of the gyroscope sensor measurement $^{S}\boldsymbol{Y}_k$ (with units deg/s) is:

$$^{S}\boldsymbol{Y}_k = \boldsymbol{\omega}_k + \boldsymbol{b}_k + \boldsymbol{v}_{Y,k} \tag{89}$$

where $\boldsymbol{\omega}_k$ is the true angular velocity in deg/s and $\boldsymbol{b}_k$ the gyroscope offset vector (deg/s). $\boldsymbol{v}_{Y,k}$ is the additive gyroscope noise vector (deg/s) with covariance $\boldsymbol{Q}_{vY}$ assumed to be constant over time, uncorrelated between axes (diagonal) and to have the same value Q_{vY} in each axis:

$$\boldsymbol{Q}_{vY} = E\left[\boldsymbol{v}_{Y,k}\boldsymbol{v}_{Y,k}{}^{T}\right] = Q_{vY}\boldsymbol{I} \tag{90}$$

The value of Q_{vY} is set for the 6DOF and 9DOF Kalman filter algorithms in the compile time constants `FQVY_9DOF_GBY_KALMAN` and `FQVY_6DOF_GY_KALMAN` defined in file *fusion.h*. Increasing the value of $Q_{vY,k}$ gives a lower weighting to the gyroscope orientation estimate which results in more rapid convergence to the gravity and geomagnetic vector estimate from the accelerometer and magnetometer but increased sensitivity to acceleration and magnetic disturbance noise.

The gyroscope offset $\boldsymbol{b}_k$ vector (units deg/s) is modeled as the random walk:

$$\boldsymbol{b}_k = \boldsymbol{b}_{k-1} + \boldsymbol{w}_{b,k} \tag{91}$$

where $\boldsymbol{w}_{b,k}$ is a zero mean white Gaussian noise vector with units of deg/s with covariance $\boldsymbol{Q}_{wb}$ assumed to be constant over time, uncorrelated between axes (diagonal) and to have the same value Q_{wb} in each axis:

$$\boldsymbol{Q}_{wb} = E\left[\boldsymbol{w}_{b,k}\boldsymbol{w}_{b,k}{}^{T}\right] = Q_{wb}\boldsymbol{I} \tag{92}$$

The value of Q_{wb} is set for the 6DOF and 9DOF Kalman filter algorithms in the compile time constants `FQWB_9DOF_GBY_KALMAN` and `FQWB_6DOF_GY_KALMAN` defined in file *fusion.h*. Increasing the value of Q_{wb} allows more rapid tracking of changes in the zero rate offset, including the initial estimation at power on, but has the drawback of increased sensitivity to acceleration and magnetic disturbance noise.

The *a priori* estimate of the gyroscope offset is simply the *a posteriori* estimate from the previous sample since $\boldsymbol{w}_{b,k}$ is zero mean and white:

$$\hat{\boldsymbol{b}}_k^- = \hat{\boldsymbol{b}}_{k-1}^+ \tag{93}$$

Simple algebra gives $\hat{\boldsymbol{b}}_{\varepsilon,k}^-$ as a function of $\hat{\boldsymbol{b}}_{\varepsilon,k-1}^+$:

$$\hat{\boldsymbol{b}}_{\varepsilon,k}^- = \hat{\boldsymbol{b}}_k^- - \boldsymbol{b}_k = \hat{\boldsymbol{b}}_{k-1}^+ - \boldsymbol{b}_k = \hat{\boldsymbol{b}}_{k-1}^+ - \left(\boldsymbol{b}_{k-1} + \boldsymbol{w}_{b,k}\right) = \left(\hat{\boldsymbol{b}}_{k-1}^+ - \boldsymbol{b}_{k-1}\right) - \boldsymbol{w}_{b,k} \tag{94}$$

$$\Rightarrow \hat{\boldsymbol{b}}_{\varepsilon,k}^- = \hat{\boldsymbol{b}}_{\varepsilon,k-1}^+ - \boldsymbol{w}_{b,k} \tag{95}$$

5.2 Angular Velocity Model

The *a priori estimate* $\hat{\boldsymbol{\omega}}_k^-$ of the true angular velocity $\boldsymbol{\omega}_k$ is computed from the gyroscope reading in the sensor frame ${}^S\boldsymbol{Y}_k$ by subtracting off the current *a priori* zero rate gyroscope offset $\hat{\boldsymbol{b}}_k^-$ which equals the *a posteriori* offset estimate $\hat{\boldsymbol{b}}_{k-1}^+$ from the previous iteration:

$$\hat{\boldsymbol{\omega}}_k^- = \left({}^S\boldsymbol{Y}_k - \hat{\boldsymbol{b}}_k^-\right) = \left({}^S\boldsymbol{Y}_k - \hat{\boldsymbol{b}}_{k-1}^+\right) \tag{96}$$

Substituting for ${}^S\boldsymbol{Y}_k$ gives the relationship between the error components:

$$\hat{\boldsymbol{\omega}}_k^- = \boldsymbol{\omega}_k + \boldsymbol{b}_k + \boldsymbol{v}_{Y,k} - \hat{\boldsymbol{b}}_{k-1}^+ \tag{97}$$

By definition:

$$\hat{\boldsymbol{\omega}}_k^- = \boldsymbol{\omega}_k + \hat{\boldsymbol{\omega}}_{\varepsilon,k}^- \tag{98}$$

$$\Rightarrow \hat{\boldsymbol{\omega}}_{\varepsilon,k}^- = \boldsymbol{b}_{k-1} + \boldsymbol{w}_{b,k} + \boldsymbol{v}_{Y,k} - \hat{\boldsymbol{b}}_{k-1}^+ \tag{99}$$

By definition:

$$\hat{\boldsymbol{b}}_{\varepsilon,k-1}^+ = \hat{\boldsymbol{b}}_{k-1}^+ - \boldsymbol{b}_{k-1} \tag{100}$$

$$\Rightarrow \hat{\boldsymbol{\omega}}_{\varepsilon,k}^- = -\hat{\boldsymbol{b}}_{\varepsilon,k-1}^+ + \boldsymbol{w}_{b,k} + \boldsymbol{v}_{Y,k} = -\hat{\boldsymbol{b}}_{\varepsilon,k}^- + \boldsymbol{w}_{b,k} + \boldsymbol{v}_{Y,k} \tag{101}$$

Equation (101) states that the error in the *a priori* estimate of angular velocity comprises three terms:

i) The error in the *a priori* estimate $\hat{\boldsymbol{b}}_{\varepsilon,k}^-$ of the gyroscope zero rate sensor. The minus sign results from the fact that an over-estimate of the gyroscope zero rate gyroscope leads to an under-estimate of the angular velocity.

ii) The noise $\boldsymbol{w}_{b,k}$ in the gyroscope zero rate offset drift.

iii) The additive gyroscope sensor noise $\boldsymbol{v}_{Y,k}$.

The sensor noise term $\boldsymbol{v}_{Y,k}$ can only be separated from the offset drift term $\boldsymbol{w}_{b,k}$ by observing the gyroscope over a period long enough for the drift to be measurable. On a sample by sample basis, the gyroscope offset drift term $\boldsymbol{w}_{b,k}$ is indistinguishable from the gyroscope noise term $\boldsymbol{v}_{Y,k}$.

6 Magnetic Inclination Model

6.1 Definition of the Inclination Angle

The inclination angle δ is the angle by which the geomagnetic vector dips below horizontal. Broadly speaking, δ has value -90° at the south geomagnetic pole (the geomagnetic vector points upwards),

+90° at the north geomagnetic pole (the geomagnetic vector points downwards) and approximately 0° deg at the equator (where the geomagnetic vector is horizontal).

Matters are further complicated indoors where steel building framing and steel cabinets can radically change the inclination angle. It cannot be assumed that δ is constant with respect to small position changes at any point on the earth's surface. Not only is δ not known but indoors it must be tracked with a time constant of a few seconds.

A sample by sample estimate of δ can be computed from the accelerometer and magnetometer vectors but this estimate is extremely noisy as a result of linear acceleration and magnetic disturbance. The approach taken is to track the inclination angle within the Kalman filter and use the accelerometer and noise covariances defined in section 2 to define the confidence in the instantaneous estimate and consequently the relative weighting applied by the Kalman filter.

6.2 Inclination Angle Model

The inclination angle δ_k (units deg) at iteration k is modeled as the random walk:

$$\delta_k = \delta_{k-1} + w_{\delta,k} \tag{102}$$

where $w_{\delta,k}$ is a zero mean white noise terms (with units of deg) and covariance $Q_{w\delta}$ (with units of deg^2):

$$Q_{w\delta} = E\left[{w_{\delta,k}}^2\right] \tag{103}$$

The value of $Q_{w\delta}$ is set for the 9DOF Kalman filter algorithms in the compile time constant `FQWDLT_9DOF_GBY_KALMAN` defined in file *fusion.h*. Increasing the value of $Q_{w\delta}$ allows more rapid tracking of changes in the geomagnetic inclination angle δ at the expense of increased sensitivity to acceleration and magnetic disturbance noise.

The *a priori* estimate of the inclination angle is simply the *a posteriori* estimate from the previous sample since $w_{\delta,k}$ is zero mean and white:

$$\hat{\delta}_k^- = \hat{\delta}_{k-1}^+ \tag{104}$$

Simple algebra gives $\hat{\delta}_{\varepsilon,k}^-$ as a function of $\hat{\delta}_{\varepsilon,k-1}^+$:

$$\hat{\delta}_{\varepsilon,k}^- = \hat{\delta}_k^- - \delta_k = \hat{\delta}_{k-1}^+ - \delta_k = \hat{\delta}_{k-1}^+ - \left(\delta_{k-1} + w_{\delta,k}\right) = \left(\hat{\delta}_{k-1}^+ - \delta_{k-1}\right) - w_{\delta,k} \tag{105}$$

$$\Rightarrow \hat{\delta}_{\varepsilon,k}^- = \hat{\delta}_{\varepsilon,k-1}^+ - w_{\delta,k} \tag{106}$$

6.3 Inclination Angle Error Measurement

The inclination angle δ equals 90° minus the angle between the gravity and geomagnetic vectors in the global frame and can be computed from the scalar product of the two vectors as:

Magnetic Inclination Model

$$ {}^{G}\boldsymbol{g} \cdot {}^{G}\boldsymbol{m} = |{}^{G}\boldsymbol{g}| \cdot |{}^{G}\boldsymbol{m}| cos\left(\frac{\pi}{2} - \delta\right) = |{}^{G}\boldsymbol{g}| \cdot |{}^{G}\boldsymbol{m}| sin\delta \quad \textbf{(107)} $$

The scalar product is invariant under arbitrary rotation $\boldsymbol{R}$ since rotation changes neither the magnitude of vectors nor the angle between them.

$$ {}^{S}\boldsymbol{g} \cdot {}^{S}\boldsymbol{m} = (\boldsymbol{R}^{G}\boldsymbol{g}) \cdot (\boldsymbol{R}^{G}\boldsymbol{m}) = (\boldsymbol{R}^{G}\boldsymbol{g})^{T}\boldsymbol{R}^{G}\boldsymbol{m} = ({}^{G}\boldsymbol{g})^{T}\boldsymbol{R}^{T}\boldsymbol{R}^{G}\boldsymbol{m} = ({}^{G}\boldsymbol{g})^{T\,G}\boldsymbol{m} = {}^{G}\boldsymbol{g} \cdot {}^{G}\boldsymbol{m} \quad \textbf{(108)} $$

The instantaneous (noisy) 6DOF estimate of the inclination angle δ_k^{6DOF} at iteration k is computed from the accelerometer ${}^{S}\boldsymbol{G}_k$ and calibrated magnetometer ${}^{S}\boldsymbol{B}_{c,k}$ measurements on the assumption of no acceleration and no magnetic disturbance as:

$$ {}^{S}\boldsymbol{G}_k \cdot {}^{S}\boldsymbol{B}_{c,k} = |{}^{S}\boldsymbol{G}_k|\left|{}^{S}\boldsymbol{B}_{c,k}\right| sin\,\delta_k^{6DOF} \;\; for\ Aerospace\ (NED), Windows \quad \textbf{(109)} $$

$$ \Rightarrow \delta_k^{6DOF} = sin^{-1}\left(\frac{{}^{S}\boldsymbol{G}_k \cdot {}^{S}\boldsymbol{B}_{c,k}}{|{}^{S}\boldsymbol{G}_k|\left|{}^{S}\boldsymbol{B}_{c,k}\right|}\right) for\ Aerospace\ (NED), Windows\ 8 \quad \textbf{(110)} $$

$$ {}^{S}\boldsymbol{G}_k \cdot {}^{S}\boldsymbol{B}_{c,k} = -|{}^{S}\boldsymbol{G}_k|\left|{}^{S}\boldsymbol{B}_{c,k}\right| sin\,\delta_k^{6DOF} \;\; for\ Android \quad \textbf{(111)} $$

$$ \Rightarrow \delta_k^{6DOF} = -sin^{-1}\left(\frac{{}^{S}\boldsymbol{G}_k \cdot {}^{S}\boldsymbol{B}_{c,k}}{|{}^{S}\boldsymbol{G}_k|\left|{}^{S}\boldsymbol{B}_{c,k}\right|}\right) for\ Android \quad \textbf{(112)} $$

The minus sign for the Android coordinate system results from Android being acceleration positive whereas the Aerospace and Windows 8 coordinate systems are gravity positive.

The measurement error $\delta_{z\varepsilon,k}$ in the inclination angle is defined as the difference between the *a posteriori* Kalman filter and the instantaneous 6DOF measurements as:

$$ \delta_{z\varepsilon,k} = \delta_{k-1}^{+} - \delta_k^{6DOF} = \delta_k^{-} - \delta_k^{6DOF} \quad \textbf{(113)} $$

7 Accelerometer, Magnetometer and Gyroscope Sensor Fusion Kalman Filter

7.1 Introduction

This section derives the Kalman filter equations for the sensor fusion of accelerometer, magnetometer and gyroscope data. It is also commonly referred to as a 9 degree of freedom or 9DOF sensor fusion model since each of the three sensors has three axes and provides three degrees of freedom.

7.2 Kalman Filter Process Model

The Kalman filter process state vector $\boldsymbol{x}_k$ comprises i) the orientation quaternion q_k ii) the zero rate gyroscope offset $\boldsymbol{b}_k$ and the inclination angle δ_k at iteration k:

$$\boldsymbol{x}_k = \begin{pmatrix} q_k \\ \boldsymbol{b}_k \\ \delta_k \end{pmatrix} \tag{114}$$

Instead of estimating the process state vector $\boldsymbol{x}_k$ directly, an indirect form of Kalman filter is used which estimates the error process $\boldsymbol{x}_{\varepsilon,k}$ described next.

7.3 Kalman Filter Error Process Model

The standard linear Kalman filter model is applied in the indirect Kalman filter by modeling the error process $\boldsymbol{x}_{\varepsilon,k}$ as:

$$\boldsymbol{x}_{\varepsilon,k} = \boldsymbol{A}_k \boldsymbol{x}_{\varepsilon,k-1} + \boldsymbol{w}_k \tag{115}$$

where $\boldsymbol{x}_{\varepsilon,k}$ is the 10x1 vector with components:

$$\boldsymbol{x}_{\varepsilon,k} = \begin{pmatrix} \boldsymbol{q}_{g\varepsilon,k} \\ \boldsymbol{q}_{m\varepsilon,k} \\ \boldsymbol{b}_{\varepsilon,k} \\ \delta_{\varepsilon,k} \end{pmatrix} \tag{116}$$

and where $\boldsymbol{w}_k$ is a white noise component.

The 3x1 vector $\boldsymbol{q}_{g\varepsilon,k}$ is the vector component of the quaternion $q_{g\varepsilon,k}$ which models the error in tilt angle relative to the true gravity vector and the 3x1 vector $\boldsymbol{q}_{m\varepsilon,k}$ is the vector component of the quaternion $q_{m\varepsilon,k}$ which models the error in tilt angle relative to the true geomagnetic vector. The state vector orientation quaternion q_k is then related to the estimate $\hat{q}_k$ of the orientation quaternion by:

$$q_k = \hat{q}_k \left(q_{g\varepsilon,k}\right)^* \left(q_{m\varepsilon,k}\right)^* \tag{117}$$

Strictly speaking, the order of the corrections on the right hand side of equation (117) is significant since quaternion multiplication does not, in general, commute. In the limit of small orientation corrections, both $q_{g\varepsilon,k}$ and $q_{m\varepsilon,k}$ approximate the unit quaternion and the multiplication does commute.

The 3x1 vector $\boldsymbol{b}_{\varepsilon,k}$ (deg/s) models the error in the estimate of the zero rate gyroscope offset such that the true gyroscope offset $\boldsymbol{b}_k$ is related to the estimated gyroscope offset $\widehat{\boldsymbol{b}}_k$ by:

$$\boldsymbol{b}_k = \widehat{\boldsymbol{b}}_k - \boldsymbol{b}_{\varepsilon,k} \tag{118}$$

The scalar $\delta_{\varepsilon,k}$ (deg) models the error in the estimate of the geomagnetic inclination angle such that the true geomagnetic inclination angle δ_k is related to the estimated geomagnetic inclination angle $\hat{\delta}_k$ by:

$$\delta_k = \hat{\delta}_k - \delta_{\varepsilon,k} \tag{119}$$

7.4 A Priori Estimate of the State Vector

The *a priori* estimate $\hat{q}_k^-$ of the orientation quaternion at iteration k is computed by rotating the *a posteriori* orientation estimate by the incremental rotation vector $\widehat{\boldsymbol{\omega}}_k^- \delta t$ during the Kalman filter interval δt.

The rotation vector during the period δt equals $\widehat{\boldsymbol{\omega}}_k^- \delta t$ where the *a priori* angular velocity $\widehat{\boldsymbol{\omega}}_k^-$ is given by equation (96). With the notation that $\Delta q(\widehat{\boldsymbol{\omega}}_k^- \delta t)$ the incremental rotation quaternion modeling the rotation $\widehat{\boldsymbol{\omega}}_k^- \delta t$, the *a priori* rotation quaternion $\hat{q}_k^-$ is computed from the previous iteration's *a posteriori* orientation quaternion $\hat{q}_{k-1}^+$ as:

$$\hat{q}_k^- = \hat{q}_{k-1}^+ \Delta q(\widehat{\boldsymbol{\omega}}_k^- \delta t) \tag{120}$$

From equation (93) and (104), the *a priori* estimates of the gyroscope zero rate offset $\widehat{\boldsymbol{b}}_k^-$ and geomagnetic inclination angle $\hat{\delta}_k^-$ are given by:

$$\widehat{\boldsymbol{b}}_k^- = \widehat{\boldsymbol{b}}_{k-1}^+ \tag{121}$$

$$\hat{\delta}_k^- = \hat{\delta}_{k-1}^+ \tag{122}$$

7.5 A Posteriori Error Corrections

In the indirect sensor fusion Kalman filter, the error vector $\widehat{\boldsymbol{x}}_{\varepsilon,k-1}^-$ at iteration $k-1$ is explicitly incorporated into the *a posteriori* state vector $\widehat{\boldsymbol{x}}_{k-1}^+$ via the *a posteriori* error state vector $\widehat{\boldsymbol{x}}_{\varepsilon,k-1}^+$ at iteration $k-1$.

The *a posteriori* gravity vector tilt error $\hat{q}_{g\varepsilon,k}^+$ and geomagnetic vector tilt error $\hat{q}_{m\varepsilon,k}^+$ quaternions are applied as corrections to the *a priori* orientation quaternion $\hat{q}_k^-$ as:

$$\hat{q}_k^+ = \hat{q}_k^- \left(\hat{q}_{g\varepsilon,k}^+\right)^* \left(\hat{q}_{m\varepsilon,k}^+\right)^* \tag{123}$$

The gyroscope offset error component $\hat{\boldsymbol{b}}_{\varepsilon,k}^{+}$ of the error vector $\hat{\boldsymbol{x}}_{\varepsilon,k}^{+}$ is applied as a subtractive correction to the *a priori* gyroscope offset vector $\hat{\boldsymbol{b}}_{k}^{-}$ as:

$$\hat{\boldsymbol{b}}_{k}^{+} = \hat{\boldsymbol{b}}_{k}^{-} - \hat{\boldsymbol{b}}_{\varepsilon,k}^{+} \quad \textbf{(124)}$$

The *a posteriori* geomagnetic inclination angle $\hat{\delta}_{k}^{+}$ is calculated from the *a priori* estimate by subtracting the estimate error term $\hat{\delta}_{\varepsilon,k}^{+}$:

$$\hat{\delta}_{k}^{+} = \hat{\delta}_{k}^{-} - \hat{\delta}_{\varepsilon,k}^{+} \quad \textbf{(125)}$$

A consequence of applying the *a posteriori* error corrections $\hat{\boldsymbol{x}}_{\varepsilon,k}^{+}$ to the state vector $\hat{\boldsymbol{x}}_{k}^{+}$ is that the error vector $\hat{\boldsymbol{x}}_{\varepsilon,k-1}^{-}$ at iteration $k-1$ is uncorrelated with $\hat{\boldsymbol{x}}_{\varepsilon,k}^{-}$ at iteration k. Since equation (115) also applies to the *a priori* estimate $\hat{\boldsymbol{x}}_{\varepsilon,k}^{-}$ of the process error vector $\boldsymbol{x}_{\varepsilon,k}$:

$$\hat{\boldsymbol{x}}_{\varepsilon,k}^{-} = \begin{pmatrix} \hat{\boldsymbol{q}}_{g\varepsilon,k}^{-} \\ \hat{\boldsymbol{q}}_{m\varepsilon,k}^{-} \\ \hat{\boldsymbol{b}}_{\varepsilon,k}^{-} \\ \hat{\delta}_{\varepsilon,k}^{-} \end{pmatrix} = \boldsymbol{A}_k \hat{\boldsymbol{x}}_{\varepsilon,k-1}^{-} + \boldsymbol{w}_k \quad \textbf{(126)}$$

it follows that the 10x10 linear prediction matrix $\boldsymbol{A}_k$ is zero giving:

$$\boldsymbol{A}_k = \boldsymbol{0} \quad \textbf{(127)}$$

Since the noise component $\boldsymbol{w}_k$ is white and unpredictable, the *a priori* error vector $\hat{\boldsymbol{x}}_{\varepsilon,k}^{-}$ is zero:

$$\hat{\boldsymbol{x}}_{\varepsilon,k}^{-} = \begin{pmatrix} \hat{\boldsymbol{q}}_{g\varepsilon,k}^{-} \\ \hat{\boldsymbol{q}}_{m\varepsilon,k}^{-} \\ \hat{\boldsymbol{b}}_{\varepsilon,k}^{-} \\ \hat{\delta}_{\varepsilon,k}^{-} \end{pmatrix} = \boldsymbol{0} \quad \textbf{(128)}$$

The standard Kalman filter calculates the *a posteriori* estimate of the state vector $\hat{\boldsymbol{x}}_{k}^{+}$ from a weighted sum of measurement $\boldsymbol{z}_{\varepsilon,k}$ and a priori estimate $\hat{\boldsymbol{x}}_{k}^{-}$ of the state vector as:

$$\hat{\boldsymbol{x}}_{k}^{+} = \hat{\boldsymbol{x}}_{k}^{-} + \boldsymbol{K}_k(\boldsymbol{z}_k - \boldsymbol{C}_k \hat{\boldsymbol{x}}_{k}^{-}) \quad \textbf{(129)}$$

where $\boldsymbol{K}_k$ is the Kalman filter gain matrix.

The same expression applies in the indirect Kalman filter operating on the error process $\hat{\boldsymbol{x}}_{\varepsilon,k}^{-}$ and measurement error $\boldsymbol{z}_{\varepsilon,k}$:

$$\hat{\boldsymbol{x}}_{\varepsilon,k}^{+} = \hat{\boldsymbol{x}}_{\varepsilon,k}^{-} + \boldsymbol{K}_k\left(\boldsymbol{z}_{\varepsilon,k} - \boldsymbol{C}_k \hat{\boldsymbol{x}}_{\varepsilon,k}^{-}\right) \quad \textbf{(130)}$$

Substituting $\hat{\boldsymbol{x}}_{\varepsilon,k}^{-} = \boldsymbol{0}$ gives the expression for the *a posteriori* error vector estimate $\hat{\boldsymbol{x}}_{\varepsilon,k}^{+}$ as:

$$\hat{\boldsymbol{x}}_{\varepsilon,k}^{+} = \begin{pmatrix} \hat{\boldsymbol{q}}_{g\varepsilon,k}^{+} \\ \hat{\boldsymbol{q}}_{m\varepsilon,k}^{+} \\ \hat{\boldsymbol{b}}_{\varepsilon,k}^{+} \\ \hat{\delta}_{\varepsilon,k}^{+} \end{pmatrix} = \boldsymbol{K}_k \boldsymbol{z}_{\varepsilon,k} \tag{131}$$

7.6 Kalman Filter Measurement Error Model

The 7x1 measurement error vector $\boldsymbol{z}_{\varepsilon,k}$ is defined as:

$$\boldsymbol{z}_{\varepsilon,k} = \begin{pmatrix} \boldsymbol{q}_{zg\varepsilon,k} \\ \boldsymbol{q}_{zm\varepsilon,k} \\ \delta_{z\varepsilon,k} \end{pmatrix} \tag{132}$$

where the 3x1 vectors $\boldsymbol{q}_{zg\varepsilon,k}$ and $\boldsymbol{q}_{zm\varepsilon,k}$ are defined in section 4 as the vector components of the gravity and geomagnetic vector tilt error quaternions measured between the *a priori* and the 6DOF accelerometer plus magnetometer estimates of the gravity and geomagnetic vectors.

The scalar $\delta_{z\varepsilon,k}$ is defined in section 4 as the measured difference between the *a priori* Kalman filter estimate of the inclination angle and the instantaneous 6DOF estimate.

The signs of the angle errors are defined such that a Kalman filter estimate leading the 6DOF estimate results in a positive error.

In the conventional manner for Kalman filters, the measurement error vector $\boldsymbol{z}_{\varepsilon,k}$ is modeled as being related to the error process vector $\boldsymbol{x}_{\varepsilon,k}$ through the 7x10 measurement matrix $\boldsymbol{C}_k$ plus measurement noise $\boldsymbol{v}_k$:

$$\boldsymbol{z}_{\varepsilon,k} = \begin{pmatrix} \boldsymbol{q}_{zg\varepsilon,k} \\ \boldsymbol{q}_{zm\varepsilon,k} \\ \delta_{z\varepsilon,k} \end{pmatrix} = \boldsymbol{C}_k \boldsymbol{x}_{\varepsilon,k} + \boldsymbol{v}_k = \boldsymbol{C}_k \begin{pmatrix} \boldsymbol{q}_{g\varepsilon,k} \\ \boldsymbol{q}_{m\varepsilon,k} \\ \boldsymbol{b}_{\varepsilon,k} \\ \delta_{\varepsilon,k} \end{pmatrix} + \begin{pmatrix} \boldsymbol{v}_{qzg,k} \\ \boldsymbol{v}_{qzm,k} \\ v_{\delta,k} \end{pmatrix} \tag{133}$$

The measured gravity tilt error quaternion $q_{zg\varepsilon,k}$ is related to the true gravity tilt error quaternion $q_{g\varepsilon,k}$ by:

$$q_{zg\varepsilon,k} = q_{g\varepsilon,k} q\left(\hat{\boldsymbol{\omega}}_{\varepsilon,k}^{-} \delta t\right) q(\boldsymbol{v}_k) = q_{g\varepsilon,k} q\left(\left(-\hat{\boldsymbol{b}}_{\varepsilon,k-1}^{+} + \boldsymbol{w}_{b,k} + \boldsymbol{v}_{Y,k}\right)\delta t\right) q\left(\boldsymbol{v}_{qzg,k}\right) \tag{134}$$

$$= q_{g\varepsilon,k} q\left(-\hat{\boldsymbol{b}}_{\varepsilon,k-1}^{+} \delta t\right) q\left(\boldsymbol{w}_{b,k} \delta t\right) q\left(\boldsymbol{v}_{Y,k} \delta t\right) q\left(\boldsymbol{v}_{qzg,k}\right) \tag{135}$$

With the assumption that the angular errors and noise terms are small, the scalar component of the quaternions is near unity and equation (134) can be written as:

$$\{1, \boldsymbol{q}_{zg\varepsilon,k}\} = \{1, \boldsymbol{q}_{g\varepsilon,k}\}\{1, \boldsymbol{q}(-\hat{\boldsymbol{b}}_{\varepsilon,k-1}^{+} \delta t)\}\{1, \boldsymbol{q}(\boldsymbol{w}_{b,k} \delta t)\}\{1, \boldsymbol{q}(\boldsymbol{v}_{Y,k} \delta t)\}\{1, \boldsymbol{q}(\boldsymbol{v}_{qzg,k})\} \tag{136}$$

The vector components of equation (136) then satisfy, for small rotation angles:

$$\boldsymbol{q}_{zg\varepsilon,k} \approx \boldsymbol{q}_{g\varepsilon,k} + \left(\frac{1}{2}\right)\left(\frac{\pi\delta t}{180}\right)\left(-\widehat{\boldsymbol{b}}_{\varepsilon,k-1}^{+} + \boldsymbol{w}_{b,k} + \boldsymbol{v}_{Y,k}\right) + \boldsymbol{v}_{qzg,k} \tag{137}$$

The factor $\left(\frac{1}{2}\right)\left(\frac{\pi\delta t}{180}\right)$ converts from the native units of deg/s in the gyro offset error, random walk and noise vectors to the sine of half the subtended angle (equal to half the subtended angle in radians) used in the quaternion vector.

Similarly for the measured geomagnetic tilt error vector quaternion $\boldsymbol{q}_{zm\varepsilon,k}$:

$$\boldsymbol{q}_{zm\varepsilon,k} \approx \boldsymbol{q}_{m\varepsilon,k} + \left(\frac{1}{2}\right)\left(\frac{\pi\delta t}{180}\right)\left(-\widehat{\boldsymbol{b}}_{\varepsilon,k-1}^{+} + \boldsymbol{w}_{b,k} + \boldsymbol{v}_{Y,k}\right) + \boldsymbol{v}_{qmg,k} \tag{138}$$

The measured inclination angle error $\delta_{z\varepsilon,k}$ is purely a function of the accelerometer and magnetometer measurements with no gyro dependence giving:

$$\delta_{z\varepsilon,k} = \delta_{\varepsilon,k} + v_{\delta,k} \tag{139}$$

With the definition of the constant α as:

$$\alpha = \left(\frac{\pi\delta t}{180}\right) \tag{140}$$

the 7x10 measurement matrix $\boldsymbol{C}_k$ can then be written as:

$$\boldsymbol{C}_k = \begin{pmatrix} \boldsymbol{I}_3 & \boldsymbol{0}_3 & \left(\frac{-\alpha}{2}\right)\boldsymbol{I}_3 & 0 \\ \boldsymbol{0}_3 & \boldsymbol{I}_3 & \left(\frac{-\alpha}{2}\right)\boldsymbol{I}_3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \left(\frac{-\alpha}{2}\right) & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & \left(\frac{-\alpha}{2}\right) & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \left(\frac{-\alpha}{2}\right) & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & \left(\frac{-\alpha}{2}\right) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & \left(\frac{-\alpha}{2}\right) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \left(\frac{-\alpha}{2}\right) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{141}$$

7.7 Process Error Covariance Matrix

The 10x10 error process noise covariance matrix $\boldsymbol{Q}_{w,k}$ measures the error covariance in the *a priori* linear prediction of the error state vector $\boldsymbol{x}_{\varepsilon,k}$ from one iteration to the next in equation (126). It is defined as:

$$\boldsymbol{Q}_{w,k} = E[\boldsymbol{w}_k \boldsymbol{w}_k{}^T] = \begin{pmatrix} E\left[\widehat{\boldsymbol{q}}_{g\varepsilon,k}^{-}\left(\widehat{\boldsymbol{q}}_{g\varepsilon,k}^{-}\right)^T\right] & E\left[\widehat{\boldsymbol{q}}_{g\varepsilon,k}^{-}\left(\widehat{\boldsymbol{q}}_{m\varepsilon,k}^{-}\right)^T\right] & E\left[\widehat{\boldsymbol{q}}_{g\varepsilon,k}^{-}\left(\widehat{\boldsymbol{b}}_{\varepsilon,k}^{-}\right)^T\right] & E\left[\widehat{\boldsymbol{q}}_{g\varepsilon,k}^{-}\hat{\delta}_{\varepsilon,k}^{-}\right] \\ E\left[\widehat{\boldsymbol{q}}_{m\varepsilon,k}^{-}\left(\widehat{\boldsymbol{q}}_{g\varepsilon,k}^{-}\right)^T\right] & E\left[\widehat{\boldsymbol{q}}_{m\varepsilon,k}^{-}\left(\widehat{\boldsymbol{q}}_{m\varepsilon,k}^{-}\right)^T\right] & E\left[\widehat{\boldsymbol{q}}_{m\varepsilon,k}^{-}\left(\widehat{\boldsymbol{b}}_{\varepsilon,k}^{-}\right)^T\right] & E\left[\widehat{\boldsymbol{q}}_{m\varepsilon,k}^{-}\hat{\delta}_{\varepsilon,k}^{-}\right] \\ E\left[\widehat{\boldsymbol{b}}_{\varepsilon,k}^{-}\left(\widehat{\boldsymbol{q}}_{g\varepsilon,k}^{-}\right)^T\right] & E\left[\widehat{\boldsymbol{b}}_{\varepsilon,k}^{-}\left(\widehat{\boldsymbol{q}}_{m\varepsilon,k}^{-}\right)^T\right] & E\left[\widehat{\boldsymbol{b}}_{\varepsilon,k}^{-}\left(\widehat{\boldsymbol{b}}_{\varepsilon,k}^{-}\right)^T\right] & E\left[\widehat{\boldsymbol{b}}_{\varepsilon,k}^{-}\hat{\delta}_{\varepsilon,k}^{-}\right] \\ E\left[\hat{\delta}_{\varepsilon,k}^{-}\left(\widehat{\boldsymbol{q}}_{g\varepsilon,k}^{-}\right)^T\right] & E\left[\hat{\delta}_{\varepsilon,k}^{-}\left(\widehat{\boldsymbol{q}}_{m\varepsilon,k}^{-}\right)^T\right] & E\left[\hat{\delta}_{\varepsilon,k}^{-}\left(\widehat{\boldsymbol{b}}_{\varepsilon,k}^{-}\right)^T\right] & E\left[\hat{\delta}_{\varepsilon,k}^{-}\hat{\delta}_{\varepsilon,k}^{-}\right] \end{pmatrix} \quad \textbf{(142)}$$

The gravity tilt error covariance $E\left[\widehat{\boldsymbol{q}}_{g\varepsilon,k}^{-}\left(\widehat{\boldsymbol{q}}_{g\varepsilon,k}^{-}\right)^T\right]$ is the covariance of the *a priori* estimate of the tilt error between the 6DOF accelerometer plus magnetometer and the *a priori* gyroscope estimates of the gravity vector.

$$E\left[\widehat{\boldsymbol{q}}_{g\varepsilon,k}^{-}\left(\widehat{\boldsymbol{q}}_{g\varepsilon,k}^{-}\right)^T\right] = E\left[\widehat{\boldsymbol{q}}_{g\varepsilon,k-1}^{+} + \left(\frac{\alpha}{2}\right)\left(-\widehat{\boldsymbol{b}}_{\varepsilon,k-1}^{+} + \boldsymbol{w}_{b,k} + \boldsymbol{v}_{Y,k}\right)\left\{\widehat{\boldsymbol{q}}_{g\varepsilon,k-1}^{+} + \left(\frac{\alpha}{2}\right)\left(-\widehat{\boldsymbol{b}}_{\varepsilon,k-1}^{+} + \boldsymbol{w}_{b,k} + \boldsymbol{v}_{Y,k}\right)\right\}^T\right] \quad \textbf{(143)}$$

$$= \boldsymbol{Q}_{q_{g\varepsilon}q_{g\varepsilon},k-1}^{+} + \left(\frac{\alpha}{2}\right)^2\left(\boldsymbol{Q}_{b_\varepsilon b_\varepsilon,k-1}^{+} + Q_{vY}\boldsymbol{I}_3 + Q_{wb}\boldsymbol{I}_3\right) \quad \textbf{(144)}$$

where $\boldsymbol{Q}_{q_{g\varepsilon}q_{g\varepsilon},k-1}^{+}$ is approximated by the values at iteration $k-1$ ignoring off-diagonal terms:

$$\boldsymbol{Q}_{q_{g\varepsilon}q_{g\varepsilon},k-1}^{+} = E\left[\widehat{\boldsymbol{q}}_{g\varepsilon,k-1}^{+}\left(\widehat{\boldsymbol{q}}_{g\varepsilon,k-1}^{+}\right)^T\right] \approx \begin{pmatrix} \left(\hat{q}_{gx\varepsilon,k-1}^{+}\right)^2 & 0 & 0 \\ 0 & \left(\hat{q}_{gy\varepsilon,k-1}^{+}\right)^2 & 0 \\ 0 & 0 & \left(\hat{q}_{gz\varepsilon,k-1}^{+}\right)^2 \end{pmatrix} \quad \textbf{(145)}$$

Similarly, the geomagnetic tilt error covariance $E\left[\widehat{\boldsymbol{q}}_{m\varepsilon,k}^{-}\left(\widehat{\boldsymbol{q}}_{m\varepsilon,k}^{-}\right)^T\right]$ can be written as:

$$E\left[\widehat{\boldsymbol{q}}_{m\varepsilon,k}^{-}\left(\widehat{\boldsymbol{q}}_{m\varepsilon,k}^{-}\right)^T\right] = E\left[\widehat{\boldsymbol{q}}_{m\varepsilon,k-1}^{+} + \left(\frac{\alpha}{2}\right)\left(-\widehat{\boldsymbol{b}}_{\varepsilon,k-1}^{+} + \boldsymbol{w}_{b,k} + \boldsymbol{v}_{Y,k}\right)\left\{\widehat{\boldsymbol{q}}_{m\varepsilon,k-1}^{+} + \left(\frac{\alpha}{2}\right)\left(-\widehat{\boldsymbol{b}}_{\varepsilon,k-1}^{+} + \boldsymbol{w}_{b,k} + \boldsymbol{v}_{Y,k}\right)\right\}^T\right] \quad \textbf{(146)}$$

$$= \boldsymbol{Q}_{q_{m\varepsilon}q_{m\varepsilon},k-1}^{+} + \left(\frac{\alpha}{2}\right)^2\left(\boldsymbol{Q}_{b_\varepsilon b_\varepsilon,k-1}^{+} + Q_{vY}\boldsymbol{I}_3 + Q_{wb}\boldsymbol{I}_3\right) \quad \textbf{(147)}$$

where $\boldsymbol{Q}_{q_{m\varepsilon}q_{m\varepsilon},k-1}^{+}$ is approximated by the value of iteration $k-1$ ignoring off-diagonal terms:

$$\boldsymbol{Q}^{+}_{q_{m\varepsilon}q_{m\varepsilon},k-1} = E\left[\widehat{\boldsymbol{q}}^{+}_{m\varepsilon,k-1}\left(\widehat{\boldsymbol{q}}^{+}_{m\varepsilon,k-1}\right)^{T}\right] \approx \begin{pmatrix} \left(\hat{q}^{+}_{mx\varepsilon,k-1}\right)^{2} & 0 & 0 \\ 0 & \left(\hat{q}^{+}_{my\varepsilon,k-1}\right)^{2} & 0 \\ 0 & 0 & \left(\hat{q}^{+}_{mz\varepsilon,k-1}\right)^{2} \end{pmatrix} \tag{148}$$

The gravity tilt and geomagnetic tilt errors are assumed uncorrelated and the covariance $E\left[\widehat{\boldsymbol{q}}^{-}_{g\varepsilon,k}\left(\widehat{\boldsymbol{q}}^{-}_{m\varepsilon,k}\right)^{T}\right]$ is set to zero.

The covariance $E\left[\widehat{\boldsymbol{q}}^{-}_{g\varepsilon,k}\left(\widehat{\boldsymbol{b}}^{-}_{\varepsilon,k}\right)^{T}\right]$ evaluates to:

$$E\left[\widehat{\boldsymbol{q}}^{-}_{g\varepsilon,k}\left(\widehat{\boldsymbol{b}}^{-}_{\varepsilon,k}\right)^{T}\right] = E\left[\left\{\widehat{\boldsymbol{q}}^{+}_{g\varepsilon,k-1} + \left(\frac{\alpha}{2}\right)\left(-\widehat{\boldsymbol{b}}^{+}_{\varepsilon,k-1} + \boldsymbol{w}_{b,k} + \boldsymbol{v}_{Y,k}\right)\right\}\left(\widehat{\boldsymbol{b}}^{+}_{\varepsilon,k-1} - \boldsymbol{w}_{b,k}\right)^{T}\right] \tag{149}$$

$$= \boldsymbol{Q}^{+}_{g_{\varepsilon}b_{\varepsilon},k-1} - \left(\frac{\alpha}{2}\right)Q_{wb}\boldsymbol{I}_{3} \tag{150}$$

where $\boldsymbol{Q}^{+}_{g_{\varepsilon}b_{\varepsilon},k-1}$ is approximated with the values at iteration $k-1$ ignoring off-diagonal terms:

$$\boldsymbol{Q}^{+}_{g_{\varepsilon}b_{\varepsilon},k-1} = E\left[\widehat{\boldsymbol{q}}^{+}_{g\varepsilon,k-1}\left(\widehat{\boldsymbol{b}}^{+}_{\varepsilon,k-1}\right)^{T}\right] \approx \begin{pmatrix} \hat{q}^{+}_{gx\varepsilon,k-1}\hat{b}^{+}_{x\varepsilon,k-1} & 0 & 0 \\ 0 & \hat{q}^{+}_{gy\varepsilon,k-1}\hat{b}^{+}_{y\varepsilon,k-1} & 0 \\ 0 & 0 & \hat{q}^{+}_{gz\varepsilon,k-1}\hat{b}^{+}_{z\varepsilon,k-1} \end{pmatrix} \tag{151}$$

The covariance $E\left[\widehat{\boldsymbol{q}}^{-}_{m\varepsilon,k}\left(\widehat{\boldsymbol{b}}^{-}_{\varepsilon,k}\right)^{T}\right]$ evaluates to:

$$E\left[\widehat{\boldsymbol{q}}^{-}_{m\varepsilon,k}\left(\widehat{\boldsymbol{b}}^{-}_{\varepsilon,k}\right)^{T}\right] = E\left[\left\{\widehat{\boldsymbol{q}}^{+}_{m\varepsilon,k-1} + \left(\frac{\alpha}{2}\right)\left(-\widehat{\boldsymbol{b}}^{+}_{\varepsilon,k-1} + \boldsymbol{w}_{b,k} + \boldsymbol{v}_{Y,k}\right)\right\}\left(\widehat{\boldsymbol{b}}^{+}_{\varepsilon,k-1} - \boldsymbol{w}_{b,k}\right)^{T}\right] \tag{152}$$

$$= \boldsymbol{Q}^{+}_{m_{\varepsilon}b_{\varepsilon},k-1} - \left(\frac{\alpha}{2}\right)Q_{wb}\boldsymbol{I}_{3} \tag{153}$$

where $\boldsymbol{Q}^{+}_{m_{\varepsilon}b_{\varepsilon},k-1}$ is approximated with the values at iteration $k-1$ ignoring off-diagonal terms:

$$\boldsymbol{Q}^{+}_{m_{\varepsilon}b_{\varepsilon},k-1} = E\left[\widehat{\boldsymbol{q}}^{+}_{m\varepsilon,k-1}\left(\widehat{\boldsymbol{b}}^{+}_{\varepsilon,k-1}\right)^{T}\right] \approx \begin{pmatrix} \hat{q}^{+}_{mx\varepsilon,k-1}\hat{b}^{+}_{x\varepsilon,k-1} & 0 & 0 \\ 0 & \hat{q}^{+}_{my\varepsilon,k-1}\hat{b}^{+}_{y\varepsilon,k-1} & 0 \\ 0 & 0 & \hat{q}^{+}_{mz\varepsilon,k-1}\hat{b}^{+}_{z\varepsilon,k-1} \end{pmatrix} \tag{154}$$

The covariance $E\left[\widehat{\boldsymbol{b}}^{-}_{\varepsilon,k}\left(\widehat{\boldsymbol{b}}^{-}_{\varepsilon,k}\right)^{T}\right]$ evaluates to:

$$E\left[\hat{\boldsymbol{b}}_{\varepsilon,k}^{-}\left(\hat{\boldsymbol{b}}_{\varepsilon,k}^{-}\right)^{T}\right] = E\left[\left(\hat{\boldsymbol{b}}_{\varepsilon,k-1}^{+} - \boldsymbol{w}_{b,k}\right)\left(\hat{\boldsymbol{b}}_{\varepsilon,k-1}^{+} - \boldsymbol{w}_{b,k}\right)^{T}\right] = \boldsymbol{Q}_{b_\varepsilon b_\varepsilon,k-1}^{+} + Q_{wb}\boldsymbol{I}_3 \tag{155}$$

where $\boldsymbol{Q}_{b_\varepsilon b_\varepsilon,k-1}^{+}$ is approximated with the values at iteration $k-1$ ignoring off-diagonal terms

$$\boldsymbol{Q}_{b_\varepsilon b_\varepsilon,k-1}^{+} = E\left[\hat{\boldsymbol{b}}_{\varepsilon,k-1}^{+}\left(\hat{\boldsymbol{b}}_{\varepsilon,k-1}^{+}\right)^{T}\right] \approx \begin{pmatrix} \left(\hat{b}_{x\varepsilon,k-1}^{+}\right)^2 & 0 & 0 \\ 0 & \left(\hat{b}_{y\varepsilon,k-1}^{+}\right)^2 & 0 \\ 0 & 0 & \left(\hat{b}_{z\varepsilon,k-1}^{+}\right)^2 \end{pmatrix} \tag{156}$$

The covariance $E\left[\hat{\delta}_{\varepsilon,k}^{-}\hat{\delta}_{\varepsilon,k}^{-}\right]$ evaluates to:

$$E\left[\hat{\delta}_{\varepsilon,k}^{-}\hat{\delta}_{\varepsilon,k}^{-}\right] = E\left[\left(\hat{\delta}_{\varepsilon,k-1}^{+} - w_{\delta,k}\right)\left(\hat{\delta}_{\varepsilon,k-1}^{+} - w_{\delta,k}\right)\right] = Q_{\delta_\varepsilon\delta_\varepsilon,k-1}^{+} + Q_{w\delta} \tag{157}$$

where:

$$Q_{\delta_\varepsilon\delta_\varepsilon,k-1}^{+} = E\left[\left(\hat{\delta}_{\varepsilon,k-1}^{+}\right)^2\right] \approx \left(\hat{\delta}_{\varepsilon,k-1}^{+}\right)^2 \tag{158}$$

The covariance $E\left[\hat{\boldsymbol{b}}_{\varepsilon,k}^{-}\hat{\delta}_{\varepsilon,k}^{-\,T}\right]$ evaluates to:

$$E\left[\hat{\boldsymbol{b}}_{\varepsilon,k}^{-}\hat{\delta}_{\varepsilon,k}^{-\,T}\right] = E\left[\left(\hat{\boldsymbol{b}}_{\varepsilon,k-1}^{+} - \boldsymbol{w}_{b,k}\right)\left(\hat{\delta}_{\varepsilon,k-1}^{+} - w_{\delta,k}\right)^{T}\right] = \boldsymbol{0} \tag{159}$$

7.8 Measurement Error Covariance Matrix

The measurement noise vector $\boldsymbol{v}_k$ has size 7x1 with covariance matrix defined as:

$$\boldsymbol{Q}_{v,k} = E\left[\boldsymbol{v}_k\boldsymbol{v}_k^{T}\right] = E\left[\begin{pmatrix} \boldsymbol{v}_{qzg,k} \\ \boldsymbol{v}_{qzm,k} \\ v_{\delta,k} \end{pmatrix}\begin{pmatrix} \boldsymbol{v}_{qzg,k} \\ \boldsymbol{v}_{qzm,k} \\ v_{\delta,k} \end{pmatrix}^{T}\right] \tag{160}$$

$$= \begin{pmatrix} E\left[\boldsymbol{v}_{qzg,k}\left(\boldsymbol{v}_{qzg,k}\right)^{T}\right] & E\left[\boldsymbol{v}_{qzg,k}\left(\boldsymbol{v}_{qzm,k}\right)^{T}\right] & E\left[\boldsymbol{v}_{qzg,k}v_{\delta,k}\right] \\ E\left[\boldsymbol{v}_{qzm,k}\left(\boldsymbol{v}_{qzg,k}\right)^{T}\right] & E\left[\boldsymbol{v}_{qzm,k}\left(\boldsymbol{v}_{qzm,k}\right)^{T}\right] & E\left[\boldsymbol{v}_{qzm,k}v_{\delta,k}\right] \\ E\left[v_{\delta,k}\left(\boldsymbol{v}_{qzg,k}\right)^{T}\right] & E\left[v_{\delta,k}\left(\boldsymbol{v}_{qzm,k}\right)^{T}\right] & E\left[\left(v_{\delta,k}\right)^2\right] \end{pmatrix} \tag{161}$$

The measurement quaternion vector $\boldsymbol{q}_{zg\varepsilon,k}$ is proportional to the sine of *half* the rotation angle between the *a priori* and 6DOF measurements of the gravity vector. Its noise term $\boldsymbol{v}_{qzg,k}$ therefore includes i) the accelerometer sensor noise plus acceleration noise and ii) the gyroscope sensor and zero rate offset noise. It is unaffected by magnetometer noise and magnetic disturbance.

Similarly, the measurement quaternion vector $\boldsymbol{q}_{zm\varepsilon,k}$ is the sine of *half* the rotation angle between the *a priori* and 6DOF measurements of the geomagnetic vector and its noise term $\boldsymbol{v}_{qzm,k}$ therefore includes i) the magnetometer sensor noise plus magnetic disturbance noise and ii) the gyroscope sensor and zero rate offset noise. It is unaffected by accelerometer noise and acceleration.

The noise in the measurement $\delta_{z\varepsilon,k}$ of the geomagnetic inclination angle has no dependence on the gyroscope sensor but its noise term $v_{\delta,k}$ is the sum of i) the accelerometer sensor noise plus acceleration noise and ii) the magnetometer sensor noise plus magnetic disturbance noise.

With a small angle approximation and, remembering that the native units of the vector quaternion are radians and the native units of the gyroscope are deg/s, then the terms in $\boldsymbol{Q}_{v,k}$ are:

$$E\left[\boldsymbol{v}_{qzg,k}\left(\boldsymbol{v}_{qzg,k}\right)^T\right] = \left\{\left(\frac{1}{4}\right)\left(Q_{vG,k} + Q_{a,k}\right) + \left(\frac{\alpha^2}{4}\right)\left(Q_{vY,k} + Q_{wb,k}\right)\right\}\boldsymbol{I}_3 \quad \textbf{(162)}$$

$$E\left[\boldsymbol{v}_{qzm,k}\left(\boldsymbol{v}_{qzm,k}\right)^T\right] = \left\{\left(\frac{1}{4B^2}\right)\left(Q_{vB,k} + Q_{d,k}\right) + \left(\frac{\alpha^2}{4}\right)\left(Q_{vY,k} + Q_{wb,k}\right)\right\}\boldsymbol{I}_3 \quad \textbf{(163)}$$

$$E\left[\left(v_{\delta,k}\right)^2\right] = \left(\frac{180}{\pi}\right)^2\left(\left(Q_{vG,k} + Q_{a,k}\right) + \left(\frac{Q_{vB,k} + Q_{d,k}}{B^2}\right)\right) \quad \textbf{(164)}$$

The remaining terms are assumed to be uncorrelated:

$$E\left[\boldsymbol{v}_{qzg,k}\left(\boldsymbol{v}_{qzg,k}\right)^T\right] = E\left[\boldsymbol{v}_{qzg,k}v_{\delta,k}\right] = E\left[\boldsymbol{v}_{mzg,k}v_{\delta,k}\right] = \boldsymbol{0} \quad \textbf{(165)}$$

7.9 A Priori Covariance Matrix Update

The Kalman filter updates the *a priori* covariance matrix $\boldsymbol{P}_k^-$ as:

$$\boldsymbol{P}_k^- = \boldsymbol{A}_k\boldsymbol{P}_{k-1}^+\boldsymbol{A}_k{}^T + \boldsymbol{Q}_{w,k} \quad \textbf{(166)}$$

Since the linear prediction matrix $\boldsymbol{A}_k$ is zero, $\boldsymbol{P}_k^-$ need not be updated and simply equals the process error covariance $\boldsymbol{Q}_{w,k}$:

$$\boldsymbol{P}_k^- = \boldsymbol{Q}_{w,k} \quad \textbf{(167)}$$

7.10 A Posteriori Covariance Matrix Update

The Kalman filter updates the *a posteriori* covariance matrix $\boldsymbol{P}_k^+$ as:

$$\boldsymbol{P}_k^+ = (\boldsymbol{I} - \boldsymbol{K}_k \boldsymbol{C}_k)\boldsymbol{P}_k^- \quad \textbf{(168)}$$

The only purpose of $\boldsymbol{P}_k^+$ is to allow the calculation of $\boldsymbol{P}_k^-$ in equation (166) but since $\boldsymbol{P}_k^+$ is multiplied by $\boldsymbol{A}_k = \boldsymbol{0}$, there is no need to calculate $\boldsymbol{P}_k^+$. The *a priori* and *a posteriori* covariance matrices $\boldsymbol{P}_k^-$ and $\boldsymbol{P}_k^+$ are therefore both eliminated from this form of Kalman filter.

7.11 Kalman Filter Gain Matrix Update

The Kalman filter updates the 10x7 Kalman filter gain matrix $\boldsymbol{K}_k$ as:

$$\boldsymbol{K}_k = \boldsymbol{P}_k^- \boldsymbol{C}_k{}^T \left(\boldsymbol{C}_k \boldsymbol{P}_k^- \boldsymbol{C}_k{}^T + \boldsymbol{Q}_{v,k}\right)^{-1} \quad \textbf{(169)}$$

Since $\boldsymbol{P}_k^- = \boldsymbol{Q}_{w,k}$ this simplifies to:

$$\boldsymbol{K}_k = \boldsymbol{Q}_{w,k} \boldsymbol{C}_k{}^T \left(\boldsymbol{C}_k \boldsymbol{Q}_{w,k} \boldsymbol{C}_k{}^T + \boldsymbol{Q}_{v,k}\right)^{-1} \quad \textbf{(170)}$$

7.12 Compile Time Constants

These compile time constants are implemented with #define in file *fusion.h.*

The constants `FQVY_9DOF_GBY_KALMAN`, `FQWB_9DOF_GBY_KALMAN` and `FQWDLT_9DOF_GBY_KALMAN` define covariances Q_{vY}, Q_{wb} and $Q_{w\delta}$ respectively.

The constants `FGYRO_OFFSET_MIN_9DOF_GBY_KALMAN` and `FGYRO_OFFSET_MIN_9DOF_GBY_KALMAN` limit the permissible range of the gyroscope zero rate offset $\boldsymbol{b}_k$. The default range is -5 deg/s to +5 deg/s.

8 Accelerometer and Gyroscope Sensor Fusion Kalman Filter

8.1 Introduction

This section derives the Kalman filter equations for the sensor fusion of accelerometer and gyroscope data. It is also commonly referred to as a 6 degree of freedom or 6DOF sensor fusion model since each of the two sensors has three axes and provides three degrees of freedom. This Kalman filter is a simplified version of the 9DOF filter described in section 7.

8.2 Kalman Filter Process Model

The Kalman filter process state vector $\boldsymbol{x}_k$ comprises i) the orientation quaternion q_k ii) the zero rate gyroscope offset $\boldsymbol{b}_k$ and the inclination angle δ_k at iteration k:

$$\boldsymbol{x}_k = \begin{pmatrix} q_k \\ \boldsymbol{b}_k \end{pmatrix} \quad \textbf{(171)}$$

8.3 Kalman Filter Error Process Model

The Kalman filter model of the error process $\boldsymbol{x}_{\varepsilon,k}$ is:

$$\boldsymbol{x}_{\varepsilon,k} = \boldsymbol{A}_k \boldsymbol{x}_{\varepsilon,k-1} + \boldsymbol{w}_k \quad \textbf{(172)}$$

where $\boldsymbol{x}_{\varepsilon,k}$ is the 6x1 vector with components:

$$\boldsymbol{x}_{\varepsilon,k} = \begin{pmatrix} \boldsymbol{q}_{g\varepsilon,k} \\ \boldsymbol{b}_{\varepsilon,k} \end{pmatrix} \quad \textbf{(173)}$$

and where $\boldsymbol{w}_k$ is a white noise component.

The 3x1 vector $\boldsymbol{q}_{g\varepsilon,k}$ is the vector component of the quaternion $q_{g\varepsilon,k}$ which models the error in tilt angle relative to the gravity vector. The 3x1 vector $\boldsymbol{b}_{\varepsilon,k}$ (deg/s) models the error in the estimate of the zero rate gyroscope offset.

The state vector orientation quaternion q_k is then related to the estimate $\hat{q}_k$ of the orientation quaternion by:

$$q_k = \hat{q}_k \left(q_{g\varepsilon,k}\right)^* \quad \textbf{(174)}$$

8.4 A Priori Estimate of the State Vector

The *a priori* rotation quaternion $\hat{q}_k^-$ is computed from the previous iteration's *a posteriori* orientation quaternion $\hat{q}_{k-1}^+$ as:

$$\hat{q}_k^- = \hat{q}_{k-1}^+ \Delta q(\widehat{\boldsymbol{\omega}}_k^- \delta t) \quad \textbf{(175)}$$

The *a priori* estimate of the gyroscope zero rate offset $\hat{\boldsymbol{b}}_k^-$ is:

$$\hat{\boldsymbol{b}}_k^- = \hat{\boldsymbol{b}}_{k-1}^+ \tag{176}$$

8.5 A Posteriori Error Corrections

The *a posteriori* gravity vector tilt error $\hat{q}_{g\varepsilon,k}^+$ quaternion is applied as a correction to the *a priori* orientation quaternion $\hat{q}_k^-$ as:

$$\hat{q}_k^+ = \hat{q}_k^- \left(\hat{q}_{g\varepsilon,k}^+\right)^* \tag{177}$$

The gyroscope offset error component $\hat{\boldsymbol{b}}_{\varepsilon,k}^+$ of the error vector $\hat{\boldsymbol{x}}_{\varepsilon,k}^+$ is applied as a subtractive correction to the *a priori* gyroscope offset vector $\hat{\boldsymbol{b}}_k^-$ as:

$$\hat{\boldsymbol{b}}_k^+ = \hat{\boldsymbol{b}}_k^- - \hat{\boldsymbol{b}}_{\varepsilon,k}^+ \tag{178}$$

Since the a posteriori error corrections are applied to the state vector $\hat{\boldsymbol{x}}_k^+$, the linear prediction matrix $\boldsymbol{A}_k = \boldsymbol{0}$.

The *a posteriori* error vector is estimated from the measurement error $\boldsymbol{z}_{\varepsilon,k}$ as:

$$\hat{\boldsymbol{x}}_{\varepsilon,k}^+ = \begin{pmatrix} \hat{\boldsymbol{q}}_{g\varepsilon,k}^+ \\ \hat{\boldsymbol{b}}_{\varepsilon,k}^+ \end{pmatrix} = \boldsymbol{K}_k \boldsymbol{z}_{\varepsilon,k} \tag{179}$$

8.6 Kalman Filter Measurement Error Model

The 3x1 measurement error vector $\boldsymbol{z}_{\varepsilon,k}$ is defined in terms of the gravity tilt error quaternion discussed in section 4 as:

$$\boldsymbol{z}_{\varepsilon,k} = \boldsymbol{q}_{zg\varepsilon,k} \tag{180}$$

The measurement error vector $\boldsymbol{z}_{\varepsilon,k}$ is modeled as being related to the error process vector $\boldsymbol{x}_{\varepsilon,k}$ through the 3x6 measurement matrix $\boldsymbol{C}_k$ plus measurement noise $\boldsymbol{v}_k$:

$$\boldsymbol{z}_{\varepsilon,k} = \boldsymbol{q}_{zg\varepsilon,k} = \boldsymbol{C}_k \boldsymbol{x}_{\varepsilon,k} + \boldsymbol{v}_k = \boldsymbol{C}_k \begin{pmatrix} \boldsymbol{q}_{g\varepsilon,k} \\ \boldsymbol{b}_{\varepsilon,k} \end{pmatrix} + \boldsymbol{v}_{qzg,k} \tag{181}$$

where:

$$\boldsymbol{C}_k = \begin{pmatrix} \boldsymbol{I}_3 & \left(\frac{-\alpha}{2}\right) \boldsymbol{I}_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \left(\frac{-\alpha}{2}\right) & 0 & 0 \\ 0 & 1 & 0 & 0 & \left(\frac{-\alpha}{2}\right) & 0 \\ 0 & 0 & 1 & 0 & 0 & \left(\frac{-\alpha}{2}\right) \end{pmatrix} \tag{182}$$

8.7 Process Error Covariance Matrix

The 6x6 error process noise covariance matrix $\boldsymbol{Q}_{w,k}$ is defined as:

$$\boldsymbol{Q}_{w,k} = E[\boldsymbol{w}_k \boldsymbol{w}_k{}^T] = \begin{pmatrix} E\left[\widehat{\boldsymbol{q}}_{g\varepsilon,k}^{-}\left(\widehat{\boldsymbol{q}}_{g\varepsilon,k}^{-}\right)^T\right] & E\left[\widehat{\boldsymbol{q}}_{g\varepsilon,k}^{-}\left(\widehat{\boldsymbol{b}}_{\varepsilon,k}^{-}\right)^T\right] \\ E\left[\widehat{\boldsymbol{b}}_{\varepsilon,k}^{-}\left(\widehat{\boldsymbol{q}}_{g\varepsilon,k}^{-}\right)^T\right] & E\left[\widehat{\boldsymbol{b}}_{\varepsilon,k}^{-}\left(\widehat{\boldsymbol{b}}_{\varepsilon,k}^{-}\right)^T\right] \end{pmatrix} \tag{183}$$

where:

$$E\left[\widehat{\boldsymbol{q}}_{g\varepsilon,k}^{-}\left(\widehat{\boldsymbol{q}}_{g\varepsilon,k}^{-}\right)^T\right] = \boldsymbol{Q}_{q_{g\varepsilon}q_{g\varepsilon},k-1}^{+} + \left(\frac{\alpha}{2}\right)^2 \left(\boldsymbol{Q}_{b_\varepsilon b_\varepsilon,k-1}^{+} + Q_{vY}\boldsymbol{I}_3 + Q_{wb}\boldsymbol{I}_3\right) \tag{184}$$

$$E\left[\widehat{\boldsymbol{q}}_{g\varepsilon,k}^{-}\left(\widehat{\boldsymbol{b}}_{\varepsilon,k}^{-}\right)^T\right] = \boldsymbol{Q}_{g_\varepsilon b_\varepsilon,k-1}^{+} - \left(\frac{\alpha}{2}\right) Q_{wb}\boldsymbol{I}_3 \tag{185}$$

$$E\left[\widehat{\boldsymbol{b}}_{\varepsilon,k}^{-}\left(\widehat{\boldsymbol{b}}_{\varepsilon,k}^{-}\right)^T\right] = E\left[\left(\widehat{\boldsymbol{b}}_{\varepsilon,k-1}^{+} - \boldsymbol{w}_{b,k}\right)\left(\widehat{\boldsymbol{b}}_{\varepsilon,k-1}^{+} - \boldsymbol{w}_{b,k}\right)^T\right] = \boldsymbol{Q}_{b_\varepsilon b_\varepsilon,k-1}^{+} + Q_{wb}\boldsymbol{I}_3 \tag{186}$$

8.8 Measurement Error Covariance Matrix

The measurement noise vector $\boldsymbol{v}_k$ has size 3x1 with covariance matrix defined as:

$$\boldsymbol{Q}_{v,k} = E[\boldsymbol{v}_k \boldsymbol{v}_k{}^T] = E\left[\boldsymbol{v}_{qzg,k}\left(\boldsymbol{v}_{qzg,k}\right)^T\right] \tag{187}$$

where:

$$E\left[\boldsymbol{v}_{qzg,k}\left(\boldsymbol{v}_{qzg,k}\right)^T\right] = \left\{\left(\frac{1}{4}\right)\left(Q_{vG,k} + Q_{a,k}\right) + \left(\frac{\alpha^2}{4}\right)\left(Q_{vY,k} + Q_{wb,k}\right)\right\}\boldsymbol{I}_3 \tag{188}$$

8.9 A Priori Covariance Matrix Update

The Kalman filter updates the *a priori* covariance matrix $\boldsymbol{P}_k^-$ as:

$$\boldsymbol{P}_k^- = \boldsymbol{A}_k \boldsymbol{P}_{k-1}^+ \boldsymbol{A}_k{}^T + \boldsymbol{Q}_{w,k} \tag{189}$$

Since the linear prediction matrix $\boldsymbol{A}_k$ is zero, $\boldsymbol{P}_k^-$ need not be updated and simply equals the process error covariance $\boldsymbol{Q}_{w,k}$:

$$\boldsymbol{P}_k^- = \boldsymbol{Q}_{w,k} \tag{190}$$

8.10 A Posteriori Covariance Matrix Update

The Kalman filter updates the *a posteriori* covariance matrix $\boldsymbol{P}_k^+$ as:

$$\boldsymbol{P}_k^+ = (\boldsymbol{I} - \boldsymbol{K}_k\boldsymbol{C}_k)\boldsymbol{P}_k^- \quad \textbf{(191)}$$

The only purpose of $\boldsymbol{P}_k^+$ is to allow the calculation of $\boldsymbol{P}_k^-$ in equation (189) but since $\boldsymbol{P}_k^+$ it is multiplied by $\boldsymbol{A}_k = \boldsymbol{0}$, there is no need to calculate $\boldsymbol{P}_k^+$.

8.11 Kalman Filter Gain Matrix Update

The Kalman filter updates the 6x3 Kalman filter gain matrix $\boldsymbol{K}_k$ as:

$$\boldsymbol{K}_k = \boldsymbol{Q}_{w,k}\boldsymbol{C}_k{}^T\left(\boldsymbol{C}_k\boldsymbol{Q}_{w,k}\boldsymbol{C}_k{}^T + \boldsymbol{Q}_{v,k}\right)^{-1} \quad \textbf{(192)}$$

8.12 Compile Time Constants

The compile time constants below are implemented with #define in file *fusion.h.*

The constants `FQVY_6DOF_GY_KALMAN` and `FQWB_6DOF_GY_KALMAN` define the covariances Q_{vY} and Q_{wb}.

9 References

1. Freescale Application Note (AN5018) *Basic Kalman Filter Theory*

10 Revision History

Table 2. Revision history

Rev. No.	Date	Description
1	9/2015	Initial release

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