Freescale Semiconductor
APPLICATION NOTE

Document Number: AN5019
Rev. 1, 9/2015

Magnetic Calibration Algorithms

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Contents

1 Introduction

1.1 Summary

This application note documents the mathematics underlying the functions in Table 1 located in the file *magnetic.c.* These functions determine the calibration of the magnetometer sensor and the hard and soft-iron magnetic interference from the circuit board.

1.2 Software Functions

Table 1. Sensor Fusion Library software functions

Function	Description	Section
`void fInvertMagCal (struct MagSensor *pthisMag, struct MagCalibration *pthisMagCal)`	Computes calibrated magnetometer measurements by applying the calibration coefficients to uncalibrated measurements.	3.1
`void fUpdateCalibration4INV (struct MagCalibration *pthisMagCal, struct MagneticBuffer *pthisMagBuffer, struct MagSensor *pthisMag)`	Determines the coefficients of the 4 element calibration model.	4
`void fUpdateCalibration7EIG(struct MagCalibration *pthisMagCal, struct MagneticBuffer *pthisMagBuffer, struct MagSensor *pthisMag);`	Determines the coefficients of the 7 element calibration model.	5
`void fUpdateCalibration10EIG(struct MagCalibration *pthisMagCal, struct MagneticBuffer *pthisMagBuffer, struct MagSensor *pthisMag);`	Determines the coefficients of the 10 element calibration model.	6

2 Least Squares Optimization

2.1 Linear Measurement Model

The general linear model relating independent variables $X_j[i]$ to dependent variable $Y[i]$ at measurement i via fitted model parameters β_j is:

$$Y[i] = \beta_0 X_0[i] + \beta_1 X_1[i] + \cdots + \beta_{N-1} X_{N-1}[i] \tag{1}$$

The fit to the model will not be perfectly accurate and will result in an error residual $r[i]$ defined as:

$$r[i] = Y[i] - \beta_0 X_0[i] - \beta_1 X_1[i] - \cdots - \beta_{N-1} X_{N-1}[i] \tag{2}$$

For a series of M measurements, equation (2) can be written in the form:

$$\begin{pmatrix} r[0] \\ r[1] \\ \dots \\ r[M-1] \end{pmatrix} = \begin{pmatrix} Y[0] \\ Y[1] \\ \dots \\ Y[M-1] \end{pmatrix} - \begin{pmatrix} X_0[0] & X_1[0] & \dots & X_{N-1}[0] \\ X_0[1] & X_1[1] & \dots & X_{N-1}[1] \\ \dots & \dots & \dots & \dots \\ X_0[M-1] & X_1[M-1] & \dots & X_{N-1}[M-1] \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \\ \dots \\ \beta_{N-1} \end{pmatrix} \quad \textbf{(3)}$$

With the definitions that $\boldsymbol{r}$ is the column vector of residuals:

$$\boldsymbol{r} = \begin{pmatrix} r[0] \\ r[1] \\ \dots \\ r[M-1] \end{pmatrix} \quad \textbf{(4)}$$

$\boldsymbol{Y}$ is the $M \times 1$ column vector of M measurements on the dependent variable:

$$\boldsymbol{Y} = \begin{pmatrix} Y[0] \\ Y[1] \\ \dots \\ Y[M-1] \end{pmatrix} \quad \textbf{(5)}$$

$\boldsymbol{X}$ is the $M \times N$ matrix of M measurements of the independent variable:

$$\boldsymbol{X} = \begin{pmatrix} X_0[0] & X_1[0] & \dots & X_{N-1}[0] \\ X_0[1] & X_1[1] & \dots & X_{N-1}[1] \\ \dots & \dots & \dots & \dots \\ X_0[M-1] & X_1[M-1] & \dots & X_{N-1}[M-1] \end{pmatrix} \quad \textbf{(6)}$$

$\boldsymbol{\beta}$ is the $N \times 1$ column vector of unknown model coefficients β_0 to β_{N-1} to be determined:

$$\boldsymbol{\beta} = \begin{pmatrix} \beta_0 \\ \beta_1 \\ \dots \\ \beta_{N-1} \end{pmatrix} \quad \textbf{(7)}$$

then equation (3) can be written in the form:

$$\boldsymbol{r} = \boldsymbol{Y} - \boldsymbol{X\beta} \quad \textbf{(8)}$$

If there are more measurements M than there are unknowns N, then the equations are typically solved in a least squares sense by minimizing the error function E defined as the modulus squared of the error vector $\boldsymbol{r}$ defined in equation (5):

$$E = \sum_{i=0}^{M-1} r[i]^2 = \boldsymbol{r}^T\boldsymbol{r} = \|\boldsymbol{r}\|^2 = (\boldsymbol{Y} - \boldsymbol{X\beta})^T(\boldsymbol{Y} - \boldsymbol{X\beta}) = \|\boldsymbol{Y} - \boldsymbol{X\beta}\|^2 \quad \textbf{(9)}$$

The error function E is proportional to the number of measurements M and has dimensions of the square of the elements $r[]$.

A useful normalized error measure ε is:

$$\varepsilon = \frac{1}{2}\sqrt{\frac{E}{M}} \tag{10}$$

2.2 Normal Equations Solution in Non-homogeneous Case

If the measurement vector on the dependent vector $\boldsymbol{Y}$ is not zero, then the equations are termed non-homogeneous. The error function E will be a minimum when it is stationary with respect to any perturbation $\delta\boldsymbol{\beta}$ about the optimal least squares solution $\boldsymbol{\beta}$:

$$lim_{\delta\boldsymbol{\beta}\to\boldsymbol{0}}\{E(\boldsymbol{\beta}+\delta\boldsymbol{\beta})\} - E(\boldsymbol{\beta}) = 0 \; for \; all \; \delta\boldsymbol{\beta} \tag{11}$$

Substituting equation (8) into (11) and ignoring second order terms gives:

$$\left(\boldsymbol{Y} - \boldsymbol{X}(\boldsymbol{\beta}+\delta\boldsymbol{\beta})\right)^T\left(\boldsymbol{Y} - \boldsymbol{X}(\boldsymbol{\beta}+\delta\boldsymbol{\beta})\right) - (\boldsymbol{Y}-\boldsymbol{X\beta})^T(\boldsymbol{Y}-\boldsymbol{X\beta}) = 0 \tag{12}$$

$$\Rightarrow -\boldsymbol{Y}^T\boldsymbol{X}\delta\boldsymbol{\beta} + (\boldsymbol{X\beta})^T\boldsymbol{X}\delta\boldsymbol{\beta} - (\boldsymbol{X}\delta\boldsymbol{\beta})^T\,(\boldsymbol{Y}-\boldsymbol{X\beta}) = 0 \tag{13}$$

$$\Rightarrow -\boldsymbol{Y}^T\boldsymbol{X}\delta\boldsymbol{\beta} + (\boldsymbol{X\beta})^T\boldsymbol{X}\delta\boldsymbol{\beta} - \delta\boldsymbol{\beta}^T\boldsymbol{X}^T\boldsymbol{Y} + \delta\boldsymbol{\beta}^T\boldsymbol{X}^T\boldsymbol{X\beta} \; = 0 \tag{14}$$

Because $\delta\boldsymbol{\beta}^T\boldsymbol{X}^T\boldsymbol{Y}$ and $\delta\boldsymbol{\beta}^T\boldsymbol{X}^T\boldsymbol{X\beta}$ are scalars their values are unchanged by the transpose operation and equation (14) can be rewritten as:

$$(-\boldsymbol{Y}^T\boldsymbol{X} + \boldsymbol{\beta}^T\boldsymbol{X}^T\boldsymbol{X})\delta\boldsymbol{\beta} = 0 \; for \; all \; \delta\boldsymbol{\beta} \tag{15}$$

$$\Rightarrow \boldsymbol{\beta}^T\boldsymbol{X}^T\boldsymbol{X} = \boldsymbol{Y}^T\boldsymbol{X} \Rightarrow \boldsymbol{X}^T\boldsymbol{X\beta} = \boldsymbol{X}^T\boldsymbol{Y} \tag{16}$$

$$\Rightarrow \boldsymbol{\beta} = (\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T\boldsymbol{Y} \tag{17}$$

Equation (17) is termed the Normal Equations solution for $\boldsymbol{\beta}$ in the nonhomogeneous case.

The error function E at the optimum solution equals:

$$E = \boldsymbol{r}^T\boldsymbol{r} = \{\boldsymbol{Y}-\boldsymbol{X\beta}\}^T\{\boldsymbol{Y}-\boldsymbol{X\beta}\} = (\boldsymbol{Y}^T - \boldsymbol{\beta}^T\boldsymbol{X}^T)(\boldsymbol{Y}-\boldsymbol{X\beta}) = \boldsymbol{Y}^T\boldsymbol{Y} - \boldsymbol{Y}^T\boldsymbol{X\beta} - \boldsymbol{\beta}^T\boldsymbol{X}^T\boldsymbol{Y} + \boldsymbol{\beta}^T\boldsymbol{X}^T\boldsymbol{X\beta} \tag{18}$$

$$= \boldsymbol{Y}^T\boldsymbol{Y} - \boldsymbol{\beta}^T\boldsymbol{X}^T\boldsymbol{Y} - (\boldsymbol{\beta}^T\boldsymbol{X}^T\boldsymbol{Y})^T + \boldsymbol{\beta}^T\boldsymbol{X}^T\boldsymbol{X\beta} \tag{19}$$

Because each term of equation (19) is a scalar and equal to its transpose. the error function can be written as:

$$E = \boldsymbol{Y}^T\boldsymbol{Y} - 2\boldsymbol{\beta}^T(\boldsymbol{X}^T\boldsymbol{Y}) + \boldsymbol{\beta}^T(\boldsymbol{X}^T\boldsymbol{X})\boldsymbol{\beta} \qquad \textbf{(20)}$$

Substituting the expression for the solution vector $\boldsymbol{\beta}$ into equation (20) gives:

$$E = \boldsymbol{Y}^T\boldsymbol{Y} - 2((\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T\boldsymbol{Y})^T(\boldsymbol{X}^T\boldsymbol{Y}) + ((\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T\boldsymbol{Y})^T(\boldsymbol{X}^T\boldsymbol{X})(\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T\boldsymbol{Y} \qquad \textbf{(21)}$$

$$= \boldsymbol{Y}^T\boldsymbol{Y} - \boldsymbol{Y}^T\boldsymbol{X}\{(\boldsymbol{X}^T\boldsymbol{X})^{-1}\}^T\boldsymbol{X}^T\boldsymbol{Y} \qquad \textbf{(22)}$$

For the special case where the number of measurements equals the number of model parameters to be fitted, the matrix $\boldsymbol{X}$ is square and the transpose and inversion operators commute:

$$(\boldsymbol{X}^T\boldsymbol{X})^{-1} = \boldsymbol{X}^{-1}(\boldsymbol{X}^T)^{-1} \qquad \textbf{(23)}$$

In this special case:

$$E = \boldsymbol{Y}^T\boldsymbol{Y} - \boldsymbol{Y}^T\boldsymbol{X}\{\boldsymbol{X}^{-1}(\boldsymbol{X}^T)^{-1}\}^T\boldsymbol{X}^T\boldsymbol{Y} = \boldsymbol{Y}^T\boldsymbol{Y} - \boldsymbol{Y}^T\boldsymbol{X}\boldsymbol{X}^{-1}(\boldsymbol{X}^{-1})^T\boldsymbol{X}^T\boldsymbol{Y} = \boldsymbol{Y}^T\boldsymbol{Y} - \boldsymbol{Y}^T(\boldsymbol{X}^T)^{-1}\boldsymbol{X}^T\boldsymbol{Y} = 0 \qquad \textbf{(24)}$$

Equation (24) states the expected result that the error function is zero and the fit is perfect when the number of model parameters to be fitted equals the number of measurements.

2.3 Eigenvector Solution in Homogeneous Case

If the dependent measurement vector $\boldsymbol{Y}$ is zero, then the equations are termed homogeneous. The model being fitted in a least squares sense is now:

$$\boldsymbol{X}\boldsymbol{\beta} = 0 \qquad \textbf{(25)}$$

The error function E to be minimized simplifies to:

$$E = \|\boldsymbol{r}\|^2 = \|\boldsymbol{X}\boldsymbol{\beta}\|^2 = (\boldsymbol{X}\boldsymbol{\beta})^T\boldsymbol{X}\boldsymbol{\beta} = \boldsymbol{\beta}^T\boldsymbol{X}^T\boldsymbol{X}\boldsymbol{\beta} \qquad \textbf{(26)}$$

Unfortunately, using the Normal Equations solution given by equation (17) for the non-homogeneous case gives the zero vector solution for $\boldsymbol{\beta}$ when $\boldsymbol{Y}$ is the null vector.

$$\boldsymbol{\beta} = (\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T\boldsymbol{Y} = 0 \qquad \textbf{(27)}$$

This is a valid solution but not terribly useful. A solution method is required that minimizes the error function E in equation (26) subject to the constraint that $\boldsymbol{\beta}$ has nonzero magnitude. Because equation (25) is linear, the solution vector $\boldsymbol{\beta}$ can be scaled to have unit magnitude:

$$1 - \boldsymbol{\beta}^T\boldsymbol{\beta} = 0 \qquad \textbf{(28)}$$

The error function is unaffected by adding multiples of equation (28) and can be rewritten as:

$$E(\boldsymbol{\beta}) = (\boldsymbol{X\beta})^T \boldsymbol{X\beta} + \lambda(1 - \boldsymbol{\beta}^T \boldsymbol{\beta}) = \boldsymbol{\beta}^T \boldsymbol{X}^T \boldsymbol{X\beta} + \lambda(1 - \boldsymbol{\beta}^T \boldsymbol{\beta}) \tag{29}$$

This is equivalent to method of Lagrange Multipliers for constrained optimization. Applying the stationary constraint that $E(\boldsymbol{\beta} + \delta\boldsymbol{\beta}) = E(\boldsymbol{\beta})$ to equation (29) gives:

$$(\boldsymbol{\beta} + \delta\boldsymbol{\beta})^T \boldsymbol{X}^T \boldsymbol{X}(\boldsymbol{\beta} + \delta\boldsymbol{\beta}) + \lambda\left(1 - (\boldsymbol{\beta} + \delta\boldsymbol{\beta})^T(\boldsymbol{\beta} + \delta\boldsymbol{\beta})\right) = \boldsymbol{\beta}^T \boldsymbol{X}^T \boldsymbol{X\beta} + \lambda(1 - \boldsymbol{\beta}^T \boldsymbol{\beta}) \tag{30}$$

Ignoring second order terms gives:

$$\delta\boldsymbol{\beta}^T \boldsymbol{X}^T \boldsymbol{X\beta} + \boldsymbol{\beta}^T \boldsymbol{X}^T \boldsymbol{X}\delta\boldsymbol{\beta} - \lambda(\boldsymbol{\beta}^T \delta\boldsymbol{\beta} + \delta\boldsymbol{\beta}^T \boldsymbol{\beta}) = 0 \tag{31}$$

Because each term in equation (31) is a scalar and equal to its transpose, the solution for the optimum $\boldsymbol{\beta}$ which constrains the performance function is:

$$\delta\boldsymbol{\beta}^T(\boldsymbol{X}^T \boldsymbol{X\beta} - \lambda\boldsymbol{\beta}) = 0 \tag{32}$$

$$\Rightarrow \boldsymbol{X}^T \boldsymbol{X\beta} = \lambda\boldsymbol{\beta} \tag{33}$$

Equation (33) states that the required solution vector $\boldsymbol{\beta}$ is an eigenvector of the product matrix $\boldsymbol{X}^T\boldsymbol{X}$ associated with eigenvalue λ.

Substituting equation (33) into equation (26) for the error function E_i associated with eigenvalue λ_i and eigenvector $\boldsymbol{\beta_i}$ gives:

$$E_i = \boldsymbol{\beta}_i{}^T \boldsymbol{X}^T \boldsymbol{X\beta}_i = \lambda_i \boldsymbol{\beta}_i{}^T \boldsymbol{\beta}_i = \lambda_i \tag{34}$$

The error function E_i for the i^{th} eigenvector solution equals the eigenvalue λ_i of $\boldsymbol{X}^T\boldsymbol{X}$ and the minimum error function is equal to the smallest eigenvalue λ_{min}. The required solution $\boldsymbol{\beta}$ is then the eigenvector associated with the smallest eigenvalue λ_{min}.

2.4 Eigenvectors and Eigenvalues of Symmetric Matrices

The measurement matrix $\boldsymbol{X}^T\boldsymbol{X}$ in equation (33) is symmetric because:

$$(\boldsymbol{X}^T\boldsymbol{X})^T = \boldsymbol{X}^T\boldsymbol{X} \tag{35}$$

The eigenvectors of equation (35) satisfy:

$$\left(\boldsymbol{\beta}_j\right)^T \boldsymbol{X}^T \boldsymbol{X}\, \boldsymbol{\beta}_k = \lambda_k \left(\boldsymbol{\beta}_j\right)^T \boldsymbol{\beta}_k \tag{36}$$

Transposing equation (36) gives:

$$\left\{\left(\boldsymbol{\beta}_j\right)^T \boldsymbol{X}^T \boldsymbol{X}\, \boldsymbol{\beta}_k\right\}^T = \left\{\lambda_k \left(\boldsymbol{\beta}_j\right)^T \boldsymbol{\beta}_k\right\}^T \tag{37}$$

$$\Rightarrow (\boldsymbol{\beta}_k)^T \boldsymbol{X}^T \boldsymbol{X\beta}_j = \lambda_k (\boldsymbol{\beta}_k)^T \boldsymbol{\beta}_j \tag{38}$$

$$\Rightarrow \lambda_j (\boldsymbol{\beta}_k)^T \boldsymbol{\beta}_j = \lambda_k (\boldsymbol{\beta}_k)^T \boldsymbol{\beta}_j \quad \textbf{(39)}$$

$$\Rightarrow (\lambda_j - \lambda_k)(\boldsymbol{\beta}_k)^T \boldsymbol{\beta}_j = 0 \quad \textbf{(40)}$$

$$\Rightarrow (\boldsymbol{\beta}_k)^T \boldsymbol{\beta}_j = 0 \; if \; \lambda_j \neq \lambda_k \quad \textbf{(41)}$$

The eigenvectors of a symmetric matrix are therefore orthogonal if the eigenvalues are distinct.

The definition of a positive semi-definite matrix $\boldsymbol{A}$ is that it satisfies for all nonzero vectors $\boldsymbol{v}_j$:

$$(\boldsymbol{v}_j)^T \boldsymbol{A} \boldsymbol{v}_j \geq 0 \quad \textbf{(42)}$$

Setting $j = k$ in equation (36) gives:

$$(\boldsymbol{\beta}_j)^T \boldsymbol{X}^T \boldsymbol{X} \, \boldsymbol{\beta}_j = \lambda_j (\boldsymbol{\beta}_j)^T \boldsymbol{\beta}_j \quad \textbf{(43)}$$

$$\Rightarrow |\boldsymbol{X} \, \boldsymbol{\beta}_j|^2 = \lambda_j |\boldsymbol{\beta}_j|^2 \quad \textbf{(44)}$$

The left-hand side of equation (44) is non negative. For nonzero $\boldsymbol{\beta}_j$ it therefore follows that symmetric matrices are positive semi-definite and have non-negative eigenvalues if the associated eigenvector is non-zero.

3 Hard and Soft-Iron Magnetic Model

3.1 General Linear Model

Ignoring hard and soft iron magnetic distortion, the magnetic field at the location of the magnetometer sensor is the local geomagnetic field $\boldsymbol{B}_r$ rotated by the orientation matrix $\boldsymbol{R}$ describing the orientation of the magnetometer. The geomagnetic vector $\boldsymbol{B}_r$ is a fixed vector in the global reference frame (pointing northwards and downwards in the northern hemisphere) and the multiplication by the circuit board orientation matrix $\boldsymbol{R}$ is an example of a vector transformation from the global coordinate frame to the sensor coordinate frame:

The most general linear model for the magnetometer measurement $\boldsymbol{B}_s$ in the sensor as a consequence of hard and soft-iron distortion to the true applied field $\boldsymbol{B}_c$ is:

$$\boldsymbol{B}_s = \boldsymbol{W} \boldsymbol{B}_c + \boldsymbol{V} = \boldsymbol{W} \boldsymbol{R} \boldsymbol{B}_r + \boldsymbol{V} \quad \textbf{(45)}$$

where $\boldsymbol{V}$ is a 3×1 vector and $\boldsymbol{W}$ is a 3×3 matrix.

The vector $\boldsymbol{V}$ is termed the *hard-iron offset* and the matrix $\boldsymbol{W}$ is termed the *soft-iron matrix*. The hard-iron offset models the sensor's intrinsic zero field offset plus the effects of permanently magnetized components on the circuit board. The soft-iron matrix models the directional effect of induced magnetic fields and differing sensitivities in the three axes of the magnetometer sensor.

The calibration algorithms derived in this document estimate the hard-iron offset $\boldsymbol{V}$ and the soft-iron matrix $\boldsymbol{W}$ from magnetometer measurements stored in the magnetometer buffer and then invert equation (45) to give the calibrated magnetometer measurement in the sensor frame $\boldsymbol{B}_c$ as:

$$\boldsymbol{B}_c = \boldsymbol{W}^{-1}(\boldsymbol{B}_s - \boldsymbol{V}) \tag{46}$$

Substituting equation (45) shows that the calibrated magnetometer measurement $\boldsymbol{B}_c$ then equals the geomagnetic vector rotated from the global into the sensor frame:

$$\boldsymbol{B}_c = \boldsymbol{R}\boldsymbol{B}_r \tag{47}$$

Equation (45) is a normally excellent model for the magnetometer measurements but will become less accurate when the linearity assumption starts to break down. The most common reason for deviations from equation (45) is the presence of magnetic hysteresis which is, by definition, a nonlinear path-dependent magnetic distortion.

3.2 Measurement Locus

Under arbitrary rotation of the device, the locus of the calibrated magnetometer reading $\boldsymbol{B}_c = \boldsymbol{R}\boldsymbol{B}_r$ lies on the surface of a sphere with radius equal to B the local geomagnetic field strength:

$$(\boldsymbol{B}_c)^T\boldsymbol{B}_c = (\boldsymbol{R}\boldsymbol{B}_r)^T\boldsymbol{R}\boldsymbol{B}_r = \boldsymbol{B}_r{}^T\boldsymbol{R}^T\boldsymbol{R}\boldsymbol{B}_r = B^2 \tag{47}$$

In the presence of hard and soft-iron effects, the locus of uncalibrated magnetometer reading $\boldsymbol{B}_s$ lies on the surface defined using equation (46) to be:

$$\{\boldsymbol{W}^{-1}(\boldsymbol{B}_s - \boldsymbol{V})\}^T\{\boldsymbol{W}^{-1}(\boldsymbol{B}_s - \boldsymbol{V})\} = (\boldsymbol{R}\boldsymbol{B}_r)^T\boldsymbol{R}\boldsymbol{B}_r = \boldsymbol{B}_r{}^T\boldsymbol{R}^T\boldsymbol{R}\boldsymbol{B}_r = B^2 \tag{48}$$

$$\Rightarrow (\boldsymbol{B}_s - \boldsymbol{V})^T(\boldsymbol{W}^{-1})^T\boldsymbol{W}^{-1}(\boldsymbol{B}_s - \boldsymbol{V}) = B^2 \tag{49}$$

The general expression for the locus of a vector $\boldsymbol{u}$ lying on the surface of an ellipsoid with center at $\boldsymbol{u}_0$ is:

$$(\boldsymbol{u} - \boldsymbol{u}_0)^T\boldsymbol{A}(\boldsymbol{u} - \boldsymbol{u}_0) = const \tag{50}$$

where $\boldsymbol{A}$ is a symmetric matrix defining the shape of the ellipsoid.

Equations (49) and (50) are of the same form because it can be easily proven that the matrix $\boldsymbol{A} = \{\boldsymbol{W}^{-1}\}^T\boldsymbol{W}^{-1}$ is a symmetric matrix:

$$\boldsymbol{A}^T = \{\{\boldsymbol{W}^{-1}\}^T\boldsymbol{W}^{-1}\}^T = \{\boldsymbol{W}^{-1}\}^T\{\{\boldsymbol{W}^{-1}\}^T\}^T = \{\boldsymbol{W}^{-1}\}^T\boldsymbol{W}^{-1} = \boldsymbol{A} \tag{51}$$

The hard and soft-iron distortions, therefore, force the geomagnetic vector, as measured in the magnetometer reference frame, to lie on the surface on an ellipsoid centered at the hard-iron offset $\boldsymbol{V}$ with its shape determined by the transposed product of the inverse soft-iron matrix with itself $\{\boldsymbol{W}^{-1}\}^T\boldsymbol{W}^{-1}$. The precise point on the ellipsoid surface where any measurement falls is determined by the orientation of the magnetometer.

Once the hard-iron offset $\boldsymbol{V}$ and the soft-iron matrix $\boldsymbol{W}$ have been determined by the calibration algorithms, the mapping of the raw distorted magnetometer measurement $\boldsymbol{B}_s$ onto the calibrated measurement $\boldsymbol{B}_c$ using equation (46) equates to mapping measurements from the surface of the ellipsoid to the surface of a sphere with radius B.

3.3 Example Calibration Surfaces

Figure 1 shows measurements taken from a simple sensor demonstration board with uncalibrated measurements in red and calibrated measurements in blue. The soft-iron matrix $\boldsymbol{W}$ is close to the identity matrix and the hard-iron vector is dominated by a 100 µT offset in the z-axis. This type of PCB could be calibrated for hard-iron offset only using the algorithm described in Section 4. The more sophisticated soft-iron algorithms of Sections 5 and 6 could also be used with little, if any, additional performance improvement.

In this particular example, the calibration mapping is approximately the translation:

$$\boldsymbol{B}_c \approx \boldsymbol{B}_s - V_z\widehat{\boldsymbol{k}} \approx \boldsymbol{B}_p - 100\mu T\widehat{\boldsymbol{k}} \quad \textbf{(52)}$$

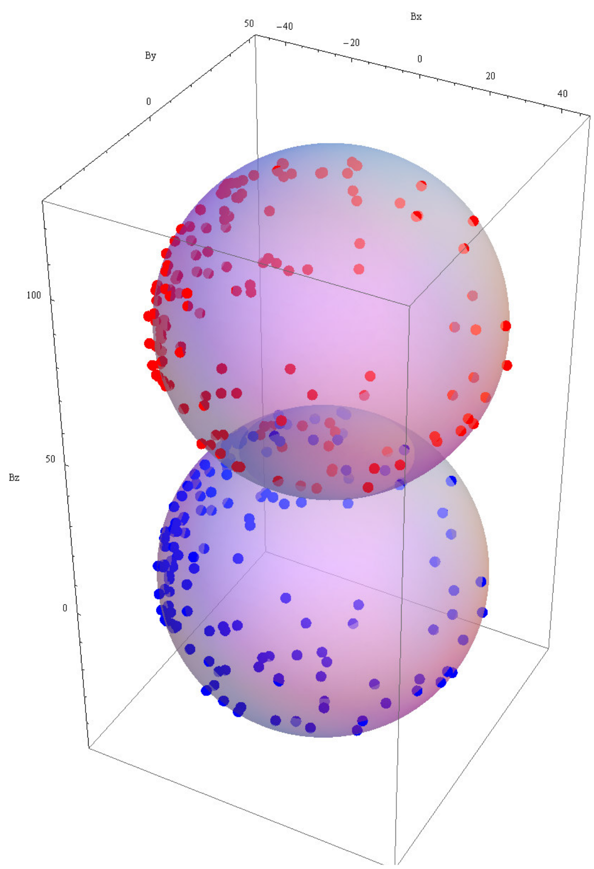

Figure 1. Uncalibrated (red) and calibrated (blue) measurements from a simple hard-iron environment

Figure 2 shows measurements taken from an Android tablet with strong hard and soft-iron distortions resulting from ferromagnetic components on the PCB. This type of distortion must be calibrated using the more sophisticated hard and soft-iron algorithms described in Sections 5 and 6.

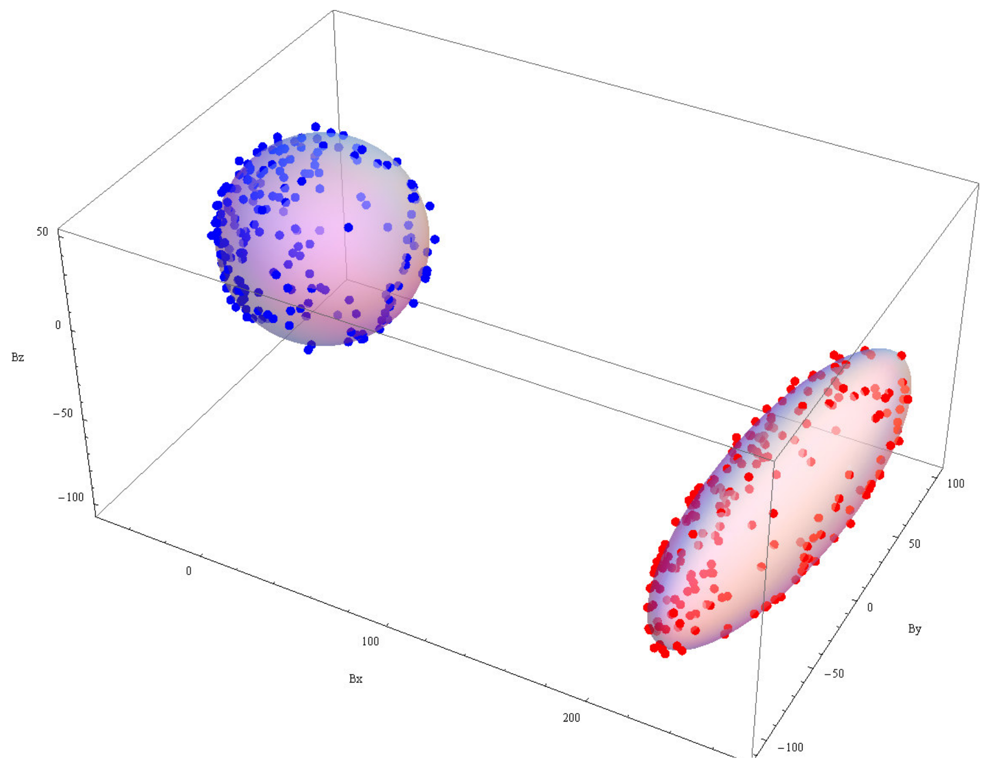

Figure 2. Uncalibrated (red) and calibrated (blue) measurements from a simple hard-iron environment

4 Four-Parameter Magnetic Calibration Model

4.1 Construction of Four-Parameter Linear Model

This section documents the simplest magnetic calibration algorithm implemented in function `fUpdateCalibration4INV` which calculates the four parameters comprising the hard-iron offset vector $\boldsymbol{V}$ and geomagnetic field strength B. The soft-iron matrix $\boldsymbol{W}$ is assumed to be the identity matrix $\boldsymbol{I}$. This model provides reasonable performance with high simplicity on simple circuit boards which tend to have limited soft-iron distortion.

$$\boldsymbol{B}_s = \boldsymbol{B}_c + \boldsymbol{V} = \boldsymbol{R}\boldsymbol{B}_r + \boldsymbol{V} \tag{54}$$

Equation (45) for the case of a PCB subject to hard-iron interference only simplifies to:

$$(\boldsymbol{B}_s - \boldsymbol{V})^T(\boldsymbol{B}_s - \boldsymbol{V}) = B^2 \Rightarrow \boldsymbol{B}_s{}^T\boldsymbol{B}_s - 2\boldsymbol{B}_s{}^T\boldsymbol{V} + \boldsymbol{V}^T\boldsymbol{V} - B^2 = 0 \tag{53}$$

Equation (53) models the locus of the magnetometer measurements $\boldsymbol{B}_s$ as lying on the surface of a sphere with radius B offset from the origin by $\boldsymbol{V}$.

The residual error $r[i]$ for the i^{th} observation is:

$$r[i] = B_{sx}[i]^2 + B_{sy}[i]^2 + B_{sz}[i]^2 - 2B_{sx}[i]V_x - 2B_{sy}[i]V_y - 2B_{sz}[i]V_z + V_x{}^2 + V_y{}^2 + V_z{}^2 - B^2 \tag{54}$$

Simplifying and returning to matrix format gives:

$$r[i] = \left(B_{sx}[i]^2 + B_{sy}[i]^2 + B_{sz}[i]^2\right) - \begin{pmatrix} B_{sx}[i] \\ B_{sy}[i] \\ B_{sz}[i] \\ 1 \end{pmatrix}^T \begin{pmatrix} 2V_x \\ 2V_y \\ 2V_z \\ B^2 - {V_x}^2 - {V_y}^2 - {V_z}^2 \end{pmatrix} \quad \textbf{(55)}$$

The dependent measurement variable $y[i]$ can be defined to be:

$$y[i] = B_{sx}[i]^2 + B_{sy}[i]^2 + B_{sz}[i]^2 \quad \textbf{(56)}$$

and the solution vector $\boldsymbol{\beta}$ as:

$$\boldsymbol{\beta} = \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} = \begin{pmatrix} 2V_x \\ 2V_y \\ 2V_z \\ B^2 - {V_x}^2 - {V_y}^2 - {V_z}^2 \end{pmatrix} \quad \textbf{(57)}$$

The error residual $r[i]$ is now given by:

$$r[i] = B_{sx}[i]^2 + B_{sy}[i]^2 + B_{sz}[i]^2 - \begin{pmatrix} B_{sx}[i] \\ B_{sy}[i] \\ B_{sz}[i] \\ 1 \end{pmatrix}^T \begin{pmatrix} 2V_x \\ 2V_y \\ 2V_z \\ B^2 - {V_x}^2 - {V_y}^2 - {V_z}^2 \end{pmatrix} \quad \textbf{(58)}$$

$$= B_{sx}[i]^2 + B_{sy}[i]^2 + B_{sz}[i]^2 - \begin{pmatrix} B_{sx}[i] \\ B_{sy}[i] \\ B_{sz}[i] \\ 1 \end{pmatrix}^T \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} \quad \textbf{(59)}$$

Equation (59) can be expanded to represent M measurements as:

$$\begin{pmatrix} r[0] \\ r[1] \\ \dots \\ r[M-1] \end{pmatrix} = \begin{pmatrix} B_{sx}[0]^2 + B_{sy}[0]^2 + B_{sz}[0]^2 \\ B_{sx}[1]^2 + B_{sy}[1]^2 + B_{sz}[1]^2 \\ \dots \\ B_{sx}[M-1]^2 + B_{sy}[M-1]^2 + B_{sz}[M-1]^2 \end{pmatrix} - \begin{pmatrix} B_{sx}[0] & B_{sy}[0] & B_{sz}[0] & 1 \\ B_{sx}[1] & B_{sy}[1] & B_{sz}[1] & 1 \\ \dots & \dots & \dots & 1 \\ B_{sx}[M-1] & B_{sy}[M-1] & B_{sz}[M-1] & 1 \end{pmatrix} \begin{pmatrix} 2V_x \\ 2V_y \\ 2V_z \\ B^2 - {V_x}^2 - {V_y}^2 - {V_z}^2 \end{pmatrix} \quad \textbf{(60)}$$

With the definitions of the error residual vector $\boldsymbol{r}$ as:

$$\boldsymbol{r} = \begin{pmatrix} r[0] \\ r[1] \\ \dots \\ r[M-1] \end{pmatrix} \quad \textbf{(61)}$$

and $\boldsymbol{Y}$ the vector of dependent variables:

$$\boldsymbol{Y} = \begin{pmatrix} B_{sx}[0]^2 + B_{sy}[0]^2 + B_{sz}[0]^2 \\ B_{sx}[1]^2 + B_{sy}[1]^2 + B_{sz}[1]^2 \\ \dots \\ B_{sx}[M-1]^2 + B_{sy}[M-1]^2 + B_{sz}[M-1]^2 \end{pmatrix} \quad \textbf{(62)}$$

and $\boldsymbol{X}$ the $M \times 4$ measurement matrix:

$$\boldsymbol{X} = \begin{pmatrix} B_{sx}[0] & B_{sy}[0] & B_{sz}[0] & 1 \\ B_{sx}[1] & B_{sy}[1] & B_{sz}[1] & 1 \\ \dots & \dots & \dots & 1 \\ B_{sx}[M-1] & B_{sy}[M-1] & B_{sz}[M-1] & 1 \end{pmatrix} \quad \textbf{(63)}$$

then equation (60) can be written as:

$$\boldsymbol{r} = \boldsymbol{Y} - \boldsymbol{X\beta} \quad \textbf{(64)}$$

The model being fitted has the nonhomogeneous form $\boldsymbol{r} = \boldsymbol{Y} - \boldsymbol{X\beta}$ and can be solved using the Normal Equations method documented in Section 2.

The matrices $\boldsymbol{X}^T\boldsymbol{X}$, $\boldsymbol{X}^T\boldsymbol{Y}$ and $\boldsymbol{Y}^T\boldsymbol{Y}$ have values:

$$\boldsymbol{X}^T\boldsymbol{X} = \begin{pmatrix} B_{sx}[0] & B_{sy}[0] & B_{sz}[0] & 1 \\ B_{sx}[1] & B_{sy}[1] & B_{sz}[1] & 1 \\ \dots & \dots & \dots & 1 \\ B_{sx}[M-1] & B_{sy}[M-1] & B_{sz}[M-1] & 1 \end{pmatrix}^T \begin{pmatrix} B_{sx}[0] & B_{sy}[0] & B_{sz}[0] & 1 \\ B_{sx}[1] & B_{sy}[1] & B_{sz}[1] & 1 \\ \dots & \dots & \dots & 1 \\ B_{sx}[M-1] & B_{sy}[M-1] & B_{sz}[M-1] & 1 \end{pmatrix} \quad \textbf{(65)}$$

$$= \sum_{i=0}^{M-1} \begin{pmatrix} B_{sx}[i]^2 & B_{sx}[i]B_{sy}[i] & B_{sx}[i]B_{sz}[i] & B_{sx}[i] \\ B_{sx}[i]B_{sy}[i] & B_{sy}[i]^2 & B_{sy}[i]B_{sz}[i] & B_{sy}[i] \\ B_{sx}[i]B_{sz}[i] & B_{sy}[i]B_{sz}[i] & B_{sz}[i]^2 & B_{sz}[i] \\ B_{sx}[i] & B_{sy}[i] & B_{sz}[i] & 1 \end{pmatrix} \quad \textbf{(66)}$$

$$\boldsymbol{X}^T\boldsymbol{Y} = \begin{pmatrix} B_{sx}[0] & B_{sy}[0] & B_{sz}[0] & 1 \\ B_{sx}[1] & B_{sy}[1] & B_{sz}[1] & 1 \\ \dots & \dots & \dots & 1 \\ B_{sx}[M-1] & B_{sy}[M-1] & B_{sz}[M-1] & 1 \end{pmatrix}^T \begin{pmatrix} B_{sx}[0]^2 + B_{sy}[0]^2 + B_{sz}[0]^2 \\ B_{sx}[1]^2 + B_{sy}[1]^2 + B_{sz}[1]^2 \\ \dots \\ B_{sx}[M-1]^2 + B_{sy}[M-1]^2 + B_{sz}[M-1]^2 \end{pmatrix} \quad \textbf{(67)}$$

$$= \sum_{i=0}^{M-1} \begin{pmatrix} B_{sx}[i]\left(B_{sx}[i]^2 + B_{sy}[i]^2 + B_{sz}[i]^2\right) \\ B_{sy}[i]\left(B_{sx}[i]^2 + B_{sy}[i]^2 + B_{sz}[i]^2\right) \\ B_{sz}[i]\left(B_{sx}[i]^2 + B_{sy}[i]^2 + B_{sz}[i]^2\right) \\ B_{sx}[i]^2 + B_{sy}[i]^2 + B_{sz}[i]^2 \end{pmatrix} \tag{68}$$

$$\boldsymbol{Y}^T\boldsymbol{Y} = \begin{pmatrix} B_{sx}[0]^2 + B_{sy}[0]^2 + B_{sz}[0]^2 \\ B_{sx}[1]^2 + B_{sy}[1]^2 + B_{sz}[1]^2 \\ \dots \\ B_{sx}[M-1]^2 + B_{sy}[M-1]^2 + B_{sz}[M-1]^2 \end{pmatrix}^T \begin{pmatrix} B_{sx}[0]^2 + B_{sy}[0]^2 + B_{sz}[0]^2 \\ B_{sx}[1]^2 + B_{sy}[1]^2 + B_{sz}[1]^2 \\ \dots \\ B_{sx}[M-1]^2 + B_{sy}[M-1]^2 + B_{sz}[M-1]^2 \end{pmatrix} \tag{69}$$

$$= \sum_{i=0}^{M-1} \left(B_{sx}[i]^2 + B_{sy}[i]^2 + B_{sz}[i]^2\right)^2 \tag{70}$$

The solution vector $\boldsymbol{\beta}$ is then given by equation (17) as:

$$\boldsymbol{\beta} = \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} = \left\{ \sum_{i=0}^{M-1} \begin{pmatrix} B_{sx}[i]^2 & B_{sx}[i]B_{sy}[i] & B_{sx}[i]B_{sz}[i] & B_{sx}[i] \\ B_{sx}[i]B_{sy}[i] & B_{sy}[i]^2 & B_{sy}[i]B_{sz}[i] & B_{sy}[i] \\ B_{sx}[i]B_{sz}[i] & B_{sy}[i]B_{sz}[i] & B_{sz}[i]^2 & B_{sz}[i] \\ B_{sx}[i] & B_{sy}[i] & B_{sz}[i] & 1 \end{pmatrix} \right\}^{-1} \sum_{i=0}^{M-1} \begin{pmatrix} B_{sx}[i]\left(B_{sx}[i]^2 + B_{sy}[i]^2 + B_{sz}[i]^2\right) \\ B_{sy}[i]\left(B_{sx}[i]^2 + B_{sy}[i]^2 + B_{sz}[i]^2\right) \\ B_{sz}[i]\left(B_{sx}[i]^2 + B_{sy}[i]^2 + B_{sz}[i]^2\right) \\ B_{sx}[i]^2 + B_{sy}[i]^2 + B_{sz}[i]^2 \end{pmatrix} \tag{71}$$

4.2 Hard-iron Vector

The hard-iron solution vector is given directly by equation (57) as:

$$\begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} = \left(\frac{1}{2}\right) \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{pmatrix} \tag{72}$$

4.3 Soft-iron Matrix

The soft-iron matrix $\boldsymbol{W}$ is always the identity matrix in the four-parameter magnetic calibration model.

4.4 Geomagnetic Field Strength

The geomagnetic field strength is computed from the last component of equation (57) as:

$$B^2 = \beta_3 + {V_x}^2 + {V_y}^2 + {V_z}^2 \Rightarrow B = \sqrt{\beta_3 + {V_x}^2 + {V_y}^2 + {V_z}^2} \tag{73}$$

4.5 Fit Error

The residuals $r[i]$ of equation (58) have dimensions of magnetic field strength squared. The error function E is proportional to the number of measurements M and has dimensions of the residuals squared or the fourth power of the geomagnetic field strength B. The dimensionless measure of fit error ε used is defined to be:

$$\varepsilon = \frac{1}{2B^2}\sqrt{\frac{E}{M}} \tag{74}$$

5 Seven-Parameter Magnetic Calibration Model

5.1 Construction of Seven-Parameter Linear Model

This section documents the magnetic calibration algorithm implemented in function `fUpdateCalibration7EIG` which extends the four-parameter model of the previous section with the addition of three gain terms of the diagonal of the soft-iron matrix $\boldsymbol{W}$ giving a total of seven magnetic calibration parameters. This model gives a significant improvement when either the magnetometer sensor has differing gains in its three channels or when the PCB has differing magnetic impedances along its three Cartesian axes. The diagonal form of $\boldsymbol{W}$ means that the magnetic distribution ellipsoid is modeled as having its principal axes aligned with the PCB's Cartesian axes.

The magnetometer measurement $\boldsymbol{B}_s$ in the presence of arbitrary orientation and hard- and soft-iron interference is modeled as:

$$\boldsymbol{B}_s = \boldsymbol{W}\boldsymbol{B}_c + \boldsymbol{V} = \boldsymbol{W}\boldsymbol{R}\boldsymbol{B}_r + \boldsymbol{V} \tag{75}$$

where, for the seven-parameter magnetic calibration model, the soft-iron matrix $\boldsymbol{W}$ is diagonal.

The locus of the magnetometer measurements is:

$$\{\boldsymbol{W}^{-1}(\boldsymbol{B}_s - \boldsymbol{V})\}^T\boldsymbol{W}^{-\boldsymbol{1}}(\boldsymbol{B}_s - \boldsymbol{V}) = (\boldsymbol{B}_s - \boldsymbol{V})^T(\boldsymbol{W}^{-1})^T\boldsymbol{W}^{-1}(\boldsymbol{B}_s - \boldsymbol{V}) = (\boldsymbol{B}_s - \boldsymbol{V})^T\boldsymbol{A}(\boldsymbol{B}_s - \boldsymbol{V}) = B^2 \tag{76}$$

The manipulations that follow derive an expression for the error residual $r[i]$ from the i^{th} measurement defined as:

$$r[i] = |\boldsymbol{W}^{-1}(\boldsymbol{B}_s[i] - \boldsymbol{V})|^2 - B^2 = |\boldsymbol{B}_c[i]|^2 - B^2 \tag{77}$$

The parameter $r[i]$ is defined as the difference between the squared modulus of the calibrated magnetometer measurement $\boldsymbol{B}_c[i]$ and the square of the radius of the geomagnetic sphere. Therefore, $r[i]$ has dimensions of B^2, as was also the case for the four-element calibration model.

Expanding equation (77) gives:

$$r[i] = \boldsymbol{B}_s{}^T\boldsymbol{A}\boldsymbol{B}_s - \boldsymbol{B}_s{}^T\boldsymbol{A}\boldsymbol{V} - \boldsymbol{V}^T\boldsymbol{A}\boldsymbol{B}_s + \boldsymbol{V}^T\boldsymbol{A}\boldsymbol{V} - B^2 \tag{78}$$

Because $\boldsymbol{B}_s{}^T\boldsymbol{AV}$ is a scalar:

$$\left(\boldsymbol{B}_s{}^T\boldsymbol{AV}\right)^T = \boldsymbol{V}^T\boldsymbol{AB}_s = \boldsymbol{B}_s{}^T\boldsymbol{AV} \tag{79}$$

Substituting equation (79) into equation (78) and rearranging gives:

$$r[i] = \boldsymbol{B}_s{}^T\boldsymbol{AB}_s - 2\boldsymbol{B}_s{}^T\boldsymbol{AV} + \boldsymbol{V}^T\boldsymbol{AV} - B^2 \tag{80}$$

Expanding equation (80) for the i^{th} measurement gives:

$$\begin{aligned} r[i] = {} & A_{xx}B_{sx}[i]^2 + A_{yy}B_{sy}[i]^2 + A_{zz}B_{sz}[i]^2 - 2B_{sx}[i]A_{xx}V_x - 2B_{sy}[i]A_{yy}V_y - 2B_{sz}[i]A_{zz}V_z \\ & + A_{xx}{V_x}^2 + A_{yy}{V_y}^2 + A_{zz}{V_z}^2 - B^2 \end{aligned} \tag{81}$$

Simplifying and returning to matrix format gives:

$$r[i] = \begin{pmatrix} B_{sx}[i]^2 \\ B_{sy}[i]^2 \\ B_{sz}[i]^2 \\ B_{sx}[i] \\ B_{sy}[i] \\ B_{sz}[i] \\ 1 \end{pmatrix}^T \begin{pmatrix} A_{xx} \\ A_{yy} \\ A_{zz} \\ -2A_{xx}V_x \\ -2A_{yy}V_y \\ -2A_{zz}V_z \\ A_{xx}{V_x}^2 + A_{yy}{V_y}^2 + A_{zz}{V_z}^2 - B^2 \end{pmatrix} \tag{82}$$

Defining the right-hand side of equation (82) to be the solution vector $\boldsymbol{\beta}$ gives:

$$\boldsymbol{\beta} = \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ \beta_5 \\ \beta_6 \end{pmatrix} = \begin{pmatrix} A_{xx} \\ A_{yy} \\ A_{zz} \\ -2A_{xx}V_x \\ -2A_{yy}V_y \\ -2A_{zz}V_z \\ A_{xx}{V_x}^2 + A_{yy}{V_y}^2 + A_{zz}{V_z}^2 - B^2 \end{pmatrix} \tag{83}$$

Equation (81) for the error residual $r[i]$ whose squared sum is to be minimized is now:

$$r[i] = \begin{pmatrix} B_{sx}[i]^2 \\ B_{sy}[i]^2 \\ B_{sz}[i]^2 \\ B_{sx}[i] \\ B_{sy}[i] \\ B_{sz}[i] \\ 1 \end{pmatrix}^T \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ \beta_5 \\ \beta_6 \end{pmatrix} \tag{84}$$

With the definition of the error residual vector $\boldsymbol{r}$ from M measurements as:

$$\boldsymbol{r} = \begin{pmatrix} r[0] \\ r[1] \\ \ldots \\ r[M-1] \end{pmatrix} \tag{85}$$

and $\boldsymbol{X}$ defined as is the $M \times 7$ measurement matrix:

$$\boldsymbol{X} = \begin{pmatrix} B_{sx}[0]^2 & B_{sy}[0]^2 & B_{sz}[0]^2 & B_{sx}[0] & B_{sy}[0] & B_{sz}[0] & 1 \\ B_{sx}[1]^2 & B_{sy}[1]^2 & B_{sz}[1]^2 & B_{sx}[1] & B_{sy}[1] & B_{sz}[1] & 1 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & 1 \\ B_{sx}[M-1]^2 & B_{sy}[M-1]^2 & B_{sz}[M-1]^2 & B_{sx}[M-1] & B_{sy}[M-1] & B_{sz}[M-1] & 1 \end{pmatrix} \tag{86}$$

Equation (82) can be expanded to represent M measurements as:

$$\boldsymbol{r} = \boldsymbol{X\beta} \tag{87}$$

The model being fitted is the homogeneous model $\boldsymbol{X\beta} = 0$ which can be solved for $\boldsymbol{\beta}$ using the eigen-decomposition approach described in Section 2.

The 7×7 product matrix $\boldsymbol{X}^T\boldsymbol{X}$ whose eigenvectors and eigenvalues are to be determined evaluates to:

$$\boldsymbol{X}^T\boldsymbol{X} = \begin{pmatrix} B_{sx}[0]^2 & B_{sx}[1]^2 & \ldots & B_{sx}[M-1]^2 \\ B_{sy}[0]^2 & B_{sy}[1]^2 & \ldots & B_{sy}[M-1]^2 \\ B_{sz}[0]^2 & B_{sz}[1]^2 & \ldots & B_{sz}[M-1]^2 \\ B_{sx}[0] & B_{sx}[1] & \ldots & B_{sx}[M-1] \\ B_{sy}[0] & B_{sy}[1] & \ldots & B_{sy}[M-1] \\ B_{sz}[0] & B_{sz}[1] & \ldots & B_{sz}[M-1] \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} B_{sx}[0]^2 & B_{sy}[0]^2 & B_{sz}[0]^2 & B_{sx}[0] & B_{sy}[0] & B_{sz}[0] & 1 \\ B_{sx}[1]^2 & B_{sy}[1]^2 & B_{sz}[1]^2 & B_{sx}[1] & B_{sy}[1] & B_{sz}[1] & 1 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & 1 \\ B_{sx}[M-1]^2 & B_{sy}[M-1]^2 & B_{sz}[M-1]^2 & B_{sx}[M-1] & B_{sy}[M-1] & B_{sz}[M-1] & 1 \end{pmatrix} \tag{88}$$

$$= \sum_{i=0}^{M-1} \begin{pmatrix} B_{sx}[i]^4 & B_{sx}[i]^2B_{sy}[i]^2 & B_{sx}[i]^2B_{sz}[i]^2 & B_{sx}[i]^3 & B_{sx}[i]^2B_{sy}[i] & B_{sx}[i]^2B_{sz}[i] & B_{sx}[i]^2 \\ B_{sy}[i]^2B_{sx}[i]^2 & B_{sy}[i]^4 & B_{sy}[i]^2B_{sz}[i]^2 & B_{sy}[i]^2B_{sx}[i] & B_{sy}[i]^3 & B_{sy}[i]^2B_{sz}[i] & B_{sy}[i]^2 \\ B_{sz}[i]^2B_{sx}[i]^2 & B_{sz}[i]^2B_{sy}[i]^2 & B_{sz}[i]^4 & B_{sz}[i]^2B_{sx}[i] & B_{sz}[i]^2B_{sy}[i] & B_{sz}[i]^3 & B_{sz}[i]^2 \\ B_{sx}[i]^3 & B_{sx}[i]B_{sy}[i]^2 & B_{sx}[i]B_{sz}[i]^2 & B_{sx}[i]^2 & B_{sx}[i]B_{sy}[i] & B_{sx}[i]B_{sz}[i] & B_{sx}[i] \\ B_{sy}[i]B_{sx}[i]^2 & B_{sy}[i]^3 & B_{sy}[i]B_{sz}[i]^2 & B_{sy}[i]B_{sx}[i] & B_{sy}[i]^2 & B_{sy}[i]B_{sz}[i] & B_{sy}[i] \\ B_{sz}[i]B_{sx}[i]^2 & B_{sz}[i]B_{sy}[i]^2 & B_{sz}[i]^3 & B_{sz}[i]B_{sx}[i] & B_{sz}[i]B_{sy}[i] & B_{sz}[i]^2 & B_{sz}[i] \\ B_{sx}[i]^2 & B_{sy}[i]^2 & B_{sz}[i]^2 & B_{sx}[i] & B_{sy}[i] & B_{sz}[i] & 1 \end{pmatrix} \tag{89}$$

Because the eigenvalues of $\boldsymbol{X}^T\boldsymbol{X}$ are equal to the fit errors associated with the seven candidate eigenvector solutions, the required solution vector $\boldsymbol{\beta}$ is the eigenvector associated with the smallest eigenvalue λ_{min}.

5.2 Ellipsoid Fit Matrix

The ellipsoid fit matrix $\boldsymbol{A}$ is obtained directly from the first three rows of the solution vector $\boldsymbol{\beta}$ in equation (83):

$$\boldsymbol{A} = \begin{pmatrix} A_{xx} & 0 & 0 \\ 0 & A_{yy} & 0 \\ 0 & 0 & A_{zz} \end{pmatrix} = \begin{pmatrix} \beta_0 & 0 & 0 \\ 0 & \beta_1 & 0 \\ 0 & 0 & \beta_2 \end{pmatrix} \tag{90}$$

The solution eigenvector $\boldsymbol{\beta}$ is undefined within a multiplicative factor of ±1 (assuming it is normalized to unit magnitude). A test must therefore be performed on the determinant of the ellipsoid matrix $\boldsymbol{A}$ defined in equation (120) and the entire solution vector $\boldsymbol{\beta}$ negated if the determinant is negative. Negating the solution vector $\boldsymbol{\beta}$ changes the sign of $\boldsymbol{A}$ and ensures a positive determinant.

The ellipsoid matrix $\boldsymbol{A}$ is then normalized to have unit determinant:

$$\begin{vmatrix} A_{xx} & 0 & 0 \\ 0 & A_{yy} & 0 \\ 0 & 0 & A_{zz} \end{vmatrix} = A_{xx}A_{yy}A_{zz} = 1 \tag{91}$$

The justification for the normalization in equation (91) is that it is impossible to separate out the geomagnetic field strength B from the soft-iron magnetic matrix gain terms. A 25 µT geomagnetic field strength with no soft-iron gain terms gives the same magnetometer measurement as a 50 µT geomagnetic field strength attenuated 50 percent by magnetic shielding. The solution taken in the Freescale software sets the determinant of the soft-iron matrices to one and assigns the magnitude of the calibrated measurement to the geomagnetic field strength B.

5.3 Hard-iron Vector

The hard-iron model is given by equation (83) as:

$$\begin{pmatrix} -2A_{xx}V_x \\ -2A_{yy}V_y \\ -2A_{zz}V_z \end{pmatrix} = \begin{pmatrix} \beta_3 \\ \beta_4 \\ \beta_5 \end{pmatrix} \Rightarrow \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} = \begin{pmatrix} \left(\frac{-\beta_3}{2A_{xx}}\right) \\ \left(\frac{-\beta_4}{2A_{yy}}\right) \\ \left(\frac{-\beta_5}{2A_{zz}}\right) \end{pmatrix} = \begin{pmatrix} \left(\frac{-\beta_3}{2\beta_0}\right) \\ \left(\frac{-\beta_4}{2\beta_1}\right) \\ \left(\frac{-\beta_5}{2\beta_2}\right) \end{pmatrix} \tag{92}$$

5.4 Inverse Soft-iron Matrix

The inverse soft-iron matrix can be found from the square root of the diagonal ellipsoid matrix as:

$$\boldsymbol{W}^{-1} = \begin{pmatrix} W_{xx} & 0 & 0 \\ 0 & W_{yy} & 0 \\ 0 & 0 & W_{zz} \end{pmatrix} = \sqrt{\boldsymbol{A}} = \begin{pmatrix} \sqrt{\beta_0} & 0 & 0 \\ 0 & \sqrt{\beta_1} & 0 \\ 0 & 0 & \sqrt{\beta_2} \end{pmatrix} \quad \textbf{(93)}$$

5.5 Geomagnetic Field Strength

The geomagnetic field strength B is given by the last component of equation (83):

$$\beta_6 = A_{xx}{V_x}^2 + A_{yy}{V_y}^2 + A_{zz}{V_z}^2 - B^2 \quad \textbf{(94)}$$

$$\Rightarrow B^2 = A_{xx}{V_x}^2 + A_{yy}{V_y}^2 + A_{zz}{V_z}^2 - \beta_6 \quad \textbf{(95)}$$

Because the solution vector and ellipsoid matrix elements may be negated in software to force a positive determinant, it is necessary to compute the geomagnetic field from the absolute value of the right-hand side of equation (95).

$$B = \sqrt{\left|A_{xx}{V_x}^2 + A_{yy}{V_y}^2 + A_{zz}{V_z}^2 - \beta_6\right|} \quad \textbf{(96)}$$

5.6 Fit Error

The error function E equals the smallest eigenvalue λ_{min} of the product matrix $\boldsymbol{X}^T\boldsymbol{X}$ but is not normalized to the number of measurement points M nor is it normalized to the geomagnetic field strength B. Because $E = \boldsymbol{r}^T\boldsymbol{r}$ and $\boldsymbol{r}$ has M elements with each element having dimensions B^2, a suitable normalized calibration fit error measurement ε is:

$$\varepsilon = \frac{1}{2B^2}\sqrt{\frac{\lambda_{min}}{M}} \quad \textbf{(97)}$$

6 Ten-Parameter Magnetic Calibration Model

6.1 Construction of Ten-Parameter Linear Model

This section documents the magnetic calibration algorithm implemented in function `fUpdateCalibration10EIG`, which extends the seven-parameter model of the previous section with the addition of three off-diagonal soft-iron matrix terms to $\boldsymbol{W}$ to give a total of 10 magnetic calibration parameters. This model gives an improvement over the seven-element model when the PCB's magnetic impedances steer the geomagnetic field in directions that are not aligned with the PCB's Cartesian axes giving a rotated magnetic ellipsoid.

The magnetometer measurement $\boldsymbol{B}_s$ in the presence of arbitrary orientation and hard and soft-iron interference is modeled as:

$$\boldsymbol{B}_s = \boldsymbol{W}\boldsymbol{B}_c + \boldsymbol{V} = \boldsymbol{W}\boldsymbol{R}\boldsymbol{B}_r + \boldsymbol{V} \quad \textbf{(98)}$$

where, for the 10-parameter magnetic calibration model, the soft-iron matrix $\boldsymbol{W}$ is symmetric.

The locus of the magnetometer measurements is:

$$\{\boldsymbol{W}^{-1}(\boldsymbol{B}_s - \boldsymbol{V})\}^T \boldsymbol{W}^{-\boldsymbol{1}}(\boldsymbol{B}_s - \boldsymbol{V}) = (\boldsymbol{B}_s - \boldsymbol{V})^T (\boldsymbol{W}^{-1})^T \boldsymbol{W}^{-1}(\boldsymbol{B}_s - \boldsymbol{V}) = (\boldsymbol{B}_s - \boldsymbol{V})^T \boldsymbol{A}(\boldsymbol{B}_s - \boldsymbol{V}) = B^2 \quad \textbf{(99)}$$

Equation (99) models the locus of the magnetometer measurements $\boldsymbol{B}_s$ as lying on the surface of an ellipsoid with arbitrary dimensions and directions of its axes and offset from the origin by the hard-iron vector $\boldsymbol{V}$.

The following manipulations derive an expression for the error residual $r[i]$ from the i^{th} measurement defined as:

$$r[i] = |\boldsymbol{W}^{-1}(\boldsymbol{B}_s[i] - \boldsymbol{V})|^2 - B^2 = |\boldsymbol{B}_c[i]|^2 - B^2 \quad \textbf{(100)}$$

$r[i]$ is defined as the difference between the squared modulus of the calibrated magnetometer measurement $\boldsymbol{B}_c[i]$ and the square of the radius of the geomagnetic sphere. Therefore, $r[i]$ has dimensions of B^2.

Expanding equation (100) gives:

$$r[i] = \boldsymbol{B}_s{}^T \boldsymbol{A}\boldsymbol{B}_s - \boldsymbol{B}_s{}^T \boldsymbol{A}\boldsymbol{V} - \boldsymbol{V}^T \boldsymbol{A}\boldsymbol{B}_s + \boldsymbol{V}^T \boldsymbol{A}\boldsymbol{V} - B^2 \quad \textbf{(101)}$$

Because $\boldsymbol{B}_s{}^T \boldsymbol{A}\boldsymbol{V}$ is a scalar:

$$\left(\boldsymbol{B}_s{}^T \boldsymbol{A}\boldsymbol{V}\right)^T = \boldsymbol{V}^T \boldsymbol{A}\boldsymbol{B}_s = \boldsymbol{B}_s{}^T \boldsymbol{A}\boldsymbol{V} \quad \textbf{(102)}$$

Substituting equation (102) into equation (101) and re-arranging gives:

$$r[i] = \boldsymbol{B}_s{}^T \boldsymbol{A}\boldsymbol{B}_s - 2\boldsymbol{B}_s{}^T \boldsymbol{A}\boldsymbol{V} + \boldsymbol{V}^T \boldsymbol{A}\boldsymbol{V} - B^2 \quad \textbf{(103)}$$

Expanding equation (103) for the i^{th} measurement gives:

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$$r[i] = \begin{pmatrix} B_{sx}[i] \\ B_{sy}[i] \\ B_{sz}[i] \end{pmatrix}^T \begin{pmatrix} A_{xx} & A_{xy} & A_{xz} \\ A_{xy} & A_{yy} & A_{yz} \\ A_{xz} & A_{yz} & A_{zz} \end{pmatrix} \begin{pmatrix} B_{sx}[i] \\ B_{sy}[i] \\ B_{sz}[i] \end{pmatrix} - 2 \begin{pmatrix} B_{sx}[i] \\ B_{sy}[i] \\ B_{sz}[i] \end{pmatrix}^T \begin{pmatrix} A_{xx} & A_{xy} & A_{xz} \\ A_{xy} & A_{yy} & A_{yz} \\ A_{xz} & A_{yz} & A_{zz} \end{pmatrix} \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} + \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix}^T \begin{pmatrix} A_{xx} & A_{xy} & A_{xz} \\ A_{xy} & A_{yy} & A_{yz} \\ A_{xz} & A_{yz} & A_{zz} \end{pmatrix} \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} - B^2 \quad \textbf{(104)}$$

The first term in equation (104) expands to:

$$\begin{pmatrix} B_{sx}[i] \\ B_{sy}[i] \\ B_{sz}[i] \end{pmatrix}^T \begin{pmatrix} A_{xx} & A_{xy} & A_{xz} \\ A_{xy} & A_{yy} & A_{yz} \\ A_{xz} & A_{yz} & A_{zz} \end{pmatrix} \begin{pmatrix} B_{sx}[i] \\ B_{sy}[i] \\ B_{sz}[i] \end{pmatrix} = \begin{pmatrix} B_{sx}[i] \\ B_{sy}[i] \\ B_{sz}[i] \end{pmatrix}^T \begin{pmatrix} A_{xx}B_{sx}[i] + A_{xy}B_{sy}[i] + A_{xz}B_{sz}[i] \\ A_{xy}B_{sx}[i] + A_{yy}B_{sy}[i] + A_{yz}B_{sz}[i] \\ A_{xz}B_{sx}[i] + A_{yz}B_{sy}[i] + A_{zz}B_{sz}[i] \end{pmatrix} \quad \textbf{(105)}$$

$$= A_{xx}B_{sx}[i]^2 + A_{yy}B_{sy}[i]^2 + A_{zz}B_{sz}[i]^2 + 2A_{xy}B_{sx}[i]B_{sy}[i] + 2A_{xz}B_{sx}[i]B_{sz}[i] + 2A_{yz}B_{sy}[i]B_{sz}[i] \quad \textbf{(106)}$$

The second term in equation (104) expands to:

$$-2\begin{pmatrix} B_{sx}[i] \\ B_{sy}[i] \\ B_{sz}[i] \end{pmatrix}^T \begin{pmatrix} A_{xx} & A_{xy} & A_{xz} \\ A_{xy} & A_{yy} & A_{yz} \\ A_{xz} & A_{yz} & A_{zz} \end{pmatrix} \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} = -2\begin{pmatrix} B_{sx}[i] \\ B_{sy}[i] \\ B_{sz}[i] \end{pmatrix}^T \begin{pmatrix} A_{xx}V_x + A_{xy}V_y + A_{xz}V_z \\ A_{xy}V_x + A_{yy}V_y + A_{yz}V_z \\ A_{xz}V_x + A_{yz}V_y + A_{zz}V_z \end{pmatrix} \quad \textbf{(107)}$$

$$= -2B_{sx}[i]A_{xx}V_x - 2B_{sx}[i]A_{xy}V_y - 2B_{sx}[i]A_{xz}V_z - 2B_{sy}[i]A_{xy}V_x - 2B_{sy}[i]A_{yy}V_y - 2B_{sy}[i]A_{yz}V_z - 2B_{sz}[i]A_{xz}V_x - 2B_{sz}[i]A_{yz}V_y - 2B_{sz}[i]A_{zz}V_z \quad \textbf{(108)}$$

The third term in equation (104) expands to:

$$\begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix}^T \begin{pmatrix} A_{xx} & A_{xy} & A_{xz} \\ A_{xy} & A_{yy} & A_{yz} \\ A_{xz} & A_{yz} & A_{zz} \end{pmatrix} \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} = \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix}^T \begin{pmatrix} A_{xx}V_x + A_{xy}V_y + A_{xz}V_z \\ A_{xy}V_x + A_{yy}V_y + A_{yz}V_z \\ A_{xz}V_x + A_{yz}V_y + A_{zz}V_z \end{pmatrix} \quad \textbf{(109)}$$

$$= A_{xx}{V_x}^2 + A_{yy}{V_y}^2 + A_{zz}{V_z}^2 + 2A_{xy}V_xV_y + 2A_{xz}V_xV_z + 2A_{yz}V_yV_z \quad \textbf{(110)}$$

The full equation for the residual error $r[i]$ from the *i-th* observation is then:

$r[i] = A_{xx}B_{sx}[i]^2 + A_{yy}B_{sy}[i]^2 + A_{zz}B_{sz}[i]^2$

$+2A_{xy}B_{sx}[i]B_{sy}[i] + 2A_{xz}B_{sx}[i]B_{sz}[i] + 2A_{yz}B_{sy}[i]B_{sz}[i]$

$-2B_{sx}[i]A_{xx}V_x - 2B_{sx}[i]A_{xy}V_y - 2B_{sx}[i]A_{xz}V_z$

$-2B_{sy}[i]A_{xy}V_x - 2B_{sy}[i]A_{yy}V_y - 2B_{sy}[i]A_{yz}V_z$

$-2B_{sz}[i]A_{xz}V_x - 2B_{sz}[i]A_{yz}V_y - 2B_{sz}[i]A_{zz}V_z$

$$+A_{xx}{V_x}^2 + A_{yy}{V_y}^2 + A_{zz}{V_z}^2 + 2A_{xy}V_xV_y + 2A_{xz}V_xV_z + 2A_{yz}V_yV_z - B^2 \tag{111}$$

Simplifying and returning to matrix format gives:

$$r[i] = \begin{pmatrix} B_{sx}[i]^2 \\ 2B_{sx}[i]B_{sy}[i] \\ 2B_{sx}[i]B_{sz}[i] \\ B_{sy}[i]^2 \\ 2B_{sy}[i]B_{sz}[i] \\ B_{sz}[i]^2 \\ B_{sx}[i] \\ B_{sy}[i] \\ B_{sz}[i] \\ 1 \end{pmatrix}^T \begin{pmatrix} A_{xx} \\ A_{xy} \\ A_{xz} \\ A_{yy} \\ A_{yz} \\ A_{zz} \\ -2A_{xx}V_x - 2A_{xy}V_y - 2A_{xz}V_z \\ -2A_{xy}V_x - 2A_{yy}V_y - 2A_{yz}V_z \\ -2A_{xz}V_x - 2A_{yz}V_y - 2A_{zz}V_z \\ A_{xx}{V_x}^2 + 2A_{xy}V_xV_y + 2A_{xz}V_xV_z + A_{yy}{V_y}^2 + 2A_{yz}V_yV_z + A_{zz}{V_z}^2 - B^2 \end{pmatrix} \tag{112}$$

Defining the right-hand side of equation (112) to be the solution vector $\boldsymbol{\beta}$ gives:

$$\boldsymbol{\beta} = \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ \beta_5 \\ \beta_6 \\ \beta_7 \\ \beta_8 \\ \beta_9 \end{pmatrix} = \begin{pmatrix} A_{xx} \\ A_{xy} \\ A_{xz} \\ A_{yy} \\ A_{yz} \\ A_{zz} \\ -2A_{xx}V_x - 2A_{xy}V_y - 2A_{xz}V_z \\ -2A_{xy}V_x - 2A_{yy}V_y - 2A_{yz}V_z \\ -2A_{xz}V_x - 2A_{yz}V_y - 2A_{zz}V_z \\ A_{xx}{V_x}^2 + 2A_{xy}V_xV_y + 2A_{xz}V_xV_z + A_{yy}{V_y}^2 + 2A_{yz}V_yV_z + A_{zz}{V_z}^2 - B^2 \end{pmatrix} \tag{113}$$

$$= \begin{pmatrix} A_{xx} \\ A_{xy} \\ A_{xz} \\ A_{yy} \\ A_{yz} \\ A_{zz} \\ -2\beta_0V_x - 2\beta_1V_y - 2\beta_2V_z \\ -2\beta_1V_x - 2\beta_3V_y - 2\beta_4V_z \\ -2\beta_2V_x - 2\beta_4V_y - 2\beta_5V_z \\ A_{xx}{V_x}^2 + 2A_{xy}V_xV_y + 2A_{xz}V_xV_z + A_{yy}{V_y}^2 + 2A_{yz}V_yV_z + A_{zz}{V_z}^2 - B^2 \end{pmatrix} \tag{114}$$

Equation (112) for the error residual $r[i]$ whose squared sum is to be minimized is then:

$$r[i] = \begin{pmatrix} B_{sx}[i]^2 \\ 2B_{sx}[i]B_{sy}[i] \\ 2B_{sx}[i]B_{sz}[i] \\ B_{sy}[i]^2 \\ 2B_{sy}[i]B_{sz}[i] \\ B_{sz}[i]^2 \\ B_{sx}[i] \\ B_{sy}[i] \\ B_{sz}[i] \\ 1 \end{pmatrix}^T \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ \beta_5 \\ \beta_6 \\ \beta_7 \\ \beta_8 \\ \beta_9 \end{pmatrix} \tag{115}$$

With the definition of the error residual vector $\boldsymbol{r}$ from M measurements as:

$$\boldsymbol{r} = \begin{pmatrix} r[0] \\ r[1] \\ \dots \\ r[M-1] \end{pmatrix} \tag{116}$$

and $\boldsymbol{X}$ defined as the $M \times 10$ measurement matrix:

$$\boldsymbol{X} = \begin{pmatrix} B_{sx}[0]^2 & 2B_{sx}[0]B_{sy}[0] & \dots & B_{sy}[0] & B_{sz}[0] & 1 \\ B_{sx}[1]^2 & 2B_{sx}[1]B_{sy}[1] & \dots & B_{sy}[1] & B_{sz}[1] & 1 \\ \dots & \dots & \dots & \dots & \dots & 1 \\ B_{sx}[M-1]^2 & 2B_{sx}[M-1]B_{sy}[M-1] & \dots & B_{sy}[M-1] & B_{sz}[M-1] & 1 \end{pmatrix} \tag{117}$$

then equation (115) can be expanded to represent M measurements as:

$$\boldsymbol{r} = \boldsymbol{X\beta} \tag{118}$$

The model being fitted is the homogeneous model $\boldsymbol{X\beta} = 0$ which can be solved for $\boldsymbol{\beta}$ using the eigen-decomposition approach described in Section 2.

The 10×10 product matrix $\boldsymbol{X}^T\boldsymbol{X}$ whose eigenvectors and eigenvalues are to be determined evaluates to:

$$\boldsymbol{X}^T\boldsymbol{X} = \sum_{i=0}^{M-1} \begin{pmatrix} B_{sx}[i]^4 & 2B_{sx}[i]^3B_{sy}[i] & 2B_{sx}[i]^3B_{sz}[i] & \dots & B_{sx}[i]^2B_{sz}[i] & B_{sx}[i]^2 \\ 2B_{sx}[i]^3B_{sy}[i] & 4B_{sx}[i]^2B_{sy}[i]^2 & 4B_{sx}[i]^2B_{sy}[i]B_{sz}[i] & \dots & 2B_{sx}[i]B_{sy}[i]B_{sz}[i] & 2B_{sx}[i]B_{sy}[i] \\ 2B_{sx}[i]^3B_{sz}[i] & 4B_{sx}[i]^2B_{sy}[i]B_{sz}[i] & 4B_{sx}[i]^2B_{sz}[i]^2 & \dots & 2B_{sx}[i]B_{sz}[i]^2 & 2B_{sx}[i]B_{sz}[i] \\ B_{sx}[i]^2B_{sy}[i]^2 & 2B_{sx}[i]B_{sy}[i]^3 & 2B_{sx}[i]B_{sy}[i]^2B_{sz}[i] & \dots & B_{sy}[i]^2B_{sz}[i] & B_{sy}[i]^2 \\ 2B_{sx}[i]^2B_{sy}[i]B_{sz}[i] & 4B_{sx}[i]B_{sy}[i]^2B_{sz}[i] & 4B_{sx}[i]B_{sy}[i]B_{sz}[i]^2 & \dots & 2B_{sy}[i]B_{sz}[i]^2 & 2B_{sy}[i]B_{sz}[i] \\ B_{sx}[i]^2B_{sz}[i]^2 & 2B_{sx}[i]B_{sy}[i]B_{sz}[i]^2 & 2B_{sx}[i]B_{sz}[i]^3 & \dots & B_{sz}[i]^3 & B_{sz}[i]^2 \\ B_{sx}[i]^3 & 2B_{sx}[i]^2B_{sy}[i] & 2B_{sx}[i]^2B_{sz}[i] & \dots & B_{sx}[i]B_{sz}[i] & B_{sx}[i] \\ B_{sx}[i]^2B_{sy}[i] & 2B_{sx}[i]B_{sy}[i]^2 & 2B_{sx}[i]B_{sy}[i]B_{sz}[i] & \dots & B_{sy}[i]B_{sz}[i] & B_{sy}[i] \\ B_{sx}[i]^2B_{sz}[i] & 2B_{sx}[i]B_{sy}[i]B_{sz}[i] & 2B_{sx}[i]B_{sz}[i]^2 & \dots & B_{sz}[i]^3 & B_{sz}[i] \\ B_{sx}[i]^2 & 2B_{sx}[i]B_{sy}[i] & 2B_{sx}[i]B_{sz}[i] & \dots & B_{sz}[i] & 1 \end{pmatrix} \tag{119}$$

Because the eigenvalues of $\boldsymbol{X}^T\boldsymbol{X}$ are equal to the fit errors associated with the 10-parameter eigenvector solutions, the required solution vector $\boldsymbol{\beta}$ is the eigenvector associated with the smallest eigenvalue λ_{min}.

6.2 Ellipsoid Fit Matrix

The ellipsoid fit matrix $\boldsymbol{A}$ is obtained directly from the first six rows of the solution vector $\boldsymbol{\beta}$:

$$\boldsymbol{A} = \begin{pmatrix} A_{xx} & A_{xy} & A_{xz} \\ A_{xy} & A_{yy} & A_{yz} \\ A_{xz} & A_{yz} & A_{zz} \end{pmatrix} = \begin{pmatrix} \beta_0 & \beta_1 & \beta_2 \\ \beta_1 & \beta_3 & \beta_4 \\ \beta_2 & \beta_4 & \beta_5 \end{pmatrix} \quad \textbf{(120)}$$

The solution eigenvector $\boldsymbol{\beta}$ is undefined within a multiplicative factor of ±1 (assuming it is normalized to unit magnitude). A test must therefore be performed on the determinant of the ellipsoid matrix $\boldsymbol{A}$ defined in equation (120) and the entire solution vector $\boldsymbol{\beta}$ negated if the determinant is negative. Negating the solution vector $\boldsymbol{\beta}$ changes the sign of $\boldsymbol{A}$ and ensures a positive determinant.

For the same reasons as for the seven element calibration algorithm, the determinant of $\boldsymbol{A}$ is set to 1.0.

6.3 Hard-iron Vector

The last three rows of equation (113) can be written as:

$$\begin{pmatrix} \beta_6 \\ \beta_7 \\ \beta_8 \end{pmatrix} = -2 \begin{pmatrix} \beta_0 & \beta_1 & \beta_2 \\ \beta_1 & \beta_3 & \beta_4 \\ \beta_2 & \beta_4 & \beta_5 \end{pmatrix} \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} = -2\boldsymbol{A} \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} \quad \textbf{(121)}$$

The solution for the hard-iron vector $\boldsymbol{V}$ is:

$$\boldsymbol{V} = \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} = -\left(\frac{1}{2}\right) \begin{pmatrix} \beta_0 & \beta_1 & \beta_2 \\ \beta_1 & \beta_3 & \beta_4 \\ \beta_2 & \beta_4 & \beta_5 \end{pmatrix}^{-1} \begin{pmatrix} \beta_6 \\ \beta_7 \\ \beta_8 \end{pmatrix} = -\left(\frac{1}{2}\right) \boldsymbol{A}^{-1} \begin{pmatrix} \beta_6 \\ \beta_7 \\ \beta_8 \end{pmatrix} \quad \textbf{(122)}$$

The solution for the hard-iron vector $\boldsymbol{V}$ is independent of any sign change or other scaling of the solution vector $\boldsymbol{\beta}$ as a consequence of the multiplication by the inverse soft-iron matrix which cancels the scaling.

6.4 Inverse Soft-iron Matrix

The inverse soft-iron matrix $\boldsymbol{W}^{-1}$ is computed from the square root of the symmetric matrix $\boldsymbol{A}$ whose components are provided in elements β_1 to β_6 of the solution vector $\boldsymbol{\beta}$:

$$W^{-1} = \sqrt{A} = \begin{pmatrix} A_{xx} & A_{xy} & A_{xz} \\ A_{xy} & A_{yy} & A_{yz} \\ A_{xz} & A_{yz} & A_{zz} \end{pmatrix}^{\frac{1}{2}} = \begin{pmatrix} \beta_0 & \beta_1 & \beta_2 \\ \beta_1 & \beta_3 & \beta_4 \\ \beta_2 & \beta_4 & \beta_5 \end{pmatrix}^{\frac{1}{2}} \quad \textbf{(123)}$$

The matrix square root is calculated using a further eigen-decomposition but this time as part of the 3×3 ellipsoid matrix $\boldsymbol{A}$. By definition, the 3×3 matrix $\boldsymbol{Q}$ of the eigenvectors and the 3×3 diagonal matrix $\boldsymbol{\Lambda}$ of the eigenvalues of $\boldsymbol{A}$ are related by:

$$\boldsymbol{AQ} = \boldsymbol{Q\Lambda} \Rightarrow \boldsymbol{A} = \boldsymbol{Q\Lambda Q}^{-1} \Rightarrow \boldsymbol{\Lambda} = \boldsymbol{Q}^{-1}\boldsymbol{AQ} \quad \textbf{(124)}$$

The matrix $\boldsymbol{Q}\sqrt{\boldsymbol{\Lambda}}\boldsymbol{Q}^{-1}$ can be shown to be the required square root of $\boldsymbol{A}$ by simple multiplication and using the standard result that the eigenvectors of a symmetric matrix are orthonormal:

$$\left(\boldsymbol{Q}\sqrt{\boldsymbol{\Lambda}}\boldsymbol{Q}^{-1}\right)\left(\boldsymbol{Q}\sqrt{\boldsymbol{\Lambda}}\boldsymbol{Q}^{-1}\right) = \boldsymbol{Q}\sqrt{\boldsymbol{\Lambda}}\boldsymbol{Q}^{-1}\boldsymbol{Q}\sqrt{\boldsymbol{\Lambda}}\boldsymbol{Q}^{-1} = \boldsymbol{Q\Lambda Q}^{-1} = \boldsymbol{A} \quad \textbf{(125)}$$

The required square-root solution for the inverse soft-iron matrix is then:

$$\boldsymbol{W}^{-1} = \sqrt{\boldsymbol{A}} = \boldsymbol{Q}\sqrt{\boldsymbol{\Lambda}}\boldsymbol{Q}^{-1} = \boldsymbol{Q}\sqrt{\boldsymbol{\Lambda}}\boldsymbol{Q}^{T} \quad \textbf{(126)}$$

6.5 Geomagnetic Field Strength

The geomagnetic field strength can be computed from the last component of equation (113):

$$B^2 = A_{xx}{V_x}^2 + 2A_{xy}V_xV_y + 2A_{xz}V_xV_z + A_{yy}{V_y}^2 + 2A_{yz}V_yV_z + A_{zz}{V_z}^2 - \beta_9 \quad \textbf{(127)}$$

Because the right-hand side of equation (94) may be negated to force a positive determinant for $\boldsymbol{A}$, it is important that the geomagnetic field calculation is taken from the absolute value of the right-hand side.

$$B = \sqrt{\left|A_{xx}{V_x}^2 + 2A_{xy}V_xV_y + 2A_{xz}V_xV_z + A_{yy}{V_y}^2 + 2A_{yz}V_yV_z + A_{zz}{V_z}^2 - \beta_9\right|} \quad \textbf{(128)}$$

6.6 Fit Error

The error function E equals the smallest eigenvalue λ_{min} of the product matrix $\boldsymbol{X}^T\boldsymbol{X}$ but is not normalized to the number of measurement points M nor is it normalized to the geomagnetic field strength B. Because $E = \boldsymbol{r}^T\boldsymbol{r}$ and $\boldsymbol{r}$ has M elements with each element having dimensions B^2, a suitable normalized calibration fit error measure ε is:

$$\varepsilon = \frac{1}{2B^2}\sqrt{\frac{\lambda_{min}}{M}} \tag{129}$$

7 Revision History

Table 2. Revision history

Rev. No.	Date	Description
1	9/2015	Initial public release

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Revision **Error! No text of specified style in document.**, 9/2015