# Fully connected NN

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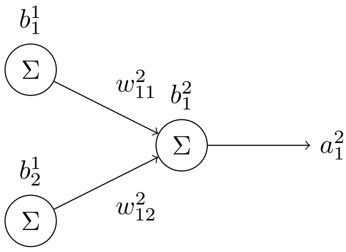
## 1 Notations

- $w_{jk}^l$ is the weight of the $k^{th}$ neuron in the layer $(l-1)$ to the $j^{th}$ neuron in the layer $l$
- $b_j^l$ is the bias of the $j^{th}$ neuron on the layer $l$.
- $a_j^l$ is the activation of the $j^{th}$ neuron on the layer $l$.
- $act$ is the activation function.
- $cost$ is the cost function.
- $\delta_j^l$ is the error of the $j^{th}$ neuron on the layer $l$.
- $L$ is the last layer.



## 2 Definitions

Transition from the layer $(l-1)$ to the layer $l$:

$$z_j^l = \sum_k w_{jk}^l a_k^{l-1} + b_j^l \tag{1}$$

$$a_j^l = act(z_j^l) \tag{2}$$

The error of a neuron can be define as:

$$\delta_j^l \equiv \frac{\partial cost}{\partial z_j^l} \tag{3}$$

We define $\delta_j^L$ the error on the output layer as the following:

$$\delta_j^L = \frac{\partial cost}{\partial a_j^l} act'(z_j^L) \tag{4}$$

Then we have $\delta^L$ which is a vector that contains the errors of all the output neurons:

$$\delta^L = \nabla_a cost \odot act'(z^L) \tag{5}$$

Here, we can deduce the value of $\delta^l$:

$$\delta^l = ((w^{l+1})^T \delta^{l+1}) \odot act'(z^l) \tag{6}$$

Now that we have this, we can define the error in terms of $w$ and $b$:

$$\frac{\partial cost}{\partial b_j^l} = \delta_j^l \tag{7}$$

$$\frac{\partial cost}{\partial w_{jk}^l} = a_k^{l-1} \delta_j^l \tag{8}$$

# 3 Algorithms

First we wan to apply the model to the input by making the data flow threw the network using the feed forward algorithm.

**Algorithm 1** FeedForward

**procedure** FEEDFORWARD($weights$, $biases$, $input$)
  $a = input$
  $as = [a]$
  $zs = []$

  **for** $w$, $b$ in ($weights$, $biases$) **do**
    $z = w \times a + b$
    append($zs$, $z$)
    $a = \text{act}(z)$
    append($as$, $a$)
  **end for**
  **return** $as$, $zs$
**end procedure**

Then we compute the gradiant of all the nodes using the back propagation (propagate the result error back in the network).

**Algorithm 2** Back propagation algorithm

**procedure** BACKPROPAGATE($as$, $zs$, $weights$, $biases$)
  $deltas = \text{cost'}(as[len(as) - 1], expected\_solution) \odot \text{act'}(zs[len(zs) - 1])$
  $grads\_b[L - 1] = deltas$
  $grads\_w[L - 1] = \text{matmul}(deltas, as[len(as) - 2])$

  **for** $l$ in $2.. < L$ **do**
    $deltas = \text{matmul}(\text{T}(weights[L - l + 1]), deltas) \odot \text{act'}(zs[len(zs) - l])$
    $grads\_b[L - l] = deltas$
    $grads\_w[L - l] = \text{matmul}(deltas, as[len(as) - l - 1])$
  **end for**
  **return** $grads\_w$, $grads\_b$
**end procedure**

The last component we need is the optimize function that will update the weights and the biases. There are various ways to do this, here is an example for a stochastic gradiant descant:

**Algorithm 3** Optimization for a stochastic gradiant descant

**procedure** OPTIMIZE($weights$, $biases$, $grads\_w$, $grads\_b$, $l\_rate$)
  **for** $l$ in $0.. < L$ **do**
    $weights[l]$ -= $l\_rate \times grads\_w[l]$
    $biases[l]$ -= $l\_rate \times grads\_b[l]$
  **end for**
  **return** $weights$, $biases$
**end procedure**

To train, we usually use minibatches. This means that we compute the average error on all the minibatches before updating the weights and biases.

**Algorithm 4** Update the weights using a minibatch

```
procedure UpdateMinibatch(weights, biases, minibatches, l_rate)
    grads_w_total = 0
    grads_b_total = 0

    for minibatch in minibatches do
        as, zs = FeedForward(weights, biases, minibatch)
        grads_w, grads_b = BackPropagate(as, zs, weights, biases)
        grads_w_total += grads_w
        grads_b_total += grads_b
    end for
    grads_w = grads_w_total / size(minibatches)
    grads_b = grads_b_total / size(minibatches)
    weights, biases = Optimize(weights, biases, grads_w, grads_b, l_rate)

    return weights, biases
end procedure
```

# 4 Proofs

### 4.0.1 Error on the output layer $\delta_j^L$

Let's prove the following:

$$\delta_j^L = \frac{\partial cost}{\partial a_j^L} act'(z_j^L) \tag{9}$$
$$= \frac{\partial cost}{\partial a_j^L} \frac{\partial act}{\partial z_j^L} \tag{10}$$

The definition of the error is:

$$\delta_j^L = \frac{\partial cost}{\partial z_j^L} \tag{11}$$
$$= \sum_k \frac{\partial cost}{\partial a_k^L} \frac{\partial a_k^L}{\partial z_j^L} \tag{12}$$

Since $a_k^L = act(z_k^L)$, if $j \neq k$, then $\frac{\partial a_k^L}{\partial z_j^L} = 0$. Which gives the result expression:

$$\delta_j^L = \frac{\partial cost}{\partial a_j^L} \frac{\partial a_j^L}{\partial z_j^L} \tag{13}$$

### 4.0.2 Error on internal layers $\delta_j^l$

$\delta_j^l$ is defined as:

$$\delta_j^l = \frac{\partial cost}{\partial z_j^l} \tag{14}$$
$$= \sum_k \frac{\partial cost}{\partial z_k^{l+1}} \frac{\partial z_k^{l+1}}{\partial z_j^l} \tag{15}$$
$$= \sum_k \delta_k^{l+1} \frac{\partial z_k^{l+1}}{\partial z_j^l} \tag{16}$$

We can then define $z_k^{l+1}$ as:

$$z_k^{l+1} = \sum_j w_{kj}^{l+1} act(z_j^l) + b_k^{l+1} \tag{17}$$

so the derivative is:

$$\frac{z_k^{l+1}}{z_j^l} = w_{kj}^{l+1} act'(z_j^l) \tag{18}$$

Substituing back to the original formula:

$$\delta_j^l = \sum_k w_{kj}^{l+1} \delta_k^{l+1} act'(z_j^l) \tag{19}$$

### 4.0.3 Bias gradiant $\frac{\partial cost}{\partial b_j^l}$

We prove the following:

$$\frac{\partial cost}{\partial b_j^l} = \delta_j^l \tag{20}$$

We can rewrite it like the following:

$$\frac{\partial cost}{\partial b_j^l} = \sum_k \frac{\partial cost}{\partial z_k^{l+1}} \frac{\partial z_k^{l+1}}{\partial z_j^l} \frac{\partial z_j^l}{\partial b_j^l} \tag{21}$$

since $\frac{\partial z_j^l}{\partial b_j^l} = 1$, then:

$$\begin{align} \frac{\partial cost}{\partial b_j^l} &= \sum_k \frac{\partial cost}{\partial z_k^{l+1}} \frac{\partial z_k^{l+1}}{\partial z_j^l} \tag{22} \\ &= \delta_j^l \tag{23} \end{align}$$

### 4.0.4 Bias gradiant $\frac{\partial cost}{\partial w_{jk}^l}$

We prove the following:

$$\frac{\partial cost}{\partial w_{jk}^l} = a_k^{l-1} \delta_j^l \tag{24}$$

We can rewrite it like the following:

$$\frac{\partial cost}{\partial w_{jk}^l} = \sum_k \frac{\partial cost}{\partial z_k^{l+1}} \frac{\partial z_k^{l+1}}{\partial z_j^l} \frac{\partial z_j^l}{\partial w_{jk}^l} \tag{25}$$

since $\frac{\partial z_j^l}{\partial w_{jk}^l} = a_k^{l-1}$, then:

$$\begin{align} \frac{\partial cost}{\partial w_{jk}^l} &= \sum_k \frac{\partial cost}{\partial z_k^{l+1}} \frac{\partial z_k^{l+1}}{\partial z_j^l} a_k^{l-1} \tag{26} \\ &= \delta_j^l a_k^{l-1} \tag{27} \end{align}$$