Fully connected NN

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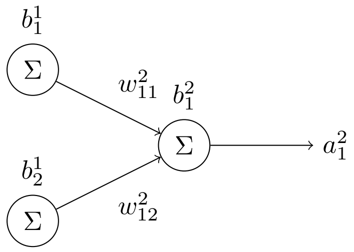
1 Notations

- w^l_{jk} is the weight of the k^{th} neuron in the layer $(l-1)$ to the j^{th} neuron in the layer l
- b^l_j is the bias of the j^{th} neuron on the layer l.
- a^l_j is the activation of the j^{th} neuron on the layer l.
- act is the activation function.
- $cost$ is the cost function.
- δ^l_j is the error of the j^{th} neuron on the layer l.
- L is the last layer.



2 Definitions

Transition from the layer $(l-1)$ to the layer l:

$$z_j^l = \sum_k w_{jk}^l a_k^{l-1} + b_j^l \tag{1}$$

$$a_j^l = act(z_j^l) \tag{2}$$

The error of a neuron can be define as:

$$\delta_j^l \equiv \frac{\partial cost}{\partial z_j^l} \tag{3}$$

We define δ_j^L the error on the output layer as the following:

$$\delta_j^L = \frac{\partial cost}{\partial a_j^l} act'(z_j^L) \tag{4}$$

Then we have δ^L which is a vector that contains the errors of all the output neurons:

$$\delta^L = \nabla_a cost \odot act'(z^L) \tag{5}$$

Here, we can deduce the value of δ^l:

$$\delta^l = ((w^{l+1})^T \delta^{l+1}) \odot act'(z^l) \tag{6}$$

Now that we have this, we can define the error in terms of w and b:

$$\frac{\partial cost}{\partial b_j^l} = \delta_j^l \tag{7}$$

$$\frac{\partial cost}{\partial w_{jk}^l} = a_k^{l-1} \delta_j^l \tag{8}$$

3 Algorithms

First we wan to apply the model to the input by making the data flow threw the network using the feed forward algorithm.

Algorithm 1 FeedForward

procedure FEEDFORWARD($weights$, $biases$, $input$)
  $a = input$
  $as = [a]$
  $zs = []$

  for w, b in ($weights$, $biases$) **do**
    $z = w \times a + b$
    append(zs, z)
    $a = \text{act}(z)$
    append(as, a)
  end for
  return as, zs
end procedure

Then we compute the gradiant of all the nodes using the back propagation (propagate the result error back in the network).

Algorithm 2 Back propagation algorithm

procedure BACKPROPAGATE(as, zs, $weights$, $biases$)
  $deltas = \text{cost'}(as[L-1], expected\_solution) \odot \text{act'}(zs[L-1])$
  $grads\_b[L-1] = deltas$
  $grads\_w[L-1] = \text{matmul}(deltas, \text{act'}(a[L-2]))$

  for l in $(L-2)..=1$ **do**
    $deltas = \text{matmul}(weights[l+1], deltas) \odot \text{act'}(zs[l])$
    $grads\_b[l] = deltas$
    $grads\_w[l] = \text{matmul}(deltas, \text{act'}(a[l-1]))$
  end for
  return $grads\_w$, $grads\_b$
end procedure

The last component we need is the optimize function that will update the weights and the biases. There are various ways to do this, here is an example for a stochastic gradiant descant:

Algorithm 3 Optimization for a stochastic gradiant descant

procedure OPTIMIZE($weights$, $biases$, $grads\_w$, $grads\_b$, l_rate)
  for l in 0..¡L **do**
    $weights[l]$ -= $l\_rate \times grads\_w[l]$
    $biases[l]$ -= $l\_rate \times grads\_b[l]$
  end for
  return $weights$, $biases$
end procedure

To train, we usually use minibatches. This means that we compute the average error on all the minibatches before updating the weights and biases.

Algorithm 4 Update the weights using a minibatch

```
procedure UpdateMinibatch(weights, biases, minibatches, l_rate)
    grads_w_total = 0
    grads_b_total = 0

    for minibatch in minibatches do
        as, zs = FeedForward(weights, biases, minibatch)
        grads_w, grads_b = BackPropagate(as, zs, weights, biases)
        grads_w_total += grads_w
        grads_b_total += grads_b
    end for
    grads_w = grads_w_total / size(minibatches)
    grads_b = grads_b_total / size(minibatches)
    weights, biases = Optimize(weights, biases, grads_w, grads_b, l_rate)

    return weights, biases
end procedure
```

4 Proofs

4.0.1 Error on the output layer δ_j^L

Let's prove the following:

$$\delta_j^L = \frac{\partial cost}{\partial a_j^L} act'(z_j^L) \tag{9}$$

$$= \frac{\partial cost}{\partial a_j^L} \frac{\partial act}{\partial z_j^L} \tag{10}$$

The definition of the error is:

$$\delta_j^L = \frac{\partial cost}{\partial z_j^L} \tag{11}$$

$$= \sum_k \frac{\partial cost}{\partial a_k^L} \frac{\partial a_k^L}{\partial z_j^L} \tag{12}$$

Since $a_k^L = act(z_k^L)$, if $j \neq k$, then $\frac{\partial a_k^L}{\partial z_j^L} = 0$. Which gives the result expression:

$$\delta_j^L = \frac{\partial cost}{\partial a_j^L} \frac{\partial a_j^L}{\partial z_j^L} \tag{13}$$

4.0.2 Error on internal layers δ_j^l

δ_j^l is defined as:

$$\delta_j^l = \frac{\partial cost}{\partial z_j^l} \tag{14}$$

$$= \sum_k \frac{\partial cost}{\partial z_k^{l+1}} \frac{\partial z_k^{l+1}}{\partial z_j^l} \tag{15}$$

$$= \sum_k \delta_k^{l+1} \frac{\partial z_k^{l+1}}{\partial z_j^l} \tag{16}$$

We can then define z_k^{l+1} as:

$$z_k^{l+1} = \sum_j w_{kj}^{l+1} act(z_j^l) + b_k^{l+1} \tag{17}$$

so the derivative is:

$$\frac{z_k^{l+1}}{z_j^l} = w_{kj}^{l+1} act'(z_j^l) \tag{18}$$

Substituing back to the original formula:

$$\delta_j^l = \sum_k w_{kj}^{l+1} \delta_k^{l+1} act'(z_j^l) \tag{19}$$

4.0.3 Bias gradiant $\frac{\partial cost}{\partial b_j^l}$

We prove the following:

$$\frac{\partial cost}{\partial b_j^l} = \delta_j^l \tag{20}$$

We can rewrite it like the following:

$$\frac{\partial cost}{\partial b_j^l} = \sum_k \frac{\partial cost}{\partial z_k^{l+1}} \frac{\partial z_k^{l+1}}{\partial z_j^l} \frac{\partial z_j^l}{\partial b_j^l} \tag{21}$$

since $\frac{\partial z_j^l}{\partial b_j^l} = 1$, then:

$$\begin{align} \frac{\partial cost}{\partial b_j^l} &= \sum_k \frac{\partial cost}{\partial z_k^{l+1}} \frac{\partial z_k^{l+1}}{\partial z_j^l} \tag{22} \\ &= \delta_j^l \tag{23} \end{align}$$

4.0.4 Bias gradiant $\frac{\partial cost}{\partial w_{jk}^l}$

We prove the following:

$$\frac{\partial cost}{\partial w_{jk}^l} = a_k^{l-1} \delta_j^l \tag{24}$$

We can rewrite it like the following:

$$\frac{\partial cost}{\partial w_{jk}^l} = \sum_k \frac{\partial cost}{\partial z_k^{l+1}} \frac{\partial z_k^{l+1}}{\partial z_j^l} \frac{\partial z_j^l}{\partial w_{jk}^l} \tag{25}$$

since $\frac{\partial z_j^l}{\partial w_{jk}^l} = a_k^{l-1}$, then:

$$\begin{align} \frac{\partial cost}{\partial w_{jk}^l} &= \sum_k \frac{\partial cost}{\partial z_k^{l+1}} \frac{\partial z_k^{l+1}}{\partial z_j^l} a_k^{l-1} \tag{26} \\ &= \delta_j^l a_k^{l-1} \tag{27} \end{align}$$