Chapter 3

The Fundamental Theorem of Calculus

In this chapter we will formulate one of the most important results of calculus, the Fundamental Theorem. This result will link together the notions of an integral and a derivative. Using this result will allow us to replace the technical calculations of Chapter 2 by much simpler procedures involving antiderivatives of a function.

3.1 The definite integral

In Chapter 2, we defined the definite integral, I, of a function $f(x) > 0$ on an interval $[a, b]$ as the area under the graph of the function over the given interval $a \le x \le b$. We used the notation

$$I = \int_a^b f(x)dx$$

to represent that quantity. We also set up a technique for computing areas: the procedure for calculating the value of I is to write down a sum of areas of rectangular strips and to compute a limit as the number of strips increases:

$$I = \int_a^b f(x)dx = \lim_{N\to\infty} \sum_{k=1}^{N} f(x_k)\Delta x, \tag{3.1}$$

where N is the number of strips used to approximate the region, k is an index associated with the k'th strip, and $\Delta x = x_{k+1} - x_k$ is the width of the rectangle. As the number of strips increases ($N \to \infty$), and their width decreases ($\Delta x \to 0$), the sum becomes a better and better approximation of the true area, and hence, of the definite integral, I. Example of such calculations (tedious as they were) formed the main theme of Chapter 2 .

We can generalize the definite integral to include functions that are not strictly positive, as shown in Figure 3.1. To do so, note what happens as we incorporate strips corresponding to regions of the graph below the x axis: These are associated with negative values of the function, so that the quantity $f(x_k)\Delta x$ in the above sum would be negative for each rectangle in the "negative" portions of the function. This means that regions of the graph below the x axis will contribute negatively to the net value of I.

If we refer to A_1 as the area corresponding to regions of the graph of $f(x)$ above the x axis, and A_2 as the total area of regions of the graph under the x axis, then we will find that the value of the definite integral I shown above will be

$$I = A_1 - A_2.$$

Thus the notion of "area under the graph of a function" must be interpreted a little carefully when the function dips below the axis.

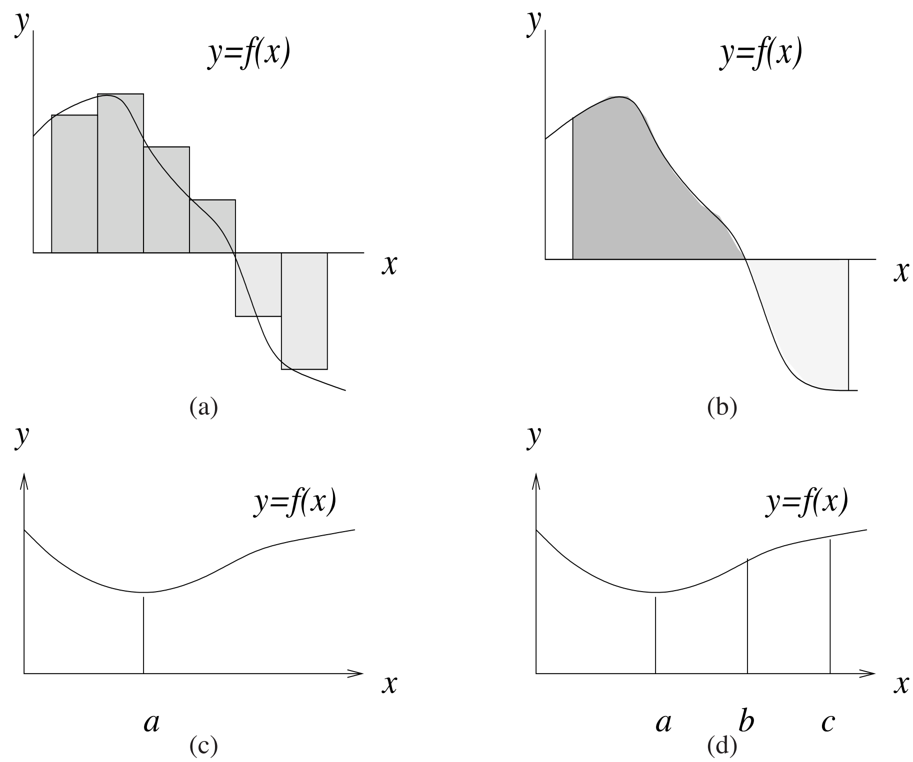

Figure 3.1. *(a) If $f(x)$ is negative in some regions, there are terms in the sum (3.1) that carry negative signs: this happens for all rectangles in parts of the graph that dip below the x axis. (b) This means that the definite integral $I = \int_a^b f(x)dx$ will correspond to the difference of two areas, $A_1 - A_2$ where A_1 is the total area (dark) of positive regions minus the total area (light) of negative portions of the graph. Properties of the definite integral: (c) illustrates Property 1. (d) illustrates Property 2.*

3.2 Properties of the definite integral

The following properties of a definite integral stem from its definition, and the procedure for calculating it discussed so far. For example, the fact that summation satisfies the distributive property means that an integral will satisfy the same the same property. We illustrate some of these in Fig 3.1.

1. $\displaystyle\int_a^a f(x)dx = 0,$

2. $\displaystyle\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx,$

3. $\displaystyle\int_a^b Cf(x)dx = C\int_a^b f(x)dx,$

4. $\displaystyle\int_a^b (f(x) + g(x))dx = \int_a^b f(x) + \int_a^b g(x)dx,$

5. $\displaystyle\int_a^b f(x)dx = -\int_b^a f(x)dx.$

Property 1 states that the "area" of a region with no width is zero. Property 2 shows how a region can be broken up into two pieces whose total area is just the sum of the individual areas. Properties 3 and 4 reflect the fact that the integral is actually just a sum, and so satisfies properties of simple addition. Property 5 is obtained by noting that if we perform the summation "in the opposite direction", then we must replace the previous "rectangle width" given by $\Delta x = x_{k+1} - x_k$ by the new "width" which is of opposite sign: $x_k - x_{k+1}$. This accounts for the sign change shown in Property 5.

3.3 The area as a function

In Chapter 2, we investigated how the area under the graph of a function changes as one of the endpoints of the interval moves. We defined a function that represents the area under the graph of a function f, from some fixed starting point, a to an endpoint x.

$$A(x) = \int_a^x f(t)\,dt.$$

This endpoint is considered as a variable[12], i.e. we will be interested in the way that this area changes as the endpoint varies (Figure 3.2(a)). We will now investigate the interesting connection between $A(x)$ and the original function, $f(x)$.

We would like to study how $A(x)$ changes as x is increased ever so slightly. Let $\Delta x = h$ represent some (very small) increment in x. (*Caution: do not confuse h with height here. It is actually a step size along the x axis.*) Then, according to our definition,

$$A(x+h) = \int_a^{x+h} f(t)\,dt.$$

[12]Recall that the "dummy variable" t inside the integral is just a "place holder", and is used to avoid confusion with the endpoint of the integral (x in this case). Also note that the value of $A(x)$ does not depend in any way on t, so any letter or symbol in its place would do just as well.

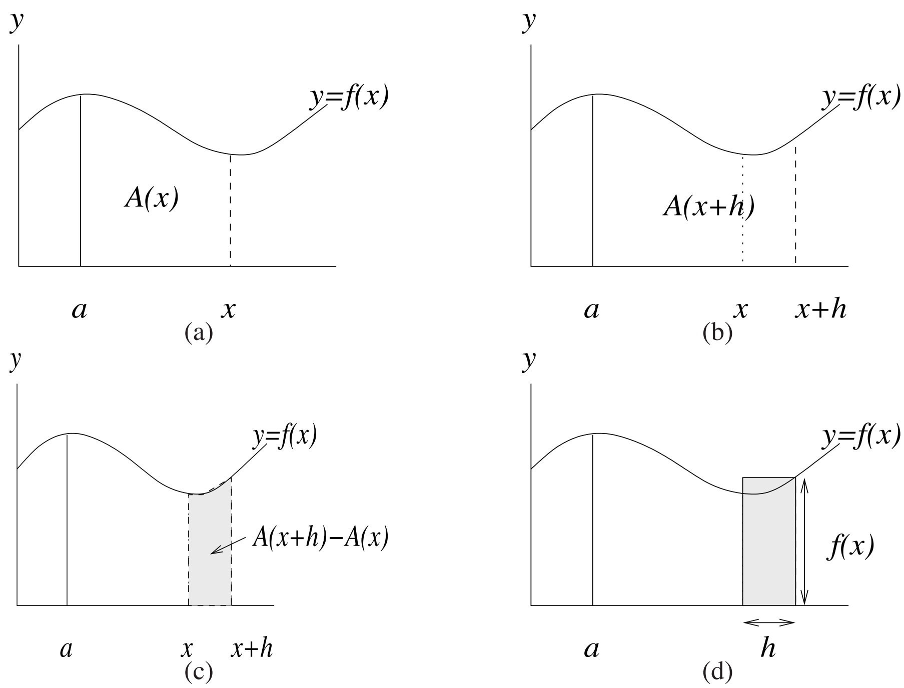

Figure 3.2. *When the right endpoint of the interval moves by a distance h, the area of the region increases from $A(x)$ to $A(x+h)$. This leads to the important Fundamental Theorem of Calculus, given in Eqn. (3.2).*

In Figure 3.2(a)(b), we illustrate the areas represented by $A(x)$ and by $A(x+h)$, respectively. The difference between the two areas is a thin sliver (shown in Figure 3.2(c)) that looks much like a rectangular strip (Figure 3.2(d)). (Indeed, if h is small, then the approximation of this sliver by a rectangle will be good.) The height of this sliver is specified by the function f evaluated at the point x, i.e. by $f(x)$, so that the area of the sliver is approximately $f(x) \cdot h$. Thus,

$$A(x+h) - A(x) \approx f(x)h$$

or

$$\frac{A(x+h) - A(x)}{h} \approx f(x).$$

As h gets small, i.e. $h \to 0$, we get a better and better approximation, so that, in the limit,

$$\lim_{h \to 0} \frac{A(x+h) - A(x)}{h} = f(x).$$

The ratio above should be recognizable. It is simply the derivative of the area function, i.e.

$$f(x) = \frac{dA}{dx} = \lim_{h \to 0} \frac{A(x+h) - A(x)}{h}. \tag{3.2}$$

We have just given a simple argument in support of an important result, called the *Fundamental Theorem of Calculus*, which is restated below..

3.4 The Fundamental Theorem of Calculus

3.4.1 Fundamental theorem of calculus: Part I

Let $f(x)$ be a bounded and continuous function on an interval $[a, b]$. Let

$$A(x) = \int_a^x f(t)\, dt.$$

Then for $a < x < b$,

$$\frac{dA}{dx} = f(x).$$

In other words, this result says that $A(x)$ is an "antiderivative" of the original function, $f(x)$[13].

Proof

See above argument. and Figure 3.2.

3.4.2 Example: an antiderivative

Recall the connection between functions and their derivatives. Consider the following two functions:

$$g_1(x) = \frac{x^2}{2}, \quad g_2 = \frac{x^2}{2} + 1.$$

Clearly, both functions have the same derivative:

$$g_1'(x) = g_2'(x) = x.$$

We would say that $x^2/2$ is an "antiderivative" of x and that $(x^2/2) + 1$ is also an "antiderivative" of x. In fact, *any* function of the form

$$g(x) = \frac{x^2}{2} + C \quad \text{where } C \text{ is any constant}$$

is also an "antiderivative" of x.

This example illustrates that adding a constant to a given function will not affect the value of its derivative, or, stated another way, antiderivatives of a given function are defined only up to some constant. We will use this fact shortly: if $A(x)$ and $F(x)$ are both antiderivatives of some function $f(x)$, then $A(x) = F(x) + C$.

[13] We often write "antiderivative", with no hyphen.

3.4.3 Fundamental theorem of calculus: Part II

Let $f(x)$ be a continuous function on $[a, b]$. Suppose $F(x)$ is *any* antiderivative of $f(x)$. Then for $a \le x \le b$,

$$A(x) = \int_a^x f(t)\,dt = F(x) - F(a).$$

Proof

From comments above, we know that a function $f(x)$ could have many different antiderivatives that differ from one another by some additive constant. We are told that $F(x)$ is an antiderivative of $f(x)$. But from Part I of the Fundamental Theorem, we know that $A(x)$ is also an antiderivative of $f(x)$. It follows that

$$A(x) = \int_a^x f(t)\,dt = F(x) + C, \quad \text{where } C \text{ is some constant.} \tag{3.3}$$

However, by property 1 of definite integrals,

$$A(a) = \int_a^a f(t) = F(a) + C = 0.$$

Thus,

$$C = -F(a).$$

Replacing C by $-F(a)$ in equation 3.3 leads to the desired result. Thus

$$A(x) = \int_a^x f(t)\,dt = F(x) - F(a).$$

Remark 1: Implications

This theorem has tremendous implications, because it allows us to use a powerful new tool in determining areas under curves. Instead of the drudgery of summations in order to compute areas, we will be able to use a shortcut: find an antiderivative, evaluate it at the two endpoints a, b of the interval of interest, and subtract the results to get the area. In the case of elementary functions, this will be very easy and convenient.

Remark 2: Notation

We will often use the notation

$$F(t)|_a^x = F(x) - F(a)$$

to denote the difference in the values of a function at two endpoints.

3.5 Review of derivatives (and antiderivatives)

By remarks above, we see that integration is related to "anti-differentiation". This motivates a review of derivatives of common functions. Table 3.1 lists functions $f(x)$ and their derivatives $f'(x)$ (in the first two columns) and functions $f(x)$ and their antiderivatives $F(x)$ in the subsequent two columns. These will prove very helpful in our calculations of basic integrals.

function	derivative		function	antiderivative
$f(x)$	$f'(x)$		$f(x)$	$F(x)$
Cx	C		C	Cx
x^n	nx^{n-1}		x^m	$\frac{x^{m+1}}{m+1}$
$\sin(ax)$	$a\cos(ax)$		$\cos(bx)$	$(1/b)\sin(bx)$
$\cos(ax)$	$-a\sin(ax)$		$\sin(bx)$	$-(1/b)\cos(bx)$
$\tan(ax)$	$a\sec^2(ax)$		$\sec^2(bx)$	$(1/b)\tan(bx)$
e^{kx}	ke^{kx}		e^{kx}	e^{kx}/k
$\ln(x)$	$\frac{1}{x}$		$\frac{1}{x}$	$\ln(x)$
$\arctan(x)$	$\frac{1}{1+x^2}$		$\frac{1}{1+x^2}$	$\arctan(x)$
$\arcsin(x)$	$\frac{1}{\sqrt{1-x^2}}$		$\frac{1}{\sqrt{1-x^2}}$	$\arcsin(x)$

Table 3.1. *Common functions and their derivatives (on the left two columns) also result in corresponding relationships between functions and their antiderivatives (right two columns). In this table, we assume that* $m \neq -1, b \neq 0, k \neq 0$. *Also, when using* $\ln(x)$ *as antiderivative for* $1/x$, *we assume that* $x > 0$.

As an example, consider the polynomial

$$p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

This polynomial could have many other terms (or even an infinite number of such terms, as we discuss much later, in Chapter 10). Its antiderivative can be found easily using the "power rule" together with the properties of addition of terms. Indeed, the antiderivative is

$$F(x) = C + a_0 x + \frac{a_1}{2}x^2 + \frac{a_2}{3}x^3 + \frac{a_3}{4}x^4 + \dots$$

This can be checked easily by differentiation[14].

3.6 Examples: Computing areas with the Fundamental Theorem of Calculus

3.6.1 Example 1: The area under a polynomial

Consider the polynomial

$$p(x) = 1 + x + x^2 + x^3.$$

(Here we have taken the first few terms from the example of the last section with coefficients all set to 1.) Then, computing

$$I = \int_0^1 p(x)\, dx$$

leads to

$$I = \int_0^1 (1 + x + x^2 + x^3)\, dx = (x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4)\Big|_0^1 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \approx 2.083.$$

3.6.2 Example 2: Simple areas

Determine the values of the following definite integrals by finding antiderivatives and using the Fundamental Theorem of Calculus:

1. $I = \displaystyle\int_0^1 x^2\, dx,$

2. $I = \displaystyle\int_{-1}^1 (1 - x^2)\, dx,$

3. $I = \displaystyle\int_{-1}^1 e^{-2x}\, dx,$

4. $I = \displaystyle\int_0^\pi \sin\left(\frac{x}{2}\right)\, dx,$

Solutions

1. An antiderivative of $f(x) = x^2$ is $F(x) = (x^3/3)$, thus

$$I = \int_0^1 x^2 dx = F(x)\Big|_0^1 = (1/3)(x^3)\Big|_0^1 = \frac{1}{3}(1^3 - 0) = \frac{1}{3}.$$

[14] In fact, it is very good practice to perform such checks.

2. An antiderivative of $f(x) = (1 - x^2)$ is $F(x) = x - (x^3/3)$, thus

$$I = \int_{-1}^{1} (1-x^2)\,dx = F(x)\Big|_{-1}^{1} = \left(x - (x^3/3)\right)\Big|_{-1}^{1} = \left(1 - (1^3/3)\right) - \left((-1) - ((-1)^3/3)\right) = 4/3$$

See comment below for a simpler way to compute this integral.

3. An antiderivative of e^{-2x} is $F(x) = (-1/2)e^{-2x}$. Thus,

$$I = \int_{-1}^{1} e^{-2x}\,dx = F(x)\Big|_{-1}^{1} = (-1/2)(e^{-2x})\Big|_{-1}^{1} = (-1/2)(e^{-2} - e^{2}).$$

4. An antiderivative of $\sin(x/2)$ is $F(x) = -\cos(x/2)/(1/2) = -2\cos(x/2)$. Thus

$$I = \int_{0}^{\pi} \sin\left(\frac{x}{2}\right)\,dx = -2\cos(x/2)\Big|_{0}^{\pi} - 2(\cos(\pi/2) - \cos(0)) = -2(0-1) = 2.$$

Comment: The evaluation of Integral 2. in the examples above is tricky only in that signs can easily get garbled when we plug in the endpoint at -1. However, we can simplify our work by noting the symmetry of the function $f(x) = 1 - x^2$ on the given interval. As shown in Fig 3.3, the areas to the right and to the left of $x = 0$ are the same for the interval $-1 \le x \le 1$. This stems directly from the fact that the function considered is **even**[15]. Thus, we can immediately write

$$I = \int_{-1}^{1} (1 - x^2)\,dx = 2\int_{0}^{1} (1 - x^2)\,dx = 2\left(x - (x^3/3)\right)\Big|_{0}^{1} = 2\left(1 - (1^3/3)\right) = 4/3$$

Note that this calculation is simpler since the endpoint at $x = 0$ is trivial to plug in.

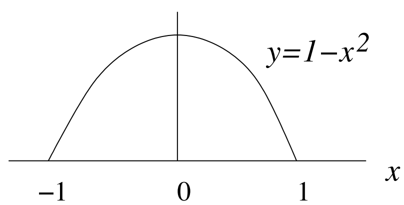

Figure 3.3. *We can exploit the symmetry of the function $f(x) = 1 - x^2$ in the second integral of Examples 3.6.2. We can integrate over $0 \le x \le 1$ and double the result.*

We state the general result we have obtained, which holds true for any function with even symmetry integrated on a symmetric interval about $x = 0$:

If $f(x)$ is an **even** function, then

$$\int_{-a}^{a} f(x)\,dx = 2\int_{-a}^{a} f(x)\,dx \tag{3.4}$$

[15]Recall that a function $f(x)$ is **even** if $f(x) = f(-x)$ for all x. A function is **odd** if $f(x) = -f(-x)$.

3.6.3 Example 3: The area between two curves

The definite integral is an area of a somewhat special type of region, i.e., an axis, two vertical lines ($x = a$ and $x = b$) and the graph of a function. However, using additive (or subtractive) properties of areas, we can generalize to computing areas of other regions, including those bounded by the graphs of two functions.

(a) Find the area enclosed between the graphs of the functions $y = x^3$ and $y = x^{1/3}$ in the first quadrant.

(b) Find the area enclosed between the graphs of the functions $y = x^3$ and $y = x$ in the first quadrant.

(c) What is the relationship of these two areas? What is the relationship of the functions $y = x^3$ and $y = x^{1/3}$ that leads to this relationship between the two areas?

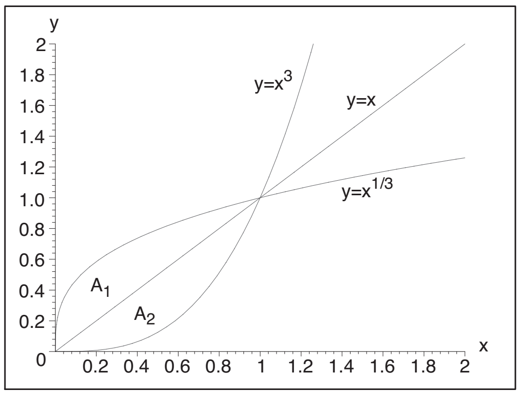

Figure 3.4. *In Example 3, we compute the areas A_1 and A_2 shown above.*

Solution

(a) The two curves, $y = x^3$ and $y = x^{1/3}$, intersect at $x = 0$ and at $x = 1$ in the first quadrant. Thus the interval that we will be concerned with is $0 < x < 1$. On this interval, $x^{1/3} > x^3$, so that the area we want to find can be expressed as:

$$A_1 = \int_0^1 \left(x^{1/3} - x^3\right) dx.$$

Thus,

$$A_1 = \left.\frac{x^{4/3}}{4/3}\right|_0^1 - \left.\frac{x^4}{4}\right|_0^1 = \frac{3}{4} - \frac{1}{4} = \frac{1}{2}.$$

(b) The two curves $y = x^3$ and $y = x$ also intersect at $x = 0$ and at $x = 1$ in the first quadrant, and on the interval $0 < x < 1$ we have $x > x^3$. The area can be represented as

$$A_2 = \int_0^1 \left(x - x^3\right) dx.$$

$$A_2 = \left.\frac{x^2}{2}\right|_0^1 - \left.\frac{x^4}{4}\right|_0^1 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}.$$

(c) The area calculated in (a) is twice the area calculated in (b). The reason for this is that $x^{1/3}$ is the inverse of the function x^3, which means geometrically that the graph of $x^{1/3}$ is the mirror image of the graph of x^3 reflected about the line $y = x$. Therefore, the area A_1 between $y = x^{1/3}$ and $y = x^3$ is twice as large as the area A_2 between $y = x$ and $y = x^3$ calculated in part (b): $A_1 = 2A_2$ (see Figure 3.4).

3.6.4 Example 4: Area of land

Find the exact area of the piece of land which is bounded by the y axis on the west, the x axis in the south, the lake described by the function $y = f(x) = 100 + (x/100)^2$ in the north and the line $x = 1000$ in the east.

Solution

The area is

$$A = \int_0^{1000} \left(100 + \left(\frac{x}{100}\right)^2\right) dx. = \int_0^{1000} \left(100 + \left(\frac{1}{10000}\right) x^2\right) dx.$$

Note that the multiplicative constant (1/10000) is not affected by integration. The result is

$$A = 100x\Big|_0^{1000} + \left.\frac{x^3}{3}\right|_0^{1000} \cdot \left(\frac{1}{10000}\right) = \frac{4}{3}10^5.$$

3.7 Qualitative ideas

In some cases, we are given a sketch of the graph of a function, $f(x)$, from which we would like to construct a sketch of the associated function $A(x)$. This sketching skill is illustrated in the figures shown in this section.

Suppose we are given a function as shown in the top left hand panel of Figure 3.5. We would like to assemble a sketch of

$$A(x) = \int_a^x f(t)dt$$

which corresponds to the area associated with the graph of the function f. As x moves from left to right, we show how the "area" accumulated along the graph gradually changes. (See $A(x)$ in bottom panels of Figure 3.5): We start with no area, at the point $x = a$ (since, by definition $A(a) = 0$) and gradually build up to some net positive amount, but then we encounter a portion of the graph of f below the x axis, and this subtracts from the amount accrued. (Hence the graph of $A(x)$ has a little peak that corresponds to the point at which $f = 0$.) Every time the function $f(x)$ crosses the x axis, we see that $A(x)$ has either a maximum or minimum value. This fits well with our idea of $A(x)$ as the antiderivative of $f(x)$: Places where $A(x)$ has a critical point coincide with places where $dA/dx = f(x) = 0$.

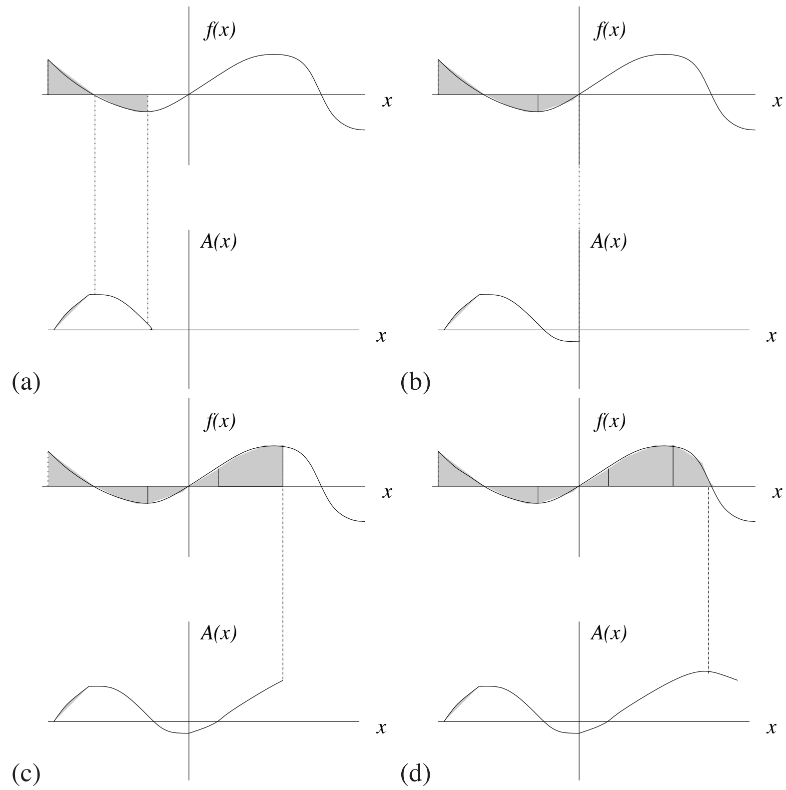

Figure 3.5. *Given a function $f(x)$, we here show how to sketch the corresponding "area function" $A(x)$. (The relationship is that $f(x)$ is the derivative of $A(x)$*

Sketching the function $A(x)$ is thus analogous to sketching a function $g(x)$ when we are given a sketch of its derivative $g'(x)$. Recall that this was one of the skills we built up in learning the connection between functions and their derivatives in a first semester calculus course.

Remarks

The following remarks may be helpful in gaining confidence with sketching the "area" function $A(x) = \int_a^x f(t)\,dt$, from the original function $f(x)$:

1. The endpoint of the interval, a on the x axis indicates the place at which $A(x) = 0$. This follows from Property 1 of the definite integral, i.e. from the fact that $A(a) = \int_a^a f(t)\,dt = 0$.

2. Whenever $f(x)$ is positive, $A(x)$ is an increasing function - this follows from the fact that the area continues to accumulate as we "sweep across" positive regions of $f(x)$.

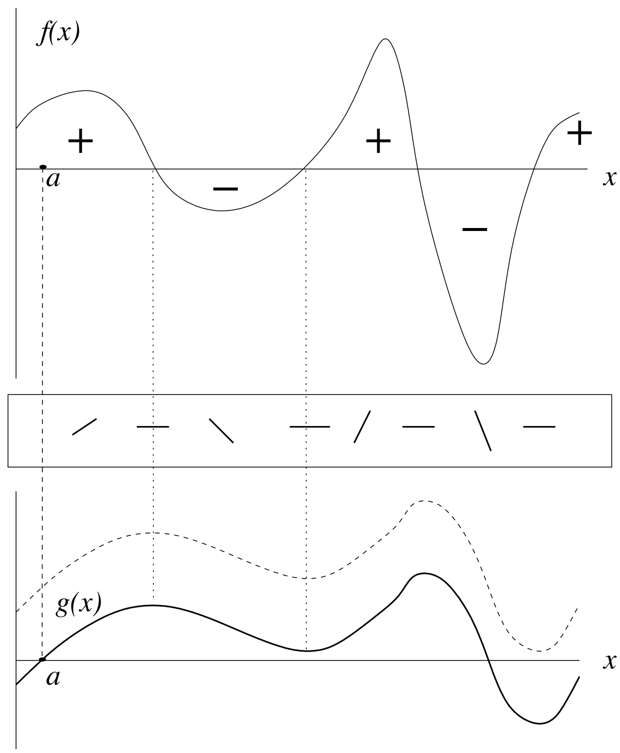

Figure 3.6. *Given a function $f(x)$ (top, solid line), we assemble a plot of the corresponding function $g(x) = \int_a^x f(t)dt$ (bottom, solid line). $g(x)$ is an antiderivative of $f(x)$. Whether $f(x)$ is positive (+) or negative (-) in portions of its graph, determines whether $g(x)$ is increasing or decreasing over the given intervals. Places where $f(x)$ changes sign correspond to maxima and minima of the function $g(x)$ (Two such places are indicated by dotted vertical lines). The box in the middle of the sketch shows configurations of tangent lines to $g(x)$ based on the sign of $f(x)$. Where $f(x) = 0$, those tangent lines are horizontal. The function $g(x)$ is drawn as a smooth curve whose direction is parallel to the tangent lines shown in the box. While the function $f(x)$ has many antiderivatives (e.g., dashed curve parallel to $g(x)$), only one of these satisfies $g(a) = 0$ as required by Property 1 of the definite integral. (See dashed vertical line at $x = a$). This determines the height of the desired function $g(x)$.*

3. Wherever $f(x)$, changes sign, the function $A(x)$ has a local minimum or maximum. This means that either the area stops increasing (if the transition is from positive to negative values of f), or else the area starts to increase (if f crosses from negative to positive values).

4. Since $dA/dx = f(x)$ by the Fundamental Theorem of Calculus, it follows that (tak-

ing a derivative of both sides) $d^2A/dx^2 = f'(x)$. Thus, when $f(x)$ has a local maximum or minimum, (i.e. $f'(x) = 0$), it follows that $A''(x) = 0$. This means that at such points, the function $A(x)$ would have an inflection point.

Given a function $f(x)$, Figure 3.6 shows in detail how to sketch the corresponding function

$$g(x) = \int_a^x f(t)dt.$$

3.7.1 Example: sketching $A(x)$

Consider the $f(x)$ whose graph is shown in the top part of Figure 3.7. Sketch the corresponding function $g(x) = \int_a^x f(x)dx$.

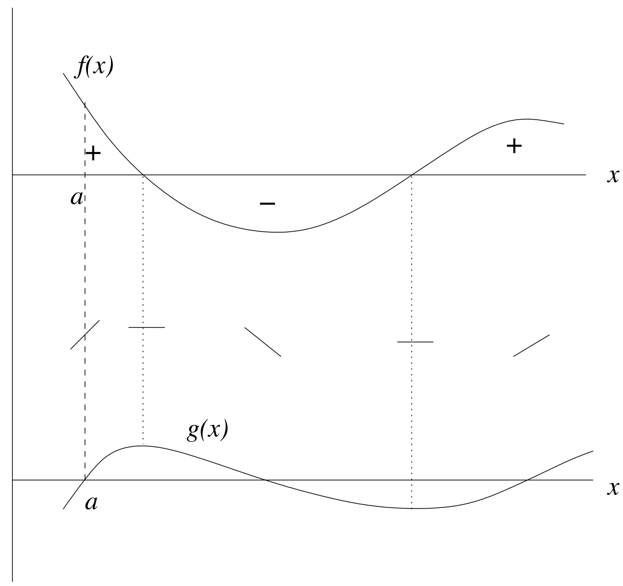

Figure 3.7. *The original functions, $f(x)$ is shown above. The corresponding functions $g(x)$ is drawn below.*

Solution

See Figure 3.7

3.8 Some fine print

The Fundamental Theorem has a number of restrictions that must be satisfied before its results can be applied. In this section we look at some examples in which care must be used.

3.8.1 Function unbounded I

Consider the definite integral

$$\int_0^2 \frac{1}{x}\, dx.$$

The function $f(x) = \frac{1}{x}$ is undefined at $x = 0$, and unbounded on any interval that contains the point $x = 0$. Hence, we cannot evaluate this integral using the Fundamental theorem, and indeed, we say that "*this integral does not exist*".

3.8.2 Function unbounded II

Consider the definite integral

$$\int_{-1}^1 \frac{1}{x^2} dx.$$

This function is also undefined (and hence not continuous) at $x = 0$. The Fundamental Theorem of Calculus cannot be applied. Technically, although one can "go through the motions" of computing an antiderivative, evaluating it at both endpoints, and getting a numerical answer, the result so obtained would be simply wrong. We say that his integral does not exist.

3.8.3 Example: Function discontinuous or with distinct parts

Suppose we are given the integral

$$I = \int_{-1}^2 |x|\, dx.$$

This function is actually made up of two distinct parts, namely

$$f(x) = \begin{cases} x & \text{if } x > 0 \\ -x & \text{if } x < 0. \end{cases}$$

The integral I must therefore be split up into two parts, namely

$$I = \int_{-1}^2 |x|\, dx = \int_{-1}^0 (-x)\, dx + \int_0^2 x\, dx.$$

We find that

$$I = -\left.\frac{x^2}{2}\right|_{-1}^0 + \left.\frac{x^2}{2}\right|_0^2 = -\left[0 - \frac{1}{2}\right] + \left[\frac{4}{2} - 0\right] = 2.5$$

3.8.4 Function undefined

Now let us examine the integral

$$\int_{-1}^1 x^{1/2}\, dx.$$

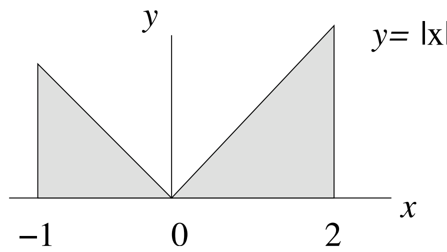

Figure 3.8. *In this example, to compute the integral over the interval $-1 \leq x \leq 2$, we must split up the region into two distinct parts.*

We see that there is a problem here. Recall that $x^{1/2} = \sqrt{x}$. Hence, the function is not defined for $x < 0$ and the interval of integration is inappropriate. Hence, this integral does not make sense.

3.8.5 Infinite domain ("improper integral")

Consider the integral

$$I = \int_0^b e^{-rx}\,dx, \quad \text{where } r > 0, \text{ and } b > 0 \text{ are constants.}$$

Simple integration using the antiderivative in Table 3.1 (for $k = -r$) leads to the result

$$I = \left.\frac{e^{-rx}}{-r}\right|_0^b = -\frac{1}{r}\left(e^{-rb} - e^0\right) = \frac{1}{r}\left(1 - e^{-rb}\right).$$

This is the area under the exponential curve between $x = 0$ and $x = b$. Now consider what happens when b, the upper endpoint of the integral increases, so that $b \to \infty$. Then the value of the integral becomes

$$I = \lim_{b\to\infty}\int_0^b e^{-rx}\,dx = \lim_{b\to\infty}\frac{1}{r}\left(1 - e^{-rb}\right) = \frac{1}{r}(1-0) = \frac{1}{r}.$$

(We used the fact that $e^{-rb} \to 0$ as $b \to \infty$.) We have, in essence, found that

$$I = \int_0^\infty e^{-rx}\,dx = \frac{1}{r}. \tag{3.5}$$

An integral of the form (3.5) is called an **improper integral**. Even though the domain of integration of this integral is infinite, $(0, \infty)$, observe that the value we computed is finite, so long as $r \neq 0$. Not all such integrals have a bounded finite value. Learning to distinguish between those that do and those that do not will form an important theme in Chapter 10.

Regions that need special treatment

So far, we have learned how to compute areas of regions in the plane that are bounded by one or more curves. In all our examples so far, the basis for these calculations rests on imagining rectangles whose heights are specified by one or another function. Up to now, all the rectangular strips we considered had bases (of width Δx) on the x axis. In Figure 3.9 we observe an example in which it would not be possible to use this technique. We are

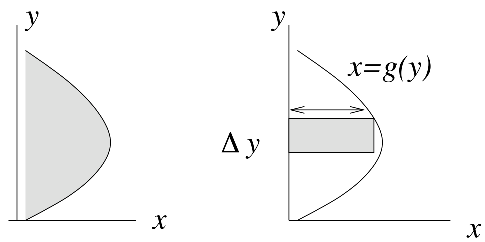

Figure 3.9. *The area in the region shown here is best computed by integrating in the y direction. If we do so, we can use the curved boundary as a single function that defines the region. (Note that the curve cannot be expressed in the form of a function in the usual sense, $y = f(x)$, but it can be expressed in the form of a function $x = f(y)$.)*

asked to find the area between the curve $y^2 - y + x = 0$ and the y axis. However, one and the same curve, $y^2 - y + x = 0$ forms the boundary from both the top and the bottom of the region. We are unable to set up a series of rectangles with bases along the x axis whose heights are described by this curve. This means that our definite integral (which is really just a convenient way of carrying out the process of area computation) has to be handled with care.

Let us consider this problem from a "new angle", i.e. with rectangles based on the y axis, we can achieve the desired result. To do so, let us express our curve in the form

$$x = g(y) = y - y^2.$$

Then, placing our rectangles along the interval $0 < y < 1$ on the y axis (each having base of width Δy) leads to the integral

$$I = \int_0^1 g(y)\, dy = \int_0^1 (y - y^2) dy = \left(\frac{y^2}{2} - \frac{y^3}{3} \right) \Bigg|_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}.$$

3.9 Summary

In this chapter we first recapped the definition of the definite integral in Section 3.1, recalled its connection to an area in the plane under the graph of some function $f(x)$, and examined its basic properties.

If one of the endpoints, x of the integral is allowed to vary, the area it represents, $A(x)$, becomes a function of x. Our construction in Figure 3.2 showed that there is a connection between the derivative $A'(x)$ of the area and the function $f(x)$. Indeed, we showed that $A'(x) = f(x)$ and argued that this makes $A(x)$ an antiderivative of the function $f(x)$.

This important connection between integrals and antiderivatives is the crux of Integral Calculus, forming the Fundamental Theorem of Calculus. Its significance is that finding areas need not be as tedious and labored as the calculation of Riemann sums that formed the bulk of Chapter 2. Rather, we can take a shortcut using antidifferentiation.

Motivated by this very important result, we reviewed some common functions and derivatives, and used this to relate functions and their antiderivatives in Table 3.1. We used these antiderivatives to calculate areas in several examples. Finally, we extended the treatment to include qualitative sketches of functions and their antiderivatives.

As we will see in upcoming chapters, the ideas presented here have a much wider range of applicability than simple area calculations. Indeed, we will shortly show that the same concepts can be used to calculate net changes in continually varying processes, to compute volumes of various shapes, to determine displacement from velocity, mass from densities, as well as a host of other quantities that involve a process of accumulation. These ideas will be investigated in Chapters 4, and 5.