Lab 4: RC and RL circuits: Time Domain Response

Reference Reading: Chapter 2, Sections 2.6, 2.7 and 2.8.

Goals:

1 Characterize the exponential response to step input voltage changes in RC and RL circuits.
2 Understand the meaning of a "characteristic time": $\tau = RC$ or $\tau = L/R$.
3 Understand two rules of thumb:
   a) One cannot instantaneously change the voltage across a capacitor.
   b) One cannot instantaneously change the current through an inductor.

Expectations:

1. As noted during lecture, you should show the full derivation of at least one of the time-dependent voltage equations
2. You essentially collect four data samples (downloaded to your computer) for this lab. You are expected to present this data in graphical form.
3. You are expected to do a detailed analysis of each of your data sets. This means that relevant equations and derivations should be in your lab book. An appropriate linearized plot for each data set should be presented, and fits to the plot should be presented.
4. A comparison of the results of your measurements and the expected results should be presented. If things are radically "wrong", you should look carefully to understand what is going on.

4.1 Introduction

We introduce two new circuit elements in this laboratory: capacitors and inductors. Together with resistors, these complete the list of two-terminal, linear devices which are commonly used in electronics.

In this and the following lab, we will study circuits with various combinations of resistors, capacitors, and inductors from two different points of view: the time domain and the frequency domain. In the time domain, we examine transient responses to sudden changes in applied voltages (the closing of a switch or the application of a step change in voltage). In the frequency domain, we will study the response to sinusoidal applied voltages as a function of frequency. The two points of view turn out to be entirely equivalent: complete knowledge of the behavior in one domain implies (with appropriate theory) complete knowledge of behavior in the other. Both points of view are useful throughout electronics, as well as being applicable to a wide variety of other physical systems.

4.1.1 RC circuit analysis

The voltage across a resistor-capacitor pair connected in series (see Figure 1) must equal the voltage across the capacitor plus the voltage across the resistor. If we write the voltages as a function of time (as opposed to functions of angular frequency ω), then we are using a time-domain treatment of the problem. The equation for the voltages in the RC circuit is

Equation 1 $$v(t) = v_C\,(t) + v_R\,(t).$$

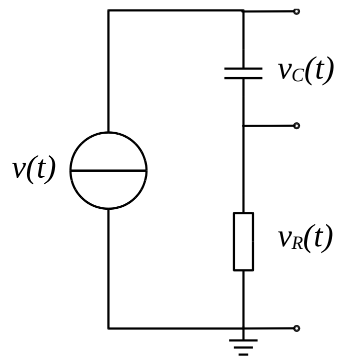

Figure 1 An AC source, v(t), driving an RC circuit. We can measure the output voltage across the capacitor, $v_C(t)$ ot the resistor, $v_R(t)$.

Using the relations $v_C(t) = q(t)/C$ and $v_R(t) = i(t)R$, this can be rewritten as:

Equation 2 $$v(t) = q(t)/C + i(t)R.$$

The charge, *q(t)*, on the capacitor and the current, *i(t)*, in the circuit loop are related by

Equation 3 $$q(t) = \int_0^t i(t')dt',$$

or

Equation 4 $$i(t) = dq(t)/dt.$$

Combining Equation 3 or Equation 4 with Equation 2, a variety of interesting limits can be found. For example, by taking the derivative of Equation 2, we get

$$dv(t)/dt = i(t)/C + R\,di(t)/dt.$$

We can express the right-hand side in terms of a single time-varying quantity, $v_R\,(t) = i(t)/R$

$$dv(t)/dt = (v_R(t))/RC + (dv_R(t))/dt.$$

If the voltage across the resistor changes slowly enough,

$$\frac{dv_R(t)}{dt} \ll \frac{v_R(t)}{RC},$$

or

$$\frac{1}{v_R(t)}\frac{dv_R(t)}{dt} \ll \frac{1}{RC},$$

we can neglect the second term on the right-hand side. Referring back to Equation 2, this amounts to requiring the voltage across *R*, $i(t)R$, to be small, so most of $v(t)$ appears across *C*. Under this assumption, we get

$$\frac{dv(t)}{dt} \approx \frac{v_R(t)}{RC}$$

Equation 5 $$v_{R(t)} \approx RC\ [dv(t)/dt]$$

Thus, for slowly-varying signals, the voltage across the resistor is proportional to the derivative of the input voltage. This is called a differentiating circuit since it will differentiate slowly varying input voltages.

As noted in our textbook, the quantity

Equation 6 $$\tau_{RC} \equiv RC$$

has units of time. We define this to be the *characteristic time*, τ_{RC}, of an *RC* circuit. This is a quantity that we will see many times during this course. This allows us to rewrite Equation 5 as

Equation 7 $$v_R(t) \approx \tau_{RC}\frac{dv}{dt}.$$

One can also take the integral of Equation 2 to get

$$\int_0^t v(t')dt' = \frac{1}{C}\int_o^t q(t')dt' + R\int_0^t i(t')dt',$$

and since the integral of the current is just the charge,

Equation 8
$$\int_0^t v(t')dt' = \frac{1}{C}\int_o^t q(t')dt' + q(t)R,$$

Looking back at Equation 2, we see that if the voltage on *C*, $q(t)/C$, is small, we can neglect the first term on the right. This means

$$\frac{q(t)}{C} \ll Ri(t)$$

To see what this inequality corresponds to, we differentiate it with respect to t and use $dq(t)/dt = i(t)$. This gives

$$\frac{di(t)}{dt} \gg \frac{i(t)}{RC}.$$

For this approximation to hold, the current (and thus the input signal) should be rapidly varying. Then neglecting the first term on the right-hand side in Equation 8, and dividing by RC, we get that

$$\frac{q(t)}{C} \approx \frac{1}{RC}\int_0^t v(t')dt',$$

or

$$v_C(t) \approx \frac{1}{RC}\int_0^t v(t')dt'.$$

This can be written in terms of the characteristic time as

Equation 9

$$v_C(t) \approx \frac{1}{\tau_{RC}}\int_0^t v(t')dt'.$$

Thus, we see that for rapidly varying input signals, the voltage across the capacitor, $v_C(t)$, is proportional to the integral of the input voltage, $v(t)$.

In this lab, we are going to consider a step-function input voltage where at time $t = 0$, the voltage instantaneously goes from zero to some value, V_0, and then remains at V_0 for all future times. Given this as an input voltage, we can solve the differential equation and find the following solutions for the voltages across the resistor and the capacitor for times $t > 0$.

Equation 10

$$v_R(t) = V_0 e^{-t/\tau_{RC}} \qquad (t > 0)$$

Equation 11

$$v_C(t) = V_0\left(1 - e^{-\frac{t}{\tau_{RC}}}\right) \qquad (t > 0)$$

We also note that, as expected, $v_R(t) + v_C(t) = V_0$ for any times $t > 0$. For a step-function input, we can also look at Equation 7 and Equation 9, which indicate that the voltage across the resistor should behave like the derivative of our step function, while that across the capacitor should behave like the integral. At first glance, this seems a bit crazy to claim that Equation 10 and Equation 11 behave like this. However, we have a couple of caveats on how rapidly the signal can be changing for these to work. In the case of the derivative, if we look at Equation 10 on time scales much larger than τ_{RC}, the exponential decay will look like a spike, which is the same shape as the derivative of the input signal. Similarly, if we look at Equation 11 on time scales small compared to τ_{RC}, then the output will look like a linearly rising voltage, which is the integral of the input signal with respect to time. Both of these are consistent with what we expect.

4.1.2 Analysis of RL circuits

The voltage across an inductor-resistor pair in series can be written as

$$v(t) = v_L(t) + v_R(t)$$

and can expand to yield

Equation 12

$$v(t) = L\frac{di(t)}{dt} + i(t)R\,.$$

Differentiating this equation, we get:

$$\frac{dv(t)}{dt} = L\frac{d^2i(t)}{dt^2} + R\frac{di(t)}{dt}$$

An analysis similar to that above for the capacitor yields integrating and differentiating circuits built using an inductor. For slowly varying inputs, most of the voltage is across the resistor,

$$L\frac{d^2i}{dt^2} \ll R\frac{di}{dt},$$

and the remaining voltage across the inductor satisfies the expression

$$v_L(t) = L\frac{di}{dt}$$

$$v_L(t) \approx \frac{L}{R}\frac{dv(t)}{dt}\,.$$

Dimensional analysis implies that

Equation 13

$$\tau_{LR} \equiv \frac{L}{R}$$

is the characteristic time constant for this *LR* circuit. Thus, we can write our voltage across the inductor as

Equation 14

$$v_L(t) \approx \tau_{LR} \frac{dv(t)}{dt} .$$

For rapidly varying inputs, most of the voltage is across the inductor,

$$L \frac{di}{dt} \gg iR ,$$

and the voltage across the resistor can be written as

Equation 15

$$v_R(t) = Ri(t)$$

Equation 16

$$v_R(t) \approx \frac{1}{\tau_{LR}} \int_0^t v(t')dt'$$

As we did with the capacitor, we can also consider a step-function input to this circuit. In the limit of a pure inductor, we would find a solution similar to what we did for the capacitor. As the current is changing rapidly at time $t = 0$, we would initially find all the voltage across the inductor, and none across the resistor. It would then decay away from the inductor and build up on the resistor, In the case of a physical inductor, there is not only an inductance L, but an internal resistance R_L as well. This means that at long times, we have a voltage divider with resistors R and R_L. To solve the equations, we define the characteristic time to be

Equation 17

$$\tau_{LR} = \frac{L}{R + R_L} .$$

Thus, we find

Equation 18

$$v_R(t) = \frac{R}{R + R_L} V_0 (1 - e^{-t/\tau_{LR}})$$

Equation 19

$$v_L(t) = \frac{R_L}{R + R_L} V_0 + \frac{R}{R + R_L} V_0 e^{-t/\tau_{LR}}$$

We note the subtle difference between these expressions and the corresponding ones for the capacitor, Equation 10 and Equation 11. In particular, we see the impact of the two resistors, *R* and R_L, leading to terms that come from the voltage-divider expression. The long-time limit of the voltage across the resistor is

$$v_R(t = \infty) = \frac{R}{R + R_L} V_0,$$

while for that across the inductor is

$$v_L(t = \infty) = \frac{R_L}{R + R_L} V_0 \, .$$

As with the capacitor, we have $v_R(t) + v_L(t) = V_0$ at all times.

4.2 Preliminary Lab Questions

The work in this section must be completed and signed off by an instructor before you start working on the lab. Do this work in your lab book.

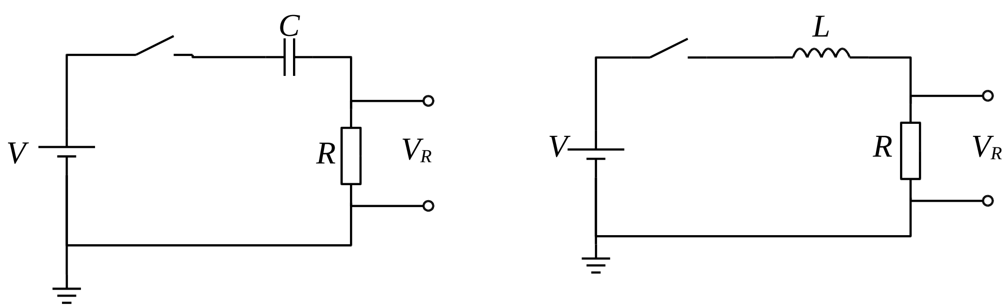

Figure 2 Figure for the pre-lab questions.

1) What are the characteristic time constants of the circuits shown above? Take R = 4.7 kΩ, C = 0.1 μF, L = 1 mH.
2) Of what order are the differential equations that govern the behavior of the above circuits? (The order of a differential equation is the order of the highest derivative in the equation, e.g. d^2i/dt^2 is second order.)
3) After a long time, which element (C or R) in the left-hand circuit will have a significant voltage across it? In the right-hand circuit? What about at the instant the switches are closed?

4.3 Equipment and Parts

In this lab, we will utilize the following equipment. This equipment is located at your lab station.

1) A Jameco DC power supply.
2) A Rigol DS1054Z digital oscilloscope.
3) Two probes for the oscilloscope.
4) One USB memory stick (bring your own).
5) A Rigol DG1022Z signal generator.
6) One BNC-to-alligator cable.
7) An Amprobe 37XR-A digital meter.
8) The Jameco proto-board.

You will also need the following components in order to carry out this lab. It makes more sense to get them as you need them, rather than all at once before the start of the lab.

1. Ones each of the following resistors: 470 Ω and 1 kΩ.
2. One additional resistor of your choice to match the resistance of the inductor.
3. One capacitor in the range of 20 pF to 30 pF.
4. One 0.33 μF capacitor.
5. One 500 mH inductor.

4.4 Procedure

4.4.1 Gotcha!

1. Is the DG1022Z set to be “High-Z”?
2. Is the voltage offset of the DG1022Z set to 0 V?
3. Is the “current limit” turned to the maximum value on your DC power supply?
4. Do you want to use AC or DC coupling on your oscilloscope? If you are interested in seeing a DC offset, then you need to DC couple. If you want to only see the time-varying part of a signal, then you want to AC couple.
5. Recall from Lab 2 that at very-low and very-high frequencies, AC coupling may attenuate your signals.

4.4.2 Time-domain response of RC circuits

Voltage across the resistor. Investigate the output characteristics of the circuit shown in Figure 3. In order to observe the complete transient response, you want to apply a step voltage and then hold this applied voltage for a time that is long compared to τ_{RC} (i.e., until the transient has died away). A ’transient’ is a temporary signal which exists as a system moves from one stable mode to another. In this case, the transient exists as the system moves from an uncharged capacitor and zero applied voltage across the capacitor to a constant applied voltage and a steady charge on the capacitor.

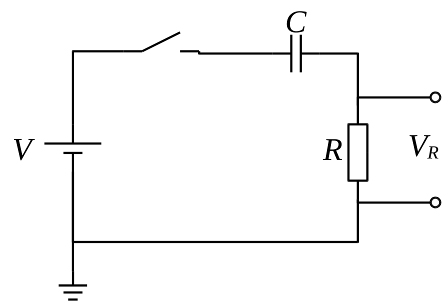

Figure 3 The setup to measure the time response of an RC circuit.

1) Set up the circuit shown using $C = 0.33\mu F$ and $R = 1k\Omega$. Be sure to measure the exact values of the components first.
2) Calculate the numerical value of the time constant, τ_{RC} for this circuit.
3) As the supply voltage, use 5 V from the DC supply at your experimental station.
4) Instead of using a switch, you can obtain a much better result by using the ends of two stripped wires as your "switch". You close your "switch" by rapidly brushing one of the stripped ends of the wire across the other wire. A variation that also works well is to brush the stripped end of the wire across the leg of one of the components (*R* or *C*) in your circuit.
5) Because the capacitor will charge up each time you close your switch, you will need to discharge it between each measurement. This is accomplished by shorting the two legs of the capacitor together with a wire.
6) You will have to set up the oscilloscope to take a "single shot" measurement of the transient. This will require some trial and error to make the triggering work and to be sure that you get the horizontal and vertical scales right.
   a) Set the time (horizontal) scale of the oscilloscope to a value that is of similar size to your expected τ_{RC}.
   b) Set the voltage (vertical) scale on channel one of your scope to something comparable to the input voltage that you are using.
   c) Adjust the trigger level on channel one of your scope to be a bit above zero.
   d) Press the single button on your oscilloscope. You should see the word "Ready" in yellow at the top of the display on the scope.
   e) Close your "switch" and the scope should capture an image.
   f) Adjust the time and voltage scales so your image reasonably fills the screen.
7) Continue trying to get a signal until you have one that is clean. This will likely take you several tries.
8) Once you have obtained a clean transient signal, download the waveform to your lab notebook. The Lab 04 Notebook Template.ipynb has code demonstrating how to do this.
9) It is smart to check the data you have downloaded by plotting it in your notebook.
10) Verify that the expected functional form and time constant are observed.
    a) To verify the functional form, make a plot that will transform the expected form to a straight line. To do this, note that if

$$v(t) = Ae^{-t/\tau_{RC}},$$

then

$$\ln[v(t)] = \ln[A] - \frac{t}{\tau_{RC}}.$$

b) The above equation implies a "natural" way to plot your data. Find a way to plot your data so that your data points form a straight line with slope $-1/\tau_{RC}$. What happens if you use log10(...) instead of ln(...) (log10 is a more customary way to plot data because humans have 10 fingers instead of 2.7182818...!)?

The input function to your circuit is a step function that turns on when you close the switch. When we are looking at the response of an RC circuit, we talk about times either long or short compared to the time $\tau_{RC} = RC$.

Question 4.1

What does the derivative of a step function look like? What part(s) of the observed output signal mimics the derivative of the input "signal"? Is this consistent with the above equations?

Voltage across the capacitor.

1. Reverse the positions of *R* and *C* in your circuit and measure the transient voltage response across *C*.
2. Graph and fit the voltage across *C* as a function of time, see Equation 20 and Equation 21.

Question 4.2

What does the integral of a step function look like? What part(s) of the observed output signal mimics the integral of the input "signal"? (Hint: think about τ_{RC}*).*

Measuring a fast transient. To see how well the scope works, try measuring the transient in a circuit with $R = 470\ \Omega$ and $C = 27$ pf (or similar values).

Question 4.3

What is the calculated time constant? Recall that the speed of light is 1 foot per nanosecond (excuse the non-metric units!)

1 With this value of a time constant, you aren't likely to be able to close a mechanical switch fast enough to make these voltage measurements. Instead, you should replace the D.C. power supply and your "wire switch" with the square-wave output of your DG1022Z Function Generator. This will serve to charge and discharge the capacitor repeatedly. From the RC time (τ_{RC}) of your circuit, choose square-wave frequency so that your circuit has a long time to reach equilibrium before the square-wave voltage reverses.

2 However, even the "square wave" can't make transitions on a time scale which is fast compared to RC in this case. Look at the shape of your square wave on the scope and

determine how long it takes the voltage to “ramp up” to a reasonably stable level. You only expect the simple RC behavior in the circuit once the input voltage has stabilized.

3 Now the circuit is not floating, because the DG1022Z output is grounded on one side. You’ll have to arrange the circuit so you can measure the desired voltage while still having the ground connection of the scope (black alligator clip) connected to the same part of the circuit as the ground connection of the function generator.

4 When you analyze the data, you will want to ignore the first part of the transient, taken when the input voltage was changing. Knowing the “ramp up” time allows you to decide which part of the data to discard.

5 Again, you should measure, plot, and fit the transient voltages across the resistor and across the capacitor. Compare your fitted value of τ_{RC} with what you expect from your measured component values.

Data Analysis: In this experiment, the primary data consist of a scope trace showing the time dependence of the voltage signal across a given component. The rather large amount of data from these traces are downloaded into your notebook and then manipulated. However, in doing this analysis, it is necessary to look at the data. There will be some part of the data that corresponds to t < 0, where the voltage across the components is zero. This is not useful for our analysis. There will also be part of the signal for large times where the voltage has fallen below the noise level. In this region, it will appear that the output voltage has become constant and no longer follows an exponential decay. These data should also be discarded from your analysis.

In the case of the resistor, the voltage should follow Equation 10, and it is easy to linearize this by taking the natural log of both sides. For the case of the capacitor, we have Equation 11. Here, simply taking the natural log of both sides leads to a mess. We first need to isolate the exponential piece of this equation.

Equation 20

$$v_C(t) = V_0(1 - e^{-t/\tau\_RC})$$

Equation 21

$$\frac{V_0 - v_C(t)}{V_0} = e^{-t/\tau_{RC}}$$

Then, taking the natural log of the latter equation will yield an equation linear in time, and allow us to fit for the slope, which is related to τ_{RC}.

4.4.3 Time-domain response of RL circuits

In this part of the lab, the circuit shown in Figure 4.4 is to be investigated. However, real inductors introduce a complication: they have an internal resistance, RL, that is generally non-negligible so we have to write

$$v(t) = v_L(t) + v_R(t)$$

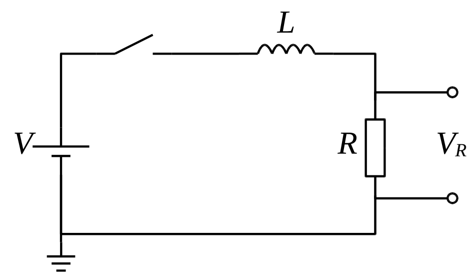

Figure 4 The setup to measure the time response of an RL circuit.

Equation 22

$$v(t) = \left[L\frac{di(t)}{dt} + i(t)R_L\right] + i(t)R$$

The internal resistance, R_L, is usually not shown in circuit diagrams, but must always be kept in mind. For example, you cannot directly measure the voltage across just the inductor; the apparent voltage is across the series combination of L and R_L. In fact, R_L is not just the DC resistance of the coiled wire either: The effective R_L includes all dissipative effects which remove energy from the circuit. This includes inductive heating in any magnetic core material around which the coil is wound and, at high frequencies, radiative effects as well. So, you will only measure across the resistor, R.

1. Measure the resistance and inductance of your inductor.
2. Set up the circuit shown in Figure 4. Use a series resistance (R in the diagram) that has a value comparable to R_L.
3. Measure the functional form and time constant of the voltage across R in the same way you did for the RC circuit using your “wire switch” and the DC power supply. Also, note the final voltage across R. Note that an inductor does not “charge up” like a capacitor, so there is nothing to discharge between measurements.
4. Save your data to your USB flash drive and verify that it has been saved. Fit your saved data and find the characteristic time of the circuit. Compare your fitted value of τ_{RL} with the expected value.

Question 4.4

What are the initial and final voltages across L?

5. What is the value of R_L? There are (at least) two ways to determine this from the data: from the time constant and from the final voltage across R. Use both of these methods to measure R_L.

Question 4.5

Do these measurements of R_L agree with each other? Do they agree with our ohmmeter measurement of R_L?

4.5 Additional Problems

After completing this lab, you should be able to answer the following questions.

1. Let us consider the ideal R_L circuit shown in Figure 5. The inductor has inductance L and is assumed to be an ideal inductor, meaning its internal resistance is zero. The resistor has resistance R, and the circuit is connected through a switch to a DC source whose voltage is V_0.

(a) At time t = 0, the switch is closed and current begins to flow through the circuit. Sketch the voltage between the points A and B as a function of time, $v_{AB}(t)$.

(b) In terms of the instantaneous current $i(t)$ in the circuit, what is the voltage across the resistor, $v_R(t)$, and across the inductor, $v_L(t)$?

(c) Our expressions in part (b) lead to the first-order differential equation for the current

$$0 = \frac{d}{dt}i(t) + \frac{R}{L}i(t) - \frac{V_0}{L}.$$

Consider a solution of the form

$$i(t) = i_0(1 - e^{-\alpha t}),$$

where i_0 and α are constants. What must i_0 and α be in terms of R, L and V_0 for the solution to work?

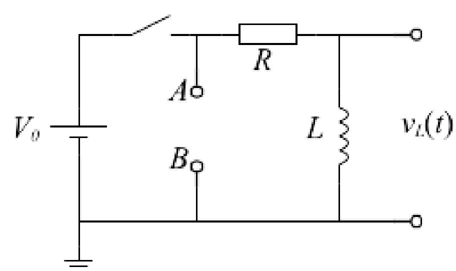

Figure 5 The circuit for problem 1.

2. *RC* circuits can be used as the timing mechanism in an electronic clock, where the resistor and capacitor are chosen to yield an appropriate τ_{RC}. For the circuit shown in Figure 6, we saw previously that the voltage across the resistor is given as

$$v_R(t) = V_0\, e^{-t/\tau_{RC}} \ ,$$

where *t* measures the time from when the switch is closed.

(a) In terms of τ_{RC}, how long does it take for the voltage across the resistor to fall from 2/3 of its maximum value to 1/3 of its maximum value?

(b) We would like to design a clock that has a period $T = 0.05$ s, and we are told that the time we found in part (a) represents ½ of the period. If we are using a 1 kΩ resistor, what value should we choose for our capacitor?

(c) What would you do if the only capacitors that are available are twice the value that you need?

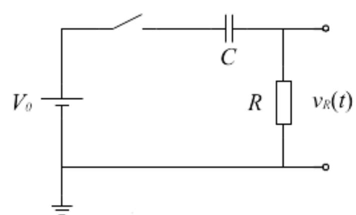

Figure 6 The circuit for problem 2.