Technische Universität München

MIMO Systems Programming Project 1

Instructions

Michael Newinger, M.Sc.
Dr.-Ing. Michael Joham

WS 2018/19

Fakultät für Elektrotechnik und Informationstechnik
Professur für Methoden der Signalverarbeitung
Univ.-Prof. Dr.-Ing. Wolfgang Utschick

Professur für Methoden der Signalverarbeitung
Technische Universität München
80290 München

http://www.msv.ei.tum.de

Contents

Chapter 1

Power Allocation for Point-to-Point MIMO Channels

In the first project, we analyze a point-to-point *multiple-input multiple-output* (MIMO) system as depicted in Fig. 1.1. An N-antenna transmitter sends the signal $\boldsymbol{x} \in \mathbb{C}^N$ via the channel $\boldsymbol{H} \in \mathbb{C}^{M \times N}$ to the M-antenna receiver. The distorted channel output is perturbed by zero-mean circularly symmetric complex additive Gaussian noise $\boldsymbol{n} \sim \mathcal{N}_{\mathbb{C}}(\mathbf{0}_M, \boldsymbol{C}_{\boldsymbol{n}})$ with covariance $\boldsymbol{C}_{\boldsymbol{n}} \in \mathbb{C}^{M \times M}$. The M-dimensional complex received signal reads as

$$\boldsymbol{y} = \boldsymbol{H}\boldsymbol{x} + \boldsymbol{n}. \tag{1.1}$$

For mutual information maximization with limited transmit power, a zero-mean transmit signal $\boldsymbol{x}$ with covariance matrix $\boldsymbol{Q} \in \mathbb{C}^N$ is optimal. The *eigenvalue decomposition* (EVD) of the transmit covariance reads as

$$\boldsymbol{Q} = \boldsymbol{V}\boldsymbol{\Psi}\boldsymbol{V}^{\mathrm{H}} = \boldsymbol{V}\operatorname{diag}(\psi_1, \ldots, \psi_N)\boldsymbol{V}^{\mathrm{H}}, \tag{1.2}$$

where the unitary matrix $\boldsymbol{V} \in \mathbb{C}^{N \times N}$ stems from the EVD

$$\boldsymbol{H}^{\mathrm{H}}\boldsymbol{C}_{\boldsymbol{n}}^{-1}\boldsymbol{H} = \boldsymbol{V}\boldsymbol{\Phi}\boldsymbol{V}^{\mathrm{H}} = \boldsymbol{V}\operatorname{diag}(\phi_1, \ldots, \phi_N)\boldsymbol{V}^{\mathrm{H}}. \tag{1.3}$$

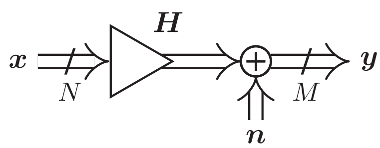

Figure 1.1: Point-to-Point MIMO System

The diagonal elements ψ_i, $i = 1, \ldots, N$, are the power allocations for the individual data streams. Their optimal values depend on the applied performance metric and the imposed transmit power requirements.

The specific aim of this programming project is a numerical analysis and comparison of various power allocation strategies. This comparison is based on the analysis in Problem 2.2 of the MIMO Systems tutorials. Afterwards, different transmission schemes are compared to a SISO system and the diversity gain that can be achieved is analyzed.

1.1 Comparison of Power Allocation Schemes

1.1.1 Mutual Information Maximization – Waterfilling Solution

The mutual information maximization for above MIMO point-to-point system is formulated as

$$\max_{\boldsymbol{Q}\succeq\boldsymbol{0}} \log_2 \det(\boldsymbol{I}_N + \boldsymbol{H}^{\mathrm{H}}\boldsymbol{C}_{\boldsymbol{n}}^{-1}\boldsymbol{H}\boldsymbol{Q}) \quad \text{s.t.} \quad \mathrm{tr}(\boldsymbol{Q}) \leq P_{\mathrm{Tx}}, \tag{1.4}$$

where P_{Tx} denotes the maximum average transmit power per channel use. The mutual information maximizing transmit covariance is derived in Section 2.2 of the lecture notes and has the structure as given in (1.2). The solution for ψ_i, $i = 1, \ldots, N$, is given by the waterfilling principle:

$$\begin{aligned} \psi_i &= \max\left(0, \mu' - \frac{1}{\phi_i}\right), \quad i = 1, \ldots, N \\ \sum_{i=1}^{N} \psi_i &= P_{\mathrm{Tx}}. \end{aligned} \tag{1.5}$$

The 'waterlevel' μ' depends on the transmit power, i.e., $\mu' = \frac{1}{K}\left(P_{\mathrm{Tx}} + \sum_{i=1}^{K}\frac{1}{\phi_i}\right)$, where K denotes the number of active streams, that is, $\psi_i > 0$ for $i \in \{1, \ldots, K\}$ and $\psi_i = 0$ for $i > K$, if the eigenvalues ϕ_i, $i = 1, \ldots, N$, are non-increasingly ordered ($\phi_i \geq \phi_j$ for $i \leq j$). The capacity of the MIMO channel can then be found as

$$C = \sum_{i=1}^{N} \log_2(1 + \phi_i \psi_i). \tag{1.6}$$

PROGRAMMING TASK 1

Implement a function that takes the eigenvalues $\phi_1, \ldots, \phi_N$ and the available sum transmit power P_{Tx} as inputs and that computes the optimal waterfilling solution given in (1.5).

1 **Deliverables (Matlab code file):** `waterfilling.m`
- Function definition:
  `function [psi,mu,K] = waterfilling(phi,Ptx)`

2 **Input Specifiations:**
- `phi`: vector of eigenvalues $\phi_1, \ldots, \phi_N$
- `Ptx`: available transmit power

3 **Output Specifications:**
- `psi`: vector of optimal power allocations $\psi_1^\star, \ldots, \psi_N^\star$
- `mu`: value of the optimal waterlevel $\mu'^\star$
- `K`: number of non-zero data streams K

4 **Hint:** Make sure that the ordering of the output vector `psi` corresponds to that of the input vector `phi`.

For a comparison with other power allocation strategies, the values for P_{Tx} shall be determined where the waterfilling solution switches from K to $K+1$ active streams.

QUESTION 1

Give an expression for P_{Tx} as a function of $\phi_i, i = 1, \ldots, N$, where the waterfilling solution switches from K to $K+1$ active streams. What is the optimal power allocation of the waterfilling solution for $P_{\text{Tx}} \to \infty$?

PROGRAMMING TASK 2

Implement a function that takes the eigenvalues $\phi_1, \ldots, \phi_N$ and calculates the power values P_{Tx} where the waterfilling solution switches from K to $K+1$ active streams for $K = 1, \ldots, N-1$.

1 **Deliverables (Matlab code file):** `activeStreams_waterfilling.m`
- `function [Ptx] = activeStreams_waterfilling(phi)`

2 **Input Specifiations:** `phi`: vector of eigenvalues $\phi_1, \ldots, \phi_N$

3 **Output Specifications:** `Ptx`: $(N-1) \times 1$ array containing power values where the allocation switches from K to $K+1$ active streams.

For the next measurement task, consider a system with $N = M = 4$ transmit and receive antennas as given in the file `example_channels.mat` and use the function `activeStreams_waterfilling.m`.

QUESTION 2

Give the transmit powers P_{Tx} where the waterfilling solution switches from K to $K+1$ active streams via calling `[Ptx] = activeStreams_waterfilling(phi)` for $K = 1, \dots, 3$.

1.1.2 Optimal Uniform Power Allocation

Besides the waterfilling power allocation, the simple but suboptimal uniform power allocation scheme was considered in Problem 2.2 of the MIMO Systems tutorials. In this scheme, the transmit power is uniformly distributed among the K largest effective channel eigenvalues, i.e.,

$$\psi_i = \begin{cases} P_{\mathrm{Tx}}/K & i \in \{1, \dots, K\}, \\ 0 & i \in \{K+1, \dots, N\}. \end{cases} \tag{1.7}$$

The number of active streams K shall be chosen to maximize the achievable rate.

QUESTION 3

Rewrite the achievable rate R_K for K active data streams, i.e.

$$R_K = \log_2 \det(\boldsymbol{I}_N + \boldsymbol{H}^{\mathrm{H}} \boldsymbol{C_n}^{-1} \boldsymbol{H}\boldsymbol{Q}),$$

in terms of ϕ_i and P_{Tx} when the uniform power allocation from (1.7) is used.

A natural approach to find the rate maximizing K is to test all possibilities in a predefined order. For example, start with $K = 1$, determine whether $R_K > R_{K+1}$ and properly adjust K for the next iteration if the eigenvalues ϕ_i, $i = 1, \dots, N$ are decreasingly ordered.

PROGRAMMING TASK 3

Implement the uniform power allocation function that takes the values of the eigenvalues $\phi_1, \dots, \phi_N$ and the available sum transmit power P_{Tx} as inputs and computes the rate maximizing power allocation based on (1.7).

1 Deliverables: `uniform_rate.m`

- Function definitions:
  `function [psi,K] = uniform_rate(phi,Ptx)`

2 Input Specifications:

- `phi`: vector of eigenvalues $\phi_1, \ldots, \phi_N$
- `Ptx`: available transmit power

3 Output Specifications:

- `psi`: vector of optimal power allocations $\psi_1^\star, \ldots, \psi_N^\star$
- `K`: number of non-zero data streams K

To compare the general behavior of the uniform power allocations with the waterfilling solution, measure the range of P_{Tx} where K active streams are optimal in terms of the mutual information maximization. To this end, consider the system given in `example_channels.mat` with $M = N = 4$ transmit and receive antennas and use the provided function `maxpower_Kstreams.m`.[1]

QUESTION 4

Measure the transmit powers when the function `uniform_rate.m`, switches from K to $K+1$ streams by calling `maxpower_Kstreams(phi,K,'uniform_rate')`. Compare the results with those of the waterfilling solution and answer the following questions:

- Which of the two power allocations switches earlier from K to $K+1$ active streams?
- Is the rate optimal uniform power allocation asymptotically optimal, i.e., does the uniform power allocation achieve the same rate as the waterfilling solution for $P_{\text{Tx}} \to \infty$? Justify your answer.

1.1.3 MSE Optimal Power Allocation with Transmit Filter Design

Consider the Point-to-Point MIMO system depicted in Fig. 1.2 with the i.i.d. Gaussian distributed data signal $\boldsymbol{s} \sim \mathcal{N}_{\mathbb{C}}(\mathbf{0}, \mathbf{I})$ that is precoded with the precoder $\boldsymbol{T} \in \mathbb{C}^{N \times B}$ and transmitted via the channel $\boldsymbol{H} \in \mathbb{C}^{B \times N}$. The received signal is perturbed by Gaussian noise $\boldsymbol{n} \sim \mathcal{N}_{\mathbb{C}}(\mathbf{0}, \boldsymbol{C}_{\boldsymbol{n}})$. The receive filter $\boldsymbol{G} = g\mathbf{I}$ with $g \in \mathbb{C}$ performs

[1] The function `maxpower_Kstreams.m` iteratively performs the power allocation strategy to find the maximum P_{Tx} for the desired K active streams.

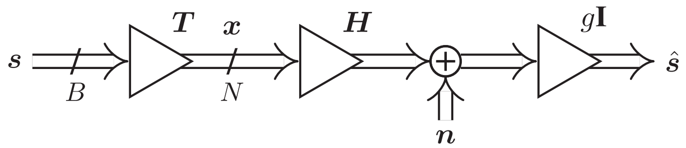

Figure 1.2: Point-to-Point MIMO System

a scaling of the received signal. The noise and data signals are uncorrelated, i.e. $\mathrm{E}\left[\boldsymbol{s}\boldsymbol{n}^{\mathrm{H}}\right] = \mathbf{0}$.

In this problem, the *mean squared error* (MSE) $\varepsilon(\boldsymbol{T}, g) = \mathrm{E}\left[\|\boldsymbol{s} - \hat{\boldsymbol{s}}\|_2^2\right]$ will be minimized with respect to the precoder and equalizer subject to a power constraint

$$\min_{\boldsymbol{T},g} \mathrm{E}\left[\|\boldsymbol{s} - \hat{\boldsymbol{s}}\|_2^2\right] \quad \text{s.t.} \quad \mathrm{E}\left[\|\boldsymbol{x}\|_2^2\right] \leq P_{\mathrm{Tx}}. \tag{1.8}$$

QUESTION 5

Give an expression for $\hat{\boldsymbol{s}}$ as a function $\boldsymbol{s}$ and $\boldsymbol{n}$ and calculate $\varepsilon(\boldsymbol{T}, g)$ as a function of $\boldsymbol{T}$, g, and $\boldsymbol{C}_{\boldsymbol{n}}$.

For a given precoder $\boldsymbol{T}$, the optimal scaling factor g can be found via the unconstrained optimization problem

$$\min_{g} \varepsilon(\boldsymbol{T}, g). \tag{1.9}$$

QUESTION 6

Calculate the optimal choice for g.
Hint: $\varepsilon(\boldsymbol{T}, g)$ is convex in g.

For a given equalizer $g\mathbf{I}$, the optimal precoder $\boldsymbol{T}$ can be found by solving the constrained optimization problem

$$\min_{\boldsymbol{T}} \varepsilon(\boldsymbol{T}, g) \quad \text{s.t.} \quad \mathrm{E}\left[\|\boldsymbol{x}\|_2^2\right] \leq P_{\mathrm{Tx}}. \tag{1.10}$$

The optimization problem in (1.10) can be shown to be (hidden) convex. The *Karush-Kuhn-Tucker* (KKT) conditions are therefore necessary and sufficient.

QUESTION 7

Rewrite $\mathrm{E}\left[\|\boldsymbol{x}\|_2^2\right]$ as a function of the precoder $\boldsymbol{T}$ only and give the Lagrangian function $\mathcal{L}(\boldsymbol{T}, \mu)$ for the optimization problem in (1.10), where $\mu \geq 0$ is the Lagrangian multiplier corresponding to the power constraint. State the KKT conditions.

QUESTION 8

Give an expression for $\boldsymbol{T}$ as a function of g and μ.
Hint: Assume that $\mu > 0$.

In order to find μ, the dual feasibility condition can be rewritten to

$$-|g|^2 \operatorname{tr}(\boldsymbol{C_n}) + \mu \operatorname{tr}(\boldsymbol{T}\boldsymbol{T}^{\mathrm{H}}) = 0. \tag{1.11}$$

QUESTION 9

Use (1.11) to find an expression for μ as a function of $\boldsymbol{C_n}$, g, and P_{Tx}, only. Show that $\boldsymbol{T}$ can be written as

$$\boldsymbol{T} = \frac{1}{g}\left(\boldsymbol{H}^{\mathrm{H}}\boldsymbol{H} + \frac{\operatorname{tr}(\boldsymbol{C_n})}{P_{\mathrm{Tx}}}\mathbf{I}\right)^{-1}\boldsymbol{H}^{\mathrm{H}}.$$

Use the power constraint to find an expression for g as a function of $\boldsymbol{H}$, $\boldsymbol{C_n}$, and P_{Tx}.
Hint: Assume that the power constraint $\mathrm{E}\left[\|\boldsymbol{x}\|_2^2\right] \leq P_{\mathrm{Tx}}$ is fulfilled with equality.

PROGRAMMING TASK 4

Implement a function that takes a channel $\boldsymbol{H}$, the noise covariance matrix $\boldsymbol{C_n}$, and the available sum transmit power P_{Tx} as inputs and that computes the achievable rate for the precoder and receive filter determined in Tasks 6 and 9.

1 Deliverables (Matlab code file): `tf_mmseallocation.m`

- Function definition:

```
function [R ] = tf_mmseallocation(H,Cn,Ptx)
```

2 Input Specifiations:

- `H`: Channel
- `Cn`: Noise covariance matrix
- `Ptx`: available transmit power

3 Output Specifications:

- `R`: Achievable rate

1.1.4 Graphical Comparison of Power Allocation Schemes

To visualize the achievable performance of the discussed power allocation schemes, a script shall be created in the next programming task for plotting the achievable mutual information, versus the average total transmit power P_{Tx} in *decibels* (dB).

PROGRAMMING TASK 5

Modify the Matlab script `rate_visualization.m` in order to plot the achievable rate versus P_{Tx} in dB for above power allocation schemes. For the waterfilling and uniform power allocation, mark those values for P_{Tx} where the power allocation switches from K to $K+1$ active streams in the plots.

1 Deliverables: `rate_visualization.m`

2 Input Specifications:

- `N`: number of transmit/receive antennas
- `Ptx_dB`: array of available transmit power in dB

3 Internal Specifications:

- `R_waterfilling`: array of achievable rates for the waterfilling solution allocation
- `R_uniform`: array of achievable rates for the uniform power allocation
- `R_tf`: array of achievable rates for the MSE minimizing power allocation with transmit filter design.

4 Hint(s):

- First, transform the array of dB values for P_{Tx} into the actual transmit powers. Use the channel from `example_channels.mat` and determine $\phi_1, \ldots, \phi_N$ and calculate $\psi_1, \ldots, \psi_N$ for the various schemes. Based on these results, calculate the achievable rates. Finally, calculate the P_{Tx} values (in dB) and the corresponding rates where the waterfilling scheme and the uniform power allocation increase their number of active streams and mark these points in the plots.

Based on the plots for these scripts, the achieved rate curves of the various power allocation schemes shall be compared for the described system in `example_channels.mat`.

> **QUESTION 10**
>
> Run the Matlab script `rate_visualization.m` to create a plot for the achievable rate with the various power allocation schemes and answer the following questions:
>
> - What are the slopes of the waterfilling and the MMSE power allocation curves for P_{Tx} equal to 30 dB?
> - What is the distance in dB between the waterfilling curve and the MMSE curve for a rate of 20 bits per channel use?
> - Explain the high SNR behavior of the uniform power allocation and the MMSE solution with transmit filter design, only.

1.2 Average Mutual Information Maximization

An important question for the system design of MIMO point-to-point systems is what average rate is achievable for a certain channel model, e.g., a Rayleigh fading channel. That is, the focus is on the maximization of the average achievable rate $\bar{R} = \mathrm{E}[R(\boldsymbol{H}, \boldsymbol{Q}(\boldsymbol{H}))]$, where

$$R(\boldsymbol{H}, \boldsymbol{Q}(\boldsymbol{H})) = \log_2 \det\left(\boldsymbol{I}_N + \boldsymbol{H}^{\mathrm{H}} \boldsymbol{C}_{\boldsymbol{n}}{}^{-1} \boldsymbol{H} \boldsymbol{Q}(\boldsymbol{H})\right) \tag{1.12}$$

is a function of the realization of $\boldsymbol{H}$ and the choice of $\boldsymbol{Q}(\boldsymbol{H})$. The optimization is compared for two types of transmit power limitations. An average power constraint that reads as

$$\mathrm{E}\big[\mathrm{tr}\big(\boldsymbol{Q}(\boldsymbol{H})\big)\big] \le \bar{P}_{\text{Tx}} \tag{1.13}$$

where the expectation is over the channel realizations, and the instantaneous, i.e., per channel realization, transmit power constraint

$$\mathrm{tr}\big(\boldsymbol{Q}(\boldsymbol{H})\big) \le P_{\text{Tx}} = \bar{P}_{\text{Tx}}. \tag{1.14}$$

1.2.1 Average versus Instantaneous Transmit Power Constraint

For an average transmit power limitation as in (1.13), the average mutual information maximization problem reads as

$$\max_{\boldsymbol{Q}(\boldsymbol{H}) \succeq \boldsymbol{0}} \quad \mathrm{E}[R(\boldsymbol{H}, \boldsymbol{Q}(\boldsymbol{H}))] \quad \text{s.t.} \quad \mathrm{E}\big[\mathrm{tr}\big(\boldsymbol{Q}(\boldsymbol{H})\big)\big] \le \bar{P}_{\text{Tx}}. \tag{1.15}$$

This optimization problem is a variational problem and difficult to solve in closed form. For the programming project, the expectation evaluation shall be computed via numerical integration by a Monte-Carlo method. That is, we draw a large number L of realizations $\boldsymbol{H}_l$, $l = 1, \dots, L$ from the probability density function $f_{\boldsymbol{H}}(\boldsymbol{H})$ and approximate the expectation operations via the sample means. The resulting approximate problem formulation reads as

$$\max_{\boldsymbol{Q}(\boldsymbol{H}_l)\succeq \boldsymbol{0}\ \forall l=1,\dots,L} \quad \frac{1}{L}\sum_{l=1}^{L} R(\boldsymbol{H}_l, \boldsymbol{Q}(\boldsymbol{H}_l)) \quad \text{s.t.} \quad \frac{1}{L}\sum_{l=1}^{L} \operatorname{tr}\big(\boldsymbol{Q}(\boldsymbol{H}_l)\big) \le \bar{P}_{\text{Tx}}. \tag{1.16}$$

The optimal choice for the transmit covariance matrices is

$$\boldsymbol{Q}(\boldsymbol{H}_l) = \boldsymbol{V}_l \operatorname{diag}(\psi_{1,l}, \dots, \psi_{N,l}) \boldsymbol{V}_l^{\mathrm{H}}, \tag{1.17}$$

where the unitary matrix $\boldsymbol{V}_l \in \mathbb{C}^{N\times N}$ is the modal matrix of the eigenvalue decomposition $\boldsymbol{H}_l^{\mathrm{H}} \boldsymbol{C}_{\boldsymbol{n}}^{-1} \boldsymbol{H}_l = \boldsymbol{V}_l \operatorname{diag}(\phi_{1,l}, \dots, \phi_{N,l}) \boldsymbol{V}_l^{\mathrm{H}}$.

QUESTION 11

Use (1.12) and (1.17) to rewrite the optimization problem in (1.16) in terms of the eigenvalues $\phi_{i,l}$ and $\psi_{i,l}$. Use the similarities to the mutual information maximization problem in Section 1.1.1 to find the optimal solution for $\psi_{i,l}$

PROGRAMMING TASK 6

Implement a function that takes the values of the eigenvalues $\phi_{1,1}, \dots, \phi_{N,L}$ of the L channel realizations and the available average transmit power $\bar{P}_{\text{Tx}}$ as inputs and that computes the optimal waterfilling solution obtained in Question 11, the used per channel transmit powers $P_{\text{Tx},l} = \sum_{i=1}^{N} \psi_{i,l}$, and the maximum average achievable rate.

1 Deliverables (Matlab code file): `averageRate_averagePtx.m`

- Function definition:
```
function [averageRate,powers]
      = averageRate_averagePtx(Phi,averagePtx)
```

2 Input Specification:

- `Phi`: $N \times L$ matrix of eigenvalues $\phi_{1,1}, \dots, \phi_{N,l}$; the l-th column contains the values $\phi_{1,l}, \dots, \phi_{N,l}$

- `averagePtx`: available average transmit power $\bar{P}_{\text{Tx}}$

3 Output Specification:

- `average_rate`: maximum average achievable rate $\bar{R}$
- `powers`: vector of used transmit powers per channel realization $P_{\text{Tx},1}, \ldots, P_{\text{Tx},L}$

4 Hint(s):

- Use the function `waterfilling.m` to calculate the optimal choice of $\psi_{1,1}, \ldots, \psi_{N,L}$. The function shall also work for $N = 1$.

For the instantaneous transmit power limitation as given in (1.14), the average rate maximization problem reads as

$$\max_{\boldsymbol{Q}(\boldsymbol{H}_l)\succeq\boldsymbol{0}\ \forall l=1,\ldots,L} \quad \mathrm{E}\big[R(\boldsymbol{H}_l, \boldsymbol{Q}(\boldsymbol{H}_l))\big] \quad \text{s.t.} \quad \operatorname{tr}\big(\boldsymbol{Q}(\boldsymbol{H}_l)\big) \leq \bar{P}_{\text{Tx}}, \quad l = 1, \ldots, L. \tag{1.18}$$

When evaluating the expectation via the Monte-Carlo method with L randomly drawn channel realizations, (1.18) is approximated as

$$\max_{\boldsymbol{Q}(\boldsymbol{H}_l)\succeq\boldsymbol{0}\ \forall l=1,\ldots,L} \quad \frac{1}{L}\sum_{l=1}^{L} R(\boldsymbol{H}_l, \boldsymbol{Q}(\boldsymbol{H}_l)) \quad \text{s.t.} \quad \operatorname{tr}\big(\boldsymbol{Q}(\boldsymbol{H}_l)\big) \leq \bar{P}_{\text{Tx}}, \quad l = 1, \ldots, L. \tag{1.19}$$

Since there is no joint constraint on the covariance matrices $\boldsymbol{Q}_l(\boldsymbol{H}_l)$, $l = 1, \ldots, L$, the sample mean rate is maximized via separately maximizing each summand $R(\boldsymbol{H}_l, \boldsymbol{Q}(\boldsymbol{H}_l))$ in the objective function. That is, the optimal transmit covariance matrices are $\boldsymbol{Q}(\boldsymbol{H}_l) = \boldsymbol{V}_l \mathrm{diag}(\psi_{1,l}, \ldots, \psi_{N,l})\boldsymbol{V}_l^{\mathrm{H}}$, where the unitary matrix $\boldsymbol{V}_l \in \mathbb{C}^{N\times N}$ is the modal matrix of the eigenvalue decomposition $\boldsymbol{H}_l^{\mathrm{H}}\boldsymbol{C}_{\boldsymbol{n}}^{-1}\boldsymbol{H}_l = \boldsymbol{V}_l \mathrm{diag}(\phi_{1,l}, \ldots, \phi_{N,l})\boldsymbol{V}_l^{\mathrm{H}}$ and the diagonal elements $\psi_{i,l}$, $i = 1, \ldots, N$ are the results of the standard waterfilling principle in (1.5).

PROGRAMMING TASK 7

Write a function that takes the values $\phi_{1,1}, \ldots, \phi_{N,L}$ and the available average transmit power $\bar{P}_{\text{Tx}}$ as inputs and that solves the approximate mutual information maximization with instantaneous transmit power constraints in (1.19) and computes the average achievable rate.

1 Deliverables (Matlab code file): `averageRate_perchannelPtx.m`

- Function definition:

```
function [averageRate] =
     averageRate_perchannelPtx(Phi,averagePtx)
```

2 Input Specification:

- `Phi`: $N \times L$ matrix of eigenvalues $\phi_{1,1}, \ldots, \phi_{N,L}$; the l-th column contains the values $\phi_{1,l}, \ldots, \phi_{N,l}$
- `averagePtx`: available average transmit power $\bar{P}_{\text{Tx}}$

3 Output Specification:

- `average_rate`: maximum average achievable rate $\bar{R}$

4 Hint(s):

- Use the function `waterfilling.m` to calculate the optimal choice of $\psi_{1,1}, \ldots, \psi_{N,L}$. The function shall also work for $N = 1$.

For a comparison of the achievable average rates in (1.16) and (1.19) with average and instantaneous power constraints, respectively, a set of random channels can be generated with the provided function `ChannelsForAverageRateMaximization.m`. The function takes an array containing the number of transmit and receive antennas for which the channels shall be generated and number of channels to be generated as input and returns a cell array. With the script `averageRateMaximization.m`, a comparison shall be performed. Therein, the implemented functions are evaluated for four point-to-point systems with $N = 2, 4, 8, 16$ transmit and receive antennas and transmit power limitations -10 dB $\leq \bar{P}_{\text{Tx}} \leq 20$ dB.

QUESTION 12

Run the Matlab script `averageRateMaximization.m` to create a plot for the average achievable rates over $\bar{P}_{\text{Tx}}$ in dB. What is the difference in bits per channel use between the average rate curves for the average and the instantaneous transmit power constraints for $\bar{P}_{\text{Tx}} = 10\,\text{dB}$ and $N = 2, 4, 8, 16$? What conclusion can be drawn from this observation for $\operatorname{tr}\big(\boldsymbol{Q}(\boldsymbol{H}_l)\big)$, $l = 1, \ldots, L$, of the average rate maximization with average transmit power constraint in (1.15)?

To verify the drawn conclusion, the empirical *cumulative distribution function* (CDF) of the resulting per channel transmit powers $\operatorname{tr}\big(\boldsymbol{Q}(\boldsymbol{H}_l)\big)$, $l = 1, \ldots, L$ of the solution to (1.16) shall be plotted and compared for various choices of N. The following Matlab script, that needs to be modified accordingly, shall create the plot.

PROGRAMMING TASK 8

Modify the Matlab script `CDFplot_perChannelPtx.m` that takes $\bar{P}_{\mathrm{Tx}}$, L, and the column vector $\boldsymbol{N}_{\mathrm{arr}}$ that combines the integer values for N as parameters, and calculates the transmit powers $\bar{P}_{\mathrm{Tx},l} = \mathrm{tr}\big(\boldsymbol{Q}(\boldsymbol{H}_l)\big)$, $l = 1, \ldots, L$ for $\boldsymbol{Q}(\boldsymbol{H}_l)$ being the solution to (1.16).

1 Deliverables (Matlab code file): `CDFplot_perChannelptx.m`

2 Parameter Specification:

- `Narr`: vector for number of transmit/receive antennas $\boldsymbol{N}_{\mathrm{arr}}$
- `L`: number of Monte-Carlo channel realizations L
- `meanPtx_dB`: available average transmit power $\bar{P}_{\mathrm{Tx}}$ in dB

3 Internal Specification:

- `meanPtx`: available average transmit power $\bar{P}_{\mathrm{Tx}}$ (not in dB)
- `P`: $L \times 8$ array of the resulting per channel transmit powers $\bar{P}_{\mathrm{Tx},l}$

4 Hint(s):

- Use the implemented function `averageRate_averagePtx.m` for this task.

QUESTION 13

Run the Matlab script `CDFplot_perChannelPtx.m` for $\bar{P}_{\mathrm{Tx}} = 10\,\mathrm{dB}$, and $\boldsymbol{N}_{\mathrm{arr}} = [2, 4, 8, 16]^{\mathrm{T}}$ to create the empirical CDF plots for the used per channel transmit powers $\bar{P}_{\mathrm{Tx},l}$. What can be observed for increasing N?

1.2.2 Large System Approximation

In the following, instead of the given channels, a specific channel model is considered. Namely, the square $N \times N$ channel matrix is assumed to be Rayleigh fading with i.i.d. circularly symmetric complex Gaussian distributed elements with zero mean and variance $1/N$, i.e.,

$$[\boldsymbol{H}]_{i,j} \sim \mathcal{N}_{\mathbb{C}}(0, 1/N), i, j = 1, \ldots, N, \tag{1.20}$$

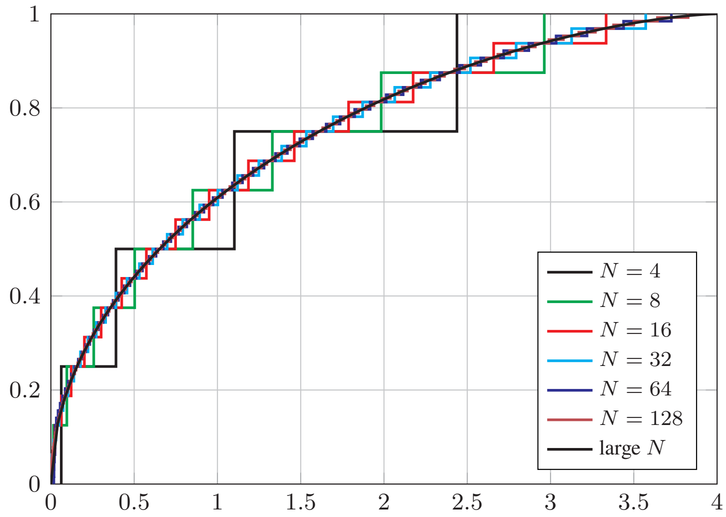

Figure 1.3: CDF of the eigenvalues of $\boldsymbol{H}^{\mathrm{H}}\boldsymbol{H}$ with exemplary realizations

and the noise covariance matrix is the identity matrix, i.e., $\boldsymbol{C}_{\boldsymbol{n}} = \mathbf{I}_N$. For the large system limit $N \to \infty$ the empirical eigenvalue distribution of $\boldsymbol{H}^{\mathrm{H}}\boldsymbol{H}$ converges almost surely to a non-random limiting distribution with density (this law is called *Marčenko-Pastur Law* in the literature)

$$f_\phi(x) = \begin{cases} \frac{\sqrt{x(4-x)}}{2\pi x} & \text{for } 0 < x < 4 \\ 0 & \text{for } x \leq 0 \text{ or } x \geq 4. \end{cases} \tag{1.21}$$

The corresponding CDF of the eigenvalues is given by

$$F_\phi(x) = \int_{-\infty}^{x} f_\phi(t)\mathrm{d}t = \begin{cases} 0 & \text{for } x \leq 0 \\ \frac{1}{2\pi}\left(\sqrt{x(4-x)} + 4\arcsin\left(\sqrt{x}/2\right)\right) & \text{for } 0 < x \leq 4 \\ 1 & \text{for } x > 4. \end{cases} \tag{1.22}$$

The CDF is depicted in Fig. 1.3, together with the empirical CDF's for the eigenvalues of $\boldsymbol{H}^{\mathrm{H}}\boldsymbol{H}$ with exemplary channel realizations for various N. As can be seen, the eigenvalue realizations for $N = 64$ and $N = 128$ result in empirical distributions that are close to the CDF for $N \to \infty$.

The CDF in (1.22) is applied in the research literature amongst others for a sharp approximation of the average rate (or the average MMSE) of point-to-point MIMO channels. To this end, a large but finite N is considered and the channel eigenvalues ϕ_i, $i = 1, \ldots, N$ are fixed as the unique roots of the function $F_\phi(x) - \frac{1}{N}(k - 1/2)$, $k \in \{1, \ldots, N\}$.

PROGRAMMING TASK 9

Write a function that computes the unique roots of $F_\phi(x) - \frac{1}{N}(k - 1/2)$, $k \in \{1, \ldots, N\}$ for a large system approximation with a given number of channel eigenmodes N, where the CDF is given in (1.22).

1 Deliverables (Matlab code file): `largeSystem.m`

- Function definition:
  `function [phi]=largeSystem(N)`

2 Input Specification:

- `N`: positive integer number of channel eigenmodes N

3 Output Specification:

- `phi`: vector of eigenvalues $\phi_1, \ldots, \phi_N$

4 Hint(s):

- Use the internal Matlab function `fzero.m` for finding the individual roots with the initial lower and upper bound given by $x_0 = 0$ and $x_1 = 4$.

When using the fixed choice for ϕ_i, $i = 1, \ldots, N$ for an approximation of the average rate maximization in (1.15), the average rate maximizing power allocation can easily be determined via waterfilling. The proposed achievable average rate shall be determined and compared with the calculated rates from the Monte-Carlo simulations.

PROGRAMMING TASK 10

Create a Matlab script `averageRateMaximizationRayleigh.m` as a copy of `averageRateMaximization.m` and modify it such that the channels are randomly drawn with the distribution given in (1.20). Extend the script to calculate the large system approximation of the average achievable rate for every number of antennas N and every realization of $\bar{P}_{\text{Tx}}$. Add a plot command to also plot the average achievable rate curves for the large system approximation.

1 Deliverables (Matlab code file): `averageRateMaximizationRayleigh.m`

2 Additional Variable Specification:

- `meanRate_largeN`: array that contains the proposed average achievable rates

QUESTION 14

Run the script `averageRateMaximizationRayleigh.m` and compare the curves of the large system approximation with the curves of the Monte-Carlo approximation for the average power constraint. What is the difference between the maximum average rate of the Monte-Carlo approach and the large system approach for $\bar{P}_{\mathrm{Tx}} = 0\,\mathrm{dB}$ and $N = 2, 4, 8, 16$?