Technische Universität München

MIMO Systems Programming Project 3

Instructions

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Chapter 3

Broadcast Channel

The third project considers the *K-user broadcast channel* (BC), where one transmitter sends information to K receivers. A BC with input $\boldsymbol{x}$, outputs $\boldsymbol{y}_1$, $\boldsymbol{y}_2$, ... $\boldsymbol{y}_K$, and the transition probability density function $f_{\boldsymbol{y}_1,\ldots,\boldsymbol{y}_K|\boldsymbol{x}}(\boldsymbol{y}_1,\ldots,\boldsymbol{y}_K|\boldsymbol{x})$ is depicted in Fig. 3.1.

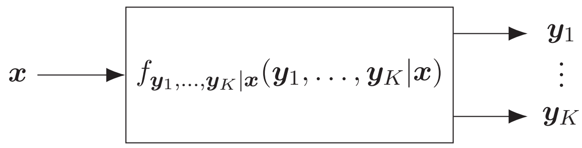


Figure 3.1: K-user Broadcast Channel

In the following tasks, the Gaussian MIMO BC is considered, where we analyze the capacity region, which depends only on the conditional marginals $f_{\boldsymbol{y}_k|\boldsymbol{x}}(\boldsymbol{y}_k|\boldsymbol{x})$, $\forall k$.

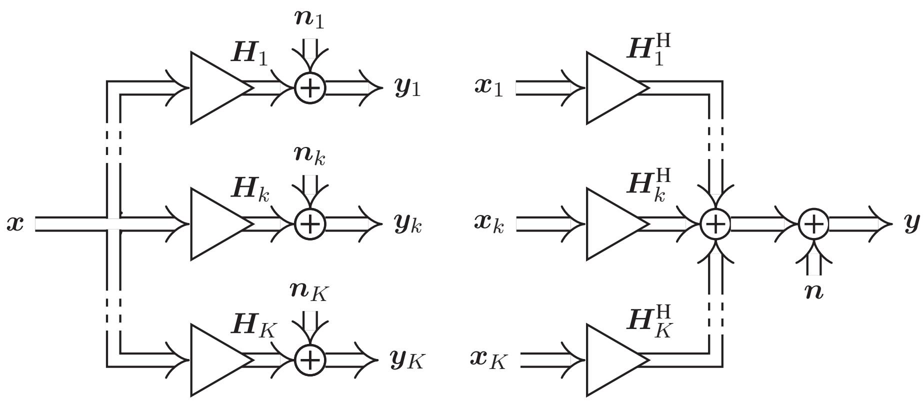
3.1 Capacity of the MIMO Broadcast Channel

A model of the Gaussian MIMO BC is shown in Fig. 3.2a, where the transmit signal $\boldsymbol{x} \in \mathbb{C}^N$ is sent to K receivers. The received signals read as

$$\boldsymbol{y}_k = \boldsymbol{H}_k\boldsymbol{x} + \boldsymbol{n}_k \ \in \mathbb{C}^M, \quad k \in \{1,\ldots,K\},$$

and the transmit power is limited to $\mathrm{E}[\|\boldsymbol{x}\|_2^2] \leq P_{\mathrm{Tx}}$. Throughout this project, we assume white Gaussian noise, i.e.,

$$\boldsymbol{n}_k \sim \mathcal{N}_{\mathbb{C}}(\mathbf{0}, \mathbf{I}_M).$$



(a) MIMO Broadcast Channel (b) Dual Multiple Access Channel

Figure 3.2: Gaussian MIMO Broadcast and Dual Multiple Access Channel

The capacity region of the MIMO BC is achieved via the vector extension of dirty paper coding. To this end, let the transmitted signal $\boldsymbol{x}$ be the superposition of the K signals $\boldsymbol{s}_k$ intended for the respective receiver k, i.e.,

$$\boldsymbol{x} = \sum_{k=1}^{K} \boldsymbol{s}_k.$$

To derive the MIMO BC capacity we will assume that the encoding order corresponds to the user index for the sake of notational brevity, i.e., user 1 is encoded first, followed by user 2, etc. When choosing a codeword for the signal $\boldsymbol{s}_k$ for user k, the transmitter therefore has knowledge of the $k-1$ previously encoded signals and the achievable rate of receiver k is the same as if $\boldsymbol{s}_i$ for $i<k$ would be known to the receiver and

$$R_k^{\mathrm{BC}} \leq I(\boldsymbol{s}_k; \boldsymbol{y}_k | \boldsymbol{s}_1, \ldots, \boldsymbol{s}_{k-1}). \tag{3.1}$$

By the introduction of a one-to-one mapping from user index to encoding order $k \mapsto \pi(k)$ the following results can be extended to any arbitrary encoding order. Since the capacity of the Gaussian BC can be achieved with Gaussian signaling, we furthermore assume

$$\boldsymbol{s}_k \sim \mathcal{N}_{\mathbb{C}}(\boldsymbol{0}, \boldsymbol{S}_k). \tag{3.2}$$

QUESTION 1

Calculate the achievable rate R_k^{BC} of user k in terms of $\boldsymbol{S}_i$, $i \in \{1, \ldots, K\}$, assuming that dirty paper coding is employed and users $i < k$ have been encoded prior to user k.

In the following, we will examine the sum rate optimization for this K-user MIMO Gaussian BC, i.e.,

$$\max \sum_{k=1}^{K} R_k^{\mathrm{BC}} \quad \text{s.t.} \quad \mathrm{E}[\|\boldsymbol{x}\|_2^2] \leq P_{\mathrm{Tx}}. \tag{3.3}$$

The key idea for solving this problem is the transformation to a dual Gaussian MIMO MAC that inherits the same sum power constraint. The resulting MAC problem is convex and can be solved with standard convex optimization tools.

3.1.1 Dual MIMO Multiple Access Channel

In the dual MIMO MAC depicted in Fig. 3.2b, the roles of the transmitter and receivers are reversed compared to the BC (see Fig. 3.2a) with complex-conjugate transpose channels. The received signal reads as

$$\boldsymbol{y} = \sum_{k=1}^{K} \boldsymbol{H}_k^{\mathrm{H}} \boldsymbol{x}_k + \boldsymbol{n} \in \mathbb{C}^N,$$

with Gaussian noise $\boldsymbol{n} \sim \mathcal{N}_{\mathbb{C}}(\mathbf{0}, \mathbf{I})$. While a successive encoding scheme is used in the Gaussian MIMO BC, successive decoding is used in the MIMO MAC, where the MAC decoding order is the *reversed* BC encoding order, i.e., the achievable rate is upper bounded by

$$R_k^{\mathrm{MAC}} \leq I(\boldsymbol{x}_k, \boldsymbol{y} | \boldsymbol{x}_{k+1}, \ldots, \boldsymbol{x}_K). \tag{3.4}$$

Similar to the BC, we assume Gaussian distributed input signals with

$$\boldsymbol{x}_k \sim \mathcal{N}_{\mathbb{C}}(\mathbf{0}, \boldsymbol{Q}_k). \tag{3.5}$$

QUESTION 2

Determine the achievable rate R_k^{MAC} of user k in terms of $\boldsymbol{Q}_i$, $i \in \{1, \ldots, K\}$, assuming that successive decoding is employed and that users $i > k$ have been decoded prior to user k.

PROGRAMMING TASK 1

Implement a function that calculates the achievable rates of the MAC and BC for given channels $\boldsymbol{H}_k$ and transmit covariance matrices $\boldsymbol{Q}_k$, $\boldsymbol{S}_k$ and a given BC encoding order. Note that the MAC decoding order corresponds to the reversed BC encoding order.

1 Deliverables (Matlab code file): `MAC_BC_rates.m`

- Function definition:
  `function [R_BC, R_MAC] = MAC_BC_rates(H,Q,S,order)`

2 Input Specification:

- `H`: $K \times 1$ cell array with channel $\boldsymbol{H}_k \in \mathbb{C}^{M\times N}$ per cell
- `Q`: $K \times 1$ cell array with matrix $\boldsymbol{Q}_k \in \mathbb{C}^{M\times M}$ per cell
- `S`: $K \times 1$ cell array with matrix $\boldsymbol{S}_k \in \mathbb{C}^{N\times N}$ per cell
- `order`: $1 \times K$ vector in $\{1, \ldots, K\}^K$ with the BC encoding order

3 Output Specification:

- `R_BC`: $K \times 1$ array with achievable rates of the BC: R_k^{BC}
- `R_MAC`: $K \times 1$ array with achievable rates of the MAC: R_k^{MAC}

Point-to-Point Duality Principle

The rate duality between the Gaussian MIMO BC and the corresponding MAC is based on the point-to-point duality principle, i.e., the same rates can be achieved for MIMO point-to-point systems with the channels $\boldsymbol{H}$ and $\boldsymbol{H}^{\mathrm{H}}$ and additive white Gaussian noise if the same transmit power is available. To this end, we construct $\boldsymbol{S} \succeq \boldsymbol{0}$ such that

$$\log_2 \det \left(\boldsymbol{I}_M + \boldsymbol{H}\boldsymbol{S}\boldsymbol{H}^{\mathrm{H}}\right) \overset{!}{=} \log_2 \det \left(\boldsymbol{I}_N + \boldsymbol{H}^{\mathrm{H}}\boldsymbol{Q}\boldsymbol{H}\right) \tag{3.6}$$

is satisfied and $\mathrm{tr}(\boldsymbol{S}) \leq \mathrm{tr}(\boldsymbol{Q})$ for given $\boldsymbol{Q} \succeq \boldsymbol{0}$, where $\boldsymbol{H} \in \mathbb{C}^{M\times N}$ denotes the effective channel. Vice versa, a covariance matrix $\boldsymbol{Q} \succeq \boldsymbol{0}$ could be constructed for given $\boldsymbol{S} \succeq \boldsymbol{0}$ with $\mathrm{tr}(\boldsymbol{Q}) \leq \mathrm{tr}(\boldsymbol{S})$ such that (3.6) is fulfilled. In the following, we assume that $\boldsymbol{H}$ is full rank, i.e., $r = \mathrm{rank}(\boldsymbol{H}) = \min(M, N)$.

Let the reduced *singular value decomposition* (SVD) of the effective channel be

$$\boldsymbol{H} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^{\mathrm{H}},$$

with the square diagonal matrix $\boldsymbol{\Sigma} \in \mathbb{R}_+^{r\times r}$ and the (sub-)unitary matrices $\boldsymbol{U} \in \mathbb{C}^{M\times r}$ and $\boldsymbol{V} \in \mathbb{C}^{N\times r}$. Then, the duality requirement in (3.6) can be rewritten as

$$\det \left(\boldsymbol{I}_M + \boldsymbol{\Sigma}\boldsymbol{V}^{\mathrm{H}}\boldsymbol{S}\boldsymbol{V}\boldsymbol{\Sigma}\right) \overset{!}{=} \det \left(\boldsymbol{I}_N + \boldsymbol{\Sigma}\boldsymbol{U}^{\mathrm{H}}\boldsymbol{Q}\boldsymbol{U}\boldsymbol{\Sigma}\right)$$

and equality for a given $\boldsymbol{Q}$ holds if

$$\boldsymbol{S} = \boldsymbol{V}\boldsymbol{U}^{\mathrm{H}}\boldsymbol{Q}\boldsymbol{U}\boldsymbol{V}^{\mathrm{H}}.$$

For $r = M$, this choice leads to $\mathrm{tr}(\boldsymbol{S}) = \mathrm{tr}(\boldsymbol{U}\boldsymbol{U}^{\mathrm{H}}\boldsymbol{Q}) = \mathrm{tr}(\boldsymbol{Q})$ since $\boldsymbol{U}$ is unitary in this case. For $r = N$, we represent $\boldsymbol{Q}$ as

$$\boldsymbol{Q} = \begin{bmatrix} \boldsymbol{U} & \boldsymbol{U}' \end{bmatrix} \begin{bmatrix} \boldsymbol{Q}_{11} & \boldsymbol{Q}_{12} \\ \boldsymbol{Q}_{12}^{\mathrm{H}} & \boldsymbol{Q}_{22} \end{bmatrix} \begin{bmatrix} \boldsymbol{U}^{\mathrm{H}} \\ \boldsymbol{U}'^{,\mathrm{H}} \end{bmatrix},$$

where $[\boldsymbol{U}, \boldsymbol{U}'] \in \mathbb{C}^{M\times M}$ is unitary and $\boldsymbol{Q}_{11}, \boldsymbol{Q}_{22}$ are positive semidefinite.

QUESTION 3

Use above representation to show that $\mathrm{tr}(\boldsymbol{S}) \leq \mathrm{tr}(\boldsymbol{Q})$.

PROGRAMMING TASK 2

Implement the point-to-point transformation to construct $\boldsymbol{S}$ for given $\boldsymbol{Q}$ and $\boldsymbol{H}$.

1 Deliverables (Matlab code file): `ptpTransform.m`
- Function definition: `function [S] = ptpTransform(Q,Heff)`

2 Input Specification:
- `Q`: $M \times M$ covariance matrix $\boldsymbol{Q}$
- `Heff`: effective channel matrix $\boldsymbol{H}$

3 Output Specification:
- `S`: $N \times N$ transmit covariance matrix $\boldsymbol{S}$

MAC-BC Transformation Principle

Based on the duality transformation for the point-to-point case, the duality between the BC and the MAC in Fig. 3.2 can be established. The duality implies that the same rates are achievable in the BC and the MAC, i.e., $R_k^{\mathrm{BC}} = R_k^{\mathrm{MAC}}$, using the same transmit powers $\sum_{k=1}^{K} \mathrm{tr}(\boldsymbol{S}_k) = \sum_{k=1}^{K} \mathrm{tr}(\boldsymbol{Q}_k)$. In other words, a rate tuple $(R_1, \ldots, R_K)$ is achievable in the MIMO BC if and only if it is achievable in its dual MIMO MAC.

For the transformation, again assume that the MAC decoding order is given by $(K, K-1, \ldots, 1)$. We introduce the matrix $\boldsymbol{X}_k = \mathbf{I} + \sum_{i<k} \boldsymbol{H}_i^{\mathrm{H}}\boldsymbol{Q}_i\boldsymbol{H}_i$, and an invertible Hermitian matrix $\boldsymbol{F}_k$. With these definitions the achievable rate of user k in the MAC can be rewritten to

$$R_k^{\mathrm{MAC}} = \log_2 \det\Big(\mathbf{I} + \underbrace{\boldsymbol{X}_k^{-\frac{1}{2}}\boldsymbol{H}_k^{\mathrm{H}}\boldsymbol{F}_k^{-\frac{1}{2}}}_{\boldsymbol{H}_{k,\mathrm{eff}}^{\mathrm{H}}}\underbrace{\boldsymbol{F}_k^{\frac{1}{2}}\boldsymbol{Q}_k\boldsymbol{F}_k^{\frac{\mathrm{H}}{2}}}_{\boldsymbol{Q}_{k,\mathrm{eff}}}\underbrace{\boldsymbol{F}_k^{-\frac{\mathrm{H}}{2}}\boldsymbol{H}_k\boldsymbol{X}_k^{-\frac{\mathrm{H}}{2}}}_{\boldsymbol{H}_{k,\mathrm{eff}}}\Big), \tag{3.7}$$

with the effective channel and transmit covariance $\boldsymbol{H}_{k,\text{eff}}$ and $\boldsymbol{Q}_{k,\text{eff}}$, respectively. From above point-to-point duality we know that the same rate can be achieved if the channel matrix $\boldsymbol{H}_{k,\text{eff}}$ is flipped, i.e.

$$R_k^{\text{BC}} = \log_2 \det \left(\mathbf{I} + \boldsymbol{F}_k^{-\frac{\text{H}}{2}} \boldsymbol{H}_k \boldsymbol{X}_k^{-\frac{\text{H}}{2}} \boldsymbol{S}_{k,\text{eff}} \boldsymbol{X}_k^{-\frac{1}{2}} \boldsymbol{H}_k^{\text{H}} \boldsymbol{F}_k^{-\frac{1}{2}} \right). \tag{3.8}$$

We again define the singular value decomposition

$$\boldsymbol{H}_{k,\text{eff}} = \boldsymbol{F}_k^{-\frac{\text{H}}{2}} \boldsymbol{H}_k \boldsymbol{X}_k^{-\frac{\text{H}}{2}} = \boldsymbol{U}_k \boldsymbol{\Sigma}_k \boldsymbol{V}_k^{\text{H}}. \tag{3.9}$$

QUESTION 4

Use the point-to-point duality to show that by choosing

$$\boldsymbol{S}_k = \boldsymbol{X}_k^{-\frac{\text{H}}{2}} \boldsymbol{V}_k \boldsymbol{U}_k^{\text{H}} \boldsymbol{Q}_{k,\text{eff}} \boldsymbol{U}_k \boldsymbol{V}_k^{\text{H}} \boldsymbol{X}_k^{-\frac{1}{2}} \tag{3.10}$$

the same rates are achieved in the MAC and BC, i.e., $R_k^{\text{MAC}} = R_k^{\text{BC}}$. Give an expression for $\boldsymbol{F}_k$ such that R_k^{BC} is equal to your result from Question 1.

As can be seen above, the eigenvalue decomposition depends on $\boldsymbol{F}_k$ which itself is a function of $\boldsymbol{S}_i$, $i > k$. The transmit covariances therefore have to be computed successively starting with the user that is encoded last.

PROGRAMMING TASK 3

Write a function that calculates the BC transmit covariance matrices $\boldsymbol{S}_k$, $\forall k$ for given matrices $\boldsymbol{Q}_k$, $\forall k$ and a given encoding order in the BC.

1 Deliverables (Matlab code file): `MACtoBCtransform.m`
- Function definition:
  `function [S] = MACtoBCtransform(Q,H,order)`

2 Input Specification:
- `Q`: $K \times 1$ cell array with matrix $\boldsymbol{Q}_k \in \mathbb{C}^{M \times M}$ per cell
- `H`: $K \times 1$ cell array with channel $\boldsymbol{H}_k \in \mathbb{C}^{M \times N}$ per cell
- `order`: $1 \times K$ vector in $\{1, \ldots, K\}^K$ with the given BC encoding order

3 Output Specification:
- `S`: $K \times 1$ cell array with matrix $\boldsymbol{S}_k \in \mathbb{C}^{N \times N}$ per cell

4 Hint(s):

- Use the function `ptpTransform.m` for the transformation.
- Use the proper order for calculating $\boldsymbol{S}_k$ dependent on the BC encoding order.

QUESTION 5

Run the `MACtoBCtransform.m` for the exemplary channels and MAC transmit covariance matrices given in `exampleMIMOBC.mat` and calculate the achieved rates in the BC and the MAC to test your function for the two encoding orders $(1, 2, \ldots, K)$ and $(K, K-1, \ldots, 1)$. Compare the powers of the transmit signals in the BC and the dual MAC, i.e., $\mathrm{tr}(\boldsymbol{Q}_i)$ with $\mathrm{tr}(\boldsymbol{S}_i)$, $i \in \{1, \ldots K\}$.

3.1.2 Sum-Rate optimization for the MIMO BC

Due to the BC-MAC duality, the sum rate maximization problem in (3.3) can alternatively be formulated as a problem in the dual MAC domain, i.e.,

$$\max \sum_{k=1}^{K} R_k^{\mathrm{MAC}} \quad \text{s.t.} \quad \sum_{k=1}^{K} \mathrm{E}[\|\boldsymbol{x}_k\|_2^2] \leq P_{\mathrm{Tx}}. \tag{3.11}$$

The solution to this problem can then be transformed to the BC domain to obtain the sum rate maximizing transmit covariance matrices for the BC.

In the dual Gaussian MIMO MAC, the sum-rate is bounded by the mutual information corresponding to joint decoding, i.e.,

$$\sum_{k=1}^{K} R_k^{\mathrm{MAC}} \leq I(\boldsymbol{y}; \boldsymbol{x}_1, \ldots, \boldsymbol{x}_k) = \log_2 \det\Big(\boldsymbol{I}_M + \sum_{k=1}^{K} \boldsymbol{H}_k^{\mathrm{H}} \boldsymbol{Q}_k \boldsymbol{H}_k\Big), \tag{3.12}$$

and the transmit power is $\sum_{k=1}^{K} \mathrm{tr}(\boldsymbol{Q}_k) \leq P_{\mathrm{Tx}}$.

The sum rate maximization in (3.11) can equivalently be written as

$$\max_{\boldsymbol{Q}_k \succeq \boldsymbol{0}} \log_2 \det\Big(\boldsymbol{I}_N + \sum_{k=1}^{K} \boldsymbol{H}_k^{\mathrm{H}} \boldsymbol{Q}_k \boldsymbol{H}_k\Big) \quad \text{s.t.} \quad \sum_{k=1}^{K} \mathrm{tr}(\boldsymbol{Q}_k) \leq P_{\mathrm{Tx}}. \tag{3.13}$$

The objective of this problem is known to be concave (see Chapter 2). Note, however, that instead of individual power constraints for each receiver, this optimization problem is subject to a sum power constraint.

QUESTION 6

Show that the sum-power constraint and the positive semidefiniteness constraints define a convex constraint set.

Hence, (3.13) is a convex problem and can be solved via standard semidefinite programming such as interior-point methods or a projected gradient method. In this project we will again use YALMIP with SDPT3. Refer to Project 2 for instructions on the usage of YALMIP.

PROGRAMMING TASK 4

Implement a function that computes the sum capacity and the optimal transmit covariances in the dual MAC $\boldsymbol{Q}_k$ for a joint transmit power constraint with the help of YALMIP and SDPT3. Use the hints given in Listing 3.1.

1 Deliverables (Matlab code file): `DualMACSumRateMaximization.m`

- Function definition:
  `function [Q,Csum] = DualMACSumRateMaximization(H,Ptx)`

2 Input Specification:

- `H`: $K \times 1$ cell array with channel $\boldsymbol{H} \in \mathbb{C}^{M \times N}$ per cell
- `Ptx`: joint transmit power P_{Tx}

3 Output Specification:

- `Q`: $K \times 1$ cell array with matrix $\boldsymbol{Q} \in \mathbb{C}^{M \times M}$ per cell
- `Csum`: sum capacity C_{sum}

```matlab
% Use cell arrays to define multiple optimization variables
Q = cell(K,1);
for k=1:K
  Q{k} = sdpvar(M,M,'hermitian','complex')
end

% It is possible to define functions of optimization variables in loops
Z = zeros(M,M);
for k=1:K
  Z = Z + Q{k};
end
% The same is possible to define constraints.
```

Listing 3.1: Hints for optimization with YALMIP

QUESTION 7

Run `DualMACSumRateMaximization.m` for the exemplary channels given in `exampleMIMOBC.mat` and $P_{\mathrm{Tx}} = 10$ dB and give the sum capacity C_{sum}.

QUESTION 8

Use the function `MACtoBCtransform.m` to calculate the sum rate maximizing transmit covariance matrices in the BC, i.e., $\boldsymbol{S}_k$ for the decoding orders $(1, 2, \ldots, K)$ and $(K, K-1, \ldots, 1)$. What are the powers allocated to the transmit signals in the BC, i.e., $\mathrm{tr}(\boldsymbol{S}_k)$ for $k = 1, \ldots, K$, at the sum capacity for the two possible encoding orders?

PROVIDED PROGRAM

The provided function `plotSumRateBC.m` runs the function `DualMACSumRateMaximization.m` to calculate the sum capacities C_{sum} for given transmit powers P_{Tx} in dB and plots these rate values over P_{Tx}.

1 Matlab file: `plotSumRateBC.m`

- Function definition:
  `function [fig] = plotSumRateBC(H,Ptx,fig)`

2 Input Specification:

- `H`: $K \times 1$ cell array with channels $\boldsymbol{H}_k^{M \times N}$ per cell
- `Ptx`: vector of P_{Tx} values in dB
- `fig` (optional): figure handle for plotting the sum rates into a given figure

3 Output Specification:

- `fig`: returned handle of the plotted figure

QUESTION 9

Run the function `plotSumRateBC.m` to plot the sum capacity C_{sum} versus P_{Tx} in dB for the given exemplary system in `exampleMIMOBCs.mat` with P_{Tx} from -15 dB to 30 dB (in steps of 5 dB) and save the figure as `CsumBCvsPtx.fig`. What is the slope of the curve in [bits/channel use] per [dB] power between 25 dB and 30 dB?

PROGRAMMING TASK 5

Extend the function `plotSumRateBC.m` such that it also calculates and plots the sum capacity curves and each users' rate curves that are obtained for the corresponding BC transmit covariance matrices. To this end, first transform the MAC transmit covariance matrices into BC transmit covariance matrices and calculate the corresponding rates. Consider the encoding orders $(1, 2, \ldots, K)$ and $(K, K-1, \ldots, 1)$ for the transformation and the rate calculation.

1 Deliverables (Matlab code file): `plotSumRateBC.m`
- Function definition:
  `function [fig] = plotSumRateBC(H,Ptx,fig)`

2 Input Specification:
- `H`: $K \times 1$ cell array with channels $\boldsymbol{H}_k^{M \times N}$ per cell
- `Ptx`: vector of P_{Tx} values in dB
- `fig` (optional): figure handle for plotting the sum rates into a given figure

3 Output Specification:
- `fig`: returned handle of the plotted figure

QUESTION 10

Run the function `plotSumRateBC.m` again to plot the sum capacities C_{sum} versus P_{Tx} in dB for the given exemplary systems in `exampleMIMOBCs.mat` P_{Tx} from -15 dB to 30 dB (in steps of 5 dB) and save the figure as `CsumBCvsEtx_rates.fig`. Qualitatively describe the differences between the observed individual rates for the two encoding orders. Comment on your observation.

3.1.3 Sato's Bound

In this section, we will numerically illustrate that the result obtained in the previous section is in fact capacity achieving by evaluating Sato's upper bound which is given by

$$\sum_{k=1}^{K} R_k \leq \min_{\substack{f_{\boldsymbol{y}_1,\ldots,\boldsymbol{y}_K|\boldsymbol{x}}(\boldsymbol{y}_1,\ldots,\boldsymbol{y}_K|\boldsymbol{x}) \text{ with} \\ \text{marginals } f_{\boldsymbol{y}_1|\boldsymbol{x}}(\boldsymbol{y}_1|\boldsymbol{x}),\ldots,f_{\boldsymbol{y}_K|\boldsymbol{x}}(\boldsymbol{y}_K|\boldsymbol{x})}} I(\boldsymbol{y}_1, \ldots, \boldsymbol{y}_K; \boldsymbol{x}), \tag{3.14}$$

where $I(\boldsymbol{y}_1, \ldots, \boldsymbol{y}_K; \boldsymbol{x})$ corresponds to the mutual information if both receivers cooperate and perform joint decoding. Since separate decoding, which is assumed in the broadcast channel, can never outperform joint decoding, Sato's upper bound is found by minimizing $I(\boldsymbol{y}_1, \boldsymbol{y}_2; \boldsymbol{x})$ with respect to the joint PDF $f_{\boldsymbol{y}_1,\ldots,\boldsymbol{y}_K|\boldsymbol{x}}(\boldsymbol{y}_1, \ldots, \boldsymbol{y}_K|\boldsymbol{x})$ under the constraint that the corresponding marginals are given by $f_{\boldsymbol{y}_k|\boldsymbol{x}}(\boldsymbol{y}_k|\boldsymbol{x})$, characterizing the BC.

Since for Sato's bound, the broadcast channel is interpreted as a Point-to-Point MIMO Channel, we define the stacked variables

$$\boldsymbol{y} = \begin{bmatrix} \boldsymbol{y}_1 \\ \vdots \\ \boldsymbol{y}_K \end{bmatrix}, \qquad \hat{\boldsymbol{H}} = \begin{bmatrix} \boldsymbol{H}_1 \\ \vdots \\ \boldsymbol{H}_K \end{bmatrix}, \qquad \boldsymbol{n} = \begin{bmatrix} \boldsymbol{n}_1 \\ \vdots \\ \boldsymbol{n}_K \end{bmatrix}. \tag{3.15}$$

Assuming Gaussian signals, the joint distribution $\boldsymbol{y}$ unconditioned and given $\boldsymbol{x}$ read as

$$\boldsymbol{y} \sim \mathcal{N}_{\mathbb{C}}\left(\mathbf{0}, \hat{\boldsymbol{H}}\boldsymbol{C}\hat{\boldsymbol{H}}^{\mathrm{H}} + \boldsymbol{Z}\right), \qquad \boldsymbol{y}|\boldsymbol{x} \sim \mathcal{N}_{\mathbb{C}}\left(\hat{\boldsymbol{H}}\boldsymbol{x}, \boldsymbol{Z}\right),$$

with $\mathrm{E}[\boldsymbol{x}\boldsymbol{x}^{\mathrm{H}}] = \boldsymbol{C}$ and $\mathrm{E}[\boldsymbol{n}\boldsymbol{n}^{\mathrm{H}}] = \boldsymbol{Z}$. The joint mutual information from (3.14) is then given by

$$I(\boldsymbol{y}; \boldsymbol{x}) = \log_2 \det(\mathbf{I} + \boldsymbol{Z}^{-1}\hat{\boldsymbol{H}}\boldsymbol{C}\hat{\boldsymbol{H}}^{\mathrm{H}}).$$

The Sato bound is now found by minimizing this mutual information with respect to the joint PDF of $\boldsymbol{y}|\boldsymbol{x}$, while keeping the marginal distributions fixed. As these marginal distributions are defined by the noise covariances of $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$, which correspond to the diagonal blocks of $\boldsymbol{Z}$ (cf. (3.15)), we define the set $\mathcal{Z}$ of admissible noise covariance matrices as

$$\mathcal{Z} = \left\{\boldsymbol{Z} \in \mathbb{C}^{2M\times 2M} : \boldsymbol{Z} \succeq \mathbf{0},\ \boldsymbol{S}_k\boldsymbol{Z}\boldsymbol{S}_k^{\mathrm{T}} \preceq \mathbf{I}, \forall k\right\}. \tag{3.16}$$

The matrices

$$\boldsymbol{S}_k = [\ \underbrace{\mathbf{0}}_{M\times(k-1)M}, \mathbf{I}_M, \underbrace{\mathbf{0}}_{M\times(K-k)M}\ ] \tag{3.17}$$

are selection matrices that cut out the respective diagonal block of $\boldsymbol{Z}$. Sato's bound is then given by

$$R_{\text{Sato}} = \frac{1}{\ln(2)} \min_{\boldsymbol{Z}\in\mathcal{Z}} \max_{\substack{\boldsymbol{C}\succeq\mathbf{0} \\ \mathrm{tr}(\boldsymbol{C})\leq P_{\mathrm{Tx}}}} \ln\det(\mathbf{I} + \boldsymbol{Z}^{-1}\hat{\boldsymbol{H}}\boldsymbol{C}\hat{\boldsymbol{H}}^{\mathrm{H}}). \tag{3.18}$$

Above problem is difficult to solve in its current form, however, it can be transformed into an equivalent convex optimization problem via Lagrangian duality. To this end, we first consider the inner maximization problem only. From the point-to-point duality in Section 3.1.1 we know that the same rates can be achieved for MIMO point-to-point systems with the channels $\boldsymbol{H}$ and $\boldsymbol{H}^{\mathrm{H}}$. The following optimization problems are therefore equivalent

$$\begin{aligned}&\max_{\substack{\boldsymbol{C}\succeq\mathbf{0}\\ \mathrm{tr}(\boldsymbol{C})\leq P_{\mathrm{Tx}}}} \ln\det(\mathbf{I}+\boldsymbol{Z}^{-1/2}\hat{\boldsymbol{H}}\boldsymbol{C}\hat{\boldsymbol{H}}^{\mathrm{H}}\boldsymbol{Z}^{-\mathrm{H}/2})\\ =&\max_{\substack{\boldsymbol{Q}\succeq\mathbf{0}\\ \mathrm{tr}(\boldsymbol{Q})\leq P_{\mathrm{Tx}}}} \ln\det(\mathbf{I}+\hat{\boldsymbol{H}}^{\mathrm{H}}\boldsymbol{Z}^{-\mathrm{H}/2}\boldsymbol{Q}\boldsymbol{Z}^{-1/2}\hat{\boldsymbol{H}}).\end{aligned} \tag{3.19}$$

We rewrite (3.19) by introducing an auxiliary variable $\boldsymbol{X}$

$$\begin{aligned}\max_{\boldsymbol{X},\boldsymbol{Q}} \ln\det(\boldsymbol{X}) \quad \text{s.t.} \quad & \mathrm{tr}(\boldsymbol{Q})\leq P_{\mathrm{Tx}},\ \boldsymbol{Q}\succeq\mathbf{0}\\ & \boldsymbol{X}=\mathbf{I}+\hat{\boldsymbol{H}}^{\mathrm{H}}\boldsymbol{Z}^{-\mathrm{H}/2}\boldsymbol{Q}\boldsymbol{Z}^{-1/2}\hat{\boldsymbol{H}}\end{aligned} \tag{3.20}$$

QUESTION 11

Determine the Lagrangian function $\mathcal{L}(\boldsymbol{X},\boldsymbol{Q},\boldsymbol{K},\boldsymbol{\Lambda},\mu)$ for above optimization problem, where $\boldsymbol{K}$, $\boldsymbol{\Lambda}\succeq\mathbf{0}$, and $\mu\geq 0$ correspond to the Lagrangian multipliers for the equality, non-negative definiteness, and power constraint, respectively.

Since above optimization problem is convex, the KKT conditions must hold at the optimum. Specifically, the dual feasibility conditions need to hold at the optimum, i.e., the derivatives of the Lagrangian function with respect to the optimization variables are equal to zero.

QUESTION 12

Determine $\frac{\partial}{\partial\boldsymbol{X}^{\mathrm{T}}}\mathcal{L}(\boldsymbol{X},\boldsymbol{Q},\boldsymbol{K},\boldsymbol{\Lambda},\mu)$ and $\frac{\partial}{\partial\boldsymbol{Q}^{\mathrm{T}}}\mathcal{L}(\boldsymbol{X},\boldsymbol{Q},\boldsymbol{K},\boldsymbol{\Lambda},\mu)$.
Hint: $\frac{\partial}{\partial\boldsymbol{X}^{\mathrm{T}}}\det(\boldsymbol{X})=\det(\boldsymbol{X})\boldsymbol{X}^{-1}$

QUESTION 13

Use the dual feasibility conditions to show that the Lagrangian function can be written as

$$\mathcal{L}(\boldsymbol{K},\mu)=-\ln\det(\boldsymbol{K})+\mathrm{tr}(\boldsymbol{K})+\mu P_{\mathrm{Tx}}-\mathrm{tr}(\mathbf{I}_N).$$

Since (3.19) is convex, strong duality holds. The maximization of (3.19) is therefore equivalent to minimizing the Lagrangian dual function with respect to the dual variables

$$\min_{\boldsymbol{K},\boldsymbol{\Lambda},\mu} \left\{ \max_{\boldsymbol{Q},\boldsymbol{X}} \mathcal{L}(\boldsymbol{X},\boldsymbol{Q},\boldsymbol{K},\boldsymbol{\Lambda},\mu) \right\} \quad \text{s.t.} \quad \boldsymbol{\Lambda} \succeq \mathbf{0}, \mu \geq 0.$$

Note that, with help of the dual feasibility condition, we have rewritten the Lagrangian function such that it is independent of $\boldsymbol{Q}$, $\boldsymbol{X}$, and $\boldsymbol{\Lambda}$, making the maximization w.r.t. $\boldsymbol{Q}$ and $\boldsymbol{X}$ superfluous. However, for the maximum to exist, we have to make sure that the dual feasibility conditions hold by adding additional constraints. The resulting dual problem is then given by

$$\begin{aligned} \min_{\boldsymbol{K},\mu} \; -\ln\det(\boldsymbol{K}) + \mathrm{tr}(\boldsymbol{K}) + \mu P_{\mathrm{Tx}} - N \quad \text{s.t.} \quad & \boldsymbol{K} \succeq \mathbf{0},\ \mu \geq 0 \\ & \mu \boldsymbol{Z} \succeq \hat{\boldsymbol{H}} \boldsymbol{K} \hat{\boldsymbol{H}}^{\mathrm{H}}. \end{aligned} \tag{3.21}$$

This minimization problem is equivalent to the inner maximization of (3.18). We substitute this result back into (3.18) and introduce the auxiliary variable $\hat{\boldsymbol{Z}} = \mu \boldsymbol{Z}$, yielding

$$\begin{aligned} R_{\mathrm{Sato}} = \frac{1}{\ln(2)} \min_{\hat{\boldsymbol{Z}},\boldsymbol{K},\mu} \; & -\ln\det(\boldsymbol{K}) + \mathrm{tr}(\boldsymbol{K}) + \mu P_{\mathrm{Tx}} - N \\ \text{s.t.}\; & \hat{\boldsymbol{Z}} \succeq \mathbf{0},\ \boldsymbol{K} \succeq \mathbf{0},\ \mu \geq 0 \\ & \hat{\boldsymbol{Z}} \succeq \hat{\boldsymbol{H}} \boldsymbol{K} \hat{\boldsymbol{H}}^{\mathrm{H}} \\ & \boldsymbol{S}_i \hat{\boldsymbol{Z}} \boldsymbol{S}_i^{\mathrm{T}} \preceq \mu \mathbf{I},\ \forall i \in \{1,\ldots,K\}. \end{aligned} \tag{3.22}$$

This optimization problem is convex and can be solved with standard optimization tools.

PROGRAMMING TASK 6

Modify the function `SatoBound.m` to determine R_{Sato}. Use YALMIP with SDPT3 (cf. Project 2) to solve (3.22).

QUESTION 14

Modify the script `plotSumRateBC.m` to additionally plot the Sato bound for the given exemplary system in `exampleMIMOBCs.mat` for P_{Tx} from -15 dB to 30 dB. Verify that the Sato bound is equal to the sum capacity of the broadcast channel.

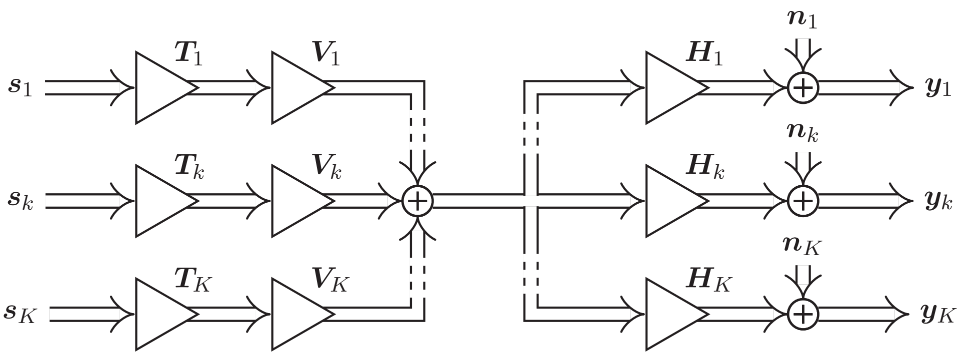


Figure 3.3: Linear Block Diagonalization Precoding

3.2 Block Diagonalization Precoding

Due to the high complexity of dirty paper coding, linear processing is often preferred in practical systems. In this section, a linear block diagonalization precoder will be derived and the performance compared to the BC capacity. To this end, consider the BC depicted in Fig. 3.3 with channels $\boldsymbol{H}_k \in \mathbb{C}^{M \times N}$ and $N \geq KM$. The noise is assumed Gaussian distributed according to $\boldsymbol{n}_k \sim \mathcal{N}_{\mathbb{C}}(\boldsymbol{0}, \boldsymbol{C}_{\boldsymbol{n}_k})$. The input signals are given as $\boldsymbol{s}_k \sim \mathcal{N}_{\mathbb{C}}(\boldsymbol{0}, \mathbf{I}) \in \mathbb{C}^M$. All noise and input signals are assumed to be mutually independent.

The precoding is performed in two steps, where $\boldsymbol{V}_k \in \mathbb{C}^{N \times M}$ ensures that the signal $\boldsymbol{s}_k$ is only received at user k and does not cause interference at other users. The transmitted signal intended for user k therefore has to lie in the nullspace of the channels $\boldsymbol{H}_i$, $i \neq k$. This matrix $\boldsymbol{V}_k$ can be obtained via an orthogonal projection matrix $\boldsymbol{P}_k$

$$\boldsymbol{P}_k = \mathbf{I} - \bar{\boldsymbol{H}}_k^{\mathrm{H}} (\bar{\boldsymbol{H}}_k \bar{\boldsymbol{H}}_k^{\mathrm{H}})^{-1} \bar{\boldsymbol{H}}_k \tag{3.23}$$

where

$$\bar{\boldsymbol{H}}_k^{\mathrm{H}} = \left[\begin{array}{c} \boldsymbol{H}_1^{\mathrm{H}} \cdots \boldsymbol{H}_{k-1}^{\mathrm{H}} \boldsymbol{H}_{k+1}^{\mathrm{H}} \cdots \boldsymbol{H}_K^{\mathrm{H}} \end{array} \right]. \tag{3.24}$$

Assuming full rank channel matrices of size $\boldsymbol{H}_k \in \mathbb{C}^{M \times N}$, $\boldsymbol{P}_k$ has rank M. With help of an eigenvalue decomposition, $\boldsymbol{P}_k$ can therefore be written as

$$\boldsymbol{P}_k = \boldsymbol{V}_k \boldsymbol{V}_k^{\mathrm{H}} \tag{3.25}$$

where $\boldsymbol{V}_k \in \mathbb{C}^{N \times M}$ contains the eigenvectors corresponding to the M non-zero eigenvalues of $\boldsymbol{P}_k$.

With this choice of the precoder $\boldsymbol{V}_k$, the transmissions from $\boldsymbol{s}_k$ to $\boldsymbol{y}_k$ become independent of the input signals $\boldsymbol{s}_i$, $i \neq k$ and we can formulate k independent point-to-point channels

$$\boldsymbol{y}_k = \boldsymbol{H}_k \boldsymbol{V}_k \boldsymbol{T}_k \boldsymbol{s}_k + \boldsymbol{n}_k = \hat{\boldsymbol{H}}_k \boldsymbol{T}_k \boldsymbol{s}_k + \boldsymbol{n}_k. \tag{3.26}$$

A suboptimal, yet simple choice to now allocate power among the users is to divide the available transmit power P_{Tx} equally among all users.

QUESTION 15

Formulate an optimization problem to maximize the achievable rate of the point-to-point MIMO system given in (3.26) with the assumption that the available transmit power is divided equally among all users. Give the optimal choice for $\boldsymbol{T}_k$.

PROGRAMMING TASK 7

Write a function that computes the maximum achievable rate of a MIMO BC with block diagonalization for given channels $\boldsymbol{H}_k$, noise covariances $\boldsymbol{C}_{\boldsymbol{n}_k}$, assuming that the available transmit power P_{Tx} is divided equally among all users for.

1 Deliverables (Matlab code file): `BlockDiagBCEqualPower.m`
- Function definition:
  `function [Rsum] = BlockDiagBCEqualPower(H,C,Ptx)`

2 Input Specification:
- `H`: $K \times 1$ cell array with channel $\boldsymbol{H}_k \in \mathbb{C}^{M\times N}$ per cell
- `C`: $K \times 1$ cell array with noise covariance $\boldsymbol{C}_k \in \mathbb{C}^{M\times M}$ per cell
- `Ptx`: Available transmit power P_{Tx}

3 Output Specification:
- `Rsum`: Achievable sum rate with zero forcing precoding.

The achievable rate of this system can still be improved by also optimizing the transmit power that is allocated to each user. To this end, we now define the composite matrices

$$\boldsymbol{H} = \begin{bmatrix} \boldsymbol{H}_1 \\ \vdots \\ \boldsymbol{H}_K \end{bmatrix} \qquad \boldsymbol{V} = \begin{bmatrix} \boldsymbol{V}_1 \dots \boldsymbol{V}_K \end{bmatrix} \tag{3.27}$$

as well as the stacked vectors

$$\boldsymbol{s} = \begin{bmatrix} \boldsymbol{s}_1 \\ \vdots \\ \boldsymbol{s}_K \end{bmatrix} \qquad \boldsymbol{n} = \begin{bmatrix} \boldsymbol{n}_1 \\ \vdots \\ \boldsymbol{n}_K \end{bmatrix} \qquad \boldsymbol{y} = \begin{bmatrix} \boldsymbol{y}_1 \\ \vdots \\ \boldsymbol{y}_K \end{bmatrix}. \tag{3.28}$$

With above definitions we can find that the product $\boldsymbol{H}\boldsymbol{V}$ as well as the covariance matrix of $\boldsymbol{n}$ become block diagonal, i.e.,

$$\begin{aligned}\hat{\boldsymbol{H}} &= \boldsymbol{H}\boldsymbol{V} = \mathrm{blockdiag}(\boldsymbol{H}_1\boldsymbol{V}_1, \ldots, \boldsymbol{H}_K\boldsymbol{V}_K) \\ \boldsymbol{C} &= \mathrm{E}[\boldsymbol{n}\boldsymbol{n}^{\mathrm{H}}] = \mathrm{blockdiag}(\boldsymbol{C}_{\boldsymbol{n}_1}, \ldots, \boldsymbol{C}_{\boldsymbol{n}_K})\end{aligned} \tag{3.29}$$

We can therefore give an equivalent point-to-point channel

$$\boldsymbol{y} = \hat{\boldsymbol{H}}\boldsymbol{T}\boldsymbol{s} + \boldsymbol{n}, \tag{3.30}$$

where also $\boldsymbol{T}$ is block diagonal with entries $\boldsymbol{T}_1, \ldots, \boldsymbol{T}_K$. Due to the block diagonal structure of the equivalent channel as well as the noise covariance, separate decoding of each block $\boldsymbol{y}_k$ is optimal for this system.

QUESTION 16

Formulate an optimization problem to maximize the achievable rate of the point-to-point MIMO system given in (3.30). What is the optimal choice for $\boldsymbol{T}$?

PROGRAMMING TASK 8

Write a function that computes the maximum achievable rate of a MIMO BC with block diagonalization for given channels $\boldsymbol{H}_k$, noise covariances $\boldsymbol{C}_{\boldsymbol{n}_k}$ and transmit power P_{Tx}.

1 **Deliverables (Matlab code file):** `BlockDiagBC.m`
- Function definition:
```
function [Rsum] = BlockDiagBC(H,C,Ptx)
```

2 **Input Specification:**
- `H`: $K \times 1$ cell array with channel $\boldsymbol{H}_k \in \mathbb{C}^{M\times N}$ per cell
- `C`: $K \times 1$ cell array with noise covariance $\boldsymbol{C}_k \in \mathbb{C}^{M\times M}$ per cell
- `Ptx`: Available transmit power P_{Tx}

3 **Output Specification:**
- `Rsum`: Achievable sum rate with zero forcing precoding.

Again, assume that the noise at each receiver is given by $\boldsymbol{n}_k \sim \mathcal{N}_{\mathbb{C}}(\mathbf{0}, \mathbf{I})$.

PROGRAMMING TASK 9

Write a Matlab script `plotBlogDiagBC.m` to plot the achievable rates obtained with `BlockDiagBCEqualPower.m`, `BlockDiagBC.m`, and the sum capacity of the BC versus P_{Tx} for the exemplary channels given in `exampleBlockDiagMIMOBC.mat`.

QUESTION 17

Run the Matlab script `plotBlockDiagBC.m` for the exemplary channels given in `exampleBlockDiagMIMOBC.m` and P_{Tx} from -10 dB to 40 dB (in steps of 2 dB) and save the figure as `BlockDiagVSCapacity.fig`. Comment on your observations.