Technical University of Munich, Professorship of Signal Processing

Project 2: Chromatic Dispersion Compensation Using Complex-Valued All-Pass Filters

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1 Problem formulation

We start with the following chromatic dispersion channel model of a single mode optical fiber of length L:

$$H_{CD}(\omega) = \exp\Big(-j\frac{\lambda_0^2 DL}{4\pi c}\omega^2\Big)\,, \tag{1.1}$$

where $\omega = 2\pi f$ is the baseband angular frequency, $\lambda_0 = c/f_0$ the operating wavelength, D the fiber dispersion parameter and c the speed of light.
Obviously the channel (**??**) only influences the phase of the input signal. Therefore, our objective is to compensate the nonlinear phase distortion caused by the physical-optical effect of dispersion by filtering with an infinite impulse response equalization filter. The design is of the equalizer is to follow the algorithm described in the reading material.

2 Time-Discretization of the Channel Model

Since we want to process all signals digitally, we first need to transform from the continuous- to the discrete-time domain through sampling (A/D-conversion). The relationship between the continuous-time angular frequency ω (i.e. the Fourier Transform, the FT) and the discrete-time angular frequency Ω (i.e. the Discrete-Time Fourier Transform, the DTFT) is

$$\Omega = \omega T_S\,, \tag{2.1}$$

with the sampling period $T_S = 1/F_S$. This frequency normalization (**??**) is the result of the sampling process:

$$t = nT_S \,. \tag{2.2}$$

With a sampling rate $F_S = B$ and (**??**), we obtain the discretized channel model as

$$\begin{aligned} H_{CD}(\Omega) &= H_{CD}(\omega)\Big|_{\omega=\Omega B} = \exp\Big(-j\frac{\lambda_0^2 DL}{4\pi c}(\Omega B)^2\Big) \\ &= \exp(-j\alpha\Omega^2)\,, \end{aligned} \tag{2.3}$$

with the normalized angular frequency $\Omega = 2\pi f/F_S \in [-\pi, \pi)$ and the channel characterizing constant

$$\alpha = \lambda_0^2 DLB^2/(4\pi c)\,. \tag{2.4}$$

3 Equalizer Design

3.1 Ideal Equalizer

The ideal equalizer for the channel (**??**) fully compensates the phase distortion such that

$$H_{CD}(\Omega)G_{Ideal}(\Omega) = 1\,. \tag{3.1}$$

Plugging in (**??**) and solving for $G_{Ideal}(\Omega)$, we get

$$G_{Ideal}(\Omega) = \exp(j\alpha\Omega^2)\,. \tag{3.2}$$

3.2 IIR Filter

We realize the ideal equalization filter (**??**) with an IIR filter:

$$G_{IIR}(z) = \prod_{i=1}^{N_{IIR}} \frac{-\rho_i e^{-j\theta_i} + z^{-1}}{1 - \rho_i e^{j\theta_i} z^{-1}}\,, \tag{3.3}$$

where ρ_i and θ_i are the radius and angle of the i^{th} pole location in the complex z-domain. Consequently, its zeros are located at $1/\rho_i e^{j\theta_i}$. N_{IIR} represents the number of filter taps.

3.3 Group Delay

The group delay of a filter is defined as

$$\tau(\Omega) = -\frac{\partial}{\partial\Omega}\phi(\Omega)\,, \tag{3.4}$$

with the phase response $\phi(\Omega) = \arg\{G(\Omega)\}$. For the ideal filter (**??**), we therefore obtain

$$\tau_{Ideal}(\Omega) = -2\alpha\Omega\,. \tag{3.5}$$

To derive the group delay of the IIR filter (**??**), we consider its frequency response:

$$
\begin{aligned}
G_{IIR}(\Omega) &= G_{IIR}(z)\Big|_{z=e^{j\Omega}} = \prod_{i=1}^{N_{IIR}} \frac{-\rho_i e^{-j\theta_i} + e^{-j\Omega}}{1-\rho_i e^{j\theta_i} e^{-j\Omega}} \\
&= \prod_{i=1}^{N_{IIR}} e^{-j\Omega} \frac{\big(1-\rho_i \mathrm{cos}(\omega-\theta_i) + j\rho_i \mathrm{sin}(\omega-\theta_i)\big)^2}{1+\rho_i^2 - 2\rho_i \mathrm{cos}(\theta_i-\omega)}\,. \qquad (3.6)
\end{aligned}
$$

Thus, we can express its phase response as

$$
\phi_{IIR}(\Omega) = \sum_{i=1}^{N_{IIR}} \left[-\Omega - 2\mathrm{arctan}\Big(\frac{\rho_i \mathrm{sin}(\Omega-\theta_i)}{1-\rho_i\mathrm{cos}(\Omega-\theta_i)}\Big)\right]. \qquad (3.7)
$$

Applying (**??**), we get the group delay

$$
\begin{aligned}
\tau_{IIR}(\Omega) &= -\frac{\partial}{\partial\Omega}\phi_{IIR}(\Omega) \\
&= \sum_{i=1}^{N_{IIR}} \left[1 + \frac{2}{1+\frac{\rho_i^2 \mathrm{sin}^2(\Omega-\theta_i)}{(1-\rho_i\mathrm{cos}(\Omega-\theta_i))^2}} \frac{\rho_i\mathrm{cos}(\Omega-\theta_i) - \rho_i^2(\mathrm{cos}^2(\Omega-\theta_i)+\mathrm{sin}^2(\Omega-\theta_i))}{(1-\rho_i\mathrm{cos}(\Omega-\theta_i))^2}\right] \\
&= \sum_{i=1}^{N_{IIR}} \frac{1-\rho_i^2}{1+\rho_i^2-2\rho_i\mathrm{cos}(\Omega-\theta_i)}\,. \qquad (3.8)
\end{aligned}
$$

3.4 Group Delay Extrema

We now consider (**??**) for a single tap $N_{IIR} = 1$:

$$
\tau_{IIR_1}(\Omega) = \frac{1-\rho^2}{1+\rho^2-2\rho\mathrm{cos}(\Omega-\theta)}\,. \qquad (3.9)
$$

Obviously, (**??**) becomes maximum for $\Omega = \theta$:

$$
\max_{\Omega}\{\tau_{IIR_1}(\Omega)\} = \tau_{IIR_1}(\theta) = \frac{1+\rho}{1-\rho}\,, \qquad (3.10)
$$

and minimum for $\Omega = \theta + \pi$:

$$
\min_{\Omega}\{\tau_{IIR_1}(\Omega)\} = \tau_{IIR_1}(\theta+\pi) = \frac{1-\rho}{1+\rho}\,. \qquad (3.11)
$$

Since the group delay cannot be negative, (**??**) implies $-1 \le \rho < 1$ and (**??**) $-1 < \rho \le 1$. Combined altogether with the knowledge that the radius must be $\rho \ge 0$, this leads to

$$
0 \le \rho < 1\,, \qquad (3.12)
$$

which guarantees that the pole $\rho e^{j\theta}$ lies within the unit circle and thus ensures stability.

3.5 Integer Factor

It has been shown that approximating the ideal negative group delay (**??**) with (**??**) leads to unstable poles with $\rho_i > 1$. Thus, we slightly modify (**??**) to ensure positivity by adding a integer constant β:

$$\tau_{Desired}(\Omega) = -2\alpha\Omega + \beta\,. \tag{3.13}$$

We choose β such that $\tau'_{Ideal}(\Omega) > 0 \quad \forall\Omega \in [-\pi, \pi)$. Obviously, the minimum of (**??**) occurs at $\Omega = \pi$. In order for this minimum $-2\alpha\pi + \beta$ to be positive, we therefore set

$$\beta = \lceil 2\alpha\pi \rceil > 2\alpha\pi\,. \tag{3.14}$$

3.6 Desired Phase Response

From (**??**), we can derive the desired phase response which our IIR filter should ideally meet:

$$-2\alpha\Omega + \beta = -\frac{\partial}{\partial\Omega}\phi_{Desired}\,. \tag{3.15}$$

Applying the separation of variables method to the ODE (**??**), one obtains

$$\begin{aligned}
\int 2\alpha\Omega - \beta d\Omega \quad &= \quad \int d\phi_{Desired} \\
\Longleftrightarrow \quad \alpha\Omega^2 - \beta\Omega \quad &= \quad \phi_{Desired}(\Omega) + \phi_0\,, \quad \phi_0 \in \mathbb{R} \\
\Longleftrightarrow \quad \phi_{Desired}(\Omega) \quad &= \quad \alpha\Omega^2 - \beta\Omega - \phi_0\,.
\end{aligned} \tag{3.16}$$

4 Implementation of the Practical Examples

In the following, as stated in the task formulation, we will consider the following infrared-light scenario:

$$\begin{aligned}
B &= 56\text{GHz} \\
\lambda_0 &= 1550\text{nm} \\
D &= 16\text{ps}/(\text{nm}\times\text{km}) \\
L &= 23\text{km}\,.
\end{aligned} \tag{4.1}$$

4.1 Number of Taps

According to the reading material, equalization with an FIR filter would require a tap number of

$$N_{FIR} \approx 2\left\lfloor \frac{\lambda_0^2}{2c} D B^2 L \right\rfloor + 1 = 9\,. \tag{4.2}$$

In contrast to the FIR approach, the IIR filters complexity is considerably smaller:

$$N_{IIR} = \left\lceil \frac{\lambda_0^2}{2c} D B^2 L \right\rceil = \lceil 2\alpha\pi \rceil = 5\,. \tag{4.3}$$

4.2 Optimization Framework

The four-step optimization framework proposed in the reading material has the goal of deriving an optimal IIR all-pass equalization filter (**??**), i.e. finding the optimal ρ_i and θ_i as well as the phase correction ϕ_0 (**??**), such that ideally

$$H_{CD}(\Omega)G_{IIR}(\Omega) = e^{-j(\phi_0+\beta\Omega)} \,. \tag{4.4}$$

Defining the mean square error (MSE) of the transfer function containing the phase information as

$$MSE_{trans.phase} = \int_{-\pi}^{\pi} \left| H_{CD}(\Omega)G_{IIR}(\Omega) - e^{-j(\phi_0+\beta\Omega)} \right|^2 , \tag{4.5}$$

the coefficients of (**??**) are found by solving the following non-convex and non-linear optimization problem:

$$\Psi_{trans.phase} = \min_{\substack{\rho_i,\theta_i,\phi_0 \\ s.t. \rho_i<1 \forall i=1,\dots,N_{IIR}}} \left\{ MSE_{trans.phase} \right\} , \tag{4.6}$$

In order to solve (**??**) without getting stuck in local minima, we introduce another sub-optimization problem with the goal of minimizing the mean square error of the group delay first:

$$\Psi_{GD} = \min_{\substack{\rho_i,\theta_i \\ s.t. \rho_i<1 \forall i=1,\dots,N_{IIR}}} \left\{ MSE_{GD} \right\} , \tag{4.7}$$

where

$$MSE_{GD} = \int_{-\pi}^{\pi} \left| \tau_{Desired}(\Omega) - \tau_{IIR}(\Omega) \right|^2 , \tag{4.8}$$

and $\tau_{IIR}(\Omega)$ from (**??**).

4.2.1 Abel-Smith Algorithm

First step is the finding of initial estimates ρ_i and θ_i for the sub-optimization (**??**). This is done via the Abel-Smith Algorithm as described in the reading material. The segmentation of $\tau_{Desired}(\Omega)$ into 2π-area frequency bands is done by integration:

$$\begin{aligned} \int_{\Omega_j}^{\Omega_{j+1}} \tau_{Desired}(\Omega) d\Omega &= 2\pi \,, \quad \forall j = 0, ..., N_{IIR} \\ \iff \Big[-\alpha\Omega^2 + \beta\Omega \Big]_{\Omega_j}^{\Omega_{j+1}} &= 2\pi \\ \iff -\alpha\Omega_{j+1}^2 + \beta\Omega_{j+1} + \alpha\Omega_j^2 - \beta\Omega_j &= 2\pi \,, \end{aligned} \tag{4.9}$$

and solving the resulting quadratic equation with the midnight formula:

$$\Omega_{j+1} = \frac{\beta}{2\alpha} - \sqrt{\frac{\beta^2}{4\alpha^2} - \frac{2\pi}{\alpha} - \frac{\beta}{\alpha} + \Omega_j^2} \,, \tag{4.10}$$

with $\Omega_0 = -\pi$ and $\Omega_{N_{IIR}} = \pi$. We choose the "$-$"-sign since $\Omega_{j+1} < \pi$ and $\beta/(2\alpha) \geq \pi$.

4.2.2 Non-linear Optimizations using Numerical Integration Method: Trapezoidal Rule

All of the follwoing three non-linear optimizations involve the approximation of mean square error integrals (**??**) and (**??**) via the trapezoidal rule. We split the integration interval from $a = \pi$ to $b = \pi$ into 2^{14} equally spaced intervals, resulting in the step width of

$$h = \frac{b-a}{n}\,, \tag{4.11}$$

with $n = 2^{14} - 1$.
Then, we approximate the integrals according to

$$\int_a^b f(x)dx = h\Big(\frac{1}{2}f(a) + \frac{1}{2}f(b) + \sum_{k=1}^{n-1} f(a+kh)\Big)\,. \tag{4.12}$$

4.3 Chromatic Dispersion Compensation

Lastly, we are to put our designed IIR equalization filter into practice for a physical realistic chromatic dispersion channel (**??**).

4.3.1 Frequency Sampling

We derive the impulse response of the channel model (**??**) in the discrete-time domain by exploiting the relationship between the Discrete Time Fourier Transform (DTFT) and Discrete Fourier Transform (DFT), which is given by sampling the DTFT at $\Omega = 2\pi k/N$, $k = 0, ..., N-1$. The time-domain impulse response can then be obtained by the IDFT of the sampled (**??**), e.g. for $N = 256$:

$$h_{CD}[n] = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{-j\alpha 4\pi^2 k^2/N^2} e^{j2\pi kn/N}\,, \quad n = 0, ..., N-1\,. \tag{4.13}$$

In comparison, would could also perform the Inverse Fourier Transform of (**??**):

$$h_{CD}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-j\frac{\lambda_0 DL}{4\pi c}\Omega^2} e^{j\Omega t} d\Omega\,, \tag{4.14}$$

which using the Gaussian identity

$$1 = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(\Omega-\mu)^2}{2\sigma^2}} d\Omega\,, \tag{4.15}$$

for $\sigma^2 = 2\pi c/(j\lambda_0^2 DL)$ and $\mu = 2\pi c/(\lambda_0^2 DL)$, leads to the time-continuous channel impulse response

$$h_{CD}(t) = \sqrt{\frac{B^2}{j4\pi\alpha}} e^{j\frac{B^2 t^2}{4\alpha}}\,. \tag{4.16}$$

Sampling (**??**) with rate B results in:

$$h_{CD}[n] = \sqrt{\frac{B^2}{j4\pi\alpha}} e^{j\frac{n^2}{4\alpha}}\,. \tag{4.17}$$

4.3.2 Equalization Implementation

For the equalization, we first filter the received channel output with our derived IIR filter (**??**) with the coefficients representing the solution of (**??**). Afterwards, keeping (**??**) we still have to correct the phase of the equalization filters output by multiplying with $e^{j\phi_0}$ and then time-shift by $+\beta$ in order to compensate the term $e^{j\beta\Omega}$ since a multiplication with the latter in frequency-domain corresponds to the convolution with the IDFT

$$e^{j\beta\Omega} \xrightarrow{IDFT} \delta[n+\beta] \tag{4.18}$$

in the time domain. However, due to various implementation issues with Matlab functions zp2sos(z,p,k) and zp2tf(z,p,k), it was not possible to simulate the equalization for either (**??**) or (**??**) with the function filter() as required. When passing the impulse response to the equalization filter it should result, ideally, in a dirac as the output of the compensation filter (input = dirac convolved with channel generates impulse response as the channel output). This, at least, is also the case for our implementation. The bit-error-rate simulation causes errors, though.