CTA 200: Thermal Evolution

Problem 1: The purpose of this problem is to write a python script that determines the time evolution of a species of massive particles as the universe expands. This species interacts with a thermal bath at temperature T which tries to keep them in thermal equilibrium. The time evolution of the number density is described by

$$\dot{n} + 3Hn = -\sigma(n^2 - n_{\rm eq}^2) \tag{0.1}$$

where σ is a constant and $H \propto T^{-2}$ and $n_{\rm eq}$ is the equilibrium distribution at temperature T

$$n_{\rm eq} = \int \frac{d^3p}{(2\pi)^3} e^{-E/T} = \frac{1}{2\pi^2} \int_0^\infty p^2 dp e^{-E/T} \tag{0.2}$$

where $E^2 = p^2 + m^2$ is the energy of a single particle.

As the universe expands and cool, the temperature evolves according to the equation

$$\dot{T} = -HT \tag{0.3}$$

so by a change of variables, $Y = n/T^3$ and $x = m/T$ we can rewrite this equation in terms of x and Y as

$$\partial_x Y = -\frac{\lambda}{x^2}(Y^2 - Y_{\rm eq}^2) \tag{0.4}$$

$$Y_{\rm eq}(x) = \frac{1}{2\pi^2} x^3 \int_0^\infty \tilde{p}^2 d\tilde{p} e^{-x\sqrt{\tilde{p}^2+1}} \tag{0.5}$$

where λ is a constant.

(a) Write a function that computes $Y_{\rm eq}(x)$ numerically. Sample x from 10^{-10} to 10^{10} and make a log-log plot. If your function does not give values for all x, make a new function that gives accurate output over that range.

(b) Write a script that solves for the evolution of Y from some initial x_0 to $x_f = 1000$. What are good values for x_0 and how do you pick $Y(x_0)$ (hint : at high temperature, thermal equilibrium should be a good approximation)? For $\lambda = 10, 100, 1000$, make a plot of the percentage differences between solutions as you vary x_0 away from a reference value $x_0 = 10^{-5}$.

(c) Write a script to sample various values of λ and make a plot of $Y(1000)$ as a function of λ. To make this run quickly, use an interpolating function to avoid re-calculating $Y_{\rm eq}$ unnecessarily.

Problem 2: Now suppose we have two different types of particles with two masses m and m' and densities n and n'. We will also assume that the particle n' decays into n with rate Γ. This is now defined by the system of equations

$$\dot{n}' + 3Hn' = -\sigma'(n'^2 - {n'_{\rm eq}}^2) - \Gamma n' \tag{0.6}$$

$$\dot{n} + 3Hn = -\sigma(n^2 - n_{\rm eq}^2) + \Gamma n' \ . \tag{0.7}$$

By the same change of variables picking $x = m/T$

$$\begin{aligned}
\partial_x Y' &= -\frac{\lambda'}{x^2}(Y'^2 - {Y'_{\mathrm{eq}}}^2) - x\gamma Y' && (0.8)\\
\partial_x Y &= -\frac{\lambda}{x^2}(Y^2 - Y_{\mathrm{eq}}^2) + x\gamma Y' && (0.9)\\
Y_{\mathrm{eq}}(x) &= \frac{1}{2\pi^2}x^3 \int \tilde{p}^2 d\tilde{p} e^{-x\sqrt{\tilde{p}^2+1}} && (0.10)\\
Y'_{\mathrm{eq}}(x) &= Y_{\mathrm{eq}}(\beta x) && (0.11)
\end{aligned}$$

where $\beta = m'/m$.

(a) Write a script that solves for Y and Y' using the same approach to initial conditions as you used in problem one.

(b) For some fixed $\beta = 10$ and $\lambda = \lambda' = 1000$, show the evolution of Y and Y' as a function of x for various values of γ. Show that the scripts from problem 1 and problem 2 give the same results for $Y(x)$ when $\gamma = 0$. Pick a final value of x, x_f, such that both Y and Y' are nearly constant for all choices of γ.

(c) For $\beta = 10$, make a contour plot of $Y(x_f)$ sampling over $\lambda = \lambda'$ and γ.