$\{\varphi_1\} \vdash \varphi_2$ și $\{\varphi_2\} \vdash \varphi_1$ valide

Ex 118 $\quad \varphi_1 \dashv\vdash \varphi_2$ <u>ddacă</u> $\varphi_1 \equiv \varphi_2$

"$\Longrightarrow$" Pp. $\varphi_1 \dashv\vdash \varphi_2$ $\iff$ $\begin{cases} \{\varphi_1\} \vdash \varphi_2 \text{ valide } \xrightarrow{\text{Th corect}} \{\varphi_1\} \vDash \varphi_2 \quad (1) \\ \text{și} \\ \{\varphi_2\} \vdash \varphi_1 \text{ valide } \xrightarrow{\text{Th corect}} \{\varphi_2\} \vDash \varphi_1 \quad (2) \end{cases}$

Dem $\varphi_1 \equiv \varphi_2$

(1) $\{\varphi_1\} \vDash \varphi_2 \iff$ pt orice $\tau: A \to B$ a.î. $\hat{\tau}(\varphi_1) = 1$ avem și $\hat{\tau}(\varphi_2) = 1$. $(*)$

(2) $\{\varphi_2\} \vDash \varphi_1 \iff$ pt orice $\tau: A \to B$ a.î. $\hat{\tau}(\varphi_2) = 1$ avem și $\hat{\tau}(\varphi_1) = 1$ $(**)$

$\varphi_1 \equiv \varphi_2$ ddacă pt orice $\tau: A \to B$ avem <u>$\hat{\tau}(\varphi_1) = \hat{\tau}(\varphi_2)$</u> (tb dem)

Fie $\tau: A \to B$ arbitrar

Caz 1 $\quad$ Dacă $\hat{\tau}(\varphi_1) = 1 \xrightarrow{\text{Din } (*)} \hat{\tau}(\varphi_2) = 1 \Rightarrow \hat{\tau}(\varphi_1) = \hat{\tau}(\varphi_2)$ pt caz 1.

Caz 2 $\quad$ Dacă $\hat{\tau}(\varphi_1) = 0$

pp. RA $\hat{\tau}(\varphi_2) = 1 \xrightarrow{\text{Din } (**)} \hat{\tau}(\varphi_1) = 1 \Rightarrow$ contradicție $\Rightarrow$

$\Rightarrow \hat{\tau}(\varphi_2) = 0 \Rightarrow \hat{\tau}(\varphi_1) = \hat{\tau}(\varphi_2)$ pt caz 2

Din caz 1 și 2 $\Rightarrow$ $\hat{\tau}(\varphi_1) = \hat{\tau}(\varphi_2)$ pt orice $\tau: A \to B \Rightarrow \varphi_1 \equiv \varphi_2$
(acoperă toate τ)

"$\Longleftarrow$" Pp. $\varphi_1 \equiv \varphi_2 \iff$ pt orice $\tau: A \to B$, $\hat{\tau}(\varphi_1) = \hat{\tau}(\varphi_2)$ (3) ipoteză

$\varphi_1 \dashv\vdash \varphi_2$

Dem. $\varphi_1 \dashv\vdash \varphi_2 \iff \begin{cases} \{\varphi_1\} \vdash \varphi_2 \\ \{\varphi_2\} \vdash \varphi_1 \end{cases}$

Dacă dem $\varphi_1 \vDash \varphi_2 \xrightarrow{\text{Th. compl}} \varphi_1 \vdash \varphi_2$

$\varphi_2 \vDash \varphi_1 \xrightarrow{\text{Th complet}} \varphi_2 \vdash \varphi_1$

$\varphi_1 \vDash \varphi_2$ ddacă pt orice $\tau : A \to B$ a.î. $\hat{\tau}(\varphi_1) = 1$ avem și $\hat{\tau}(\varphi_2) = 1$ $\Rightarrow$

Din (3)

$\Rightarrow$ "A" $\Rightarrow \varphi_1 \vDash \varphi_2 \xRightarrow{\text{Th complet}} \varphi_1 \vdash \varphi_2$ (4)

$\varphi_2 \vDash \varphi_1$ ddacă pt orice $\tau : A \to B$ a.î. $\hat{\tau}(\varphi_2) = 1$ avem și $\hat{\tau}(\varphi_1) = 1$ "A" $\Rightarrow$

Din (3)

$\Rightarrow \varphi_2 \vDash \varphi_1 \xRightarrow{\text{Th complet}} \varphi_2 \vdash \varphi_1$ (5)

Din (4) și (5) $\Rightarrow \varphi_1 \dashv\vdash \varphi_2$

Ex 116 Th corect Dacă $\Gamma \vdash \varphi$ validă atunci $\Gamma \vDash \varphi$

$\Gamma \vdash \varphi$ validă $\Rightarrow$ există o dem formală (pt $\Gamma \vdash \varphi$)

1.

2.

3. $\Gamma_3 \vdash \varphi_3 \Longrightarrow \Gamma_3 \vDash \varphi_3$

⋮

n. $\Gamma \vdash \varphi$ (...)

Dem prin inducție după lungimea demonstrației,

pp. Th corectitudine are loc pt toate secvențele $\Gamma_k \vdash \varphi_k$ cu dem formale cu $< n$ linii

Caz $\vee_e$

⋮

i_1. $\Gamma \vdash (\varphi_1 \vee \varphi_2) \xrightarrow{\text{ip. ind}} \Gamma \vDash (\varphi_1 \vee \varphi_2)$ (1)

$$\vee_e \frac{\Gamma \vdash (\varphi_1 \vee \varphi_2) \quad \Gamma, \varphi_1 \vdash \varphi \quad \Gamma, \varphi_2 \vdash \varphi}{\Gamma \vdash \varphi}$$

⋮

i_2. $\Gamma, \varphi_1 \vdash \varphi \xrightarrow{\text{ip ind}} \Gamma, \varphi_1 \vDash \varphi$ (2)

⋮

i_3. $\Gamma, \varphi_2 \vdash \varphi \xrightarrow{\text{ip ind}} \Gamma, \varphi_2 \vDash \varphi$ (3)

n: $\Gamma \vdash \varphi$ ($\vee_e$, i_1, i_2, i_3)

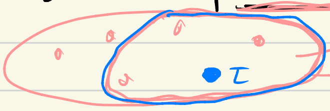
Tb dem că $\Gamma \models \varphi \iff$ pt orice $\tau: A \to B$ a.î. pt orice $\psi \in \Gamma$, $\hat{\tau}(\psi)=1$

avem și $\hat{\tau}(\varphi)=1$

Fie $\tau : A \to B$ arbitrar (fixat) a.î. orice $\psi \in \Gamma$, $\hat{\tau}(\psi)=1$ (*)

(1) $\Gamma \models (\varphi_1 \vee \varphi_2) \iff$ pt orice $\tau_1 : A \to B$ a.î. orice $\psi \in \Gamma$, $\hat{\tau}_1(\psi)=1$ (*) $\Rightarrow$

avem și $\hat{\tau}_1((\varphi_1 \vee \varphi_2))=1$



$\hat{\tau}_1(\psi)=1$

$\Rightarrow \hat{\tau}((\varphi_1 \vee \varphi_2))=1 \Rightarrow \hat{\tau}(\varphi_1)+\hat{\tau}(\varphi_2)=1$ "A"

Caz 1: $\hat{\tau}(\varphi_1)=1$ ($\tau : A \to B$ a.î. orice $\psi \in \Gamma$, $\hat{\tau}(\psi)=1$ și $\hat{\tau}(\varphi_1)=1$)

(**) orice $\psi \in \Gamma \cup \{\varphi_1\}$, $\hat{\tau}(\psi)=1$

(2) $\Gamma, \varphi_1 \models \varphi \iff$ pt orice $\tau_2 : A \to B$ a.î. orice $\psi \in \Gamma \cup \{\varphi_1\}$, $\hat{\tau}_2(\psi)=1$ (**) $\Rightarrow$

avem și $\hat{\tau}_2(\varphi)=1$

$\Rightarrow \hat{\tau}(\varphi)=1$

Caz 2: $\hat{\tau}(\varphi_2)=1$ ($\tau : A \to B$ a.î. orice $\psi \in \Gamma \cup \{\varphi_2\}$, $\hat{\tau}(\psi)=1$) (***)

(3) $\Gamma, \varphi_2 \models \varphi \iff$ pt orice $\tau_3 : A \to B$ a.î. orice $\psi \in \Gamma \cup \{\varphi_2\}$, $\hat{\tau}_3(\psi)=1$ $\Rightarrow$

avem și $\hat{\tau}_3(\varphi)=1$

$\Rightarrow \hat{\tau}(\varphi)=1$

Caz 1, 2 $\Rightarrow \hat{\tau}(\varphi)=1$ pt orice (arbitrar) $\tau : A \to B$ a.î. orice $\psi \in \Gamma$, $\hat{\tau}(\psi)=1$

$\Rightarrow \Gamma \models \varphi$

Caz $\neg i$

<n

k. $\Gamma, \varphi \vdash \bot \xrightarrow{\text{ip.}} \Gamma, \varphi \models \bot$

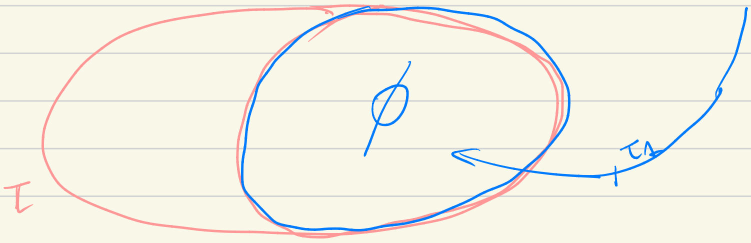
$\neg i \; \dfrac{\Gamma, \varphi \vdash \bot}{\Gamma \vdash \neg\varphi}$

n. $\Gamma \vdash \neg\varphi$ ($\neg i$, k)

$\Gamma, \varphi \models \bot$ ddacă pt orice $\tau_1 : A \to B$ a.î. $\hat{\tau}_1(\psi) = 1$ pt orice $\psi \in \Gamma \cup \{\varphi\}$

avem și $\hat{\tau}_1(\bot) = 1$

imposibil !!!



ddacă nu există $\tau_1 : A \to B$ a.î. $\hat{\tau}_1(\psi) = 1$ pt orice $\psi \in \Gamma \cup \{\varphi\}$ (*)

Arătăm că $\Gamma \models \neg\varphi$ ddacă pt orice $\tau : A \to B$ a.î. $\hat{\tau}(\psi) = 1$ pt orice $\psi \in \Gamma$

avem și $\hat{\tau}(\neg\varphi) = 1$ $\Leftrightarrow \hat{\tau}(\varphi) = 0$

Fie $\tau : A \to B$ fixat arbitrar a.î. $\hat{\tau}(\psi) = 1$ pt orice $\psi \in \Gamma$

Pp. RA. că $\hat{\tau}(\varphi) = 1$ ($\tau : A \to B$ $\tau(\psi) = 1$ pt orice $\psi \in \Gamma \cup \{\varphi\}$)

Din (*) știm că nu există astfel de atribuire $\Rightarrow$

$\Rightarrow \hat{\tau}(\varphi) \neq 1 \Rightarrow \hat{\tau}(\varphi) = 0 \Rightarrow \hat{\tau}(\neg\varphi) = 1$.

Ex 141

φ FNC $\quad \varphi = C_1 \wedge C_2 \wedge \ldots \wedge C_m \quad m \geq 1$

$C_i = L_{i1} \vee L_{i2} \vee \ldots \vee L_{in_i} \quad n_i \geq 1$

$L_{ij} = a$ sau $L_{ij} = \neg a$

$$\varphi_1 = ((\boxed{\neg\neg p} \vee q) \wedge \neg\neg p) \equiv ((p \vee q) \wedge \neg\neg p)$$

$\neg\neg p \equiv p$

5) $$((\underbrace{p \wedge q}_{\varphi_1}) \vee (\underbrace{\neg p}_{\varphi_2} \wedge \underbrace{\neg q}_{\varphi_3})) \equiv \boxed{((\overset{\varphi_1}{p} \wedge \overset{\varphi_2}{q}) \vee \overset{\varphi_3}{\neg p})} \wedge ((p \wedge q) \vee \neg q) \equiv$$

$$\varphi_1 \vee (\varphi_2 \wedge \varphi_3) \equiv (\varphi_1 \vee \varphi_2) \wedge (\varphi_1 \vee \varphi_3)$$

$$(\varphi_1 \wedge \varphi_2) \vee \varphi_3 \equiv (\varphi_1 \vee \varphi_3) \wedge (\varphi_2 \vee \varphi_3)$$

$$\equiv \left((p \vee \neg p) \wedge (q \vee \neg p) \right) \wedge \boxed{\left((p \wedge q) \vee \neg q \right)} \equiv$$

$$\equiv \underbrace{\left((p \vee \neg p) \wedge (q \vee \neg p) \right)}_{\varphi_1} \wedge \left(\underbrace{(p \vee \neg q)}_{\varphi_2} \wedge \underbrace{(q \vee \neg q)}_{\varphi_3} \right) \equiv$$

$$(\varphi_1 \wedge \varphi_2) \wedge \varphi_3 \overset{\text{not}}{\equiv} \varphi_1 \wedge \varphi_2 \wedge \varphi_3$$

$$\to \underline{\varphi_1 \wedge (\varphi_2 \wedge \varphi_3)} \equiv (\varphi_1 \wedge \varphi_2) \wedge \varphi_3$$

$$\equiv \cancel{(}\left((p \vee \neg p) \wedge (q \vee \neg p) \right) \wedge (p \vee \neg q)\cancel{)} \wedge (q \vee \neg q) \equiv$$

$$\equiv \cancel{(}\underbrace{(p \vee \neg p)}_{\varphi_1} \wedge \underbrace{(q \vee \neg p)}_{\varphi_2}\cancel{)} \wedge \underbrace{(p \vee \neg q) \wedge (q \vee \neg q)}_{\varphi_3} \equiv$$

$$\equiv (p \vee \neg p) \wedge (q \vee \neg p) \wedge (p \vee \neg q) \wedge (q \vee \neg q)$$ FNC

⑦ $\neg\left(\underbrace{\neg(p \wedge q)} \vee \underbrace{(p \vee q)} \right) \equiv \underbrace{\neg\neg(p \wedge q)} \wedge \neg(p \vee q) \equiv$

$\neg(\varphi_1 \vee \varphi_2) \equiv \neg\varphi_1 \wedge \neg\varphi_2$ $\qquad \neg\neg\varphi \equiv \varphi$

$$\equiv (p \wedge q) \wedge \underbrace{\neg(p \vee q)} \equiv \underbrace{(p \wedge q)}_{\varphi_1} \wedge (\underbrace{\neg p}_{\varphi_2} \wedge \underbrace{\neg q}_{\varphi_3}) \equiv$$

$$\equiv \cancel{(}(p \wedge q) \wedge \neg p\cancel{)} \wedge \neg q \equiv \cancel{(}p \wedge q\cancel{)} \wedge \neg p \wedge \neg q \equiv \overset{C_1}{p} \wedge \overset{C_2}{q} \wedge \overset{C_3}{\neg p} \wedge \overset{C_4}{\neg q}$$

FNC

⑨ $\left(p \leftrightarrow \left(\underbrace{q \to (\neg p \wedge \neg q)} \right) \right) \equiv \left(p \leftrightarrow \left(\neg q \vee (\neg p \wedge \neg q) \right) \right) \equiv$

$(\varphi_1 \to \varphi_2) \equiv \neg\varphi_1 \vee \varphi_2$ $\qquad (\varphi_1 \leftrightarrow \varphi_2) \equiv (\varphi_1 \to \varphi_2) \wedge (\varphi_2 \to \varphi_1)$

$$\equiv \left(p \to (\neg q \vee (\neg p \wedge \neg q)) \right) \wedge \left((\neg q \vee (\neg p \wedge \neg q)) \to p \right) \equiv$$

$$\equiv \boxed{\left(\neg p \vee (\neg q \vee (\neg p \wedge \neg q)) \right)} \wedge \left(\boxed{\neg(\neg q \vee (\neg p \wedge \neg q))} \vee p \right) \equiv$$

$$\varphi_1 \vee (\varphi_2 \vee \varphi_3) \equiv (\varphi_1 \vee \varphi_2) \vee \varphi_3 \overset{\text{not}}{\equiv} \varphi_1 \vee \varphi_2 \vee \varphi_3$$

$$\equiv \Big(\underbrace{\underbrace{\neg p \vee \neg q}_{\varphi_1} \vee (\underbrace{\neg p}_{\varphi_2} \wedge \underbrace{\neg q}_{\varphi_3})}\Big) \wedge \Big(\big(\boxed{\neg\neg q} \wedge \boxed{\neg(\neg p \wedge \neg q)}\big) \vee p\Big) \equiv$$

$$\varphi_1 \vee (\varphi_2 \wedge \varphi_3) \equiv \ldots$$

$$\equiv \Big(\underbrace{(\neg p \vee \neg q \vee \neg p)} \wedge \underbrace{(\neg p \vee \neg q \vee \neg q)}\Big) \wedge \underbrace{\Big(\big(\boxed{q} \wedge \boxed{(\neg\neg p \vee \neg\neg q)}\big) \vee p\Big)} \equiv$$

$$(\varphi_1 \wedge \varphi_2) \wedge \varphi_3 \overset{\text{not}}{\equiv} \varphi_1 \wedge \varphi_2 \wedge \varphi_3$$

$$\equiv (\neg p \vee \neg q \vee \neg p) \wedge (\neg p \vee \neg q \vee \neg q) \wedge \big((\underbrace{\boxed{q}}_{\varphi_1} \wedge \underbrace{\boxed{(p \vee q)}}_{\varphi_2}) \vee \underbrace{p}_{\varphi_3}\big) \equiv$$

$$\equiv \underbrace{(\neg p \vee \neg q \vee \neg p) \wedge (\neg p \vee \neg q \vee \neg q)}_{\varphi_1} \wedge \Big(\underbrace{(q \vee p)}_{\varphi_2} \wedge \overset{\varphi_3}{\big((p \vee q) \vee p\big)}\Big) \equiv$$

$$(\varphi_1 \vee \varphi_2) \vee \varphi_3 \overset{\text{not}}{\equiv} \varphi_1 \vee \varphi_2 \vee \varphi_3$$

$$\varphi_1 \wedge (\varphi_2 \wedge \varphi_3) \equiv (\varphi_1 \wedge \varphi_2) \wedge \varphi_3$$

$$(q \wedge (p \vee q)) \vee p \quad -$$

$$(\varphi_1 \wedge \varphi_2) \vee \varphi_3 \equiv (\varphi_1 \vee \varphi_3) \wedge (\varphi_2 \vee \varphi_3)$$

$$\equiv \Big(\underbrace{(\neg p \vee \neg q \vee \neg p) \wedge (\neg p \vee \neg q \vee \neg q)}_{\varphi_1} \wedge \underbrace{(q \vee p)}_{\varphi_2}\Big) \wedge \underbrace{(p \vee q \vee p)}_{\varphi_3} \equiv$$

$$\equiv (\neg p \vee \neg q \vee \neg p) \wedge (\neg p \vee \neg q \vee \neg q) \wedge (q \vee p) \wedge (p \vee q \vee p) \qquad \text{FNC.}$$

(11) $(p_1 \wedge q_1) \vee (p_2 \wedge q_2) \vee \ldots \vee (p_n \wedge q_n)$

$n=2$:

$$\underbrace{(p_1 \wedge q_1)} \vee (\underbrace{p_2} \wedge \underbrace{q_2}) \equiv \big(\underbrace{(p_1 \wedge q_1) \vee p_2}\big) \wedge \big(\underbrace{(p_1 \wedge q_1) \vee q_2}\big) \equiv$$

$$\equiv \underbrace{\big((p_1 \vee p_2) \wedge (q_1 \vee p_2)\big)}_{\varphi_1} \wedge \big(\underbrace{(p_1 \vee q_2)}_{\varphi_2} \wedge \underbrace{(q_1 \vee q_2)}_{\varphi_3}\big) \equiv$$

$$\equiv \Big(\underbrace{\big((p_1 \vee p_2) \wedge (q_1 \vee p_2)\big) \wedge (p_1 \vee q_2)}\Big) \wedge (q_1 \vee q_2) \equiv$$

$$\equiv (p_1 \vee p_2) \wedge (q_1 \vee p_2) \wedge (p_1 \vee q_2) \wedge (q_1 \vee q_2) \quad (*) = \varphi_1$$

$n=3$: $\underbrace{(p_1 \wedge q_1) \vee (p_2 \wedge q_2)}_{\equiv \varphi_1} \vee (p_3 \wedge q_3) \equiv$

$$\equiv \Big(\underbrace{(p_1 \vee p_2)}_{\varphi_1} \wedge \underbrace{(q_1 \vee p_2) \wedge (p_1 \vee q_2) \wedge (q_1 \vee q_2)}_{\varphi_2}\Big) \vee \underbrace{(p_3 \wedge q_3)}_{\varphi_3} \equiv$$

$$\equiv \Big((p_1 \vee p_2) \vee (p_3 \wedge q_3)\Big) \wedge \Big(\big(\underbrace{(q_1 \vee p_2)}_{\varphi_1} \wedge \underbrace{(p_1 \vee q_2) \wedge (q_1 \vee q_2)}_{\varphi_2}\big) \vee \underbrace{(p_3 \wedge q_3)}_{\varphi_3}\Big) \equiv$$

$$\equiv \Big(\underbrace{((p_1 \vee p_2) \vee p_3)}_{(\varphi_1 \vee \varphi_2) \vee \varphi_3} \wedge ((p_1 \vee p_2) \vee q_3)\Big) \wedge \Big((q_1 \vee p_2) \vee (p_3 \wedge q_3)\Big) \wedge \underbrace{\Big(\big((p_1 \vee q_2) \wedge (q_1 \vee q_2)\big) \vee (p_3 \wedge q_3)\Big)}_{\text{TO DO}}$$

$$\equiv (p_1 \vee p_2 \vee p_3) \wedge (p_1 \vee p_2 \vee q_3) \wedge (q_1 \vee p_2 \vee p_3) \wedge (q_1 \vee p_2 \vee q_3) \wedge$$

$$\wedge (p_1 \vee q_2 \vee p_3) \wedge (p_1 \vee q_2 \vee q_3) \wedge (q_1 \vee q_2 \vee p_3) \wedge (q_1 \vee q_2 \vee q_3)$$

$$(p_1 \vee p_2 \vee p_3 \vee \ldots \vee \underbrace{p_n}_{q_n})$$

$$\bigwedge_{\substack{l_i \in \{p_i, q_i\} \\ i=1,\ldots,n}} (l_1 \vee l_2 \vee l_3 \vee \ldots \vee l_n)$$

Ex 136 pt orice $\varphi \in LP_{\neg,\wedge,\vee}$ avem $\underbrace{\varphi^c \equiv \neg\varphi}_{P(\varphi)}$

Fie φ arbitrar fixat $\varphi \in LP_{\neg,\wedge,\vee}$

Dem prin ind structurală

CB: $\varphi = a \in A$ $\quad \varphi^c = \neg\varphi \equiv \neg\varphi \Rightarrow P(\varphi)$ "A".

CI 1: $\varphi = \neg\varphi_1$ $\quad$ pp. $P(\varphi_1)$ "A" $\Leftrightarrow \varphi_1^c \equiv \neg\varphi_1$ "A".

$$\frac{P(\varphi_1)}{P(\varphi)}$$

$P(\varphi)$: $\varphi^c \equiv \neg\varphi \iff (\neg\varphi_1)^c \equiv \neg\neg\varphi_1 \iff$

$\iff \varphi_1 \equiv \neg\neg\varphi_1$ "A".

CI2: $\varphi = (\varphi_1 \wedge \varphi_2)$

$$\frac{\text{pp. } \begin{array}{ll} P(\varphi_1) \text{ "A"} & \iff \varphi_1^c \equiv \neg\varphi_1 \quad (*) \\ P(\varphi_2) \text{ "A"} & \iff \varphi_2^c \equiv \neg\varphi_2 \quad (**) \end{array}}{P(\varphi) \text{ "A"}}$$

$P(\varphi)$: $(\varphi_1 \wedge \varphi_2)^c \equiv \neg(\varphi_1 \wedge \varphi_2)$

$$(\varphi_1 \wedge \varphi_2)^c = \boxed{\varphi_1^c} \vee \varphi_2^c \overset{*}{\equiv} \neg\varphi_1 \vee \boxed{\varphi_2^c} \overset{**}{\equiv} \neg\varphi_1 \vee \neg\varphi_2 \equiv \neg(\varphi_1 \wedge \varphi_2)$$

$\varphi_1^c \equiv \neg\varphi_1$ + Th înlocuire

$\Rightarrow P(\varphi)$ "A".

CI3 TO DO.