1. Se consideră operatorul

$$T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$

$$T(x_1, x_2, x_3) = (3x_1 + x_2,\ 6x_1 + 2x_2 + 2x_3,\ mx_1 + 3x_2 + 4x_3)$$
$$m \in \mathbb{R}$$

a) Calculați $T(1, -3, 0)$

b) Determinați m a.î. T să nu fie injectiv

c) Pt m determinat anterior, calculați

i) Im T, rang T, o bază în Im T

ii) Ker T, def T, o bază în Ker T

d) Pt m determinat la b, calculați

i) valorile proprii ale lui T

ii) subspațiile proprii asociate și dimensiunile acestora

e) Determinați pt m determinat la b

i) dacă T este diagonalizabil

ii) forma diagonală a lui T

iii) o bază în care T are forma diagonală

iv) verificați identitatea pt schimbarea de bază a matricii operatorului.

f) Ortonormați baza de la e ii prin procedeul Gramm-Schmidt.

a) Scriem matricea operatorului

$$A = \begin{pmatrix} 3 & 1 & 0 \\ 6 & 2 & 2 \\ m & 3 & 4 \end{pmatrix}$$

$$T\begin{pmatrix} 1 \\ -3 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 & 1 & 0 \\ 6 & 2 & 2 \\ m & 3 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ -3 \\ 0 \end{pmatrix} =$$

$$\begin{pmatrix} 3\cdot 1 + 1\cdot(-3) + 0\cdot 0 \\ 6\cdot 1 + 2\cdot(-3) + 2\cdot 0 \\ m\cdot 1 + 3\cdot(-3) + 4\cdot 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ m-9 \end{pmatrix}$$

b) T este injectiv $\Leftrightarrow \operatorname{Ker} T = \{0\}$

$\Leftrightarrow \dim(\operatorname{Ker} T) = 0 \Leftrightarrow$

$\operatorname{def} T = 0 \Leftrightarrow$

$\operatorname{rang} T = 3 \Leftrightarrow$

$\operatorname{rang} A = 3 \Leftrightarrow \det A \neq 0$

T este surjectiv $\Leftrightarrow \operatorname{Im} T = \mathbb{R}^3 \Leftrightarrow \dim(\operatorname{Im} T) = 3$

$\Updownarrow$

$\operatorname{rang} T = 3$

$\Updownarrow$

$\operatorname{rang} A = 3$

T nu este injectiv $\Leftrightarrow \det A = 0$

$$\det A = \begin{vmatrix} 3 & 1 & 0 \\ 6 & 2 & 2 \\ m & 3 & 4 \end{vmatrix} \qquad \underline{\underline{C_1 = C_1 - 3C_2}}$$

$$\begin{vmatrix} 0 & 1 & 0 \\ 0 & 2 & 2 \\ m-9 & 3 & 4 \end{vmatrix} =$$

$$(m-9)\cdot(-1)^{3+1}\begin{vmatrix} 1 & 0 \\ 2 & 2 \end{vmatrix} =$$

$$(m-9)\cdot 2$$

$\det A = 0 \iff \underline{m = 9}$

Altă situație

$T: \mathbb{R}^3 \longrightarrow \mathbb{R}^4 \qquad A \in \mathcal{M}_{4\times 3}(\mathbb{R})$

T surjectiv $\iff \operatorname{Im} T = \mathbb{R}^4 \iff$

$\dim(\operatorname{Im} T) = 4 \iff$

$\operatorname{rang} T = 4 \iff$

$\operatorname{rang} A = 4$ F

T nu poate fi surjectiv dacă

$\rightarrow$ dim domeniului < dim codomeniului

$T: \mathbb{R}^3 \longrightarrow \mathbb{R}^2 \quad A \in \mathcal{M}_{2\times 3}(\mathbb{R})$

T injectiv $\Leftrightarrow$ $\operatorname{Ker} T = \{0\}$ $\Leftrightarrow$

dim $\operatorname{Ker} T = 0$ $\Leftrightarrow$

dim $(\operatorname{Im} T) = 3$ $\Leftrightarrow$

rang $T = 3$ $\Leftrightarrow$

rang $A = 3$ F

T nu poate fi injectiv dacă

dim domeniului > dim codomeniului

Dacă dim domeniului = dim codomeniului

$\Rightarrow$ T injectiv $\Leftrightarrow$ T surjectiv $\Leftrightarrow$ T bijectiv

c) $A = \begin{pmatrix} 3 & 1 & 0 \\ 6 & 2 & 2 \\ 9 & 3 & 4 \end{pmatrix}$

$\det A = 0$

i) $\operatorname{Im}(T) = \operatorname{Lin}(T(e_1), T(e_2), T(e_3))$

$$= \operatorname{Lin}\left(\begin{pmatrix} 3 \\ 6 \\ 9 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 4 \end{pmatrix} \right)$$

Verif dacă $\begin{pmatrix} 3 \\ 6 \\ 9 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 4 \end{pmatrix}$ sunt lin dep.

$$\left.\text{Știm că } \det A = 0 \right\} \Rightarrow$$

$T(e_1), T(e_2), T(e_3)$ l. dep.

Fie $\Delta = \begin{vmatrix} 1 & 0 \\ 2 & 2 \end{vmatrix} = 2 \neq 0$

$\Rightarrow$ $T(e_2), T(e_3)$ l. indep

$$\Rightarrow \operatorname{Im}(T) = \operatorname{Lin}\left(T(e_2), T(e_3)\right) =$$

$$\left\{ \alpha\, \underline{T(e_2)} + \beta\, \underline{T(e_3)} \;\middle|\; \alpha, \beta \in \mathbb{R} \right\}$$

$$\left\{ \alpha \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 2 \\ 4 \end{pmatrix} \;\middle|\; \alpha, \beta \in \mathbb{R} \right\}$$

$$\left\{ \begin{pmatrix} \alpha \\ 2\alpha + 2\beta \\ 3\alpha + 4\beta \end{pmatrix} \;\middle|\; \alpha, \beta \in \mathbb{R} \right\}$$

$$\dim \operatorname{Im}(T) = \operatorname{rang} T = 2$$

O bază în $\operatorname{Im} T = \{T(e_2), T(e_3)\} =$

$$\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 4 \end{pmatrix} \right\}$$

ii) Ker T = ?

$$A\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{array}{ccc} x_1 & x_2 & x_3 \end{array}$$
$$\underbrace{\begin{pmatrix} 3 & 1 & 0 \\ 6 & 2 & 2 \\ 9 & 3 & 4 \end{pmatrix}}_{\det \neq 0}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$x_1 = \alpha$ nec sec

x_2, x_3 nec. pp

$$\begin{pmatrix} 1 & 0 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} x_2 \\ x_3 \end{pmatrix} = -\alpha\begin{pmatrix} 3 \\ 6 \end{pmatrix}$$

$\Rightarrow$ $x_2 = -3\alpha$

$2x_2 + 2x_3 = -6\alpha$

$-6\alpha + 2x_3 = -6\alpha$

$\Rightarrow x_3 = 0$

$\Rightarrow \operatorname{Ker} T = \{ (\alpha, -3\alpha, 0) \mid \alpha \in \mathbb{R} \}$

def $T = \dim \operatorname{Ker} T = 1 = 3 - \operatorname{rang} T$

O bază în $\operatorname{Ker} T = \{(1, -3, 0)\}$

(am ales $\alpha = 1$)

d)

1. $A = \begin{pmatrix} 3 & 1 & 0 \\ 6 & 2 & 2 \\ 9 & 3 & 4 \end{pmatrix}$

2. $\det(A - \lambda I_3) = \begin{vmatrix} 3-\lambda & 1 & 0 \\ 6 & 2-\lambda & 2 \\ 9 & 3 & 4-\lambda \end{vmatrix}$

$\underline{\underline{C_1 = C_1 - 3C_2}}$

$3C_2 = \begin{pmatrix} 3 \\ 6-3\lambda \\ 9 \end{pmatrix}$

$$\begin{vmatrix} -\lambda & 1 & 0 \\ 3\lambda & 2-\lambda & 2 \\ 0 & 3 & 4-\lambda \end{vmatrix} =$$

$$\lambda \begin{vmatrix} -1 & 1 & 0 \\ 3 & 2-\lambda & 2 \\ 0 & 3 & 4-\lambda \end{vmatrix} \quad \underline{\underline{C_2 = C_2 + C_1}}$$

$$\lambda \begin{vmatrix} -1 & 0 & 0 \\ 3 & 5-\lambda & 2 \\ 0 & 3 & 4-\lambda \end{vmatrix} =$$

$$\lambda(-1) \begin{vmatrix} 5-\lambda & 2 \\ 3 & 4-\lambda \end{vmatrix} =$$

$$-\lambda \left[(5-\lambda)(4-\lambda) - 6 \right] =$$

$$-\lambda \left(\lambda^2 - 9\lambda + 20 - 6 \right) =$$

$$-\lambda \left(\lambda^2 - 9\lambda + 14 \right)$$

3. i) $\lambda(\lambda^2 - 9\lambda + 14) = 0$

$\lambda_1 = 0 \qquad m_{\lambda_1} = 1$

$\lambda^2 - 9\lambda + 14 = 0$

$$\lambda_{2,3} = \frac{9 \pm \sqrt{81 - 4 \cdot 14}}{2} = \frac{9 \pm \sqrt{25}}{2}$$

$$\lambda_2 = \frac{9+5}{2} = 7 \qquad m_{\lambda_2} = 1$$

$$\lambda_3 = \frac{9-5}{2} = 2 \qquad m_{\lambda_3} = 1$$

ii) λ_1

$$\left(A - 0 \cdot I_3\right) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Dacă 0 este valoare proprie

$V_0 = \operatorname{Ker} T$

$$V_0 = \{(\alpha, -3\alpha, 0) \mid \alpha \in \mathbb{R}\}$$

dim $V_0 = 1$

$$v_0 = (1, -3, 0)$$

$$\lambda_2 = 7$$

$$(A - 7I_3)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 3-7 & 1 & 0 \\ 6 & 2-7 & 2 \\ 9 & 3 & 4-7 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} -4 & 1 & 0 \\ 6 & -5 & 2 \\ 9 & 3 & -3 \end{pmatrix}}_{\det = 0} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{vmatrix} 1 & 0 \\ 3 & -3 \end{vmatrix} = -3 \neq 0$$

x_2, x_3 nec pp

$x_1 = \alpha$ nec sec

$$\begin{pmatrix} 1 & 0 \\ 3 & -3 \end{pmatrix} \begin{pmatrix} x_2 \\ x_3 \end{pmatrix} = -\alpha \begin{pmatrix} -4 \\ 9 \end{pmatrix}$$

$$x_2 = 4\alpha$$

$$3x_2 - 3x_3 = -9\alpha$$

$$12\alpha + 9\alpha = 3x_3$$

$$x_3 = 7\alpha$$

$$V_{\lambda_2} = \{(\alpha, 4\alpha, 7\alpha) \mid \alpha \in \mathbb{R}\}$$

$$v_{\lambda_2} = (1, 4, 7) \qquad \dim V_{\lambda_2} = 1$$

λ_3

$$(A - 2I_3)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 3-2 & 1 & 0 \\ 6 & 2-2 & 2 \\ 9 & 3 & 4-2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} 1 & 1 & 0 \\ 6 & 0 & 2 \\ 9 & 3 & 2 \end{pmatrix}}_{\det = 0} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Delta = \begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix}$$

x_1 var sec $\quad x_1 = \alpha$

x_2, x_3 var pp

$$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x_2 \\ x_3 \end{pmatrix} = -\alpha \begin{pmatrix} 1 \\ 6 \end{pmatrix}$$

$x_2 = -\alpha$

$2x_3 = -6\alpha \rightarrow x_3 = -3\alpha$

$V_{\lambda_3} = \{(\alpha, -\alpha, -3\alpha) \mid \alpha \in \mathbb{R}\}$

$v_{\lambda_3} = (1, -1, -3)$

$\dim V_{\lambda_3} = 1$

e)

i) • $\underbrace{m_{\lambda_1}}_{1} + \underbrace{m_{\lambda_2}}_{1} + \underbrace{m_{\lambda_3}}_{1} = \dim \mathbb{R}^3$

$1 + 1 + 1 = 3$ A

• $m_{\lambda_i} = \dim V_{\lambda_i} \quad \forall i = \overline{1,3}$

$m_{\lambda_i} = 1 = \dim V_{\lambda_i}$ A

$\Rightarrow T$ diagonalizabil

ii) $A_D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

iii) $D = \{(1,-3,0),\ (1,4,7),\ (1,-1,-3)\}$

iv) $A_D = (S_{C|D})^{-1} \cdot A \cdot S_{C|D}$

$$S_{C|D} = \begin{pmatrix} 1 & 1 & 1 \\ -3 & 4 & -1 \\ 0 & 7 & -3 \end{pmatrix}$$

$$S_{C|D} \quad \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ -3 & 4 & -1 & 0 & 1 & 0 \\ 0 & 7 & -3 & 0 & 0 & 1 \end{array}\right)$$

$\underbrace{\qquad\qquad}_{I_3}$

$L_2 = L_2 + 3L_1$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 7 & 2 & 3 & 1 & 0 \\ 0 & 7 & -3 & 0 & 0 & 1 \end{pmatrix}$$

$L_3 = L_3 - L_2$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 7 & 2 & 3 & 1 & 0 \\ 0 & 0 & -5 & -3 & -1 & 1 \end{pmatrix} \qquad L_3 = \frac{L_3}{-5}$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 7 & 2 & 3 & 1 & 0 \\ 0 & 0 & 1 & \frac{3}{5} & \frac{1}{5} & -\frac{1}{5} \end{pmatrix} \qquad L_2 = \frac{L_2}{7}$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & \frac{2}{7} & \frac{3}{7} & \frac{1}{7} & 0 \\ 0 & 0 & 1 & \frac{3}{5} & \frac{1}{5} & -\frac{1}{5} \end{pmatrix} \qquad L_1 = L_1 - L_2$$

$$\begin{pmatrix} 1 & 0 & \frac{5}{7} & \frac{4}{7} & -\frac{1}{7} & 0 \\ 0 & 1 & \frac{2}{7} & \frac{3}{7} & \frac{1}{7} & 0 \\ 0 & 0 & 1 & \frac{3}{5} & \frac{1}{5} & -\frac{1}{5} \end{pmatrix}$$

$$L_1 = L_1 - L_3 \cdot \frac{5}{7}$$

$$L_2 = L_2 - L_3 \cdot \frac{2}{7}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{7} & -\frac{2}{7} & \frac{1}{7} \\ 0 & 1 & 0 & \frac{9}{35} & \frac{3}{35} & \frac{2}{35} \\ 0 & 0 & 1 & \frac{3}{5} & \frac{1}{5} & -\frac{1}{5} \end{array}\right)$$

$$\left(S_{CID}\right)^{-1} = \frac{1}{35}\begin{pmatrix} 5 & -10 & 5 \\ 9 & 3 & 2 \\ 21 & 7 & -7 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 2 \end{pmatrix} = \frac{1}{35}\begin{pmatrix} 5 & -10 & 5 \\ 9 & 3 & 2 \\ 21 & 7 & -7 \end{pmatrix}\begin{pmatrix} 3 & 1 & 0 \\ 6 & 2 & 2 \\ 9 & 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ -3 & 4 & -1 \\ 0 & 7 & -3 \end{pmatrix}$$

$15 \quad -60 + 45$

$$\frac{1}{35}\begin{pmatrix} 0 & 0 & 0 \\ 63 & 21 & 14 \\ 42 & 14 & -14 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ -3 & 4 & -1 \\ 0 & 7 & -3 \end{pmatrix}$$

$$\frac{1}{5}\begin{pmatrix} 0 & 0 & 0 \\ 9 & 3 & 2 \\ 6 & 2 & -2 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ -3 & 4 & -1 \\ 0 & 7 & -3 \end{pmatrix}$$

$$\frac{1}{5}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 35 & 0 \\ 0 & 0 & 10 \end{pmatrix} =$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$\begin{array}{c}\begin{pmatrix} 1 & 1 & 1 \\ -3 & 4 & -1 \\ 0 & 7 & -3 \end{pmatrix} \\ \begin{array}{ccc} \underbrace{}_{v_{\lambda_1}} & \underbrace{}_{v_{\lambda_2}} & v_{\lambda_3} \end{array}\end{array}$$

$$b_1 = v_{\lambda_1} = (1, -3, 0)$$

$$b_2 = v_{\lambda_2} + \alpha b_1$$

$$\langle b_2, b_1 \rangle = 0$$

$$\Downarrow$$

$$\langle v_{\lambda_2} + \alpha b_1 , b_1 \rangle = 0$$

$$\langle v_{\lambda_2}, b_1 \rangle + \alpha \langle b_1, b_1 \rangle = 0$$

$$\Rightarrow \alpha = - \frac{\langle v_{\lambda_2}, b_1 \rangle}{\langle b_1, b_1 \rangle}$$

$$b_2 = v_{\lambda_2} - \frac{\langle v_{\lambda_2}, b_1 \rangle}{\langle b_1, b_1 \rangle} b_1$$

$$(1,4,7) - \frac{\langle (1,4,7),(1,-3,0)\rangle}{\langle (1,-3,0),(1,-3,0)\rangle}(1,-3,0)$$

$$(1,4,7) - \frac{1-12}{1+9}(1,-3,0)$$

$$(1,4,7) - \frac{-11}{10}(1,-3,0) =$$

$$\left(\frac{21}{10}, \frac{7}{10}, \frac{70}{10}\right) =$$

$$\frac{7}{10}(3, 1, 10)$$

$$b_3 = v_3 + \beta b_1 + \mu b_2$$

$$b_3 \perp b_1 \rightarrow \langle b_3, b_1\rangle = 0$$

$$b_3 \perp b_2 \rightarrow \langle b_3, b_2\rangle = 0$$

$$\langle v_{\lambda_3} + \beta b_1 + \mu b_2 , b_1 \rangle = 0$$

$$\langle v_{\lambda_3}, b_1 \rangle + \beta \langle b_1, b_1 \rangle + \underbrace{\mu \langle b_2, b_1 \rangle}_{0} = 0$$

$$\beta = - \frac{\langle v_{\lambda_3}, b_1 \rangle}{\langle b_1, b_1 \rangle}$$

$$\langle v_{\lambda_3} + \beta b_1 + \mu b_2 , b_2 \rangle = 0$$

$$\langle v_{\lambda_3}, b_2 \rangle + \underbrace{\beta \langle b_1, b_2 \rangle}_{0} + \mu \langle b_2, b_2 \rangle = 0$$

$$\mu = - \frac{\langle v_{\lambda_3}, b_2 \rangle}{\langle b_2, b_2 \rangle}$$

$$\rightarrow b_3 = v_{\lambda_3} - \frac{\langle v_{\lambda_3}, b_1 \rangle}{\langle b_1, b_1 \rangle} b_1 - \frac{\langle v_{\lambda_3}, b_2 \rangle}{\langle b_2, b_2 \rangle} b_2$$

$$b_j = v_j - \sum_{i=1}^{j-1} \frac{\langle v_j, b_i \rangle}{\langle b_i, b_i \rangle} b_i$$

$$b_3 = (1,-1,-3) - \frac{\langle (1,-1,-3),(1,-3,0)\rangle}{\langle (1,-3,0),(1,-3,0)\rangle}(1,-3,0)$$

$$- \frac{\cancel{\frac{7}{10}}\langle (1,-1,-3),(3,1,10)\rangle}{\cancel{\left(\frac{7}{10}\right)^{2}}\langle (3,1,10),(3,1,10)\rangle} \cdot \cancel{\frac{7}{10}}(3,1,10)$$

$$b_3 = (1,-1,-3) - \frac{4}{10}(1,-3,0) - \frac{-28}{110}(3,1,10)$$

55 11

$$(1,-1,-3) - \frac{2}{5}(1,-3,0) + \frac{14}{55}(3,1,10)$$

$$\frac{33 + 42}{75} \qquad \frac{-165 +}{140}$$

$$\left(\frac{75}{55}, \frac{25}{55}, \frac{-25}{55}\right) =$$

$$\frac{5}{11}\left(3, 1, -1\right)$$

$$\overline{b_1} = (1, -3, 0)$$

$$\overline{b_2} = \frac{7}{10}\left(3, 1, 10\right)$$

$$b_3 = \frac{5}{11}\left(3, 1, -1\right)$$

$$\overline{f_1} = \frac{1}{\|\overline{b_1}\|} \cdot \overline{b_1} = \frac{1}{\sqrt{10}}(1, -3, 0)$$

$$\|\overline{b_1}\| = \sqrt{1^2 + (-3)^2 + 0^2} = \sqrt{10}$$

$$\overline{f_1} = \frac{1}{\sqrt{10}}\left(1, -3, 0\right)$$

$$\|b_2\| = \frac{7}{10}\sqrt{3^2 + 1^2 + 10^2} = \frac{7\sqrt{110}}{10}$$

$$\overline{f_2} = \frac{\cancel{10}}{\cancel{7}\sqrt{110}} \cdot \frac{\cancel{7}}{\cancel{10}} (3, 1, 10) =$$

$$\frac{1}{\sqrt{110}} (3, 1, 10)$$

$$\overline{f_3} = \frac{1}{\|\overline{b_3}\|} \cdot \overline{b_3} =$$

$$\|\overline{b_3}\| = \frac{5}{11} \cdot \sqrt{3^2 + 1^2 + 1^2} = \frac{5}{11} \cdot \sqrt{11}$$

$$\overline{f_3} = \frac{\cancel{11}}{\cancel{5}} \cdot \frac{1}{\sqrt{11}} \frac{\cancel{5}}{\cancel{11}} (3, 1, -1) \cdot$$

$$\frac{1}{\sqrt{11}} (3, 1, -1)$$

2. Pe clasa seriilor convergente cu termeni nenuli se consideră următoarea relație

$\sum_{n=0}^{\infty} a_n \; f \; \sum_{n=0}^{\infty} b_n$ dacă

$$\lim_{n\to\infty} \left|\frac{a_n}{b_n}\right| \in (0,\infty)$$

Să se studieze dacă ρ este relație de echivalență. Dar de ordine?

Reflexivitate.

Fie $\sum a_n$ o serie convergentă

$$\lim_{n\to\infty} \left|\frac{a_n}{a_n}\right| = \lim_{n\to\infty} |1| = \lim_{n\to\infty} 1 =$$

$$1 \in (0,\infty) \Rightarrow$$

$$\sum a_n \,\rho\, \sum a_n$$

$\Rightarrow \rho$ este relație reflexivă

Simetrica

Fie $\sum a_n, \sum b_n$ două serie convergente a.î $\sum a_n \mathcal{S} \sum b_n$

$\Downarrow$

$\lim_{n\to\infty} \left|\frac{a_n}{b_n}\right| \in (0,\infty)$

$\Rightarrow \exists\, l \in (0,\infty)$ a.î

$\lim_{n\to\infty} \left|\frac{a_n}{b_n}\right| = l$

$\lim_{n\to\infty} \left|\frac{b_n}{a_n}\right| = \frac{1}{l} \Rightarrow$

$0 < l < \infty$

$0 < \frac{1}{l} < \infty$

$\sum b_n \mathcal{S} \sum a_n$

$\Rightarrow \mathcal{S}$ este relație simetrică

Tranzitivă Fie $\sum a_n \mathcal{S} \sum b_n$

$$\sum b_n \, f \, \sum c_n$$

$$\lim_{n\to\infty} \left|\frac{a_n}{b_n}\right| = l_1 \in (0,\infty)$$

$$\lim_{n\to\infty} \left|\frac{b_n}{c_n}\right| = l_2 \in (0,\infty)$$

$$\Rightarrow \lim_{n\to\infty} \left|\frac{a_n}{c_n}\right| = \lim_{n\to\infty} \left|\frac{a_n}{b_n} \cdot \frac{b_n}{c_n}\right|$$

$$= \lim_{n\to\infty} \left|\frac{a_n}{b_n}\right| \cdot \left|\frac{b_n}{c_n}\right| \overset{\text{Dacă } \exists}{=}$$

$$\lim_{n\to\infty} \left|\frac{a_n}{b_n}\right| \lim_{n\to\infty} \left|\frac{b_n}{c_n}\right| =$$

$$l_1 \cdot l_2 \in (0,\infty)$$

$$\Rightarrow \sum a_n \, f \sum c_n$$

$\rightarrow$ f este tranzitivă

$\Rightarrow$ $\int$ relație de echivalență

Fie $\sum a_n \int \sum b_n$

$\Downarrow$

$\sum b_n \int \sum a_n$

$a_n = \frac{1}{n^2}$ $\qquad \left|\frac{a_n}{b_n}\right| = \frac{\frac{1}{n^2}}{\frac{1}{n^2+1}} =$

$b_n = \frac{1}{n^2+1}$

$\frac{n^2+1}{n^2} \longrightarrow 1 \in (0, \infty)$

$\sum a_n$ C

$\sum b_n$ C $\qquad \left|\frac{b_n}{a_n}\right| = \frac{\frac{1}{n^2+1}}{\frac{1}{n^2}} =$

$\frac{n^2}{n^2+1} \longrightarrow 1 \in (0, \infty)$

$\left.\begin{array}{l} a_n \int b_n \\ b_n \int a_n \end{array}\right\} \not\Rightarrow a_n = b_n$

$\Rightarrow$ $\int$ nu este relație antisimetrică

$\to f$ nu e relație de ordine

3. Studiați seria

$$\sum_{n=0}^{\infty} \underbrace{\frac{2n^{\alpha}+1}{3n^{3}+1}}_{a_n} \left(\underbrace{\frac{x-1}{x+1}}_{z}\right)^{n} \qquad x \neq -1$$

$\underline{\alpha > 0}$

Discutați convergența și absoluta convergență a seriei

$$\frac{a_{n+1}}{a_n} = \frac{\dfrac{2(n+1)^{\alpha}+1}{3(n+1)^{3}+1}}{\dfrac{2n^{\alpha}+1}{3n^{3}+1}} =$$

$$\underbrace{\frac{2(n+1)^{\alpha}+1}{2n^{\alpha}+1}}_{\downarrow \, 1} \cdot \underbrace{\frac{3n^{3}+1}{3(n+1)^{3}+1}}_{\downarrow \, 1} = 1$$

$\to$ Raza $= \frac{1}{1} = 1$

$\Rightarrow$ Seria $\begin{cases} AC \text{ pt } |z|<1 \\ D \text{ pt } |z|>1 \\ ? \text{ pt } |z|=1 \end{cases}$

Pt $|z|=1$

1. $z=1$.

$$\sum a_n = \sum \frac{2n^{\alpha}+1}{3n^3+1}$$

$$b_n = \frac{1}{n^{3-\alpha}}$$

$$\frac{a_n}{b_n} = \frac{\frac{2n^{\alpha}+1}{3n^3+1}}{\frac{1}{n^{3-\alpha}}} = \frac{(2n^{\alpha}+1)\cdot n^{3-\alpha}}{3n^3+1} =$$

$$\frac{2n^3+n^{3-\alpha}}{3n^3+1} \to \frac{2}{3} \in (0,\infty)$$

$\alpha > 0 \Rightarrow 3-\alpha < 3$

$\xRightarrow{CC\,III} \sum a_n \sim \sum b_n$

$$\sum b_n \begin{cases} C \text{ pt } 3-\alpha > 1 \\ D \text{ pt } 3-\alpha \le 1 \end{cases}$$

$$\begin{cases} C \text{ pt } 2 > \alpha \\ \hline D \text{ pt } 2 \le \alpha \end{cases}$$

2. $z = -1$

$\sum a_n (-1)^n$

Dacă studiez modulul

$\Rightarrow$ Pt $2 > \alpha$ seria convergentă

Ce se întâmplă pt $2 \le \alpha$

$$\sum \underbrace{\frac{2n^\alpha + 1}{3n^3 + 1}}_{>0} \cdot (-1)^n$$

Pt $\alpha < 3$ $\quad \frac{2n^\alpha + 1}{3n^3 + 1} \longrightarrow 0$

$$\frac{a_{n+1}}{a_n} = \frac{\dfrac{2(n+1)^{\alpha}+1}{3(n+1)^3+1}}{\dfrac{2n^{\alpha}+1}{3n^3+1}} < 1$$

$$\frac{2(n+1)^{\alpha}+1}{3(n+1)^3+1} < \frac{2n^{\alpha}+1}{3n^3+1}$$

$$\left[2(n+1)^{\alpha}+1\right](3n^3+1) <$$

$$(2n^{\alpha}+1)\left[3(n+1)^3+1\right]$$

$$6(n+1)^{\alpha}\cdot n^3 + 3n^3 + 2(n+1)^{\alpha} + \cancel{1}$$

$$< 6n^{\alpha}(n+1)^3 + 3(n+1)^3 + 2n^{\alpha} + \cancel{1}$$

$$6(n+1)^{\alpha}\cdot n^3 + \cancel{3n^3} + 2(n+1)^{\alpha}$$

$$< 6n^{\alpha}(n+1)^3 + \cancel{3n^3} + 3n^2$$

$$+ 3n + 1 +$$

$$2n^{\alpha}$$

$$\ldots$$

Fie $f : \mathbb{R}_+ \longrightarrow \mathbb{R}$

$$f(x) = \frac{2x^{\alpha}+1}{3x^{3}+1}$$

$$f'(x) = \frac{(2\alpha x^{\alpha-1})(3x^3+1) - (2x^{\alpha}+1)\, 9x^2}{(3x^3+1)^2}$$

$$f'(x) = \frac{6\alpha x^{\alpha+2} + 2\alpha x^{\alpha-1} - 18x^{\alpha+2} - 9x^2}{(3x^3+1)^2}$$

$f'(x) < 0$

$6\alpha < 18 \Rightarrow 6\alpha x^{\alpha+2} - 18x^{\alpha+2} < 0$

$2\alpha < 9 \Rightarrow 2x\alpha^{\alpha-1} < 9x^2$

$\Rightarrow f \searrow$

$\Rightarrow a_{n+1} < a_n$

pt $\alpha < 3 \xrightarrow{\text{Leibniz}}$

$$\sum (-1)^n a_n \quad C$$

pt $\alpha \geq 3 \quad a_n \not\to 0$

$$\to \sum (-1) a_n \quad D$$

$$\sum a_n z^n \begin{cases} AC \text{ pt } |z| < 1 \\ D \text{ pt } |z| > 1 \\ AC \text{ pt } |z| = 1 \text{ și } \alpha < 2 \\ \cancel{D \text{ pt } z = 1 \text{ și } \alpha \geq 2} \\ SC \text{ pt } z = -1 \text{ și } \alpha \in [2,3) \\ D \text{ pt } z = -1 \text{ și } \alpha > 3 \end{cases}$$

$$\frac{x-1}{x+1} = 1 \iff x - 1 = x + 1 \quad F$$

$$\frac{x-1}{x+1} = -1 \iff x - \cancel{1} = -x - \cancel{1}$$

$$x = -x \Rightarrow$$

$x = 0$

$$-1 < \frac{x-1}{x+1} < 1$$

$$\frac{x-1}{x+1} - 1 < 0$$

$$\frac{\cancel{x}-1-\cancel{x}-1}{x+1} < 0$$

$$\frac{-2}{x+1} < 0$$

$$\frac{2}{x+1} > 0$$

$x+1 > 0$

$x > -1$ și

$$\frac{x-1}{x+1} + 1 > 0$$

$$\frac{x-1+x+1}{x+1} > 0$$

$$\frac{2x}{x+1} > 0$$

$2x(x+1) > 0$

Fie $x < -1$

Fie $x > 0$

A C pt $x > 0$

A C pt $x = 0$ și $x < ?$

[illegible]

SC pt $x=0$ și $\alpha \in [2,3)$

D pt $x=0$ și $\alpha > 3$

D pt $x<0$