Serie cu termeni oarecare $\sum x_n$

Ne interesează că absolută conv

$\frac{(-1)^n}{n}$ C

$\left|\frac{(-1)^n}{n}\right| = \frac{1}{n}$ D

1. AC $\Leftrightarrow$ $\sum |x_n|$ C $\Rightarrow$ $\sum x_n$ C

2. $\sum |x_n|$ D $\sum x_n$ C mai spunem $\sum x_n$ SC

3. $\sum x_n$ D

Seminar 4

*Exerciţii recomandate:** 4.1(a-f), 4.2(a), 4.3(a-f)

*Rezerve:** 4.1(g,j,k,l), 4.3(g,i,k)

S4.1 Folosind diverse criterii de convergenţă, să se stabilească natura seriilor:

a) $\displaystyle\sum_{n=1}^{\infty} \underbrace{\frac{(-1)^{n-1} \cdot n}{2^{n-1}}}_{x_n}$;

Studiem seria modulelor $|x_n| \geq 0$ $|x_n| = \frac{n}{2^{n-1}}$

Aplic criteriul raportului:

$$\frac{|x_{n+1}|}{|x_n|} = \frac{\frac{n+1}{2^n}}{\frac{n}{2^{n-1}}} = \frac{n+1}{2^{\cancel{n}}} \cdot \frac{\cancel{2^{n-1}}}{n} = \frac{n+1}{2n} \to \frac{1}{2} \in (0,1)$$

$\xRightarrow{\text{crit rap}}$ $\sum |x_n|$ C $\Rightarrow$ $\sum x_n$ AC (în part $\sum x_n$ C)

b) $\displaystyle\sum_{n=0}^{\infty} \underbrace{(-1)^n \frac{\ln 2 + 3^n}{\ln 3 + 2^n}}_{x_n}$;

Studiem $|x_n| = \frac{\ln 2 + 3^n}{\ln 3 + 2^n} = \frac{\overbrace{\frac{\ln 2}{3^n}}^{\to 0} + 1}{\underbrace{\frac{\ln 3}{3^n}}_{\to 0} + \underbrace{\left(\frac{2}{3}\right)^n}_{\to 0}} \to \infty$

$|x_n| \longrightarrow \infty \Rightarrow x_n \not\to 0 \Rightarrow$

$\sum x_n$ D

Indiciu pt a folosi Leibniz

c) $\underbrace{\sum_{n=0}^{\infty}(-1)^{n+1}\frac{\sqrt{n}}{n+1}}_{x_n}$; $x_n = (-1)^{n+1} \cdot \underbrace{\frac{\sqrt{n}}{n+1}}_{z_n}$

Studiem $|x_n| = \frac{\sqrt{n}}{n+1}$

$y_n = \frac{1}{n^{1-\frac{1}{2}}} = \frac{1}{n^{\frac{1}{2}}} = \frac{1}{\sqrt{n}}$ $\sum \frac{1}{n^\alpha} \begin{cases} C, & \alpha \geq 1 \\ D, & \alpha < 1 \end{cases}$

$\frac{|x_n|}{y_n} = \frac{\frac{\sqrt{n}}{n+1}}{\frac{1}{\sqrt{n}}} = \frac{n}{n+1} \to 1 \xRightarrow{CCIII} \sum |x_n| \sim \sum y_n \; D$

$\Rightarrow \sum |x_n| \; D$

$z_n > 0$

$\frac{z_{n+1}}{z_n} = \frac{\frac{\sqrt{n+1}}{n+2}}{\frac{\sqrt{n}}{n+1}} = \frac{\sqrt{(n+1)^3}}{\sqrt{n(n+2)^2}} = \sqrt{\frac{n^3+3n^2+3n+1}{n^3+4n^2+4n}} < 1$

$z_n = \frac{\sqrt{n}}{n+1} = \frac{1}{\sqrt{n}+\frac{1}{\sqrt{n}}} \to 0 \xRightarrow{Leibniz} \sum x_n \; C$

d) $\sum_{n=1}^{\infty} \frac{\sin n \cdot \cos n^2}{\sqrt{n}}, n \in \mathbb{N}^*$;

Serie cu termeni oarecare în care apar fct. trig $\Rightarrow$ Indiciu pt a folosi Dirichlet

$x_n = \frac{\sin n \cos n^2}{\sqrt{n}} = \underbrace{\sin n \cos n^2}_{y_n} \cdot \underbrace{\frac{1}{\sqrt{n}}}_{z_n}$

$z_n \longrightarrow 0 \quad (2)$

$\left.\begin{array}{l} \sqrt{n+1} > \sqrt{n} \\ \frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}} \end{array}\right\} \Rightarrow z_n$ descrescător (1)

Studiem șirul sumelor parțiale pt y_n

$$S_n = \sum_{k=1}^{n} \sin k \cos k^2$$

$\sin a \cos b = \frac{1}{2}\left[\sin(a+b) + \sin(a-b)\right]$

$$= \frac{1}{2}\sum_{k=1}^{n}\left[\sin(k+k^2) + \sin(k-k^2)\right]$$

sin

fct

impară

$$= \frac{1}{2}\sum_{k=1}^{n}\left[\sin(k+k^2) - \sin(k^2-k)\right]$$

$$= \frac{1}{2}\sum_{k=1}^{n}\left[\sin k(k+1) - \sin k(k-1)\right]$$

$$= \frac{1}{2}\sum_{k=1}^{n}\sin k(k+1) - \frac{1}{2}\sum_{k=1}^{n}\sin k(k-1)$$

$l = k-1$

$$= \frac{1}{2}\sum_{k=1}^{n}\sin k(k+1) - \frac{1}{2}\sum_{l=0}^{n-1}\sin l(l+1) =$$

$$\frac{1}{2}\left(\sin n(n+1) - \sin 0\right) = \frac{1}{2}\sin n(n+1)$$

$|\sin n(n+1)| \le 1$

$\Rightarrow |S_n| \le \frac{1}{2}$ (3)

$(1), (2), (3) \xRightarrow{\text{Dirichlet}} \sum x_n$ C

$\sum x_n$ C $\Rightarrow x_n \to 0$

$P \to Q \iff \neg Q \to \neg P$

Dacă $x_n \not\to 0 \Rightarrow \sum x_n$ D

e) $\displaystyle\sum_{n=1}^{\infty} \underbrace{(-1)^n \frac{(2n+1)!!}{2^n \cdot n!}}_{x_n}$;

$$2^n \cdot n! = \underbrace{2\cdot 2\cdot 2 \cdots 2}_{n \text{ ori}} \cdot \underbrace{1\cdot 2\cdot 3 \cdots n}_{n \text{ factori}} = (2\cdot 1)\cdot(2\cdot 2)\cdot(2\cdot 3)\cdots(2\cdot n) = 2\cdot 4\cdot 6 \cdots (2n)$$

$$2^n \cdot n! = (2n)!!$$

$$2^n \cdot 1\cdot 2 \cdots n = (2\cdot 1)\cdot(2\cdot 2)\cdots(2n) = (2n)!!$$

$$|x_n| = \frac{(2n+1)!!}{(2n)!!} = 1\cdot\frac{3}{2}\cdot\frac{5}{4}\cdot\frac{7}{6}\cdots\frac{2n+1}{2n} > 1$$

$$|x_n| > 1 \Rightarrow x_n \not\to 0 \Rightarrow \sum x_n \text{ D}$$

$$2^5 \cdot 5! = 2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 1\cdot 2\cdot 3\cdot 4\cdot 5 = (2\cdot 1)\cdot(2\cdot 2)\cdot(2\cdot 3)\cdot(2\cdot 4)\cdot(2\cdot 5) = 2\cdot 4\cdot 6\cdot 8\cdot 10 = 10!!$$

$\sum \frac{1}{n}$ D deși $\frac{1}{n} \to 0$

f) $\displaystyle\sum_{n=1}^{\infty} \underbrace{(-1)^{n-1} \ln\left(\frac{n^2+2}{n^2+1}\right)}_{x_n}$;

$$|x_n| = \ln\left(\frac{n^2+2}{n^2+1}\right) = \ln\left(1+\frac{1}{n^2+1}\right) = \frac{n^2+1}{n^2+1}\ln\left(1+\frac{1}{n^2+1}\right) = \frac{1}{n^2+1}\underbrace{\underbrace{\underbrace{\ln\left(1+\frac{1}{n^2+1}\right)^{n^2+1}}_{\downarrow\, e}}_{1}}_{<\,2}$$

$$< \frac{2}{n^2+1}$$

$$\sum \frac{2}{n^2+1} \sim \sum \underbrace{\frac{1}{n^2+1}}_{y_n} \qquad z_n = \frac{1}{n^2}$$

$$\frac{y_n}{z_n} = \frac{\frac{1}{n^2+1}}{\frac{1}{n^2}} = \frac{n^2}{n^2+1} \to 1 \Rightarrow \sum y_n \sim \sum z_n \text{ C}$$

$\to \sum y_n$ C

$\left.\begin{array}{l}|x_n| < y_n \\ \sum y_n \text{ C}\end{array}\right\} \xRightarrow{\text{CCT}} \begin{array}{l}\sum |x_n| \text{ C} \\ \sum x_n \text{ A C}\end{array}$

g) (R) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n (n+1)^{n+1}}{n^{n+2}}$;

h) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\ln(n+1)}$;

i) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n - \ln(n)}$;

j) (R) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[n+1]{\ln(n+1)}}$;

k) (R) $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{n} \cdot \frac{\sin \frac{n\pi}{6}}{\sqrt{n^3+1}}$;

l) (R) $\displaystyle\sum_{n=0}^{\infty} \frac{a^n + \operatorname{sh} n}{3^n} \cdot b^n, a, b \in \mathbb{R}$;

m) $\displaystyle\sum_{n=1}^{\infty} \operatorname{tg}^n \left(a + \frac{b}{n}\right), a, b \in \left(0, \frac{\pi}{2}\right)$;

n) $\sum_{n=1}^{\infty}(-1)^{n-1}n^{\alpha}\left(\ln\left(\frac{n+2}{n}\right)\right)^{\beta}, \alpha, \beta \in \mathbb{R}.$

S4.2 Să studieze convergenţa produsului Cauchy al următoarelor serii:

a) $\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$ şi $\sum_{n=1}^{\infty}\frac{n}{2^n}$;

Serii de puteri

Eventual renotez un arg cu x

$\sum a_n \cdot x^n$

$\frac{|a_{n+1}|}{|a_n|} \to \frac{1}{R}$

R = raza de conv a seriei

$$\begin{cases} AC \text{ pt } |x| < R \\ D \text{ pt } |x| > R \\ ? \text{ pt } |x| = R \to \text{se studiază separat} \end{cases}$$

Eventual revin la notația inițială

b) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{\sqrt{n+1}}$ şi $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{\sqrt{n+1}}$.

S4.3 Să se studieze natura următoarelor serii de puteri:

a) $\sum_{n=0}^{\infty}\underbrace{[2+(-1)^n]}_{a_n}x^n,\ x \in \mathbb{R}$;

$\sqrt[n]{a_n} = \sqrt[n]{2+(-1)^n}$

$a_{2n} = 3 \qquad \sqrt[2n]{a_{2n}} = \sqrt[2n]{3} = \sqrt[n]{\sqrt{3}} \longrightarrow 1$

$a_{2n+1} = 1$

$\lim_{n\to\infty}\sqrt[n]{x_n}$ Dacă $\lim_{n\to\infty}\frac{x_{n+1}}{x_n}$

$\limsup_{n\to\infty}\sqrt[n]{a_n} = 1 \Longrightarrow R = 1$

$$\sum a_n x^n \begin{cases} AC \text{ pt } |x|<1 \\ D \text{ pt } |x|>1 \end{cases}$$

$x = 1$ $\quad \sum 2+(-1)^n \quad D$

$x=-1$ $\quad \sum \left(2+(-1)^n\right)(-1)^n \quad D$

b) $\sum_{n=0}^{\infty} \underbrace{\frac{n+1}{\sqrt{n^4+n^3+1}}}_{a_n} \left(\frac{x+1}{2x+3}\right)^n, \ x \in \mathbb{R} \setminus \left\{\frac{3}{2}\right\};$ $\qquad \sum a_n z^n$

$z = \frac{x+1}{2x+3} \qquad a_n > 0$

$$\frac{a_{n+1}}{a_n} = \frac{\frac{n+2}{\sqrt{(n+1)^4+(n+1)^3+1}}}{\frac{n+1}{\sqrt{n^4+n^3+1}}} = \underbrace{\frac{n+2}{n+1}}_{\downarrow \, 1} \cdot \underbrace{\sqrt{\frac{n^4+n^3+1}{(n+1)^4+(n+1)^3+1}}}_{\downarrow \, 1} \to 1$$

$$R = 1 \Rightarrow \begin{cases} AC \text{ pt } |z| < 1 \\ D \text{ pt } |z| > 1 \end{cases}$$

I $z = 1$

$$\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} \frac{n+1}{\sqrt{n^4+n^3+1}} \qquad b_n = \frac{1}{n^{\frac{4}{2}-1}} = \frac{1}{n}$$

$$\frac{a_n}{b_n} = \frac{\frac{n+1}{\sqrt{n^4+n^3+1}}}{\frac{1}{n}} = \frac{n^2+n}{\sqrt{n^4+n^3+1}} \to 1$$

$$\Rightarrow \sum a_n \sim \sum b_n \ D$$

$$\Rightarrow \sum a_n \ D \quad \text{pt } z = 1 \ \sum a_n z^n \ D$$

II $z = -1$

$$\sum_{n=0}^{\infty} a_n (-1)^n \qquad a_n = \frac{n+1}{\sqrt{n^4+n^3+1}} = \frac{1+\frac{1}{n}}{\sqrt{n^2+n+\frac{1}{n^2}}} \to 0$$

$$\frac{a_{n+1}}{a_n} = \frac{\frac{n+2}{\sqrt{(n+1)^4+(n+1)^3+1}}}{\frac{n+1}{\sqrt{n^4+n^3+1}}} =$$

d) $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{(x-4)^n}{n \cdot 3^n}, x \in \mathbb{R};$

$$\frac{(n+2)\sqrt{n^4+n^3+1}}{(n+1)\sqrt{(n+1)^4+(n+1)^3+1}} < 1$$

$$(n+2)\sqrt{n^4+n^3+1} < (n+1)\sqrt{(n+1)^4+(n+1)^3+1} \quad \Big|^2$$

$$(n^2+4n+4)(n^4+n^3+1) < (n^2+2n+1)\left[(n+1)^4+(n+1)^3+1\right]$$

$$n^6+4n^5+4n^4+ \quad < \quad (n^2+2n+1)(n^4+4n^3+6n^2+4n+1+$$
$$n^5+4n^4+4n^3+ \qquad\qquad n^3+3n^2+3n+1+1)$$
$$n^2+4n+4$$

$$n^6+5n^5+8n^4+4n^3+n^2+4n+4 < n^6+2n^5+n^4+$$
$$4n^5+8n^4+4n^3+\dots$$
$$+\dots$$

(A)

$\Rightarrow a_n$ descrescător

$a_n \longrightarrow 0$

$\Big\}\overset{\text{Leibniz}}{\Rightarrow} \sum (-1)^n a_n$ C

AC pt $|z|<1$

D pt $|z|>1$

D pt $z=1$

SC pt $z=-1$

$$f = \frac{x+1}{2x+3} \qquad \left|\frac{x+1}{2x+3}\right| > 1$$

$$-1 < \frac{x+1}{2x+3} < 1$$

$$\begin{cases} -1 < \dfrac{x+1}{2x+3} \\ \\ \dfrac{x+1}{2x+3} < 1 \end{cases} \Leftrightarrow \begin{cases} \dfrac{x+1}{2x+3} + 1 > 0 \\ \\ \dfrac{x+1}{2x+3} - 1 < 0 \end{cases}$$

$$\begin{cases} \dfrac{x+1+2x+3}{2x+3} > 0 \\ \\ \dfrac{x+1-2x-3}{2x+3} < 0 \end{cases} \Leftrightarrow \begin{cases} \dfrac{3x+4}{2x+3} > 0 \\ \\ \dfrac{-x-2}{2x+3} < 0 \\ \dfrac{x+2}{2x+3} > 0 \end{cases}$$

		-2		$-\frac{3}{2}$		$-\frac{4}{3}$	
$x+2$	$-$ $-$	0	$+$		$+$		$+$
$2x+3$	$-$ $-$	$-$	$-$ $-$	0	$+$		$+$ $+$
$3x+4$	$-$ $-$	$-$	$-$ $-$	$-$	$-$	0	$+$ $+$

$x \in (-\infty, -2) \cup (-\frac{4}{3}, \infty)$ $\sum a_n x^n$ AC

$z = 1 \Rightarrow \frac{x+1}{2x+3} = 1 \Rightarrow x + 1 = 2x + 3$

$x + 2 = 0 \Rightarrow$

$x = -2$ $\sum a_n x^n$ D

$z = -1$ $\frac{x+1}{2x+3} = -1 \Rightarrow x + 1 = -2x - 3$

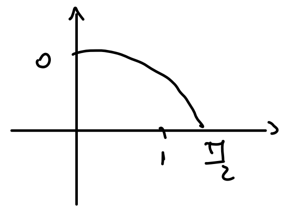
$3x + 4 = 0 \Rightarrow$ $\sum a_n x^n$ SC

$x = -\frac{4}{3}$

$\Downarrow$

$x \in [-2, -\frac{4}{3})$ $\sum a_n x^n$ D

c) $\sum_{n=1}^{\infty} \underbrace{\left(\cos\frac{1}{n}\right)^{\frac{n^2+2}{n+2}}}_{a_n} \cdot x^n, x \in \mathbb{R};$



$$\sqrt[n]{a_n} = \left[\left(\cos\frac{1}{n}\right)^{\frac{n^2+2}{n+2}}\right]^{\frac{1}{n}} =$$

$$\underbrace{\left(\cos \underbrace{\frac{1}{n}}_{\to 0}\right)}_{1}^{\underbrace{\frac{n^2+2}{n^2+2n}}_{\to 1}} \longrightarrow 1$$

$$\Rightarrow R = \frac{1}{1} = 1$$

$$\sum a_n x^n \begin{cases} AC \text{ pt } |x|<1 \\ D \text{ pt } |x|>1 \\ D \text{ pt } |x|=1 \end{cases}$$

$x=1 \quad \sum a_n$

$\cos a - \cos b =$
$-2 \sin\frac{a+b}{2} \sin\frac{a-b}{2}$

$$\left(\cos\frac{1}{n}\right)^{\frac{n^2+2}{n+2}} \overset{1^\infty}{=}$$

$$\left[\left(1+\underbrace{\cos\frac{1}{n}-1}_{\to 0}\right)^{\frac{1}{\cos\frac{1}{n}-1}}\right]^{\left(\cos\frac{1}{n}-1\right)\cdot\frac{n^2+2}{n+2}}$$

$1 = \cos 0$

$$= \left[\left(1+\underbrace{\cos\frac{1}{n}-1}_{\to 0}\right)^{\frac{1}{\cos\frac{1}{n}-1}}\right]^{-2 \sin\frac{1}{2n} \sin\frac{1}{2n} \cdot \frac{n^2+2}{n+2}}$$

$$= \underbrace{\left[\left(1+\cos\frac{1}{n}-1\right)^{\frac{1}{\cos\frac{1}{n}-1}}\right]}_{\to e}^{-2 \underbrace{\frac{\sin^2\frac{1}{2n}}{\left(\frac{1}{2n}\right)^2}}_{\to 1} \cdot \underbrace{\left(\frac{1}{2n}\right)^2 \cdot \frac{n^2+2}{n+2}}_{\to 0}}$$

$\rightarrow e^0 = 1$

$\Rightarrow \sum a_n \ D$

$x = -1 \quad \sum a_n \cdot (-1)^n \Rightarrow \sum a_n \cdot (-1)^n \ D$

$a_n \cdot (-1)^n \not\rightarrow 0$

d) $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(x-4)^n}{n\cdot 3^n}, x\in\mathbb{R};$

$$\frac{x-4}{3}=z$$

$$\| $$

$$\underbrace{\frac{(-1)^{n-1}}{n}}_{a_n}\cdot z^n$$

Studiez $\frac{|a_{n+1}|}{|a_n|}=\frac{\frac{1}{n+1}}{\frac{1}{n}}=\frac{n}{n+1}\to 1$

$\Rightarrow$ Serie este $\begin{cases} AC \text{ pt } |z|<1 \\ D \text{ pt } |z|>1 \\ ? \text{ pt } |z|=1 \end{cases}$ D pt $z=-1$
SC pt $z=1$

$\underline{I}$ $z=1$ $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$

$\left.\begin{array}{l}\frac{1}{n}\to 0 \\ \frac{1}{n} \text{ descrescător}\end{array}\right\} \xRightarrow{\text{Leibniz}} \sum\frac{(-1)^{n-1}}{n}$ C

$\underline{II}$ $z=-1$ $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\cdot(-1)^n=$

$$\sum_{n=1}^{\infty}\frac{(-1)^{2n-1}}{n}=\sum_{n=1}^{\infty}\frac{-1}{n}=$$

$$-\sum_{n=1}^{\infty}\frac{1}{n} \quad D$$

$$\left|\frac{x-4}{3}\right| < 1 \iff$$

$$-1 < \frac{x-4}{3} < 1 \iff$$

$$-3 < x-4 < 3 \iff$$

$$1 < x < 7 \iff$$

$$\frac{x-4}{3} = 1 \Rightarrow x-4 = 3 \iff x = 7$$

$$\frac{x-4}{3} = -1 \iff x-4 = -3 \iff x = 1$$

$$\begin{cases} AC \text{ pt } x \in (1,7) \quad C \text{ pt } x \in (1,7] \\ SC \text{ pt } x = 7 \\ D \text{ pt } x \in (-\infty, 1] \cup (7, \infty) \end{cases}$$

$$\sum_{n=1}^{\infty} (-1)^n \quad D$$

$$-1+1+(-1)+1+(-1)+1+\ldots$$

$$(-1+1)+((-1)+1)+((-1)+1)$$

$$0 + 0 + 0$$

greşit pt că seria nu e AC

e) $\sum_{n=1}^{\infty} \frac{x^n}{n^p}, \ p \in \mathbb{R};$

$$\underbrace{\frac{1}{n^p}}_{a_n} \cdot x^n$$

$$\frac{a_{n+1}}{a_n} = \frac{\frac{1}{(n+1)^p}}{\frac{1}{n^p}} = \left(\frac{n}{n+1}\right)^p \longrightarrow 1$$

$$\Rightarrow R = \frac{1}{1} = 1 \qquad \begin{cases} AC \text{ pt } |x|<1 \\ D \text{ pt } |x|>1 \\ ? \text{ pt } |x|=1 \end{cases}$$

$$\text{Pt } x=1 \qquad \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n^p} \begin{cases} D \text{ pt } p \le 1 \\ C \text{ pt } p>1 \end{cases}$$

$$\text{Pt } x=-1 \qquad \sum_{n=1}^{\infty} a_n \cdot (-1)^n$$

Dacă $a_n \to 0$ seria converge

$$\frac{1}{n^p} \to 0 \text{ pt } p>0$$

$$\text{Pt } p \le 0 \qquad \frac{1}{n^p} \not\to 0 \Rightarrow \sum_{n=1}^{\infty} a_n (-1)^n \ D$$

$$\Rightarrow (-1)^n \cdot \frac{1}{n^p} \not\to 0$$

$x = -1$ $\begin{cases} D \text{ pt } p \le 0 \\ C \text{ pt } p > 0 \end{cases}$

$$\begin{cases} AC \text{ pt } & x \in (-1, 1) \\ AC \text{ pt } & x = \pm 1 \quad p > 1 \\ SC \text{ pt } & x = -1 \quad p \in (0, 1] \\ D \text{ pt } & x \in (-\infty, -1) \cup (1, \infty) \\ & x = \pm 1, \quad p \le 0 \end{cases}$$

f) $\sum_{n=2}^{\infty} \frac{x^n}{3^n \cdot n \cdot \ln n}$ $x \in \mathbb{R}$; $= \underbrace{\left(\frac{x}{3}\right)^n}_{z^n} \cdot \underbrace{\frac{1}{n \ln n}}_{a_n}$

$a_n > 0$

$$\frac{a_{n+1}}{a_n} = \frac{\frac{1}{(n+1)\ln(n+1)}}{\frac{1}{n \ln n}} = \underbrace{\frac{n}{n+1}}_{\downarrow \atop 1} \cdot \underbrace{\frac{\ln n}{\ln(n+1)}}_{\downarrow \atop 1} \longrightarrow 1$$

Considerăm fct atașată

$f : \mathbb{R} \to \mathbb{R}$ $\quad f(x) = \frac{\ln x}{\ln(x+1)}$

$\lim_{x \to \infty} \frac{\ln x}{\ln(x+1)}$ $\underset{\exists}{\overset{\text{Dacă}}{=}}$ $\lim_{x \to \infty} \frac{\frac{1}{x}}{\frac{1}{x+1}} = \lim_{x \to \infty} \frac{x+1}{x} = 1$

$\Rightarrow R = \frac{1}{1} = 1$

$$\Rightarrow \sum a_n z^n \begin{cases} AC & \text{pt } |z|<1 \\ D & \text{pt } |z|>1 \\ ? & \text{pt } |z|=1 \end{cases}$$

$\underline{\text{I}}$ $z=1$

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n} \sim \sum_{n=2}^{\infty} \cancel{2^n} \frac{1}{\cancel{2^n} \ln 2^n} = \sum_{n=2}^{\infty} \frac{1}{n \ln 2}$$

$\left(\frac{1}{n \ln n}\right)$ descrescător $= \frac{1}{\ln 2} \sum_{n=2}^{\infty} \frac{1}{n}$ D

$\underline{\text{II}}$ $z=-1$

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$$

$$\left.\begin{array}{l} \left(\frac{1}{n \ln n}\right) \searrow \\ \frac{1}{n \ln n} \longrightarrow 0 \end{array}\right\} \xRightarrow{\text{Leibniz}} \sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n} \text{ C}$$

$$\begin{cases} \text{AC pt } |z|<1 \\ \text{SC pt } z=-1 \\ \text{D pt } z=1 \\ \text{D pt } |z|>1 \end{cases}$$

$$\begin{cases} \text{AC pt } |x|<3 \\ \text{SC } x=-3 \\ \text{D pt } x=3 \\ |x|>3 \end{cases}$$

g) (R) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n (x+1)^{2n}}{(4n+1)^2}, \ x \in \mathbb{R};$

$$\underbrace{\frac{1}{(4n+1)^2}}_{a_n} \underbrace{\left[2 \cdot (x+1)^2\right]}_{z}{}^{n}$$

$$\frac{a_{n+1}}{a_n} = \frac{(4n+1)^2}{(4n+5)^2} \longrightarrow 1 \Rightarrow R = \frac{1}{1} = 1$$

$\sum a_n z^n$ AC pt $|z| < 1$

D pt $|z| > 1$

? pt $|z| = 1$

$\frac{1}{n}$ pt - pt numitor numărător

$z = 1$ $\quad \sum a_n = \sum \frac{1}{(4n+1)^2}$ $\quad b_n = \frac{1}{n^2}$

$$\frac{a_n}{b_n} = \frac{\frac{1}{(4n+1)^2}}{\frac{1}{n^2}} = \frac{n^2}{(4n+1)^2} \longrightarrow \frac{1}{16}$$

$\rightarrow \sum a_n \sim \sum b_n$ C

$$\left|2(x+1)^2\right| \le 1$$

$$(x+1)^2 \le \frac{1}{2} \Rightarrow$$

$$-\frac{1}{\sqrt{2}} \le x+1 \le \frac{1}{\sqrt{2}}$$

$$-\frac{1}{\sqrt{2}} - 1 \le x \le \frac{1}{\sqrt{2}} - 1$$

AC pt $|z| \le 1$ pt $x \in \left[-\frac{1}{\sqrt{2}} - 1, \frac{1}{\sqrt{2}} - 1\right]$

D pt $|z| > 1$ în rest

$z = -1$ $\quad \sum (-1)^n \cdot a_n$

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$\Rightarrow |z| = 1$ AC

h) $\displaystyle\sum_{n=1}^{\infty} \left(\sqrt{n} - 1\right)^n \cdot x^n, x \in \mathbb{R};$

i) (R) $\displaystyle\sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^n \left(\frac{1-x}{1-2x}\right)^n, x \in \mathbb{R} \setminus \left\{\frac{1}{2}\right\};$

j) $\sum_{n=1}^{\infty}(-1)^n \frac{1}{\ln n}\left(\frac{1-x^2}{1+x^2}\right)^n, x \in \mathbb{R};$

k) (R) $\sum_{n=0}^{\infty}(-1)^n \frac{1}{3^{\frac{n}{2}}\sqrt{1+n^2}}\mathrm{tg}^n x,\ x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right);$

l) $\sum_{n=1}^{\infty}\frac{n!}{(a+1)(a+2)\cdot \ldots \cdot (a+n)}x^n, a > 0, x \in \mathbb{R}.$

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