Seminar 3

***Exerciţii recomandate:** 3.1, 3.2(a,b,c,f,k), 3.3(a,c), 3.4

***Rezerve:** 3.2(g,i,o), 3.3(e), 3.6

Breviar teoretic

Serii de numere

$$\sum_{n=n_0}^{\infty} x_n \qquad \sum_{n\geq n_0}$$

x_n = termenul general

$S_n = \sum_{k=n_0}^{n} x_k$ $\quad (S_n)$ = şirul sumelor parţiale

<u>def</u>: Dacă (S_n) este convergent, atunci seria este convergentă.

<u>Prop</u>: Dacă seria este convergentă, atunci termenul general $x_n \longrightarrow 0$.

Propoziţie $p \rightarrow q \Leftrightarrow \neg q \rightarrow \neg p$

Dacă $x_n \not\rightarrow 0$, atunci seria diverge.

Dacă $\exists \lim_{n\to\infty} S_n$, această limită este suma seriei

Trei tipuri de exerciții de bază

→ suma seriei poate fi calculată (sumă telescopică / progresie)

→ pot spune din criterii de convergență dacă seria e convergentă / nu

→ termenul general nu tinde la 0

$$S = \sum_{i=1}^{n} x_i \qquad \text{for (int } i=1;\ i<=n;\ i++) \quad S += x_i$$

$\sum_{n=1}^{\infty} x_n$ ← sumă formală

Studiez $S_n = \sum_{i=1}^{n} x_i$

Dacă $\exists\ S = \lim_{n\to\infty} S_n$

$$\Rightarrow \sum_{n=1}^{\infty} x_n = S$$

S3.1 Stabiliţi natura următoarelor serii, iar în caz de convergenţă, determinaţi sumele lor.

a) $\sum_{n=1}^{\infty} \frac{1}{n^2+4n+3}$

Studiem şirul sumelor parţiale

$\sum \frac{1}{pol(n)}$ desc pol

$$S_n = \sum_{k=1}^{n} \frac{1}{k^2+4k+3} = \sum_{k=1}^{n} \frac{1}{(k+1)(k+3)} = \frac{1}{2}\sum_{k=1}^{n} \frac{2}{(k+1)(k+3)} =$$

$k^2+4k+3=0$

$k_{1,2} = \frac{-4 \pm \sqrt{16-4\cdot 3\cdot 1}}{2} = \frac{-4\pm 2}{2}$ → -1, -3

$$\frac{1}{2}\sum_{k=1}^{n} \frac{(k+3)-(k+1)}{(k+1)(k+3)} = \frac{1}{2}\sum_{k=1}^{n}\left(\frac{\cancel{k+3}}{(k+1)\cancel{(k+3)}} - \frac{\cancel{k+1}}{\cancel{(k+1)}(k+3)}\right)$$

$$= \frac{1}{2}\left(\sum_{k=1}^{n} \frac{1}{k+1} - \sum_{k=1}^{n} \frac{1}{k+3}\right) \xrightarrow[l=k+3]{j=k+1}$$

$$\frac{1}{2}\left(\sum_{j=2}^{n+1} \frac{1}{j} - \sum_{l=4}^{n+3} \frac{1}{l}\right) = \frac{1}{2}\left(\frac{1}{2}+\frac{1}{3}-\underbrace{\frac{1}{n+2}}_{\to 0}-\underbrace{\frac{1}{n+3}}_{\to 0}\right)$$

$$\to \frac{1}{2}\left(\frac{1}{2}+\frac{1}{3}\right) = \frac{1}{2}\cdot\frac{5}{6} = \frac{5}{12}$$

Seria converge la $\frac{5}{12}$

$$\frac{1}{2}+\frac{1}{3}+\cancel{\frac{1}{4}}+\dots+\cancel{\frac{1}{n}}+\cancel{\frac{1}{n+1}}$$
$\ominus$
$$\cancel{\frac{1}{4}}+\dots+\cancel{\frac{1}{n}}+\cancel{\frac{1}{n+1}}+\frac{1}{n+2}+\frac{1}{n+3}$$

$$\frac{1}{2}+\frac{1}{3}-\frac{1}{n+2}-\frac{1}{n+3}$$

b) $\sum_{n=1}^{\infty} \ln\frac{n+1}{n}$;

$\ln\frac{a}{b} = \ln a - \ln b$

$a, b > 0$

Studiem şirul sumelor parţiale

$$S_n = \sum_{k=1}^{n} \ln\frac{k+1}{k} = \sum_{k=1}^{n}\left[\ln(k+1)-\ln(k)\right] =$$

$$\sum_{k=1}^{n} \ln(k+1) - \sum_{k=1}^{n} \ln(k) \xrightarrow{k+1=j}$$

$$\sum_{j=2}^{n+1} \ln j - \sum_{k=1}^{n} \ln k = \ln(n+1) - \underbrace{\ln 1}_{0} =$$

$$\ln(n+1) \longrightarrow \infty$$

$\Rightarrow$ Seria diverge la ∞

2 ——— $n+1$ $\ominus$

1 ——— n

$n! = 1\cdot 2\cdot \ldots \cdot n \qquad 0! = 1$

$(n+1)! = (n-1)!\cdot n\cdot (n+1)$

c) $\sum_{n=1}^{\infty} \frac{n^2+n-1}{(n+1)!}$

$x_n = \frac{n^2+n-1}{(n+1)!} = \ldots = \frac{1}{(n-1)!} - \frac{1}{(n+1)!}$

Studiem şirul sumelor parţiale

$$s_n = \sum_{k=1}^{n} \frac{k^2+k-1}{(k+1)!} = \sum_{k=1}^{n} \left[\frac{k^2+k}{(k+1)!} - \frac{1}{(k+1)!}\right] =$$

$$= \sum_{k=1}^{n} \left[\frac{k(k+1)}{(k+1)!} - \frac{1}{(k+1)!}\right] = \sum_{k=1}^{n} \left[\frac{\cancel{k(k+1)}}{(k-1)!\cdot \cancel{k(k+1)}} - \frac{1}{(k+1)!}\right] =$$

$$\sum_{k=1}^{n} \left(\frac{1}{(k-1)!} - \frac{1}{(k+1)!}\right) = \sum_{k=1}^{n} \frac{1}{(k-1)!} - \sum_{k=1}^{n} \frac{1}{(k+1)!} \qquad \frac{j=k-1}{l=k+1}$$

$$\sum_{j=0}^{n-1} \frac{1}{j!} - \sum_{l=2}^{n+1} \frac{1}{l!} = \frac{1}{0!} + \frac{1}{1!} - \underbrace{\frac{1}{n!}}_{\downarrow\, 0} - \underbrace{\frac{1}{(n+1)!}}_{\downarrow\, 0} \longrightarrow 2$$

$0 \quad 1 \quad 2 \quad \ldots \quad n-1$

$ 2 \quad \ldots \quad n-1 \quad n \quad n+1$

seria converge la 2

d) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{2n+1}-\sqrt{2n-1}}$

$$x_n = \frac{1\cdot(\sqrt{2n+1}+\sqrt{2n-1})}{\sqrt{2n+1}-\sqrt{2n-1}} = \frac{\sqrt{2n+1}+\sqrt{2n-1}}{(\cancel{2n}+1)-(\cancel{2n}-1)} = \frac{\sqrt{2n+1}+\sqrt{2n-1}}{2} \to \infty$$

seria diverge la ∞.

$$\sum_{k=0}^{n} x^k = x^0 + \dots + x^n = \frac{x^{n+1}-1}{x-1}$$

$x \neq 1$

e) $\displaystyle\sum_{n=1}^{\infty} \frac{3^{n-1}+2^{n+1}}{6^n}$

şirul sumelor parţiale

$$S_n = \sum_{k=1}^{n} \frac{3^{k-1}+2^{k+1}}{6^k} = \sum_{k=1}^{n} \frac{3^{k-1}}{6^k} + \sum_{k=1}^{n} \frac{2^{k+1}}{6^k} =$$

$$\frac{1}{6}\sum_{k=1}^{n} \frac{3^{k-1}}{6^{k-1}} + 2\sum_{k=1}^{n} \frac{2^k}{6^k} = \frac{1}{6}\sum_{k=1}^{n}\left(\frac{1}{2}\right)^{k-1} + 2\sum_{k=1}^{n}\left(\frac{1}{3}\right)^k =$$

$$\frac{1}{6}\cdot\frac{\left(\frac{1}{2}\right)^n - 1}{\frac{1}{2}-1} + 2\cdot\frac{1}{3}\cdot\frac{\left(\frac{1}{3}\right)^n - 1}{\frac{1}{3}-1} \to \frac{1}{\cancel{6}_3}\cdot\frac{-1}{-\frac{1}{\cancel{2}}} + \frac{\cancel{2}}{\cancel{3}}\cdot\frac{-1}{-\frac{\cancel{2}}{\cancel{3}}}$$

$$\frac{1}{3} + 1 = \frac{4}{3}$$

$a^n \longrightarrow 0$ pt $|a| < 1$

Seria converge la $\frac{4}{3}$

$$\sum_{k=1}^{n}\left(\frac{1}{3}\right)^k = \frac{1}{3^1} + \frac{1}{3^2} + \dots + \frac{1}{3^n} =$$

$$\frac{1}{3}\left(\frac{1}{3^0} + \frac{1}{3^1} + \dots + \frac{1}{3^{n-1}}\right) =$$

$$\frac{1}{3}\cdot\frac{\left(\frac{1}{3}\right)^n - 1}{\frac{1}{3}-1}$$

f) $\displaystyle\sum_{n=1}^{\infty} \ln\left(1+\frac{2}{n(n+3)}\right)$

şirul sumelor parţiale

$$S_n = \sum_{k=1}^{n} \ln\left(1+\frac{2}{k(k+3)}\right) = \sum_{k=1}^{n} \ln\frac{k(k+3)+2}{k(k+3)} =$$

$$\sum_{k=1}^{n} \ln\frac{k^2+3k+2}{k(k+3)} = \sum_{k=1}^{n} \ln\frac{(k+1)(k+2)}{k(k+3)} =$$

$k^2+3k+2=0$

$$k_{1,2} = \frac{-3 \pm \sqrt{9-4\cdot 2}}{2} =$$

$$\frac{-3 \pm 1}{2} \to -2, \; -1$$

$$\sum_{k=1}^{n}\left[\ln(k+1) + \ln(k+2) - \ln k - \ln(k+3)\right]$$

$k+1=a$
$k+2=b$
$k+3=c$

$$\sum_{a=2}^{n+1} \ln a + \sum_{b=3}^{n+2} \ln b - \sum_{k=1}^{n} \ln k - \sum_{c=4}^{n+3} \ln c =$$

$$\ln(n+1) - \underbrace{\ln 1}_{0} + \ln 3 - \ln(n+3) =$$

$$\ln \frac{n+1}{n+3} + \ln 3 = \ln \frac{3(n+1)}{n+3} \xrightarrow[\text{conv}]{\text{ln}} \ln 3$$

Seria converge cu limită la $\ln 3$

g) $\sum_{n=0}^{\infty} \operatorname{arctg} \frac{1}{n^2+n+1}$;

Șirul sumelor parțiale

$$S_n = \sum_{k=0}^{n} \operatorname{arctg} \frac{1}{k^2+k+1} = \sum_{k=0}^{n} \operatorname{arctg} \frac{1}{k(k+1)+1} = \sum_{k=0}^{n} \operatorname{arctg} \frac{(k+1)-k}{k(k+1)+1}$$

$$\operatorname{tg}(a-b) = \frac{\operatorname{tg} a - \operatorname{tg} b}{\operatorname{tg} a \operatorname{tg} b + 1} \qquad a = \operatorname{arctg} x, \quad b = \operatorname{arctg} y$$

$$\operatorname{tg}(\operatorname{arctg} x - \operatorname{arctg} y) = \frac{\operatorname{tg}(\operatorname{arctg} x) - \operatorname{tg}(\operatorname{arctg} y)}{\operatorname{tg}(\operatorname{arctg} x)\operatorname{tg}(\operatorname{arctg} y) + 1}$$

$$\operatorname{tg}(\operatorname{arctg} x - \operatorname{arctg} y) = \frac{x-y}{xy+1}$$

$$\operatorname{arctg}(\operatorname{tg}(\operatorname{arctg} x - \operatorname{arctg} y)) = \operatorname{arctg} \frac{x-y}{xy+1}$$

$$\operatorname{arctg} x - \operatorname{arctg} y = \operatorname{arctg} \frac{x-y}{xy+1}$$

$$\Rightarrow \sum_{k=0}^{n} \left[\operatorname{arctg}(k+1) - \operatorname{arctg} k\right] =$$

$$\sum_{k=0}^{n} \operatorname{arctg}(k+1) - \sum_{k=0}^{n} \operatorname{arctg} k \overset{j=k+1}{=}$$

$$\sum_{j=1}^{n+1} \operatorname{arctg} j - \sum_{k=0}^{n} \operatorname{arctg} k =$$

$$\operatorname{arctg}(n+1) - \operatorname{arctg} \overset{0}{0} =$$

$$\operatorname{arctg}(n+1) \longrightarrow \frac{\pi}{2}$$

Seria converge la $\frac{\pi}{2}$

h) $\sum_{n=1}^{\infty} (\sqrt{n+2} - 2\sqrt{n+1} + \sqrt{n})$.

Șirul sumelor parțiale

$$S_n = \sum_{k=1}^{n} \left[\sqrt{k+2} - \sqrt{k+1} + \sqrt{k} - \sqrt{k+1}\right]$$

$$\overset{\substack{k+2=j \\ k+1=l}}{=} \sum_{j=3}^{n+2} \sqrt{j} - \sum_{l=2}^{n+1} \sqrt{l} + \sum_{k=1}^{n} \sqrt{k} - \sum_{l=2}^{n+1} \sqrt{l}$$

$$= \sqrt{n+2} - \sqrt{2} + \sqrt{1} - \sqrt{n+1}$$

$$= 1 - \sqrt{2} + \sqrt{n+2} - \sqrt{n+1}$$

$$= 1 - \sqrt{2} + \underbrace{\frac{\cancel{n}+2-(\cancel{n}+1)}{\sqrt{n+2}+\sqrt{n+1}}}_{\downarrow\, 0}$$

$$\longrightarrow 1 - \sqrt{2}$$

Seria converge la $1-\sqrt{2}$

Seria armonică generalizată

$$\sum \frac{1}{n^\alpha} \quad \alpha \begin{cases} \alpha > 1 & C \\ \alpha \leq 1 & D \end{cases}$$

$$\sum \frac{1}{n} \quad D$$

$$\frac{1}{n} \to 0$$

Dacă termenul tinde la 0 nu înseamnă că seria este convergentă

S3.2 Folosind diverse criterii de convergenţă, să se stabilească natura următoarelor serii:

a) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n(n+1)(n+2)}}$;

First idea

$$\frac{1}{\sqrt{n(n+1)(n+2)}} = \sqrt{\frac{1}{n(n+1)(n+2)}} = \sqrt{\frac{2(n+1)-(n+2)-n}{n(n+1)(n+2)}} =$$

$$\sqrt{\frac{2}{n(n+2)} - \frac{1}{n(n+1)} - \frac{1}{(n+1)(n+2)}}$$

$\rightarrow$ Won't work because I can't get rid of $\sqrt{\ }$

Dacă avem un raport de puteri ale lui n $\frac{n^p}{n^q}$ p puterea maximă a lui n numărător

q puterea maximă a lui n numitor

Compar cu $\frac{1}{n^{q-p}}$

$$a_n = \frac{1}{\sqrt{n(n+1)(n+2)}}$$

$$b_n = \frac{1}{n^{\frac{3}{2}}}$$

$$\sqrt{n(n+1)(n+2)} > n^{\frac{3}{2}} \Leftrightarrow$$

$$\sqrt{n^3+3n^2+2n} > \sqrt{n^3}$$

$$\Rightarrow \frac{1}{\sqrt{n(n+1)(n+2)}} < \frac{1}{n^{\frac{3}{2}}}$$

$$\left.\begin{array}{l} a_n < b_n \\ a_n, b_n > 0 \\ \sum b_n \ C \end{array}\right\} \overset{C\ C\ \mathrm{I}}{\Rightarrow}$$

$$\sum a_n \ C$$

Serii cu termeni pozitive

$$a_n \le b_n$$

$$C \Leftarrow C$$

$$D \Rightarrow D$$

b) $\sum_{n=3}^{\infty} \frac{4n-3}{n(n^2-4)}$; $= \sum_{n=3}^{\infty} \frac{4n-3}{n^3-4n}$

$a_n = \frac{4n-3}{n^3-4n}$

$b_n = \frac{1}{n^{3-1}} = \frac{1}{n^2}$

$\frac{a_n}{b_n} = \frac{\frac{4n-3}{\cancel{n}(n^2-4)}}{\frac{1}{n^{\cancel{2}}}} = \frac{4n-3}{n^2-4} \cdot n =$

$\frac{4n^2-3n}{n^2-4} \to 4 \in (0,\infty)$ $\Big\}$ CC(II) $\Rightarrow$

$a_n, b_n > 0$

$\sum a_n \sim \sum b_n$ au aceeași natură $\Big\}$

$\sum b_n$ C

$\sum a_n$ C

* $a_n = \frac{4n-3}{n(n-2)(n+2)}$

Orice fracție de polinoame se poate descompune în fracții simple

$\frac{4n-3}{n(n-2)(n+2)} = \frac{A}{n} + \frac{B}{n+2} + \frac{C}{n-2}$

(amplificări: $(n+2)(n-2)$, $n(n-2)$, $n(n+2)$)

$4n-3 = A(n+2)(n-2) + B\,n(n-2) + C\,n(n+2)$

2 moduri $\to$ fac calcule în stânga și obțin egalitate de polinoame

$\to$ înlocuiesc și în dreapta și în stânga cu n anulând polinoamele de la numitor

$n=0 \quad -3 = A \cdot 2 \cdot (-2) \Rightarrow A = \frac{3}{4} = \frac{6}{8}$

$n=2 \quad 5 = C \cdot 8 \Rightarrow C = \frac{5}{8}$

$n=-2 \quad -11 = B \cdot 8 \Rightarrow B = -\frac{11}{8}$

$$a_n = \frac{\frac{6}{8}}{n} + \frac{\frac{5}{8}}{n-2} - \frac{\frac{11}{8}}{n+2} =$$

$$\frac{1}{8}\left(\frac{6}{n} - \frac{6}{n+2} + \frac{5}{n-2} - \frac{5}{n+2}\right)$$

şirul sumelor parţiale:

$$\sum_{k=3}^{n} a_k = \frac{1}{8}\left(\sum_{k=3}^{n}\frac{6}{k} - \sum_{k=3}^{n}\frac{6}{k+2} + \sum_{k=3}^{n}\frac{5}{k-2} - \sum_{k=3}^{n}\frac{5}{k+2}\right) \underset{k-2=b}{\overset{k+2=a}{=\!=}}$$

$$\frac{1}{8}\left(\sum_{k=3}^{n}\frac{6}{k} - \sum_{a=5}^{n+2}\frac{6}{a} + \sum_{b=1}^{n-2}\frac{5}{b} - \sum_{a=5}^{n+2}\frac{5}{a}\right) =$$

$$\frac{1}{8}\left(\frac{6}{3} + \frac{6}{4} - \frac{6}{n+1} - \frac{6}{n+2} + \frac{5}{1} + \frac{5}{2} + \frac{5}{3} + \frac{5}{4} - \frac{5}{n-1} - \frac{5}{n} - \frac{5}{n+1} - \frac{5}{n+2}\right)$$

Am un n finit de fracţii 6 care tind la 0

$$\rightarrow \frac{1}{8}\left(\frac{11}{3} + \frac{11}{4} + \frac{15}{2}\right) = \frac{1}{8} \cdot \frac{77+90}{12} = \frac{167}{96}$$

c) $\sum_{n=1}^{\infty}\left(\frac{1^3+2^3+\ldots+n^3}{n^3}-\frac{n}{4}\right)^n$;

$$a_n = \left(\frac{1^3+2^3+\ldots+n^3}{n^3}-\frac{n}{4}\right)^n = \left(\frac{\cancel{n^2}(n+1)^2}{4n^{\cancel{3}}}-\frac{n}{4}\right)^n$$

$$= \left(\frac{\cancel{n^2}+2n+1-\cancel{n^2}}{4n}\right)^n = \left(\frac{2n+1}{4n}\right)^n \quad a_n>0$$

$$\sqrt[n]{a_n} = \frac{2n+1}{4n} \longrightarrow \frac{1}{2}\in(0,1) \xRightarrow[\text{rad}]{\text{crit}} \sum a_n \; C$$

ex: Dacă $\sqrt[n]{a_n} = \frac{4n}{2n+1} \longrightarrow 2>1 \Rightarrow \sum a_n \; D$

d) $\sum_{n=1}^{\infty}\operatorname{arctg}\frac{1}{2n^2}$;

e) $\sum_{n=2}^{\infty}\frac{1}{(\ln n)^{\ln n}}$;

f) $\sum_{n=1}^{\infty}\frac{1^2\cdot5^2\cdot9^2\cdot\ldots\cdot(4n-3)^2}{3^2\cdot7^2\cdot11^2\cdot\ldots\cdot(4n-1)^2}$;

Prin inducție: $\Rightarrow \sum a_n \; D$

$a_n \geq \frac{1}{9n}$ | $a_n>0$

$$a_n = \frac{1^2\cdot5^2\cdot9^2\cdot\ldots\cdot(4n-3)^2}{3^2\cdot7^2\cdot11^2\cdot\ldots\cdot(4n-1)^2}$$

Încerc să aplic crit raportului

$$\frac{a_{n+1}}{a_n} = \frac{\dfrac{\cancel{1^2\cdot5^2\cdot\ldots\cdot(4n-3)^2}\cdot(4n+1)^2}{\cancel{3^2\cdot7^2\cdot11^2\cdot\ldots\cdot(4n-1)^2}\cdot(4n+3)^2}}{\dfrac{\cancel{1^2\cdot5^2\cdot\ldots\cdot(4n-3)^2}}{\cancel{3^2\cdot7^2\cdot11^2\cdot\ldots\cdot(4n-1)^2}}} =$$

$$\frac{(4n+1)^2}{(4n+3)^2} \longrightarrow 1$$

Nu pot aplica criteriul raportului

Încerc Raabe-Duhamel

$$n\left(\frac{a_n}{a_{n+1}}-1\right)=n\left(\frac{(4n+3)^2}{(4n+1)^2}-1\right)=$$

$$n\,\frac{(4n+3)^2-(4n+1)^2}{(4n+1)^2}=$$

$$n\,\frac{(\cancel{4n}+3-\cancel{4n}-1)(4n+3+4n+1)}{16n^2+8n+1}=$$

$$\frac{2n(8n+4)}{16n^2+8n+1}=\frac{16n^2+8n}{16n^2+8n+1}\to 1$$

Nu pot aplica Raabe-Duhamel

Vream să aplic Gauss

$$\frac{a_n}{a_{n+1}}\overset{?}{=}\underbrace{\lambda+\frac{\mu}{n}+\frac{x_n}{n^{\alpha+1}}}_{}$$

$\lambda,\mu\in\mathbb{R}$

$\alpha>0$

x_n mărg

$$\frac{a_n}{a_{n+1}}=\frac{(4n+3)^2}{(4n+1)^2}=$$

$$\frac{16n^2+24n+9}{16n^2+8n+1} =$$

$$\frac{16n^2+8n+1+16n+8}{16n^2+8n+1} =$$

$$1 + \frac{16n+8}{16n^2+8n+1} =$$

$$1 + \frac{16n}{16n^2+8n+1} + \frac{8}{16n^2+8n+1} =$$

$$1 + \frac{16}{16n+8+\frac{1}{n}} + \frac{8}{16n^2+8n+1}$$

$$1 + \frac{1=\mu}{n+\frac{1}{2}+\frac{1}{2n}} + \frac{8 = x_n}{16n^2+8n+1}$$

$$1 = \lambda$$

$\xrightarrow[\text{Gauss}]{\text{Crit}}$ Seria diverge

g) (R) $\displaystyle\sum_{n=1}^{\infty} \frac{1! + 2! + ... + n!}{(n+2)!}$;

h) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$;

i) (R) $\displaystyle\sum_{n=1}^{\infty} (\sqrt{n^4 + 3n^2 + 1} - n^2)$.

j) $\displaystyle\sum_{n=1}^{\infty}\left(\frac{n!}{n^n}\right)^2$;

k) $\displaystyle\sum_{n=1}^{\infty}\frac{(2n-1)!!}{(2n)!!}\cdot\frac{1}{2n+1}$, unde $(2n-1)!! = 1\cdot 3\cdot \ldots \cdot (2n-1)$;

$$a_n = \frac{(2n-1)!!}{(2n)!!}\cdot\frac{1}{2n+1}$$

Crit raportului nu dă nimic

$$\frac{a_{n+1}}{a_n} = \frac{\dfrac{(2n+1)!!}{(2n+2)!!}\cdot\dfrac{1}{2n+3}}{\dfrac{(2n-1)!!}{(2n)!!}\cdot\dfrac{1}{2n+1}} = \frac{\dfrac{2n+1}{2n+2}\cdot\dfrac{1}{2n+3}}{\dfrac{1}{2n+1}} = \frac{(2n+1)^2}{(2n+2)(2n+3)} \to 1$$

$$n\left(\frac{a_n}{a_{n+1}}-1\right) = n\cdot\left(\frac{(2n+2)(2n+3)}{(2n+1)^2}-1\right) =$$

$$n\,\frac{4n^2+10n+6-4n^2-4n-1}{4n^2+4n+1} =$$

$$\frac{n(6n+5)}{4n^2+4n+1} \to \frac{3}{2} > 1$$

$\Downarrow$ Raabe

$\sum a_n$ C

l) $\displaystyle\sum_{n=1}^{\infty}\arcsin\frac{1}{n\sqrt[3]{n}+5}$

m) $\displaystyle\sum_{n=1}^{\infty}n^2\ln\Big(1+\frac{1}{n^2}\Big)$;

n) $\displaystyle\sum_{n=1}^{\infty}\Big(\frac{\pi}{2}-\operatorname{arctg} n\Big)^n$;

o) (R) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{e\cdot\sqrt{e}\cdot\sqrt[3]{e}\cdot\ldots\cdot\sqrt[n]{e}}$;

p) $\sum_{n=1}^{\infty} \frac{2^n + 3^{n+1} - 6^{n-1}}{12^n}$;

q) $\sum_{n=1}^{\infty} \left(\frac{1}{n} + \ln \frac{n}{n+1} \right)$;

S3.3 Precizaţi natura seriilor următoare în funcţie de parametrii corespunzători.

a) $\sum_{n=2}^{\infty} \frac{\sqrt{n+2} - \sqrt{n-2}}{n^{\alpha}}, \alpha \in \mathbb{R}$;

$$a_n = \frac{\sqrt{n+2} - \sqrt{n-2}}{n^{\alpha}} = \frac{(\cancel{n}+2) - (\cancel{n}-2)}{n^{\alpha}(\sqrt{n+2} + \sqrt{n-2})} =$$

$$\frac{4}{n^{\alpha}(\sqrt{n+2} + \sqrt{n-2})}$$

$$b_n = \frac{1}{n^{\alpha + \frac{1}{2}}} \qquad \sqrt{n}$$

$$\frac{a_n}{b_n} = \frac{\frac{4}{\cancel{n^{\alpha}}(\sqrt{n+2} + \sqrt{n-2})}}{\frac{1}{\cancel{n^{\alpha+\frac{1}{2}}}}} = \frac{4}{\sqrt{1+\frac{2}{n}} + \sqrt{1-\frac{2}{n}}} \to 2$$

$$2 \in (0, \infty)$$

$\mathrm{CC}\,\underline{\overline{\mathrm{III}}} \Rightarrow$

$$\sum a_n \sim \sum b_n \begin{cases} C & \alpha + \frac{1}{2} > 1 \\ D & \alpha + \frac{1}{2} \leq 1 \end{cases}$$

b) $\sum_{n=1}^{\infty} \frac{\operatorname{arctg}(n\alpha)}{(\ln 3)^n}, \alpha \in \mathbb{R}$;

c) $\sum_{n=1}^{\infty} \underbrace{\frac{\alpha^n}{\sqrt[n]{n!}}}_{a_n}, \alpha \in \mathbb{R}_+^*$;

$$\sum a_n \begin{cases} C & \alpha > \frac{1}{2} \\ D & \alpha \leq \frac{1}{2} \end{cases}$$

Se poate arăte că

$$n > \sqrt[n]{n!} \geq \sqrt{n}$$

Dacă $\alpha \geq 1 \Rightarrow \alpha^n \geq 1$

$$\left.\begin{matrix} \frac{\alpha^n}{\sqrt[n]{n!}} \geq \frac{1}{n} \\ \sum \frac{1}{n}\ D \end{matrix}\right\} \Rightarrow \sum a_n\ D$$

Dacă $\alpha < 1 \Rightarrow \frac{\alpha^n}{\sqrt[n]{n!}} \le \frac{\alpha^n}{\sqrt{n}} \Rightarrow \sum a_n\ C$

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d) $\sum_{n=2}^{\infty} \left(\sqrt{n+1} - \sqrt{n}\right)^a \ln\left(\frac{n+1}{n-1}\right), a \in \mathbb{R};$

e) $*\sum_{n=1}^{\infty} \frac{\alpha(\alpha+1)\cdot\ldots\cdot(\alpha+n-1)}{n!n^\beta}, \alpha \in \mathbb{R}_+^*, \beta \in \mathbb{R}.$

S3.4 Utilizând criteriul logaritmului să se studieze convergenţa următoarei serii: $\sum_{n=1}^{\infty} \left(\frac{1}{n^3 - n + 3} \right)^{\ln(n+1)}$.

S3.5* Fie $\sum_{n=1}^{\infty} u_n$ o serie convergentă din $\mathbb{R}$, cu $u_n \geq 0$, $\forall\ n \in \mathbb{N}^*$. Ce se poate spune despre natura seriei $\sum_{n=1}^{\infty} \left(\frac{u_n}{1 + u_n} \right)^{\alpha}$, unde α este un număr real?

S3.6* Să se analizeze seria cu termenul general

$$\arccos \frac{n(n+1) + \sqrt{(n+1)(n+2)(3n+1)(3n+4)}}{(2n+1)(2n+3)}, n \in \mathbb{N}^*$$

şi, în caz de convergenţă, să i se afle suma.

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