# Seminar 3

***Exerciţii recomandate:** 3.1, 3.2(a,b,c,f,k), 3.3(a,c), 3.4

***Rezerve:** 3.2(g,i,o), 3.3(e), 3.6

## Breviar teoretic

$\sum_{n\geq k}^{\infty} x_n$ $\quad (x_n)$ = termen general

$S_n = \sum_{j=k}^{n} x_j$ = şirul sumelor parţiale

Dacă $S_n$ (C), atunci seria este convergentă.

Dacă seria este convergentă, atunci $x_n \to 0$

$p \to q \iff \neg q \to \neg p$

Dacă $(x_n) \not\to 0 \Rightarrow$ seria este divergentă

<u>Observaţie</u>: Se poate întâmpla ca $x_n \to 0$ şi seria să fie divergentă. <u>ex</u>: $\frac{1}{n}$

Dacă $\exists \lim_{n\to\infty} S_n$ atunci ea se numeşte suma seriei.

$$x_1 = 0^1 = 0 \quad \dots \quad x_n = 0^n = 0$$

$$x_2 = 0^2 = 0$$

**S3.1** Stabiliţi natura următoarelor serii, iar în caz de convergenţă, determinaţi sumele lor.

a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+4n+3}$

$$k^2+4k+3=0 \Rightarrow \begin{matrix} -1 \\ -3 \end{matrix}$$

Şirul sumelor parţiale

$$S_n = \sum_{k=1}^{n} \frac{1}{k^2+4k+3} = \sum_{k=1}^{n} \frac{1}{(k+1)(k+3)} = \frac{1}{2}\sum_{k=1}^{n} \frac{2}{(k+1)(k+3)} =$$

$$\frac{1}{2}\sum_{k=1}^{n} \frac{(k+3)-(k+1)}{(k+1)(k+3)} = \frac{1}{2}\sum_{k=1}^{n}\left[\frac{\cancel{k+3}}{(k+1)\cancel{(k+3)}} - \frac{\cancel{k+1}}{\cancel{(k+1)}(k+3)}\right] =$$

$$\frac{1}{2}\sum_{k=1}^{n}\left(\frac{1}{k+1} - \frac{1}{k+3}\right) \overset{k+1=j,\ k+3=l}{=} \frac{1}{2}\left(\sum_{j=2}^{n+1}\frac{1}{j} - \sum_{l=4}^{n+3}\frac{1}{l}\right) = \frac{1}{2}\left(\frac{1}{2}+\frac{1}{3} - \frac{1}{n+2} - \frac{1}{n+3}\right) \longrightarrow \frac{1}{2}\left(\frac{1}{2}+\frac{1}{3}\right) = \frac{1}{2}\cdot\frac{5}{6} = \frac{5}{12}$$

$$\frac{1}{n+2} \xrightarrow{n\to\infty} 0, \qquad \frac{1}{n+3} \xrightarrow{n\to\infty} 0$$

$\Rightarrow$ Seria converg cu suma $\frac{5}{12}$

b) $\displaystyle\sum_{n=1}^{\infty} \ln\frac{n+1}{n}$; $\displaystyle = \sum_{n=1}^{\infty}\left(\ln(n+1) - \ln(n)\right)$

Şirul sumelor parţiale

$$S_n = \sum_{k=1}^{n}\left(\ln(k+1) - \ln(k)\right) = \sum_{k=1}^{n}\ln(k+1) - \sum_{k=1}^{n}\ln(k) \overset{k+1=j}{=}$$

$$\sum_{j=2}^{n+1}\ln(j) - \sum_{k=1}^{n}\ln(k) = \ln(n+1) - \ln(1) = \underline{\ln(n+1)} \xrightarrow{n\to\infty} \infty$$

$\Rightarrow$ Seria divergentă cu suma $\infty$

$$(k+1)! = k!\,(k+1) = (k-1)!\,k(k+1)$$

$$0! = 1$$

c) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+n-1}{(n+1)!}$

Șirul sumelor parțiale

$$S_n = \sum_{k=1}^{n} \frac{k^2+k-1}{(k+1)!} = \sum_{k=1}^{n} \frac{k(k+1)-1}{(k+1)!} = \sum_{k=1}^{n} \left[ \frac{\cancel{k(k+1)}}{\underset{(k-1)!}{\cancel{(k+1)!}}} - \frac{1}{(k+1)!} \right]$$

$$\overset{\substack{j=k-1 \\ l=k+1}}{=} \sum_{j=0}^{n-1} \frac{1}{j!} - \sum_{l=2}^{n+1} \frac{1}{l!} = \frac{1}{0!} + \frac{1}{1!} - \underset{\substack{\downarrow \\ 0}}{\frac{1}{n!}} - \underset{\substack{\downarrow \\ 0}}{\frac{1}{(n+1)!}} \longrightarrow 2$$

$\Rightarrow$ Seria este convergentă cu suma 2.

d) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{2n+1}-\sqrt{2n-1}}$

Studiem termenul general

$$x_k = \frac{1}{\sqrt{2k+1}-\sqrt{2k-1}} \cdot \frac{\sqrt{2k+1}+\sqrt{2k-1}}{\sqrt{2k+1}+\sqrt{2k-1}} = \frac{\sqrt{2k+1}+\sqrt{2k-1}}{(\cancel{2k}+1)-(\cancel{2k}-1)} =$$

$$\frac{\sqrt{2k+1}+\sqrt{2k-1}}{2} \longrightarrow \infty \Rightarrow$$ seria este divergentă cu suma $\infty$

$$x^0 + x^1 + \dots + x^{n-1} = \frac{x^n - 1}{x-1}$$

$$x^0 + x^1 + \dots + x^n = \frac{x^{n+1}-1}{x-1}$$

$$x^1 + \dots + x^n = \frac{x^{n+1}-1}{x-1} - 1$$

e) $\displaystyle\sum_{n=1}^{\infty} \frac{3^{n-1} + 2^{n+1}}{6^n}$

Studiem șirul sumelor parțiale

$$S_n = \sum_{k=1}^{n} \frac{3^{k-1} + 2^{k+1}}{6^k} = \sum_{k=1}^{n} \left( \frac{3^{k-1}}{6^k} + \frac{2^{k+1}}{6^k} \right) = \sum_{k=1}^{n} \left[ \frac{1}{6} \cdot \left(\frac{3}{6}\right)^{k-1} + 2 \cdot \left(\frac{2}{6}\right)^k \right] = \frac{1}{6} \sum_{k=1}^{n} \left(\frac{1}{2}\right)^{k-1} + 2 \cdot \sum_{k=1}^{n} \left(\frac{1}{3}\right)^k = \frac{1}{6} \cdot \frac{\left(\frac{1}{2}\right)^n - 1}{\frac{1}{2} - 1} +$$

$$2 \cdot \left( \frac{\left(\frac{1}{3}\right)^{n+1} - 1}{\frac{1}{3} - 1} - 1 \right) \longrightarrow \frac{1}{6} \frac{1}{\frac{1}{2}} + 2 \left( \frac{1}{\frac{2}{3}} - 1 \right) =$$

$$\frac{1}{3} + 2 \cdot \frac{1}{2} = \frac{4}{3} \Rightarrow$$

seria converge cu suma $\frac{4}{3}$

f) $\displaystyle\sum_{n=1}^{\infty} \ln\left(1 + \frac{2}{n(n+3)}\right)$

$$k^2 + 3k + 2 = k^2 + 2k + k + 2 = k(k+2) + (k+2) = (k+2)(k+1)$$

Studiem șirul sumelor parțiale

$$S_n = \sum_{k=1}^{n} \ln\left[\frac{k(k+3)+2}{k(k+3)}\right] = \sum_{k=1}^{n} \ln \frac{k^2+3k+2}{k(k+3)} = \sum_{k=1}^{n} \ln \frac{(k+1)(k+2)}{k(k+3)}$$

$$= \ln \prod_{k=1}^{n} \frac{(k+1)(k+2)}{k(k+3)} = \ln \frac{\prod_{k=1}^{n}(k+1) \prod_{k=1}^{n}(k+2)}{\prod_{k=1}^{n} k \prod_{k=1}^{n}(k+3)} =$$

$$\ln \frac{\frac{(n+1)!}{1!} \cdot \frac{(n+2)!}{2!}}{n! \, \frac{(n+3)!}{3!}} = \ln \frac{\frac{n+1}{2}}{\frac{n+3}{6}} = \ln \frac{n+1}{\cancel{2}} \cdot \frac{\cancel{6}\,3}{n+3} =$$

$$\ln \underbrace{\frac{3(n+1)}{n+3}}_{\downarrow \, 3} \longrightarrow \ln 3 \Rightarrow$$ Seria convergentă cu suma $\ln 3$

g) $\sum_{n=0}^{\infty} \operatorname{arctg} \frac{1}{n^2+n+1}$;

Şirul sumelor parțiale

$$S_n = \sum_{k=0}^{n} \operatorname{arctg} \frac{1}{k^2+k+1} = \sum_{k=0}^{n} \operatorname{arctg} \frac{1}{k(k+1)+1} = \sum_{k=0}^{n} \operatorname{arctg} \frac{(k+1)-k}{k(k+1)+1}$$

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$$\frac{x-y}{xy+1} \qquad \operatorname{tg}(a-b) = \frac{\operatorname{tg} a - \operatorname{tg} b}{1+\operatorname{tg} a \operatorname{tg} b}$$

$a = \operatorname{arctg} x$

$b = \operatorname{arctg} y$

$$\operatorname{tg}(\operatorname{arctg} x - \operatorname{arctg} y) = \frac{x-y}{1+xy} \Big| \operatorname{arctg}$$

$$\operatorname{arctg}(\operatorname{tg}(\operatorname{arctg} x - \operatorname{arctg} y)) = \operatorname{arctg} \frac{x-y}{1+xy}$$

Recapitulați fct. trig

$$\operatorname{arctg} x - \operatorname{arctg} y = \operatorname{arctg} \frac{x-y}{1+xy}$$

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$$= \sum_{k=0}^{n} \left[\operatorname{arctg}(k+1) - \operatorname{arctg} k\right] \overset{j=k+1}{=} \sum_{j=1}^{n+1} \operatorname{arctg} j - \sum_{k=0}^{n} \operatorname{arctg} k$$

$$= \operatorname{arctg}(\underbrace{n+1}_{\downarrow\, \infty}) - \underbrace{\operatorname{arctg} 0}_{0} \longrightarrow \frac{\pi}{2} \Rightarrow \text{seria converge la } \frac{\pi}{2}$$

h) $\sum_{n=1}^{\infty} (\sqrt{n+2} - 2\sqrt{n+1} + \sqrt{n})$.

Studiem șirul sumelor parțiale

$$S_n = \sum_{k=1}^{n} \left[ \left( \sqrt{k+2} - \sqrt{k+1} \right) - \left( \sqrt{k+1} - \sqrt{k} \right) \right] =$$

$$\sum_{k=1}^{n} \left[ \frac{(k+2)-(k+1)}{\sqrt{k+2}+\sqrt{k+1}} - \frac{(k+1)-k}{\sqrt{k+1}+\sqrt{k}} \right] =$$

$$\sum_{k=1}^{n} \left[ \frac{1}{\sqrt{k+2}+\sqrt{k+1}} - \frac{1}{\sqrt{k+1}+\sqrt{k}} \right] \overset{j=k+1}{=}$$

$$\sum_{j=2}^{n+1} \frac{1}{\sqrt{j+1}+\sqrt{k}} - \sum_{k=1}^{n} \frac{1}{\sqrt{k+1}+\sqrt{k}} =$$

$\sqrt{2}-1$,

$$\underbrace{\frac{1}{\sqrt{n+2}+\sqrt{n+1}}}_{\downarrow\, 0} - \frac{1}{\sqrt{2}+\sqrt{1}} \longrightarrow -\left( \frac{1}{\sqrt{2}+\sqrt{1}} \right) =$$

$1-\sqrt{2}$

Suma converge la $1-\sqrt{2}$

**S3.2** Folosind diverse criterii de convergenţă, să se stabilească natura următoarelor serii:

a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n(n+1)(n+2)}}$;

$$\sum \frac{1}{n^{\alpha}} \begin{cases} C & \alpha > 1 \\ D & \alpha \le 1 \end{cases}$$

$$a_n = \frac{1}{\sqrt{n(n+1)(n+2)}}$$

$$b_n = \frac{1}{n^{\frac{3}{2}}}$$

$$\sqrt{n(n+1)(n+2)} > \sqrt{n^3}$$

$$\left.\begin{matrix} a_n < b_n \\ \sum b_n \ C \end{matrix}\right\} \Longrightarrow \sum a_n \ C$$

b) $\displaystyle\sum_{n=3}^{\infty} \frac{4n-3}{n(n^2-4)}$;

$$a_n = \frac{4n-3}{n(n^2-4)} \qquad a_n > 0$$

$$b_n > 0$$

$$b_n = \frac{1}{n^2} \quad C$$

$$\frac{a_n}{b_n} = \frac{\frac{4n-3}{\cancel{n}(n^2-4)}}{\frac{1}{n^{\cancel{2}}}} = \frac{(4n-3)\cdot n}{n^2-4} = \frac{4n^2-3n}{n^2-4} \to 4$$

$$4 \in (0, \infty) \Rightarrow$$

$$\sum a_n \sim \sum b_n \ C$$

$$\Rightarrow \sum a_n \ C$$

$$a_n = \frac{4n-3}{n(n-2)(n+2)} = \frac{4\cancel{n}}{\cancel{n}(n-2)(n+2)} - \frac{3}{n(n-2)(n+2)}$$

$$\frac{4}{(n-2)(n+2)} = \frac{n+2-(n-2)}{(n-2)(n+2)} = \frac{\cancel{n+2}}{(n-2)(\cancel{n+2})} - \frac{\cancel{n-2}}{(\cancel{n-2})(n+2)}$$

$$\underbrace{\frac{1}{n-2} - \frac{1}{n+2}}$$

$$\frac{3}{n(n-2)(n+2)} = \frac{A}{n} + \frac{B}{n-2} + \frac{C}{n+2} =$$

$$3 = A(n-2)(n+2) + Bn(n+2) + Cn(n-2)$$

$n=0 \quad 3 = A\cdot(-2)\cdot 2 \Rightarrow A = -\frac{3}{4}$

$n=2 \quad 3 = B\cdot 2\cdot 4 \Rightarrow B = \frac{3}{8}$

$n=-2 \quad 3 = C\cdot(-2)\cdot(-4) \Rightarrow C = \frac{3}{8}$

$$= \frac{3}{8}\left(-\frac{2}{n} + \frac{1}{n-2} + \frac{1}{n+2}\right)$$

$$a_n = \frac{1}{n-2} - \frac{1}{n+2} - \frac{3}{8}\left(-\frac{1}{n} + \frac{1}{n-2} - \frac{1}{n} + \frac{1}{n+2}\right)$$

Studiem șirul sumelor parțiale

$$S_n = \sum_{k=3}^{n} a_k = \sum_{k=3}^{n} \frac{1}{k-2} - \sum_{k=3}^{n} \frac{1}{k+2}$$

$$- \frac{3}{8}\left( - \sum_{k=3}^{n} \frac{1}{k} + \sum_{k=3}^{n} \frac{1}{k-2} - \sum_{k=3}^{n} \frac{1}{k} + \sum_{k=3}^{n} \frac{1}{k+2} \right)$$

$$\sum_{k=1}^{n-2} \frac{1}{k} - \sum_{k=5}^{n+2} \frac{1}{k} - \frac{3}{8}\left( - \sum_{k=3}^{n} \frac{1}{k} + \sum_{k=1}^{n-2} \frac{1}{k} \right.$$

$$\left. - \sum_{k=3}^{n} \frac{1}{k} + \sum_{k=5}^{n+2} \frac{1}{k} \right) =$$

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} - \underset{\substack{\downarrow \\ 0}}{\frac{1}{n-1}} - \underset{\substack{\downarrow \\ 0}}{\frac{1}{n}} - \underset{\substack{\downarrow \\ 0}}{\frac{1}{n+1}} - \underset{\substack{\downarrow \\ 0}}{\frac{1}{n+2}}$$

$$- \frac{3}{8}\left( \frac{1}{1} + \frac{1}{2} - \underset{\substack{\downarrow \\ 0}}{\frac{1}{n-1}} - \underset{\substack{\downarrow \\ 0}}{\frac{1}{n}} - \frac{1}{3} - \frac{1}{4} + \underset{\substack{\downarrow \\ 0}}{\frac{1}{n+1}} + \underset{\substack{\downarrow \\ 0}}{\frac{1}{n+2}} \right)$$

$$= \frac{167}{96}$$

c) $\sum_{n=1}^{\infty}\left(\frac{1^3+2^3+\ldots+n^3}{n^3}-\frac{n}{4}\right)^n$;

Fie termenul general $a_n=\left(\frac{1^3+\ldots+n^3}{n^3}-\frac{n}{4}\right)^n=$

$a_n=\left(\frac{\cancel{n^2}(n+1)^2}{4n^{\cancel{3}}}-\frac{n}{4}\right)^n=\left[\frac{(n+1)^2-n^2}{4n}\right]^n=$

$\left(\frac{2n+1}{4n}\right)^n$ $\qquad a_n>0$

$\sqrt[n]{a_n}=\frac{2n+1}{4n}\longrightarrow\frac{1}{2}<1\xrightarrow[\text{rădăcinii}]{\text{Crit}}$ seria converg

d) $\sum_{n=1}^{\infty}\operatorname{arctg}\frac{1}{2n^2}$;

e) $\sum_{n=2}^{\infty}\frac{1}{(\ln n)^{\ln n}}$; $\qquad 4(n+1)-3$

f) $\sum_{n=1}^{\infty}\frac{1^2\cdot5^2\cdot9^2\cdot\ldots\cdot(4n-3)^2}{3^2\cdot7^2\cdot11^2\cdot\ldots\cdot(4n-1)^2}$; $\qquad a_n>0$

Fie termenul general $a_n=\frac{1^2\cdot5^2\cdot9^2\cdot\ldots\cdot(4n-3)^2}{3^2\cdot7^2\cdot11^2\cdot\ldots\cdot(4n-1)^2}$

$\frac{a_{n+1}}{a_n}=\frac{(4n+1)^2}{(4n+3)^2}\longrightarrow1$ Criteriul raportului nu ne dă informații

$a_1=\frac{1}{9}$

$a_2=\frac{25}{9\cdot49}>\frac{25}{9\cdot50}=\frac{1}{9\cdot2}$

$a_3=\frac{25\cdot81}{9\cdot49\cdot121}=\frac{2025}{9\cdot5929}>\frac{2025}{9\cdot6075}=\frac{1}{9\cdot3}$

Vrem să dem inducție că

$P(n)$ $a_n \geq \frac{1}{9 \cdot n}$

$P(1)$ (A)

Pp $P(k)$ adev $a_k \geq \frac{1}{9k}$

$a_{k+1} \geq \frac{1}{9(k+1)}$

$$a_{k+1} = a_k \cdot \frac{(4k+1)^2}{(4k+3)^2} \geq \frac{1}{9k} \cdot \frac{(4k+1)^2}{(4k+3)^2}$$

$$\frac{(4k+1)^2}{\cancel{9}k(4k+3)^2} \geq \frac{1}{\cancel{9}(k+1)} \iff$$

$$(k+1)(4k+1)^2 \geq k(4k+3)^2$$

$$(k+1)(16k^2+8k+1) \geq k(16k^2+24k+9)$$

$$16k^3+24k^2+9k+1 \geq 16k^3+24k^2+9k$$

$\Rightarrow P(k+1)$ adev

$$\left.\begin{array}{l} P(1) \text{ (A)} \\ P(k) \to P(k+1) \end{array}\right\} \Rightarrow P(n) \text{ (A) } \forall n \geq 1$$

$a_n \geq \frac{1}{9n}$

$$\sum \frac{1}{9n} \overset{CC_{III}}{\sim} \sum \frac{1}{n} \text{ D}$$

$$\sum \frac{1}{9n} \text{ D} \Rightarrow \sum a_n \text{ D}$$

g) (R) $\sum_{n=1}^{\infty} \frac{1!+2!+...+n!}{(n+2)!}$;

Raabe - Duhamel

Seria inițial $\frac{1!+2!+\ldots+n!}{n!} \to 1$

h) $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$;

i) (R) $\sum_{n=1}^{\infty} (\sqrt{n^4+3n^2+1} - n^2)$. $\to$ raționalizare

j) $\sum_{n=1}^{\infty}\left(\frac{n!}{n^n}\right)^2$;

k) $\sum_{n=1}^{\infty}\frac{(2n-1)!!}{(2n)!!}\cdot\frac{1}{2n+1}$, unde $(2n-1)!!=1\cdot 3\cdot \ldots \cdot(2n-1)$;

$$a_n=\frac{(2n-1)!!}{(2n)!!}\cdot\frac{1}{2n+1}$$

$$\frac{a_{n+1}}{a_n}=\frac{\frac{2n+1}{2n+2}\cdot\frac{1}{2n+3}}{\frac{1}{2n+1}}=\underbrace{\frac{(2n+1)^2}{(2n+2)(2n+3)}}\to 1$$

Criteriul raportului nu oferă informații

$$\lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}}-1\right)=\lim_{n\to\infty} n\left(\frac{(2n+2)(2n+3)}{(2n+1)^2}-1\right)=$$

$$\lim_{n\to\infty} n\,\frac{\cancel{4n^2}+10n+6-(\cancel{4n^2}+4n+1)}{(2n+1)^2}=\lim_{n\to\infty}\frac{6n^2+5n}{(2n+1)^2}=\frac{6}{4}$$

$\frac{6}{4}>1 \Rightarrow \sum a_n$ C

l) $\sum_{n=1}^{\infty}\arcsin\frac{1}{n\sqrt[3]{n}+5}$

m) $\sum_{n=1}^{\infty}n^2\ln\left(1+\frac{1}{n^2}\right)$;

n) $\sum_{n=1}^{\infty}\left(\frac{\pi}{2}-\operatorname{arctg} n\right)^n$;

o) (R) $\sum_{n=1}^{\infty}\frac{1}{e\cdot\sqrt{e}\cdot\sqrt[3]{e}\cdot\ldots\cdot\sqrt[n]{e}}$;

Termenul general $a_n=\frac{1}{e^{\underbrace{1+\frac{1}{2}+\ldots+\frac{1}{n}}}}$

$x_n=1+\frac{1}{2}+\ldots+\frac{1}{n}-\ln n\longrightarrow c\in(0,1)$ $\quad a_n, b_n>0$

$$b_n=\frac{1}{e^{\ln n}}=\frac{1}{n}$$

$$\frac{a_n}{b_n}=e^{\ln n-\left(\frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{n}\right)}\to e^{-c}\in(0,\infty)$$

$\Rightarrow \sum a_n \sim \sum b_n$ D $\Rightarrow \sum a_n$ D

p) $\sum_{n=1}^{\infty} \frac{2^n + 3^{n+1} - 6^{n-1}}{12^n}$;

q) $\sum_{n=1}^{\infty} \left( \frac{1}{n} + \ln \frac{n}{n+1} \right)$;

**S3.3** Precizaţi natura seriilor următoare în funcţie de parametrii corespunzători.

a) $\sum_{n=2}^{\infty} \frac{\sqrt{n+2} - \sqrt{n-2}}{n^\alpha}, \alpha \in \mathbb{R}$;

Termen gen $a_n = \frac{\sqrt{n+2} - \sqrt{n-2}}{n^\alpha} = \frac{(\cancel{n}+2)-(\cancel{n}-2)}{(\sqrt{n+2}+\sqrt{n-2})n^\alpha} = \frac{4}{(\sqrt{n+2}+\sqrt{n-2})\,n^\alpha}$

$b_n = \frac{1}{n^{\alpha+\frac{1}{2}}}$ $\qquad a_n, b_n > 0$

$\frac{a_n}{b_n} = \frac{4 \cdot n^{\alpha+\frac{1}{2}}}{(\sqrt{n+2}+\sqrt{n-2}) \cdot \cancel{n^\alpha}} \longrightarrow 4 \in (0, \infty) \overset{CC\,III}{\Longrightarrow} \sum a_n \sim \sum b_n$

Dacă $\alpha + \frac{1}{2} > 1 \Leftrightarrow \alpha > \frac{1}{2} \Rightarrow \sum b_n\, C \Rightarrow \sum a_n\, C$

$\alpha + \frac{1}{2} \leq 1 \Leftrightarrow \alpha \leq \frac{1}{2} \Rightarrow \sum b_n\, D \Rightarrow \sum a_n\, D$

b) $\sum_{n=1}^{\infty} \frac{\mathrm{arctg}(\mathrm{n}\alpha)}{(\ln 3)^n}, \alpha \in \mathbb{R}$;

c) $\sum_{n=1}^{\infty} \frac{\alpha^n}{\sqrt[n]{n!}}, \alpha \in \mathbb{R}_+^*$;

Dacă $\alpha < 1$

Fie $k \in \{0, 1, \dots, n-1\}$

$(k+1)(n-k) \geq n$

$nk + \cancel{n} - k^2 - k \geq \cancel{n}$

$nk - k^2 \geq k$

$k(n-k) \geq k$

$$\prod_{k=0}^{n-1}(k+1)(n-k) \ge \prod_{k=0}^{n-1} n$$

$$\prod_{k=0}^{n-1}(k+1)\prod_{k=0}^{n-1}(n-k) \ge n^n$$

$n-k=j$

$$n! \cdot \prod_{j=n}^{1} j \ge n^n$$

$$(n!)^2 \ge n^n \quad \Big| \sqrt[2n]{\phantom{x}}$$

$$\sqrt[n]{n!} \ge \sqrt{n}$$

$$\frac{1}{\sqrt[n]{n!}} \le \frac{1}{\sqrt{n}}$$

$$\frac{\alpha^n}{\sqrt[n]{n!}} \le \frac{\alpha^n}{\sqrt{n}}$$

$$b_n = \frac{\alpha^n}{\sqrt{n}} > 0$$

$$\frac{b_{n+1}}{b_n} = \frac{\frac{\alpha^{\cancel{n+1}}}{\sqrt{n+1}}}{\frac{\cancel{\alpha^n}}{\sqrt{n}}} = \frac{\alpha \cdot \sqrt{n}}{\sqrt{n+1}} \to \alpha < 1$$

$\Rightarrow \sum b_n$ C

$\Rightarrow \sum \frac{\alpha^n}{\sqrt{n}}$ C

Dacă $\alpha \geq 1$

$n! < n^n$

$\frac{1}{n!} > \frac{1}{n^n}$

$\frac{1}{\sqrt[n]{n!}} > \frac{1}{n}$

$\left.\begin{array}{l} \frac{\alpha^n}{\sqrt[n]{n!}} > \frac{1}{n} \\ \sum \frac{1}{n} \text{ D} \end{array}\right\} \Rightarrow \sum \frac{\alpha^n}{\sqrt[n]{n!}}$ D

Dacă $\alpha < 1$ C

$\alpha \geq 1$ D

d) $\sum_{n=2}^{\infty} \left(\sqrt{n+1} - \sqrt{n}\right)^a \ln\left(\frac{n+1}{n-1}\right), a \in \mathbb{R};$

e) $*\sum_{n=1}^{\infty} \frac{\alpha(\alpha+1) \cdot \ldots \cdot (\alpha+n-1)}{n! n^{\beta}}, \alpha \in \mathbb{R}_{+}^{*}, \beta \in \mathbb{R}.$

**S3.4** Utilizând criteriul logaritmului să se studieze convergenţa următoarei serii: $\sum_{n=1}^{\infty}\left(\frac{1}{n^3-n+3}\right)^{\ln(n+1)}$.

$$a_n = \left(\frac{1}{n^3-n+3}\right)^{\ln(n+1)}$$

$$-\frac{\ln a_n}{\ln n} = \frac{-\ln(n+1)\left[\ln(n^3-n+3)\right]}{\ln n} =$$

$$\frac{\ln(n+1)}{\ln n}\ln(n^3-n+3) \to \infty > 1$$

$$\Rightarrow \sum a_n \; C$$

**S3.5*** Fie $\sum_{n=1}^{\infty} u_n$ o serie convergentă din $\mathbb{R}$, cu $u_n \geq 0$, $\forall\ n \in \mathbb{N}^*$. Ce se poate spune despre natura seriei $\sum_{n=1}^{\infty}\left(\frac{u_n}{1+u_n}\right)^{\alpha}$, unde $\alpha$ este un număr real?

**S3.6*** Să se analizeze seria cu termenul general

$$\arccos \frac{n(n+1)+\sqrt{(n+1)(n+2)(3n+1)(3n+4)}}{(2n+1)(2n+3)}, n \in \mathbb{N}^*$$

şi, în caz de convergenţă, să i se afle suma.

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