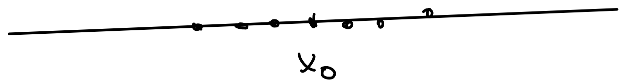
$f: \mathbb{R}^m \to \mathbb{R}^n$

Dacă $f: \mathbb{R} \to \mathbb{R}$ $\quad \lim\limits_{x \to x_0} f(x)$



Dacă $f: \mathbb{R} \to \mathbb{R}^n$

componente

$$f(x) = (f_1(x), \dots, f_n(x))$$

Pt fiecare studiem $\lim\limits_{x \to x_0} f_i(x)$ $\quad i = \overline{1,n}$

$$\lim_{x \to x_0} f(x) = \left(\lim_{x \to x_0} f_1(x), \dots, \lim_{x \to x_0} f_n(x)\right)$$

Dacă $f: \mathbb{R}^m \to \mathbb{R}^n$ $\quad x = (x_1, \dots, x_m)$

→ limite parțiale

fixem $m-1$ variabile și studiez limite

variabile m

ex: $\lim\limits_{x_m \to 0} f(\underbrace{0, 0, \dots, 0}_{m-1}, x_m)$

→ limită iterată

iau pe rând cele n limite

ex: $\lim\limits_{x_1\to 0} \lim\limits_{x_2\to 0} \dots \lim\limits_{x_n\to 0} f(x_1,\dots,x_n)$

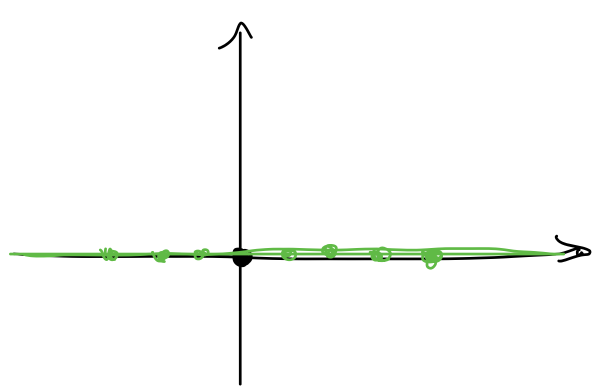
→ limită direcțională

Fixez o direcție / o dreaptă $(u_1,\dots,u_n) \neq (0,\dots,0)$

$\lim\limits_{t\to 0} f((0,0,\dots,0)+t(u_1,\dots,u_n))$

• limita globală

$$\lim_{(x_1,\dots,x_n)\to(0,\dots,0)} f(x_1,\dots,x_n)$$



limită parțială

fixez $y=0$

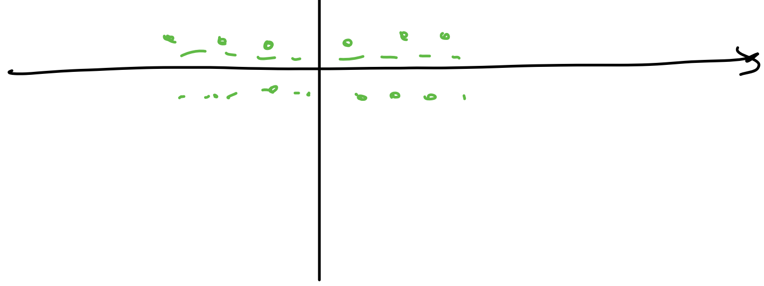
și mă duc tot mai aproape de $x=0$

$$\lim_{x\to 0}\left(\boxed{\lim_{y\to 0} f(x,y)}\right)$$

$$\lim_{x\to 0}\lim_{y\to 0}(x+y) = \lim_{x\to 0} x = 0$$

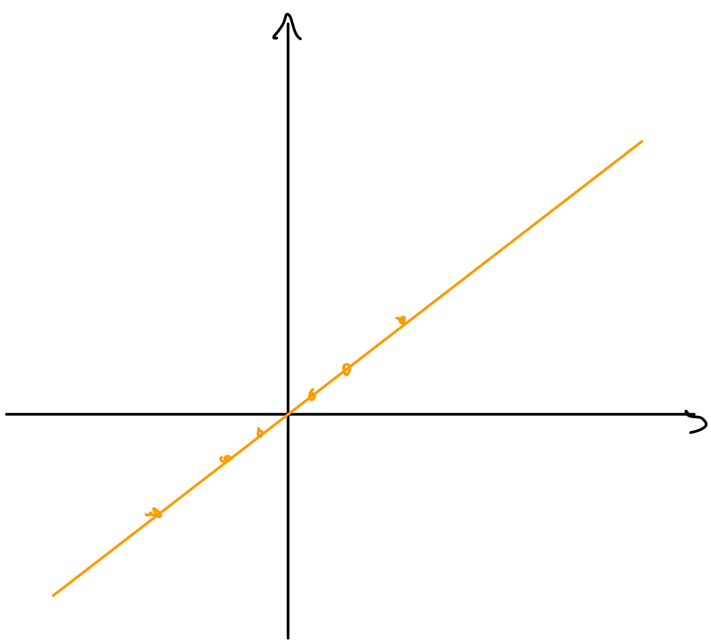
limită iterată

mă apropii cu x (y fiind fixat) și apoi mă apropii cu y



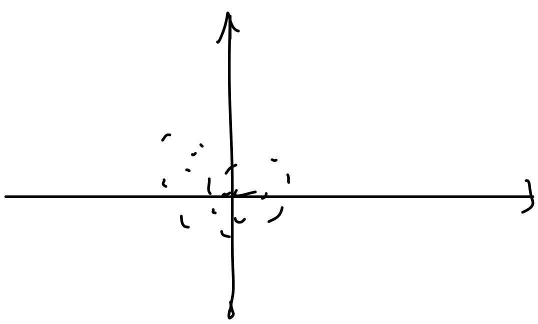
limită direcțională

aleg o direcție și mă apropii de $(x,y)=(0,0)$



limită globală

mă pot apropia în orice mod



Dacă limitele iterate/ parțiale nu sunt egale într-un punct / limita direcționate depinde de direcție $\Rightarrow$ fct nu are limită (•)

Dem pt ne ex limitei globale $\Longleftarrow$ (•)
sau
cu șiruri

O fct nu e cont dacă

$$\lim_{\substack{x \to x_0 \\ x, x_0 \in \mathbb{R}^n}} f(x) \neq f(x_0) \quad \begin{cases} \text{fie } \nexists \lim_{x \to x_0} f(x) \\ \text{fie } \lim_{x \to x_0} f(x) \neq f(x_0) \end{cases}$$

Seminar 8

Exerciţii recomandate: 8.1 b), 8.2 ii),v), 8.3 a), 8.4 b), 8.5 b), 8.6 a)
Rezerve: 8.1 a), 8.2 i),iii),iv), 8.3 b),c), 8.4 a), 8.5 a),c), 8.6 c), 8.7 a)

S8.1 Calculaţi următoarele limite:

a) $\lim\limits_{x\nearrow 1} \dfrac{\sin(p \arccos x)}{\sqrt{1-x^2}}$, $p \in \mathbb{R}^*$;

b) $\lim_{x\to 0}\left(\dfrac{\ln(1+3x^2)}{x^2}, \dfrac{e^{x^2}-\cos x}{x^2}\right)$;

c) $\lim_{x\nearrow\frac{\pi}{4}}\left((\operatorname{tg} x)^{\operatorname{tg}(2x)}, \dfrac{\sin(4x)}{\sqrt{\pi-4x}}, \dfrac{2^{\operatorname{tg} x}-2}{x-\frac{\pi}{4}}, \operatorname{tg}(2x)\operatorname{tg}\left(\dfrac{\pi}{4}-x\right), \dfrac{\ln(\operatorname{tg} x)}{\cos 2x}\right)$;

d) $\lim_{x\to\infty}\left(x-\sqrt[n]{(x-a_1)(x-a_2)\dots(x-a_n)}\right)$, $n\in\mathbb{N}^*$, $a_k\in\mathbb{R}$, $k=\overline{1,n}$;

e) $\lim\limits_{x\searrow 0}\left(\dfrac{1+\sum_{k=1}^{n}\ln(1+kx)}{\sum_{k=1}^{n} n^{kx-1}}\right)^{\frac{1}{x}}$, $n\in\mathbb{N}^*$.

$$\lim_{a\to 0}\frac{\sin a}{a} = 1$$

$$= \lim_{\substack{x\nearrow 1\\ x<1}} \frac{\sin(p\arccos x)}{\sqrt{1-x^2}} \overset{\frac{0}{0}}{=}$$

Evitați aplicarea l'Hôpital pt fct trig deoarece e un cerc vicios

$$= \lim_{x\nearrow 1} \frac{\sin(p\arccos x)}{p\arccos x}\cdot\frac{p\arccos x}{\sqrt{1-x^2}} =$$

$$= \lim_{\substack{x\nearrow 1\\ x<1}} \frac{\sin(p\arccos x)}{p\arccos x}\cdot\frac{p\arccos x}{\sin(\arccos x)} =$$

$1-x^2+x^2=1$

$$\underline{\sqrt{1-x^2}^{\,2} + x^2 = 1} \qquad (1)\ \underline{\sin^2\alpha+\cos^2\alpha=1}$$

$$x = \cos(\arccos x) \qquad \alpha = \arccos x$$

Din (1) $\Rightarrow \sin^2(\arccos x) + x^2 = 1 \Rightarrow \sin^2(\arccos x) = 1-x^2$

$$\underbrace{\sin(\arccos x) = \sqrt{1-x^2}}$$

$$= \lim_{x\to 1} \frac{\sin(p \arccos x)}{p \arccos x} \cdot p \lim_{x \to 1} \frac{\arccos x}{\sin(\arccos x)} =$$

$$1 \cdot p \cdot 1 = p$$

$$b) \lim_{x\to 0} \Big(\underbrace{\frac{\ln(1+3x^2)}{x^2}}_{f_1}, \underbrace{\frac{e^{x^2} - \cos x}{x^2}}_{f_2} \Big) =$$

$$\lim_{x\to 0} \left(\frac{\ln(1+3x^2)}{3x^2} \cdot 3 , \frac{e^{x^2}-1}{x^2} + \frac{1-\cos x}{x^2} \right)$$

$$\lim_{a\to 0} \frac{\ln(1+a)}{a} = 1$$

$$1 - \cos x = 2 \sin^2 \frac{x}{2}$$

$$\frac{0}{0}$$

$$\lim_{a\to 0} \frac{e^a - 1}{a} = 1$$

$$1 - \cos x = \cos 0 - \cos x =$$
$$- 2 \sin \frac{0+x}{2} \sin \frac{0-x}{2}$$

$$= \lim_{x\to 0} \left(\underbrace{\frac{\ln(1+3x^2)}{3x^2}}_{\downarrow 1} \cdot 3 , \underbrace{\frac{e^{x^2}-1}{x^2}}_{\downarrow 1} + \frac{\cancel{2} \sin^2 \frac{x}{2}}{\left(\frac{x}{2}\right)^2} \cdot \frac{1}{\cancel{4}_2} \right)$$

$$\left(3, \frac{3}{2}\right)$$

$$\lim_{n\to\infty} 1^n = \lim_{n\to\infty} 1 = 1$$

S8.2 Studiaţi existenţa şi, în cazul afirmativ, determinaţi limitele iterate $\lim_{x\to 0}\left(\lim_{y\to 0} f(x,y)\right)$ şi $\lim_{y\to 0}\left(\lim_{x\to 0} f(x,y)\right)$, limitele parţiale $\lim_{x\to 0} f(x,0)$, $\lim_{y\to 0} f(0,y)$ şi limita globală $\lim_{(x,y)\to(0,0)} f(x,y)$ pentru funcţiile $f : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}$ definite mai jos:

i) $f(x,y) := \dfrac{x^2-y^2}{|x|+|y|}$;

ii) $f(x,y) := \dfrac{\sqrt{1+x^2y^2}-1}{x^2+y^2}$;

iii) $f(x,y) := (x^2+y^2)^{x^2y^2}$;

iv) $f(x,y) := \dfrac{\sin(xy)}{\sqrt{x^2+y^2}}$;

v) $f(x,y) := y^2 \ln(x^2+y^2)$;

vi) $f(x,y) := \dfrac{e^{\frac{1}{x^2+y^2}}}{x^6+y^6}$.

ii) $$\lim_{x\to 0}\left(\lim_{y\to 0} \frac{\sqrt{1+x^2y^2}-1}{x^2+y^2}\right) =$$

$$\lim_{x\to 0}\left(\frac{1-1}{x^2}\right) = \lim_{x\to 0}\frac{0}{x^2} = \lim_{x\to 0} 0 = 0$$

$$\lim_{y\to 0}\left(\lim_{x\to 0} \frac{\sqrt{1+x^2y^2}-1}{x^2+y^2}\right) =$$

$$\lim_{y\to 0}\left(\frac{1-1}{y^2}\right) = \lim_{y\to 0}\frac{0}{y^2} = \lim_{y\to 0} 0 = 0$$

$$\lim_{x\to 0} f(x,0) = \lim_{x\to 0}\frac{1-1}{x^2} = \lim_{x\to 0} 0 = 0$$

$$\lim_{y\to 0} f(y,0) = \lim_{y\to 0}\frac{1-1}{y^2} = \lim_{y\to 0} 0 = 0$$

$$\sqrt{1+x^2y^2}+1)$$

$$\lim_{(x,y)\to(0,0)} \frac{\sqrt{1+x^2y^2}-1}{x^2+y^2} =$$

$$\lim_{(x,y)\to(0,0)} \frac{\cancel{1}+x^2y^2-\cancel{1}}{(x^2+y^2)(\sqrt{1+x^2y^2}+1)} \overset{(x^2y^2}{=} \lim_{(x,y)\to(0,0)} \frac{1}{\underbrace{\left(\underbrace{\frac{1}{x^2}}_{+\infty}+\underbrace{\frac{1}{y^2}}_{+\infty}\right)}_{+\infty}\underbrace{\left(\sqrt{1+x^2y^2}+1\right)}_{\to 2}}$$

$= 0$

v) $f(x, y) := y^2 \ln(x^2 + y^2)$;

$$\lim_{x \to 0} \left(\lim_{y \to 0} y^2 \ln(x^2+y^2) \right) =$$

$$\lim_{x \to 0} \left(\lim_{y \to 0} y^2 \cdot \underbrace{\lim_{y \to 0} \ln(x^2+y^2)}_{} \right) =$$

$$\lim_{x \to 0} \left(0 \cdot \ln x^2 \right)$$

$$\lim_{x \to 0} 0 = 0$$

$$\lim_{y \to 0} \left(\lim_{x \to 0} y^2 \ln(x^2+y^2) \right)$$

$$\lim_{y \to 0} \left(y^2 \lim_{x \to 0} \ln(x^2+y^2) \right)$$

$$\boxed{\lim_{y \to 0} \frac{\ln(1+y)}{y}}$$

$$\lim_{y \to 0} \left(y^2 \ln y^2 \right) =$$

$$2 \lim_{y \to 0} \left(y^2 \ln y \right) = 2 \lim_{y \to 0} \frac{\ln y}{\frac{1}{y^2}} \overset{l'H}{=} \text{Dacă } \exists$$

$$2 \lim_{y \to 0} \frac{\frac{1}{y}}{-\frac{2}{y^3}} = \lim_{y \to 0} \left(-y^2 \right) = 0$$

$$\lim_{x \to 0} f(x,0) = \lim_{x\to 0} 0 = 0$$

$$\lim_{y \to 0} f(0,y) = \lim_{y\to 0} y^2 \ln y^2 = 0$$

$$\lim_{(x,y)\to(0,0)} f(x,y) =$$

$$\lim_{(x,y)\to(0,0)} y^2 \ln(x^2+y^2)$$

3 idei pt limite globale

1. desfac în limite simple:

$$\lim_{(x,y)\to(0,0)} x+y =$$

$$\lim_{(x,y)\to(0,0)} x + \lim_{(x,y)\to(0,0)} y =$$

$$\lim_{x\to 0} x + \lim_{y \to 0} y = 0 + 0$$

2. Renotez argument

$$\lim_{(x,y)\to(0,0)} (x^2+y^2)\ln(x^2+y^2) \overset{z = x^2+y^2}{=\!=\!=}$$

$$\lim_{z\to 0} z\ln z = \_\_\_ .$$

3. Inegalități

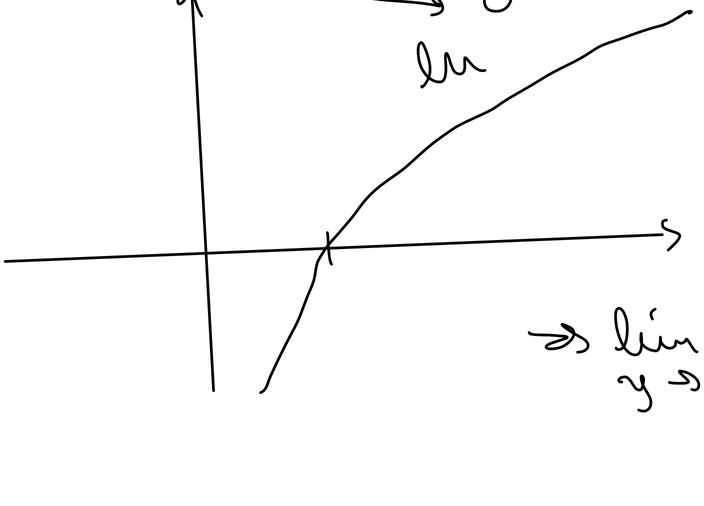
$$0 \le \lim_{(x,y)\to(0,0)} \frac{|x||y|}{|x|+|y|} \le$$

$$\sqrt{|x||y|} \le \frac{|x|+|y|}{2}$$

$$\le \lim_{(x,y)\to(0,0)} \frac{(|x|+|y|)^{\cancel{2}}}{4\cancel{(|x|+|y|)}} = 0$$

$$y^2\ln y^2 \le y^2\ln(x^2+y^2) \le 0$$

0



ln monoton

$$\ln y^2 \le \ln(x^2+y^2)$$

$$y^2\ln y^2 \le y^2\ln(x^2+y^2)$$

$$\Rightarrow \lim_{y\to 0} y^2\ln(x^2+y^2) = 0$$

S8.3 Arătaţi că următoarele funcţii nu au limită în $(0,0)$, chiar dacă au limite iterate şi/sau limite în orice direcţie admisibilă:

a) $f : \mathbb{R}^2 \backslash \{(0,0)\} \to \mathbb{R}$, $f(x,y) := \dfrac{x^2 - y^2}{x^2 + y^2}$;

b) $f : \mathbb{R}^2 \backslash \{(x,y) \mid x = y\} \to \mathbb{R}^2$, $f(x,y) := \left(\dfrac{x+y}{x-y}, \dfrac{x^2 y}{x^4 + y^2}\right)$;

c) $f : \mathbb{R}^2 \backslash \{(x,y) \mid y^2 = 2x\} \to \mathbb{R}$, $f(x,y) := \dfrac{y^2 + 2x}{y^2 - 2x}$;

d) $f : \mathbb{R}^2 \backslash \{(0,0)\} \to \mathbb{R}^2$, $f(x,y) := \left(\dfrac{x^2 y^2}{(x-y)^2 + x^2 y^2}, (1 + |xy|)^{\frac{2}{x^2+y^2}}\right)$.

a) $\lim\limits_{x\to 0}\left(\lim\limits_{y\to 0} \dfrac{x^2-y^2}{x^2+y^2}\right) = \lim\limits_{x\to 0}\left(\dfrac{x^2}{x^2}\right) = \lim\limits_{x\to 0} 1 = 1$

$\lim\limits_{y\to 0}\left(\lim\limits_{x\to 0} \dfrac{x^2-y^2}{x^2+y^2}\right) = \lim\limits_{y\to 0}\left(\dfrac{-y^2}{y^2}\right) = \lim\limits_{y\to 0} -1 = -1$

$\Rightarrow \nexists \lim\limits_{(x,y)\to(0,0)} f(x,y)$

$(u,v) \neq (0,0)$

$\lim\limits_{t\to 0} f((0,0) + t(u,v)) =$

$\lim\limits_{t\to 0} f(tu, tv) = \lim\limits_{t\to 0} \dfrac{t^2u^2 - t^2v^2}{t^2u^2 + t^2v^2}$ $(t^2$

$= \lim\limits_{t\to 0} \dfrac{u^2-v^2}{u^2+v^2} = \dfrac{u^2-v^2}{u^2+v^2}$ $\Big\} \Rightarrow$

Depinde de (u,v)

Pot spune şi de aici că $\nexists \lim\limits_{(x,y)\to(0,0)} f(x,y)$

$$f:\mathbb{R}^2\setminus\{(x,y)\mid \underline{x=y}\}\to\mathbb{R}^2,\ f(x,y):=\Big(\underbrace{\frac{x+y}{x-y}}_{f_1},\underbrace{\frac{x^2y}{x^4+y^2}}_{f_2}\Big);$$

$$\lim_{x\to0}\lim_{y\to0} f_1(x,y)=\lim_{x\to0}\lim_{y\to0}\frac{x+y}{x-y}=$$

$$\lim_{x\to0}\frac{x}{x}=\lim_{x\to0}1=1$$

$$\lim_{y\to0}\lim_{x\to0} f_1(x,y)=\lim_{y\to0}\lim_{x\to0}\frac{x+y}{x-y}=$$

$$\lim_{y\to0}\frac{y}{-y}=\lim_{y\to0}-1=-1$$

$(u,v)\neq(0,0)\longrightarrow$ se pune pt orice pct în care vrean lim

$u\neq v$

$$\lim_{t\to0} f_1((0,0)+t(u,v))=\lim_{t\to0} f_1(tu,tv)=$$

$$\lim_{t\to0}\frac{tu+tv}{tu-tv}=\lim_{t\to0}\frac{u+v}{u-v}=\frac{u+v}{u-v}$$

$\Rightarrow \nexists \lim_{(x,y)\to(0,0)} f_1(x,y)$

$\lim_{x\to 0} \lim_{y\to 0} f_2(x,y) = \lim_{x\to 0}\left(\lim_{y\to 0} \frac{x^2 y}{x^4+y^2}\right) =$

$\lim_{x\to 0} \frac{0}{x^4} = \lim_{x\to 0} 0 = 0$

$\lim_{y\to 0} \lim_{x\to 0} f_2(x,y) = \lim_{y\to 0} \left(\lim_{x\to 0} \frac{x^2 y}{x^4+y^2}\right)$

$= \lim_{y\to 0} \frac{0}{y^2} = \lim_{y\to 0} 0 = 0$

$\underline{(u,v) \neq (0,0)}$

$\lim_{t\to 0} f_2(tu,tv) = \lim_{t\to 0} \frac{t^2u^2tv}{t^4u^4+t^2v^2} = \quad | t^2$

$\lim_{t\to 0} \frac{tu^2v}{t^2u^4+v^2} \xlongequal{\text{Dacă } v\neq 0} \lim_{t\to 0} \frac{0}{v^2} = 0$

Dacă $v=0 \Rightarrow u\neq 0$

$\lim_{t\to 0} f_2(tu,tv) = \lim_{t\to 0} \frac{t^2u^2tv}{t^4u^4+t^2v^2} \xlongequal{v=0} \lim_{t\to 0} \frac{0}{t^4u^4} =$

$$\lim_{t\to 0} f_2(tu, tv) = 0$$

$$\frac{x^2 y}{x^4 + y^2}$$

$(x_n, y_n) \to (0,0)$

Deobicei)

$$(x_n, y_n) = \left(\frac{1}{n^a}, \frac{1}{n^b}\right)$$

Le "potrivesc" a.î.

coef max de la numărator

= coef max de la numitor

$$\frac{\frac{1}{n^{2a+b}}}{\frac{1}{n^{4a}} + \frac{1}{n^{2b}}}$$

$4a = 2b \to 2a = b$

$$(x_n, y_n) = \left(\frac{1}{n}, \frac{1}{n^2}\right) \to (0,0)$$

$$\lim_{n\to\infty} f_2(x_n, y_n) = \lim_{n\to\infty} \frac{\frac{1}{n^2}\cdot\frac{1}{n^2}}{\frac{1}{n^4} + \left(\frac{1}{n^2}\right)^2} =$$

$$\lim_{n\to\infty} \frac{\frac{1}{\cancel{n^4}}}{\frac{2}{\cancel{n^4}}} = \frac{1}{2}$$

$$(x_n', y_n') = \left(\frac{2}{n}, \frac{1}{n^2}\right) \to (0,0)$$

$$\lim_{n\to\infty} f_2(x_n', y_n') = \lim_{n\to\infty} \frac{\frac{4}{n^2}\cdot\frac{1}{n^2}}{\frac{16}{n^4}+\frac{1}{n^4}} =$$

$$\lim_{n\to\infty} \frac{4}{17} = \frac{4}{17}$$

$$\lim_{n\to\infty} f_2(x_n', y_n') \neq \lim_{n\to\infty} f_2(x_n, y_n)$$

$\to f_2$ nu are limită în $(0,0)$

$\to f$ nu are limită în $(0,0)$

c) $f : \mathbb{R}^2 \backslash \{(x,y) \mid y^2 = 2x\} \to \mathbb{R},\ f(x,y) := \dfrac{y^2+2x}{y^2-2x}$;

$y_n = \frac{1}{n^a}$ $\quad x_n = \frac{1}{n^b}$ $\qquad x_n = \frac{1}{n^2}$ $\quad y_n = \frac{1}{n}$

$$\frac{\frac{1}{n^{2a}} + \frac{2}{n^b}}{\frac{1}{n^{2a}} - \frac{2}{n^b}} \qquad \frac{n^b + 2n^{2a}}{n^b - 2n^{2a}}$$

$$\lim_{n\to\infty} f(x_n, y_n) = \lim_{n\to\infty} \frac{\frac{1}{n^2}+\frac{2}{n^2}}{\frac{1}{n^2}-\frac{2}{n^2}}$$

$$\lim_{n\to\infty} \frac{\frac{3}{\cancel{n^2}}}{-\frac{1}{\cancel{n^2}}} = -3$$

$$x_n' = \frac{2}{n^2} \qquad y_n' = \frac{1}{n}$$

$$\lim_{n\to\infty} f(x_n', y_n') = \lim_{n\to\infty} \frac{\frac{1}{n^2}+\frac{4}{n^2}}{\frac{1}{n^2}-\frac{4}{n^2}} = \lim_{n\to\infty} \frac{5}{-3} =$$

$$-\frac{5}{3}$$

$$\Rightarrow \nexists \lim_{(x,y)\to(0,0)} f(x_n, y_n)$$

S8.4 Determinaţi următoarele limite:

a) $\lim_{(x,y)\to(0,0)} \left(\frac{xy}{\sqrt{1+xy}-1}, \frac{\sin(x^3+y^3)}{\sqrt{x^2+y^2+1}-1} \right);$

b) $\lim_{(x,y,z)\to(0,0,0)} \left(\frac{1}{xyz} \operatorname{tg} \frac{xyz}{1+xyz}, (1+xyz)^{\frac{1}{\sqrt{x}+\sqrt{y}+\sqrt{z}}} \right);$ $x, y, z > 0$

c) $\lim_{(x,y,z)\to(0,0,0)} \left(\frac{1-\cos(1-\cos(x^2+y^2+z^2))}{(x^2+y^2+z^2)^4}, \frac{x^2y^2z^2}{(x-y)^2+(y-z)^2+(x-z)^2} \right).$

b) $\lim_{(x,y,z)\to(0,0,0)} \left(\frac{1}{xyz} \operatorname{tg} \frac{xyz}{1+xyz}, (1+xyz)^{\frac{1}{\sqrt{x}+\sqrt{y}+\sqrt{z}}} \right) =$

$\lim_{a\to0} \frac{\operatorname{tg} a}{a} = 1$

$\lim_{a\to0} (1+a)^{\frac{1}{a}} = e$

$\lim_{(x,y,z)\to(0,0,0)} \left(\underbrace{\frac{\operatorname{tg}\frac{xyz}{1+xyz}}{\frac{xyz}{1+xyz}}}_{\to 1} \cdot \underbrace{\frac{1}{\underbrace{1+xyz}_{\to 0}}}_{\to 1}, \underbrace{(1+xyz)^{\frac{1}{xyz}}}_{\to e} \cdot \underbrace{\frac{xyz}{\sqrt{x}+\sqrt{y}+\sqrt{z}}}_{\to 0} \right) = (1,1)$

$\lim_{(x,y,z)\to(0,0,0)} \frac{xyz}{\sqrt{x}+\sqrt{y}+\sqrt{z}} =$ $(xyz$

$\lim_{(x,y,z)\to(0,0,0)} \frac{1}{\underbrace{\frac{1}{yz\sqrt{x}}}_{\to+\infty} + \underbrace{\frac{1}{xz\sqrt{y}}}_{\to+\infty} + \underbrace{\frac{1}{xy\sqrt{z}}}_{\to+\infty}} = 0$

S8.5 Determinaţi mulţimile punctelor de discontinuitate ale următoarelor funcţii reale:

a) $f(x,y) := \begin{cases} \dfrac{|x|}{y} e^{-|x|y^{-2}}, & y \neq 0; \\ 1, & y = 0; \end{cases}$

b) $f(x,y) := \begin{cases} \dfrac{xy}{x+y}, & \underline{x+y \neq 0;} \\ 0, & x+y = 0; \end{cases}$

c) $f(x,y,z) := \begin{cases} (x^2+y^2+z^2)^{1/3} \ln(x^2+y^2+z^2), & (x,y,z) \neq (0,0,0); \\ 1/3, & (x,y,z) = (0,0,0). \end{cases}$

b) În afara $x+y\neq 0$, fct este raport

de fct cont, numitorul nu se anulează,

deci f cont

Fixez $x_n = x$. Aleg $x_n + y_n = \frac{1}{n} \Rightarrow$

$y_n = \frac{1}{n} - x_n =$

$\frac{1}{n} - x$

$$\lim_{n\to\infty} f(x_n, y_n) = \lim_{n\to\infty} \frac{x\cdot\left(\frac{1}{n} - x\right)}{\frac{1}{n}} =$$

$$\lim_{n\to\infty} \frac{x\,\frac{(1-nx)}{\cancel{n}}}{\frac{1}{\cancel{n}}} =$$

$$\lim_{n\to\infty} x(1-nx)$$

Pt $x \neq 0$ $\lim_{n\to\infty} f(x_n, y_n) = -\infty \Rightarrow$ $\not= f(x,y)$, $x+y=0$

$\Rightarrow f$ nu e cont

Pot arăta că f nu are limită

Fixez $x_n' = x$, $x \neq 0$. Aleg $x_n' + y_n' = -\frac{1}{n}$

$y_n' = -\frac{1}{n} - x_n' =$

$-\frac{1}{n} - x$

$$\lim_{n\to\infty} f(x_n', y_n') = \lim_{n\to\infty} \frac{x\left(-\frac{1}{n} - x\right)}{-\frac{1}{n}}$$

$$\lim_{n\to\infty} \frac{\frac{x(-1-nx)}{\cancel{n}}}{-\frac{1}{\cancel{n}}} = \lim_{n\to\infty} x(1+nx)$$

$= \infty$

Pt toți (x, y) cu $x + y = 0$

Dar $(x, y) \neq (0, 0)$

f nu are lim

Pt $(0,0)$ Aleg $x_n'' = \frac{1}{n}$

$y_n'' = -\frac{1}{n} + \frac{1}{n^2}$

$$\lim_{n\to\infty} f(x_n'', y_n'') = \lim_{n\to\infty} \frac{\frac{1}{n}\left(-\frac{1}{n}+\frac{1}{n^2}\right)}{\frac{1}{n^2}} =$$

$$\lim_{n\to\infty} \frac{\frac{1}{n}\left(\frac{1-n}{\cancel{n^2}}\right)}{\frac{1}{\cancel{n^2}}} = \lim_{n\to\infty} \frac{1-n}{n} = -1$$

$$x_n''' = \frac{1}{n}$$

$$y_n''' = -\frac{1}{n} - \frac{1}{n^2}$$

$$\lim_{n\to\infty} f(x_n''', y_n''') = \lim_{n\to\infty} \frac{\frac{1}{n}\left(-\frac{1}{n}-\frac{1}{n^2}\right)}{-\frac{1}{n^2}} =$$

$$\lim_{n\to\infty} \frac{\frac{1}{n}\left(\frac{1}{n}+\frac{1}{n^2}\right)}{\frac{1}{n^2}} =$$

$$\lim_{n\to\infty} \frac{1+\frac{1}{n}}{1} = 1$$

$$\rightarrow \nexists \lim_{(x,y)\to(0,0)} f(x,y)$$

limite parțiale

S8.6 Arătaţi că următoarele funcţii sunt continue în fiecare variabilă, dar nu global continue în punctul $(0,0)$:

a) $f(x,y) := \begin{cases} \dfrac{xy^2}{x^2+y^4}, & (x,y) \neq (0,0); \\ 0, & (x,y) = (0,0); \end{cases}$

vezi mai sus

b) $f(x,y) := \begin{cases} \dfrac{x^4 y}{x^6+y^3}, & y \neq -x^2; \\ 0, & y = -x^2; \end{cases}$

c) $f(x,y,z) := \begin{cases} \dfrac{\sin(xy+yz+xz)}{\sqrt{(x^4+y^2+z^4)}}, & (x,y,z) \neq (0,0,0); \\ 0, & (x,y,z) = (0,0,0); \end{cases}$

d) $f(x,y) := \begin{cases} \min\left\{\left|\dfrac{x}{y}\right|, \left|\dfrac{y}{x}\right|\right\}, & xy \neq 0; \\ 0, & xy = 0. \end{cases}$

S8.7 Studiaţi continuitatea următoarelor funcţii:

a) $f : [-1, +\infty) \to \mathbb{R}^2$, $f(x) := (f_1(x), f_2(x))$, unde, pentru $p \in \mathbb{R}$, f_1 şi f_2 sunt definite de

$$f_1(x) := \begin{cases} 2\,\mathrm{tg}\, x \cdot \mathrm{arctg}\, \dfrac{1}{x}, & x \in [-1, 0); \\ p, & x = 0; \\ \mathrm{e}^{\frac{1-\sqrt{1+x}}{x^2 e^x}}, & x > 0, \end{cases}$$

$$f_2(x) := \begin{cases} \mathrm{tg}\, x \cdot \sin \dfrac{1}{x}, & x \in [-1, 0); \\ p, & x = 0; \\ \cos \dfrac{1}{x} \mathrm{e}^{-\frac{1}{x}}, & x > 0. \end{cases} .$$

b) $f : A = \left\{(x, y) \in \mathbb{R}^2 \mid 0 \le x^2 + y^2 < \frac{\pi}{2}\right\} \to \mathbb{R}$, $\alpha \in \mathbb{R}$,

$$f(x, y) := \begin{cases} \dfrac{1 - \cos \sqrt{x^2 + y^2}}{\mathrm{tg}(x^2 + y^2)}, & (x, y) \in A \setminus \{(0, 0)\} \\ \alpha, & (x, y) = (0, 0). \end{cases}$$

c) $f : \mathbb{R}^3 \to \mathbb{R}^2$, $f(x, y, z) = (f_1(x, y, z), f_2(x, y, z))$, unde, pentru $p \in \mathbb{R}$, f_1 şi f_2 sunt definite de

$$f_1(x, y, z) := \begin{cases} (x^2 + y^2 + z^2)^p \cos \dfrac{1}{\sqrt{x^2 + y^2 + z^2}}, & (x, y, z) \neq (0, 0, 0); \\ 0, & (x, y, z) = (0, 0, 0), \end{cases}$$

$$f_2(x, y, z) := \begin{cases} (x^2 + y^2 + z^2)^p \mathrm{e}^{\frac{1}{\sqrt{x^2+y^2+z^2}}}, & (x, y, z) \neq (0, 0, 0); \\ 0, & (x, y, z) = (0, 0, 0). \end{cases}$$

S8.8 Stabiliţi care din următoarele funcţii sunt continue pe domeniul lor de definiţie:

a) $f:(0,1)\to\mathbb{R},\ f(x):=\sqrt{x}\sin\dfrac{2}{x}$;

b) $f:(-1,\infty)\to\mathbb{R}^2,\ f(x):=\left(\dfrac{x}{x^2+2},\dfrac{\arcsin(x+1)}{x+1}\right)$;

c) $f:\{(x,y)\in\mathbb{R}^2\mid x^2+y^2<1\}\to\mathbb{R},\ f(x,y):=\sin\dfrac{\pi}{1-x^2-y^2}$;

d) $f:[0,\pi]\times[0,\frac{\pi}{3}]\to\mathbb{R}^2,\ f(x,y):=(\sin x+\cos y,\mathrm{e}^{-x}\operatorname{tg} y-1)$.

S8.9 Calculaţi următoarele limite:

a) $\lim\limits_{x\to3}\left(\dfrac{x^x-27}{x-3},\dfrac{x\sin3-3\sin x}{x\sin x-3\sin3},\dfrac{3^x-x^3}{x-3}\right)$;

b) $\lim\limits_{x\to\infty}\left(x\arcsin\dfrac{x+2}{\sqrt{x^4-x^2+1}},\dfrac{\sqrt{x+\sqrt{x+\sqrt{x+\sqrt{x}}}}}{\sqrt{x+1}},\left(\cos\dfrac{1}{x}+\sin\dfrac{1}{x}\right)^x\right)$;

c) $\lim\limits_{(x,y)\to(0,0)}\left((x^2+y^2)\ln(x^2+y^2),(x-y)\operatorname{arctg}\dfrac{1}{2x^2+3y^2},(1+x^2y^2)^{\frac{1}{x^2+y^2}}\right)$;

d) $\lim\limits_{(x,y,z)\to(0,0,0)}\left(\dfrac{xy+yz+zx}{\sqrt{x^2+y^2+z^2}},\dfrac{2x-3y}{z},(x+y+z)\ln(1+|xyz|)\right)$.

S8.10 Analizaţi continuitatea următoarelor funcţii:

a) $f:\left(0,\frac{\pi}{2}\right)\to\mathbb{R}^2,\ f(x):=(f_1(x),f_2(x))$, unde

$$f_1(x):=\begin{cases}\operatorname{tg} x, & x\in\mathbb{Q}\cap\left(0,\frac{\pi}{2}\right)\\ \operatorname{ctg} x, & x\in(\mathbb{R}\backslash\mathbb{Q})\cap\left(0,\frac{\pi}{2}\right)\end{cases}\quad \text{şi } f_2(x):=\begin{cases}x^2+1-\dfrac{\pi}{16}, & x\in\mathbb{Q}\cap\left(0,\frac{\pi}{2}\right)\\ \dfrac{\pi}{4x}, & x\in(\mathbb{R}\backslash\mathbb{Q})\cap\left(0,\frac{\pi}{2}\right)\end{cases};$$

b) $f:\mathbb{R}^2\to\mathbb{R}^2,\ f(x,y):=(f_1(x,y),f_2(x,y))$, unde

$$f_1(x,y):=\begin{cases}\dfrac{x-\sqrt{x^2-y+2}}{y^2-4}, & \text{dacă } 2\neq y\le x^2+2\\ 2^{-3}, & \text{altfel,}\end{cases}$$

$$f_2(x,y)=\begin{cases}\dfrac{(x^4-y^2)^2}{x^6}, & \text{dacă } y^2\le x^4\neq0\\ 0, & \text{altfel.}\end{cases};$$

c) $f:\mathbb{R}^3\to\mathbb{R}$,

$$f(x,y,z)=\begin{cases}\alpha\mathrm{e}^{x+y+z}, & \text{dacă } x+y+z<0\\ \beta, & \text{dacă } x+y+z\ge0,\end{cases}$$

unde $\alpha,\beta\in\mathbb{R}$.

S8.11 Pot fi următoarele funcţii extinse prin continuitate la $\mathbb{R}^2$?

a) $f:\mathbb{R}^2\backslash\{(0,0)\}\to\mathbb{R}^2,\ f(x,y):=\left(\dfrac{\ln(1+x^2|y|)}{x^2+y^2},\left(1+\sin(x^4+y^4)\right)^{\frac{1}{x^2+y^2}}\right)$;

b) $f:\mathbb{R}^2\backslash\{(0,0)\}\to\mathbb{R}^3,\ f(x,y)=\left(\dfrac{\sin(x^2-y^2)}{|x|+|y|},(|x|+|y|)^{x^2+y^2},\dfrac{xy}{\sqrt{2x^2+3y^2}}\right).$

S8.12 Sunt următoarele funcţii continue?

a) $f:\mathbb{R}^2\to\mathbb{R}^2$,

$$f(x,y):=\begin{cases}0, & \text{dacă } (x,y)=\mathbf{0}_{\mathbb{R}^2};\\ \frac{4}{\pi}(x^2+y^2)\operatorname{arctg}\dfrac{1}{x^2+y^2}, & \text{dacă } (x,y)\in B(\mathbf{0}_{\mathbb{R}^2};1)\backslash\{\mathbf{0}_{\mathbb{R}^2}\};\\ 1, & \text{dacă } (x,y)\notin B(\mathbf{0}_{\mathbb{R}^2};1);\end{cases};$$

b) $f:\mathbb{R}^n\to\mathbb{R}$,

$$f(x,y):=\begin{cases}\mathrm{e}^{-\frac{1}{(\|x\|-1)(2-\|x\|)}}, & \text{dacă } \|x\|\in(1,2);\\ 0, & \text{altfel.}\end{cases}.$$

S8.13

a) Arătaţi că mulţimea $A:=\left\{(x,y)\in\mathbb{R}^2 \mid x=\operatorname{arctg}t,\ y=\ln(1+t^2),\ t\in[-1,1]\right\}$ este compactă.

b) Fie $f:\mathbb{R}^3\to\mathbb{R}^2$, definită de

$$f(x,y,z):=\begin{cases}\left((|x|+|y|+|z|)\ln\left(1+\dfrac{1}{|x|+|y|+|z|}\right),\sqrt{2-(|x|+|y|+|z|)}\right), & (x,y,z)\in B\\ (0,\sqrt{2}), & (x,y,z)=\mathbf{0}_{\mathbb{R}^3}\\ (\ln 2,1), & (x,y,z)\in C\end{cases},$$

unde $B:=\left\{(x,y,z)\in\mathbb{R}^3 \mid |x|+|y|+|z|\le 1\right\}\backslash\{\mathbf{0}_{\mathbb{R}^3}\}$ şi $C:=\mathbb{R}^3\backslash\left\{(x,y,z)\in\mathbb{R}^3 \mid |x|+|y|+|z|\le 1\right\}$.

De asemenea, fie $A:=\left\{(x,y,z)\in\mathbb{R}^3 \mid z\ge x^2+y^2, x+y+z\le 2\right\}$.

Arătaţi că:

i) mulţimea A este compactă;

ii) $f|_A$ este uniform continuă;

iii) $f[A]$ este compactă.

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