Ex 143 5) $((\forall x.\ \varphi_1) \vee \varphi_2) \equiv \forall x.\ (\varphi_1 \vee \varphi_2)$ dacă $x \notin free(\varphi_2)$

$((\forall x.\ \varphi_1) \vee \varphi_2) \equiv \forall x.\ (\varphi_1 \vee \varphi_2)$ ddacă

pt orice S și orice S-atr α avem

$S, \alpha \models ((\forall x.\ \varphi_1) \vee \varphi_2)$ ddacă $S, \alpha \models \forall x.\ (\varphi_1 \vee \varphi_2)$ (*)

Fie S o structură arbitrară și α o S-atr. arbitrară fixate.

$S, \alpha \models ((\forall x.\ \varphi_1) \vee \varphi_2)$ ddacă $\Longleftrightarrow$

$\alpha(x) = a$ | $\alpha' = \alpha[x \mapsto u](x) = u$
$\alpha(y) = b$ | $(y) = b$
$\alpha(z) = c$ | $(z) = c$

ddacă $\begin{cases} S, \alpha \models \forall x.\ \varphi_1 \\ \text{sau} \\ S, \alpha \models \varphi_2 \end{cases}$

ddacă $\begin{cases} \text{pt orice } u \in D \text{ avem } S, \overbrace{\alpha[x \mapsto u]}^{\alpha'} \models \varphi_1 \\ \text{sau} \\ S, \alpha \models \varphi_2 \end{cases}$ (1)

ddacă pt orice $u \in D$ avem $\begin{cases} S, \alpha[x \mapsto u] \models \varphi_1 \\ \text{sau} \\ S, \alpha \models \varphi_2 \end{cases}$ (3)

ddacă ($S, \alpha \models \varphi_2$ ddacă $S, \alpha[x \mapsto u] \models \varphi_2$)

pt orice $u \in D$ avem $\begin{cases} S, \alpha[x \mapsto u] \models \varphi_1 \\ \text{sau} \\ S, \alpha[x \mapsto u] \models \varphi_2 \end{cases}$ (2)

ddacă pt orice $u \in D$ avem $S, \alpha[x \mapsto u] \models (\varphi_1 \vee \varphi_2)$

ddacă $S, \alpha \models \forall x.\ (\varphi_1 \vee \varphi_2)$

$S, \alpha \models \varphi_2$ ddacă $S, \alpha[x \mapsto u] \models \varphi_2$?! (**)

$x \notin free(\varphi_2)$ — $x \notin vars(\varphi_2) \Rightarrow$ (**) evident

(val. lui x nu este folosită în evaluarea lui φ_2)

$x \in vars(\varphi_2)$

$\varphi_2 = \exists x \ldots P(x)$

$\forall x$

$S, \alpha[x \mapsto u] \models \varphi_2$ ddacă ...

... există $v \in D$ ... $S, \underbrace{\alpha\cancel{[x \mapsto u]}[x \mapsto v]}_{\alpha_1} \models P(x)$

orice $v \in D$

$\alpha(x) = a \quad \alpha[x \mapsto u](x) = u \quad \ldots \quad \alpha_1(x) = v$

ddacă ... există $v \in D$ ... $\underbrace{S, \alpha \ldots [x \mapsto v] \models P(x)}_{S, \alpha \models \varphi_2}$

$\Rightarrow$ (**) are loc.

$\Rightarrow$ (*) are loc pt S, α arbitrare dacă $x \notin free(\varphi_2)$

$\Rightarrow ((\forall x. \varphi_1) \vee \varphi_2) \equiv \forall x. (\varphi_1 \vee \varphi_2)$ dacă $x \notin free(\varphi_2)$

Ex 130 7)

1. $\forall x. \neg P(x), \exists x. P(x) \vdash \forall x. \neg P(x)$ (ip)

2. $\forall x. \neg P(x), \exists x. P(x) \vdash \neg P(t_x)$ ($\forall e$, 1)

3. $\forall x. \neg P(x), \exists x. P(x) \vdash \exists x. \underbrace{P(x)}_{\varphi}$ (ip)

???

t_x nu conține pe x_0

j. $\forall x. \neg P(x), \exists x. P(x), P(x_0) \vdash P(t_x)$

m. $\forall x. \neg P(x), \exists x. P(x) \vdash \underbrace{P(t_x)}_{\psi}$ ($\exists e$, 3, j)

ψ = aveam nevoie $P(x_0)$ în loc de $P(t_x)$ dar este interzis de $\exists e$

k. $\forall x. \neg P(x), \exists x. P(x) \vdash \bot$ ($\neg e$, m, 2) ?!

n. $\forall x. \neg P(x) \vdash \neg(\underbrace{\exists x. P(x)}_{\varphi})$ ($\neg i$, k)

$$\neg e\ \frac{\Gamma \vdash \varphi_1 \qquad \Gamma \vdash \neg\varphi_1}{\Gamma \vdash \bot} \qquad \neg i\ \frac{\Gamma, \varphi \vdash \bot}{\Gamma \vdash \neg\varphi}$$

$$\exists e\ \frac{\Gamma \vdash \exists x. \varphi \qquad \Gamma, \varphi[x \mapsto x_0] \vdash \psi}{\Gamma \vdash \psi}\ x_0 \notin vars(\varphi, \Gamma, \psi)$$

1. $\forall x. \neg P(x), \exists x. P(x) \vdash \exists x. \overbrace{P(x)}^{\varphi}$ (ip)

2. $\forall x. \neg P(x), \exists x. P(x), P(x_0) \vdash \overbrace{P(x_0)}^{\varphi_1}$ (ip)

3. $\forall x. \neg P(x), \exists x. P(x), P(x_0) \vdash \forall x. \neg P(x)$ (ip)

4. $\forall x. \neg P(x), \exists x. P(x), P(x_0) \vdash \neg P(x_0)$ ($\forall e$, 3)

5 ~~m~~. $\forall x. \neg P(x), \exists x. P(x), P(x_0) \vdash \bot$ ($\neg e$, 2, 4)

6 ~~k~~. $\forall x. \neg P(x), \exists x. P(x) \vdash \overset{\psi}{\bot}$ ($\exists e$, 1, 5 ~~m~~) (x_0 nu apare la linia 1 și 6)

7 ~~n~~. $\forall x. \neg P(x) \vdash \neg(\underbrace{\exists x. P(x)}_{\varphi})$ ($\neg i$, 6 ~~k~~)

φ - FNSC ddacă $\varphi = \forall x_1. \forall x_2. \ldots \forall x_n. \varphi'$

Ex 172

FNS $\begin{cases} \bullet\ \varphi' \text{ nu conține cuantif.} \\ \bullet\ free(\varphi') \subseteq \{x_1, \ldots x_n\} \quad (\varphi \text{ închis}) \\ \quad vars(\varphi') \end{cases}$

$\bullet\ \varphi'$ - FNC

$$\varphi \in LP_1 \xrightarrow[\substack{\text{Th înlocuire} \\ + \\ \text{Lem redenumire} \\ (LR)}]{FNP} \varphi_1\text{-FNP} \xrightarrow[\text{existențială}]{\text{închiderea}} \varphi_2\text{-FNP} \xrightarrow{\text{Skolem}} \varphi_3\text{-FNS}$$

$\varphi \equiv \varphi_1$

φ_2 - închisă FNP

φ_1 echisat φ_2

φ_3-FNS $\xrightarrow{\equiv \text{ (Th. înloc.)}}$ φ_4 - FNSC.

$$\text{a)} \quad \varphi = \underbrace{\left(\exists z. \underbrace{P(x, z)}_{\varphi_1}\right)}_{LR} \land \underbrace{(\forall x. P(x, z))}_{\varphi_2}$$

$(\exists \underset{y}{z}. \varphi_1) \land \varphi_2 \equiv \exists \underset{y}{z}. (\varphi_1 \land \varphi_2)$, $\underset{y}{z} \notin free(\varphi_2)$

$z \in \{z\}$

a) FNP

$$\overset{LR}{\equiv} (\exists y. P(x, y)) \land (\forall x. P(x, z))$$

$$\equiv \exists y. (\underline{P(x, y)} \land \underline{(\forall x. P(x, z))})$$

$$\equiv \exists y. \left(\underbrace{(\forall x. \underbrace{P(x, z)}_{\varphi_1})}_{LR} \land \underbrace{P(x, y)}_{\varphi_2}\right)$$

$(\forall \underset{x_1}{x}. \varphi_1) \land \varphi_2 \equiv \forall \underset{x_1}{x}. (\varphi_1 \land \varphi_2)$ $\quad \underset{x_1}{x} \notin free(\varphi_2) = \{x, y\}$

$$\overset{LR}{\equiv} \exists y. ((\forall x_1. P(x_1, z)) \land P(x, y))$$

$$\equiv \exists y. \forall x_1. (P(x_1, z) \land P(x, y)) = \varphi_1 \text{ - FNP}$$

b) închiderea exist.

$\varphi_2 = \exists z. \exists x. \exists y. \forall x_1. \left(P(x_1, z) \wedge P(x, y) \right)$ închisă φ_1

există $u \in \mathbb{N}$, există $v \in \mathbb{N}$, există $w \in \mathbb{N}$ a.î. pt orice $w_1 \in \mathbb{N}$

avem $(w_1 = u$ și $v = w)$

0

$u \leq v$

$\alpha(x) = 5$

$\alpha'(x)$ D D $P^S(u, v)$

$\varphi_2 = \exists x. \overbrace{\forall y\, P(x, y)}^{\varphi_1}$ există $u \in \mathbb{N}$ a.î. pt orice $v \in \mathbb{N}$ avem $u = v$ "A"

~~$\forall y. \exists x. P(x, y)$~~ ~~pt orice $v \in \mathbb{N}$, există $u \in \mathbb{N}$ a.î. $u = v$~~

$S, \alpha' \models \varphi_1$ pt orice $v \in D$ avem $P^S(\underbrace{\overset{a}{\alpha'(x)}}_{u}, v)$ "A"

c) Skolem

Fie a simbol funcțional nou de aritate 0 (constant)

$\Sigma = (\mathcal{P}, \mathcal{F}) \longrightarrow \Sigma' = (\mathcal{P}, \mathcal{F} \cup \{a\})$

peste Σ'

$\varphi_2' = \exists x. \exists y. \forall x_1. \left(P(x_1, a) \wedge P(x, y) \right)$

Fie b simbol constant nou

$\Sigma'' = (\mathcal{P}, \mathcal{F} \cup \{a, b\})$

$\varphi_2'' = \exists y. \forall x_1. \left(P(x_1, a) \wedge P(b, y) \right)$

Fie c simb. constant nou

$\Sigma''' = (\mathcal{P}, \mathcal{F} \cup \{a, b, c\})$

$\varphi_3 = \forall x_1. \underbrace{\left(\overbrace{P(x_1, a)}^{C_1} \wedge \overbrace{P(b, c)}^{C_2} \right)}_{\varphi' \text{ - FNC}}$ — FNS | — φ_3 - FNSC.

$g(x, y)$

$\varphi_2 = \forall x. \forall y. \exists z \left(P(x, z) \wedge P(x, y) \right)$

fie g simb fct de aritate 2

$\varphi_3 = \forall x. \forall y. \left(P(x, g(x, y)) \wedge P(x, y) \right)$

$h(x)$

$\varphi_2 = \forall x. \exists z. \forall y. P(z, x)$

fie h simb fct de aritate 1

$\varphi_3 = \forall x. \forall y. P(h(x), z)$