SEMINAR 11

Exerciţii recomandate: 11.1 a),c), 11.2, 11.6 a),b), 11.7 a)
Rezerve: 11.1 b), 11.3, 11.4, 11.5, 11.6 c),d), 11.7 b)

S11.1 Calculaţi:

a) $\displaystyle\int \frac{\cos x+\sin x}{\sqrt[5]{\sin x-\cos x}}dx$;

b) $\displaystyle\int_{-2}^{2}\min\left\{x\left(x^2-1\right),x+1\right\}dx$;

c) $\displaystyle\int_{e^2}^{e^3}\frac{dx}{x\cdot\ln x\cdot\ln(\ln x)}$;

d) $\displaystyle\int_0^{1/2}x\ln\frac{1+x}{1-x}dx+\int_0^1\frac{x\operatorname{arctg}x}{\sqrt{1+x^2}}dx+\int_0^{\frac{\pi}{2}}\sin^5 x dx.$

S11.2 Fie funcţia $f:\mathbb{R}\to\mathbb{R}$, definită prin

$$f(x)=\frac{x^2-1}{x^4+x^3+3x^2+x+1},\ x\in\mathbb{R}.$$

Găsiţi o primitivă a funcţiei $f|_{\mathbb{R}_+^*}$, prin substituţia $t=x+\dfrac{1}{x}$. Determinaţi atunci o primitivă a lui f pe $\mathbb{R}$.

S11.3 Arătaţi că dacă $f:[0,1]\to\mathbb{R}$ este o funcţie continuă astfel încât

$$\int_0^1 f^2(x)dx\le 3\left(\int_0^1 F(x)dx\right)^2,$$

unde F este o primitivă a lui f, pentru care $F(1)=0$, atunci f este liniară.

S11.4 Determinaţi $\alpha\in\mathbb{R}$ pentru care funcţia $f:\mathbb{R}\to\mathbb{R}$, definită de

$$f(x)=\begin{cases}\dfrac{1-e^{\sin x}}{(1+x)^{1/x}-e}, & x\neq 0\\ \alpha, & x=0\end{cases},$$

admite primitive.

S11.5 Arătaţi că

$$\int_0^a e^{x^2}dx\cdot\int_0^a e^{-x^2}dx\ge a^2,\ \forall a\in\mathbb{R}_+.$$

S11.6 Calculaţi:

a) $\displaystyle\iint_D\left(x^2+y\right)dxdy$, unde D este domeniul mărginit de curbele $x=y^2$ şi $y=x^2$;

b) $\displaystyle\iint_D\frac{x^2}{\sqrt{x^2+y^2}}dxdy$, unde D este domeniul mărginit de curbele $x^2+y^2=1$ şi $x^2+y^2=4$;

c) $\displaystyle\iint_D\frac{x^2}{y^2}dxdy$, unde D este domeniul mărginit de curbele $y=\dfrac{1}{x}$ şi $y=x$, cu $x\in[1,2]$;

d) $\displaystyle\iint_D\frac{x}{x^2+y^2}dxdy$, unde D este domeniul mărginit de curbele $y=x$ şi $y=\dfrac{x^2}{4}$.

$$D'=\begin{cases}0\le y\le\frac{\pi}{2}-z\\ 0\le z\le\frac{\pi}{2}\end{cases}$$

S11.7 Calculaţi:

a) $\displaystyle\iiint_D\frac{1}{(x+y+z)^3}dxdydz$, unde $D=[1,2]\times[1,2]\times[1,2]$;

b) $\displaystyle\iiint_D xyz\sin(x+y+z)dxdydz$, unde D este domeniul mărginit de planele $x=0$, $y=0$, $z=0$ and $x+y+z=\dfrac{\pi}{2}$.

$$\iint_{D'}\int_0^{\frac{\pi}{2}-(y+z)} xyz\sin(x+y+z)\,dxdydz$$

$$\int_0^{\frac{\pi}{3}} z\left[\int_0^{\frac{\pi}{2}-z} y\left(\int_0^{\frac{\pi}{2}-(y+z)} x \sin(x+y+z)\,dx\right)dy\right]dz$$

S11.8 Calculaţi aria mărginită de curba $(a_1x + b_1y + c_1)^2 + (a_2x + b_2y + c_2)^2 = 1$, unde $a_1, a_2, b_1, b_2, c_1, c_2 \in \mathbb{R}$ sunt astfel încât $a_1b_2 - a_2b_1 \neq 0$.

S11.9 Calculaţi aria mărginită de curba

$$\left|\frac{x}{a}\right|^p + \left|\frac{y}{b}\right|^p = 1,$$

unde a, b şi p sunt parametri din $\mathbb{R}_+^*$.

$$\frac{\pi^4}{384} - \frac{\pi^2}{8} + 1$$

S11.10 Utilizând coordonatele cilindrice, calculaţi volumul mărginit de suprafeţele

$$x^2 + y^2 - 3z = 0 \text{ şi } \left(x^2 + y^2\right)^2 = 9\left(x^2 - y^2\right).$$

S11.11 În ce proporţie este împărţit volumul sferei

$$x^2 + y^2 + z^2 = 4z$$

de suprafaţa $x^2 + y^2 + z = 4$?

Bibliografie recomandată

[1] G. Bucur, E. Câmpu, S. Găină, *Culegere de probleme de calcul diferenţial şi integral*, Ed. Tehnică, Bucureşti, 1966.
[2] C. Ciobotaru, R. Miculescu, A. Roşioru, ş.a., *Probleme de calcul integral,* Editura Gil, Zalău, 2005.
[3] A. Fulga, I. Radomir, *Analiză matematică. Culegere de probleme*, Ed. Albastră, Cluj-Napoca, 2005.
[4] M. Hidegkuti, *Lecture Notes and Worksheets on Improper Integrals,* City Colleges of Chicago, 2015.
[5] John K. Hunter, *An Introduction to Real Analysis,* Univ. of California Publ., 2013.
[6] M. Postolache (coord.), D. Cioroboiu, A. Pitea, *Calcul integral.Exerciţii şi probleme,* Editura "Fair Partners", Bucureşti, 2010.

$$\int \frac{x^4}{x^3+2x^2-x-2}\,dx \rightarrow \text{integrală raţională}$$

$$R(x) = \frac{P(x)}{Q(x)}$$

P, Q polinoame

1. Studiem dacă $\deg P(x) \geq \deg Q(x)$

În caz afirmativ,

$\exists\ C(x),\ \exists\ r(x)$ a.î.

$$P(x) = Q(x)\cdot C(x) + r(x)$$

$$\deg r(x) < \deg Q(x)$$

$$\int R(x)\,dx = \int \frac{P(x)}{Q(x)}\,dx = \int \frac{Q(x)\,C(x) + r(x)}{Q(x)}\,dx$$

$$= \int C(x) + \frac{r(x)}{Q(x)}\,dx$$

Pe est nsobre :

$$P(x) = x^4 \qquad \deg P(x) = 4$$

$$Q(x) = x^3 + 2x^2 - x - 2$$

$$\deg Q(x) = 3$$

$$\begin{array}{l|l}
x^4 & x^3 + 2x^2 - x - 2 \\ \cline{2-2}
 & x - 2 \\
-x^4 - 2x^3 + x^2 + 2x & \\ \hline
/\ \ -2x^3 + x^2 + 2x & \\
\quad\ \ 2x^3 + 4x^2 - 2x - 4 & \\ \hline
\qquad\quad 5x^2 - 4 &
\end{array}$$

$$\int \frac{x^4}{x^3+2x^2-x-2}\,dx = \int \left(x - 2 + \frac{5x^2-4}{x^3+2x^2-x-2} \right) dx$$

$$= \frac{x^2}{2} - 2x + \int \frac{5x^2-4}{\underline{x^3+2x^2-x-2}}\,dx \qquad (1)$$

2. Descompunem numitorul în factori ireductibili: $(ax+b)^n$

$(ax^2+bx+c)^m \quad \Delta < 0$

$$x^3+2x^2-x-2 = (x-1)(1\cdot x^2 + 3\cdot x + 2)$$

	x^3	x^2	x	1
	1	2	−1	−2
1	1	$1\cdot 1+2=3$	$1\cdot 3+-1=2$	$1\cdot 2+-2=0$

$$x^2+3x+2=0$$

$$x_{1,2}=\frac{-3\pm\sqrt{9-4\cdot 2}}{2}=$$

$$\frac{-3\pm 1}{2} \quad \begin{cases} -2 \\ -1 \end{cases}$$

$$x^2+3x+2=(x-(-2))(x-(-1))$$

$$=(x+2)(x+1)$$

$$x^3+2x^2-x-2=(x-1)(x+2)(x+1)$$

3. Descompunem $\frac{r(x)}{Q(x)}$ în

fracții simple:

$$\frac{A}{(ax+b)^n}, \quad \frac{Bx+C}{(ax^2+bx+c)^m}$$

$$(x+2)(x+1) \qquad (x-1)(x+1) \qquad (x-1)(x+2)$$

$$\frac{5x^2-4}{(x-1)(x+2)(x+1)} = \frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{x+1}$$

$$5x^2-4 = A(x+2)(x+1) + \underline{B(x-1)(x+1)} +$$
$$\underline{C(x-1)(x+2)}$$

Met $\underline{I}$

$x = 1$

$$1 = A(1+2)(1+1)$$

$$6A = 1 \Rightarrow A = \frac{1}{6}$$

$x = -2$

$$16 = 3B \Rightarrow B = \frac{+6}{3}$$

$x = -1$

$$1 = -2C \Rightarrow C = -\frac{1}{2}$$

Met II

$$5x^2-4 = A(x^2+3x+2) + B(x^2-1) + C(x^2+x-2)$$

$$5x^2-4 = x^2\underline{(A+B+C)} + x(3A+C) + (2A-B-2C)$$

$$\begin{cases} A+B+C=5 \\ 3A+C=0 \Rightarrow C=-3A \\ 2A-B-2C=-4 \end{cases}$$

$A-3A+B=5 \Rightarrow$

$B=5+2A$

$\Rightarrow A = \frac{C}{-3}$

$\cancel{2A}-5-\cancel{2A}-2C=-4$

$-2C=1$

$C=-\frac{1}{2}$

$A=\frac{1}{6}$

$B=5+\frac{2}{6}=5+\frac{1}{3}=\frac{16}{3}$

$$\int \frac{(5x^2-4)\,dx}{(x-1)(x+2)(x+1)} =$$

$$\int \left(\frac{\frac{1}{6}}{x-1} + \frac{\frac{16}{3}}{x+2} + \frac{-\frac{1}{2}}{x+1} \right) dx$$

$$= \frac{1}{6} \ln|x-1| + \frac{16}{3} \ln|x+2| -$$

$$\frac{1}{2} \ln|x+1| + C$$

(1) $$\int \frac{x^4}{x^3+2x^2-x-2}\,dx =$$

$$\frac{x^2}{2} - 2x + \frac{1}{6} \ln|x-1| +$$

$$\frac{16}{3} \ln|x+2| +$$

$$\left(-\frac{1}{2}\right) \ln|x+1| + C$$

a) $I=\int \frac{\cos x+\sin x}{\sqrt[5]{\sin x-\cos x}}\,dx;$

$$\sqrt[5]{\sin x-\cos x}=t$$

$$\sin x-\cos x=t^5$$

$$(\cos x+\sin x)\,dx=5t^4\,dt$$

$$I=\int \frac{5t^4\,dt}{t}=5\int t^3\,dt=\frac{5}{4}t^4+C$$

Revenim la var inițială

$$=\frac{5}{4}\left(\sqrt[5]{\sin x-\cos x}\right)^4+\underline{C}$$

c) $I=\int_{e^2}^{e^3} \frac{dx}{x\cdot \ln x\cdot \ln(\ln x)};$

Met I $\underline{\ln}(\ln x)=t$

$$\frac{1}{\ln x}\cdot\frac{1}{x}\,dx=dt$$

$$\frac{dx}{x\ln x} = dt$$

$$x = e^2 \Rightarrow t = \ln(\ln e^2) = \ln 2$$

$$x = e^3 \Rightarrow t = \ln(\ln e^3) = \ln 3$$

$$I = \int_{\ln 2}^{\ln 3} \frac{dt}{t} = \ln|t| \Big|_{t=\ln 2}^{t=\ln 3} =$$

$$\ln(\ln 3) - \ln(\ln 2) = \ln \frac{\ln 3}{\ln 2}$$

Met II $\ln x = u$

$$\frac{1}{x}dx = du \qquad x = e^2 \Rightarrow u = 2$$

$$x = e^3 \Rightarrow u = 3$$

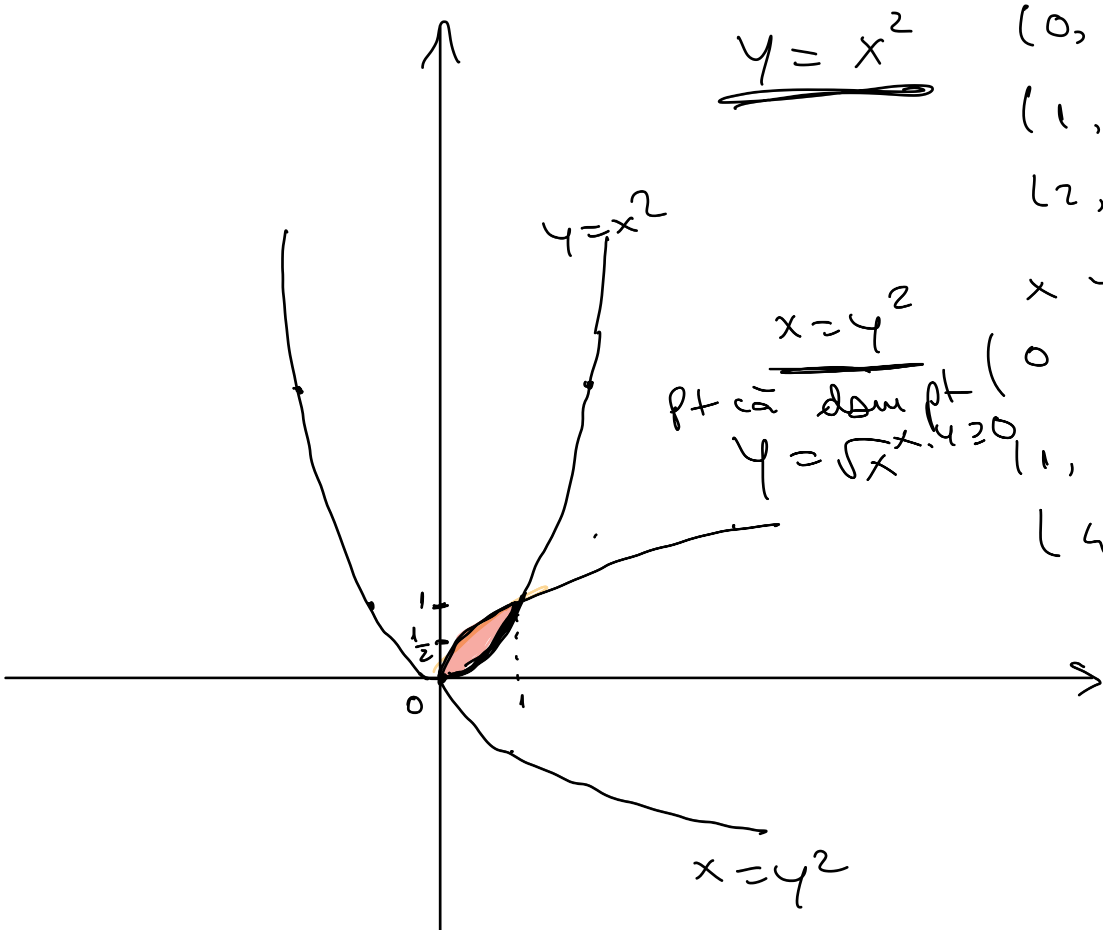
$$I = \int_2^3 \frac{du}{u\ln u} =$$

$$\ln u = v \qquad u = 2 \Rightarrow v = \ln 2$$

$$\frac{1}{u}du = dv \qquad u = 3 \Rightarrow v = \ln 3$$

$$I = \int_{\ln 2}^{\ln 3} \frac{dv}{v} = \ldots = \ln \frac{\ln 3}{\ln 2}$$

$I = \iint_D \left(x^2 + y\right) dxdy$, unde D este domeniul mărginit de curbele $x = y^2$ şi $y = x^2$;



$y = x^2$

x	y
0	0
1	1
2	4

$x = y^2$

pt că dom pt $x, y \geq 0$

$y = \sqrt{x}$

x	y
0	0
1	1
4	2

$$\begin{cases} y = x^2 \\ x = y^2 \end{cases}$$

$$\begin{cases} y = x^2 \\ x = (x^2)^2 \end{cases} \Rightarrow$$

Dacă $x \in [0,1]$

$0 \le x \le 1$ $\quad 0 \le x^{\frac{3}{2}} \le 1$

$x = x^4 \Rightarrow$

$x^4 - x = 0 \Rightarrow$

$x(x^3 - 1) = 0$

$x^2 \boxed{\le} \sqrt{x} \quad \Big| \quad \sqrt{x} =$

$x\sqrt{x} \boxed{\le} 1$

$x^{\frac{3}{2}} \le 1$

$\Rightarrow x = 0 \Rightarrow y = 0 \quad (0,0)$

sau

$x^3 = 1 \Rightarrow x = 1 \Rightarrow y = 1 \quad (1,1)$

Domeniu simplu în raport cu Oy

$$\varphi(x) \le y \le \psi(x)$$

$$a \le x \le b$$

$$I = \int_a^b \int_{\varphi(x)}^{\psi(x)} f(x,y)\, dy\, dx$$

$$x^2 \le y \le \sqrt{x}$$

$$0 \le x \le 1$$

$$I = \int_0^1 \int_{x^2}^{\sqrt{x}} (x^2 + y)\, dy\, dx =$$

$$\int_0^1 \left[\int_{x^2}^{\sqrt{x}} (x^2 + y)\, dy \right] dx =$$

$$\int_0^1 \left(x^2 y + \frac{y^2}{2} \right) \Bigg|_{y=x^2}^{y=\sqrt{x}} dx =$$

$$\int_0^1 \left[\left(x^2 \sqrt{x} + \frac{\sqrt{x}^2}{2} \right) - \left(\underbrace{x^2 \cdot x^2 + \frac{(x^2)^2}{2}}_{\frac{3x^4}{2}} \right) \right] dx$$

$$= \int_0^1 \left(x^{\frac{5}{2}} + \frac{x}{2} - \frac{3x^4}{2} \right) dx =$$

$$\left(\frac{x^{\frac{7}{2}}}{\frac{7}{2}} + \frac{x^2}{4} - \frac{3x^5}{10} \right) \Bigg|_{x=0}^{x=1} =$$

$$\int x^a \, dx = \frac{x^{a+1}}{a+1} + C$$

$$\left(\frac{1}{\frac{7}{2}} + \frac{1}{4} - \frac{3}{10} \right) =$$

$$\left(\frac{2}{7} + \frac{1}{4} - \frac{3}{10} \right) = \frac{75-42}{140} = \frac{33}{140}$$

20) 35) 14)

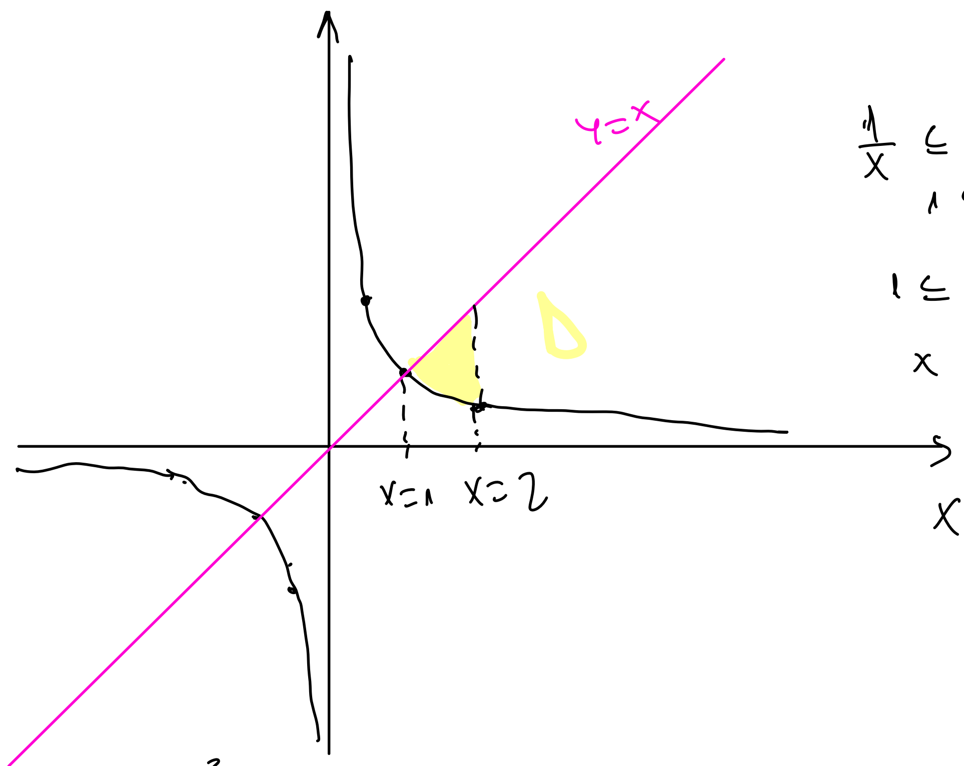
$I = \iint_D \frac{x^2}{y^2} dxdy$, unde D este domeniul mărginit de curbele $y = \frac{1}{x}$ şi $y = x$, cu $x \in [1, 2]$;

Domeniu simplu în raport cu Oy

$$1 \le x \le 2$$

$$x = 1 \qquad \frac{1}{x} = 1 \qquad x = 1$$

$$x = 2 \qquad \frac{1}{x} = \frac{1}{2} \qquad x = 2$$



$$\frac{1}{x} \le y \le x$$

$$1 \le x^2 \le 4$$

$$1 \le x \le 2$$

$$x \boxed{>} \frac{1}{x} \quad | \cdot x$$

$$x^2 \boxed{>} 1$$

$$I = \int_1^2 \int_{\frac{1}{x}}^{x} \frac{x^2}{y^2} dy dx =$$

$$\int_1^2 x^2 \left(\int_{\frac{1}{x}}^{x} \frac{1}{y^2} \, dy \right) dx$$

$$\int_1^2 x^2 \cdot \left(-\frac{1}{y} \right) \Big|_{y=\frac{1}{x}}^{y=x} dx$$

$$\int_1^2 x^2 \left(-\frac{1}{x} - \left(- \frac{1}{\frac{1}{x}} \right) \right) dx$$

$$\int_1^2 x^2 \left(-\frac{1}{x} + x \right) dx =$$

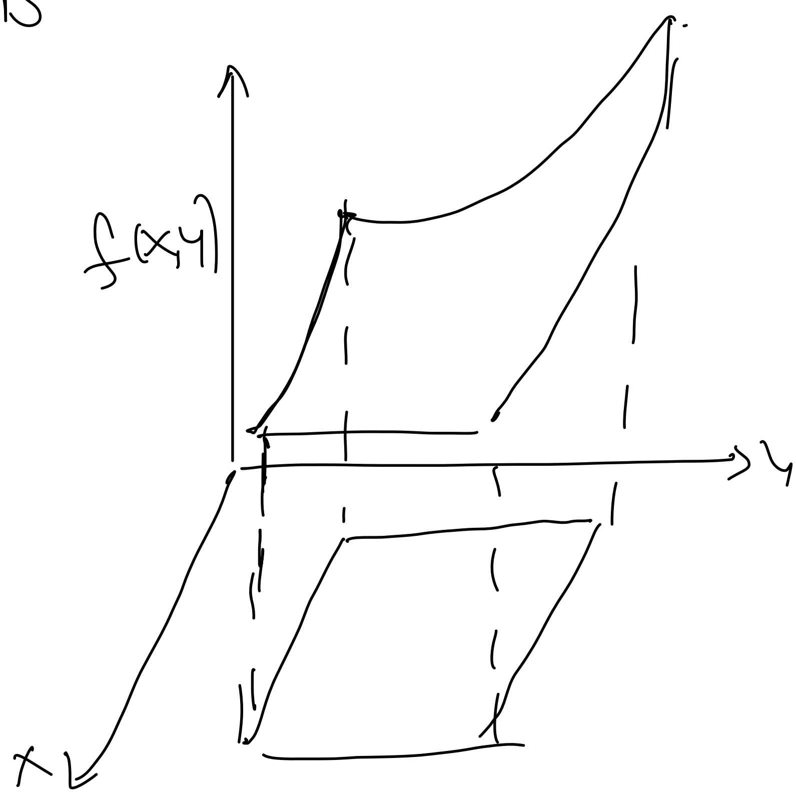
$$\int_1^2 \left(-x + x^3 \right) dx$$

$$\left(-\frac{x^2}{2} + \frac{x^4}{4} \right) \Big|_{x=1}^{x=2}$$

$$\left(-\frac{4}{2} + \frac{16}{4} \right) - \left(-\frac{1}{2} + \frac{1}{4} \right) =$$

$$(-2+4)-\left(-\frac{1}{4}\right)=\frac{9}{4}$$

$$\iint_D dx\,dy = A_D$$



$$\iint_D f\,dx\,dy = V$$

$I = \iint_D \frac{x}{x^2 + y^2} dxdy$, unde D este domeniul mărginit de curbele $y = x$ şi $y = \frac{x^2}{4}$.

$\Rightarrow$ Domeniu simplu în raport cu y

Pct de int

$$\begin{cases} y = x \\ y = \frac{x^2}{4} \end{cases} \quad \begin{cases} y = x \\ x = \frac{x^2}{4} \end{cases} \quad \begin{cases} y = x \\ 4x = x^2 \end{cases}$$

$$\begin{cases} y = x \\ x(x-4) = 0 \end{cases}$$

I $x = 0 \quad y = 0$

$(0,0)$

II $x = 4, y = 4$

$(4,4)$

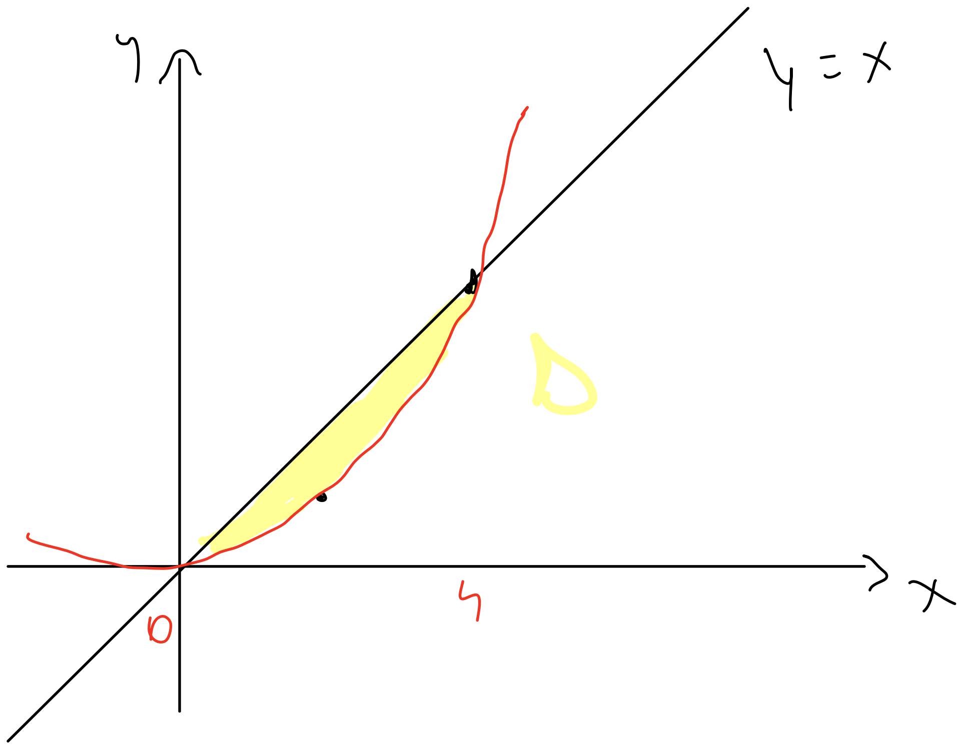
$\underline{0 \leq x \quad \leq 4}$

$x \boxed{\geq} \frac{x^2}{4} \quad | : x$

$1 \boxed{\geq} \frac{x}{4} \quad | \cdot 4$

$4 \boxed{\geq} x$

$$I = \int_0^4 \int_{\frac{x^2}{4}}^{x} \frac{x}{x^2+y^2}\,dy\,dx =$$



$$\int_0^4 x\left(\int_{\frac{x^2}{4}}^{x} \frac{1}{x^2+y^2}\,dy\right)dx =$$

$$\int_0^4 \cancel{x}\left[\frac{1}{\cancel{x}}\,\text{arctg}\,\frac{y}{x}\right]_{y=\frac{x^2}{4}}^{y=x} dx =$$

$$\int_0^4 \left(\operatorname{arctg} \frac{x}{x} - \operatorname{arctg} \frac{\frac{x^2}{4}}{x}\right) dx$$

$$\int_0^4 \left(\underbrace{\operatorname{arctg} 1}_{\frac{\pi}{4}} - \operatorname{arctg} \frac{x}{4}\right) dx$$

$$\int_0^4 \left(\frac{\pi}{4} - \operatorname{arctg} \frac{x}{4}\right) dx =$$

$$\int \operatorname{arctg} \frac{x}{4} \cdot 1\, dx = \int \underbrace{\operatorname{arctg} \frac{x}{4}}_{f} \cdot \underbrace{(x')}_{g'} dx$$

$$\int f g' = fg - \int f' g$$

$$= \operatorname{arctg} \frac{x}{4} \cdot x - \int \left(\operatorname{arctg} \frac{x}{4}\right)' \cdot x\, dx$$

$$x \operatorname{arctg} \frac{x}{4} - \int \frac{1}{4} \cdot \frac{1}{\left(\frac{x}{4}\right)^2 + 1} \cdot x\, dx$$

$$x \operatorname{arctg} \frac{x}{4} - \int \frac{\cancel{16}\ 4}{\cancel{4}(x^2+16)}\, x\, dx$$

$$x \operatorname{arctg} \frac{x}{4} - \int \frac{4x}{x^2+16}\, dx$$

$$t = x^2+16 \qquad dt = 2x\,dx$$

$$x \operatorname{arctg} \frac{x}{4} - \int \frac{2\,dt}{t} =$$

$$x \operatorname{arctg} \frac{x}{4} - 2\ln(x^2+16)$$

$$\frac{\pi}{4}x - x \operatorname{arctg} \frac{x}{4} + 2\ln(x^2+16) \Big|_{x=0}^{x=4}$$

$$\frac{\pi}{\cancel{4}} \cdot \cancel{4} - 4 \underbrace{\operatorname{arctg} \frac{4}{4}}_{\frac{\pi}{4}} + 2\ln(16+16) -$$

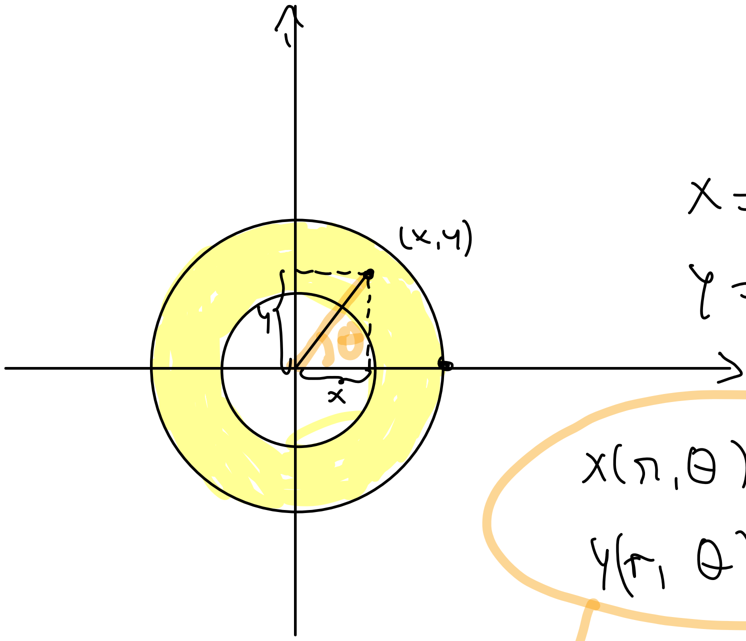
$$2\ln 16 =$$

$$\cancel{\pi} - \cancel{\pi} + 2\ln 32 - 2\ln 16 =$$

$$2\ln 2$$

) $I = \iint_D \frac{x^2}{\sqrt{x^2+y^2}}\,dxdy$, unde D este domeniul mărginit de curbele $x^2+y^2=1$ şi $x^2+y^2=4$;

$x^2+y^2=R^2$



$x = r\cos\theta$

$y = r\sin\theta$ $\quad r \ge 0$

Coord polare

$x(r,\theta) = r\cos\theta$

$y(r,\theta) = r\sin\theta$

$x^2+y^2 = r^2\cos^2\theta + r^2\sin^2\theta$
$= r^2$

$1 \le r^2 \le 4$

$1 \le r \le 2$

$\theta \in [0, 2\pi]$

Dacă ambele sunt între nr, nu cont ordinea int

$I = \int_{r=1}^{r=2} \int_{\theta=0}^{\theta=2\pi} \frac{r^2\cos^2\theta}{r} J \, dr \, d\theta$

$$J = \begin{vmatrix} \frac{\partial x}{\partial r}(r,\theta) & \frac{\partial x}{\partial \theta}(r,\theta) \\ \frac{\partial y}{\partial r}(r,\theta) & \frac{\partial y}{\partial \theta}(r,\theta) \end{vmatrix} =$$

$$\begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} =$$

$$r\cos^2\theta - (-r\sin^2\theta) =$$

$$r(\cos^2\theta + \sin^2\theta) = r$$

Jacobianul transf e r

$$I = \int_1^2 \int_0^{2\pi} \frac{r^2\cos^2\theta}{\cancel{r}} \cdot \cancel{r}\, dr\, d\theta$$

$$\underline{\int_1^2 r^2\, dr} \quad \underline{\int_0^{2\pi} \cos^2\theta\, d\theta}$$

$$= \frac{r^3}{3}\Big|_1^2 \cdot \int_0^{2\pi} \frac{1+\cos 2\theta}{2}\, d\theta$$

$$\frac{r^3}{3}\Big|_1^2 \cdot \left(\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right)\Big|_0^{2\pi}$$

$$\left(\frac{8}{3} - \frac{1}{3}\right) \cdot \left(\frac{2\pi}{2} + \frac{\sin 4\pi}{4} - 0 - \sin 0\right)$$

$$\frac{7}{3}\pi$$

$$\iiint_D \frac{1}{(x+y+z)^3}\,dxdydz, \text{ unde } D = [1,2] \times [1,2] \times [1,2];$$

$$\int_1^2 \left(\int_1^2 \left(\int_1^2 \frac{1}{(x+y+z)^3}\, dx \right) dy \right) dz$$

$$\int_1^2 \left[\int_1^2 \left(\int_1^2 (x+y+z)^{-3}\,dx\right) dy\right] dz$$

$$\boxed{\int (x+ct)^a = \frac{(x+ct)^{a+1}}{a+1} + C}$$

$$\int_1^2 \int_1^2 \frac{(x+y+z)^{-2}}{-2} \Bigg|_{x=1}^{x=2} dy\,dz =$$

$$-\frac{1}{2}\int_1^2 \int_1^2 \left[(2+y+z)^{-2} - (1+y+z)^{-2}\right] dy\,dz$$

$$-\frac{1}{2}\int_1^2 \left[\frac{(2+y+z)^{-1}}{-1} - \frac{(1+y+z)^{-1}}{-1}\right]\Bigg|_{y=1}^{y=2} dz$$

$$\frac{1}{2}\int_1^2 \left((2+y+z)^{-1} - (1+y+z)^{-1}\right)\Bigg|_{y=1}^{y=2} dz$$

$$\frac{1}{2}\int_1^2 \left((4+z)^{-1} - (3+z)^{-1} - \right.$$

$$(3+z)^{-1} + (2+z)^{-1}\Big)\, dz$$

$$\frac{1}{2}\left(\int_1^2 \frac{1}{4+z} - \frac{2}{3+z} + \frac{1}{2+z}\right) dz$$

$$\frac{1}{2}\Big[\ln(4+z) - 2\ln(3+z) + \ln(2+z)\Big]_{z=1}^{z=2}$$

$$\frac{1}{2}\Big[\big(\ln 6 - 2\ln 5 + \ln 4\big) -$$

$$\big(\ln 5 - 2\ln 4 + \ln 3\big)\Big]$$

$$\frac{1}{2}\Big(\ln 6 - \ln 5^2 + \ln 4 -$$

$$\ln 5 + \ln 4^2 - \ln 3\Big)$$

$$\frac{1}{2} \ln \frac{\cancel{6}^2 \cdot 4^3}{5^3 \cdot \cancel{3}}$$

$$\ln \sqrt{\frac{128}{125}}$$