SEMINAR 10

Exerciţii recomandate: 10.1 b),d),f), 10.2 a),c), 10.3
Rezerve: 10.2 b), 10.4 a), 10.5 a)

S10.1 Determinaţi extremele locale ale următoarelor funcţii:

a) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \cos x \cdot \mathrm{e}^{\sin^2 x}$;
b) $f : \mathbb{R}_+^* \to \mathbb{R}$, $f(x) = x(\ln x)^2$;
c) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 2\cos x + x^2$;
d) $f : \underline{\mathbb{R}_+^* \times \mathbb{R}_+^*} \to \mathbb{R}$, $f(x, y) = xy + \frac{4}{x} + \frac{2}{y} - 3$;
e) $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $f(x, y) = \dfrac{2 - xy}{x^2 + y^2 + 1}$;
f) $f : \mathbb{R}_+^* \times \mathbb{R}_+^* \times \mathbb{R}_+^* \to \mathbb{R}_+^*$, $f(x, y, z) = x + \frac{y^2}{4x} + \frac{z^2}{y} + \frac{2}{z}$.

S10.2 Găsiţi extremele cu legături pentru fiecare din cazurile următoare:

a) $f(x, y) = x^2 + y^2$, cu legătura $3x + 2y - 6 = 0$;
b) $f(x, y, z) = x - 2y + 2z$, cu legătura $x^2 + y^2 + z^2 - 9 = 0$;
c) $f(x, y, z) = xy^2z^3$, cu legătura $x + y + z = 12$, $x > 0$, $y > 0$, $z > 0$;
d) $f(x, y, z) = xyz$, cu legăturile $x + y + z = 5$ şi $xy + yz + zx = 8$.

S10.3 Determinaţi cum trebuie tăiată o bară metalică pentru a confecţiona un acvariu paralelipipedic de capacitate maximă.

S10.4 Găsiţi extremele următoarelor funcţii:

a) $f_1(x, y) = x^2\mathrm{e}^{-x^4 - y^2}$, $(x, y) \in \mathbb{R}^2$;
b) $f_2(x, y, z) = xy^2z^3(1 - x - 2y - 3z)$, $(x, y, z) \in \mathbb{R}^3$.

S10.5 Găsiţi extremele cu legături ale următoarelor funcţii:

a) $f(x, y, z) = xy + xz + yz$, pentru $xyz = 1$, $x > 0$, $y > 0$, $z > 0$;
b) $f(p_1, p_2, \dots, p_n) = \log_2\left(\sum_{i=1}^n p_i^2\right)$, cu $\sum_{i=1}^n p_i = 1$ şi $p_i \in (0, 1)$, $\forall 1 \le i \le n$ (*entropia Rényi*).

S10.6 Găsiţi extremele globale pentru următoarele funcţii pe submulţimile K descrise mai jos.

a) $f(x, y) = 5x^2 + 3xy + y^2$, $(x, y) \in K = \left\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \le 1\right\}$;
b) $f(x, y) = \sin x + \sin y + \sin(x + y)$, $(x, y) \in K = \left\{(x, y) \in \mathbb{R}^2 \mid 0 \le x \le \frac{\pi}{2}, 0 \le y \le \frac{\pi}{2}\right\}$.

Bibliografie recomandată

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b) $f : \mathbb{R}_+^* \to \mathbb{R}, f(x) = x(\ln x)^2$;

1. Calculăm derivata

$$f'(x) = x'(\ln x)^2 + x\left[(\ln x)^2\right]' =$$

$$(\ln x)^2 + \cancel{x} \cdot 2\ln x \cdot \frac{1}{\cancel{x}} = (\ln x)^2 + 2\ln x$$

$$(\ln x)(\ln x + 2)$$

2. Determinăm pct critice

$$f'(x) = 0 \Rightarrow (\ln x)(\ln x + 2) = 0$$

I $\ln x = 0 \Rightarrow x = e^0 = 1$

II $\ln x + 2 = 0 \Rightarrow \ln x = -2 \Rightarrow x = e^{-2} = \frac{1}{e^2}$

3. Var I: Specifică pt $f : \mathbb{R}$

Tablou de variație:

x	0		$\frac{1}{e^2}$		1		∞
$f'(x)$		$+$	0	$-$	0	$+$	
$f(x)$	0	$\nearrow$	$\frac{4}{e^2}$	$\searrow$	0	$\nearrow$	∞

Pt $x > 1$
$\ln x > 0$
$f'(x) > 0$

$f\left(\frac{1}{e^2}\right) =$
$\frac{1}{e^2} \cdot (-2)^2 =$
$\frac{4}{e^2}$

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{(\ln x)^2}{\frac{1}{x}} \overset{l'H}{=}$$

Dată: $f(1) = 1 \cdot 0^2 = 0$

$$\lim_{x \to 0} \frac{2\ln x \cdot \frac{1}{\cancel{x}}}{-\frac{1}{x^{\cancel{2}}}} = -2\lim_{x\to 0} \frac{\ln x}{\frac{1}{x}} \overset{l'H}{=} \text{dacă } \exists$$

$$-2\lim_{x\to 0} \frac{\frac{1}{\cancel{x}}}{-\frac{1}{x^{\cancel{2}}}} = -2\lim_{x\to 0}(-x) = 0$$

$\Rightarrow \frac{1}{e^2}$, 1 pct de extrem local

$\frac{1}{e^2}$ max local

1 min chiar global din analiza val

Var II Calculăm hessiana

$$H(f)(x) = f''(x) = 2\ln x \cdot \frac{1}{x} + \frac{2}{x} =$$

$$\frac{2}{x}(\ln x + 1)$$

$Hf\left(\frac{1}{e^2}\right) = 2e^2 \cdot (-2+1) = -2e^2 < 0 \Rightarrow$

$\frac{1}{e^2}$ pct de maxim local

$Hf(1) = 2 \cdot 1 = 2 > 0 \Rightarrow 1$ pct de minim local

$f : \mathbb{R}_+^* \times \mathbb{R}_+^* \to \mathbb{R},\ f(x,y) = xy + \frac{4}{x} + \frac{2}{y} - 3;$

$f(2,1) =$

$2 + 2 + 2 - 3 =$

3

1. Derivate parțiale de ordin I

$\frac{\partial f}{\partial x} = y - \frac{4}{x^2}$

$\frac{\partial f}{\partial y} = x - \frac{2}{y^2}$

2. Pct critice / staționare

$\begin{cases} \frac{\partial f}{\partial x}(x,y) = 0 \\ \frac{\partial f}{\partial y}(x,y) = 0 \end{cases} \Rightarrow \begin{cases} y - \frac{4}{x^2} = 0 \\ x - \frac{2}{y^2} = 0 \end{cases} \Rightarrow$

$\begin{cases} x^2 y = 4 \\ x y^2 = 2 \end{cases} \quad x, y \neq 0 \Rightarrow \begin{cases} y = \frac{4}{x^2} \\ x \cdot \left(\frac{4}{x^2}\right)^2 = 2 \end{cases}$

$$\begin{cases} y = \frac{4}{x^2} \\ x \cdot \frac{16}{x^4} = 2 \end{cases} \qquad \begin{cases} y = \frac{4}{x^2} \\ \frac{16}{x^3} = 2 \Rightarrow x^3 = 8 \Rightarrow x = 2 \end{cases} \Rightarrow y = 1$$

Pct critic $(2,1) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$

3. Hessiana

$$\frac{\partial^2 f}{\partial y \partial x}(x,y) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}(x,y)\right) = \frac{\partial}{\partial y}\left(y - \frac{4}{x^2}\right) = 1$$

$$\frac{\partial^2 f}{\partial x^2}(x,y) = \frac{8}{x^3}$$

$$\frac{\partial^2 f}{\partial x \partial y}(x,y) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}(x,y)\right) = \frac{\partial}{\partial x}\left(x - \frac{2}{y^2}\right) = 1$$

$$\frac{\partial^2 f}{\partial y^2}(x,y) = \frac{4}{y^3}$$

$$\frac{\partial^2 f}{\partial x \partial y}(x,y) = 1 = \frac{\partial^2 f}{\partial y \partial x}(x,y)$$

$$H(f)(x,y) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2}(x,y) & \frac{\partial^2 f}{\partial x \partial y}(x,y) \\ \frac{\partial^2 f}{\partial y \partial x}(x,y) & \frac{\partial^2 f}{\partial y^2}(x,y) \end{pmatrix}$$

$$= \begin{pmatrix} \frac{8}{x^3} & 1 \\ 1 & \frac{4}{y^3} \end{pmatrix}$$

$$H(f)(2,1) = \begin{pmatrix} \frac{8}{8} & 1 \\ 1 & \frac{4}{1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix}$$

$\det H = 3 > 0 \Rightarrow$

$(2,1)$ pct de minim local

$\frac{y^2}{4} x^{-1}$

$f : \mathbb{R}_+^* \times \mathbb{R}_+^* \times \mathbb{R}_+^* \to \mathbb{R}_+^*$, $f(x,y,z) = x + \frac{y^2}{4x} + \frac{z^2}{y} + \frac{2}{z}$.

1. $\frac{\partial f}{\partial x}(x,y,z) = 1 - \frac{y^2}{4x^2}$

$\frac{\partial f}{\partial y}(x,y,z) = \frac{\cancel{2}y}{\cancel{4}_2 x} - \frac{z^2}{y^2} = \frac{y}{2x} - \frac{z^2}{y^2}$

$\frac{\partial f}{\partial z}(x,y,z) = \frac{2z}{y} - \frac{2}{z^2}$

2. $$\begin{cases} \frac{\partial f}{\partial x}(x,y,z) = 0 \\ \frac{\partial f}{\partial y}(x,y,z) = 0 \\ \frac{\partial f}{\partial z}(x,y,z) = 0 \end{cases} \Rightarrow \begin{cases} y^2 = 4x^2 \\ \frac{y}{2x} = \frac{z^2}{y^2} \\ \frac{2z}{y} = \frac{2}{z^2} \end{cases}$$

$x, y, z \in (\mathbb{R}^*_+)^3 \Longrightarrow$ deci $x, y, z > 0$

$$\begin{cases} y = 2x \\ \Downarrow \\ z^2 = y^2 \to z = y \\ 2 = \frac{2}{z^2} \to z = 1 \end{cases} \to (x, y, z) = \left(\frac{1}{2}, 1, 1\right)$$

Unicul pct critic $= \left(\frac{1}{2}, 1, 1\right)$

3. $\frac{\partial^2 f}{\partial x^2}(x, y, z) = \frac{\partial}{\partial x}\left(1 - \frac{y^2}{4} \cdot x^{-2}\right) =$

$$- \frac{y^2}{4} \cdot (-2) x^{-3} = \frac{2y^2}{4x^3} = \frac{y^2}{2x^3}$$

$$\frac{\partial^2 f}{\partial y^2}(x, y, z) = \frac{\partial}{\partial y}\left(\frac{y}{2x} - z^2 y^{-2}\right) =$$

$$\frac{1}{2x} + \frac{2z^2}{y^3}$$

$$\frac{\partial^2 f}{\partial z^2}(x, y, z) = \frac{\partial}{\partial z}\left(\frac{2z}{y} - 2z^{-2}\right) =$$

$$\frac{2}{y} + \frac{4}{z^3}$$

$$\frac{\partial^2 f}{\partial x \partial y}(x,y,z) = \frac{\partial}{\partial x}\left(\frac{y}{2x} - \frac{z^2}{y^2}\right) =$$

$$-\frac{y}{2x^2}$$

$$\frac{\partial^2 f}{\partial z \partial y}(x,y,z) = \frac{\partial}{\partial z}\left(\frac{y}{2x} - \frac{z^2}{y^2}\right)$$

$$= \frac{-2z}{y^2}$$

$$\frac{\partial^2 f}{\partial x \partial z}\left(\frac{2z}{y} - 2z^{-2}\right) = 0$$

Derivatele mixte egale

$$H(f)(x,y,z) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial x \partial z} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} & \frac{\partial^2 f}{\partial y \partial z} \\ \frac{\partial^2 f}{\partial z \partial x} & \frac{\partial^2 f}{\partial z \partial y} & \frac{\partial^2 f}{\partial z^2} \end{pmatrix}$$

$$H(f)(x,y,z) = \begin{pmatrix} \frac{y^2}{2x^3} & -\frac{y}{2x^2} & 0 \\ -\frac{y}{2x^2} & \frac{1}{2x}+\frac{2z^2}{y^3} & -\frac{2z}{y^2} \\ 0 & -\frac{2z}{y^2} & 2y+4z^3 \end{pmatrix}$$

$$Hf\left(\frac{1}{2},1,1\right) = \begin{pmatrix} \frac{1}{\cancel{2}\cdot\frac{1}{\cancel{8}4}} & -\frac{1}{2\cdot\frac{1}{4}} & 0 \\ -\frac{1}{2\cdot\frac{1}{4}} & \frac{1}{\cancel{2}\cdot\frac{1}{\cancel{2}}}+\frac{2\cdot 1}{1} & -\frac{2\cdot 1}{1} \\ 0 & -2\cdot\frac{1}{1} & 2+4 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & -2 & 0 \\ -2 & 3 & -2 \\ 0 & -2 & 6 \end{pmatrix}$$

/

$\Delta_1 = 4 > 0$

$\Delta_2 = \begin{vmatrix} 4 & -2 \\ -2 & 3 \end{vmatrix} = 12 - 4 = 8 > 0$

$\Delta_3 = \begin{vmatrix} 4 & -2 & 0 \\ -2 & 3 & -2 \\ 0 & -2 & 6 \end{vmatrix} \overset{L_1 = \frac{1}{2}L_1}{=} 2 \begin{vmatrix} 2 & -1 & 0 \\ -2 & 3 & -2 \\ 0 & -2 & 6 \end{vmatrix} =$

$L_2 = L_2 + L_1 \quad = 2 \begin{vmatrix} 2 & -1 & 0 \\ 0 & 2 & -2 \\ 0 & -2 & 6 \end{vmatrix} =$

$= 2 \cdot 2 \begin{vmatrix} 2 & -2 \\ -2 & 6 \end{vmatrix} = 2 \cdot 2 \cdot 8 =$

$32 > 0$

$\Rightarrow H(f)(\frac{1}{2}, 1, 1)$ poz definită $\Rightarrow$

$(\frac{1}{2}, 1, 1)$ pct de minim local.

$f(x, y, z) = x + \frac{y^2}{4x} + \frac{z^2}{y} + \frac{2}{z}.$ $\quad f(\frac{1}{2}, 1, 1) = \frac{1}{2} + \frac{1}{\cancel{4}^2 \cdot \frac{1}{2}} + 1 + 2 = 5\frac{1}{2}$

102.

a) $f(x, y) = x^2 + y^2$, cu legătura $3x + 2y - 6 = 0$;

$g(x,y) = 3x+2y-6$

Construim $L(x,y;\lambda) = f(x,y) + \lambda g(x,y)$

$L(x,y;\lambda) = x^2+y^2+\lambda(3x+2y-6)$

Aplicăm pașii de la problema de extrem pt $L(x,y;\lambda)$

1. $\frac{\partial L}{\partial x}(x,y,\lambda) = 2x+3\lambda$

$\frac{\partial L}{\partial y}(x,y,\lambda) = 2y+2\lambda$

$\frac{\partial L}{\partial \lambda}(x,y,z) = 3x+2y-6$

2. $\begin{cases} \frac{\partial L}{\partial x}(x,y,\lambda)=0 \\ \frac{\partial L}{\partial y}(x,y,\lambda)=0 \\ \frac{\partial L}{\partial z}(x,y,\lambda)=0 \end{cases} \Rightarrow \begin{cases} 2x+3\lambda=0 \\ 2y+2\lambda=0 \\ 3x+2y-6=0 \end{cases}$

$$\Rightarrow \begin{cases} x = -\dfrac{3\lambda}{2} \\ y = -\lambda \\ -\dfrac{9\lambda}{2} + -2\lambda = 6 \end{cases}$$

$\Rightarrow -13\lambda = 12$

$\Rightarrow \lambda = -\dfrac{12}{13}$

$\Rightarrow x = -\dfrac{3}{\cancel{2}} \cdot \dfrac{-\cancel{12}^{6}}{13} = \dfrac{18}{13}$

$y = \dfrac{12}{13}$

Calculăm Hessiana pt $L\left(x, y, -\dfrac{12}{13}\right)$ $\Rightarrow$ vàzută ca f-ct. de 2 var

$$\frac{\partial^2 L}{\partial x^2}\left(x, y, -\frac{12}{13}\right) = 2$$

$$\frac{\partial^2 L}{\partial y^2}\left(x, y, -\frac{12}{13}\right) = 2$$

$$\frac{\partial^2 L}{\partial x \partial y}\left(x, y, -\frac{12}{13}\right) = 0$$

$$H\left(L\left(x,y;-\frac{12}{13}\right)\right)(x,y)=\begin{pmatrix}2&0\\0&2\end{pmatrix}$$

$$H\left(L\left(x,y,-\frac{12}{13}\right)\right)\left(\frac{18}{13},\frac{12}{13}\right)=\begin{pmatrix}2&0\\0&2\end{pmatrix}$$

$\Delta_1=2>0$

$\Delta_2=4>0$

$\rightarrow (x,y)=\left(\frac{18}{13},\frac{12}{13}\right)$

pct de minim local

$$f\left(\frac{18}{13},\frac{12}{13}\right)=\left(\frac{18}{13}\right)^2+\left(\frac{12}{13}\right)^2=$$

$$\left(\frac{6}{13}\right)^2\left(3^2+2^2\right)=$$

$$\frac{36}{13^{\cancel{2}}} \cdot \cancel{13} = \frac{36}{13}$$

c) $f(x, y, z) = xy^2z^3$, cu legătura $x + y + z = 12$, $x > 0$, $y > 0$, $z > 0$;

$$g(x,y,z) = x + y + z - 12$$

$$L(x,y,z;\lambda) = f(x,y,z) + \lambda\, g(x,y,z) = xy^2z^3 + \lambda(x+y+z-12)$$

$$\frac{\partial L}{\partial x}(x,y,z;\lambda) = y^2z^3 + \lambda$$

$$\frac{\partial L}{\partial y}(x,y,z;\lambda) = 2xyz^3 + \lambda$$

$$\frac{\partial L}{\partial z}(x,y,z;\lambda) = 3xy^2z^2 + \lambda$$

$$\frac{\partial L}{\partial \lambda}(x,y,z;\lambda) = x + y + z - 12$$

$$\begin{cases} y^2 z^3 + \lambda = 0 \\ 2xyz^3 + \lambda = 0 \\ 3xy^2z^2 + \lambda = 0 \\ x + y + z - 12 = 0 \end{cases}$$

$$\begin{cases} y^2 z^3 = -\lambda \\ 2xyz^3 = -\lambda \\ 3xy^2z^2 = -\lambda \\ x + y + z = 12 \end{cases}$$

$:$

$:$

$$\begin{cases} \frac{y}{2x} = 1 \\ \frac{2z}{3y} = 1 \\ 3xy^2z^2 = -\lambda \\ x + y + z = 12 \end{cases}$$

$$\begin{cases} y = 2x \to x = \frac{y}{2} \\ 2z = 3y \to z = \frac{3y}{2} \\ 3 \cdot \frac{y}{2} \cdot y^2 \cdot \frac{9y^2}{4} = -\lambda \\ \frac{y}{2} + y + \frac{3y}{2} = 12 \end{cases}$$

$3y = 12 \to$

$y = 4$

$$\begin{cases} x = 2 \\ y = 4 \\ z = 6 \\ \lambda = -3 \cdot 2 \cdot 16 \cdot 9 \cdot \frac{16}{4} = -3456 \end{cases}$$

$$H\left(L(x,y,z;-3456)\right)$$

$$\frac{\partial L}{\partial x}(x,y,z;\lambda) = y^2z^3 + \lambda$$

$$\frac{\partial L}{\partial y}(x,y,z;\lambda) = 2xyz^3 + \lambda$$

$$\frac{\partial L}{\partial z}(x,y,z;\lambda) = 3xy^2z^2 + \lambda$$

$$\frac{\partial^2 L}{\partial x^2}(x,y,z;-3456) = \frac{\partial}{\partial x}\left(y^2z^3 - 3456\right)$$

$$= 0$$

$$\frac{\partial^2 L}{\partial y^2}(x,y,z;-3456) = \frac{\partial}{\partial y}\left(2xyz^3 - 3456\right)$$

$$= 2xz^3$$

$$\frac{\partial^2 L}{\partial z^2}(x,y,z;-3456) = \frac{\partial}{\partial z}\left(3xy^2z^2 - 3456\right)$$

$$= 6xy^2z$$

$$\frac{\partial^2 L}{\partial y \partial x}(x, y, z; -3456) = \frac{\partial}{\partial y}\left(y^2 z^3 - 3456\right)$$

$$= 2yz^3$$

$$\frac{\partial^2 L}{\partial z \partial x}(x, y, z; -3456) = \frac{\partial}{\partial z}\left(y^2 z^3 - 3456\right)$$

$$= 3y^2 z^2$$

$$\frac{\partial^2 L}{\partial y \partial z}(x, y, z; -3456) = \frac{\partial}{\partial y}\left(3xy^2 z^2 - 3456\right)$$

$$= 6xyz^2$$

$$H(L(x,y,z,-3456))(x,y,z) = \begin{pmatrix} 0 & 2yz^3 & 3y^2z^2 \\ 2yz^3 & 2xz^3 & 6xyz^2 \\ 3y^2z^2 & 6xyz^2 & 6xy^2z \end{pmatrix}$$

$$H(L(x,y,z;-3456))(2,4,6) = \begin{pmatrix} 0 & 2\cdot 4\cdot 6^3 & 3\cdot 4^2\cdot 6^2 \\ 2\cdot 4\cdot 6^3 & 2\cdot 2\cdot 6^3 & 6\cdot 2\cdot 4\cdot 6^2 \\ 3\cdot 4^2\cdot 6^2 & 6\cdot 2\cdot 4\cdot 6^2 & 6\cdot 2\cdot 4^2\cdot 6 \end{pmatrix} 2$$

$$\begin{pmatrix} 0 & 2^3 \cdot 2^3 \cdot 3^3 & 3 \cdot 2^4 \cdot 2^2 \cdot 3^2 \\ 2^3 \cdot 2^3 \cdot 3^3 & 2 \cdot 2 \cdot 2^3 \cdot 3^3 & 2^3 \cdot 3^3 \cdot 2^3 \\ 3 \cdot 2^4 \cdot 2^2 \cdot 3^2 & 2^3 \cdot 2^3 \cdot 3^3 & 2^5 \cdot 2^2 \cdot 3^2 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 2^6 \cdot 3^3 & 2^6 \cdot 3^3 \\ 2^6 \cdot 3^3 & 2^5 \cdot 3^3 & 2^6 \cdot 3^3 \\ 2^6 \cdot 3^3 & 2^6 \cdot 3^3 & 2^7 \cdot 3^2 \end{pmatrix}$$

$$= 2^5 \cdot 3^2 \begin{pmatrix} 0 & 6 & 6 \\ 6 & 3 & 6 \\ 6 & 6 & 4 \end{pmatrix}$$

$\Delta_1 = 2^5 \cdot 3^2 \cdot 0 \rightarrow$ Nu poate merge mai departe

Trick: Putem schimba ordinea var

$H(L(y, x, z; -3456))(4, 2, 6)$

$$\begin{array}{cc} & \begin{array}{ccc} y & x & z \end{array} \\ 2^5 \cdot 3^2 & \begin{pmatrix} 3 & 6 & 6 \\ 6 & 0 & 6 \\ 6 & 6 & 4 \end{pmatrix} \end{array} \begin{array}{c} y \\ \cancel{x} \\ z \end{array}$$

$$\Delta_1 = 2^5 \cdot 3^2 \cdot 3 > 0$$

$$\Delta_2 = \left(2^5 \cdot 3^2\right)^2 \left(3 \cdot 0 - 6^2\right) =$$

$$-36 \left(2^5 \cdot 3^2\right)^2 < 0$$

$$\Delta_3 = \left(2^5 \cdot 3^2\right)^3 \begin{vmatrix} 3 & 6 & 6 \\ 6 & 0 & 6 \\ 6 & 6 & 4 \end{vmatrix}$$

$\underline{\underline{L_1 = L_1 - L_3}}$ $\quad \left(2^5 \cdot 3^2\right)^3 \begin{vmatrix} -3 & 0 & 2 \\ 6 & 0 & 6 \\ 6 & 6 & 4 \end{vmatrix}$

$$\left(2^5 \cdot 3^2\right)^3 \cdot 6 \cdot (-1)^{2+3} \begin{vmatrix} -3 & 2 \\ 6 & 6 \end{vmatrix}$$

$$= -6(2^5 \cdot 3^2) \cdot |-30| > 0$$

Dacă toți $\Delta_i > 0 \Rightarrow$ pct minim

Dacă $(-1)^{j+1} \Delta_j < 0 \Rightarrow$ pct maxim

$$(-1)^2 \Delta_j < 0$$

Dacă $\underset{\forall j}{\Delta_j \geq 0}$ sau $\underset{\forall j}{(-1)^{j+1} \Delta_j \leq 0}$

și $\exists$ un rang pt care

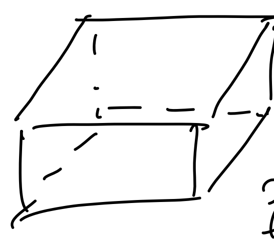
$$\Delta_i = 0$$

$\Rightarrow$ nu pot decide

3. Acvariu: din x, y, z

$f(x,y,z) = xyz$

$g(x,y,z) = 4(x+y+z) - \ell$



$x, y, z > 0$
$\ell > 0$

$$L(x,y,z;\lambda) = xyz + \lambda\left(4(x+y+z) - \ell\right)$$

Dp

$$\frac{\partial L}{\partial x}(x,y,z;\lambda) = yz + 4\lambda$$

$$\frac{\partial L}{\partial y}(x,y,z;\lambda) = xz + 4\lambda$$

$$\frac{\partial L}{\partial z}(x,y,z;\lambda) = xy + 4\lambda$$

$$\frac{\partial L}{\partial \lambda}(x,y,z;\lambda) = 4(x+y+z) - \ell$$

Pti critici

$$\begin{cases} yz + 4\lambda = 0 \\ xz + 4\lambda = 0 \\ xy + 4\lambda = 0 \\ 4(x+y+z) - \ell = 0 \end{cases} \qquad \begin{cases} yz = -4\lambda \\ xz = -4\lambda \\ xy = -4\lambda \\ 4(x+y+z) = \ell \end{cases}$$

$$\begin{cases} \frac{y}{x} = 1 \\ \frac{z}{y} = 1 \\ xy = -4\lambda \\ 4(x+y+z) = \ell \end{cases} \qquad \Rightarrow x = y = z$$

$$\frac{\ell^2}{144} = -4\lambda \Rightarrow \lambda = -\frac{\ell^2}{576}$$

$$12x = \ell \Rightarrow x = \frac{\ell}{12}$$

$$\frac{\partial^2 L}{\partial x^2}\left(x, y, z; -\frac{\ell^2}{576}\right) = 0$$

$$\frac{\partial^2 L}{\partial y^2}\left(x, y, z; -\frac{\ell^2}{576}\right) = 0$$

$$\frac{\partial^2 L}{\partial z^2}\left(x, y, z; -\frac{\ell^2}{576}\right) = 0$$

$$\frac{\partial^2 L}{\partial x \partial y}\left(x, y, z; -\frac{\ell^2}{576}\right) = z$$

$$\frac{\partial^2 L}{\partial y \partial z}\left(x, y, z; -\frac{\ell^2}{576}\right) = x$$

$$\frac{\partial^2 L}{\partial z \partial x}\left(x, y, z; -\frac{\ell^2}{576}\right) = y$$

$$H\left(L\left(x, y, z; -\frac{12}{576}\right)\right) =$$

$$\begin{pmatrix} 0 & z & y \\ z & 0 & x \\ y & x & 0 \end{pmatrix}$$

$\Delta_1 = 0$

$\Delta_2 = -z^2 \quad < 0$ Nu pot decide

$\Delta_3 = 2xyz \quad > 0$

Noua problemă:

$f(x, y) = x^2 y$

$g(x, y) = 8x + 4y - l$

$$L = f(x,y) + \lambda g(x,y)$$

$$\begin{cases} \frac{\partial L}{\partial x}(x,y;\lambda) = \underline{2xy + 8\lambda} \\ \frac{\partial L}{\partial y}(x,y;\lambda) = x^2 + 4\lambda \\ \frac{\partial L}{\partial \lambda}(x,y;\lambda) = 8x + 4y - \ell \end{cases}$$

$$\begin{cases} 2xy + 8\lambda = 0 \rightarrow xy = -4\lambda \\ x^2 + 4\lambda = 0 \rightarrow x^2 = -4\lambda \\ 8x + 4y - \ell = 0 \rightarrow 8x + 4y = \ell \end{cases}$$

$\lambda = -36$

$\rightarrow x = y$

$12x = \ell \rightarrow x = \frac{\ell}{12}$

$$\frac{\partial^2 L}{\partial x^2}(x,y;\lambda) = 2y$$

$$\frac{\partial^2 L}{\partial y^2}(x, y; \lambda) = 0$$

$$\frac{\partial^2 L}{\partial x \partial y} = 2x$$

$$H = \begin{pmatrix} 2y & 2x \\ 2x & 0 \end{pmatrix}$$

$$H\left(\frac{\ell}{12}, \frac{\ell}{12}\right) = \begin{pmatrix} \frac{\ell}{6} & \frac{\ell}{6} \\ \frac{\ell}{6} & 0 \end{pmatrix}$$

$$\Delta_1 = \frac{\ell}{6} > 0$$

$$\Delta_2 = -\frac{\ell^2}{36} < 0$$

Continuare problema 3

$$H\left(x, y, z, -\frac{\ell^2}{576}\right) = \begin{array}{c|ccc} & dx & dy & dz \\ \hline dx & & & \\ dy & & & \\ dz & & & \end{array} \begin{pmatrix} 0 & z & y \\ z & 0 & x \\ y & x & 0 \end{pmatrix}$$

$$g(x, y, z) = 4(x+y+z) - \ell$$

$$dg = 0 \Rightarrow 4\,dx + 4\,dy + 4\,dz = 0$$

$$dx + dy + dz = 0$$

$$\Rightarrow \underline{dz = -(dx + dy)}$$

hessiana este matricea formei pătratice d^2L

$$d^2L\left(x, y, z, -\frac{\ell^2}{576}\right) = 0\,dx^2 + 0\,dy^2 + 0\,dz^2 + 2z\,dy\,dx + 2y\,dz\,dx + 2x\,dz\,dy$$

$$= 2\,(z\,dy\,dx + y\,dz\,dx + x\,dz\,dy)$$

$$2\,(z\,dy\,dx + y\,(-(dx+dy))\,dx$$

$$x\,(-(dx+dy)\,dy)$$

$$= 2\,(z\,dy\,dx - y\,dx^2 - y\,dy\,dx$$

$$- x\,dx\,dy - x\,dy^2)$$

Pct critic

$$d^2L\left(\frac{l}{12}, \frac{l}{12}, \frac{l}{12}\right) =$$

$$2\,\Big(\frac{l}{12}\,\cancel{dy\,dx} - \frac{l}{12}\,dx^2 -$$

$$\cancel{\frac{l}{12}\,dx\,dy} - \frac{l}{12}\,dx\,dy -$$

$$\frac{l}{12}\,dy^2\Big)$$

$$= -\frac{2l}{12}\,(dx^2 + dx\,dy + dy^2)$$

$$= -\frac{l}{12}\left(2dx^2 + \underline{2dxdy} + 2dy^2\right)$$

$$-\frac{l}{12}\left(dx^2 + dx^2 + 2dxdy + dy^2 + dy^2\right) =$$

$$-\frac{l}{12}\left(dx^2 + (dx+dy)^2 + dy^2\right)$$

$$-\frac{l}{12}\left(dx^2 + dz^2 + dy^2\right) < 0$$

$\Rightarrow$ d^2L negativ definită

i.e. e 0 doar pt

$dx = dy = dz = 0$

$\Rightarrow$ pct critic e pct de maxim