Seminar 7

Exerciţii recomandate: 7.1 i), 7.2 a), e), 7.3
Rezerve: 7.1 ii), iv), 7.2 b), f), 7.4, 7.5 a), b), f)

S7.1 i) Fie $g : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ definită de

$$g(\mathbf{x}, \mathbf{y}) = 2x_1y_1 + 3x_2y_2 + \frac{28}{5}x_3y_3 - x_1y_2 - 2x_1y_3 - x_2y_1 + 4x_2y_3 - 2x_3y_1 + 4x_3y_2,$$

pentru $\mathbf{x} = (x_1, x_2, x_3), \mathbf{y} = (y_1, y_2, y_3) \in \mathbb{R}^3$.

a) Arătaţi că aplicaţia g este o formă biliniară simetrică pe $\mathbb{R}^3$.

b) Găsiţi matricea lui g în raport cu baza canonică a lui $\mathbb{R}^3$. Determinaţi discriminantul lui g şi $\operatorname{rang} g$.

c) Determinaţi $\operatorname{Ker}(g)$.

B

d) Găsiţi matricea lui g în raport cu baza $\{(1, 1, 1), (2, -1, 2), (1, 3, -3)\}$.

e) Scrieţi forma pătratică h corespunzătoare lui g şi stabiliţi o formă normală a lui h. Determinaţi signatura lui h şi deduceţi forma biliniară corespunzătoare formei normale a lui h.

f) Determinaţi o bază a lui $\mathbb{R}^3$ în raport cu care h are forma normală de mai sus. Caracterizaţi dintr-un punct de vedere geometric nucleul lui h.

Repetaţi acest exerciţiu pentru:

ii) $g_1(\mathbf{x}, \mathbf{y}) = x_1y_1 + 5x_2y_2 + x_3y_3 + x_1y_2 + 3x_1y_3 + x_2y_1 + x_2y_3 + 3x_3y_1 + x_3y_2$;

iii) $g_2(\mathbf{x}, \mathbf{y}) = x_1y_1 + x_2y_2 + 4x_3y_3 + x_1y_2 + 2x_1y_3 + x_2y_1 + 2x_2y_3 + 2x_3y_1 + 2x_3y_2$;

iv) $g_3(\mathbf{x}, \mathbf{y}) = 2x_1y_1 + x_2y_2 + 2x_3y_3 - x_1y_2 - x_2y_1 + x_2y_3 + x_3y_2$,

unde $\mathbf{x} = (x_1, x_2, x_3), \mathbf{y} = (y_1, y_2, y_3) \in \mathbb{R}^3$.

i) a)
g simetrică

$$g(y,x) = 2y_1x_1 + 3y_2x_2 + \frac{28}{5}y_3x_3 - y_1x_2 - 2y_1x_3 - y_2x_1 + 4y_2x_3 - 2y_3x_1 + 4y_3x_2$$

$$g(\mathbf{x}, \mathbf{y}) = 2x_1y_1 + 3x_2y_2 + \frac{28}{5}x_3y_3 - x_1y_2 - 2x_1y_3 - x_2y_1 + 4x_2y_3 - 2x_3y_1 + 4x_3y_2,$$

$\Rightarrow g(y,x) = g(x,y) \quad \forall x, y \in \mathbb{R}^3$

$\Rightarrow g$ simetrică

$$g(x, y+z) = 2x_1(y_1+z_1) + 3x_2(y_2+z_2) + \frac{28}{5}x_3(y_3+z_3) - x_1(y_2+z_2) - 2x_1(y_3+z_3) - x_2(y_1+z_1) + 4x_2(y_3+z_3) - 2x_3(y_1+z_1) + 4x_3(y_2+z_2) =$$

$$2x_1y_1 + 2x_1z_1 + 3x_2y_2 + 3x_2z_2 + \frac{28}{5}x_3y_3 + \frac{28}{5}x_3z_3 - x_1y_2 - x_1z_2 - 2x_1y_3 - 2x_1z_3 - x_2y_1 - x_2z_1 + 4x_2y_3 + 4x_2z_3 - 2x_3y_1 - 2x_3z_1 + 4x_3y_2 + 4x_3z_2 =$$

$$g(x,y) + g(x,z)$$

$$\Rightarrow g(x, y+z) = g(x,y) + g(x,z) \quad (\bullet) \quad \forall x, y, z \in \mathbb{R}^3$$

$$g(x, \alpha y) = 2x_1\alpha y_1 + 3x_2\alpha y_2 + \frac{28}{5}x_3\alpha y_3 - x_1\alpha y_2 - 2x_1\alpha y_3 - x_2\alpha y_1 + 4x_2\alpha y_3 - 2x_3\alpha y_1 + 4x_3\alpha y_2 =$$

$$= \alpha(2x_1y_1 + 3x_2y_2 + \frac{28}{5}x_3y_3$$

$$- x_1y_2 - 2x_1y_3 - x_2y_1 +$$

$$4x_2y_3 - 2x_3y_1 + 4x_3y_2)$$

$$= \alpha g(x,y)$$

$$g(x, \alpha y) = \alpha g(x,y) \quad (\circ\circ)$$

$$\forall x, y \in \mathbb{R}^3$$

Din $(\circ)$, $(\circ\circ)$ $\Rightarrow$ g este liniară
în al doilea argument $\Big\}\Rightarrow$
Dar g simetrică

g liniară în primul argument

$\Rightarrow$ g biliniară

Dacă (g omogenă toți termenii au grad 2)

$g(x,y) = x^T A y \quad \forall x,y \in \mathbb{R}^3$

$\Rightarrow g$ biliniară $\quad A \in \mathcal{M}_{3\times 3}(\mathbb{R})$

Dacă A simetrică $\Rightarrow$ g simetrică

$$g(\mathbf{x}, \mathbf{y}) = 2x_1y_1 + 3x_2y_2 + \frac{28}{5}x_3y_3 - x_1y_2 - 2x_1y_3 - x_2y_1 + 4x_2y_3 - 2x_3y_1 + 4x_3y_2,$$

$$A = \begin{pmatrix} 2 & -1 & -2 \\ -1 & 3 & 4 \\ -2 & 4 & \frac{28}{5} \end{pmatrix}$$

$a_{ij} = a_{ji} \quad \forall i,j \in \overline{1,3}$

$\Rightarrow$ A simetrică

sau

$A = A^T \Rightarrow$ A simetrică

$\Rightarrow$ g biliniară

$g(x,y) = x^T A y$

$$g(x,y) = \underbrace{\underbrace{\begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix}}_{1\times 3} \underbrace{\begin{pmatrix} 2 & -1 & -2 \\ -1 & 3 & 4 \\ -2 & 4 & \frac{28}{5} \end{pmatrix}}_{3\times 3} \underbrace{\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}}_{3\times 1}}_{1\times 1}$$

$$\begin{pmatrix} 2x_1 - x_2 - 2x_3 & -x_1 + 3x_2 + 4x_3 & -2x_1 + 4x_2 + \frac{28}{5}x_3 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} =$$

$$(2x_1 - x_2 - 2x_3)y_1 + (-x_1 + 3x_2 + 4x_3)y_2 + (-2x_1 + 4x_2 + \tfrac{28}{5}x_3)y_3 =$$

$$2x_1y_1 - x_2y_1 - 2x_3y_1 - x_1y_2 + 3x_2y_2 + 4x_3y_2 - 2x_1y_3 + 4x_2y_3 + \frac{28}{5}x_3y_3$$

$$g(\mathbf{x}, \mathbf{y}) = 2x_1y_1 + 3x_2y_2 + \frac{28}{5}x_3y_3 - x_1y_2 - 2x_1y_3 - x_2y_1 + 4x_2y_3 - 2x_3y_1 + 4x_3y_2,$$

b)

$$A_{Bc} = \begin{pmatrix} 2 & -1 & -2 \\ -1 & 3 & 4 \\ -2 & 4 & \frac{28}{5} \end{pmatrix}$$

discriminantul lui g = det A

$$\begin{vmatrix} 2 & -1 & -2 \\ -1 & 3 & 4 \\ -2 & 4 & \frac{28}{5} \end{vmatrix}$$

$L_3 = L_1 + L_3$

$L_1 = L_1 + 2L_2$

$$\begin{vmatrix} 0 & 5 & 6 \\ -1 & 3 & 4 \\ 0 & 3 & \frac{18}{5} \end{vmatrix}$$

$L_3 = \frac{5}{3} L_3$

$$\frac{3}{5}\begin{vmatrix} 0 & 5 & 6 \\ -1 & 3 & 4 \\ 0 & 5 & 6 \end{vmatrix} = 0$$

rangul lui g = rang A

$$\Delta = \begin{vmatrix} 2 & -1 \\ -1 & 3 \end{vmatrix} = 6 - 1 = 5 \neq 0$$

$\Rightarrow$ rang $A = 2 \Rightarrow$

rang $g = 2$

c) g simetrică $\to$ $\ker g_s = \ker g_d = \ker g$

$= \ker (A\,y)$

$$A \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\det A = 0$

$$\Delta = \begin{vmatrix} 2 & -1 \\ -1 & 3 \end{vmatrix} \neq 0$$

nec p|p x_1, x_2

$x_3 = \alpha$ nec sec

$$\begin{pmatrix} 2 & -1 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = -\alpha \begin{pmatrix} -2 \\ 4 \end{pmatrix}$$

$$g\begin{cases} 2x_1 - x_2 = 2\alpha & |\cdot 3 \\ -x_1 + 3x_2 = -4\alpha & |\cdot 2 \end{cases}$$

$$6x_1 - 3x_2 = 6\alpha$$

$$5x_1 = 2\alpha \Rightarrow x_1 = \frac{2}{5}\alpha$$

$$-2x_1 + 6x_2 = -8\alpha$$

$$5x_2 = -6\alpha$$

$$\Rightarrow x_2 = \frac{-6}{5}\alpha$$

$$\Rightarrow \ker g = \left\{\left(\frac{2}{5}\alpha, -\frac{6}{5}\alpha, \alpha\right) \middle| \alpha \in \mathbb{R}\right\}$$

$$= \{(2\alpha, -6\alpha, 5\alpha) \mid \alpha \in \mathbb{R}\}$$

$$\dim \ker g = 1$$

$\{(1, 1, 1), (2, -1, 2), (1, 3, -3)\}.$

$$A_B = S_{CB}^t \, A \, S_{CB}$$

$$S_{CB} = \begin{pmatrix} 1 & 2 & 1 \\ 1 & -1 & 3 \\ 1 & 2 & -3 \end{pmatrix} \qquad A_{Bc} = \begin{pmatrix} 2 & -1 & -2 \\ -1 & 3 & 4 \\ -2 & 4 & \frac{28}{5} \end{pmatrix}$$

$$A_B = \begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & 2 \\ 1 & 3 & -3 \end{pmatrix} \begin{pmatrix} 2 & -1 & -2 \\ -1 & 3 & 4 \\ -2 & 4 & \frac{28}{5} \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \\ 1 & -1 & 3 \\ 1 & 2 & -3 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 6 & \frac{38}{5} \\ 1 & 3 & \frac{16}{5} \\ 5 & -4 & -\frac{34}{5} \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \\ 1 & -1 & 3 \\ 1 & 2 & -3 \end{pmatrix} =$$

$$\frac{1}{5} \begin{pmatrix} 63 & 36 & -29 \\ 36 & 27 & 2 \\ -29 & 2 & 67 \end{pmatrix}$$

e) Forma pătratică asociată

$h : \mathbb{R}^3 \to \mathbb{R}$

$h(x) = g(x, x)$

$$g(\mathbf{x},\mathbf{y}) = 2x_1y_1 + 3x_2y_2 + \frac{28}{5}x_3y_3 - x_1y_2 - 2x_1y_3 - x_2y_1 + 4x_2y_3 - 2x_3y_1 + 4x_3y_2,$$

$$h(x) = 2x_1x_1 + 3x_2x_2 + \frac{28}{5}x_3x_3 - \underline{x_1x_2} - \underline{2x_1x_3} - \underline{x_2x_1} + 4x_2x_3 - \underline{2x_3x_1} + 4x_3x_2$$

$$= 2x_1^2 + 3x_2^2 + \frac{28}{5}x_3^2 - 2\underline{x_1x_2} - 4\underline{x_1x_3} + 8x_2x_3$$

Aducerea la forma normală

I Gauss

$$h(x) = 2x_1^2 + 3x_2^2 + \frac{28}{5}x_3^2 - 2x_1x_2 - 4x_1x_3 + 8x_2x_3$$

$$(a+b)^2 = a^2 + 2ab + b^2$$

$$(a-b)^2 = a^2 - 2ab + b^2$$ binom

$$(a+b+c)^2 = \underbrace{a^2 + b^2 + c^2 + 2ab + 2bc + 2ca}_{\text{trinom}}$$

Pornesc cu x_1. Iau toți termenii care îl conțin pe x_1

$$2x_1^2 - 2x_1x_2 - 4x_1x_3 =$$

$$2(x_1^2 - x_1x_2 - 2x_1x_3)$$

Ne apar termeni cu x_2, x_3.

Formăm un pătrat folosind trinomul

$$\underbrace{x_1^2}_{a^2} - x_1x_2 - 2x_1x_3 = \underbrace{x_1^2}_{a^2} - \underbrace{2x_1}_{2a}\underbrace{\frac{x_2}{2}}_{b} - \underbrace{2x_1x_3}_{+2ac}$$

$$b = -\frac{x_2}{2}$$

$$c = -x_3$$

$$+\left(-\frac{x_2}{2}\right)^2 + (-x_3)^2 + 2\left(-\frac{x_2}{\cancel{2}} \cdot (-x_3)\right)$$

$$+\frac{x_2^2}{4} + x_3^2 + x_2x_3$$

$$h(x) = 2x_1^2 - 2x_1x_2 - 4x_1x_3 + 3x_2^2 + \frac{28}{5}x_3^2 + 8x_2x_3$$

$$= 2\left(x_1^2 - x_1x_2 - 2x_1x_3 + \underline{\frac{x_2^2}{4}} + \underline{x_3^2} + \underline{x_2x_3}\right) - 2\frac{x_2^2}{4} - 2x_3^2 - 2x_2x_3 + 3x_2^2 + \frac{28}{5}x_3^2 + 8x_2x_3$$

$$= 2\left(x_1 - \frac{x_2}{2} - x_3\right)^2 + \frac{5}{2}x_2^2 + \frac{18}{5}x_3^2 + \underline{6x_2x_3}$$

$$\frac{5}{2} \cdot ? = \frac{18}{5} \qquad \frac{5}{2} \cdot ? = 6$$

$$2\left(x_1 - \frac{x_2}{2} - x_3\right)^2 + \frac{5}{2}\left(x_2^2 + \frac{36}{25}x_3^2 + \frac{12}{5}x_2x_3\right)$$

$$2\left(x_1 - \frac{x_2}{2} - x_3\right)^2 + \frac{5}{2}\left(x_2^2 + \left(\frac{6}{5}x_3\right)^2 + 2 \cdot x_2 \cdot \frac{6}{5}x_3\right)$$

$$2\left(x_1 - \frac{x_2}{2} - x_3\right)^2 + \frac{5}{2}\left(x_2 + \frac{6}{5}x_3\right)^2$$

Forma canonică a lui h

$$y_1 = x_1 - \frac{x_2}{2} - x_3$$

$$y_2 = x_2 + \frac{6}{5}x_3$$

$$y_3 = x_3 \qquad \text{(Nu mi apare ca pătrat)}$$

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & -\frac{1}{2} & -1 \\ 0 & 1 & \frac{6}{5} \\ 0 & 0 & 1 \end{pmatrix}}_{S_{CD}^{-1}} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$h(y) = 2y_1^2 + \frac{5}{2} y_2^2$$

$$\begin{cases} z_1 = \sqrt{2}\, y_1 \\ z_2 = \sqrt{\frac{5}{2}}\, y_2 \\ z_3 = y_3 \end{cases}$$

$$S_{DN}^{-1} = \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{\frac{5}{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Forma normală:

$$h(z) = z_1^2 + z_2^2$$

Forma pătratică inițială → Forma canonică → Forma normală

h(x) = pătrate + produse — Am doar pătrate — Am doar pătrate cu coef ± 1

$$C \xrightarrow{S_{CD}} D \xrightarrow{S_{DN}} N$$

$$\text{signatura lui } h = (\underbrace{2}_{\text{nr de pătrate cu coef} > 0},\ \underbrace{0}_{\text{nr de pătrate cu coef} < 0},\ \underbrace{1}_{\text{din sp- primele 2}})$$

Forma biliniară corsp lui h în forma normală

$$\tilde{g}(x,y) = \frac{1}{2}\left(h(x+y) - h(x) - h(y)\right) =$$

$$\frac{1}{2}\left((x_1+y_1)^2 + (x_2+y_2)^2 - (x_1^2 + x_2^2) - (y_1^2 + y_2^2)\right)$$

$$= \frac{1}{2}\left(\cancel{x_1^2} + \cancel{y_1^2} + 2x_1y_1 + \cancel{x_2^2} + \cancel{y_2^2} + 2x_2y_2 - \cancel{x_1^2} - \cancel{x_2^2} - \cancel{y_1^2} - \cancel{y_2^2}\right) =$$

$$\frac{1}{2}(2x_1y_1+2x_2y_2)=$$

$$x_1y_1+x_2y_2$$

$$\begin{tikzcd} C \arrow[r, "S_{CD}"] \arrow[rr, bend right] & D \arrow[r, "S_{DN}"] & \underline{N} \end{tikzcd}$$

$$S_{CN}=S_{CD}\cdot S_{DN}$$

$$C=\{e_1,e_2,e_3\}$$

$$D=\{d_1,d_2,d_3\}$$

$$N=\{n_1,n_2,n_3\}$$

$$(d_1\ \ d_2\ \ d_3)=(e_1\ \ e_2\ \ e_3)S_{CD}$$

$$(n_1\ \ n_2\ \ n_3)=(d_1\ \ d_2\ \ d_3)S_{DN}$$

$$(n_1\ \ n_2\ \ n_3)=(e_1\ \ e_2\ \ e_3)(S_{CD}\cdot S_{DN})$$

$$S_{CN}=S_{CD}\cdot S_{DN}$$

$$S_{CN})^{-1}=(S_{CD}\cdot S_{DN})^{-1}=S_{DN}^{-1}\cdot S_{CD}^{-1}$$

$$\begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{\frac{1}{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -\frac{1}{2} & -1 \\ 0 & 1 & \frac{6}{5} \\ 0 & 0 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} \sqrt{2} & -\frac{\sqrt{2}}{2} & -\sqrt{2} \\ 0 & \sqrt{\frac{5}{2}} & 3\sqrt{\frac{2}{5}} \\ 0 & 0 & 1 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} \sqrt{2} & -\frac{\sqrt{2}}{2} & -\sqrt{2} & 1 & 0 & 0 \\ 0 & \sqrt{\frac{5}{2}} & 3\sqrt{\frac{2}{7}} & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\underline{L_1 = \frac{1}{\sqrt{2}} L_1}$$

$$L_2 = \sqrt{\frac{2}{5}}\, L_2$$

$$\begin{pmatrix} 1 & -\frac{1}{2} & -1 & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 1 & \frac{6}{5} & 0 & \sqrt{\frac{2}{5}} & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

$L_1 = L_1 + L_3$

$$\begin{pmatrix} 1 & -\frac{1}{2} & 0 & \frac{1}{\sqrt{2}} & 0 & 1 \\ 0 & 1 & \frac{6}{5} & 0 & \sqrt{\frac{2}{5}} & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

$L_3 = L_3 - \frac{6}{5} L_3$

$$\begin{pmatrix} 1 & -\frac{1}{2} & 0 & \frac{1}{\sqrt{2}} & 0 & 1 \\ 0 & 1 & 0 & 0 & \sqrt{\frac{2}{5}} & -\frac{6}{5} \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

$$L_1 = L_1 + \frac{1}{2} L_2$$

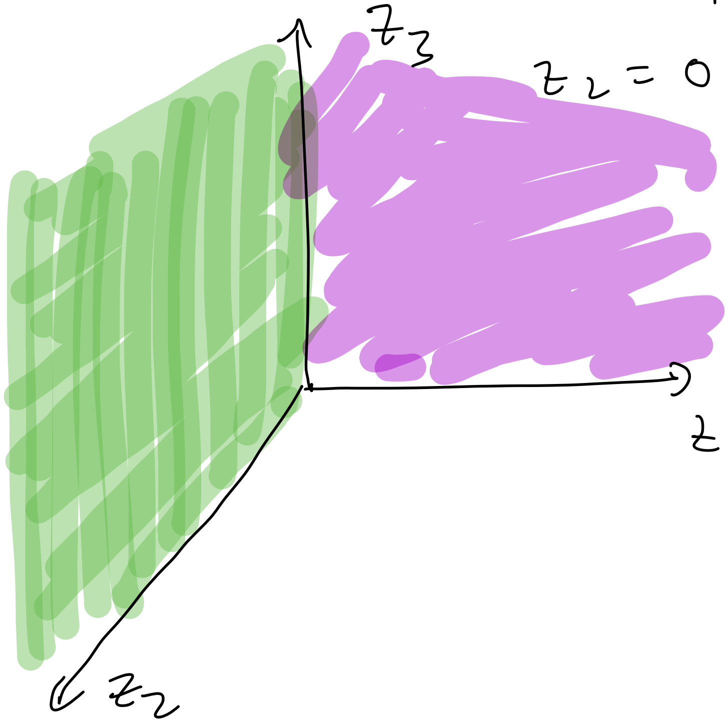
$$\begin{pmatrix} 1 & 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{2}\sqrt{\frac{2}{5}} & \frac{2}{5} \\ 0 & 1 & 0 & 0 & \sqrt{\frac{2}{5}} & -\frac{6}{5} \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

$$S_{CN} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{10}} & \frac{2}{5} \\ 0 & \sqrt{\frac{2}{5}} & -\frac{6}{5} \\ 0 & 0 & 1 \end{pmatrix}$$

$$N = \left\{ \left(\frac{1}{\sqrt{2}}, 0, 0 \right), \left(\frac{1}{\sqrt{10}}, \sqrt{\frac{2}{5}}, 0 \right), \left(\frac{2}{5}, -\frac{6}{5}, 1 \right) \right\}$$

$h(z) = z_1^2 + z_2^2$
$h(z) = 0$ $\Bigg\} \Rightarrow$ $z_1 = z_2 = 0$

$z_1 = 0 \Rightarrow$ plan
$z_2 = 0 \Rightarrow$ plan $\Bigg\} \Rightarrow$



Ker h =
o dreaptă

$z_3 \in \mathbb{R}$

S7.2 Stabiliţi ce este, dintr-un punct de vedere geometric, nucleul fiecăreia dintre formele pătratice neomogene:

a) $h_1(\mathbf{x}) = 4x_1^2 + 6x_1x_2 - 4x_2^2 - 26x_1 + 18x_2 - 39$, $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$.

b) $h_2(\mathbf{x}) = 5x_1^2 + 8x_1x_2 + 5x_2^2 - 18x_1 - 18x_2 + 9$, $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$.

c) $h_3(\mathbf{x}) = x_1^2 - 4x_1x_2 + 4x_2^2 - 14x_1 - 2x_2 + 3$, $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$.

d) $h_4(\mathbf{x}) = x_1^2 + 2x_2^2 - 3x_3^2 + 12x_1x_2 - 8x_1x_3 - 4x_2x_3 + 14x_1 + 16x_2 - 12x_3 - 33$, $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$.

e) $h_5(\mathbf{x}) = 4x_1^2 + 2x_2^2 + 3x_3^2 + 4x_1x_3 - 4x_2x_3 + 6x_1 + 4x_2 + 8x_3 + 2$, $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$.

f) $h_6(\mathbf{x}) = x_1^2 + x_2^2 + x_3^2 + 2x_1x_2 - 2x_1x_3 - 2x_2x_3 + 2x_1 + 2x_2 - 2x_3 - 3$, $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$.

g) $h_7(\mathbf{x}) = x_1^2 - 2x_1x_2 - 2x_1x_3 + 2x_1 - 4x_2 + 4$, $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$.

h) $h_8(\mathbf{x}) = x_1x_2 - x_1x_3 + x_2x_3 - 1$, $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$.

$$h_1(x) = 4x_1^2 + 6x_1x_2 - 4x_2^2 - 26x_1 + 18x_2 - 39$$

$$4x_1^2 + 6x_1\underset{\sim}{x_2} - 26x_1$$

$$(\underbrace{2x_1}_{a})^2 + 2\cdot(2x_1)\cdot \underbrace{3\frac{x_2}{2}}_{b} - 2\cdot 2x_1 \cdot \underbrace{\frac{13}{2}}_{-c}$$

$$a^2 + 2ab + 2ac + b^2 + c^2 + 2bc$$

$$+ \frac{9x_2^2}{4} + \frac{13^2}{4} + 2\cdot\frac{3}{2}x_2\cdot\left(-\frac{13}{2}\right)$$

$$h_1(x) = \left((2x_1)^2 + 2\cdot(2x_1)\cdot 3\frac{x_2}{2} - 2\cdot 2x_1\cdot\frac{13}{2} + \frac{9x_2^2}{4} + \frac{13^2}{4} + 2\cdot\frac{3}{2}x_2\left(-\frac{13}{2}\right)\right)$$

$$-\frac{9x_2^2}{4} - \frac{13^2}{4} + \cancel{2}\cdot\frac{3}{\cancel{2}}x_2\cdot\frac{13}{2} -$$

$$4)\quad 4x_2^2 + 18x_2 - 39$$

$$\begin{array}{r} 36+ \\ 39 \\ \hline \cancel{7}5 \end{array}$$

$$= \left(2x_1 + \frac{3x_2}{2} - \frac{13}{2}\right)^2$$

$$\frac{169}{4} + 39$$

$$-\frac{25}{4}x_2^2 + \frac{75}{2}x_2 - \frac{325}{4}$$

$$\begin{array}{r} 156+ \\ 169 \\ \hline 325 \end{array}$$

$$= \left(2x_1 + \frac{3x_2}{2} - \frac{13}{2}\right)^2 -$$

$$\frac{25}{4}\left(x_2^2 - \underset{2\cdot x_2\cdot 3}{6x_2} + 9\right)$$

$$- \frac{100}{4}$$

$$\left(2x_1 + \frac{3x_2}{2} - \frac{13}{2}\right)^2 -$$

$$\frac{25}{4}(x_2-3)^2 - 25$$

$h_1(x) = 0 \Rightarrow$

$$\left(2x_1 + \frac{3x_2}{2} - \frac{13}{2}\right)^2 - \frac{25}{4}(x_2 - 3)^2 - 25 = 0$$

$$\frac{\left(2x_1 + \frac{3x_2}{2} - \frac{13}{2}\right)^2}{25} - \frac{(x_2 - 3)^2}{4} - 1 = 0$$

$$y_1 = 2x_1 + \frac{3x_2}{2} - \frac{13}{2}$$

$$y_2 = x_2 - 3$$

$$\frac{y_1^2}{5^2} - \frac{y_2^2}{2^2} - 1 = 0$$

$\Rightarrow$ hiperbolă

e) $h_5(\mathbf{x}) = 4x_1^2 + 2x_2^2 + 3x_3^2 + 4x_1x_3 - 4x_2x_3 + 6x_1 + 4x_2 + 8x_3 + 2$, $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$.

$$4x_1^2 + 4x_1x_3 + 6x_1$$

$$= \underbrace{(2x_1)^2}_{a} + 2 \cdot \underbrace{2x_1}_{a} \cdot \underbrace{x_3}_{b} + 2 \cdot \underbrace{2x_1}_{a} \cdot \underbrace{\frac{3}{2}}_{c}$$

$$+ x_3^2 + \frac{9}{4} + \cancel{2} \cdot x_3 \cdot \frac{3}{\cancel{2}}$$

$$h_5(x) = \left((2x_1)^2 + 2x_1x_3 + 2 \cdot 2x_1 \cdot \frac{3}{2} + x_3^2 + \frac{9}{4} + 3x_3\right)$$

$$- x_3^2 - \frac{9}{4} - 3x_3 + 2x_2^2 + 3x_3^2$$

$$- 4x_2x_3 + 4x_2 + 8x_3 + 2$$

$$= \left(2x_1 + x_3 + \frac{3}{2}\right)^2 +$$

$$2x_3^2 + 5x_3 + \underline{2x_2^2} - 4x_2x_3 + 4x_2 - \frac{1}{4}$$

$$\left(2x_1+x_3+\frac{3}{2}\right)^2 +$$

$$2\left(x_2^2-2x_2x_3+2x_2\right)$$

$$+2x_3^2+5x_3-\frac{1}{4}=$$

$$\underbrace{x_2}_{a}{}^2-2\underbrace{x_2}_{a}\underbrace{x_3}_{-b}+2\underbrace{x_2}_{a}\cdot\underbrace{1}_{c}+$$

$$x_3^2+1-\underbrace{2x_3}_{2bc}$$

$$\left(2x_1+x_3+\frac{3}{2}\right)^2+2\left(x_2^2-2x_2x_3+2x_2+x_3^2+1-2x_3\right)$$

$$\cancel{-2x_3^2}-2+4x_3+\cancel{2x_3^2}+5x_3-\frac{1}{4}=$$

$$\left(2x_1+x_3+\frac{3}{2}\right)^2+2\left(x_2-x_3+1\right)^2$$

$$+9x_3-\frac{9}{4}$$

$$\left(2x_1 + x_3 + \frac{3}{2}\right)^2 + 2\left(x_2 - x_3 + 1\right)^2 +$$

$$9\left(x_3 - \frac{1}{4}\right)$$

$$y_1 = 2x_1 + x_3 + \frac{3}{2}$$

$$y_2 = x_2 - x_3 + 1 \qquad\qquad y^2 = 2x$$

$$y_3 = x_3 - \frac{1}{4}$$

$$h_5(y) = 0$$

$$y_1^2 + 2y_2^2 + 9y_3 = 0$$

$$\underbrace{y_1^2 + 2y_2^2}\ \underbrace{- 9(-y_3) = 0}$$

paraboloid eliptic

S7.3 Fie $g : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ forma biliniară definită de

$$g(\mathbf{x}, \mathbf{y}) = 3x_1y_1 + 13x_2y_2 + 14x_3y_3 - 3x_1y_2 + 2x_1y_3 - 3x_2y_1 - 12x_2y_3 + 2x_3y_1 - 12x_3y_2,$$

pentru $\mathbf{x} = (x_1, x_2, x_3), \mathbf{y} = (y_1, y_2, y_3) \in \mathbb{R}^3$.

a) Arătaţi că g determină pe $\mathbb{R}^3$ o structură de spaţiu prehilbertian, pentru care vectorii $(1, 0, 0)$ şi $(1, 1, 0)$ sunt ortogonali.

b) Ortonormalizaţi sistemul $\{\underbrace{(1, 0, 0)}_{u_1}, \underbrace{(1, 1, 0)}_{u_2}, \underbrace{(1, 1, 1)}_{u_3}\}$ în raport cu structura de mai sus.

g este produs scalar $\begin{cases} g \text{ biliniară} \\ g \text{ simetrică} \\ g \text{ pozitiv definită} \end{cases}$

$$\left.\begin{array}{l} g(x,y) = x^T A y \\ \text{coef lui } x_i y_j = a_{ij} \\ A = \begin{pmatrix} 3 & -3 & 2 \\ -3 & 13 & -12 \\ 2 & -12 & 14 \end{pmatrix} \end{array}\right\} \Rightarrow g \text{ biliniară } (1)$$

A simetrică $\Rightarrow$ g simetrică (2)

$\Delta_1 = 3 > 0$ $\quad \Delta_2 = \begin{vmatrix} 3 & -3 \\ -3 & 13 \end{vmatrix} = 39 - 9 = 30 > 0$

$$\Delta_3 = \begin{vmatrix} 3 & -3 & 2 \\ -3 & 13 & -12 \\ 2 & -12 & 14 \end{vmatrix} = 80 > 0$$

$\Delta_1, \Delta_2, \Delta_3 > 0 \Rightarrow g$ pozitiv (3) definită

(1), (2), (3) $\Rightarrow g$ produs scalar

$(1, 1, 0) \perp (1, 0, 0)$

$\Leftrightarrow g((1, 1, 0), (1, 0, 0))$

$g(\mathbf{x}, \mathbf{y}) = 3x_1y_1 + 13x_2y_2 + 14x_3y_3 - 3x_1y_2 + 2x_1y_3 - 3x_2y_1 - 12x_2y_3 + 2x_3y_1 - 12x_3y_2,$

$x_1, x_2, x_3), \mathbf{y} = (y_1, y_2, y_3) \in \mathbb{R}^3.$

$g((1,1,0), (1,0,0)) =$

$3 + 0 + 0 - 0 + 0 - 3 - 0 + 0 - 0 = 0$

$\Rightarrow (1,1,0) \perp (1,0,0)$

S7.4 Fie $g : \mathbb{R}^4 \times \mathbb{R}^4 \to \mathbb{R}$ forma biliniară definită de

$$g(\mathbf{x}, \mathbf{y}) = 5x_1y_1 + x_2y_2 + 4x_3y_3 - x_4y_4 + x_1y_2 - x_1y_3 - 2x_1y_4 + x_2y_1 + 2x_2y_3 - x_2y_4 - x_3y_1 + 2x_3y_2 + x_3y_4 - 2x_4y_1 - x_4y_2 + x_4y_3,$$

pentru $\mathbf{x} = (x_1, x_2, x_3, x_4), \mathbf{y} = (y_1, y_2, y_3, y_4) \in \mathbb{R}^4$.

a) Arătaţi că g este simetrică şi determinaţi $\ker(g)$.

b) Găsiţi matricea lui g în raport cu baza canonică a lui $\mathbb{R}^4$. Determinaţi discriminantul lui g şi $\operatorname{rang} g$.

c) Scrieţi forma pătratică a lui h corespunzătoare lui g. Stabiliţi forma normală a lui h şi o bază a lui $\mathbb{R}^4$ în raport cu care h are această formă normală.

d) Determinaţi signatura lui h.

S7.5 Stabiliţi ce este, dintr-un punct de vedere geometric, nucleul fiecăreia dintre formele pătratice neomogene:

a) $h(\mathbf{x}) = 3x_1 - 4x_2 + x_3 - x_4 + 2$, $\mathbf{x} = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$.

b) $h(\mathbf{x}) = x_1 - x_3$, $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$.

c) $h(\mathbf{x}) = 7x_1^2 - 8x_1x_2 + x_2^2 - 6x_1 - 12x_2 - 9$, $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$.

d) $h(\mathbf{x}) = 2x_1^2 + 5x_2^2 + 11x_3^2 - 20x_1x_2 + 4x_1x_3 + 16x_2x_3 - 24x_1 - 6x_2 - 6x_3 - 180$, $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$.

e) $h(\mathbf{x}) = 2x_1^2 + 3x_1x_2 + x_2^2 - x_1 - 1$, $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$.

f) $h(\mathbf{x}) = x_2^2 - x_3^2 + 4x_1x_2 - 4x_1x_3 - 3$, $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$.

S7.6* Pe spaţiul euclidian $\mathbb{R}^3$ considerăm două forme pătratice h_1 şi $h_2 : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$, definite de:

1. $h_1(\mathbf{x}) = x_1^2 + x_2^2 + x_3^2 + 2x_1x_3$;
2. $h_2(\mathbf{x}) = x_1^2 + x_2^2 + x_3^2 - \sqrt{2}x_1x_2 + \sqrt{2}x_1x_3$,

pentru $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$. Aduceţi cele două forme pătratice la forma canonică printr-o schimbare ortogonală de bază.

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