Seminar 8

Exerciţii recomandate: ~~8.1 b), 8.2 ii), v), 8.3 a), 8.4 b), 8.5 b), 8.6 a)~~
Rezerve: 8.1 a), 8.2 i),iii),iv), 8.3 b),c), 8.4 a), 8.5 a),c), 8.6 c), 8.7 a)

S8.1 Calculaţi următoarele limite:

a) $\lim\limits_{x \nearrow 1} \dfrac{\sin(p \arccos x)}{\sqrt{1-x^2}}$, $p \in \mathbb{R}^*$;

b) $\lim_{x\to 0}\left(\dfrac{\ln(1+3x^2)}{x^2}, \dfrac{e^{x^2}-\cos x}{x^2}\right)$;

c) $\lim_{x\nearrow \frac{\pi}{4}}\left((\operatorname{tg} x)^{\operatorname{tg}(2x)}, \dfrac{\sin(4x)}{\sqrt{\pi-4x}}, \dfrac{2^{\operatorname{tg} x}-2}{x-\frac{\pi}{4}}, \operatorname{tg}(2x)\operatorname{tg}\left(\dfrac{\pi}{4}-x\right), \dfrac{\ln(\operatorname{tg} x)}{\cos 2x}\right)$;

d) $\lim_{x\to\infty}\left(x - \sqrt[n]{(x-a_1)(x-a_2)\dots(x-a_n)}\right)$, $n \in \mathbb{N}^*$, $a_k \in \mathbb{R}$, $k = \overline{1,n}$;

e) $\lim\limits_{x \searrow 0}\left(\dfrac{1+\sum_{k=1}^n \ln(1+kx)}{\sum_{k=1}^n n^{kx-1}}\right)^{\frac{1}{x}}$, $n \in \mathbb{N}^*$.

a) $\lim\limits_{x\nearrow 1} \dfrac{\sin(p \arccos x)}{\sqrt{1-x^2}} \overset{0/0}{=}$

$\lim\limits_{y\to 0} \dfrac{\sin y}{y} = 1$

$\sin(\arccos x) = \sqrt{1-x^2}$

$\lim\limits_{x\nearrow 1} \dfrac{\sin(p \arccos x)}{p \arccos x} \cdot \dfrac{p \arccos x}{\sin(\arccos x)} =$

$\cos(\arccos x) = x$

$\sin^2(\arccos x) + \underbrace{\cos^2(\arccos x)}_{x^2} = 1$

$\Rightarrow \sin^2(\arccos x) = 1 - x^2$

$\lim\limits_{x\nearrow 1} \dfrac{\sin(p \arccos x)}{p \arccos x} \cdot$

$p \cdot \lim\limits_{x\nearrow 1} \dfrac{\arccos x}{\sin(\arccos x)} = 1 \cdot p \cdot 1 = p$

$\lim\limits_{a\to 0} \dfrac{\ln(1+a)}{a} = 1$

$\lim\limits_{a\to 0} \dfrac{e^a - 1}{a} = 1$

b) $\lim\limits_{x\to 0} \left(\dfrac{\ln(1+3x^2)}{x^2}, \dfrac{e^{x^2}-\cos x}{x^2}\right)$

$= \lim\limits_{x\to 0}\left(3\dfrac{\ln(1+3x^2)}{3x^2}, \dfrac{e^{x^2}-1}{x^2} + \dfrac{1-\cos x}{x^2}\right)$

$1 - \cos x = 2\sin^2\dfrac{x}{2}$

$$= \lim_{x\to 0}\left(3\,\underbrace{\frac{\ln(1+3x^2)}{3x^2}}_{\downarrow\atop 1},\ \underbrace{\frac{e^{x^2}-1}{x^2}}_{\downarrow\atop 1} + \underbrace{\frac{\cancel{2}\sin^2\frac{x}{2}}{\left(\frac{x}{2}\right)^2}}_{\downarrow\atop 1}\cdot\frac{1}{\substack{\cancel{4}\\ 2}}\right)$$

$$\left(3,\ 1+\frac{1}{2}\right)=\left(3,\ \frac{3}{2}\right)$$

S8.2 Studiaţi existenţa şi, în cazul afirmativ, determinaţi limitele iterate $\lim_{x\to 0}\left(\lim_{y\to 0} f(x,y)\right)$ şi $\lim_{y\to 0}\left(\lim_{x\to 0} f(x,y)\right)$, limitele parţiale $\lim_{x\to 0} f(x,0)$, $\lim_{y\to 0} f(0,y)$ şi limita globală $\lim_{(x,y)\to(0,0)} f(x,y)$ pentru funcţiile $f:\mathbb{R}^2\setminus\{(0,0)\}\to\mathbb{R}$ definite mai jos:

i) $f(x,y) := \dfrac{x^2-y^2}{|x|+|y|}$;

ii) $f(x,y) := \dfrac{\sqrt{1+x^2y^2}-1}{x^2+y^2}$;

iii) $f(x,y) := (x^2+y^2)^{x^2y^2}$;

iv) $f(x,y) := \dfrac{\sin(xy)}{\sqrt{x^2+y^2}}$;

v) $f(x,y) := y^2\ln(x^2+y^2)$;

vi) $f(x,y) := \dfrac{e^{\frac{1}{x^2+y^2}}}{x^6+y^6}$.

Limite parțiale

ii) $\lim_{x\to 0} f(x,0) = \lim_{x\to 0}\dfrac{\sqrt{1+x^2\cdot 0^2}-1}{x^2+0^2} = \lim_{x\to 0}\dfrac{\sqrt{1}-1}{x^2} = \lim_{x\to 0}\dfrac{0}{x^2}$

$= \lim_{x\to 0} 0 = 0$

$\lim_{y\to 0} f(0,y) = \lim_{y\to 0}\dfrac{\sqrt{1+0^2\cdot y^2}-1}{0^2+y^2} = \lim_{y\to 0}\dfrac{\sqrt{1}-1}{y^2} = \lim_{y\to 0}\dfrac{0}{y^2}$

$= \lim_{y\to 0} 0 = 0$

$\sqrt{1+x^2y^2}+1$)

Limite iterate

$$\lim_{x\to 0}\left(\lim_{y\to 0} f(x,y)\right) = \lim_{x\to 0}\left(\lim_{y\to 0}\frac{\sqrt{1+x^2y^2}-1}{x^2+y^2}\right)$$

$$\lim_{x\to 0}\left(\lim_{y\to 0}\frac{\cancel{1}+x^2y^2-\cancel{1}}{(x^2+y^2)(\sqrt{1+x^2y^2}+1)}\right)$$

$$\lim_{x\to 0}\left(\lim_{y\to 0}\frac{1}{\left(\underbrace{\frac{1}{y^2}}_{+\infty}+\frac{1}{x^2}\right)\underbrace{\left(\sqrt{1+x^2y^2}+1\right)}_{>1}}\right) = \lim_{x\to 0} 0 = 0$$

$$\lim_{y\to 0}\left(\lim_{x\to 0} f(x,y)\right) = \lim_{y\to 0}\left(\lim_{x\to 0}\frac{\sqrt{1+x^2y^2}-1}{x^2+y^2}\right) =$$

$$\lim_{y\to 0}\left(\lim_{x\to 0}\frac{\cancel{1}+x^2y^2\cancel{-1}}{(x^2+y^2)\left(\sqrt{1+x^2y^2}+1\right)}\right) =$$

$$\lim_{y\to 0}\left(\lim_{x\to 0}\frac{1}{\left(\frac{1}{y^2}+\frac{1}{x^2}\right)\left(\sqrt{1+x^2y^2}+1\right)}\right) = \lim_{y\to 0} 0 = 0$$

$$\lim_{(x,y)\to(0,0)}\frac{\sqrt{1+x^2y^2}-1}{x^2+y^2} =$$

$$\lim_{(x,y)\to(0,0)}\frac{\cancel{1}+x^2y^2\cancel{-1}}{(x^2+y^2)\left(\sqrt{1+x^2y^2}+1\right)} =$$

$$\lim_{(x,y)\to(0,0)}\frac{1}{\underbrace{\left(\frac{1}{x^2}+\frac{1}{y^2}\right)}_{\downarrow\ \infty}\underbrace{\left(\sqrt{1+x^2y^2}+1\right)}_{\downarrow\ 2}} = 0$$

v) $$\lim_{x\to 0}\left(\lim_{y\to 0}\left(y^2\ln(x^2+y^2)\right)\right) =$$

$$\lim_{x\to 0}\left(\lim_{y\to 0} y^2 \cdot \lim_{y\to 0}\left(\ln(x^2+y^2)\right)\right) =$$

$$\lim_{x \to 0} \left(0 \cdot \ln x^2 \right) = \lim_{x \to 0} 0 = 0$$

$$\lim_{y \to 0} \left(\lim_{x \to 0} \left(y^2 \ln \left(x^2 + y^2 \right) \right) \right) =$$

$$\lim_{y \to 0} \left(y^2 \lim_{x \to 0} \ln \left(x^2 + y^2 \right) \right)$$

$$\boxed{\ln a^b = b \ln a}$$

$$\lim_{y \to 0} \left(y^2 \ln y^2 \right)$$

$$= 2 \lim_{y \to 0} \underline{\left(y^2 \ln y \right)} = 2 \lim_{y \to 0} \frac{\ln y}{\frac{1}{y^2}} \underset{\text{Dacă } \exists}{\overset{l'H}{=\!=}}$$

$$2 \lim_{y \to 0} \frac{\frac{1}{\cancel{y}}}{-2 \frac{1}{y^{\cancel{3}2}}} = -2 \lim_{y \to 0} y^2 = 0$$

$$\lim_{x \to 0} f(x, 0) = \lim_{x \to 0} \left(0 \cdot \ln \left(x^2 + 0 \right) \right) =$$

$$\lim_{x \to 0} 0 = 0$$

$$\lim_{y \to 0} f(0, y) = \lim_{y \to 0} \left(y^2 \ln \left(0 + y^2 \right) \right) =$$

$$\lim_{y \to 0} y^2 \ln y^2 \overset{\text{vezi}}{\underset{\text{mai sus}}{=}} 0$$

$$y^2 \ln y^2 \le y^2 \underline{\ln(x^2+y^2)} \le 0$$

$\to 0$

$x^2 \ge 0$

$x^2+y^2 \ge y^2$

$\ln(x^2+y^2) \ge \ln y^2$ (ln monoton)

$y^2 \ln(x^2+y^2) \ge y^2 \ln y^2$ ($y^2 \ge 0$)

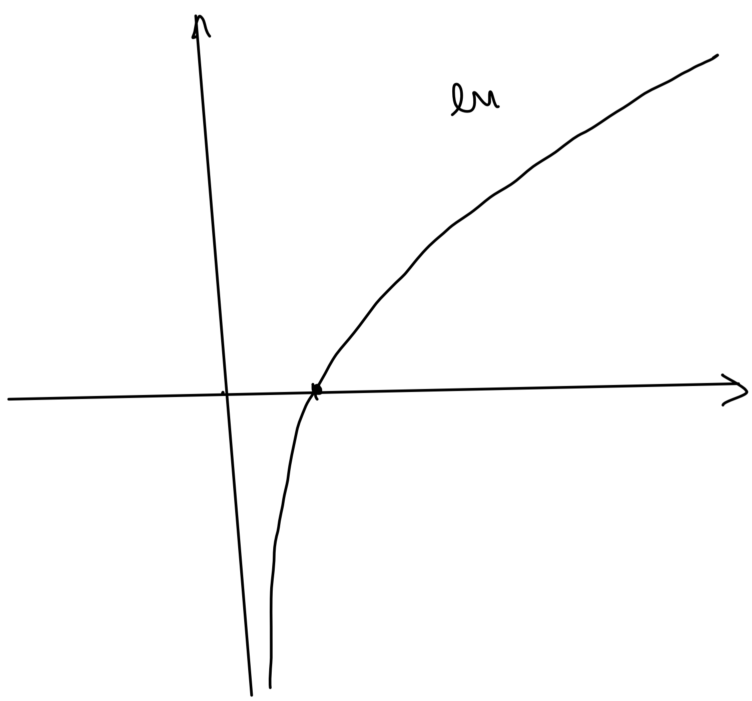
$x^2 \ge 0$

$x^2+y^2 \ge y^2$

$(x^2+y^2)\ln(x^2+y^2) \ge$

$y^2 \ln(x^2+y^2)$

$\ln(x^2+y^2)$



$y^2 \ln y^2 \le y^2 \ln(x^2+y^2) \le 0$ pt $(x,y)\to(0,0)$

$\to 0$

$\Rightarrow \lim\limits_{(x,y)\to(0,0)} y^2 \ln(x^2+y^2) = 0$

y

S8.3 Arătaţi că următoarele funcţii nu au limită în $(0,0)$, chiar dacă au limite iterate şi/sau limite în orice direcţie admisibilă:

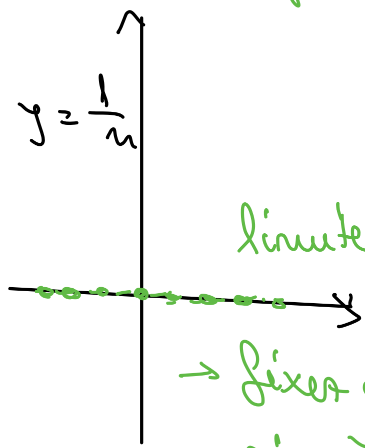
a) $f:\mathbb{R}^2\backslash\{(0,0)\}\to\mathbb{R},\ f(x,y):=\dfrac{x^2-y^2}{x^2+y^2}$;

b) $f:\mathbb{R}^2\backslash\{(x,y)\mid x=y\}\to\mathbb{R}^2,\ f(x,y):=\left(\dfrac{x+y}{x-y},\dfrac{x^2y}{x^4+y^2}\right)$;

c) $f:\mathbb{R}^2\backslash\{(x,y)\mid y^2=2x\}\to\mathbb{R},\ f(x,y):=\dfrac{y^2+2x}{y^2-2x}$;

d) $f:\mathbb{R}^2\backslash\{(0,0)\}\to\mathbb{R}^2,\ f(x,y):=\left(\dfrac{x^2y^2}{(x-y)^2+x^2y^2},(1+|xy|)^{\frac{2}{x^2+y^2}}\right)$.

În $\mathbb{R}$ un şir
tip de lim

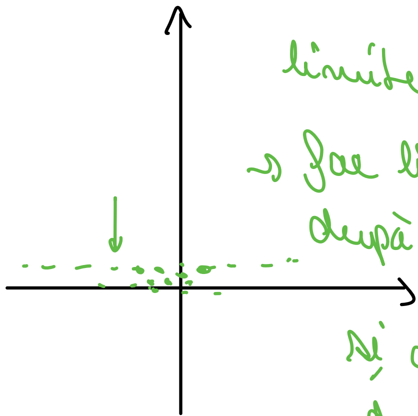


$x_n=\frac{k}{n^2}$ $\quad y=\frac{1}{n}$

limite parţiale:
→ fixez o var
şi mă apropii
cu cealaltă

$$\lim_{x\to 0}\left(\lim_{y\to 0}\frac{x^2-y^2}{x^2+y^2}\right)=$$

$$\lim_{x\to 0}\left(\frac{x^2}{x^2}\right)=\lim_{x\to 0}1=1$$



limite iterate:
→ fac limita
după o var
şi apoi
după cealaltă

$$\lim_{y\to 0}\left(\lim_{x\to 0}\frac{x^2-y^2}{x^2+y^2}\right)=$$

$$\lim_{y\to 0}\left(-\frac{y^2}{y^2}\right)=\lim_{y\to 0}(-1)=-1$$

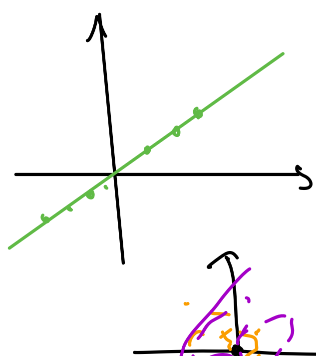
$(u,v)\neq(0,0)$

$$\lim_{t\to 0} f((0,0)+t(u,v))=$$

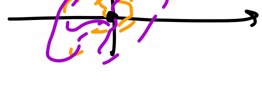
limita direcţională
→ îmi aleg o
direcţie/dreaptă
şi mă apropii pe
ea

$$\lim_{t\to 0} f(tu,tv)=$$

$$\lim_{t\to 0}\frac{t^2u^2-t^2v^2}{t^2u^2+t^2v^2}=\ (t^2$$



$$\lim_{t\to 0}\frac{u^2-v^2}{u^2+v^2}=\frac{u^2-v^2}{u^2+v^2}$$



$$\lim_{x\to 0}\left(\lim_{y\to 0} f(x,y)\right) \neq \lim_{y\to 0}\left(\lim_{x\to 0} f(x,y)\right)$$

sau limite direcționale depinde de direcție

f nu are limită globală

$f:\mathbb{R}^2 \to \mathbb{R}^2$

b) $$\lim_{x\to 0}\left(\lim_{y\to 0}\left(\frac{x+y}{x-y}, \frac{x^2 y}{x^4+y^2}\right)\right) = \left| \begin{array}{l} f(x,y) = \\ (f_1(x,y), \\ f_2(x,y)) \end{array}\right.$$

$$\lim_{x\to 0}\left(\frac{x}{x}, \frac{0}{x^4}\right) = \lim_{x\to 0}(1,0) = (1,0)$$

$$\lim_{y\to 0}\left(\lim_{x\to 0}\left(\frac{x+y}{x-y}, \frac{x^2 y}{x^4+y^2}\right)\right) =$$

$$\lim_{y\to 0}\left(\frac{y}{-y}, \frac{0}{y^2}\right) = \lim_{y\to 0}(-1,0) = (-1,0)$$

$\boxed{(u,v) \neq (0,0)}$ tot timpul

$u \neq v \leftarrow$ din faptul că $x \neq y$

$$l = \lim_{t\to 0} f((0,0) + t(u,v)) = \lim_{t\to 0} f(tu,tv) =$$

$$\lim_{t\to 0}\left(\frac{tu+tv}{tu-tv}, \frac{t^2u^2\, tv}{t^4u^4+t^2v^2}\right) =$$

$$\lim_{t\to 0}\left(\frac{u+v}{u-v},\ \frac{t^3u^2v}{t^2(t^2u^4+v^2)}\right)$$

$$\lim_{t\to 0}\left(\frac{u+v}{u-v},\ \frac{tu^2v}{t^2u^4+v^2}\right) =$$

$$\lim_{t\to 0}\left(\frac{u+v}{u-v},\ \frac{t}{v^2}\right)$$

Dacă $v \neq 0 \Rightarrow l = \left(\frac{u+v}{u-v},\ 0\right)$

Dacă $v = 0$ $\quad u \neq 0)$

$$l = \lim_{t\to 0}\left(\frac{tu}{tu},\ \frac{0}{t^4u^4}\right) = (1,\ 0)$$

$$l = \left(\frac{u+v}{u-v},\ 0\right)$$

Pt prima componentă limita direcțională depinde de $(u,v) \Rightarrow \nexists$ limită globală.

Pt. a doua componentă limita direcțională e mereu 0. Totuși $\nexists$ limită globală.

Ajungem la acest rezultat cu șiruri

$$(x_n, y_n) = \left(\frac{1}{n}, \frac{1}{n^2}\right)$$

$$\lim_{n\to\infty} f(x_n, y_n) = \lim_{n\to\infty} \left(\frac{\frac{1}{n} + \frac{1}{n^2}}{\frac{1}{n} - \frac{1}{n^2}}, \frac{\left(\frac{1}{n}\right)^2 \cdot \frac{1}{n^2}}{\left(\frac{1}{n}\right)^4 + \left(\frac{1}{n^2}\right)^2} \right)$$

$$= \lim_{n\to\infty} \left(\frac{\frac{n+1}{\cancel{n^2}}}{\frac{n-1}{\cancel{n^2}}}, \frac{\frac{1}{n^4}}{\frac{1}{n^4} + \frac{1}{n^4}} \right)$$

$$\lim_{n\to\infty} \left(\frac{n+1}{n-1}, \frac{1}{2} \right) = \left(1, \frac{1}{2}\right)$$

$$(x_n', y_n') = \left(\frac{2}{n}, \frac{1}{n^2}\right)$$

$$\lim_{n\to\infty} f(x_n', y_n') = \lim_{n\to\infty} \left(\frac{\frac{2}{n} + \frac{1}{n^2}}{\frac{2}{n} - \frac{1}{n^2}}, \frac{\left(\frac{2}{n}\right)^2 \cdot \frac{1}{n^2}}{\left(\frac{2}{n}\right)^4 + \left(\frac{1}{n^2}\right)^2} \right)$$

$$= \lim_{n\to\infty} \left(\frac{2n+1}{2n-1}, \frac{\frac{4}{\cancel{n^4}}}{\frac{17}{\cancel{n^4}}} \right) =$$

$$\left(1, \frac{4}{17}\right)$$

$$\lim_{n\to\infty} f_2(x_n, y_n) = \frac{1}{2}$$
$$\lim_{n\to\infty} f_2(x_n', y_n') = \frac{4}{17}$$
$$\Rightarrow \nexists \lim_{(x,y)\to(0,0)} f_2(x,y)$$

c) $$\lim_{n\to\infty} f(x_n, y_n) = \lim_{n\to\infty} \frac{\frac{1}{n^2} + \frac{2k}{n^2}}{\frac{1}{n^2} - \frac{2k}{n^2}} =$$

$$= \frac{1+2k}{1-2k}$$

depinde de $k \Rightarrow$
$\nexists \lim$

S8.4 Determinaţi următoarele limite:

a) $\lim\limits_{(x,y)\to(0,0)} \left(\frac{xy}{\sqrt{1+xy}-1}, \frac{\sin(x^3+y^3)}{\sqrt{x^2+y^2+1}-1} \right);$

b) $\lim\limits_{(x,y,z)\searrow(0,0,0)} \left(\frac{1}{xyz} \operatorname{tg} \frac{xyz}{1+xyz}, (1+xyz)^{\frac{1}{\sqrt{x}+\sqrt{y}+\sqrt{z}}} \right);$ $\qquad x, y, z > 0$

c) $\lim\limits_{(x,y,z)\to(0,0,0)} \left(\frac{1-\cos(1-\cos(x^2+y^2+z^2))}{(x^2+y^2+z^2)^4}, \frac{x^2y^2z^2}{(x-y)^2+(y-z)^2+(x-z)^2} \right).$

$$\lim_{(x,y,z)\to(0,0,0)} \left(\frac{1}{xyz} \operatorname{tg} \frac{xyz}{1+xyz}, (1+xyz)^{\frac{1}{\sqrt{x}+\sqrt{y}+\sqrt{z}}} \right)$$

$$\boxed{\lim_{a\to 0} \frac{\operatorname{tg} a}{a} = 1} \qquad \lim_{a\to 0} (1+a)^{\frac{1}{a}} = e$$

$$\lim_{(x,y,z)\to(0,0,0)} \left(\underbrace{\frac{\operatorname{tg}\frac{xyz}{1+xyz}}{\frac{xyz}{1+xyz}}}_{\to 1} \cdot \underbrace{\frac{1}{1+xyz}}_{\to 1}, \underbrace{(1+xyz)^{\frac{1}{xyz}}}_{e} \cdot \underbrace{\frac{xyz}{\sqrt{x}+\sqrt{y}+\sqrt{z}}}_{0} \right) = (1, 1)$$

$(xyz$

$$\lim_{(x,y,z)\to(0,0,0)} \frac{xyz}{\sqrt{x}+\sqrt{y}+\sqrt{z}}$$

$$= \lim_{(x,y,z)\to(0,0,0)} \frac{1}{\underbrace{\frac{1}{yz\sqrt{x}}}_{+\infty} + \underbrace{\frac{1}{xz\sqrt{y}}}_{+\infty} + \underbrace{\frac{1}{xy\sqrt{z}}}_{+\infty}} = 0$$

S8.5 Determinaţi mulţimile punctelor de discontinuitate ale următoarelor funcţii reale:

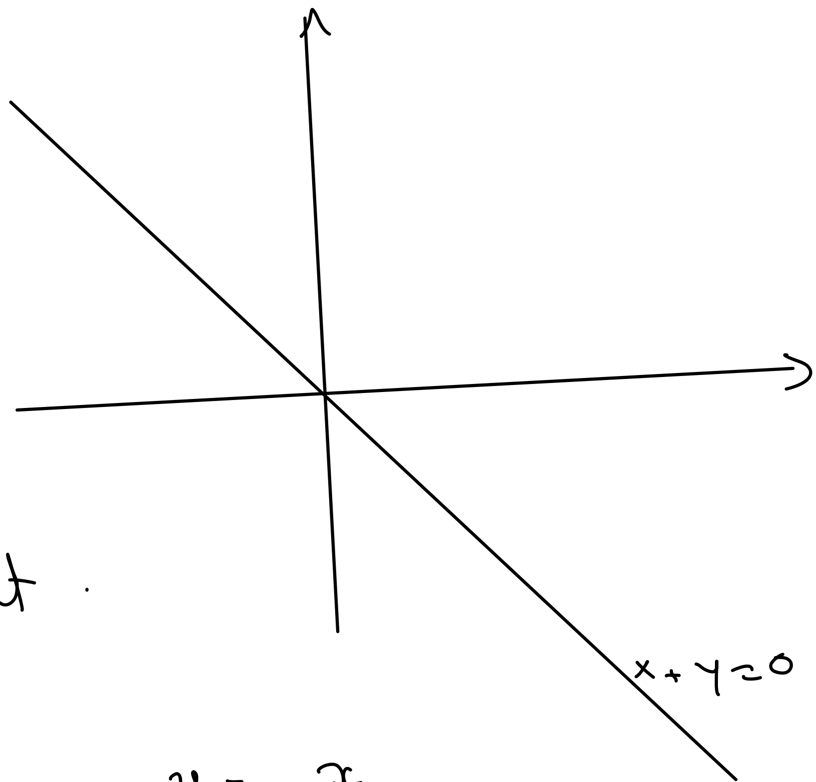
a) $f(x,y) := \begin{cases} \dfrac{|x|}{y} e^{-|x| y^{-2}}, & y \neq 0; \\ 1, & y = 0; \end{cases}$

b) $f(x,y) := \begin{cases} \dfrac{xy}{x+y}, & x+y \neq 0; \\ 0, & x+y = 0; \end{cases}$

c) $f(x,y,z) := \begin{cases} (x^2+y^2+z^2)^{1/3} \ln(x^2+y^2+z^2), & (x,y,z) \neq (0,0,0); \\ 1/3, & (x,y,z) = (0,0,0). \end{cases}$

$x + y = 0$

$y = -x$



În afara $x + y = 0$

f e raport de fct continue, deci cont.

Pe $\underline{x + y = 0}$

Fixez $x \neq 0$

$y = -x$

$$\lim_{y \to -x} \frac{xy}{x+y} = \pm\infty$$

Vreau să mă apropii de $y = -x$

$x_n = x$ ct

Fie (x_n, y_n) $\quad y_n = \frac{1}{n} - x \quad x \neq 0$

$$\lim_{n \to \infty} \frac{x_n y_n}{x_n + y_n} = \lim_{n \to \infty} \frac{x\left(\frac{1}{n} - x\right)}{\cancel{x} + \frac{1}{n} - \cancel{x}} =$$

$$\lim_{n \to \infty} \frac{x\frac{(1-nx)}{\cancel{n}}}{\frac{1}{\cancel{n}}} = -\infty \neq$$

$-(x, \frac{1}{n} - x)$

Pt $x+y=0$ $x\neq 0$ f nu e cont.

Ce se întâmplă în $(0,0)$?

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x+y}$$

Dacă $x, y > 0$

$$\sqrt{xy} \leq \frac{x+y}{2} \to$$

$$xy \leq \frac{(x+y)^2}{4}$$

$$0 \leq \lim_{(x,y)\to(0,0)} \frac{xy}{x+y} \leq$$

$$\lim_{(x,y)\to(0,0)} \frac{(x+y)^{\cancel{2}}}{4\cancel{(x+y)}} =$$

$$\lim_{(x,y)\to(0,0)} \frac{x+y}{4} = 0 = f(0,0)$$

Studiați pt (x,y) oarecare

Intuiesc că limitele se schimbă

pt x, y de semne contrare

$(x_n, y_n) \to (0, 0)$

$y_n = -\frac{1}{n}$

$x_n + y_n = \frac{1}{n^2} \Rightarrow x_n = \frac{1}{n^2} + \frac{1}{n}$

$$\lim_{n \to \infty} \frac{x_n y_n}{x_n + y_n} = \lim_{n \to \infty} \frac{\left(\frac{1}{n^2} + \frac{1}{n}\right) \cdot \left(-\frac{1}{n}\right)}{\frac{1}{n^2}} =$$

$$\lim_{n \to \infty} \frac{-\frac{1}{n^3} - \frac{1}{n^2}}{\frac{1}{n^2}} =$$

$$\lim_{n \to \infty} \frac{-\frac{1}{n} - 1}{1} = -1 \quad (\omega)$$

$$y_n' = -\frac{1}{n}$$

$$x_n' + y_n' = -\frac{1}{n^2} \Rightarrow x_n' = -\frac{1}{n^2} + \frac{1}{n}$$

$$\lim_{n\to\infty} \frac{x_n' y_n'}{x_n' + y_n'} = \lim_{n\to\infty} \frac{\left(-\frac{1}{n^2} + \frac{1}{n}\right)\left(-\frac{1}{n}\right)}{-\frac{1}{n^2}}$$

$n^2)$

$$\lim_{n\to\infty} \frac{\frac{1}{n^3} - \frac{1}{n^2}}{-\frac{1}{n^2}} =$$

$$\lim_{n\to\infty} \frac{\frac{1}{n} - 1}{-1} = 1 \quad (\cdot\cdot)$$

$$\lim_{n\to\infty} f(x_n, y_n) \Rightarrow$$

$$\lim_{n\to\infty} f(x_n', y_n') \Rightarrow$$

f nu are limită în $(0,0)$
$\Rightarrow$ f nu e cont în $(0,0)$

S8.6 Arătaţi că următoarele funcţii sunt continue în fiecare variabilă, dar nu global continue în punctul $(0,0)$:

a) $f(x,y) := \begin{cases} \dfrac{xy^2}{x^2+y^4}, & (x,y) \neq (0,0); \\ 0, & (x,y) = (0,0); \end{cases}$

b) $f(x,y) := \begin{cases} \dfrac{x^4 y}{x^6+y^3}, & y \neq -x^2; \\ 0, & y = -x^2; \end{cases}$

c) $f(x,y,z) := \begin{cases} \dfrac{\sin(xy+yz+xz)}{\sqrt{(x^4+y^2+z^4)}}, & (x,y,z) \neq (0,0,0); \\ 0, & (x,y,z) = (0,0,0); \end{cases}$

d) $f(x,y) := \begin{cases} \min\left\{\left|\dfrac{x}{y}\right|, \left|\dfrac{y}{x}\right|\right\}, & xy \neq 0; \\ 0, & xy = 0. \end{cases}$

$$\lim_{x \to 0} f(x,0) = \lim_{x\to 0} \frac{x \cdot 0}{x^2+0} = \lim_{x \to 0} 0 = 0 = f(0,0)$$

$$\lim_{y \to 0} f(0,y) = \lim_{y\to 0} \frac{0 \cdot y^2}{0+y^4} = \lim_{y \to 0} 0 = 0 = f(0,0)$$

$\Rightarrow f$ cont în x în $(0,0)$

y în $(0,0)$

$(x_n, y_n) = \left(\frac{k}{n^2}, \frac{1}{n}\right)$ k parametru

$$\lim_{n\to\infty} f(x_n,y_n) = \lim_{n\to\infty} \frac{\frac{k}{n^2}\cdot\frac{1}{n^2}}{\frac{k^2}{n^4}+\frac{1}{n^4}} =$$

$$\lim_{n\to\infty} \frac{\frac{k}{\cancel{n^4}}}{\frac{k^2+1}{\cancel{n^4}}} = \underline{\underline{\frac{k}{k^2+1}}} \Rightarrow f \text{ nu are limită în } (0,0) \Rightarrow$$

f nu e cont în (0,0)

$k=2 \quad \lim_{n\to\infty} f(x_n,y_n) = \frac{2}{5}$ $\not\parallel$

$k=3 \quad \lim_{n\to\infty} f(x_n,y_n) = \frac{3}{10}$

---

f nu are limită $\Longleftarrow$ limite direcț. depind de direcție sau șiruri

$\Downarrow$

f nu e continuă

$\Uparrow$

f are limită $\neq$ val în pct

S8.7 Studiaţi continuitatea următoarelor funcţii:

a) $f : [-1, +\infty) \to \mathbb{R}^2$, $f(x) := (f_1(x), f_2(x))$, unde, pentru $p \in \mathbb{R}$, f_1 şi f_2 sunt definite de

$$f_1(x) := \begin{cases} 2\operatorname{tg} x \cdot \operatorname{arctg} \dfrac{1}{x}, & x \in [-1, 0); \\ p, & x = 0; \\ \mathrm{e}^{\frac{1-\sqrt{1+x}}{x^2 e^x}}, & x > 0, \end{cases}$$

$$f_2(x) := \begin{cases} \operatorname{tg} x \cdot \sin \dfrac{1}{x}, & x \in [-1, 0); \\ p, & x = 0; \\ \cos \dfrac{1}{x} \mathrm{e}^{-\frac{1}{x}}, & x > 0. \end{cases} .$$

b) $f : A = \left\{(x, y) \in \mathbb{R}^2 \mid 0 \le x^2 + y^2 < \frac{\pi}{2}\right\} \to \mathbb{R}$, $\alpha \in \mathbb{R}$,

$$f(x, y) := \begin{cases} \dfrac{1 - \cos\sqrt{x^2 + y^2}}{\operatorname{tg}(x^2 + y^2)}, & (x, y) \in A \setminus \{(0, 0)\} \\ \alpha, & (x, y) = (0, 0). \end{cases}$$

c) $f : \mathbb{R}^3 \to \mathbb{R}^2$, $f(x, y, z) = (f_1(x, y, z), f_2(x, y, z))$, unde, pentru $p \in \mathbb{R}$, f_1 şi f_2 sunt definite de

$$f_1(x, y, z) := \begin{cases} (x^2 + y^2 + z^2)^p \cos \dfrac{1}{\sqrt{x^2 + y^2 + z^2}}, & (x, y, z) \neq (0, 0, 0); \\ 0, & (x, y, z) = (0, 0, 0), \end{cases}$$

$$f_2(x, y, z) := \begin{cases} (x^2 + y^2 + z^2)^p \mathrm{e}^{\frac{1}{\sqrt{x^2 + y^2 + z^2}}}, & (x, y, z) \neq (0, 0, 0); \\ 0, & (x, y, z) = (0, 0, 0). \end{cases}$$

S8.8 Stabiliţi care din următoarele funcţii sunt continue pe domeniul lor de definiţie:

a) $f:(0,1)\to\mathbb{R},\ f(x):=\sqrt{x}\sin\dfrac{2}{x}$;

b) $f:(-1,\infty)\to\mathbb{R}^2,\ f(x):=\left(\dfrac{x}{x^2+2},\dfrac{\arcsin(x+1)}{x+1}\right)$;

c) $f:\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2<1\}\to\mathbb{R},\ f(x,y):=\sin\dfrac{\pi}{1-x^2-y^2}$;

d) $f:[0,\pi]\times[0,\frac{\pi}{3}]\to\mathbb{R}^2,\ f(x,y):=(\sin x+\cos y, \mathrm{e}^{-x}\operatorname{tg} y-1)$.

S8.9 Calculaţi următoarele limite:

a) $\lim\limits_{x\to 3}\left(\dfrac{x^x-27}{x-3},\dfrac{x\sin 3-3\sin x}{x\sin x-3\sin 3},\dfrac{3^x-x^3}{x-3}\right)$;

b) $\lim\limits_{x\to\infty}\left(x\arcsin\dfrac{x+2}{\sqrt{x^4-x^2+1}},\dfrac{\sqrt{x+\sqrt{x+\sqrt{x+\sqrt{x}}}}}{\sqrt{x+1}},\left(\cos\dfrac{1}{x}+\sin\dfrac{1}{x}\right)^x\right)$;

c) $\lim\limits_{(x,y)\to(0,0)}\left((x^2+y^2)\ln(x^2+y^2),(x-y)\operatorname{arctg}\dfrac{1}{2x^2+3y^2},(1+x^2y^2)^{\frac{1}{x^2+y^2}}\right)$;

d) $\lim\limits_{(x,y,z)\to(0,0,0)}\left(\dfrac{xy+yz+zx}{\sqrt{x^2+y^2+z^2}},\dfrac{2x-3y}{z},(x+y+z)\ln(1+|xyz|)\right)$.

S8.10 Analizaţi continuitatea următoarelor funcţii:

a) $f:\left(0,\frac{\pi}{2}\right)\to\mathbb{R}^2,\ f(x):=(f_1(x),f_2(x))$, unde

$$f_1(x):=\begin{cases}\operatorname{tg}x, & x\in\mathbb{Q}\cap\left(0,\frac{\pi}{2}\right)\\ \operatorname{ctg}x, & x\in(\mathbb{R}\setminus\mathbb{Q})\cap\left(0,\frac{\pi}{2}\right)\end{cases}\quad \text{şi } f_2(x):=\begin{cases}x^2+1-\dfrac{\pi}{16}, & x\in\mathbb{Q}\cap\left(0,\frac{\pi}{2}\right)\\ \dfrac{\pi}{4x}, & x\in(\mathbb{R}\setminus\mathbb{Q})\cap\left(0,\frac{\pi}{2}\right)\end{cases};$$

b) $f:\mathbb{R}^2\to\mathbb{R}^2,\ f(x,y):=(f_1(x,y),f_2(x,y))$, unde

$$f_1(x,y):=\begin{cases}\dfrac{x-\sqrt{x^2-y+2}}{y^2-4}, & \text{dacă } 2\neq y\leq x^2+2\\ 2^{-3}, & \text{altfel,}\end{cases}$$

$$f_2(x,y)=\begin{cases}\dfrac{(x^4-y^2)^2}{x^6}, & \text{dacă } y^2\leq x^4\neq 0\\ 0, & \text{altfel.}\end{cases};$$

c) $f:\mathbb{R}^3\to\mathbb{R}$,

$$f(x,y,z)=\begin{cases}\alpha\mathrm{e}^{x+y+z}, & \text{dacă } x+y+z<0\\ \beta, & \text{dacă } x+y+z\geq 0,\end{cases}$$

unde $\alpha,\beta\in\mathbb{R}$.

S8.11 Pot fi următoarele funcţii extinse prin continuitate la $\mathbb{R}^2$?

a) $f:\mathbb{R}^2\setminus\{(0,0)\}\to\mathbb{R}^2,\ f(x,y):=\left(\dfrac{\ln(1+x^2|y|)}{x^2+y^2},\left(1+\sin(x^4+y^4)\right)^{\frac{1}{x^2+y^2}}\right)$;

b) $f:\mathbb{R}^2\backslash\{(0,0)\}\to\mathbb{R}^3,\ f(x,y)=\left(\dfrac{\sin\left(x^2-y^2\right)}{|x|+|y|},\left(|x|+|y|\right)^{x^2+y^2},\dfrac{xy}{\sqrt{2x^2+3y^2}}\right)$.

S8.12 Sunt următoarele funcţii continue?

a) $f:\mathbb{R}^2\to\mathbb{R}^2$,

$$f(x,y):=\begin{cases}0, & \text{dacă } (x,y)=\mathbf{0}_{\mathbb{R}^2};\\ \frac{4}{\pi}(x^2+y^2)\operatorname{arctg}\dfrac{1}{x^2+y^2}, & \text{dacă } (x,y)\in B(\mathbf{0}_{\mathbb{R}^2};1)\backslash\{\mathbf{0}_{\mathbb{R}^2}\};\\ 1, & \text{dacă } (x,y)\notin B(\mathbf{0}_{\mathbb{R}^2};1);\end{cases};$$

b) $f:\mathbb{R}^n\to\mathbb{R}$,

$$f(x,y):=\begin{cases}\mathrm{e}^{-\frac{1}{(\|x\|-1)(2-\|x\|)}}, & \text{dacă } \|x\|\in(1,2);\\ 0, & \text{altfel.}\end{cases}.$$

S8.13

a) Arătaţi că mulţimea $A:=\left\{(x,y)\in\mathbb{R}^2 \mid x=\operatorname{arctg}t,\ y=\ln\left(1+t^2\right),\ t\in[-1,1]\right\}$ este compactă.

b) Fie $f:\mathbb{R}^3\to\mathbb{R}^2$, definită de

$$f(x,y,z):=\begin{cases}\left(\left(|x|+|y|+|z|\right)\ln\left(1+\dfrac{1}{|x|+|y|+|z|}\right),\sqrt{2-\left(|x|+|y|+|z|\right)}\right), & (x,y,z)\in B\\ (0,\sqrt{2}), & (x,y,z)=\mathbf{0}_{\mathbb{R}^3}\\ (\ln 2,1), & (x,y,z)\in C\end{cases},$$

unde $B:=\left\{(x,y,z)\in\mathbb{R}^3 \mid |x|+|y|+|z|\le 1\right\}\backslash\{\mathbf{0}_{\mathbb{R}^3}\}$ şi $C:=\mathbb{R}^3\backslash\left\{(x,y,z)\in\mathbb{R}^3 \mid |x|+|y|+|z|\le 1\right\}$.

De asemenea, fie $A:=\left\{(x,y,z)\in\mathbb{R}^3 \mid z\ge x^2+y^2, x+y+z\le 2\right\}$.

Arătaţi că:

i) mulţimea A este compactă;

ii) $f|_A$ este uniform continuă;

iii) $f[A]$ este compactă.

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