1) 1. $((r_3 \land r_4) \land p), (\neg(r_3 \land r_4) \lor \neg p) \vdash (\overbrace{\neg(r_3 \land r_4)}^{\varphi_1} \lor \overbrace{\neg p}^{\varphi_2})$ (ip)

2. $((r_3 \land r_4) \land p), (\neg(r_3 \land r_4) \lor \neg p), \neg(r_3 \land r_4) \vdash ((r_3 \land r_4) \land p)$ (ip)

3. $((r_3 \land r_4) \land p), (\neg(r_3 \land r_4) \lor \neg p), \neg(r_3 \land r_4) \vdash (r_3 \land r_4)$ $(\land e_1, 2)$

4. $((r_3 \land r_4) \land p), (\neg(r_3 \land r_4) \lor \neg p), \neg(r_3 \land r_4) \vdash \neg(r_3 \land r_4)$ (ip)

5 $\cancel{m_1}$. $((r_3 \land r_4) \land p), (\neg(r_3 \land r_4) \lor \neg p), \neg(r_3 \land r_4) \vdash \bot$ $(\neg e, 3, 4)$

6. $((r_3 \land r_4) \land p), (\neg(r_3 \land r_4) \lor \neg p), \neg p \vdash ((r_3 \land r_4) \land p)$ (ip)

7. $((r_3 \land r_4) \land p), (\neg(r_3 \land r_4) \lor \neg p), \neg p \vdash p$ $(\land e_2, 6)$ $\cancel{(\land e_2, 2)}$

8. $((r_3 \land r_4) \land p), (\neg(r_3 \land r_4) \lor \neg p), \neg p \vdash \neg p$ (ip)

9 $\cancel{m_2}$. $((r_3 \land r_4) \land p), (\neg(r_3 \land r_4) \lor \neg p), \neg p \vdash \bot$ $(\neg e, 7, 8)$

10 $\cancel{k}$. $((r_3 \land r_4) \land p), (\neg(r_3 \land r_4) \lor \neg p) \vdash \bot$ $(\lor e, 1, \overset{5}{\cancel{m_1}}, \overset{9}{\cancel{m_2}})$

11 $\cancel{n}$. $((r_3 \land r_4) \land p) \vdash \neg \underbrace{(\neg(r_3 \land r_4) \lor \neg p)}_{\varphi}$ $(\neg i, 10)$

$$\neg i \ \frac{\Gamma, \varphi \vdash \bot}{\Gamma \vdash \neg \varphi}$$

$$\neg e \ \frac{\Gamma' \vdash \varphi_1 \qquad \Gamma' \vdash \neg \varphi_1}{\Gamma' \vdash \bot} \qquad\qquad \lor e \ \frac{\Gamma' \vdash \varphi_1 \lor \varphi_2 \qquad \Gamma', \varphi_1 \vdash \psi \qquad \Gamma', \varphi_2 \vdash \psi}{\Gamma' \vdash \underset{\bot}{\psi}}$$

2) φ validă ddacă pt orice structură S și orice S-atr α avem $S, \alpha \models \varphi$

a) $\varphi = (\neg(\exists y.\ Q(y))) \to (\forall y.\ \neg Q(b))$ validă ddacă pt orice S, orice α avem $S, \alpha \models \varphi$

Fie S, α arbitrar fixate.

$S, \alpha \models \varphi$ ddacă $\begin{cases} S, \alpha \not\models \neg(\exists y. Q(y)) \\ \text{sau} \\ S, \alpha \models (\forall y. \neg Q(b)) \end{cases}$

ddacă $\begin{cases} \text{nu e adevărat că } S, \alpha \models \neg(\exists y. Q(y)) \\ \text{sau} \\ S, \alpha \models (\forall y. \neg Q(b)) \end{cases}$

ddacă $\begin{cases} \text{nu e adevărat că } S, \alpha \not\models \exists y. Q(y) \\ \text{sau} \\ S, \alpha \models (\forall y. \neg Q(b)) \end{cases}$

ddacă $\begin{cases} S, \alpha \models \exists y. Q(y) \\ \text{sau} \\ S, \alpha \models (\forall y. \neg Q(b)) \end{cases}$

ddacă $\begin{cases} \text{există } u \in D \text{ a.î. } S, \alpha[y \mapsto u] \models Q(y) \\ \text{sau} \\ \text{oricare } v \in D \text{ avem } S, \alpha[y \mapsto v] \models \neg Q(b) \end{cases}$

ddacă $\begin{cases} \text{există } u \in D \text{ a.î. } Q^S(\overline{\alpha[y \mapsto u]}(y)) \\ \text{sau} \\ \text{oricare } v \in D \text{ avem } Q^S(\overline{\alpha[y \mapsto v]}(b)) \text{ nu are loc} \end{cases}$

ddacă $\begin{cases} \text{există } u \in D \text{ a.î. } Q^S(u) \\ \text{sau} \\ \text{oricare } v \in D \text{ avem } Q^S(b^S) \text{ nu are loc} \end{cases}$

ddacă $\begin{cases} \text{există } u \in D \text{ a.î. } Q^S(u) \\ \text{sau} \\ Q^S(b^S) \text{ nu are loc} \end{cases}$ (*)

"A" pt orice S, α ??

$\overline{\alpha_1}(x) = \alpha_1(x)$

$\overline{\alpha_1}(c) = c^S$

$\overline{\alpha_1}(f(t_1, \dots, t_n)) = f^S(\overline{\alpha_1}(t_1) \dots \overline{\alpha_1}(t_n))$

$S = (\underbrace{\{\_\_\_\_\_\}}_{D}, \{Q^S, \dots\}, \{\underline{b}^S, \dots\})$
$\in D$

Caz 1, dacă există $u \in D$ a.î. $Q^S(u)$ $\Rightarrow$ $(*)$ "A".

Caz 2. dacă nu există $u \in D$ a.î. $Q^S(u)$
$b^S \in D$ $\Rightarrow$ $\Rightarrow$

$\Rightarrow Q^S(b^S)$ nu are loc $\Rightarrow (*)$ "A".

$\Rightarrow$ pt orice S, α are loc $(*)$ $\Rightarrow$ φ validă

$S = (\underbrace{\{1,3,5\}}_{D}, \underbrace{\{Par\}}_{Q^S}, \underbrace{\{1\}}_{b^S})$

"F" sau "A" $\Rightarrow$ "A".

$+ : D \times D \to \underline{D}$

$c : (\ ) \to \underline{D}$

b) $\varphi = \neg(\exists x. \neg Q(b)) \vee \neg(\forall y. \neg P(y))$

validă ddacă pt orice S, α avem $S, \alpha \models \varphi$

$S, \alpha \models \varphi$ ddacă $\left[\begin{array}{l}\text{nu există } u \in D \text{ a.î. nu are loc } Q^S(b^S) \\ \text{sau} \\ \text{nu pt orice } v \in D \text{ avem nu are loc } P^S(\underline{v})\end{array}\right.$

$\alpha[y \mapsto v](y)$

ddacă $\left[\begin{array}{l}\cancel{\text{nu există } u \in D \text{ a.î.}}\ \underline{Q}^S(b^S) \text{ este fals.} \\ \text{sau} \\ \text{nu pt orice } v \in D \text{ avem } \underline{P}^S(v) \text{ este fals.}\end{array}\right.$

φ nesatisfiabilă ddacă φ nu este satisf.

ddacă NU există S și există o S-atr a.î. $S, \alpha \models \varphi$

ddacă pt orice S, α avem $S, \alpha \models \varphi$ nu are loc

(3) $$(\neg\underbrace{(p_1 \wedge p_2)}_{\varphi_1} \to \underbrace{(q_3 \wedge q_4)}_{\varphi_2}) \vdash (\underbrace{(p_1 \wedge p_2)}_{\varphi_1} \vee \underbrace{(q_3 \wedge q_4)}_{\varphi_2})$$

Notăm $\varphi_1 = (p_1 \wedge p_2)$ și $\varphi_2 = (q_3 \wedge q_4)$

1. $(\neg\varphi_1 \to \varphi_2), \neg(\varphi_1 \vee \varphi_2), \varphi_1 \vdash \varphi_1$ (ip)
2. $(\neg\varphi_1 \to \varphi_2), \neg(\varphi_1 \vee \varphi_2), \varphi_1 \vdash (\varphi_1 \vee \varphi_2)$ ($\vee i_1$, 1)
3. $(\neg\varphi_1 \to \varphi_2), \neg(\varphi_1 \vee \varphi_2), \varphi_1 \vdash \neg(\varphi_1 \vee \varphi_2)$ (ip)
4. $(\neg\varphi_1 \to \varphi_2), \neg(\varphi_1 \vee \varphi_2), \varphi_1 \vdash \perp$ ($\neg e$, 2, 3)

$\vdash \neg\varphi_1$

$\vdash \neg\varphi_1 \to \varphi_2$

$\vdash \varphi_2$

y_1 $(\neg\varphi_1 \to \varphi_2), \neg(\varphi_1 \vee \varphi_2) \vdash \neg\varphi_1$ ($\neg i$, 4)

y_2 $(\neg\varphi_1 \to \varphi_2), \neg(\varphi_1 \vee \varphi_2) \vdash (\neg\varphi_1 \to \varphi_2)$ (ip)

y_3 $(\neg\varphi_1 \to \varphi_2), \neg(\varphi_1 \vee \varphi_2) \vdash \varphi_2$ ($\to e$, y_2, y_1)

x_1. $(\neg\varphi_1 \to \varphi_2), \neg(\varphi_1 \vee \varphi_2) \vdash (\varphi_1 \vee \varphi_2)$ ($\vee i_2$, y_3)

x_2. $(\neg\varphi_1 \to \varphi_2), \neg(\varphi_1 \vee \varphi_2) \vdash \neg(\varphi_1 \vee \varphi_2)$ (ip)

k. $(\neg\varphi_1 \to \varphi_2), \neg\overbrace{(\varphi_1 \vee \varphi_2)}^{\varphi} \vdash \perp$ ($\neg e$, x_1, x_2)

m. $(\neg\varphi_1 \to \varphi_2) \vdash \neg\neg(\varphi_1 \vee \varphi_2)$ ($\neg i$, k)

n. $(\underbrace{\neg\varphi_1}_{\varphi_3} \to \varphi_2) \vdash (\varphi_1 \vee \varphi_2)$ ($\neg\neg e$, m)

$$\vee i_1 \; \frac{\Gamma \vdash \varphi_1}{\Gamma \vdash \varphi_1 \vee \varphi_2} \qquad \to e \; \frac{\Gamma \vdash \varphi_3 \to \varphi_2 \quad \Gamma \vdash \varphi_3}{\Gamma \vdash \varphi_2}$$

$$\vee i_2 \; \frac{\Gamma \vdash \varphi_2}{\Gamma \vdash \varphi_1 \vee \varphi_2} \qquad \neg i \; \frac{\Gamma, \varphi \vdash \perp}{\Gamma \vdash \neg\varphi}$$

④ $\{\varphi_1 \ldots \varphi_n\} \models \varphi$ ddacă

pt orice S și orice S-atr α a.î. dacă

$S, \alpha \models \varphi_1$, $S, \alpha \models \varphi_2$, ... $S, \alpha \models \varphi_n$

atunci $S, \alpha \models \varphi$

$\varphi_3 = P(x) \land \neg P(x) \in \{\varphi_1 \ldots \varphi_n\} \Rightarrow \{\varphi_1 \ldots \varphi_n\} \models \varphi$

$\varphi_1 = P(x)$ $\quad \varphi_2 = \neg P(x)$

" Toate dreptunghiurile roșii sunt pătrate "

Toate nr pare și impare sunt prime

$(\varphi_1 \land \varphi_2) \to \varphi$

⑤ $\varphi_1 \equiv \varphi_2$ ddacă pt orice S și orice S-atr α

avem $S, \alpha \models \varphi_1$ ddacă $S, \alpha \models \varphi_2$

φ_1 și φ_2 au aceiași valoare de adevăr în (S, α)

⑥ $\varphi = \forall y_2 \exists x \exists z' \left(\left(\neg P_1(k(a), f(k(y_2), x)) \right) \land R'(f(k(y_2), x)) \right) \to$

$\to P_1(f(y_2, z'), a))$ — FNP închisă

- skolemizarea

fie i_1 simbol funcțional nou de aritate 1. $(x \mapsto i_1(y_2))$

$$\varphi_1 = \forall y_2.\, \exists z'.\, \Big(\big(\neg P_1(k(a), f(k(y_2), i_1(y_2))) \wedge R'(f(k(y_2)), i_1(y_2)) \big) \to P_1(f(y_2, z'), a) \Big)$$

Fie j simb fct nou de aritate 1 $\quad (z' \mapsto j(y_2))$

$$\varphi_1 = \forall y_2.\, \Big(\big(\neg P_1(k(a), f(k(y_2), i_1(y_2))) \wedge R'(f(k(y_2)), i_1(y_2)) \big) \to P_1(f(y_2, j(y_2)), a) \Big)$$

7. 1. $\neg P(i(i(c)), i(i(c))) \vee \neg P(i(i(x_3)), i(x_2))$

2. $\neg I(i(i(x_3))) \vee P(i(x_3), i(i(c)))$

3. $I(h(i(c), i(x_2))) \vee \neg I(i(x_2))$

4. $\neg I(i(i(i(c)))) \vee \neg P(i(i(i(c))), i(x_2))$ RB. 2, 1

$\mathcal{P} = \{ i(x_3) \doteq i(i(c)),\ i(i(c)) \doteq i(i(c)) \} \xRightarrow{DESC}$

$\{ i(x_3) \doteq i(i(c)),\ c \doteq i(c) \} \xRightarrow{CONFL} \perp$

$\xRightarrow{STERG} \{ i(x_3) \doteq i(i(c)) \} \xRightarrow{DESC}$

$\{ x_3 \doteq i(c) \}$ – forma rez.

$\sigma_1 = \{ x_3 \mapsto i(c) \}$

8. a) $\{ z \doteq g(y, a),\ f(i(b), x) \doteq x \} \xRightarrow{ORIENT}$

$\{ z \doteq g(y, a),\ x \doteq f(i(b), x) \} \xRightarrow{OCC.CH} \perp$

b) $\mathcal{P}_2 = \{ y \doteq f(z, h(a)),\ g(i(b), z) \doteq x \} \xRightarrow{ORIENT}$

$$\{ y \doteq f(z, h(a)),\ x \doteq g(i(b), z)\}$$ – formă rez.

$$\mathrm{ungu}(\mathbb{P}_2) = \{ y \mapsto f(z, h(a)),\ x \mapsto g(i(b), z)\}$$

9) $$\left(\underbrace{(q_1 \vee q_2)}_{\varphi_1} \wedge \underbrace{(p_1 \wedge p_2)}_{\varphi_2} \right) \vee \left(\underbrace{(q_1 \vee q_2)}_{\varphi_1} \wedge \underbrace{p}_{\varphi_3} \right) \vdash \left(\underbrace{(q_1 \vee q_2)}_{\varphi_1} \wedge \left(\underbrace{(p_1 \wedge p_2)}_{\varphi_2} \vee \underbrace{p}_{\varphi_3} \right) \right)$$

1. $(\varphi_1 \wedge \varphi_2) \vee (\varphi_1 \wedge \varphi_3) \vdash (\varphi_1 \wedge \varphi_2) \vee (\varphi_1 \wedge \varphi_3)$ (ip)
2. $(\varphi_1 \wedge \varphi_2) \vee (\varphi_1 \wedge \varphi_3), (\varphi_1 \wedge \varphi_2) \vdash (\varphi_1 \wedge \varphi_2)$ (ip)
3. $\cancel{x_1}$ $(\varphi_1 \wedge \varphi_2) \vee (\varphi_1 \wedge \varphi_3), (\varphi_1 \wedge \varphi_2) \vdash \varphi_1$ $(\wedge e_1, 2)$
4. $(\varphi_1 \wedge \varphi_2) \vee (\varphi_1 \wedge \varphi_3), (\varphi_1 \wedge \varphi_2) \vdash \varphi_2$ $(\wedge e_2, 2)$
5. $\cancel{x_2}$ $(\varphi_1 \wedge \varphi_2) \vee (\varphi_1 \wedge \varphi_3), (\varphi_1 \wedge \varphi_2) \vdash (\varphi_2 \vee \varphi_3)$ $(\vee i_1, 4)$
6. $\cancel{k_1}$ $(\varphi_1 \wedge \varphi_2) \vee (\varphi_1 \wedge \varphi_3), (\varphi_1 \wedge \varphi_2) \vdash \varphi_1 \wedge (\varphi_2 \vee \varphi_3)$ $(\wedge i, x_1, x_2)$
7. $(\varphi_1 \wedge \varphi_2) \vee (\varphi_1 \wedge \varphi_3), (\varphi_1 \wedge \varphi_3) \vdash (\varphi_1 \wedge \varphi_3)$ (ip)
8. $(\varphi_1 \wedge \varphi_2) \vee (\varphi_1 \wedge \varphi_3), (\varphi_1 \wedge \varphi_3) \vdash \varphi_1$ $(\wedge e_1, 7)$
9. $(\varphi_1 \wedge \varphi_2) \vee (\varphi_1 \wedge \varphi_3), (\varphi_1 \wedge \varphi_3) \vdash \varphi_3$ $(\wedge e_2, 7)$
10. $(\varphi_1 \wedge \varphi_2) \vee (\varphi_1 \wedge \varphi_3), (\varphi_1 \wedge \varphi_3) \vdash (\varphi_2 \vee \varphi_3)$ $(\vee i_2, 9)$

k_2. $(\varphi_1 \wedge \varphi_2) \vee (\varphi_1 \wedge \varphi_3), (\varphi_1 \wedge \varphi_3) \vdash \varphi_1 \wedge (\varphi_2 \vee \varphi_3)$ $(\wedge i, 8, 10)$

n. $$\underbrace{\boxed{\overbrace{(\varphi_1 \wedge \varphi_2)}^{\varphi_1'} \vee \overbrace{(\varphi_1 \wedge \varphi_3)}^{\varphi_2'}}}_{\Gamma} \vdash \underbrace{\varphi_1 \wedge (\varphi_2 \vee \varphi_3)}_{\psi}$$ $(\vee e, 1, k_1, k_2)$

$$\wedge i\ \frac{\Gamma \vdash \varphi_1' \qquad \Gamma \vdash \varphi_2'}{\Gamma \vdash \varphi_1' \wedge \varphi_2'} \qquad \vee e\ \frac{\Gamma \vdash \varphi_1' \vee \varphi_2' \qquad \Gamma, \varphi_1' \vdash \psi \qquad \Gamma, \varphi_2' \vdash \psi}{\Gamma \vdash \psi}$$

(10)

1. $\neg(\exists x. P(x)),\ \neg(\forall z. \neg P(z)),\ P(x_0) \vdash P(x_0)$

$$\frac{\Gamma \vdash \varphi[x \mapsto t]}{\Gamma \vdash \exists x.\varphi}$$

2. $\neg(\exists x. P(x)),\ \neg(\forall z. \neg P(z)),\ P(x_0) \vdash \exists x. P(x)$ $(\exists_i, 1)$

3. $\neg(\exists x. P(x)),\ \neg(\forall z. \neg P(z)),\ P(x_0) \vdash \neg(\exists x. P(x))$ (ip)

k_1. $\neg(\exists x. P(x)),\ \neg(\forall z. \neg P(z)),\ P(x_0) \vdash \bot$ $(\neg e, 2, 3)$

x_1. $\neg(\exists x. P(x)),\ \neg(\forall z. \neg P(z)) \vdash \neg P(x_0)$ $(\neg i, k_1)$

y_1. $\neg(\exists x. P(x)),\ \neg(\forall z. \neg P(z)) \vdash \forall z. \neg P(z)$ $(\forall i, x_1)$

y_2. $\neg(\exists x. P(x)),\ \neg(\forall z. \neg P(z)) \vdash \neg(\forall z. \neg P(z))$ (ip)

n. $\neg(\exists x. P(x)),\ \underbrace{\neg(\forall z. \neg P(z))}_{\neg\varphi} \vdash \bot$ $(\neg e, y_1, y_2)$