Ex 100 : $\{(p\vee q), (p\to r), (q\to r)\} \vdash r$ valida

$\frac{\varphi_1 \in \Gamma}{\Gamma \vdash \varphi_1}$ ip

1. $\{(p\vee q), (p\to r), (q\to r)\} \vdash (\overset{\varphi_1}{p}\vee \overset{\varphi_2}{q})$ (ip)
2. $\{(p\vee q), (p\to r), (q\to r), p\} \vdash (p\to r)$ (ip)
3. $\{$ —— // —— $\} \vdash p$ (ip)
4. k_1. $\{(p\vee q), (p\to r), (q\to r), p\} \vdash r$ $(\to e, 2, 3)$

$\frac{\Gamma \vdash (\varphi_1\to\varphi_2) \quad \Gamma\vdash\varphi_1}{\Gamma\vdash\varphi_2} \to e$

5. $\{(p\vee q), (p\to r), (q\to r), q\} \vdash (q\to r)$ (ip)
6. $\{$ —— // —— $\} \vdash q$ (ip)
7. k_2. $\{(p\vee q), (p\to r), (q\to r), q\} \vdash r$ $(\to_e, 5, 6)$
8. $\cancel{r}$. $\underbrace{\{(p\vee q), (p\to r), (q\to r)\}}_{\Gamma} \vdash \underset{\psi}{r}$ $(\vee e, 1, \cancel{k_1}^{4}, \cancel{k_2}^{7})$

$\frac{\Gamma\vdash(\varphi_1\vee\varphi_2) \quad \Gamma,\varphi_1\vdash\psi \quad \Gamma,\varphi_2\vdash\psi}{\Gamma\vdash\psi} \vee e$

$p\vee q$ $\quad \Gamma\cup\{\varphi_1\}$ $\quad r$

a. $\{(p\vee q), (p\to r), (q\to r)\} \vdash (q\to r)$ (ip)

$\{(p\vee q), (p\to r), (q\to r), q\} \vdash (q\to r)$ (ext a) (ip)

Ex 101

$\boxed{\text{Ex 100} \quad k.\ \{(p\to r), (q\to r), (p\vee q)\} \vdash r \ (\text{ex } 100)}$

$n.\ \underbrace{\{(p\to r), (q\to r)\}}_{\Gamma} \vdash ((\underbrace{p\vee q}_{\varphi_1}) \to \underbrace{r}_{\varphi_2})$ $(\to_i, k)$

$\frac{\Gamma, \varphi_1 \vdash \varphi_2}{\Gamma \vdash (\varphi_1\to\varphi_2)} \to_i$

Ex 104   1) $\{(p\vee q)\} \vdash \neg(\neg p \wedge \neg q)$ validă

1. $\{(p\vee q), (\neg p\wedge\neg q)\} \vdash (\overset{\varphi_1}{p}\vee\overset{\varphi_2}{q})$ (ip)

2. $\{(p\vee q),(\neg p\wedge\neg q), p\}\vdash p$ (ip)

3. $\{(p\vee q),(\neg p\wedge \neg q), p\}\vdash(\neg p\wedge\neg q)$ (ip)

4. $\{$ — // — $\}\vdash \neg p$ ($\wedge e_1$, 3)

5 ~~k_1~~. $\{(p\vee q),(\neg p\wedge\neg q), p\}\vdash \bot$ ($\neg e$, 2, 4)

6. $\{(p\vee q),(\neg p\wedge\neg q), q\}\vdash q$ (ip)

7. $\{(p\vee q),(\neg p\wedge\neg q), q\}\vdash(\neg p\wedge\neg q)$ (ip)

8. $\{(p\vee q),(\neg p\wedge\neg q), q\}\vdash \neg q$ ($\wedge e_2$, 7) ~~($\wedge e_2$, 3)~~

9 ~~k_2~~. $\{(p\vee q),(\neg p\wedge\neg q), q\}\vdash\bot$ ($\neg e$, 6, 8)

10 ~~k~~. $\underbrace{\{(p\vee q), (\neg p\wedge\neg q)\}}_{\Gamma_1}\vdash \overset{\varphi}{\bot}$ ($\vee e$, 1, $\overset{5}{\cancel{k_1}}$, $\overset{9}{\cancel{k_2}}$)

11 ~~n~~. $\underbrace{\{(p\vee q)\}}_{\Gamma}\vdash\neg\underbrace{(\neg p\wedge\neg q)}_{\varphi}$ ($\neg i$, $\underset{10}{\cancel{k}}$)

$$\neg e\ \frac{\Gamma_1\vdash\overset{p}{\varphi_1}\qquad\Gamma_1\vdash\neg\varphi_1}{\Gamma_1\vdash\bot}$$

$$\neg i\ \frac{\Gamma,\varphi\vdash\bot}{\Gamma\vdash\neg\varphi}$$

$$\vee e\ \frac{\Gamma_1\vdash(\varphi_1\vee\varphi_2)\qquad\Gamma_1,\varphi_1\vdash\psi\qquad\Gamma_1,\varphi_2\vdash\psi}{\Gamma_1\vdash\psi}\ \bot$$

$$\cancel{\frac{\Gamma_1,\overset{\neg p}{\varphi}\vdash\varphi_1}{\Gamma\vdash\varphi_1}}$$

1. $\Gamma_1\vdash(p\vee q)$
2. $\Gamma_1\vdash(\neg p\wedge\neg q)$
3. $\Gamma_1\vdash\neg p$ ($\wedge e_1$, 2)
4. $\Gamma_1\vdash\neg q$ ($\wedge e_2$, 2)
5. $\Gamma_1, p\vdash\neg p$ (ext 3)
6. $\Gamma_1, p\vdash p$ (ip)
7 ~~k_1~~. $\Gamma_1, p\vdash\bot$ ($\neg e$, 6, 5)
8. $\Gamma_1, q\vdash\neg q$ (ext 4)
9. $\Gamma_1, q\vdash q$ (ip)
10 ~~k_2~~. $\Gamma_1, q\vdash\bot$ ($\neg e$, 9, 8)

11 ~~10~~. $\underbrace{\{(p\vee q), (\neg p\wedge\neg q)\}}_{\Gamma_1}\vdash\bot$

Ex 104 4)

parte dien ex 104 1)

$j.\ \{(\neg p \wedge \neg q), (p \vee q)\} \vdash \bot$

$n.\ \{(\neg p \wedge \neg q)\} \vdash \neg\underbrace{(p \vee q)}_{\varphi_1}$ $(\neg i, j)$

$$\frac{\Gamma, \varphi_1 \vdash \bot}{\Gamma \vdash \neg\varphi_1}$$

Ex 115 2) LEM $\dfrac{\qquad}{\Gamma \vdash (\varphi \vee \neg\varphi)}$

1. $\Gamma_2 \vdash \varphi$ (ip)

2 $\cancel{x_2}$. $\Gamma_2 \vdash (\varphi \vee \neg\varphi)$ $(\vee i_1, 1)$

3 $\cancel{x_1}$. $\Gamma_2 \vdash \neg(\varphi \vee \neg\varphi)$ (ip)

4 $\cancel{j2}$. $\overbrace{\Gamma, \neg(\varphi \vee \neg\varphi), \varphi}^{\Gamma_2} \vdash \bot$ $(\neg e, \cancel{x_2}^{2}, \cancel{x_1}^{3})$

5 $\cancel{j1}$ $\Gamma, \neg(\varphi \vee \neg\varphi) \vdash \neg\varphi$ $(\neg i, \cancel{j2}^{4})$

6 $\cancel{k_2}$. $\Gamma, \neg(\varphi \vee \neg\varphi) \vdash (\varphi \vee \neg\varphi)$ $(\vee i_2, \cancel{j1}^{5})$

7 $\cancel{k_1}$. $\Gamma, \neg(\varphi \vee \neg\varphi) \vdash \neg(\varphi \vee \neg\varphi)$ (ip)

8 j. $\Gamma, \neg(\varphi \vee \neg\varphi) \vdash \bot$ $(\neg e, \cancel{k_2}^{6}, \cancel{k_1}^{7})$

9 k. $\Gamma \vdash \neg\neg(\varphi \vee \neg\varphi)$ $(\neg i, \cancel{j}^{8})$

10 $\cancel{n}$. $\Gamma \vdash \underbrace{(\varphi \vee \neg\varphi)}_{\varphi_1}$ $(\neg\neg e, \cancel{k}_{9})$ (PBC, j)

$$\neg e\ \frac{\Gamma_2 \vdash \varphi_2 \qquad \Gamma_2 \vdash \neg\varphi_2}{\Gamma_2 \vdash \bot} \qquad \varphi_2 = \varphi \vee \neg\varphi$$

$$\neg i\ \frac{\Gamma, \varphi \vdash \bot}{\Gamma \vdash \neg\varphi}$$

$$\text{PBC}\ \frac{\Gamma, \neg\varphi_1 \vdash \bot}{\Gamma \vdash \varphi_1}$$

4) $\quad \text{MT} \ \dfrac{\Gamma \vdash (\varphi \to \varphi') \qquad \Gamma \vdash \neg\varphi'}{\Gamma \vdash \neg\varphi}$

1. $\Gamma \vdash (\varphi \to \varphi')$ (data)
2. $\Gamma \vdash \neg\varphi'$ (data)
3. $\cancel{k_1}$. $\Gamma, \varphi \vdash \neg\varphi'$ (ext 2)
4. $\Gamma, \varphi \vdash (\varphi \to \varphi')$ (ext 1)
5. $\Gamma, \varphi \vdash \varphi$ (ip)
6. $\cancel{k_2}$. $\boxed{\Gamma, \varphi} \vdash \varphi'$ $(\to_e; 4, 5)$
7. j. $\underbrace{\Gamma, \varphi}_{\Gamma_1} \vdash \bot$ $(\neg e, k_2, k_1)$
8. n. $\Gamma \vdash \neg\varphi$ $(\neg i, j)$ (MT, 1, 2)

$\to_e \ \dfrac{\Gamma \vdash (\varphi_1 \to \varphi_2) \qquad \Gamma \vdash \varphi_1}{\Gamma \vdash \varphi_2}$

$\neg e \ \dfrac{\Gamma_1 \vdash \overset{\varphi'}{\varphi_1} \qquad \Gamma_1 \vdash \neg\varphi_1}{\Gamma_1 \vdash \bot}$

$\neg i \ \dfrac{\Gamma, \varphi \vdash \bot}{\Gamma \vdash \neg\varphi}$

Ex 108 3)

1. $\{\neg(\neg p \wedge \neg q), \neg(p \vee q), p\} \vdash p$ (ip)

z_3. $\{$ — // — $\} \vdash (p \vee q)$ $(\vee i_1, 1)$

(*) z_2. $\{$ — // — $\} \vdash \neg(p \vee q)$ (ip)

z_1. $\{\neg(\neg p \wedge \neg q), \neg(p \vee q), p\} \vdash \bot$ $(\neg e, z_3, z_2)$

y_1. $\{\neg(\neg p \wedge \neg q), \neg(p \vee q)\} \vdash \neg p$ $(\neg i, z_1)$

aproape identic cu (*) (q în loc de p)

y_2. $\{\neg(\neg p \wedge \neg q), \neg(p \vee q)\} \vdash \neg q$

x_2. $\{\neg(\neg p \wedge \neg q), \neg(p \vee q)\} \vdash (\neg p \wedge \neg q)$ $(\wedge i, y_1, y_2)$

x_1. $\{$ — // — $\} \vdash \neg(\neg p \wedge \neg q)$ (ip)

$\neg e \ \dfrac{\Gamma \vdash \varphi_1 \qquad \Gamma \vdash \neg\varphi_1}{\Gamma \vdash \bot}$

j. $\underbrace{\{\neg(\neg p \wedge \neg q), \neg(p \vee q)\}}_{\Gamma} \vdash \bot$ $(\neg e, x_2, x_1)$

k. $\{\neg(\neg p \wedge \neg q)\} \vdash \neg\neg(p \vee q)$ $(\neg i, j)$

n. $\{\neg(\neg p \wedge \neg q)\} \vdash (p \vee q)$ $(\neg\neg e, k)$ (PBC, j)

$\neg i \ \dfrac{\Gamma, \varphi_1 \vdash \bot}{\Gamma \vdash \underbrace{\neg\varphi_1}_{\neg\neg(p \vee q)}}$

$\Gamma, \neg\varphi \vdash \bot$
$\Gamma \vdash \neg\neg\varphi$ $(\neg i)$
$\Gamma \vdash \varphi$ $(\neg\neg e)$

$\text{PBC} \ \dfrac{\Gamma, \neg\varphi \vdash \bot}{\Gamma \vdash \varphi}$

Ex 116 Th corectitudine: pt orice $\Gamma \subseteq \mathbb{LP}_{\neg\wedge\vee\to\bot}$, orice $\varphi \in \mathbb{LP}_{\neg\wedge\vee\to\bot}$

Dacă $\underbrace{\Gamma \vdash \varphi \text{ validă, atunci } \Gamma \vDash \varphi}_{P(S)}$ $\quad S = \Gamma \vdash \varphi$

$\Gamma \vdash \varphi$ validă $\Rightarrow$ există o dem formală.

dem formală pt S_2

$\Downarrow$

S_2 validă

1. $S_1 = \Gamma_1 \vdash \varphi_1 \overset{ip.}{\Rightarrow} \Gamma_1 \vDash \varphi_1$
2. $S_2 = \Gamma_2 \vdash \varphi_2 \Rightarrow \Gamma_2 \vDash \varphi_2$
3.

⋮

n. $S_n = \Gamma \vdash \varphi$ $\quad$ (regulă, $\underbrace{S_{i_1}, \ldots S_{i_k}}$)

Dem prin inducție în funcție de lungimea dem.

→ pp. că $P(S_k)$ are loc pt toate secvențele S_k, $k < n$

$S_k = \Gamma_k \vdash \varphi_k$ validă $\Rightarrow \Gamma_k \vDash \varphi_k$

Arătăm că avem și $P(S_n)$

Caz 1 regula aplicată la linia n este $\wedge_i$

S_i este validă

i. $S_i = \Gamma \vdash \varphi_1 \xrightarrow{\text{ip Ind}} \Gamma \vDash \varphi_1$

S_j validă

j. $S_j = \Gamma \vdash \varphi_2 \xrightarrow{\text{ip Ind}} \Gamma \vDash \varphi_2$

n. $S_n = \Gamma \vdash \underbrace{(\varphi_1 \wedge \varphi_2)}_{\varphi}$ $(\wedge_i, i, j)$

$$\wedge_i \; \frac{\Gamma \vdash \varphi_1 \qquad \Gamma \vdash \varphi_2}{\underbrace{\Gamma \vdash (\varphi_1 \wedge \varphi_2)}_{n}}$$

$\Gamma \vDash \varphi_1$ ddacă pt orice $\tau : A \to B$ a.î. pt orice $\varphi' \in \Gamma$, $\hat{\tau}(\varphi') = 1$ avem și $\hat{\tau}(\varphi_1) = 1$. (1)

$\Gamma \vDash \varphi_2$ ddacă pt orice $\tau : A \to B$ a.î. orice $\varphi' \in \Gamma$, $\hat{\tau}(\varphi') = 1$ avem și $\hat{\tau}(\varphi_2) = 1$ (2)

Pt a avea $P(S_n)$, știm $\Gamma \vdash (\varphi_1 \wedge \varphi_2)$ validă și tb dem $\Gamma \vDash (\varphi_1 \wedge \varphi_2)$

$\Gamma \models (\varphi_1 \wedge \varphi_2)$ ddacă pt orice $\tau : A \to B$ a.î. $\hat{\tau}(\varphi') = 1$ pt orice $\varphi' \in \Gamma$

avem și $\hat{\tau}((\varphi_1 \wedge \varphi_2)) = 1$.

ddacă ——— // ———

avem și $\hat{\tau}(\varphi_1) * \hat{\tau}(\varphi_2) = 1$. (*)

Din (1) $\Rightarrow$ pt orice $\tau : A \to B$ a.î. $\hat{\tau}(\varphi') = 1$ pt orice $\varphi' \in \Gamma$

avem $\hat{\tau}(\varphi_1) = 1$

(2) $\Rightarrow$ ——— // ———

avem $\hat{\tau}(\varphi_2) = 1$ $\Rightarrow$

$\Rightarrow$ pt orice $\tau : A \to B$ a.î. $\hat{\tau}(\varphi') = 1$ pt orice $\varphi' \in \Gamma$

avem $\hat{\tau}(\varphi_1) * \hat{\tau}(\varphi_2) = 1 * 1 = 1$ $\Rightarrow$ (*)

$\Rightarrow \Gamma \models (\varphi_1 \wedge \varphi_2$

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orice $\varphi_1, \varphi_2 \in LP$ $(\neg\varphi_1 \vee \varphi_2)$ validă ddacă $\neg\varphi_2 \models \neg\varphi_1$

Fie $\varphi_1, \varphi_2 \in LP$ fixate arbitrar.

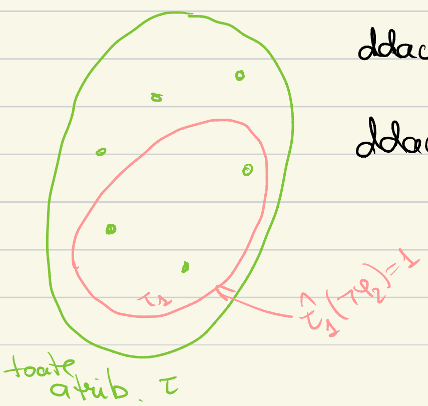
" $\Longrightarrow$ " $(\neg\varphi_1 \vee \varphi_2)$ validă ddacă pt orice $\tau : A \to B$ avem $\hat{\tau}(\neg\varphi_1 \vee \varphi_2) = 1$

$\neg\varphi_2 \models \neg\varphi_1$ ddacă —— // —— $\overline{\hat{\tau}(\varphi_1)} + \hat{\tau}(\varphi_2) = 1$ (1)

$\{\neg\varphi_2\} \models \neg\varphi_1$ ddacă pt orice $\tau_1 : A \to B$ a.î. $\hat{\tau}_1(\neg\varphi_2) = 1$ avem și $\hat{\tau}_1(\neg\varphi_1) = 1$

ddacă pt orice $\tau_1 : A \to B$ a.î. $\overline{\hat{\tau}_1(\varphi_2)} = 1$ avem și $\overline{\hat{\tau}_1(\varphi_1)} = 1$

ddacă —— // —— $\hat{\tau}_1(\varphi_2) = 0$ avem și $\hat{\tau}_1(\varphi_1) = 0$



(1) are loc pt orice $\tau_1 : A \to B$ a.î. $\hat{\tau}_1(\varphi_2) = 0$

$\Rightarrow \overline{\hat{\tau}_1(\varphi_1)} + 0 = 1 \Rightarrow \overline{\hat{\tau}_1(\varphi_1)} = 1 \Rightarrow \hat{\tau}_1(\varphi_1) = 0$

$\Rightarrow \neg\varphi_2 \models \neg\varphi_1$

"$\Longleftarrow$" $\quad \dfrac{\dfrac{\neg\varphi_2 \models \neg\varphi_1}{(\neg\varphi_1 \vee \varphi_2) \text{ validă}}}{}$ $\quad$ ddacă pt orice $\tau_1 : A \to B$ a.î. $\hat{\tau}_1(\varphi_2) = 0$ avem și $\hat{\tau}_1(\varphi_1) = 0$ (2)

$(\neg\varphi_1 \vee \varphi_2)$ validă ddacă pt orice $\tau : A \to B$ avem $\overline{\hat{\tau}(\varphi_1) + \hat{\tau}(\varphi_2)} = 1$ (*)

Fie $\tau : A \to B$ ales arbitrar

Caz 1 : Dacă $\hat{\tau}(\varphi_2) = 0$ (este din mulțimea "roz")

Din (2) $\Rightarrow$ $\hat{\tau}(\varphi_1) = 0$ / $\overline{\hat{\tau}(\varphi_1)} + \hat{\tau}(\varphi_2) = \overline{0} + 0 = 1 + 0 = 1$

$\Rightarrow$ (*) are loc în caz 1.

Caz 2 : Dacă $\hat{\tau}(\varphi_2) = 1$ $\Rightarrow$

$\overline{\hat{\tau}(\varphi_1)} + \hat{\tau}(\varphi_2) = \overline{\hat{\tau}(\varphi_1)} + 1 = 1$

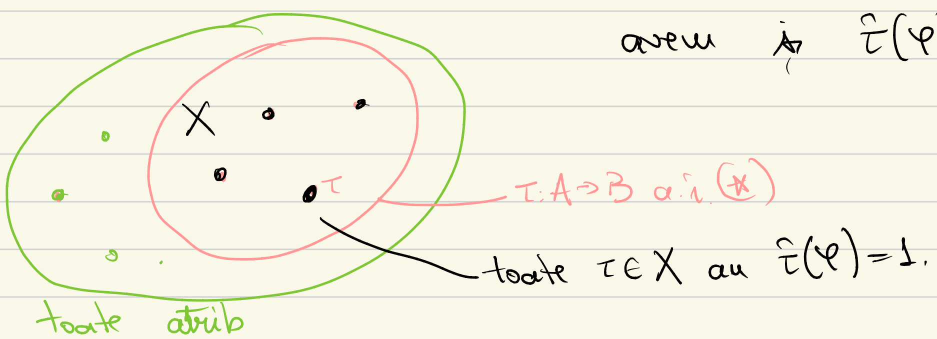
$\Rightarrow$ (*) are loc în caz 2

Caz 1 și 2 acoperă toate cazurile posibile $\Rightarrow$ (*) are loc pt orice $\tau : A \to B$

$\Rightarrow$ $(\neg\varphi_1 \vee \varphi_2)$ validă

$\{\varphi_1, \dots \varphi_n\} \models \varphi$ ddacă pt orice $\tau : A \to B$ a.î. $\hat{\tau}(\varphi_1) = \hat{\tau}(\varphi_2) = \dots \hat{\tau}(\varphi_n) = 1$ (*)

avem și $\hat{\tau}(\varphi) = 1$



$\{\varphi_1, \dots \varphi_n\} \not\models \varphi$ ddacă există $\tau : A \to B$ a.î. $\hat{\tau}(\varphi_1) = \dots = \hat{\tau}(\varphi_n) = 1$

și $\hat{\tau}(\varphi) = 0$