1. Pe $\mathbb{Z}$

Relații $xRy \iff x = y^2$

Verificați proprietățile acestei relații

Reflexivitate

Fie $x = 2$ $\quad 2 \neq 2^2 = 4 \Rightarrow R$ nu e reflexivă

Dacă vrem să arătăm că o proprietate are loc, trebuie să o arătăm pentru toate elementele

Dacă vrem să arătăm că o proprietate nu are loc, e suficient să găsim un contraexemplu

Simetria

Fie $x, y \in \mathbb{Z}$ a.î. $x R y \Rightarrow \underline{x = y^2}$

$x \geq 0$

$y = \pm\sqrt{x}$ $\quad (\bullet)$

Pt ca să aibă loc yRx ar trebui ca $y = x^2$ ceea ce pare în contradicție cu (*)

Caut 2 nr întregi a.î. $x = y^2$ dar $y \neq x^2$

$x = 4,\ y = 2$ $\quad$ $4R2$ dar $2 \not R 4$

$\Rightarrow R$ nu este simetrică

Antisimetria

Fie $x, y \in \mathbb{Z}$ a.î. $xRy \Rightarrow x = y^2$
$yRx \Rightarrow y = x^2$

$x = (x^2)^2 \Leftrightarrow$

$x = x^4 \Rightarrow$

$x^4 - x = 0$

$x(x^3 - 1) = 0$ $\Rightarrow x = 0 \Rightarrow y = 0$

$\quad\quad\quad\quad\quad\quad\quad\;\; \Rightarrow x = 1 \Rightarrow y = 1$ $\Rightarrow$

$x = y$

$\Rightarrow$ R antisimetrică

Tranzitivitatea

Fie $x, y, z \in \mathbb{Z}$

$x R y \Rightarrow x = y^2$ $\quad\quad x = (z^2)^2$

$y R z \Rightarrow y = z^2$ $\quad\quad \underline{x = z^4} \leftarrow$ Știu

$\overline{\quad\quad\quad\quad\quad\quad}$

$x R z ? \Leftrightarrow \underline{x = z^2} \leftarrow$ Vrean

$x = 16$ $\quad 16 = 4^2 \Rightarrow 16 R 4$

$y = 4$ $\quad 4 = 2^2 \Rightarrow 4 R 2$

$z = 2$

$16 \not R 2$

$\Rightarrow R$ nu e tranzitivă

Pe $\mathbb{R}$ avem relația

$x \neq y$

Fie $x, y \in \mathbb{R}$ a.î.

$\underline{x \neq y}$

și $y \neq x$

$x \neq y \Rightarrow y \neq x$ deci $\neq$ este simetrică

Pe $\mathbb{Z}$

$a << b \iff \exists\, c \in \mathbb{N}$ a.î. $b - a = c$

Dem că $<<$ este relație de ordine totală pe $\mathbb{Z}$. $\rightarrow$ 12:30

Reflexivitate: Fie $a \in \mathbb{Z}$ $\exists\, 0 \in \mathbb{N}$ a.î.

$a - a = 0 \Rightarrow a << a \ \forall a \in \mathbb{Z} \Rightarrow <<$ reflexivă

Antisimetrie Fie $a, b \in \mathbb{Z}$

$a << b \Rightarrow \exists\, c_1 \in \mathbb{N}$ a.î. $b - a = c_1$

$b << a \Rightarrow \exists\, c_2 \in \mathbb{N}$ a.î. $a - b = c_2$ (+)

$$\cancel{b} - \cancel{a} + \cancel{a} - \cancel{b} = c_1 + c_2$$

$$\left.\begin{array}{l} 0 = c_1 + c_2 \\ c_1, c_2 \in \mathbb{N} \end{array}\right\} \Rightarrow$$

$c_1 = c_2 = 0 \Rightarrow b - a = 0 \Rightarrow b = a$

$\Rightarrow$ $<<$ antisimetrică

Tranzitivitatea Fie $a, b, d \in \mathbb{Z}$

$a << b \Rightarrow \exists c_1 \in \mathbb{N}$ a.î. $b - a = c_1$

$b << d \Rightarrow \exists c_2 \in \mathbb{N}$ a.î. $d - b = c_2$ (+)

$$d - \cancel{b} + \cancel{b} - a = c_2 + c_1$$

$\Rightarrow d - a = c_2 + c_1$

$\Rightarrow \exists (c_1 + c_2) \in \mathbb{N}$ a.î. $d - a = c_1 + c_2$

$\Rightarrow a << d \Rightarrow <<$ tranzitivă

O relație este ~~totală~~ dacă oricare ar fi 2 elemente, ele pot fi puse în relație

$\forall a, b \in \mathbb{Z}$ are loc $(a << b) \vee (b << a)$

Fie $a, b \in \mathbb{Z} \Rightarrow b - a \in \mathbb{Z}$

Dacă $b - a \in \mathbb{N} \Rightarrow a << b$

$b - a \notin \mathbb{N} \Rightarrow b - a \in \mathbb{Z}_- \Rightarrow$

$a - b \in \mathbb{N}$

$\Rightarrow b << a$

$\Rightarrow <<$ este relație totală

Dacă $a \leq b \Rightarrow 0 \leq b - a \Rightarrow b - a \in \mathbb{N} \Rightarrow$
$a << b$

2\. **8.** Fie endomorfismul T: $\mathbb{R}^3 \to \mathbb{R}^3$ definit prin
$T(\overline{x}) = (2x_1 - x_2 - x_3, x_1 - 2x_2 + x_3, x_1 + x_2 - 2x_3)$, pentru
$\forall \overline{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$.

a) Să se afle matricea lui T în baza canonică B din $\mathbb{R}^3$.

b) Să se afle matricea lui T în baza $B_1 = \{\underbrace{(2, 3, -1)}_{v_1}, \underbrace{(0, -2, 1)}_{v_2}, \underbrace{(-1, -1, -1)}_{v_3}\}$.

c) Să se afle rg T şi o bază pentru Im T.

d) Să se afle def T şi o bază pentru Ker T.

a)
$$T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2x_1 - x_2 - x_3 \\ x_1 - 2x_2 + x_3 \\ x_1 + x_2 - 2x_3 \end{pmatrix} = \begin{pmatrix} 2 & -1 & -1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$A_C = \begin{pmatrix} 2 & -1 & -1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \qquad A_{B_1} = ?$$

$$C = \left\{ \bar{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \bar{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \bar{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$$B_1 = \left\{ \bar{v}_1 = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}, \bar{v}_2 = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}, \bar{v}_3 = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} \right\}$$

$$S_{CB_1} = \begin{pmatrix} 2 & 0 & -1 \\ 3 & -2 & -1 \\ -1 & 1 & -1 \end{pmatrix}$$

$$\underbrace{(\bar{v}_1 \ \bar{v}_2 \ \bar{v}_3)}_{\text{vector de vectori}} = S_{CB_1} \cdot \underbrace{(\bar{e}_1 \ \bar{e}_2 \ \bar{e}_3)}_{I_3}$$

$$T : \mathbb{R}^3 \to \mathbb{R}^3$$

$$T(\bar{x}) = y$$

$\bar{x}$ pot să îl exprim în baza canonică

$$\bar{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = (\bar{e}_1 \ \bar{e}_2 \ \bar{e}_3) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} =$$

$$x_1 \bar{e}_1 + x_2 \bar{e}_2 + x_3 \bar{e}_3 =$$

$$x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} =$$

$$\begin{pmatrix} x_1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ x_2 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ x_3 \end{pmatrix} =$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$\bar{x}$ pot să îl exprim în B,

$$\bar{x} = (\bar{v}_1 \ \bar{v}_2 \ \bar{v}_3) \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \leftarrow$$

coord lui x în B,

$$\bar{y} = (\bar{e}_1 \ \ \bar{e}_2 \ \ \bar{e}_3)\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$$\bar{y} = (\bar{v}_1 \ \ \bar{v}_2 \ \ \bar{v}_3)\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$A \, \overbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}^{x} = \overbrace{\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}}^{y}$$

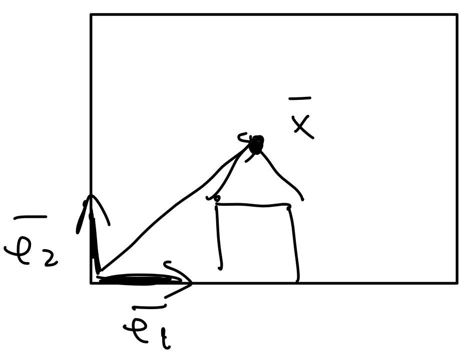
$$? \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

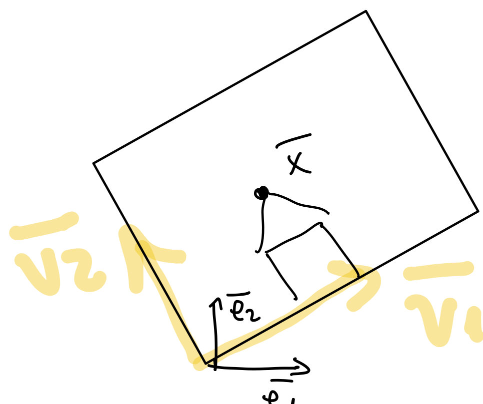
$$x = (\bar{e}_1 \ \ \bar{e}_2 \ \ \bar{e}_3)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = (\bar{v}_1 \ \ \bar{v}_2 \ \ \bar{v}_3)\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

$$(\bar{v}_1\ \ \bar{v}_2\ \ \bar{v}_3) = (\bar{e}_1\ \ \bar{e}_2\ \ \bar{e}_3)\,S_{CB_1}$$

$$(\bar{v}_1\ \ \bar{v}_2\ \ \bar{v}_3)\,S_{CB_1}^{-1} = (\bar{e}_1\ \ \bar{e}_2\ \ \bar{e}_3)$$

$$(\bar{e}_1\ \ \bar{e}_2\ \ \bar{e}_3)\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = (\bar{e}_1\ \ \bar{e}_2\ \ \bar{e}_3)\,S_{CB_1}\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}$$





$$\bar{x} = \overset{x_1}{2}\,\bar{e}_1 + \overset{x_2}{2}\,\bar{e}_2$$

$$\bar{x} = \overset{a_1}{1}\,\bar{v}_1 + \overset{a_2}{1}\cdot\bar{v}_2$$

$$(\bar{v}_1\ \ \bar{v}_2) = (\bar{e}_1\ \ \bar{e}_2)\cdot S_{CB}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = S_{CB_1} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

coord vechi = matricea schimb bază · coord noi

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = S_{CB_1} \cdot \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = A_C \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$S_{CB_1} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = A_C \cdot S_{CB_1} \cdot \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

$$\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \underbrace{\left(S_{CB_1}\right)^{-1} A_C \, S_{CB_1}}_{A_{B_1}} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

Dacă schimb baza

$$A_{B_1} = \left(S_{CB_1}\right)^{-1} \cdot A_C \, S_{CB_1}$$

$$S_{CB_1} = \begin{pmatrix} 2 & 0 & -1 \\ 3 & -2 & -1 \\ -1 & 1 & -1 \end{pmatrix}$$

$$A_C = \begin{pmatrix} 2 & -1 & -1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}$$

$T(\bar{e}_1)$ $T(\bar{e}_2)$ $T(e_3)$

$\hookrightarrow$ 13:20

$$\left(\begin{array}{ccc|ccc} 2 & 0 & -1 & 1 & 0 & 0 \\ 3 & -2 & -1 & 0 & 1 & 0 \\ -1 & 1 & -1 & 0 & 0 & 1 \end{array}\right) \quad L_1 = \frac{L_1}{2}$$

$$\sim$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 3 & -2 & -1 & 0 & 1 & 0 \\ -1 & 1 & -1 & 0 & 0 & 1 \end{array}\right) \quad \begin{array}{l} L_2 = L_2 - 3L_1 \\ L_3 = L_3 + L_1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & -2 & \frac{1}{2} & -\frac{3}{2} & 1 & 0 \\ 0 & 1 & -\frac{3}{2} & \frac{1}{2} & 0 & 1 \end{array}\right) \quad L_2 = \frac{L_2}{-2}$$

$$\sim$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & -\frac{1}{4} & \frac{3}{4} & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{3}{2} & \frac{1}{2} & 0 & 1 \end{array}\right) \quad L_3 = L_3 - L_2$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & -\frac{1}{4} & \frac{3}{4} & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{5}{4} & -\frac{1}{4} & \frac{1}{2} & 1 \end{array}\right)$$

$$L_3 = L_3 \cdot \left(-\frac{4}{5}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & -\frac{1}{4} & \frac{3}{4} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{1}{5} & -\frac{2}{5} & -\frac{4}{5} \end{array}\right)$$

$$L_1 = L_1 + \frac{1}{2} L_3$$

$$L_2 = L_2 + \frac{1}{4} L_3$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{5} & -\frac{1}{5} & -\frac{2}{5} \\ 0 & 1 & 0 & \frac{4}{5} & -\frac{3}{5} & -\frac{1}{5} \\ 0 & 0 & 1 & \frac{1}{5} & -\frac{2}{5} & -\frac{4}{5} \end{array}\right)$$

$$S_{CB_1}^{-1} = \frac{1}{5}\begin{pmatrix} 3 & -1 & -2 \\ 4 & -3 & -1 \\ 1 & -2 & -4 \end{pmatrix}$$

$$A_B = \frac{1}{5}\begin{pmatrix} 3 & -1 & -2 \\ 4 & -3 & -1 \\ 1 & -2 & -4 \end{pmatrix}\begin{pmatrix} 2 & -1 & -1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}\begin{pmatrix} 2 & 0 & -1 \\ 3 & -2 & -1 \\ -1 & 1 & -1 \end{pmatrix}$$

$$\frac{1}{5}\begin{pmatrix} 3 & -3 & 0 \\ 4 & 1 & -5 \\ -4 & -1 & 5 \end{pmatrix}\begin{pmatrix} 2 & 0 & -1 \\ 3 & -2 & -1 \\ -1 & 1 & -1 \end{pmatrix} =$$

$$\frac{1}{5}\begin{pmatrix} -3 & 6 & 0 \\ 16 & -7 & 0 \\ -16 & 7 & 0 \end{pmatrix}$$

$rg\, T = rg\, A =$ dimensiunea celui mai mare determinant

$$A_{B_1} = S_{CB_1}^{-1} \cdot A \cdot S_{CB_1}$$

$$\underbrace{\det(A_{B_1})}_{0} = \det S_{CB_1}^{-1} \cdot \det A \cdot \det S_{CB_1}$$

linia 2 = linia 3

$\Rightarrow \det A_C = 0$

Fie $\Delta = \begin{pmatrix} 2 & -1 \\ 1 & -2 \end{pmatrix} = -4+1 = -3 \neq 0$

$\Rightarrow$ rang $A_c = 2$ și

rang $T = 2$

$=$ dim Im T $\Rightarrow$

dim Im $T = 2$

Pt că minorul Δ conține col 1 și 2 care reprezintă $T(\overline{e_1}), T(\overline{e_2})$ $\Rightarrow$ O bază în Im T este $T(\overline{e_1}), T(\overline{e_2}) =$

$$\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$$

$$\mathrm{Im}\,T = \left\{ \alpha \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix} \middle| \alpha, \beta \in \mathbb{R} \right\}$$

$$T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$

$$\mathrm{Im}\,(T) \subset \mathbb{R}^3$$

Găsind rangul lui $T = 2 \Rightarrow$ Dimensiunea unei baze în $\mathrm{Im}\,T$ este 2. $\Rightarrow$ Am nevoie de 2 vectori în $\mathrm{Im}\,T$ care să fie lin indep.

$$\mathrm{Im}\,T = \{ T(\bar{x}) \mid \bar{x} \in \mathbb{R}^3 \}$$

$$A_C = \begin{pmatrix} 2 & -1 & -1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}_{\overline{e_1}} = \underbrace{\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}}_{T(\overline{e_1})}$$

$$\underbrace{}_{T(\overline{e_1})} \quad \underbrace{}_{T(\overline{e_2})} \quad T(\overline{e_3})$$

$$\overline{x} = x_1 \overline{e_1} + x_2 \overline{e_2} + x_3 \overline{e_3}$$

T liniară

$$\underline{T(\overline{x})} = T(x_1 \overline{e_1} + x_2 \overline{e_2} + x_3 \overline{e_3}) =$$

$$x_1 T(\overline{e_1}) + x_2 T(\overline{e_2}) + x_3 T(\overline{e_3})$$

element oarecare din Im T

$$\begin{pmatrix} 2 & -1 \\ 1 & -2 \end{pmatrix} \neq 0$$

$$2 \cdot (-2) - 1 \cdot (-1) \neq 0$$

$$(2, 1) \neq \alpha(-1, -2)$$

$$(2, 1, 1) \neq \alpha(-1, -2, 1)$$

$T(\bar{e_1}), T(\bar{e_2})$ l. indep

$T(\bar{e_1}), T(\bar{e_2}), T(\bar{e_3})$ generatori pt Im T

rang T = 2

Dacă det $A \neq 0 \to$ rang maximal

det $A = 0 \to$ rang < maximal

$\text{def } T = \underbrace{\text{dim sp}}_{3} - \underbrace{\text{rang } T}_{2} = 1$

$\text{Ker } T = \{ x \mid T(\bar{x}) = 0 \}$

$$\begin{pmatrix} 2 & -1 & -1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\det A = 0$

Minorul conține liniile 1 și 2 $\Rightarrow$

x_1 și x_2 nec pp și x_3 nec sec.

$x_3 = \alpha$

$$\begin{pmatrix} 2 & -1 & -1 \\ 1 & -2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} 2x_1 - x_2 - x_3 = 0 \\ x_1 - 2x_2 + x_3 = 0 \end{cases}$$

$$\begin{cases} 2x_1 - x_2 = \alpha \quad | \cdot 2 \\ x_1 - 2x_2 = -\alpha \end{cases}$$

$4x_1 - 2x_2 = 2\alpha$

$x_1 - 2x_2 = -\alpha \quad \ominus$

$3x_1 = 3\alpha \Rightarrow x_1 = \alpha$

$$\begin{pmatrix} 2 & -1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \alpha \\ -\alpha \end{pmatrix}$$

$2\alpha - x_2 = \alpha$

$+ x_2 = \alpha$

→ Cramer

→ Rezolvare reducere

→ Înmulțind cu inversa

→ Substituție

$\operatorname{Ker} T = \{(\alpha, \alpha, \alpha) \mid \alpha \in \mathbb{R}\}$

O bază în $\operatorname{Ker} T = \{(1, 1, 1)\}$

$$\begin{pmatrix} 2 & -1 & -1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}$$

$$\begin{vmatrix} 2-\lambda & -1 & -1 \\ 1 & -2-\lambda & 1 \\ 1 & 1 & -2-\lambda \end{vmatrix} \qquad \underline{\underline{L_1 = L_1 + L_2 + L_3}}$$

$$\begin{vmatrix} 4-\lambda & -2-\lambda & -2-\lambda \\ 1 & -2-\lambda & 1 \\ 1 & 1 & -2-\lambda \end{vmatrix} \qquad \underline{\underline{C_1 = C_1 + C_2 + C_3}}$$

$$\begin{vmatrix} -3\lambda & -2-\lambda & -2-\lambda \\ -\lambda & -2-\lambda & 1 \\ -\lambda & 1 & -2-\lambda \end{vmatrix} =$$

$$-\lambda \begin{vmatrix} 3 & -2-\lambda & -2-\lambda \\ 1 & -2-\lambda & 1 \\ 1 & 1 & -2-\lambda \end{vmatrix}$$

$\underline{\underline{L_2 = L_2 - L_3}}$

$$-\lambda \begin{vmatrix} 3 & -2-\lambda & -2-\lambda \\ 0 & -3-\lambda & 3+\lambda \\ 1 & 1 & -2-\lambda \end{vmatrix}$$

$$= -\lambda(3+\lambda) \begin{vmatrix} 3 & -2-\lambda & -2-\lambda \\ 0 & -1 & 1 \\ 1 & 1 & -2-\lambda \end{vmatrix}$$

$$C_3 = C_3 + C_2 = -\lambda(3+\lambda) \begin{vmatrix} 3 & -2-\lambda & -4-2\lambda \\ 0 & -1 & 0 \\ 1 & 1 & -1-\lambda \end{vmatrix}$$

$$+\lambda(3+\lambda) \begin{vmatrix} 3 & -4-2\lambda \\ 1 & -1-\lambda \end{vmatrix} =$$

$$\lambda(3+\lambda)(-3-3\lambda+4+2\lambda)=$$

$$\det(A-\lambda I_3)=\underbrace{\lambda(3+\lambda)(1-\lambda)}$$

$$(\lambda-\lambda_x)^{(i)}$$

$\lambda_1 = 0$ $\qquad m_{\lambda_1} = 1$

$\lambda_2 = -3$ $\qquad m_{\lambda_2} = 1$

$\lambda_3 = 1$ $\qquad m_{\lambda_3} = 1$

Dacă
$\lambda^2(\lambda-2)^3(\lambda+1)$
0 are mult 2
2 are mult 3
-1 are mult 1

$$m_{\lambda_1}+m_{\lambda_2}+m_{\lambda_3}=3=\dim \mathbb{R}^3$$

$$\lambda_1=0 \qquad V_{\lambda_1}=\operatorname{Ker} T=\{(\alpha,\alpha,\alpha) \mid \alpha\in\mathbb{R}\}$$

$\dim V_{\lambda_1}=1$

$v_{\lambda_1}=\underbrace{(1,1,1)}_{\text{vectori proprii}}$ bază în V_{λ_1}

$\lambda_2 = -3$

$$\begin{pmatrix} 2+3 & -1 & -1 \\ 1 & -2+3 & 1 \\ 1 & 1 & -2+3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 5 & -1 & -1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Delta' = \begin{vmatrix} 5 & -1 \\ 1 & 1 \end{vmatrix} = 5 + 1 = 6 \neq 0$$

x_1, x_2 var pp $\quad x_3 = \alpha$ var sec

$$\begin{pmatrix} 5 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = -\alpha \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$5x_1 - x_2 = \alpha$$
$$x_1 + x_2 = -\alpha \qquad +$$

$$6x_1 = 0 \Rightarrow x_1 = 0$$

$$x_2 = -\alpha$$

$$V_{\lambda_2} = \{(0, -\alpha, \alpha) \mid \alpha \in \mathbb{R}\}$$

$v_{\lambda_2} = (0, -1, 1)$ vector propriu

$\dim V_{\lambda_2} = 1$

$$\begin{pmatrix} 2-1 & -1 & -1 \\ 1 & -2-1 & 1 \\ 1 & 1 & -2-1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 & -1 \\ 1 & -3 & 1 \\ 1 & 1 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \Delta'' = \begin{vmatrix} 1 & -1 \\ 1 & -3 \end{vmatrix} \neq$$

x_1, x_2 nec pp $\quad x_3 =$ nec sec

$$\begin{pmatrix} 1 & -1 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = -\alpha \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$x_1 - x_2 = \alpha$$
$$x_1 - 3x_2 = -\alpha \quad \ominus$$

$$2x_2 = 2\alpha \Rightarrow x_2 = \alpha$$
$$x_1 = 2\alpha$$

$$V_{\lambda_3} = \{(2\alpha, \alpha, \alpha) \mid \alpha \in \mathbb{R}\}$$
$$v_{\lambda_3} = (2, 1, 1)$$
$$\dim V_{\lambda_3} = 1$$

$$m_{\lambda_1} + m_{\lambda_2} + m_{\lambda_3} = 3 = \dim \mathbb{R}^3$$

$m_{\lambda_i} = \dim V_{\lambda_i}$ pt toți i

$\Rightarrow$ T diag.

$$A_D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$D = \{ v_{\lambda_1}, v_{\lambda_2}, v_{\lambda_3} \}$

$W = \{ (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 + x_2 + 2x_3 = 0 \}$

a) $W \leq \mathbb{R}^3$

b) O bază ortonormală a lui W în rap cu < > canonic

c) $W^\perp$, o bază ortonormală în $W^\perp$

a) Met I

$\boxed{x_1 + x_2 + 2x_3 = 0}$ sistem de 1 ec și 3 nec

x_1 nec pp, x_2, x_3 nec sec.

$x_2 = \alpha \quad x_3 = \beta$

$x_1 = -\alpha - 2\beta$

$$W = \{(-\alpha - 2\beta, \alpha, \beta) \mid \alpha, \beta \in \mathbb{R}\}$$

$\forall\ x, y \in W \quad x + y \in W$

$\forall\ \mu \in \mathbb{R}\ \forall\ x \in W \quad \mu x \in W$

$x = (-\alpha_1 - 2\beta_1, \alpha_1, \beta_1)$

$y = (-\alpha_2 - 2\beta_2, \alpha_2, \beta_2)$

$x + y = (-(\underline{\alpha_1 + \alpha_2}) - 2(\underline{\beta_1 + \beta_2}),$
$\underline{(\alpha_1 + \alpha_2)}, \underline{(\beta_1 + \beta_2)}) \in W$

$\mu x = (-\mu\alpha_1 - 2\mu\beta_1, \mu\alpha_1, \mu\beta_1) \in W$

$\Rightarrow W \leq \mathbb{R}^3$

Met II

Fie $x, y \in W$

$x = (x_1, x_2, x_3)$, $x_1 + x_2 + 2x_3 = 0$

$y = (y_1, y_2, y_3)$ $y_1 + y_2 + 2y_3 = 0$

+

$x + y = (x_1 + y_1, x_2 + y_2, x_3 + y_3)$

$x + y \in W$

$(x_1 + y_1) + (x_2 + y_2) + 2(x_3 + y_3) = 0$

$\mu x = (\mu x_1, \mu x_2, \mu x_3)$

$\mu x_1 + \mu x_2 + 2\mu x_3 = 0$

$\Rightarrow \mu x \in W$

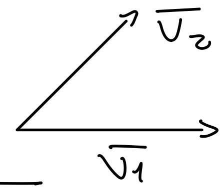
$\Rightarrow W \leq \mathbb{R}^3$

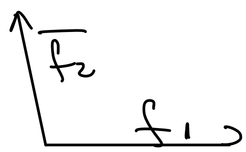
e)

$$W = \{(-\alpha - 2\beta, \alpha, \beta) \mid \alpha, \beta \in \mathbb{R}\}$$

$\alpha = 1, \beta = 0 \quad \bar{v}_1 = (-1, 1, 0)$

$\alpha = 0, \beta = 1 \quad \bar{v}_2 = (-2, 0, 1)$





$\bar{f}_1 = \bar{v}_1$

$\bar{f}_2 = \bar{v}_2 + \delta \bar{v}_1 = (-2, 0, 1) - (-1, 1, 0)$

$\underline{(-1, -1, 1)}$

$\langle \bar{f}_2, \bar{f}_1 \rangle = 0 \Rightarrow$

$\langle \bar{v}_2 + \delta \bar{v}_1, \bar{v}_1 \rangle = 0$

$\langle \bar{v}_2, \bar{v}_1 \rangle + \delta \langle \bar{v}_1, \bar{v}_1 \rangle = 0$

$$\delta = - \frac{\langle \bar{v}_2, \bar{v}_1 \rangle}{\langle \bar{v}_1, \bar{v}_1 \rangle} = - \frac{2}{2} = -1$$

$$\bar{e}_1 = \frac{1}{\|\bar{v}_1\|} \cdot \bar{v}_1 = \frac{1}{\|\bar{f}_1\|} \cdot \bar{f}_1 = \frac{1}{\sqrt{2}} (-1, 1, 0)$$

$$\bar{e}_2 = \frac{1}{\|\bar{f}_2\|} \cdot \bar{f}_2 = \frac{1}{\sqrt{3}} (-1, -1, 1)$$

c) $W^{\perp} = \{ y \mid \langle x, y \rangle = 0 \ \forall x \in W \}$

$x = (x_1, x_2, x_3)$

$y = (y_1, y_2, y_3)$

$x_1 y_1 + x_2 y_2 + x_3 y_3 = 0 \quad \forall x_1, x_2, x_3 \in W$

$(x_1, x_2, x_3) = (-\alpha - 2\beta, \alpha, \beta)$

$(-\alpha - 2\beta) y_1 + \alpha y_2 + \beta y_3 = 0$

$\alpha(\underline{-y_1 + y_2}) + \beta(\underline{-2y_1 + y_3}) = 0 \ \forall \alpha, \beta \in \mathbb{R}$

$$\begin{cases} \alpha = 1 \quad \beta = 0 \\ \underline{-y_1 + y_2 = 0} \\ \alpha = 0 \quad \beta = 1 \\ \underline{-2y_1 + y_3 = 0} \end{cases}$$

$$\begin{cases} -y_1 + y_2 = 0 \\ -2y_1 + y_3 = 0 \end{cases} \Rightarrow y_1 = y_2 \Rightarrow y_2 = \frac{\alpha}{2}$$

$$\begin{pmatrix} -1 & 1 & 0 \\ -2 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

y_3 nec sec
$y_3 = \alpha$
y_1, y_2 nec pp

$$-2y_1 + \alpha = 0 \Rightarrow$$

$$y_1 = \frac{\alpha}{2}$$

$$\Rightarrow W^{\perp} = \left\{ \left(\frac{\alpha}{2}, \frac{\alpha}{2}, \alpha \right) \middle| \alpha \in \mathbb{R} \right\}$$

O bază este $\bar{v} = \left(\frac{1}{2}, \frac{1}{2}, 1 \right)$

$$\| \bar{v} \| = \sqrt{\frac{1}{4} + \frac{1}{4} + 1} = \sqrt{\frac{3}{2}}$$

$$\bar{p} = \sqrt{\frac{2}{3}}\left(\frac{1}{2}, \frac{1}{2}, 1\right)$$

$$\sum_{n=1}^{\infty} \cos \frac{\pi\left(\sqrt{n} + \frac{1}{n}\right)}{3}$$

$$\cos \frac{\pi\left(\sqrt{n} + \frac{1}{n}\right)}{3}$$

$\cos(a \pm b) =$
$\cos a \cos b -$
$\sin a \sin b$

$n = 9k^2 \quad k \in \mathbb{N}$

$$\cos \frac{\pi\left(3k + \frac{1}{9k^2}\right)}{3} = \cos\left(\frac{3k\pi}{3} + \frac{\pi}{9k^2}\right)$$

$$\cos \frac{3k\pi}{3} \cos \frac{\pi}{9k^2} - \sin \frac{3k\pi}{3} \sin \frac{\pi}{9k^2}$$

$$\cos k\pi \cos \frac{\pi}{9k^2} - \sin k\pi \sin \frac{\pi}{9k^2} =$$

$$(-1)^k \cos\frac{\pi}{9k^2} \xrightarrow{k\to\infty}$$

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c) $$\sum_{n=20}^{\infty} \left(\underbrace{\frac{\sin\frac{n}{3}}{\ln\ln n}}_{b_n} + \underbrace{\frac{\sqrt{n+1}-\sqrt{n}}{n}}_{a_n} \right)$$

$\sqrt{n+1}+\sqrt{n}$

$$a_n = \frac{\sqrt{n+1}-\sqrt{n}}{n}$$

$$\frac{(n+1-n)}{n(\sqrt{n+1}+\sqrt{n})} = \frac{1}{n(\sqrt{n+1}+\sqrt{n})}$$

$\sum a_n$

$\sum \frac{1}{n^{\frac{3}{2}}}$

$$\frac{a_n}{n^{\frac{3}{2}}} = \frac{\frac{1}{\cancel{n}(\sqrt{n+1}+\sqrt{n})}}{\frac{1}{n^{3/2}_{1}}}$$

$$\frac{\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} \longrightarrow \frac{1}{2} \in (0,\infty)$$

$$\Rightarrow \sum a_n \sim \sum \frac{1}{n^{\frac{3}{2}}} \ C$$

$$\frac{\sin\frac{n}{3}}{\ln\ln n}$$

$$\left|\sin\frac{n}{3}\right| < 1$$

$$\ln\ln n \longrightarrow \infty$$