Polinoame $f \in K[X]$

Spunem că $a \in K[X]$ rădăcină a lui f dacă $f(a)=0$.

Dacă a rădăcină a lui f, atunci $(X-a) \mid f$.

Spunem că $a \in K[X]$ rădăcină multiplă de ordin $k \in \mathbb{N}$ dacă $(X-a)^k \mid f$ și $(X-a)^{k+1} \nmid f$.

$f = X^3 - 3X + 2$

$f(1) = 1 - 3 + 2 = 0 \Rightarrow 1$ rădăcină a lui f

$(X-1)^2 = X^2 - 2X + 1$

$$
\begin{array}{l|l}
x^3 \qquad -3x+2 & x^2-2x+1 \\ \hline
-x^3+2x^2-x & x+2 \\
\hline
/ \quad 2x^2-4x+2 & \\
\quad -2x^2+4x-2 & \\
\hline
\quad / \quad / \quad / &
\end{array}
$$

$$\Rightarrow x^3-3x+2 = (x-1)^2(x+2)$$

$$\left.\begin{array}{l}(x-1)^2 \mid f \\ (x-1)^3 \nmid f\end{array}\right\} \Rightarrow$$ 1 este rădăcină de ordin 2 a lui f

Altă formulare pt rădăcini multiple

$$(x-1)^k \mid f \Rightarrow f = (x-1)^k \cdot g, \; g \in K[X]$$

$$f' = \left[(x-1)^k\right]' \cdot g + (x-1)^k \cdot g'$$

$$k(x-1)^{k-1} \cdot g + (x-1)^k \cdot g'$$

Dacă a este rădăcină de ordin k a lui f, atunci $f(a) = f'(a) = \dots = f^{(k-1)}(a) = 0$

$$f^{(k)}(a) \neq 0$$

$$f = a_n X^n + a_{n-1} X^{n-1} + \dots + a_1 X + a_0$$

$$g = X - a$$

Schema lui Horner Exclusiv pt g cu $\deg g = 1$

$$f = g \cdot h + r \qquad h \in K[X]$$

$$\deg(g) = 1$$

$$g = b_{n-1} X^{n-1} + \dots + b_1 X + b_0$$

$$\Rightarrow \deg(r) = 0 \qquad r \in K$$

f	a_n	a_{n-1}	a_{n-2}	$\dots$	a_1	a_0
a	$a_n = b_{n-1}$	$a \cdot b_{n-1} + a_{n-1} = b_{n-2}$	$a \cdot b_{n-2} + a_{n-2} = b_{n-3}$		$a \cdot b_1 + a_1 = b_0$	$a b_0 + a_0 = r$

$$f = x^{3} - 3x + 2$$

$$g = x - 1$$

$$f = (x-1)(1 \cdot x^{2} + \boxed{1} \cdot x - 2)$$

f	$a_3 = 1$	$a_2 = 0$	$a_1 = -3$	$a_0 = 2$
1	1	$1 \cdot 1 + 0 = \boxed{1}$	$1 \cdot \boxed{1} + -3 = -2$	$1 \cdot -2 + 2 = 0$
1	1	$1 \cdot 1 + 1 = 2$	$1 \cdot 2 + -2 = 0$	

$$f = (x-1)(x-1)(x+2)$$

"Reguli de calcul"

$\infty + \infty = \infty$

$\infty \cdot \infty = \infty$

$a + \infty = \infty$

$\frac{a}{\infty} = 0$

$a^{\infty} = \begin{cases} \infty, & a > 1 \\ 0, & 0 < a < 1 \end{cases}$

Cazuri de nedeterminare

$\infty - \infty$

$n \xrightarrow{n \to \infty} \infty$
$2n \xrightarrow{n \to \infty} \infty$

$2n - n = n \to \infty$

$n - n = 0 \to 0$

$\frac{\infty}{\infty}$

$0 \cdot \infty$

1^{∞}

0^{0}

0^{∞}

$\frac{0}{0}$

Seminar 2

***Exerciţii recomandate:** 2.1(a-f), 2.2, 2.3(a,c), 2.4(b), 2.5(a,b), 2.6(a), 2.7(a), 2.10, 2.11, 2.12, 2.13, 2.16
***Rezerve:** 2.1(g,h), 2.4(a,c),2.5(b), 2.7(b), 2.14, 2.15, 2.17

S2.1 Determinaţi valorile următoarelor limite de şiruri:

a) $\lim\limits_{n\to\infty} \dfrac{2n^3-n}{n^3+n^2+1}$; b) $\lim\limits_{n\to\infty} [\ln(n^2+2n+3)-\ln(3n^2+n-6)]$; c) $\lim\limits_{n\to\infty} \dfrac{3^n+5^n}{3^{n+1}+5^{n+1}}$;

d) $\lim\limits_{n\to\infty} (3n^2+5)\ln\left(1+\dfrac{1}{n^2}\right)$; e) $\lim\limits_{n\to\infty} \dfrac{1+\frac{1}{2}+\frac{1}{2^2}+\ldots+\frac{1}{2^n}}{1+\frac{1}{3}+\frac{1}{3^2}+\ldots+\frac{1}{3^n}}$; f) $\lim\limits_{n\to\infty} \dfrac{2^2+4^2+\ldots+(2n)^2}{1^2+3^2+\ldots+(2n-1)^2}$;

g) $\lim\limits_{n\to\infty} \left(n-\sqrt[3]{n^3-3n^2+2}\right)$; h) $\lim\limits_{n\to\infty} \dfrac{\frac{1}{2}\ln 2+\frac{1}{3}\ln 3+\ldots+\frac{1}{n}\ln n}{n\sqrt{n}}$.

a) $\boxed{\lim\limits_{n\to\infty}} \dfrac{2n^3-n}{n^3+n^2+1} \overset{(n^3}{\underset{}{=}} \overset{\frac{\infty}{\infty}}{=} \lim\limits_{n\to\infty} \dfrac{2-\frac{1}{n^2}}{1+\frac{1}{n}+\frac{1}{n^3}} = 2$

$0 \quad 0 \quad 0$

$a_1 = \dfrac{2\cdot 1^3-1}{1^3+1^2+1}$ $\qquad a_2 = \dfrac{2\cdot 2^3-2}{2^3+2^2+1}$

$$\lim_{x\to 0} \frac{\sin x}{x} \overset{L'Hopital}{=} \lim_{x\to 0} \frac{\cos x}{1} = 1$$

$\sin' x = \cos x$

Se dem folosind

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

P, Q polinoame

$$\lim_{n\to\infty} \frac{P(n)}{Q(n)} = \begin{cases} 0, & \text{dacă } \deg Q > \deg P \\ \operatorname{sgn}\left(\dfrac{\text{coef dom } P}{\text{coef dom } Q}\right)\infty, & \text{dacă } \deg P > \deg Q \\ \dfrac{\text{coef dom } P}{\text{coef dom } Q}, & \text{dacă } \deg P = \deg Q \end{cases}$$

b) $\lim_{n\to\infty}\left[\ln(n^2+2n+3)-\ln(3n^2+n-6)\right] \underline{\underline{\ \ln a-\ln b=\ln\frac{a}{b}\ }}$

$$\lim_{n\to\infty} \ln \frac{n^2+2n+3}{3n^2+n-6} \underline{\underline{\ \ln \text{ fct cont}\ }} \ln \underbrace{\lim_{n\to\infty} \frac{n^2+2n+3}{3n^2+n-6}}$$

$$= \ln \frac{1}{3} = \ln 1 - \ln 3 = -\ln 3$$

c) $$\lim_{n\to\infty} \frac{3^n+5^n}{3^{n+1}+5^{n+1}} \ (5^{n+1} = \lim_{n\to\infty} \frac{\frac{3^n}{5^n}\cdot\frac{1}{5}+\frac{1}{5}}{\frac{3^{n+1}}{5^{n+1}}+1} =$$

$$\lim_{n\to\infty} \frac{\left(\frac{3}{5}\right)^n \cdot \frac{1}{5}+\frac{1}{5}}{\left(\frac{3}{5}\right)^{n+1}+1} = \frac{1}{5}$$

$\left(\frac{3}{5}\right)^n \to 0$, $\left(\frac{3}{5}\right)^{n+1} \to 0$

d) $$\lim_{n\to\infty} (3n^2+5)\ln\left(1+\frac{1}{n^2}\right) \underline{\underline{\ \infty\cdot 0\ }}$$

$a\ln b = \ln b^a$

$\frac{1}{n^2} \to 0$, $1+\frac{1}{n^2} \to 1$

$$\lim_{n\to\infty} \ln\left[\left(1+\frac{1}{n^2}\right)^{3n^2+5}\right] \underline{\underline{\ 1^\infty\ }}$$

Limită fundamentală

$$\left(1+\frac{1}{n}\right)^n \longrightarrow e$$

$$\left(1+\frac{1}{u(n)}\right)^{u(n)} \xrightarrow[u(n)\to\infty]{} e$$

$$\lim_{n\to\infty} \ln\left[\left(1+\frac{1}{n^2}\right)^{n^2\cdot\frac{3n^2+5}{n^2}}\right] \qquad a^{b\cdot c} = \left(a^b\right)^c$$

$$\lim_{n\to\infty} \ln\left\{\left[\left(1+\frac{1}{n^2}\right)^{n^2}\right]^{\frac{3n^2+5}{n^2}}\right\} =$$

$$\lim_{n\to\infty} \ln\left[\underbrace{\left(1+\frac{1}{n^2}\right)^{n^2}}\right]^{\lim\limits_{n\to\infty}\frac{3n^2+5}{n^2}}$$

Dacă lim ∃ și nu obțin nedet

$$\ln\left(e^3\right) = 3$$

e) $$\lim_{n\to\infty} \frac{1+\frac{1}{2}+\ldots+\frac{1}{2^n}}{1+\frac{1}{3}+\frac{1}{3^2}+\ldots+\frac{1}{3^n}} = $$

$$1+\frac{1}{2}+\ldots+\frac{1}{2^n} =$$

$$\left(\frac{1}{2}\right)^0+\left(\frac{1}{2}\right)^1+\ldots+\left(\frac{1}{2}\right)^n = \frac{\left(\frac{1}{2}\right)^{n+1}-1}{\frac{1}{2}-1}$$

$$= \lim_{n\to\infty} \frac{\dfrac{\left(\frac{1}{2}\right)^{n+1}-1}{\frac{1}{2}-1}}{\dfrac{\left(\frac{1}{3}\right)^{n+1}-1}{\frac{1}{3}-1}} = \lim_{n\to\infty} \frac{1-\overbrace{\left(\frac{1}{2}\right)^{n+1}}^{\to 0}}{\frac{1}{2}} \cdot$$

$$\frac{\frac{2}{3}}{1-\underbrace{\left(\frac{1}{3}\right)^{n+1}}_{\to 0}} = \frac{1}{\frac{1}{2}} \cdot \frac{\frac{2}{3}}{1} = 2 \cdot \frac{2}{3} = \frac{4}{3}$$

f) $$\lim_{n\to\infty} \frac{2^2+4^2+\dots+(2n)^2}{1^2+3^2+\dots+(2n-1)^2} \qquad \frac{(*)}{\ }$$

$$1+2+3+\dots+n = \frac{n(n+1)}{2}$$

$$\underbrace{1^2+2^2+3^2+\dots+n^2 = \frac{n(n+1)(2n+1)}{6}}$$

$$1^3+2^3+3^3+\dots+n^3 = \frac{n^2(n+1)^2}{4}$$

$$2^2+4^2+\dots+(2n)^2 =$$

$$2^2\cdot 1^2+2^2\cdot 2^2+\dots+2^2\cdot n^2 =$$

$$4(1^2+2^2+\dots+n^2) = \overset{2}{\cancel{4}}\,\frac{n(n+1)(2n+1)}{\underset{3}{\cancel{6}}}$$

$$\underbrace{1^2+3^2+\dots+(2n-1)^2+2^2+4^2+\dots+(2n)^2}$$

$$-2^2-4^2-\dots-(2n)^2 =$$

$$\frac{\cancel{2}n(2n+1)(4n+1)}{\cancel{6}\ 3} - \frac{2n(n+1)(2n+1)}{3}$$

$$= \frac{n(2n+1)}{3}\left[4n+1-2(n+1)\right] =$$

$$\frac{n(2n+1)}{3}(2n-1)$$

$$(*) = \lim_{n\to\infty} \frac{\frac{2n(n+1)(2n+1)}{\cancel{3}}}{\frac{n(2n+1)(2n-1)}{\cancel{3}}} = \frac{4}{4} = 1$$

g) $\lim_{n\to\infty}\left(n-\sqrt[3]{n^3-3n^2+2}\right) \underline{\underline{\infty-\infty}}$

$$\lim_{n\to\infty}\left(\sqrt[3]{n^3}-\sqrt[3]{n^3-3n^2+2}\right) =$$

$$\boxed{\sqrt[3]{a} - \sqrt[3]{b} = \frac{a-b}{\sqrt[3]{a^2} + \sqrt[3]{ab} + \sqrt[3]{b^2}}}$$

$$\lim_{n\to\infty} \frac{\cancel{n^3} - (\cancel{n^3} - 3n^2 + 2)}{\sqrt[3]{(n^3)^2} + \sqrt[3]{n^3(n^3-3n^2+2)} + \sqrt[3]{(n^3-3n^2+2)^2}}$$

$(n^2$

$$\lim_{n\to\infty} \frac{3n^2-2}{\sqrt[3]{n^6} + \sqrt[3]{n^6 - 3n^5 + 2n^3} + \sqrt[3]{n^6 + 9n^4 + \dots}}$$

$$\sqrt[3]{n^6} = n^2$$

$$\lim_{n\to\infty} \frac{3 - \underbrace{\frac{2}{n^2}}_{\to 0}}{\sqrt[3]{1} + \sqrt[3]{1 - \underbrace{\frac{3}{n}}_{\to 0} + \underbrace{\frac{2}{n^3}}_{\to 0}} + \sqrt[3]{1 + \underbrace{\frac{9}{n^2}}_{\to 0} + \dots}}$$

$$\frac{3}{3} = 1$$

h) $\lim_{n\to\infty} \dfrac{\frac{1}{2}\ln 2+\frac{1}{3}\ln 3+\dots+\frac{1}{n}\ln n}{n\sqrt{n}}$ $\overset{\lim}{\exists}$ $\overset{\text{Stolz}}{\underset{\text{Cesaro}}{=}}$

Stolz Cesaro

$\lim_{n\to\infty} \dfrac{x_n}{y_n}$ y_n monoton divergent

$\|$ Dacă $\exists$

$y_n \nearrow \infty$

$\lim_{n\to\infty} \dfrac{x_{n+1}-x_n}{y_{n+1}-y_n}$

$y_n \searrow -\infty$

$x_n = \frac{1}{2}\ln 2+\frac{1}{3}\ln 3+\dots+\frac{1}{n}\ln n$

$x_{n+1} = \frac{1}{2}\ln 2+\frac{1}{3}\ln 3+\dots+\frac{1}{n}\ln n+\frac{1}{n+1}\ln(n+1)$

$x_{n+1}-x_n = \frac{1}{n+1}\ln n+1$

$= \lim_{n\to\infty} \dfrac{\frac{1}{n+1}\ln(n+1)}{(n+1)\sqrt{n+1}-n\sqrt{n}} = 0$

$$\lim_{n\to\infty} \frac{1}{n+1} \ln(n+1) = \lim_{n\to\infty} \frac{\ln(n+1)}{n+1}$$

$$\frac{\text{Dacă } \exists}{\text{Stolz-Cesaro}}$$

$$\lim_{n\to\infty} \frac{\ln(n+2) - \ln(n+1)}{n+2-(n+1)} =$$

$$\lim_{n\to\infty} \ln \frac{n+2}{n+1} \quad \frac{\ln \text{ fct}}{\text{cont}} \Rightarrow$$

$$\ln \lim_{n\to\infty} \frac{n+2}{n+1} = \ln 1 = 0$$

$$\lim_{n\to\infty} \frac{1}{(n+1)\sqrt{n+1} - n\sqrt{n}}$$

$$\frac{1}{a\sqrt{a} - b\sqrt{b}} =$$

$$\lim_{n\to\infty} \frac{\sqrt{(n+1)^3} + \sqrt{n^3}}{(n+1)^3 - n^3} =$$

$$\frac{1}{\sqrt{a^3} - \sqrt{b^3}} = \frac{\sqrt{a^3} + \sqrt{b^3}}{a^3 - b^3}$$

$$\lim_{n\to\infty} \frac{\sqrt{(n+1)^3} + \sqrt{n^3}}{\cancel{n^3} + 3n^2 + 3n + 1 - \cancel{n^3}} =$$

$$\lim_{n\to\infty} \frac{\sqrt{(n+1)^3} + \sqrt{n^3}}{3n^2 + 3n + 1} = 0$$

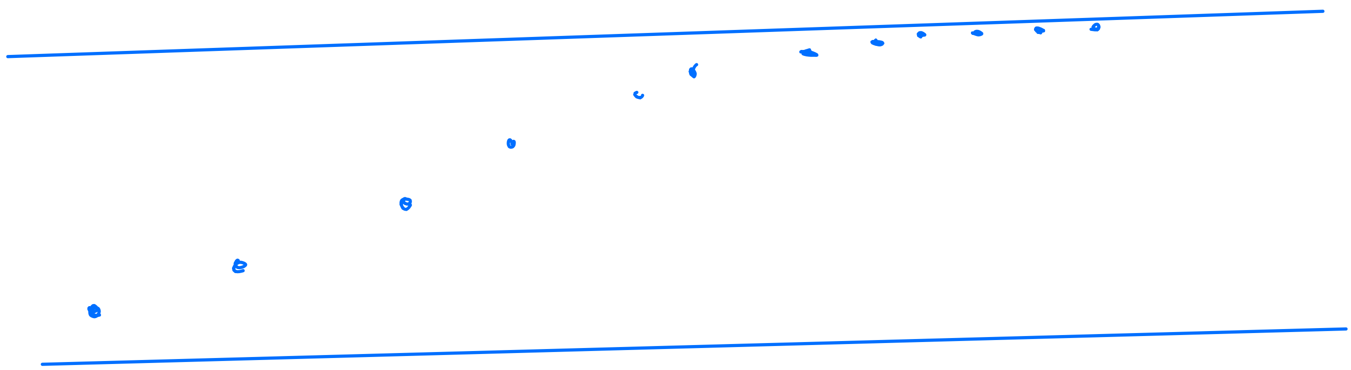
S2.2 Să se arate că şirul $(x_n)_{n\in\mathbb{N}^*} \subset \mathbb{R}$, definit prin $x_1 = 1$ şi

$$x_{n+1} = \underbrace{\underbrace{\left(1 - \frac{1}{3n^2}\right)}\, x_n}, \ \forall\, n \in \mathbb{N}^*,$$

este convergent.

$x_n > 0$ (Dem inductiv)

Weierstrass: Un şir este monoton + mărginit $\Rightarrow$ şirul este convergent



$$n \in \mathbb{N}^* \Rightarrow n \geq 1$$

$$\frac{1}{3n^2} > 0 \Rightarrow 1 - \frac{1}{3n^2} < 1$$

$$\frac{1}{3n^2} < 1 \Rightarrow 1 - \frac{1}{3n^2} > 0$$

$$\frac{x_{n+1}}{x_n} = 1 - \frac{1}{3n^2} < 1$$

şirul este descrescător | W $\Rightarrow$ şirul este convergent

$\Rightarrow x_n \in (0, 1]$

S2.3 Să se studieze convergenţa următoarelor şiruri:

a) $x_n = (1+\cos n\pi)\dfrac{n}{n+1}, n \in \mathbb{N}$; b) $x_{n+1} = \dfrac{x_n^2}{1+x_n}, n \in \mathbb{N}, x_0 = 2$;

c) $x_{n+1} = \sqrt{2+x_n}, n \in \mathbb{N}, x_0 \geq -2$; d) $x_n = \dfrac{(-1)^n n^3 - 3^{-n}}{n^3}, n \in \mathbb{N}^*$;

2.3b.

$$x_{n+1} = \frac{x_n^2}{1+x_n}, \quad x_0 = 2$$

Dem că $x_n > 0 \ \forall n \in \mathbb{N}$. $P(n)$ (A) ?

$P(0)$: $x_0 > 0$ A pt că $2 > 0$

Presupun $P(k)$: $x_k > 0$

Vreau să dem $P(k+1)$ $x_{k+1} > 0$

$$x_{k+1} = \frac{x_k^2}{1+x_k} > 0$$

$\left.\begin{array}{l} P(0) \text{ (A)} \\ P(k) \Rightarrow P(k+1) \text{ (A)} \end{array}\right\} \Rightarrow P(n) \text{ (A)} \ \forall n \in \mathbb{N}.$

$\Rightarrow x_n > 0 \ \forall n \in \mathbb{N}$

$$\frac{x_{n+1}}{x_n} = \frac{x_n}{1+x_n} < 1$$

$$\frac{x_n}{1+x_n} \boxed{<} 1 \quad \Big| \quad 1+x_n > 0$$

$$x_n \boxed{<} 1 + x_n$$

$\Rightarrow (x_n) \searrow$ descrescător.

$0 < x_n \le 2 \Rightarrow (x_n)$ mărginit $\Big\}$ W

(x_n) convergent $\Rightarrow$

$$\exists \lim_{n \to \infty} (x_n) = l$$

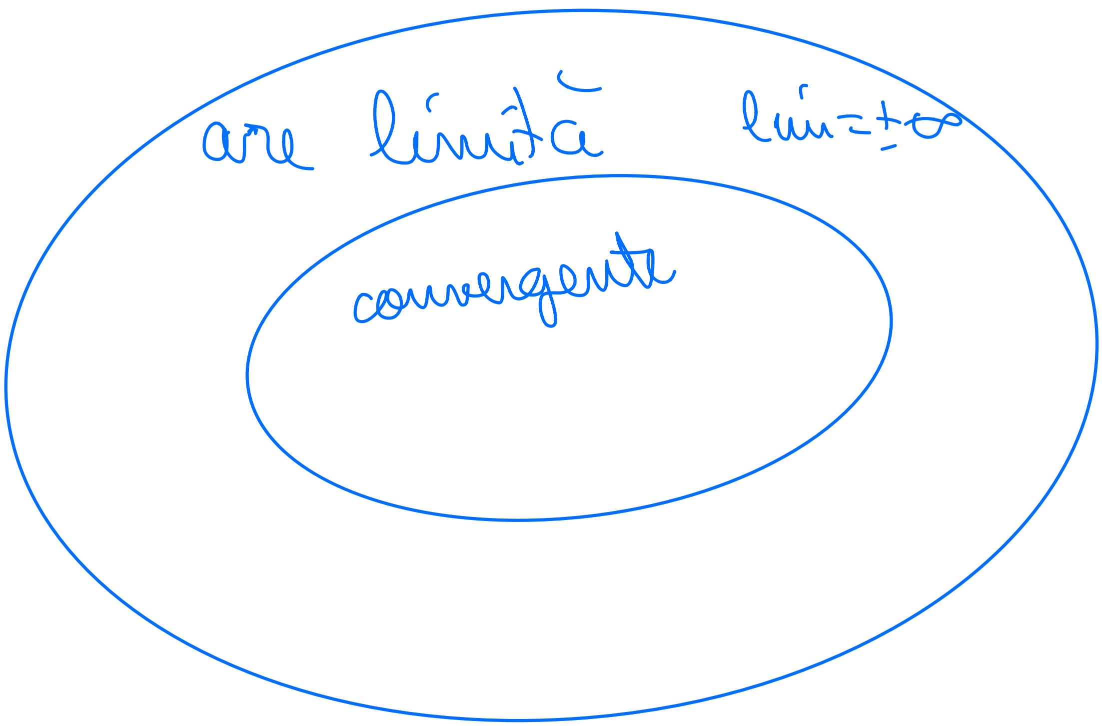
$$l = \frac{l^2}{1+l} \Leftrightarrow l(1+l) = l^2 \Leftrightarrow$$

$$l^2 + l = l^2 \Rightarrow \boxed{l = 0}$$

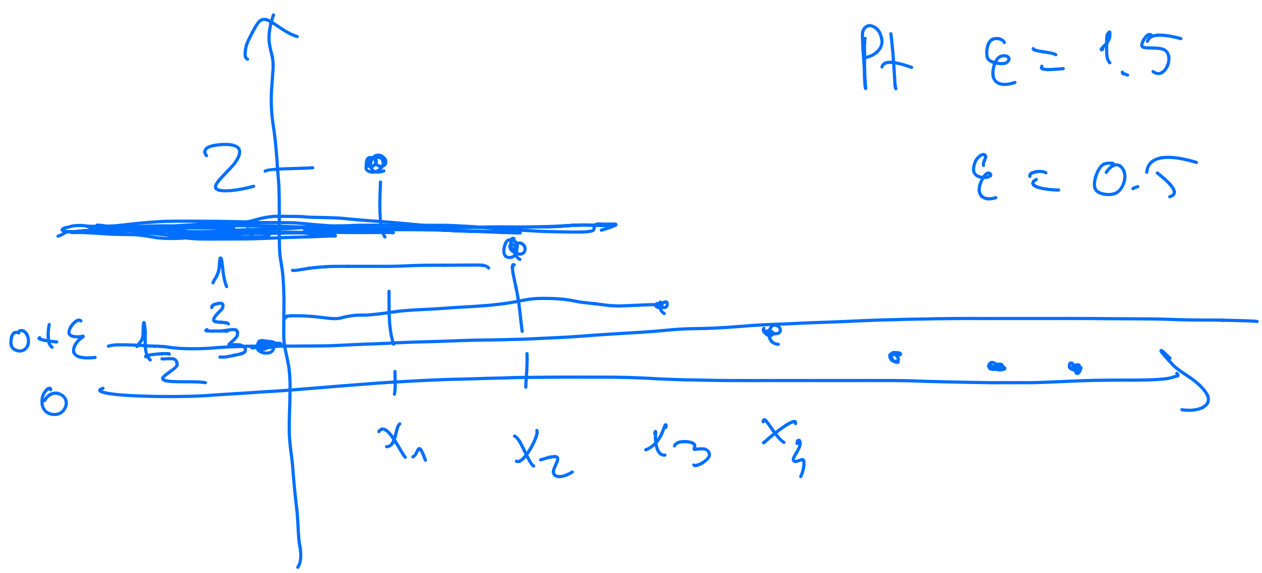
S2.4 Să se stabilească dacă următoarele şiruri sunt fundamentale:

a) $x_n = \dfrac{\cos x}{3} + \dfrac{\cos 2x}{3^2} + \ldots + \dfrac{\cos nx}{3^n}, n \in \mathbb{N}^*, x \in \mathbb{R}$; b) $x_n = 1 + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \ldots + \dfrac{1}{n^2}, n \in \mathbb{N}^*$.

c) $x_n = \dfrac{n^2}{n+1}, n \in \mathbb{N}$; d) $x_n = \dfrac{1}{1 \cdot 2} + \dfrac{1}{2 \cdot 3} + \ldots + \dfrac{1}{n \cdot (n+1)}, n \in \mathbb{N}^*$;

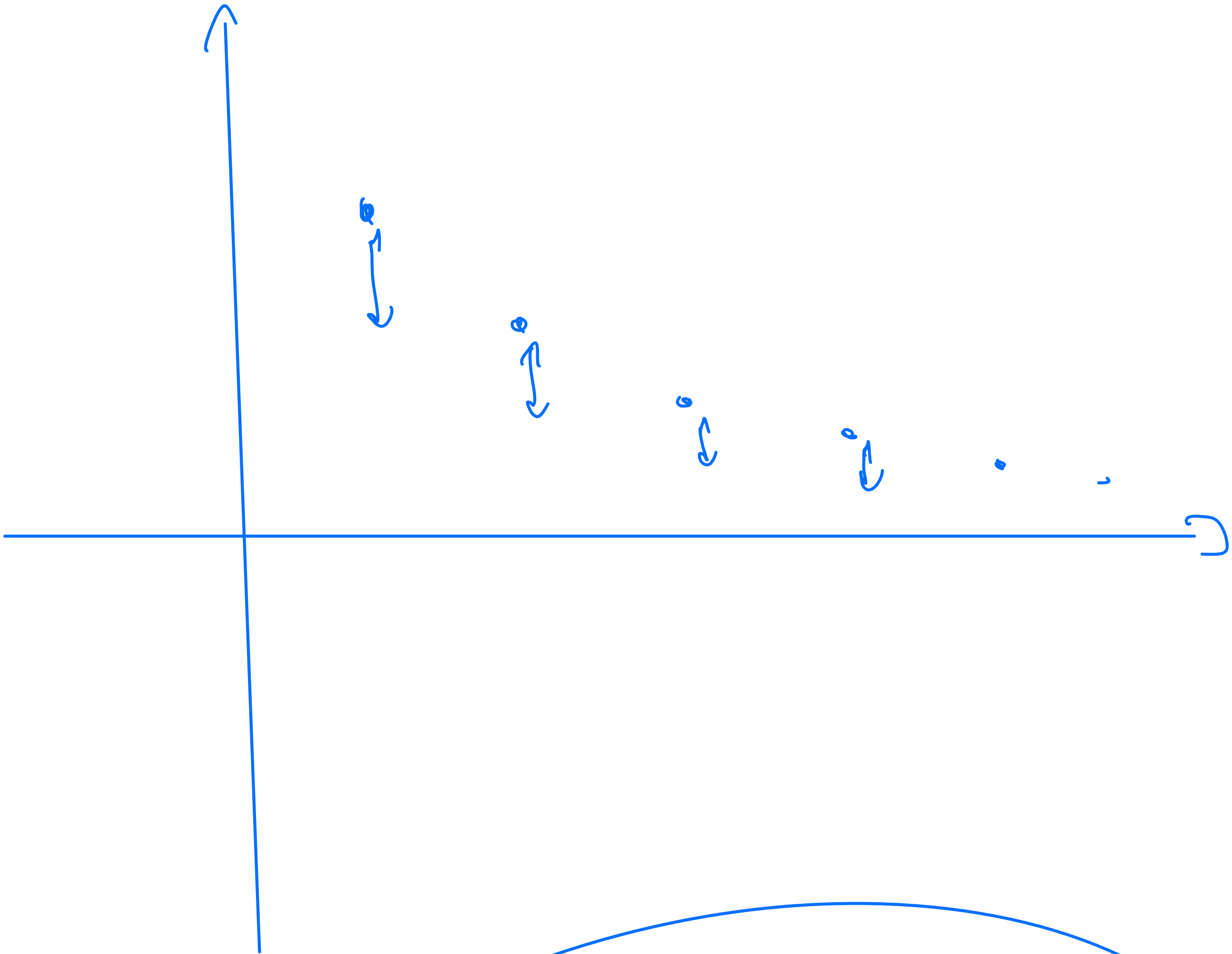


Pt $\mathbb{R}$: șir fundamental = șir convergent

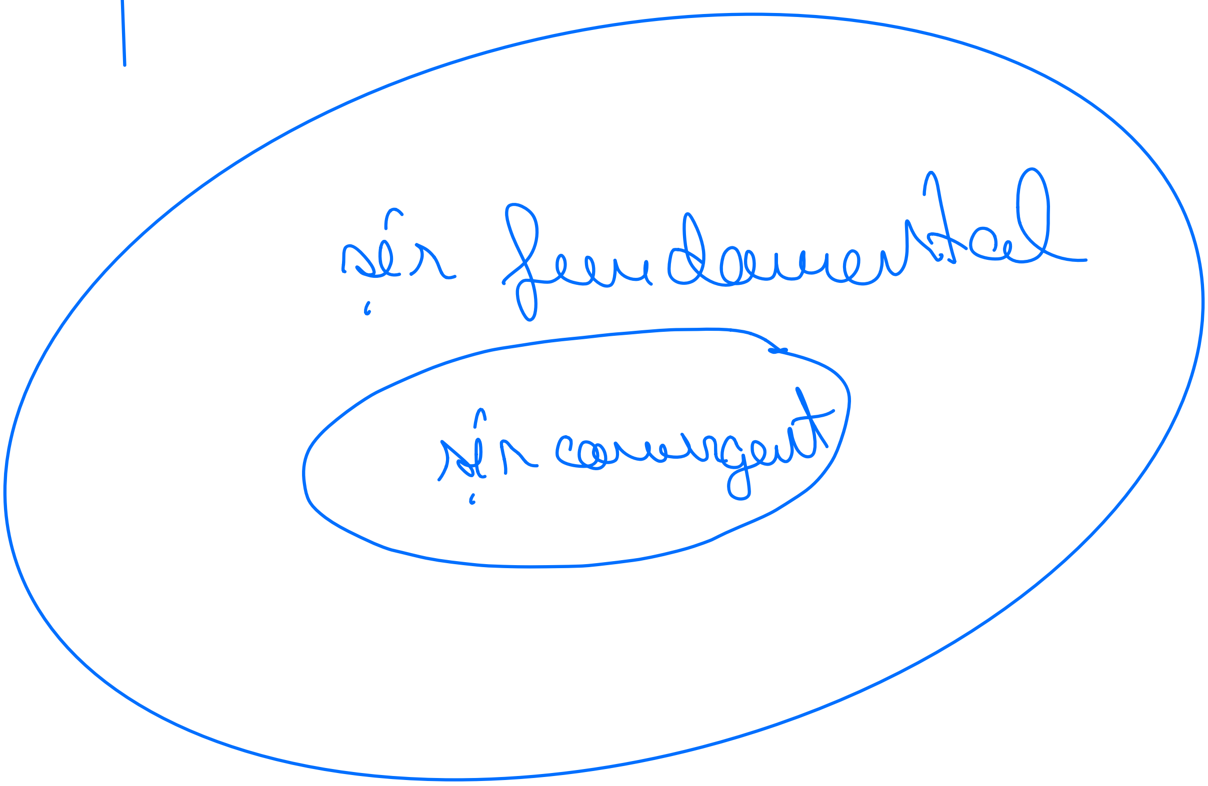


Pt $\varepsilon = 1.5$ $n_\varepsilon = 2$

$\varepsilon = 0.5$ $n_\varepsilon = 4$



$\mathbb{Q}$



$\mathbb{Q}$

$\sqrt{2}$

$\sqrt{2} \approx 1.414$

$x_1 = 1$

$x_2 = 1.4$

$x_3 = 1.41$

$\vdots$

ex:

$$\frac{1}{n} \longrightarrow 0$$

$$\left| x_{n+p} - x_n \right| = \left| \overset{n}{\frac{1}{n+p}} - \overset{n+p}{\frac{1}{n}} \right| < \frac{1}{n} \xrightarrow{n \to \infty} 0$$

$$\frac{1}{2} \quad \frac{1}{3} \quad \frac{1}{4} \quad \frac{1}{5} \quad \frac{1}{6} \ldots \longrightarrow 0$$

$$\frac{1}{6} \quad \frac{1}{12}$$

$$\left(1+\frac{1}{n}\right)^n \longrightarrow e \approx 2.71$$

$$x_1 = \left(1+\frac{1}{1}\right)^1 = 2$$

$$x_2 = \left(1+\frac{1}{2}\right)^2 = \left(\frac{3}{2}\right)^2 = \frac{9}{4} =$$

$$2.25$$

$$x_3 = \left(1+\frac{1}{3}\right)^3 = \left(\frac{4}{3}\right)^3 = \frac{64}{27} =$$

$$2.37$$

$$x_4 = \left(\frac{5}{4}\right)^4 = 2.44$$

$$\vdots$$

$$x_{n+1} - x_n < 0.001$$

$$x_n = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{n^2}$$

+ şir Cauchy

+

Fie $p \in \mathbb{N}^*$

$$\underline{|x_{n+p} - x_n|} = \left| \cancel{1} + \cancel{\frac{1}{2^2}} + \cancel{\frac{1}{3^2}} + \dots + \cancel{\frac{1}{n^2}} + \frac{1}{(n+1)^2} + \dots + \frac{1}{(n+p)^2} - \left(\cancel{1} + \cancel{\frac{1}{2^2}} + \cancel{\frac{1}{3^2}} + \dots + \cancel{\frac{1}{n^2}} \right) \right| =$$

$$\frac{1}{(n+1)^2} + \dots + \frac{1}{(n+p)^2} <$$

$$\frac{1}{n(n+1)} + \dots + \frac{1}{(n+p-1)(n+p)}$$

$$\frac{n+1-n}{n(n+1)} + \dots + \frac{n+p-(n+p-1)}{(n+p-1)(n+p)} =$$

$$\frac{\cancel{n+1}}{\cancel{n(n+1)}} - \cancel{\frac{n}{n(n+1)}} + \dots + \cancel{\frac{n+p}{(n+p-1)(n+p)}} - \cancel{\frac{n+p-1}{(n+p-1)(n+p)}} =$$

$$\frac{1}{n} - \frac{1}{n+p} < \underline{\frac{1}{n}} \to 0$$

Mulțimea pct limită

$x_n = (-1)^n$ $\quad x_{2k+1} = -1$

$x_{2k} = 1$

$x_1 = -1 \quad x_2 = 1 \quad x_3 = -1$

$L(x_n) = \{-1, 1\}$

S2.5 Să se determine $L(x_n)$ pentru fiecare din şirurile cu termenul general x_n, unde:

a) $x_n = \dfrac{(-1)^n}{1+\frac{1}{n}+e^{\frac{1}{n}}}, n \in \mathbb{N}^*$; b) $x_n = 2+(-1)^n + \sin\dfrac{n\pi}{2}, n \in \mathbb{N}$; c) $x_n = \dfrac{(-1)^n + n \operatorname{tg}\frac{n\pi}{4}}{n}, n \in \mathbb{N}^*$.

S2.6 Să se calculeze următoarele limite

$$\text{a)}\ \lim_{n\to\infty} \sqrt[n]{\frac{(3n)!}{(n!)^3}}; \quad \text{b)}\ \lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n}; \quad \text{c)}\ \lim_{n\to\infty} \frac{\sqrt[n]{(n+1)(n+2)\dots(n+n)}}{n}.$$

S2.7 Să se calculeze următoarele limite

a) $\lim\limits_{n\to\infty} \dfrac{1+\sqrt{2}+\sqrt[3]{3}+\ldots+\sqrt[n]{n}}{n}$; b) $\lim\limits_{n\to\infty} \sqrt[n]{\sin\dfrac{\pi}{2}\cdot\sin\dfrac{\pi}{3}\cdot\ldots\cdot\sin\dfrac{\pi}{n}}$; c) $\lim\limits_{n\to\infty} \left(1+\dfrac{(-1)^n}{n}\right)^{\frac{1}{\sin\left(\pi\sqrt{1+n^2}\right)}}$.

S2.8* Să se arate că şirul cu termenul general

$$x_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} - \ln n,\ \forall\, n \in \mathbb{N}^*$$

este convergent în $\mathbb{R}$ (limita sa fiind aşa numita constantă a lui Euler, $c = 0,577215... \in \mathbb{R} \setminus \mathbb{Q}$).

S2.9 Să se găsească $L(x_n)$ pentru şirul $(x_n)_{n\in\mathbb{N}^*} \subset \mathbb{R}$, unde

$$x_n = [1 + (-1)^n] \cdot n^{(-1)^n} + \cos\frac{n\pi}{6},\ \forall\, n \in \mathbb{N}^*.$$

S2.10 Se consideră polinomul de gradul al doilea $f \in \mathbb{R}[X]$, astfel încât $f(1) = 4,\ f(-1) = 7,\ f(2) = 12$.

a) Să se determine forma algebrică a polinomului f.

b) Să se determine restul împărţirii polinomului f la $X + 3$.

a) $f(x) = ax^2 + bx + c$

$$\begin{cases} a\cdot 1^2 + b\cdot 1 + c = 4 \iff a+b+c=4 \\ a\cdot(-1)^2 + b\cdot(-1) + c = 7 \iff a-b+c=7 \\ a\cdot 2^2 + b\cdot 2 + c = 12 \iff 4a+2b+c=12 \end{cases}$$

$2b = -3 \Rightarrow b = -\frac{3}{2}$

$\Rightarrow a + c = 4 + \frac{3}{2} = \frac{11}{2}$

$\Rightarrow 4a + c = 12 + 3 = 15$ $\quad (-)$

$3a = 15 - \frac{11}{2} = \frac{19}{2} \Rightarrow a = \frac{19}{6}$

$c = \frac{11}{2} - \frac{19}{6} = \frac{14}{6} = \frac{7}{3}$

$f(x) = \frac{19}{6}x^2 + \left(-\frac{3}{2}\right)x + \frac{7}{3}$

f la $\underline{X+3} = X - (-3)$

	$\frac{19}{6}$	$-\frac{3}{2}$	$\frac{7}{3}$
-3	$\frac{19}{6}$	$\underbrace{\frac{19}{\cancel{6}_2}\cdot(-\cancel{3}) - \frac{3}{2}}$ $= -\frac{19}{2} - \frac{3}{2} = -11$	$(-3)\cdot(-11) + \frac{7}{3} = 33 + \frac{7}{3} = \frac{106}{3}$

TIR

$$f = (x+3)\,g + r$$

$$f(-3) = (-3+3)\cdot g(-3) + r \Rightarrow$$

$$r = f(-3)$$

$$f(-3) = \frac{19}{6}\cdot(-3)^2 + -\frac{3}{2}\cdot(-3) + \frac{7}{3} = (-3)\Big(\underbrace{\frac{19}{6}\cdot(-3) - \frac{3}{2}}\Big) + \frac{7}{3} = \frac{106}{3}$$

S2.11 Se consideră polinoamele $f, g \in \mathbb{R}[X]$, cu $f = X^3 - 3X + a$ şi $g = X^2 - 3X + 2$, unde $a \in \mathbb{R}$.

a) Pentru $a = 2$, rezolvaţi în $\mathbb{R}$ ecuaţia $f(X) = g(X)$.

b) Să se determine valorile parametrului $a \in \mathbb{R}$ ştiind că polinomul f admite o rădăcină dublă pozitivă. Care sunt rădăcinile reale ale lui f în acest caz?

a) $X^3 - \cancel{3X} + \cancel{2} = X^2 - \cancel{3X} + \cancel{2} \Rightarrow$

$X^3 - X^2 = 0 \Rightarrow X^2(X-1) = 0$

Soluțiile sunt 0 cu ordin 2 (de multiplicitate)

1 cu ordin 1

b) Fie b rădăcina dublă

$f(b) = 0 \Rightarrow b^3 - 3b + a = 0$

$f'(b) = 0 \Rightarrow 3b^2 - 3 = 0 \Rightarrow$

$3(b^2 - 1) = 0 \Rightarrow$

$f'(X) = 3X^2 - 3$

$\left.\begin{array}{l} b \in \{\pm 1\} \\ b > 0 \end{array}\right\} \Rightarrow b = 1$

$\Rightarrow$

$1 - 3 + a = 0 \Rightarrow a = 2$

$f = X^3 - 3X + 2$ $\quad f = (X-1)^2(X+2)$

1 cu ordin 2

-2 cu ordin 1

S2.12 Se consideră polinomul $f \in \mathbb{C}[X]$, $f(X) = X^4 - 6X^3 + 18X^2 - 30X + 25$.

a) Să se demonstreze că polinomul f se divide cu $\boxed{X^2 - 2X + 5}$.

b) Să se arate că polinomul f nu are rădăcini reale.

c) Să se arate că toate rădăcinile polinomului f au acelaşi modul.

$(X^2 - 2X + 5)(\ldots$

a) $X^4 - 6X^3 + 18X^2 - 30X + 25 =$

$X^4 - 2X^3 + 5X^2 - 4X^3 + 8X^2 - 20X +$

$5X^2 - 10X + 25 =$

$\underbrace{(X^2 - 2X + 5)}\underbrace{(X^2 - 4X + 5)}$

b) rădăcinile lui f = rădăcinile lui $X^2 - 2X + 5$

$\cup$ rădăcinile lui $X^2 - 4X + 5$

$X^2 - 2X + 5 = 0 \longrightarrow x_{1,2} = \dfrac{-(-2) \pm \sqrt{(-2)^2 - 4 \cdot 5}}{2}$

$= \dfrac{2 \pm \sqrt{4 - 20}}{2} =$

$\dfrac{2 \pm \sqrt{-16}}{2}$ — $\dfrac{2+4i}{2} = 1+2i$; $\dfrac{2-4i}{2} = 1-2i$

$ax^2 + bx + c = 0$

$x_{1,2} = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$f(x) = a_m X^m + \dots + a_1 X + a_0 \in \mathbb{R}[X]$$

și $\alpha \in \mathbb{C} \setminus \mathbb{R}$ a.î. $f(\alpha) = 0$

$$f(\bar{\alpha}) = 0$$

$$x^2 - 4x + 5 = 0 \longrightarrow x_{1,2} = \frac{-(-4) \pm \sqrt{(-4)^2 - 4 \cdot 5}}{2}$$

$$\frac{4 \pm \sqrt{-4}}{2} \begin{cases} 2+i \\ 2-i \end{cases}$$

c) $|a \pm bi| = \sqrt{a^2 + b^2}$

$|1 \pm 2i| = \sqrt{1^2 + 2^2} = \sqrt{5}$

$|2 + i| = \sqrt{2^2 + 1^2} = \sqrt{5}$

S2.13 Resturile împărţirii polinomului $f \in \mathbb{R}[X]$ la binoamele $X+2$, $X+4$, $X-2$ sunt respectiv $38, 112$ şi 10.

Să se afle restul împărţirii polinomului f la $(X^2-4)(X+4)$.

S2.14 Să se determine polinomul $f \in \mathbb{R}[X]$ care satisface relaţia $2f(X) = Xf(X) - 2X^3 + 10X^2 - 16X + 8$.

S2.15 Câturile împărţirii polinomului $f \in \mathbb{R}[X]$ la $X-a$ şi $X-b$ sunt respectiv X^2-3X+4 şi X^2-4X+2. Determinaţi valorile parametrilor $a, b \in \mathbb{R}$ şi polinomul f, ştiind că termenul liber al polinomului este 1.

S2.16 Determinaţi valorile parametrilor reali a, b ştiind că polinomul $f = aX^4 + bX^3 - 3$ este divizibil cu $(X-1)^2$.

S2.17 Se consideră polinomul $f \in \mathbb{C}$, $f = (X+i)^{2020} + (X-i)^{2020}$, care are forma algebrică

$$f = a_{2020}X^{2020} + a_{2019}X^{2019} + \ldots + a_1X + a_0.$$

a) Să se calculeze $a_{2020} + a_{2019}$.

b) Să se determine restul împărţirii polinomului f la $X^2 - 1$.

c) Să se demonstreze că polinomul f are toate rădăcinile reale.

Bibliografie selectivă

1. Anca Precupanu - *Bazele analizei matematice*, (§1.5), ediţia a III-a, Ed. Polirom, Iaşi, 1998.
2. A. Croitoru, M. Durea, C. Văideanu, *Analiza matematică. Probleme*, Ed. Tehnopress, Iaşi, 2015.
3. E. Popescu - *Analiză matematică. Calcul diferenţial*, Ed. Matrix Rom, Bucureşti, 2006.
4. V. Postolică - *Eficienţă prin matematica aplicată*, Ed. Matrix Rom, Bucureşti, 2006.
5. C. Drăguşin, O. Olteanu, M. Gavrilă - *Analiză matematică. Probleme (Vol. I)*, Ed. Matrix Rom, Bucureşti, 2006.
6. S. Chiriţă - *Probleme de matematici superioare,* Editura Didactică şi Pedagogică, Bucureşti, 1989.