Descompuneri QR

$$60x_1 - 72x_2 + 43x_3 = 60$$
$$80x_1 + 154x_2 - 99x_3 = 80$$
$$75x_1 + 310x_2 + 140x_3 = 75$$

$$A = \begin{pmatrix} 60 & -72 & -43 \\ 80 & 154 & -99 \\ 75 & 310 & 140 \end{pmatrix} \qquad b = \begin{pmatrix} 60 \\ 80 \\ 75 \end{pmatrix}$$

Algoritmul Givens

Pasul 1 $\quad a_{21} = 80 \rightarrow 0 \qquad R_{12} = \begin{pmatrix} c & s & 0 \\ -s & c & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$c = \frac{60}{\sqrt{60^2+80^2}} = \frac{3}{5} \qquad s = \frac{80}{\sqrt{60^2+80^2}} = \frac{4}{5}$$

$$R_{12} * A = \begin{pmatrix} \frac{3}{5} & \frac{4}{5} & 0 \\ -\frac{4}{5} & \frac{3}{5} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 60 & -72 & -43 \\ 80 & 154 & -99 \\ 75 & 310 & 140 \end{pmatrix} = \begin{pmatrix} 100 & 80 & -105 \\ 0 & 150 & -25 \\ 75 & 310 & 140 \end{pmatrix}$$

$$u_{31} = 75 \rightarrow 0$$

$$R_{13} = \begin{bmatrix} c & 0 & s \\ 0 & 1 & 0 \\ -s & 0 & c \end{bmatrix}$$

$$c = \frac{100}{\sqrt{100^2+75^2}} = \frac{4}{5} \qquad s = \frac{75}{\sqrt{100^2+75^2}} = \frac{3}{5}$$

$$R_{13}(R_{12}A) = \begin{pmatrix} \frac{4}{5} & 0 & \frac{3}{5} \\ 0 & 1 & 0 \\ -\frac{3}{5} & 0 & \frac{4}{5} \end{pmatrix} \begin{pmatrix} 100 & 80 & -105 \\ 0 & 150 & -25 \\ 75 & 310 & 140 \end{pmatrix}$$

$$= \begin{pmatrix} 125 & 250 & 0 \\ 0 & 150 & -25 \\ 0 & 200 & 175 \end{pmatrix}$$

$$R_{12} * b = \begin{pmatrix} \frac{3}{5} & \frac{4}{5} & 0 \\ -\frac{4}{5} & \frac{3}{5} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 60 \\ 80 \\ 75 \end{pmatrix} = \begin{pmatrix} 100 \\ 0 \\ 75 \end{pmatrix}$$

$$R_{13} * (R_{12} b) = \begin{pmatrix} 4/5 & 0 & 3/5 \\ 0 & 1 & 0 \\ -3/5 & 0 & 4/5 \end{pmatrix} \begin{pmatrix} 100 \\ 0 \\ 75 \end{pmatrix} = \begin{pmatrix} 125 \\ 0 \\ 0 \end{pmatrix}$$

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$$Q^T = R_{13}\, R_{12} = \begin{pmatrix} 4/5 & 0 & 3/5 \\ 0 & 1 & 0 \\ -3/5 & 0 & 4/5 \end{pmatrix} \begin{pmatrix} 3/5 & 4/5 & 0 \\ -4/5 & 3/5 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{bmatrix} \frac{12}{25} & \frac{16}{25} & \frac{3}{5} \\ -4/5 & 3/5 & 0 \\ -9/25 & -12/25 & 4/5 \end{bmatrix}$$

<u>Pasul 2</u> : $a_{32} = 200 \longrightarrow 0 \qquad R_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c & s \\ 0 & -s & c \end{bmatrix}$

$$c = \frac{150}{\sqrt{150^2+200^2}} = \frac{3}{5} \qquad s = \frac{200}{\sqrt{150^2+200^2}} = \frac{4}{5}$$

$$R_{23}(R_{13}\, R_{12} A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3/5 & 4/5 \\ 0 & -4/5 & 3/5 \end{bmatrix} \begin{bmatrix} 125 & 250 & 0 \\ 0 & 150 & -25 \\ 0 & 200 & 175 \end{bmatrix} =$$

$$= \begin{bmatrix} 125 & 250 & 0 \\ 0 & 250 & 125 \\ 0 & 0 & 125 \end{bmatrix} = R$$

$$Q^T = R_{23} R_{13} R_{12} =$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 3/5 & 4/5 \\ 0 & -4/5 & 3/5 \end{pmatrix} \begin{pmatrix} 12/25 & 16/25 & 3/5 \\ -4/5 & 3/5 & 0 \\ -9/25 & -12/25 & 4/5 \end{pmatrix} =$$

$$= \begin{pmatrix} 0,48 & 0,64 & 0,6 \\ -0,768 & -0,024 & 0,64 \\ 0,424 & -0,768 & 0,48 \end{pmatrix}$$

$$R_{23}(R_{13} R_{12} b) = Q^T b = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3/5 & 4/5 \\ 0 & -4/5 & 3/5 \end{pmatrix} \begin{pmatrix} 125 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 125 \\ 0 \\ 0 \end{pmatrix}$$

Soluția:

$$\begin{aligned} 125x_1 + 250x_2 &= 125 \\ 250x_2 + 125x_3 &= 0 \\ 125x_3 &= 0 \end{aligned}$$

$x_3 = 0,\ x_2 = 0,\ x_1 = 1$

$$\underbrace{\begin{pmatrix} 60 & -72 & -43 \\ 80 & 154 & -99 \\ 75 & 310 & 140 \end{pmatrix}}_{A} = \underbrace{\begin{pmatrix} 0,48 & -0,768 & 0,424 \\ 0,64 & -0,024 & -0,768 \\ 0,6 & 0,64 & 0,48 \end{pmatrix}}_{Q} \underbrace{\begin{pmatrix} 125 & 250 & 0 \\ 0 & 250 & 125 \\ 0 & 0 & 125 \end{pmatrix}}_{R}$$

Algoritmul Householder

Pas r: $\quad A = P_r \cdot A$

$$P_r = I - \frac{1}{\beta} u u^T$$

$$\sigma = a_{rr}^2 + a_{r+1r}^2 + \dots + a_{nr}^2$$

$$k = -\,\text{semn}(a_{rr})\sqrt{\sigma}$$

$$\beta = \sigma - k \cdot a_{rr}$$

$$u = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ a_{rr} - k \\ a_{r+1r} \\ \vdots \\ a_{nr} \end{pmatrix}$$

$$A = \begin{pmatrix} 60 & -72 & -43 \\ 80 & 154 & -99 \\ 75 & 310 & 140 \end{pmatrix}, \quad b = \begin{pmatrix} 60 \\ 80 \\ 75 \end{pmatrix}$$

Pasul 1 $\quad \sigma = 60^2 + 80^2 + 75^2 = 15625$

$k = -\text{semn}(a_{11} = 60)\sqrt{\sigma} = -125$

$\beta = \sigma - k * a_{11} = 23125$

$$u = \begin{pmatrix} 185 \\ 80 \\ 75 \end{pmatrix} \Rightarrow P_1 = \begin{pmatrix} -0,48 & -0,64 & -0,6 \\ -0,64 & 0,7232 & -0,2595 \\ -0,6 & -0,2595 & 0,7568 \end{pmatrix}$$

$$P_1 * A = \begin{pmatrix} -125 & -250 & 0 \\ 0 & 77,027 & -80,4054 \\ 0 & 237,8378 & 157,4324 \end{pmatrix}$$

$Q^T = P_1$

$$P_1 * b = P_1 * \begin{pmatrix} 60 \\ 80 \\ 75 \end{pmatrix} = \begin{pmatrix} -125 \\ 0 \\ 0 \end{pmatrix}$$

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Pasul 2

$$\sigma = (77,027)^2 + (237,8378)^2$$
$$= 62500$$

$$k = -\text{semn}(a_{22} = 77,027)\sqrt{\sigma} = -250$$

$$\beta = \sigma - k * a_{22} = 81757$$

$$u = \begin{pmatrix} 0 \\ 327,027 \\ 237,8378 \end{pmatrix}$$

$$P_2 = I - \frac{1}{\beta} u u^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -0,3081 & -0,9514 \\ 0 & -0,9514 & 0,3081 \end{pmatrix}$$

$$P_2 * (P_1 * A) = R = \begin{pmatrix} -125 & -250 & 0 \\ 0 & -250 & -125 \\ 0 & 0 & 125 \end{pmatrix}$$

$$Q^T = P_2 * P_1 = \begin{pmatrix} -0,48 & -0,64 & -0,6 \\ 0,768 & 0,024 & -0,64 \\ 0,424 & -0,768 & 0,48 \end{pmatrix}$$

$$Q^T b = P_2 (P_1 b) = \begin{pmatrix} -125 \\ 0 \\ 0 \end{pmatrix}$$

Soluția

$$\begin{aligned} -125x_1 - 250x_2 &= -125 \\ -250x_2 - 125x_3 &= 0 \\ 125x_3 &= 0 \end{aligned}$$

$\Rightarrow x_3 = 0,\ x_2 = 0,\ x_1 = 1$

$$\underbrace{\begin{pmatrix} 60 & -72 & -43 \\ 80 & 154 & -99 \\ 75 & 310 & 140 \end{pmatrix}}_{A} = \underbrace{\begin{pmatrix} -0,48 & 0,768 & 0,424 \\ -0,64 & 0,024 & -0,768 \\ -0,6 & -0,64 & 0,48 \end{pmatrix}}_{Q} * \underbrace{\begin{pmatrix} -125 & -250 & 0 \\ 0 & -250 & -125 \\ 0 & 0 & 125 \end{pmatrix}}_{R}$$

Algoritmul Gram-Schmidt

$$A = \begin{pmatrix} 60 & -72 & -43 \\ 80 & 154 & -99 \\ 75 & 310 & 140 \end{pmatrix} = \begin{pmatrix} q^1 & q^2 & q^3 \end{pmatrix} \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ 0 & r_{22} & r_{23} \\ 0 & 0 & r_{33} \end{pmatrix}$$

$$a^1 \quad a^2 \quad a^3$$

$$a^1 = \begin{pmatrix} 60 \\ 80 \\ 75 \end{pmatrix} = r_{11} q^1$$

$$a^2 = \begin{pmatrix} -72 \\ 154 \\ 310 \end{pmatrix} = r_{12} q^1 + r_{22} q^2$$

$$a^3 = \begin{pmatrix} 43 \\ -99 \\ 140 \end{pmatrix} = r_{13} q^1 + r_{23} q^2 + r_{33} q^3$$

$$\|q^1\|_2 = \|q^2\|_2 = \|q^3\|_2 = 1$$

$$(q^1, q^2)_{\mathbb{R}^3} = (q^1, q^3)_{\mathbb{R}^3} = (q^2, q^3)_{\mathbb{R}^3} = 0$$

Pasul 1 : r_{11}, q^1

$a^1 = r_{11} q^1 \rightarrow \|a^1\|_2 = |r_{11}| \|q^1\|_2 = |r_{11}|$

$\|a^1\|_2 = \sqrt{60^2 + 80^2 + 75^2} = 125$

$r_{11} = \pm 125$ considerăm doar + de acum înainte

$r_{11} = 125$

$$q^1 = \frac{1}{r_{11}} a^1 = \frac{1}{125} \begin{pmatrix} 60 \\ 80 \\ 75 \end{pmatrix} = \begin{pmatrix} 0,48 \\ 0,64 \\ 0,6 \end{pmatrix}$$

Pasul 2 : r_{12}, r_{22}, q^2

$$(a^2, q^1)_{\mathbb{R}^3} = r_{12} \underbrace{(q^1, q^1)}_{= \|q^1\|_2^2 = 1} + r_{22} \underbrace{(q^2, q^1)}_{=0}$$

$$\Rightarrow r_{12} = (a^2, q^1)_{\mathbb{R}^3} = \left(\begin{pmatrix} -72 \\ 80 \\ 75 \end{pmatrix}, \frac{1}{125} \begin{pmatrix} 60 \\ 80 \\ 75 \end{pmatrix} \right) =$$

$= 250$

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$$r_{22}\underline{q}^2 = a^2 - r_{12}\underline{q}^1 = \begin{pmatrix} -72 \\ 154 \\ 310 \end{pmatrix} - 250 \begin{pmatrix} 0,48 \\ 0,64 \\ 0,6 \end{pmatrix} =$$

$$= \begin{pmatrix} -192 \\ -6 \\ 160 \end{pmatrix}$$

$$\Rightarrow \quad |r_{22}| \, \|\underline{q}^2\|_2 = \left\| \begin{pmatrix} -192 \\ -6 \\ 160 \end{pmatrix} \right\|_2 =$$

$$= \sqrt{(-192)^2 + (-6)^2 + 160^2} = 250$$

$$r_{22} = 250$$

$$\underline{q}^2 = \frac{1}{250} \begin{pmatrix} -192 \\ -6 \\ 160 \end{pmatrix} = \begin{pmatrix} -0,768 \\ -0,024 \\ 0,64 \end{pmatrix}$$

$$= \frac{1}{r_{22}} \begin{pmatrix} -192 \\ -6 \\ 160 \end{pmatrix}$$

$$r_{22} = 0 \Rightarrow$$

$$\det A = 0$$

Pasul 3 $\quad r_{13}, r_{23}, r_{33}, q^3$

$$a^3 = \begin{pmatrix} 43 \\ -99 \\ 140 \end{pmatrix} = r_{13} \cdot \begin{pmatrix} 0,48 \\ 0,64 \\ 0,6 \end{pmatrix} + r_{23} \begin{pmatrix} -0,768 \\ -0,024 \\ 0,64 \end{pmatrix} + $$

$$+ r_{33} q^3$$

$$(a^3, q^1)_{\mathbb{R}^3} = r_{13} \underbrace{(q^1, q^1)}_{=1} + r_{23} \underbrace{(q^2, q^1)}_{=0} +$$

$$+ r_{33} \underbrace{(q^3, q^1)}_{=0} = r_{13}$$

$$\Rightarrow r_{13} = \left(\begin{pmatrix} 43 \\ -99 \\ 140 \end{pmatrix}, \begin{pmatrix} 0,48 \\ 0,64 \\ 0,6 \end{pmatrix} \right)_{\mathbb{R}^3} = 0$$

$$(a^3, q^2) = r_{13} \underbrace{(q^1, q^2)}_{=0} + r_{23} \underbrace{(q^2, q^2)}_{=1} + r_{33} \underbrace{(q^3, q^2)}_{=0}$$

$$\Rightarrow r_{23} = \left(\begin{pmatrix} 43 \\ -99 \\ 140 \end{pmatrix}, \begin{pmatrix} -0,768 \\ -0,024 \\ 0,64 \end{pmatrix} \right)_{\mathbb{R}^3} = 125$$

$$r_{33} q^3 = a^3 - r_{13} q^1 - r_{23} q^2 =$$

$$\begin{pmatrix} -43 \\ -99 \\ 140 \end{pmatrix} - 0 \cdot \begin{pmatrix} 0{,}48 \\ 0{,}64 \\ 0{,}6 \end{pmatrix} - 125 \begin{pmatrix} -0{,}768 \\ -0{,}024 \\ 0{,}64 \end{pmatrix} =$$

$$= \begin{pmatrix} 53 \\ -96 \\ 60 \end{pmatrix}$$

$$|r_{33}| \underbrace{\|q^3\|_2}_{=1} = \left\| \begin{pmatrix} 53 \\ -96 \\ 60 \end{pmatrix} \right\|_2 = \sqrt{53^2 + (-96)^2 + 60^2} = 125$$

$$\rightarrow \quad r_{33} = 125$$

$$q^3 = \frac{1}{125} \begin{pmatrix} 53 \\ -96 \\ 60 \end{pmatrix} = \begin{pmatrix} 0{,}424 \\ -0{,}768 \\ 0{,}48 \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} 60 & -72 & -43 \\ 80 & 154 & -99 \\ 75 & 310 & 140 \end{pmatrix}}_{A} = \underbrace{\begin{pmatrix} 0{,}48 & -0{,}768 & 0{,}424 \\ 0{,}64 & -0{,}024 & -0{,}768 \\ 0{,}6 & 0{,}64 & 0{,}48 \end{pmatrix}}_{Q} * \underbrace{\begin{pmatrix} 125 & 250 & 0 \\ 0 & 250 & 125 \\ 0 & 0 & 125 \end{pmatrix}}_{R}$$