Outline of the lecture

The set of real numbers

What is $\mathbb{R}$?

Definition

A *set of real numbers* is a *Dedekind-complete ordered field*, *i.e.* a quadruplet $(\mathbb{R}, +, \cdot, \leq)$ where $\mathbb{R}$ is a set with at least two elements, $+$ (*addition*) and $\cdot$ (*multiplication*) are two *algebraic operations* on $\mathbb{R}$ and $\leq$ is a *relation* (*total order*) on $\mathbb{R}$ such that:

(F_1) $x + (y + z) = (x + y) + z, \ \forall x, y, z \in \mathbb{R}$;

(F_2) $\exists 0 \in \mathbb{R}, \ \forall x \in \mathbb{R} : x + 0 = 0 + x = x$;

(F_3) $\forall x \in \mathbb{R}, \ \exists (-x) \in \mathbb{R} : x + (-x) = (-x) + x = 0$;

(F_4) $x + y = y + x, \ \forall x, y \in \mathbb{R}$;

(F_5) $(x \cdot y) \cdot z = x \cdot (y \cdot z), \ \forall x, y, z \in \mathbb{R}$;

(F_6) $\exists 1 \in R, \ \forall x \in \mathbb{R} : x \cdot 1 = 1 \cdot x = x$;

(F_7) $\forall x \in \mathbb{R}^*, \ \exists x^{-1} \in \mathbb{R} : x \cdot x^{-1} = x^{-1} \cdot x = 1$;

(F_8) $x \cdot y = y \cdot x, \ \forall x, y \in \mathbb{R}$;

(F_9) $x \cdot (y + z) = x \cdot y + x \cdot z, \ \forall x, y, z \in \mathbb{R}$;

Definition (continuation)

(O_1) $x \leq x, \ \forall x \in \mathbb{R}$;

(O_2) $(x \leq y) \wedge (y \leq x) \Rightarrow x = y, \ \forall x, y \in \mathbb{R}$

(O_3) $(x \leq y) \wedge (y \leq z) \Rightarrow x \leq z, \ \forall x, y, z \in \mathbb{R}$

(O_4) $(x \leq y) \vee (y \leq x), \ \forall x, y \in \mathbb{R}$

(O_5) $x \leq y \Rightarrow x + z \leq y + z, \ \forall x, y, z \in \mathbb{R}$;

(O_6) $(x \leq y) \wedge (0 \leq z) \Rightarrow x \cdot z \leq y \cdot z, \ \forall x, y, z \in \mathbb{R}$;

(C) the ordered set $(\mathbb{R}, \leq)$ is *Dedekind-complete*, i.e. every non-empty and *upper bounded* set $A \subseteq \mathbb{R}$ admits a sup.

For $x, y \in \mathbb{R}$, we can define two auxiliary operations:

- *substraction*: $x - y := x + (-y), \ x, y \in \mathbb{R}$;
- *division*: $\frac{x}{y} = x/y := x \cdot (y^{-1}), \ x \in \mathbb{R}, \ y \in \mathbb{R}^*$.

Absolute value

Also, the *absolute value* of a number x is defined as

$$|x| := \begin{cases} x, & x \geq 0; \\ -x, & x < 0. \end{cases}$$

Proposition

We have:

i) $|x| \geq 0, \ \forall x \in \mathbb{R};$

ii) $|x| = 0 \Leftrightarrow x = 0, \ \forall x \in \mathbb{R};$

iii) $|xy| = |x| \cdot |y|, \ \forall x, y \in \mathbb{R};$

iv) $|x + y| \leq |x| + |y|, \ \forall x, y \in \mathbb{R}.$

Other sets of numbers

There exists a unique (up to a *homeomorphism* of ordered fields) set of real numbers, so we will call $\mathbb{R}$ *the* set of real numbers.
In fact, one can construct $\mathbb{R}$ starting with $\mathbb{N}$, then continuing with $\mathbb{Z}$, $\mathbb{Q}$.
Reciprocally, if we already have set $\mathbb{R}$, we can define

$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$$

as follows:

- $\mathbb{N} := \bigcap \{N \in \mathscr{P}(\mathbb{R}) \mid 0 \in N,\ n \in N \Rightarrow n+1 \in N,\ \forall n \in N\}$
  $= \{0, 1, 1+1, (1+1)+1, \ldots\} = \{0, 1, 2, 3, \ldots\}$;
- $\mathbb{Z} := \mathbb{N} \cup \{-n \mid n \in \mathbb{N}\}$;
- $\mathbb{Q} := \{m/n \mid m \in \mathbb{Z},\ n \in \mathbb{N}^*\}$.

Supremum and infimum

Concerning the *least upper bound* (sup) and *greatest lower bound* (inf) of a non-empty subset of $\mathbb{R}$, they can be characterized as follows:

Proposition

Let A be a non-empty subset of $\mathbb{R}$.

1. *An element $\alpha \in \mathbb{R}$ is the supremum of A if and only if:*
   - *i)* $x \leq \alpha, \ \forall x \in A$;
   - *ii)* $\forall \varepsilon > 0, \ \exists x_\varepsilon \in A : \alpha - \varepsilon < x_\varepsilon$.
2. *An element $\beta \in \mathbb{R}$ is the infimum of A if and only if:*
   - *i)* $x \geq \beta, \ \forall x \in A$;
   - *ii)* $\forall \varepsilon > 0, \ \exists x_\varepsilon \in A : \beta + \varepsilon > x_\varepsilon$.

It is now easy to check that, for $a, b \in \mathbb{R}$ with $a < b$,

- $\inf[a, b] = \inf[a, b) = \inf(a, b] = \inf(a, b) = a$;
- $\sup[a, b] = \sup[a, b) = \sup(a, b] = \sup(a, b) = b$.

Sequences of real numbers

- A *sequence* of real numbers is a function $x : \mathbb{N} \to \mathbb{R}$.
- We denote x_n instead of $x(n)$, for $n \in \mathbb{N}$.
- We denote $(x_n)_{n\in\mathbb{N}}$, $(x_n)_{n\geq 0}$ or (x_n) instead of the function x.
- Sometimes, we denote $(x_n)_{n\geq p}$ for a function $x : \{n \in \mathbb{N} \mid n \geq p\} \to \mathbb{R}$ or $(x_{m+p})_{m\geq 0}$.
- By $\{x_n\}_{n\in\mathbb{N}}$ we denote the set $\{x_n \mid n \in \mathbb{N}\}$.
- If $A \subseteq \mathbb{R}$, $(x_n)_{n\in\mathbb{N}} \subseteq A$ means (abuse of language) $\{x_n\}_{n\in\mathbb{N}} \subseteq A$, *i.e.*

$$x_n \in A, \ \forall n \in \mathbb{N}.$$

Definition

We say that a sequence $(x_n)_{n\in\mathbb{N}}$ is:

- *upper bounded* if $\{x_n\}_{n\in\mathbb{N}}$ is upper bounded, *i.e.*

$$\exists M \in \mathbb{R},\ \forall n \in \mathbb{N} : x_n \le M;$$

- *lower bounded* if $\{x_n\}_{n\in\mathbb{N}}$ is lower bounded, *i.e.*

$$\exists m \in \mathbb{R},\ \forall n \in \mathbb{N} : x_n \ge m;$$

- *bounded* if $\{x_n\}_{n\in\mathbb{N}}$ is bounded, *i.e.*

$$\exists m, M \in \mathbb{R},\ \forall n \in \mathbb{N} : m \le x_n \le M;$$

- *unbounded* if $\{x_n\}_{n\in\mathbb{N}}$ is not bounded.

Examples:

- $((-1)^n)_{n\ge1}$ is bounded (since $\{(-1)^n\}_{n\ge1} = \{-1, 1\}$);
- $(2^n)_{n\in\mathbb{N}}$ is not bounded (it is lower bounded, but not upper bounded).

Definition

We say that a sequence $(x_n)_{n\in\mathbb{N}}$ is:

- *increasing* if $x_n \leq x_{n+1}$, $\forall n \in \mathbb{N}$;
- *decreasing* if $x_n \geq x_{n+1}$, $\forall n \in \mathbb{N}$;
- *monotone* if it is increasing or decreasing;
- *strictly increasing* if $x_n < x_{n+1}$, $\forall n \in \mathbb{N}$;
- *strictly decreasing* if $x_n > x_{n+1}$, $\forall n \in \mathbb{N}$;
- *strictly monotone* if it is strictly increasing or strictly decreasing.

Examples:

- $((-1)^n)_{n\geq 1}$ is not monotone;
- $(2^n)_{n\in\mathbb{N}}$ is strictly increasing;
- $\left(\frac{1}{n}\right)_{n\geq 1}$ is strictly decreasing;
- The constant sequence $(c)_{n\in\mathbb{N}}$ (for $c \in \mathbb{R}$) is increasing and decreasing in the same time.

Convergence

Definition

We say that a sequence $(x_n)_{n\in\mathbb{N}}$ is:

- *convergent*, if there exists an element $x \in \mathbb{R}$, called a *limit* of the sequence $(x_n)_{n\in\mathbb{N}}$, such that:

$$\forall \varepsilon > 0,\ \exists n_\varepsilon \in \mathbb{N},\ \forall n \geq n_\varepsilon : |x_n - x| < \varepsilon;$$

- *divergent*, if it is not convergent.

Terminology: If $(x_n)_{n\in\mathbb{N}}$ is convergent and $x \in \mathbb{R}$ is a limit of $(x_n)_{n\in\mathbb{N}}$, we say that $(x_n)_{n\in\mathbb{N}}$ *converges* to x and we write this

$$x_n \xrightarrow[n\to\infty]{} x \quad (x_n \to x)$$

or

$$\lim_{n\to\infty} x_n = x.$$

Properties of convergent sequences

Proposition

The limit of a sequence of real numbers is unique.

Proposition

Any convergent sequence is bounded.

Therefore, any unbounded sequence does not converge (example: $(a^n)_{n\geq 1}$ with $|a| > 1$).

Examples.

- We have

$$\lim_{n\to\infty} \frac{1}{n} = 0.$$

- The constant sequence $(c)_{n\in\mathbb{N}}$ (for $c \in \mathbb{R}$) is convergent to c.
- The sequence $(a^n)_{n\geq 1}$ is convergent for $a \in (-1, 1]$ and divergent for $a \in \mathbb{R} \setminus (-1, 1]$; we have

$$\lim_{n\to\infty} a^n = \begin{cases} 0, & -1 < a < 1; \\ 1, & a = 1. \end{cases}$$

- Another well known limit is

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = \mathrm{e},$$

where e is *Euler*'s number.

Subsequences

Definition

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of real numbers. A *subsequence* of $(x_n)_{n\in\mathbb{N}}$ is a sequence $(x_{n_k})_{k\in\mathbb{N}}$ where $(n_k)_{k\in\mathbb{N}} \subseteq \mathbb{N}$ is a strictly increasing sequence of natural numbers.

Proposition

A subsequence of a convergent sequence is also convergent and it converges to the same limit.

To prove that a sequence $(x_n)_{n\in\mathbb{N}}$ does *not* converge: find $(x_{n_k})_{k\in\mathbb{N}}$ and $(x_{m_k})_{k\in\mathbb{N}}$ *converging* to different limits.

Example. $((-1)^n)_{n\geq 1}$ is not convergent, because the subsequence $\left((-1)^{2k}\right)_{k\geq 1}$ is the constant sequence converging to 1, while the subsequence $\left((-1)^{2k+1}\right)_{k\geq 0}$ is the constant sequence converging to -1.

Operations with sequences

Proposition

Let $x_n \to x \in \mathbb{R}$ *and* $y_n \to y \in \mathbb{R}$. *Then:*

i) $\lim_{n\to\infty}(x_n + y_n) = x + y$;

ii) $\lim_{n\to\infty}(x_n y_n) = xy$;

iii) $\lim_{n\to\infty}(x_n - y_n) = x - y$;

iv) *if* $y \neq 0$, $\lim_{n\to\infty}\left(\frac{x_n}{y_n}\right) = \frac{x}{y}$;

v) *if* $x_n \leq y_n$, $\forall n \in \mathbb{N}$, *then* $x \leq y$;

vi) *(squeeze theorem) if the sequence* (z_n) *is such that* $x_n \leq z_n \leq y_n$, $\forall n \in \mathbb{N}$ *and* $x = y$, *then* (z_n) *is convergent and* $\lim_{n\to\infty} z_n = x$.

Criterium of convergence: Let $(x_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$, $(\alpha_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}_+$ such that $\alpha_n \to 0$ and $x \in \mathbb{R}$. If

$$|x_n - x| \leq \alpha_n, \ \forall n \in \mathbb{N},$$

then $x_n \to x$.

Monotone convergence theorem

Theorem (Weierstrass)

Let $(x_n)_{n\in\mathbb{N}}$ *be a sequence of real numbers.*

1. *If* (x_n) *is increasing and upper bounded, then it converges to* $\sup\{x_n\}_{n\in\mathbb{N}}$.
2. *If* (x_n) *is decreasing and lower bounded, then it converges to* $\inf\{x_n\}_{n\in\mathbb{N}}$.

Bolzano–Weierstrass theorem

Theorem (Bolzano–Weierstrass)

Any bounded sequence of real numbers possesses a convergent subsequence.

For the proof of this theorem, we need the following result:

Lemma

Any sequence of real numbers has a monotone subsequence.

Cauchy sequences

Definition

We say that a sequence $(x_n)_{n\in\mathbb{N}}$ is *Cauchy* (or *fundamental*) if:

$$\forall\varepsilon > 0,\ \exists n_\varepsilon \in \mathbb{N},\ \forall n, m \geq n_\varepsilon : |x_n - x_m| < \varepsilon.$$

Theorem

A sequence of real numbers is convergent if and only if it is Cauchy.

This means that $\mathbb{R}$ is a *complete space*.

The extended real line

If we want that a variant of Weierstrass theorem to hold without the boundedness restriction, we should add the "infimum" and the "supremum" of $\mathbb{R}$. For that, we consider the *extended real line*,

$$\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\},$$

where $+\infty$ and $-\infty$ are two distinct points, $-\infty, +\infty \notin \mathbb{R}$.
The natural order on $\overline{\mathbb{R}}$ extends $\leq$ in the following manner:

- $-\infty \leq +\infty$;
- $-\infty \leq x, \; x \leq +\infty, \; \forall x \in \mathbb{R}$.

Therefore, every $A \subseteq \mathbb{R}$ has a supremum and an infimum:

- $\sup A = +\infty$ if and only if A is not upper bounded;
- $\inf A = -\infty$ if and only if A is not lower bounded;
- $\sup A = -\infty \Leftrightarrow \inf A = +\infty \Leftrightarrow A = \emptyset$.

Infinite limits

For a sequence $(x_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$, we write:

- $\lim\limits_{n\to\infty} x_n = +\infty$ or $x_n \to +\infty$ if

$$\forall a,\ \exists n_a \in \mathbb{N},\ \forall n \geq n_a,\ x_n > a;$$

- $\lim\limits_{n\to\infty} x_n = -\infty$ or $x_n \to -\infty$ if

$$\forall a,\ \exists n_a \in \mathbb{N},\ \forall n \geq n_a,\ x_n < a.$$

Proposition

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of real numbers.

1. *If $(x_n)_{n\in\mathbb{N}}$ is increasing and unbounded, then $\lim\limits_{n\to\infty} x_n = +\infty$.*
2. *If $(x_n)_{n\in\mathbb{N}}$ is decreasing and unbounded, then $\lim\limits_{n\to\infty} x_n = -\infty$.*

Therefore, any monotone sequence in $\mathbb{R}$ has a limit in $\overline{\mathbb{R}}$.

Operations on the extended real line

We introduce:

- $(-\infty)+a=a+(-\infty):=-\infty$, for $-\infty\leq a<+\infty$;
  $(+\infty)+a=a+(+\infty):=+\infty$, for $-\infty<a\leq+\infty$;
- $(-\infty)\cdot a=a\cdot(-\infty):=-\infty$, $(+\infty)\cdot a=a\cdot(+\infty):=+\infty$, for $0<a\leq+\infty$;
  $(-\infty)\cdot a=a\cdot(-\infty):=+\infty$, $(+\infty)\cdot a=a\cdot(+\infty):=-\infty$, for $-\infty\leq a<0$;
- $-(-\infty):=+\infty$, $-(+\infty):=-\infty$, $1/(-\infty)=1/(+\infty)=0$.

The operations $(-\infty)+(+\infty)$, $(+\infty)+(-\infty)$, $(-\infty)-(+\infty)$, $(+\infty)-(-\infty)$, $0\cdot(-\infty)$, $0\cdot(+\infty)$, $\frac{\pm\infty}{\pm\infty}$, $\frac{\pm\infty}{0}$ (partially) remain not defined.

If $\circledcirc$ is an operation (among $+$, $\cdot$, $-$, $/$), $(x_n),(y_n)\subseteq\mathbb{R}$ and $x,y\in\overline{\mathbb{R}}$ such that $x_n\to x$, $y_n\to y$ and $x\circledcirc y$ is defined, then

$$x_n\circledcirc y_n\to x\circledcirc y.$$

Limit points

Definition

Let $(x_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$.

- We call $x \in \overline{\mathbb{R}}$ a *limit point* of the sequence $(x_n)_{n\in\mathbb{N}}$ if there exists a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that $x_{n_k} \to x$.
- The set of the limit points of the sequence $(x_n)_{n\in\mathbb{N}}$ is denoted $\mathrm{L}_{(x_n)}$.

For any sequence $(x_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$, we have $\mathrm{L}_{(x_n)} \neq \emptyset$.

Definition

Let $(x_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$.

- We call the *inferior limit* of $(x_n)_{n\in\mathbb{N}}$ the number (in $\overline{\mathbb{R}}$):

$$\liminf_{n\to\infty} x_n = \varliminf_{n\to\infty} x_n := \inf \mathrm{L}_{(x_n)} .$$

- We call the *superior limit* of $(x_n)_{n\in\mathbb{N}}$ the number (in $\overline{\mathbb{R}}$):

$$\limsup_{n\to\infty} x_n = \varlimsup_{n\to\infty} x_n := \sup \mathrm{L}_{(x_n)} .$$

For instance, $\underline{\lim}_{n\to\infty}(-1)^n = -1$ and $\overline{\lim}_{n\to\infty}(-1)^n = 1$.

Remarks. Let $(x_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$.

- We have
$$\liminf_{n\to\infty} x_n \leqslant \limsup_{n\to\infty} x_n.$$
- If $x \in \overline{\mathbb{R}}$, then $\lim_{n\to\infty} x_n = x$ if and only if $\mathrm{L}_{(x_n)} = \{x\}$, *i.e.*
$$\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n = x.$$
- It can be shown that there exist two monotone subsequences of $(x_n)_{n\in\mathbb{N}}$ which have the limit $\underline{\lim}_{n\to\infty} x_n$, respectively $\overline{\lim}_{n\to\infty} x_n$.

Sequences of functions

Let $E \subseteq \mathbb{R}$ and $f_n : E \to \mathbb{R}$, $n \in \mathbb{N}$ be functions. We call $(f_n)_{n\in\mathbb{N}}$ a *sequence of functions*.
In fact, we deal with sequence of functions when we have a sequence of real numbers which depend on a *parameter* from E.

Example: The sequence

$$\left(\left(1+\frac{x}{n}\right)^n\right)_{n\geq 1}$$

depends on the parameter $x \in \mathbb{R}$. Then we can consider the sequence of functions $(f_n)_{n\geq 1}$, where $f_n : \mathbb{R} \to \mathbb{R}$ are defined by

$$f_n(x) := \left(1+\frac{x}{n}\right)^n, \ x \in \mathbb{R}.$$

We have $\lim_{n\to\infty} f_n(x) = \mathrm{e}^x$, $\forall x \in \mathbb{R}$.

Uniform convergence

Definition

Let $f_n : E \to \mathbb{R}$, $n \in \mathbb{N}$ and $f : E \to \mathbb{R}$. We say that:

- $(f_n)_{n\in\mathbb{N}}$ *converges pointwise* to f if $f_n(x) \to f(x)$, $\forall x \in E$ (we note $f_n \xrightarrow{\mathrm{p}} f$ or $f_n \xrightarrow[E]{\mathrm{p}} f$);
- $(f_n)_{n\in\mathbb{N}}$ *converges uniformly* to f if for any $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that:

$$|f_n(x) - f(x)| < \varepsilon, \ \forall n \geq n_\varepsilon, \ \forall x \in E.$$

(we note $f_n \xrightarrow{\mathrm{u}} f$ or $f_n \xrightarrow[E]{\mathrm{u}} f$).

Of course, if $(f_n)_{n\in\mathbb{N}}$ converges uniformly to f, it will also converge pointwise to f. Also, $f_n \xrightarrow{\mathrm{u}} f$ if and only if $\sup_{x\in E} |f_n(x) - f(x)| \in \mathbb{R}$ for n sufficiently large and

$$\lim_{n\to\infty} \sup_{x\in E} |f_n(x) - f(x)| = 0.$$

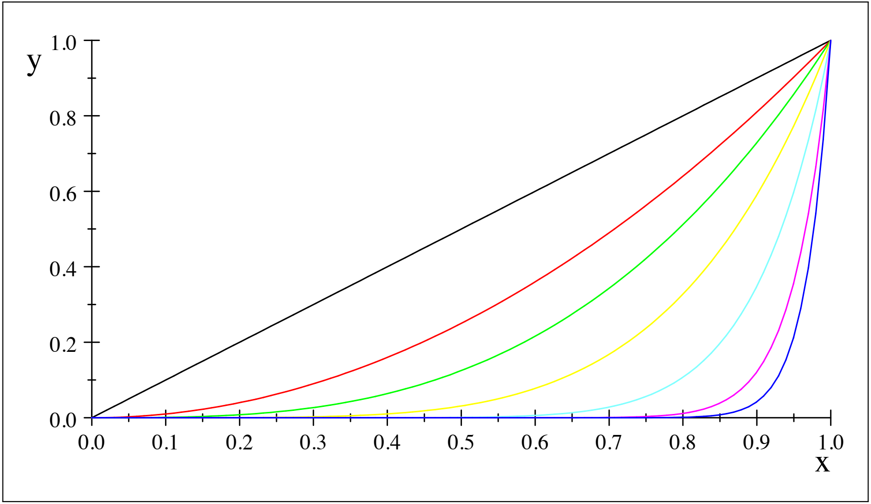
Example. Let $f_n : [0,1] \to \mathbb{R}$ be defined as $f_n(x) := x^n$, for $n \geq 1$. It is clear that

$$\lim_{n\to\infty} x^n = \begin{cases} 0, & x \in [0,1); \\ 1, & x = 1. \end{cases}$$

Therefore, $(f_n)_{n\in\mathbb{N}}$ converges pointwise to f, where $f : [0,1] \to \mathbb{R}$ is defined as

$$f(x) := \begin{cases} 0, & x \in [0,1); \\ 1, & x = 1. \end{cases}$$

Is $(f_n)_{n\in\mathbb{N}}$ converging uniformly to f?



f_n for $n = 1$, 2, 3, 5, 10, 20 and 30

The answer is **no**:

$$\sup_{x\in[0,1]} |f_n(x) - f(x)| = \sup_{x\in[0,1)} |x^n| = 1.$$

Proposition

Let $f_n : E \to \mathbb{R}$, $n \in \mathbb{N}$ *and* $f : E \to \mathbb{R}$. *If there exists a sequence* $(\alpha_n)_{n\in\mathbb{N}}$ *converging to* 0 *such that*

$$|f_n(x) - f(x)| \leq \alpha_n,\ \forall n \in \mathbb{N},\ \forall x \in E$$

then (f_n) *converges uniformly to* f.

Theorem

Let $f_n : E \to \mathbb{R}$, $n \in \mathbb{N}$. *Then there exists a function* $f : E \to \mathbb{R}$ *such that* $f_n \xrightarrow{\text{u}} f$ *if and only if* $(f_n)_{n\in\mathbb{N}}$ *is a uniform Cauchy sequence, i.e. for any* $\varepsilon > 0$, *there exists* $n_\varepsilon \in \mathbb{N}$ *such that:*

$$|f_m(x) - f_n(x)| < \varepsilon,\ \forall m, n \geq n_\varepsilon,\ \forall x \in E.$$

As we will see later, uniform convergence is closed to (keeps) properties as *boundedness*, *continuity* or *integrability*.

Let $n \in \mathbb{N}^*$, $a_1, \ldots, a_n$, $b_1, \ldots, b_n \in \mathbb{R}_+$ and $p, q \in \mathbb{R}_+^*$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then:

$$\sum_{i=1}^{n} a_i b_i \leq \left(\sum_{i=0}^{n} a_i^p \right)^{\frac{1}{p}} \cdot \left(\sum_{i=0}^{n} b_i^q \right)^{\frac{1}{q}} .$$

It is easy to prove a variant of this inequality, the *weighted Hölder inequality*:

$$\sum_{i=1}^{n} \lambda_i a_i b_i \leq \left(\sum_{i=1}^{n} \lambda_i a_i^p \right)^{\frac{1}{p}} \cdot \left(\sum_{i=1}^{n} \lambda_i b_i^q \right)^{\frac{1}{q}} ,$$

where $n \in \mathbb{N}^*$, $\lambda_1, \ldots \lambda_n$, $a_1, \ldots, a_n$, $b_1, \ldots, b_n \in \mathbb{R}_+$ and $p, q \in \mathbb{R}_+^*$ such that $\frac{1}{p} + \frac{1}{q} = 1$.

In the case $p = q = 2$ we obtain the *Cauchy-Buniakowski-Schwarz inequality*:

$$\sum_{i=1}^{n} a_i b_i \leq \left(\sum_{i=1}^{n} a_i^2 \right)^{\frac{1}{2}} \cdot \left(\sum_{i=1}^{n} b_i^2 \right)^{\frac{1}{2}} .$$

The equality holds if and only if there exist $u, v \in \mathbb{R}$ with $u^2 + v^2 \neq 0$, such that $ua_i + vb_i = 0$, $\forall\, i \in \{1, 2, ...n\}$.

Minkowski inequality

Let $n \in \mathbb{N}^*$, $a_1, \ldots, a_n$, $b_1, \ldots, b_n \in \mathbb{R}_+^*$ and $p \in \mathbb{R}_+^*$.

1. If $p \geq 1$, then
$$\left(\sum_{i=1}^{n}(a_i + b_i)^p\right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} b_i^p\right)^{\frac{1}{p}}.$$
2. If $0 < p < 1$, then
$$\left(\sum_{i=1}^{n}(a_i + b_i)^p\right)^{\frac{1}{p}} \geq \left(\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} b_i^p\right)^{\frac{1}{p}}.$$

In both cases, if $p \neq 1$, the equality holds if the n-tuples $(a_0, a_1, \ldots, a_n)$ and $(b_0, b_1, \ldots, b_n)$ are proportional.

Carleman inequality

For any $n \in \mathbb{N}^*$ and $a_1, a_2, \ldots, a_n \in \mathbb{R}_+$ it holds

$$\sum_{k=1}^{n} (a_1 a_2 \ldots a_k)^{\frac{1}{k}} \leq \mathrm{e} \sum_{k=1}^{n} a_k.$$

The equality holds if and only if $a_1 = a_2 = \ldots = a_n = 0$.