Tuesday, 2 November 2021 12:04

S.5.12

$$B_1 = \Big\{ \underbrace{\begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix}}_{A_1}, \underbrace{\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}}_{A_2}, \underbrace{\begin{pmatrix} -1 & -1 \\ 3 & 1 \end{pmatrix}}_{A_3}, \underbrace{\begin{pmatrix} -2 & 1 \\ 3 & 2 \end{pmatrix}}_{A_4} \Big\} \subseteq M_2(\mathbb{R})$$

$$B_2 = \Big\{ \underbrace{\begin{pmatrix} -2 & 0 \\ -8 & 6 \end{pmatrix}}_{C_1}, \underbrace{\begin{pmatrix} 5 & 6 \\ -1 & 6 \end{pmatrix}}_{C_2}, \underbrace{\begin{pmatrix} -2 & 5 \\ -11 & 7 \end{pmatrix}}_{C_3}, \underbrace{\begin{pmatrix} 2 & 10 \\ -10 & 5 \end{pmatrix}}_{C_4} \Big\}$$

$\dim M_2(\mathbb{R}) = 2 \cdot 2$

$$B_c = \Big\{ \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}_{E_1}, \underbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}_{E_2}, \underbrace{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}_{E_3}, \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}_{E_4} \Big\}$$ baza canonică

a) B_1, B_2 baze

B_1 bază a lui $M_2(\mathbb{R}) \to B_1$ lin ind

$\to Lin(B_1) = M_2(\mathbb{R})$

$(\forall M \in M_2(\mathbb{R}), \exists \beta_1, \beta_2, \beta_3, \beta_4$ a.î. $\sum \beta_i A_i = M)$

$\dim M_2(\mathbb{R}) = 4 = \text{card } B_1 \Rightarrow B$ este bază dacă B este lin indep.

• B lin ind

$\alpha_1 A_1 + \alpha_2 A_2 + \alpha_3 A_3 + \alpha_4 A_4 = O_{2,2} \Rightarrow \alpha_i = 0 \quad \forall i = \overline{1,4}$

$$\alpha_1 \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix} + \alpha_2 \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} + \alpha_3 \begin{pmatrix} -1 & -1 \\ 3 & 1 \end{pmatrix} + \alpha_4 \cdot \begin{pmatrix} -2 & 1 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{cases} \alpha_1 + \alpha_2 - \alpha_3 - 2\alpha_4 = 0 \\ -2\alpha_1 + \alpha_2 - \alpha_3 + \alpha_4 = 0 \\ \alpha_1 + 2\alpha_2 + 3\alpha_3 + 3\alpha_4 = 0 \\ \alpha_1 + 2\alpha_2 + \alpha_3 + 2\alpha_4 = 0 \end{cases}$$ sist lin. omogen

$$A = \begin{pmatrix} 1 & 1 & -1 & -2 \\ -2 & 1 & -1 & 1 \\ 1 & 2 & 3 & 3 \\ 1 & 2 & 1 & 2 \end{pmatrix}$$

dacă $\det A \neq 0 \Rightarrow \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$

$$\begin{vmatrix} 1 & 1 & -1 & -2 \\ -2 & 1 & -1 & 1 \\ 1 & 2 & 3 & 3 \\ 1 & 2 & 1 & 2 \end{vmatrix} \overset{C_2+C_3}{=} \begin{vmatrix} 1 & 1 & 0 & -2 \\ -2 & 1 & 0 & 1 \\ 1 & 2 & 5 & 3 \\ 1 & 2 & 3 & 2 \end{vmatrix} = 5 \cdot \begin{vmatrix} 1 & 1 & -2 \\ -2 & 1 & 1 \\ 1 & 2 & 2 \end{vmatrix} - 3 \begin{vmatrix} 1 & 1 & -2 \\ -2 & 1 & 1 \\ 1 & 2 & 3 \end{vmatrix}$$

$$= 5 \cdot \begin{vmatrix} 1 & 1 & -2 \\ 0 & 3 & -3 \\ 0 & 1 & 4 \end{vmatrix} - 3 \cdot \begin{vmatrix} 1 & 1 & -2 \\ 0 & 3 & -3 \\ 0 & 1 & 5 \end{vmatrix} = 5 \cdot (12+3) - 3 \cdot (15+3)$$

$= 5 \cdot 15 - 3 \cdot 18 \neq 0$

$= 75 - 54 = 19$

$\Rightarrow \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 \Rightarrow B_1$ lin ind

$\dim M_2 = \text{card } B_1 = 4$ $\Big| \Rightarrow B_1$ bază

b) $S_{B_1 B_2}$ – matricea de schimbare de la B_1 la B_2

$B_1 \rightsquigarrow B_2$ (scriu B_2 în fct. de B_1)

$$\begin{cases} C_1 = \alpha_{11} A_1 + \alpha_{12} A_2 + \alpha_{13} A_3 + \alpha_{14} A_4 \\ C_2 = - \quad - \quad - \quad - \\ C_3 = - \quad - \quad - \quad - \end{cases}$$

$C_4 = \alpha_{41} A_1 + \alpha_{42} A_2 + \alpha_{43} A_3 + \alpha_{44} A_4$

$$S_{B_1 B_2} = \begin{pmatrix} \alpha_{11} & \alpha_{21} & \alpha_{31} & \alpha_{41} \\ \alpha_{12} & \alpha_{22} & \alpha_{32} & \alpha_{42} \\ \alpha_{13} & \alpha_{23} & \alpha_{33} & \alpha_{43} \\ \alpha_{14} & \alpha_{24} & \alpha_{34} & \alpha_{44} \end{pmatrix}$$

$\boxed{B_2 = B_1 S_{B_1 B_2}}$

$B_2 = B_c \cdot S_{B_c B_2}$, $\quad S_{B_c B_2} = \begin{pmatrix} -2 & 5 & -2 & 2 \\ 0 & 6 & 5 & 10 \\ -8 & -1 & 11 & -10 \\ 6 & 6 & 7 & 5 \end{pmatrix}$

$$C_1 = \begin{pmatrix} -2 & 0 \\ -8 & 6 \end{pmatrix} = -2 \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 0 \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - 8 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + 6 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$= -2E_1 + 0E_2 - 8E_3 + 6E_4$

$C_2 = \begin{pmatrix} 5 & 6 \\ -1 & 6 \end{pmatrix} = 5E_1 + 6E_2 - 1E_3 + 6E_4$

$C_3 = \begin{pmatrix} -2 & 5 \\ 11 & 7 \end{pmatrix} = -2E_1 + 5E_2 + 11E_3 + 7E_4$

$C_4 = 2E_1 + 10E_2 - 10E_3 + 5E_4$

$B_1 = B_c S_{B_c B_1}$, $\quad S_{B_c B_1} = \begin{pmatrix} 1 & 1 & -1 & -2 \\ -2 & 1 & -1 & 1 \\ 1 & 2 & 3 & 3 \\ 1 & 2 & 1 & 2 \end{pmatrix}$

$A_1 = \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix} = E_1 - 2E_2 + E_3 + E_4$

$B_2 = B_1 \cdot S_{B_1 B_2} \iff B_c \cdot S_{B_c B_2} = B_c \cdot S_{B_c B_1} \cdot S_{B_1 B_2}$

$S_{B_1 B_2} = (S_{B_c B_1})^{-1} \cdot S_{B_c B_2}$

$(S_{B_c B_1})^{-1} = \dfrac{1}{\det(S_{B_c B_1})} \cdot S_{B_c B_1}^{*}$

$$S_{B_c B_1}^{T} = \begin{pmatrix} 1 & -2 & 1 & 1 \\ 1 & 1 & 2 & 2 \\ -1 & -1 & 3 & 1 \\ -2 & 1 & 3 & 2 \end{pmatrix}$$

. . .

c) coord. matricii $A = \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}$ în raport cu cele două baze.

Coord lui A în rap cu baza B_1:

$\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{R}$ a.î. $A = \alpha_1 A_1 + \alpha_2 A_2 + \alpha_3 A_3 + \alpha_4 A_4$

$$\begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix} = \alpha_1 \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix} + \alpha_2 \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} + \alpha_3 \cdot \begin{pmatrix} -1 & -1 \\ 3 & 1 \end{pmatrix} + \alpha_4 \begin{pmatrix} -2 & 1 \\ 3 & 2 \end{pmatrix}$$

$$\begin{cases} \alpha_1 + \alpha_2 - \alpha_3 - 2\alpha_4 = 1 \\ -2\alpha_1 + \alpha_2 - \alpha_3 + \alpha_4 = -1 \\ \alpha_1 + 2\alpha_2 + 3\alpha_3 + 3\alpha_4 = 1 \\ \alpha_1 + 2\alpha_2 + \alpha_3 + 2\alpha_4 = 2 \end{cases}$$

$\ominus$ $3\alpha_1 - 3\alpha_4 = 2 \Rightarrow$

$\Rightarrow 3\alpha_1 = 2 - 3 - 6\alpha_3 \Rightarrow \alpha_1 = \frac{1}{3} - 2\alpha_3$

$\ominus$ $2\alpha_3 + \alpha_4 = -1 \Rightarrow \alpha_4 = -1 - 2\alpha_3$

(1) $\alpha_2 = \alpha_3 + 2\alpha_4 - \alpha_1 + 1 \Rightarrow \alpha_2 = \alpha_3 - 2 - 4\alpha_3 + \frac{1}{3} + 2\alpha_3 + 1$

$\alpha_2 = -3\alpha_3 + 2\alpha_3 - 1 + \frac{1}{3}$

$\alpha_2 = -\alpha_3 - \frac{2}{3}$

(3) $-\frac{1}{3} - 2\alpha_3 - 2\alpha_3 - \frac{2}{3} \cdot 2 + 3\alpha_3 - 3 - 6\alpha_3 = 1$

$-7\alpha_3 = 4 + \frac{5}{3} \Rightarrow \alpha_3 = -\frac{17}{21}$

$\alpha_4 = -1 + 2 \cdot \frac{17}{21}$ . . .

* coord lui $\bar{x}$ în baza $B = \{b_1, b_2, b_3\}$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \left(S_{B_c B} \right) \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} \Rightarrow \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = S_{B_c B}^{-1} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Sau, matriceal

$A = \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}$ $\qquad S_{B_c B_2} = \begin{pmatrix} \cdots \end{pmatrix}_{4,4}$

$$\begin{pmatrix} 1 \\ -1 \\ 1 \\ 2 \end{pmatrix} = S_{B_c B_1} \cdot \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix} \Rightarrow \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix} = S_{B_c B_1}^{-1} \begin{pmatrix} 1 \\ -1 \\ 1 \\ 2 \end{pmatrix}$$

Coord lui A în baza B_1

S.5.11

i) $B = \{ \underbrace{(3,1,4)}_{b_1}, \underbrace{(-1,1,2)}_{b_2}, \underbrace{(1,3,m)}_{b_3} \}$? $m \in \mathbb{R}$ a.î. sist lin dependent

• B lin ind dacă

$\alpha_1 b_1 + \alpha_2 b_2 + \alpha_3 b_3 = (0,0,0) \Rightarrow \alpha_i = 0$

$$\begin{cases} 3\alpha_1 - \alpha_2 + \alpha_3 = 0 \\ \alpha_1 + \alpha_2 + 3\alpha_3 = 0 \\ 4\alpha_1 + 2\alpha_2 + m\alpha_3 = 0 \end{cases}$$ s. l. omog.

sau

$$\det A = \begin{vmatrix} 3 & -1 & 1 \\ 1 & 1 & 3 \\ 4 & 2 & m \end{vmatrix} \neq 0$$

$b_1^T \; b_2^T \; b_3^T$

dacă $\det A \neq 0 \Rightarrow$ s. lin. ind

$\det A = 3m - 12 + 2 - 4 - 18 + m = 4m - 32$

$\det A \neq 0 \Rightarrow 4m - 32 \neq 0 \Rightarrow m \neq 8$

• $m = 8$ sistem liniar dependent

$\det A = 0 \Rightarrow \text{rang } A < 3$

$\begin{vmatrix} 3 & -1 \\ 1 & 1 \end{vmatrix} = 3 + 1 \neq 0 \Rightarrow \text{rang } A = 2.$

$$\begin{cases} 3\alpha_1 - \alpha_2 = -\alpha_3 \\ \alpha_1 + \alpha_2 = -3\alpha_3 \end{cases} \oplus$$

$4\alpha_1 = -4\alpha_3 \Rightarrow \alpha_1 = -\alpha_3$

$\alpha_2 = -3\alpha_3 - \alpha_1 = -2\alpha_3$

$(-\alpha_3, -2\alpha_3, \alpha_3)$, $\alpha_3 \in \mathbb{R}$.

$-\alpha_3 b_1 - 2\alpha_3 b_2 + \alpha_3 b_3 = \bar{0} \quad \Big| \; \alpha_3 \neq 0$

$-b_1 - 2b_2 + b_3 = 0$

$b_3 = b_1 + 2b_2$

<u>Aplicații liniare</u>

$T: \mathbb{R}^3 \to \mathbb{R}^3$, $\quad A = \begin{pmatrix} 1 & 1 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

a) $T(1,1,1)$

b) $Ker T$, $Im T$, $def(T)$, $rang T$

a) $T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = A \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 1 & 1 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 - 2x_3 \\ x_2 + x_3 \\ 0 \end{pmatrix}$$

$T(x_1, x_2, x_3) = (x_1 + x_2 - 2x_3, \; x_2 + x_3, \; 0)$

$T(1,1,1) = (0,2,0)$

$$A \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$$

b) $Ker T = \{ \bar{u} \in \mathbb{R}^3 \mid T(\bar{u}) = 0_{\mathbb{R}^3} \}$

$T(\bar{u}) = \bar{0}$, $\quad \bar{u} = (u_1, u_2, u_3) \Rightarrow (u_1 + u_2 - 2u_3, u_2 + u_3, 0) = (0,0,0)$

$$\begin{cases} u_1 + u_2 - 2u_3 = 0 \\ u_2 + u_3 = 0 \\ 0 \cdot u_1 + 0u_2 + 0u_3 = 0 \end{cases}$$ sist lin și omogen

$$\tilde{A} = \begin{pmatrix} 1 & 1 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$\det \tilde{A} = 0 \Rightarrow \text{rang } A < 3$

$\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1 \neq 0$ → sist nedeterm.

$$\begin{cases} u_1 + u_2 = 2u_3 \Rightarrow u_1 = 2u_3 - u_2 = 3u_3 \\ u_2 = -u_3 \end{cases}$$

$u_1 = 3u_3$, $u_2 = -u_3$

$Ker T = \{ (3u_3, -u_3, u_3), u_3 \in \mathbb{R} \} = \{ u_3 (3, -1, 1), u_3 \in \mathbb{R} \}$

$= Lin(3, -1, 1)$

$\dim(Ker T) = 1 = def T$

• $Im T$ $\qquad (\text{rang } T + \underbrace{def T}_{1} = \dim \underbrace{\mathbb{R}^3}_{3}) \Rightarrow \text{rang}(T) = 2$

Imaginea lui T

$Im(T) = \{ \bar{y} \in \mathbb{R}^3 \mid \exists \bar{x} \in \mathbb{R}^3$ a.î. $T(\bar{x}) = \bar{y} \}$

$Im(T) = \langle T(e_1), T(e_2), T(e_3) \rangle \leftarrow \{T(e_1), T(e_2), T(e_3)\}$ generează $Im T$

• $T(e_1), T(e_2), T(e_3)$ lin ind

$T(e_1) = (1,0,0)$

$T(e_2) = (1,1,0)$

$T(e_3) = (-2,1,0)$

$T\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = A \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad T\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = A \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$A = \begin{pmatrix} 1 & 1 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

$\tilde{B} \{ (1,0,0), (1,1,0), (-2,1,0) \}$ lin indep?

$\tilde{B}$ lin ind dacă $\begin{vmatrix} 1 & 1 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{vmatrix} \neq 0$ (NU) $(\det = 0)$

$\Rightarrow$ sist liniar dependent

$\alpha_1 T(\bar{e}_1) + \alpha_2 T(\bar{e}_2) + \alpha_3 T(\bar{e}_3) = (0,0,0)$

$$\begin{cases} \alpha_1 + \alpha_2 - 2\alpha_3 = 0 \\ 0\alpha_1 + \alpha_2 + \alpha_3 = 0 \\ 0 = 0 \end{cases} \Rightarrow \begin{array}{l} \alpha_1 = 2\alpha_3 - \alpha_2 = 3\alpha_3 \\ \alpha_2 = -\alpha_3 \end{array}$$

$\alpha_1 = 3\alpha_3 \quad \alpha_2 = -\alpha_3 \quad \alpha_3 = \alpha_3$

$3T(e_1) - T(e_2) + T(e_3) = \bar{0}$

$T(e_2) = 3T(\bar{e}_1) + T(\bar{e}_3)$

$\{T(e_1), T(e_3)\}$ lin indep?

$\alpha T(e_1) + \beta T(e_3) = 0$

$\alpha(1,0,0) + \beta \cdot (-2,1,0) = (0,0,0)$

$\alpha - 2\beta = 0 \Rightarrow \alpha = 0$

$\beta = 0$

$\{T(\bar{e}_1), T(\bar{e}_3)\}$ lin ind

$Im T = \langle T(\bar{e}_1), T(\bar{e}_3) \rangle$

$\downarrow$

$\text{rang } T = \dim(Im T) = 2$

$\{T(e_1), T(e_3)\}$ bază

$Im(T) = \{ \alpha T(e_1) + \beta T(e_3), \alpha, \beta \in \mathbb{R} \} = \{ (\alpha - 2\beta, \beta, 0), \alpha, \beta \in \mathbb{R} \}$

$\underbrace{\dim \mathbb{R}^3}_{3} = \underbrace{\text{rang } T}_{2} + \underbrace{def T}_{1}$