Integrability in $\mathbb{R}^n$

Lecture 13

Mathematics - 1st year, English

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Multiple integrals

- Multiple integrals are a natural extension of the Riemann integral to the case of functions of several variables.
- In particular, when the function to be integrated has 2 variables: the *double integral*;
- when we deal with 3 variables: the *triple integral*.
- In this way, we can compute some numeric characteristics of 2D and 3D-objects (*area*, *volume*, *mass*, etc.).

Outline of the lecture

Jordan measure

- To some sets in $\mathbb{R}$, $\mathbb{R}^2$ or $\mathbb{R}^3$ there corresponds a number, such *length*, *area* or *volume*, respectively (or *mass*, if we think about physical objects).
- A *measure* in $\mathbb{R}^n$ generalizes this concepts: the measure of a set will be a positive number.
- We start by defining the measure of simple objects.

Definition

- Let $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n \in \mathbb{R}$ such that $a_k < b_k$, $\forall k \in \overline{1,n}$. The set

$$I_0 = [a_1, b_1] \times \cdots \times [a_n, b_n] \\ = \left\{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid a_k \leq x_k \leq b_k,\ \forall k \in \overline{1,n}\right\}$$

is called an *n-dimensional compact interval* (if $n = 2$ or $n = 3$, also called a *rectangle*, respectively a *parallelepiped* with the *edges*, respectively *faces parallel to the coordinates axes*).

Definition (continuation)

- Its (*Jordan*) *measure* is the number

$$\mu(I_0) := (b_1^0 - a_1^0)(b_2^0 - a_2^0)\dots(b_n^0 - a_n^0).$$

(if $n = 2$ or $n = 3$, this is the *area*, respectively, the *volume* of the rectangle, respectively the parallelipiped I_0).

- We call an *elementary* (*Jordan measurable*) *set* any set in $\mathbb{R}^n$ which can be written as a finite union of compact n-dimensional intervals which have no interior common points, i.e. a set

$$E = \bigcup_{\ell=1}^{q} I_\ell$$

such that $I_\ell = [a_1^\ell, b_1^\ell] \times [a_2^\ell, b_2^\ell] \times \dots \times [a_n^\ell, b_n^\ell]$, $\ell = \overline{1, q}$ and such that $\mathring{I}_j \cap \mathring{I}_\ell = \emptyset$, $\forall j, \ell \in \{1, 2, \dots, q\}$, $j \neq \ell$.

- Its *Jordan measure* of the set E is defined as

$$\mu(E) := \sum_{\ell=1}^{q} \mu(I_\ell).$$

We will denote by $\mathcal{E}_J^n$ the family of all elementary sets in $\mathbb{R}^n$.

Definition

Let $A \subseteq \mathbb{R}^n$ be a bounded set.

- We call the *Jordan interior measure* of the set A the number

$$\mu_*(A) = \sup\{\mu(E) \mid E \subseteq A,\ E \in \mathcal{E}_J^n\}$$

  (if there is no elementary set included in A, $\mu_*(A)$ is then 0).
- The *Jordan exterior measure* of the set A is the number

$$\mu^*(A) = \inf\{\mu(E) \mid E \supseteq A,\ E \in \mathcal{E}_J^n\}.$$

- We say that A is *Jordan measurable* if $\mu_*(A) = \mu^*(A)$. The commun value is called the *Jordan measure* of the set A and is denoted by $\mu_J(A)$ (it is customary to call it *area* if $n = 2$ or *volume* if $n = 3$)

It is obvious that for a bounded set $A \subseteq \mathbb{R}^n$, $\mu_*(A)$ and $\mu^*(A)$ are positive numbers satisfying $\mu_*(A) \leq \mu^*(A)$.

Remarks.

1. Any elementary set $E \in \mathcal{E}_J^n$ is Jordan measurable, by the definition.

2. Not every bounded set in $\mathbb{R}^n$ is Jordan measurable. For instance, in $\mathbb{R}^2$ let

$$A_D = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1,\ 0 \leq y \leq f_D(x)\}$$

where $f_D : \mathbb{R} \to \mathbb{R}$ is the Dirichlet function, defined by

$$f_D(x) := \begin{cases} 1, & x \in \mathbb{Q}; \\ 0, & x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Then $\mu_*(A_D) = 0$, since there is no elementary set $E \subseteq A_D$; on the other hand, $\mu^*(A_D) = 1$, since every elementary set $E \supseteq A$ has to include the rectangle $[0, 1] \times [0, 1]$. Therefore, E is not Jordan measurable.

3. There are non-elementary sets which are Jordan measurable. For instance, the subgraph of a Riemann integrable function $f : [a, b] \to \mathbb{R}_+$, *i.e.* the set

$$\Gamma_f = \left\{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq b,\ 0 \leq y \leq f(x)\right\},$$

is Jordan measurable, with $\mu_J(\Gamma_f) = \mathrm{area}(\Gamma_f) = \int\limits_a^b f(x)dx$.

4. More generally, we infer that if $f, g : [a, b] \to \mathbb{R}$ are two Riemann integrable functions on $[a, b]$ such that $f(x) \leq g(x)$, $\forall x \in [a, b]$, then the set $\Gamma_{f,g} = \left\{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq b,\ f(x) \leq y \leq g(x)\right\}$ is Jordan measurable with

$$\mu_J\left(\Gamma_{f,g}\right) = \int_a^b \left(g(x) - f(x)\right) dx.$$

Application: *area of an ellipse*
Let $\tilde{a}, \tilde{b} > 0$ and set $a := -\tilde{a}$, $b := \tilde{a}$. We define the functions $f, g : [a, b] \to \mathbb{R}$ by $f(x) := -\frac{\tilde{b}}{\tilde{a}}\sqrt{\tilde{a}^2 - x^2}$ și $g(x) := \frac{\tilde{b}}{\tilde{a}}\sqrt{\tilde{a}^2 - x^2}$, $x \in [a, b] = [-\tilde{a}, \tilde{a}]$. The union of their graphs give the boundary of the ellipse of equation $\frac{x^2}{\tilde{a}^2} + \frac{y^2}{\tilde{b}^2} - 1 = 0$; therefore, the domain bounded by this ellipse is given by

$$\Gamma_{f,g} = \left\{(x, y) \in \mathbb{R}^2 \mid -\tilde{a} \le x \le \tilde{a},\ -\frac{\tilde{b}}{\tilde{a}}\sqrt{\tilde{a}^2 - x^2} \le y \le \frac{\tilde{b}}{\tilde{a}}\sqrt{\tilde{a}^2 - x^2}\right\},$$

By computations, we find $\mu_J(\Gamma_{f,g}) = \dfrac{\tilde{b}}{\tilde{a}} \int_{-\tilde{a}}^{\tilde{a}} \sqrt{\tilde{a}^2 - x^2}dx = \pi\tilde{a}\tilde{b}$. Summarizing, the area of the ellipse with the semiaxes $\tilde{a}$ and $\tilde{b}$ is $\pi\tilde{a}\tilde{b}$.

5. By the definition, a set $B \subseteq \mathbb{R}^n$ is measurable and has null Jordan measure if for every $\varepsilon > 0$, there exists $E_\varepsilon \in \mathcal{E}_J^n$ such that $B \subseteq E_\varepsilon$ și $\mu_J(E_\varepsilon) < \varepsilon$.

Necessary and sufficient conditions of Jordan measurability

Theorem

Let $A \subseteq \mathbb{R}^n$ be a bounded set. Then the following statements are equivalent:

(i) *A is Jordan measurable;*

(ii) $\forall \varepsilon > 0,\ \exists E'_\varepsilon, E''_\varepsilon \in \mathcal{E}^n_J : E'_\varepsilon \subseteq A \subseteq E''_\varepsilon$ *and* $\mu_J(E'_\varepsilon) - \mu_J(E''_\varepsilon) < \varepsilon$;

(iii) $\partial(A)$ *is Jordan measurable and* $\mu_J(\partial(A)) = 0$;

(iv) *there exist sequences* $\left(\tilde{E}_m\right)_{m \in \mathbb{N}^*} \subseteq \mathcal{E}^n_J$ *and* $\left(\hat{E}_m\right)_{m \in \mathbb{N}^*} \subseteq \mathcal{E}^n_J$ *such that* $\tilde{E}_m \subseteq A \subseteq \hat{E}_m,\ \forall\, m \in \mathbb{N}^*$ *and* $\lim\limits_{m\to\infty} \mu_J(\tilde{E}_m) = \lim\limits_{m\to\infty} \mu_J(\hat{E}_m)$.

Remark. For a Jordan measurable set A, $\mu_J(A) \neq 0$ is equivalent to $\mathring{A} \neq \emptyset$.

Properties of the Jordan measure

Let us denote by $\mathcal{M}_J^n$ the family of all subsets of $\mathbb{R}^n$ which are Jordan measurable.

Theorem

- **i)** $\mu_J(A) \geq 0$, $\forall A \in \mathcal{M}_J^n$ *(non-negativity)*.
- **ii)** $\forall A, B \in \mathcal{M}_J^n : \mathring{A} \cap \mathring{B} = \emptyset \Rightarrow \mu_J(A \cup B) = \mu_J(A) + \mu_J(B)$ *(fin. additivity)*.
- **iii)** $\forall A, B \in \mathcal{M}_J^n : B \subseteq A \Rightarrow \mu_J(A \setminus B) = \mu_J(A) - \mu_J(B)$ *(subtraction)*.
- **iv)** $\forall A, B \in \mathcal{M}_J^n : B \subseteq A \Rightarrow \mu_J(B) \leq \mu_J(A)$ *(monotonicity)*.
- **v)** $\forall A \in \mathcal{M}_J^n$, $\forall B \subseteq \mathbb{R}^n : \mu_J(A) = 0,\ B \subseteq A \Rightarrow \mu_J(B) = 0$ *(completion)*.

Remarks.

1 We can also infer that if $A, B \in \mathcal{M}_J^n$, then $A \cup B \in \mathcal{M}_J^n$ and $A \setminus B \in \mathcal{M}_J^n$. Moreover, the subadditivity property holds: $\mu_J(A \cup B) \leq \mu_J(A) + \mu_J(B)$.

2. The graph of a continuous function $f : [a, b] \longrightarrow \mathbb{R}_+$ has null area.

3. Every set in $\mathbb{R}^2$ whose boundary can be written as a finite union of graphs of continuous functions on compact intervals is Jordan measurable. In particular, any disc in $\mathbb{R}^2$ is Jordan measurable.

Riemann multiple integral on compact sets

Let $D \subseteq \mathbb{R}^n$ be a non-empty compact (equivalently, closed and bounded) set such that $D \in \mathcal{M}_J^n$.

Definition

- We call *partition* of D any finite family $\{D_i\}_{1 \leq i \leq p}$ of subsets of D such that:
  a) $D_i \in \mathcal{M}_J^n, \forall i \in \overline{1, p}$;
  b) $\mathring{D}_i \cap \mathring{D}_j = \emptyset, \forall i, j \in \{1, \ldots, p\}$ with $i \neq j$;
  c) $D = \bigcup_{i=1}^{p} D_i$.

  We denote $\mathcal{D}(D)$ the family of all partitions of D.
- For a partition Δ we define its *norm* $\|\Delta\| := \max\limits_{1 \leq i \leq p} \{\mathrm{diam}(D_i)\}$, where $\mathrm{diam}(D_i)$ means the diameter of D_i.

Remark. By the additivity property of the Jordan measure we have
$\mu_J(D) = \sum\limits_{i=1}^{p} \mu_J(D_i)$.

Definition

Let $\Delta = \{D_i\}_{1 \leq i \leq p}$ be a partition of D

- An p-tuple $\xi_\Delta = (\xi^1, \xi^2, \dots, \xi^p) \in (\mathbb{R}^n)^p$ is called an *intermediary point system* of Δ if $\xi^i \in D_i$, $\forall i = \overline{1, n}$. The set of all intermediary point systems of Δ is denoted Ξ_Δ.
- We call the *Riemann sum* of the function $f : D \to \mathbb{R}$ with respect to Δ and an intermediary point system $\xi_\Delta = (\xi^1, \xi^2, \dots, \xi^n)$ the number

$$\sigma_f(\Delta, \xi_\Delta) = \sum_{i=1}^{n} f(\xi^i)\mu_J(D).$$

Definition

We say that the function $f : D \to \mathbb{R}$ is *Riemann integrable* if there exists $I \in \mathbb{R}$ such that for every $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$ such that for every partition $\Delta = \{D_i\}_{1 \leq i \leq p}$ of D with $\|\Delta\| < \delta_\varepsilon$ and every intermediary point system $\xi_\Delta = (\xi^1, \xi^2, \dots, \xi^p)$ of Δ we have

$$|\sigma_f(\Delta, \xi_\Delta) - I)| < \varepsilon.$$

Definition (continuation)

The number I is called the *multiple integral* (if $n = 2$ or $n = 3$, the *double*, respectively *triple integral*) and is denoted by

$$\int \cdots \int_D f(x_1, x_2, \ldots, x_n) dx_1 dx_2 \ldots dx_n.$$

- As in the one-dimensional case, it can be shown that a Riemann integrable function on a compact set is bounded.
- We can also define the *lower* and *upper Darboux sums* of a function $f : D \to \mathbb{R}$ as

$$s_f(\Delta) := \sum_{i=1}^{p} m_i \mu_J(D_i);$$

$$S_f(\Delta) := \sum_{i=1}^{p} M_i \mu_J(D_i),$$

where $\Delta = \{D_i\}_{1 \le i \le p}$ is a partition of D and $m_i := \inf_{x \in D_i} f(x)$, $M_i := \sup_{x \in D_i} f(x)$, $i = \overline{1, p}$.

- It is easy to see that the following relation holds:

$$m \cdot \mu_J(D) \leq s_f(\Delta) \leq S_f(\Delta) \leq M \cdot \mu_J(D),$$

where Δ is an arbitrary partition of D and $m := \inf\limits_{x \in D} f(x)$, $M := \sup\limits_{x \in D} f(x)$.

- If we denote $\underline{I} := \sup\limits_{\Delta \in \mathcal{D}(D)} s_f(\Delta)$ and $\overline{I} := \inf\limits_{\Delta \in \mathcal{D}(D)} s_f(\Delta)$, the *lower*, respectively the *upper Darboux integral* of f, we infer

$$m \cdot \mu_J(D) \leq \underline{I} \leq \overline{I} \leq M \cdot \mu_J(D).$$

Proposition

Let $D \subseteq \mathbb{R}^n$ be a non-empty compact set which is Jordan measurable and $f : D \to \mathbb{R}$ a bounded function. Then f is Riemann integrable if and only $\underline{I} = \overline{I}$, condition which is equivalent to

$$\forall \varepsilon > 0,\ \exists \delta_\varepsilon > 0,\ \forall \Delta \in \mathcal{D}(D) : \|\Delta\| < \delta_\varepsilon \Rightarrow S_f(\Delta_\varepsilon) - s_f(\Delta_\varepsilon) < \varepsilon.$$

In this case, $\underline{I} = \overline{I} = \int \cdots \int_D f(x_1, x_2, \ldots, x_n) dx_1 dx_2 \ldots dx_n$.

Theorem

Let $D \subseteq \mathbb{R}^n$ be a non-empty compact set which is Jordan measurable and $f : D \to \mathbb{R}$ a continuous function. Then f is Riemann integrable.

A generalization of the above result is the following:

Theorem

Let $D \subseteq \mathbb{R}^n$ be a non-empty compact set which is Jordan measurable and $f : D \to \mathbb{R}$ a function which is continuous in every element of D with the exception of a Jordan measurable set having null measure. Then f is Riemann integrable.

Properties

Proposition

Let $D \subseteq \mathbb{R}^n$ be a non-empty compact set which is Jordan measurable. Then:

i) $\int \cdots \int_D 1 \cdot dx_1 dx_2 \dots dx_n = \mu_J(D)$;

ii) *for every Riemann integrable functions $f, g : D \to \mathbb{R}$ and $\alpha, \beta \in \mathbb{R}$, $\alpha f + \beta g$ is Riemann integrable and*

$$\int \cdots \int_D (\alpha f(x_1, \dots, x_n) + \beta g(x_1, \dots, x_n)) dx_1 \dots dx_n =$$

$$\alpha \int \cdots \int_D f(x_1, \dots, x_n) dx_1 \dots dx_n + \beta \int \cdots \int_D g(x_1, \dots, x_n) dx_1 \dots dx_n;$$

iii) *for every Riemann integrable functions $f, g : D \to \mathbb{R}$ with $f(x) \leq g(x)$, $\forall x \in D$, we have:*

$$\int \cdots \int_D f(x_1, \dots, x_n) dx_1 \dots dx_n \leq \int \cdots \int_D g(x_1, \dots, x_n) dx_1 \dots dx_n;$$

Proposition (continuation)

iv) *for every Riemann integrable function* $f : D \to \mathbb{R}$, $|f|$ *is also Riemann integrable and*

$$\left|\int \cdots \int_D f(x_1, \ldots, x_n) dx_1 \ldots dx_n\right| \leq \int \cdots \int_D \left|f(x_1, \ldots, x_n)\right| dx_1 \ldots dx_n;$$

v) *for every Riemann integrable function* $f : D \to \mathbb{R}$, *there exists* $\lambda \in \left[\inf\limits_{x \in D} f(x), \sup\limits_{x \in D} f(x)\right]$ *such that:*

$$\int \cdots \int_D f(x_1, \ldots, x_n) dx_1 \ldots dx_n = \lambda \mu_J(D).$$

If, moreover, $f \in C(D)$ *and* D *is connected (i.e., it cannot be divided into two disjoint nonempty closed sets), then there exists* $\xi \in D$ *such that*

$$\int \cdots \int_D f(x_1, \ldots, x_n) dx_1 \ldots dx_n = f(\xi) \mu_J(D);$$

Proposition (continuation)

vi) *if D is the union of two non-empty compact, Jordan-measurable sets D_1 and D_2, with $\overset{\circ}{D}_1 \cap \overset{\circ}{D}_2 = \emptyset$, and f is Riemann integrable on both D_1 and D_2, then f is Riemann integrable on D and*

$$\begin{aligned}\int \cdots \int_D f(x_1, \dots, x_n) dx_1 \dots dx_n = & \int \cdots \int_{D_1} f(x_1, \dots, x_n) dx_1 \dots dx_n \\ & + \int \cdots \int_{D_2} f(x_1, \dots, x_n) dx_1 \dots dx_n;\end{aligned}$$

vii) *for every $f, g \in C(D)$ with $g(x) \geq 0$, $\forall x \in D$, there exists $\eta \in D$ such that*

$$\begin{aligned}&\int \cdots \int_D f(x_1, \dots, x_n) g(x_1, \dots, x_n) dx_1 \dots dx_n = \\ &\qquad f(\eta) \int \cdots \int_D g(x_1, \dots, x_n) dx_1 \dots dx_n.\end{aligned}$$

The double integral on compact sets

- The multiple integral in the case $n = 2$ is called *double integral*.
- If $f : D \to \mathbb{R}$ is a Riemann integrable function on a non-empty compact, Jordan measurable set $D \subseteq \mathbb{R}^2$, we will denote its double integral $\iint_D f(x,y)dx\,dy$.

Proposition (rectangle case)

Let $a, b, c, d \in \mathbb{R}$ with $a < b$, $c < d$, $D := [a,b] \times [c,d]$ and $f : D \to \mathbb{R}$ a Riemann integrable function. If, for every $x \in [a,b]$, $f(x,\cdot)$ is Riemann integrable and the function $x \mapsto \int_c^d f(x,y)dy$ is also Riemann integrable on $[a,b]$, then

$$\iint_{[a,b]\times[c,d]} f(x,y)dxdy = \int_a^b \Big(\int_c^d f(x,y)dy\Big)dx.$$

Moreover, if $f(x,y) = f_1(x)f_2(y)$ and $f_1 \in \mathcal{R}[a,b]$, $f_2 \in \mathcal{R}[c,d]$, then we have

$$\iint_{[a,b]\times[c,d]} f_1(x)f_2(y)dx\,dy = \int_a^b f_1(x)dx \cdot \int_c^d f_2(y)dy.$$

Remarks.

1. We get a similar result by inversing the roles of x and y:

$$\iint_{[a,b]\times[c,d]} f(x,y)dxdy = \int_c^d \Big(\int_a^b f(x,y)dx\Big)dy.$$

2. A sufficient condition for the conditions of the above result to be fulfilled is $f \in C([a,b] \times [c,d])$.

Definition

- A subset $D \subseteq \mathbb{R}^2$ is called *simple with respect to the axis Oy* if there exist continuous functions $\varphi, \psi : [a,b] \to \mathbb{R}$ with $\varphi(x) < \psi(x)$, $\forall x \in [a,b]$, such that
$$D = \{(x,y) \in \mathbb{R}^2 \mid a \le x \le b,\ \varphi(x) \le y \le \psi(x)\}.$$
- A subset $D \subseteq \mathbb{R}^2$ is called *simple with respect to the axis Ox* if there exist continuous functions $\gamma, \omega : [c,d] \to \mathbb{R}$ with $\gamma(y) < \omega(y)$, $\forall y \in [c,d]$, such that
$$D = \{(x,y) \in \mathbb{R}^2 \mid \gamma(y) \le x \le \omega(y),\ c \le y \le d\}.$$

Theorem

Let $D \subseteq \mathbb{R}^2$ be a simple domain with respect to the axis Oy and $f \in C(D)$. Then

$$\iint_D f(x,y)dxdy = \int_a^b \Big(\int_{\varphi(x)}^{\psi(x)} f(x,y)dy \Big) dx,$$

where the functions $\varphi, \psi : [a,b] \to \mathbb{R}$ with $\varphi(x) < \psi(x)$ are such that $D = \{(x,y) \in \mathbb{R}^2 \mid a \le x \le b,\ \varphi(x) \le y \le \psi(x)\}$.

Remark. If $f \in C(D)$, with D being simple with respect to the axis Ox, *i.e.*

$$D = \{(x,y) \in \mathbb{R}^2 \mid \gamma(y) \le x \le \omega(y),\ c \le y \le d\},$$

then the corresponding formula is the following

$$\iint_D f(x,y)dxdy = \int_c^d \left(\int_{\gamma(y)}^{\omega(y)} f(x,y)dx \right) dy.$$

Example

Let $D = \{(x, y) \in \mathbb{R}^2_+ | 1 \leq xy \leq 3,\ 1 \leq \frac{y}{x} \leq 4\}$. We will compute the area of D:

$$\text{area}(D) = \mu_J(D) = \iint_D dx\, dy.$$

Since $D = D_1 \cup D_2 \cup D_3$, with $\overset{\circ}{D}_i \cap \overset{\circ}{D}_j = \emptyset$, $\forall i, j \in \{1, 2, 3\}$, $i \neq j$, where
$D_1 = \{(x, y) \in \mathbb{R}^2 \mid \gamma_1(y) = \frac{1}{y} \leq x \leq \omega_1(y) = y, 1 \leq y \leq \sqrt{3}\}$,
$D_2 = \{(x, y) \in \mathbb{R}^2 \mid \gamma_2(y) = \frac{1}{y} \leq x \leq \omega_2(y) = \frac{3}{y}, \sqrt{3} \leq y \leq 2\}$ and
$D_3 = \{(x, y) \in \mathbb{R}^2 \mid \gamma_3(y) = \frac{y}{4} \leq x \leq \omega_3(y) = \frac{3}{y}, 2 \leq y \leq 2\sqrt{3}\}$, we get, since D_1, D_2, D_3 are simple domains with respect to the axis Ox:

$$\begin{aligned}
\mathrm{area}(D) &= \iint_D dx\,dy = \iint_{D_1} dx\,dy + \iint_{D_2} dx\,dy + \iint_{D_3} dx\,dy = \\
&= \int_1^{\sqrt{3}} \left(\int_{1/y}^{y} dx \right) dy + \int_{\sqrt{3}}^{2} \left(\int_{1/y}^{3/y} dx \right) dy + \int_2^{2\sqrt{3}} \left(\int_{y/4}^{3/y} dx \right) dy = \\
&= \int_1^{\sqrt{3}} \left(y - \frac{1}{y} \right) dy + \int_{\sqrt{3}}^{2} \frac{2}{y} dy + \int_2^{2\sqrt{3}} \left(\frac{3}{y} - \frac{y}{4} \right) dy = \\
&= \left(\frac{y^2}{2} - \ln y \right) \Big|_1^{\sqrt{3}} + 2 \ln y \Big|_{\sqrt{3}}^{2} + \left(3 \ln y - \frac{y^2}{8} \right) \Big|_2^{2\sqrt{3}} = \\
&= \frac{3}{2} - \frac{1}{2} \ln 3 - \frac{1}{2} + 2 \ln 2 - \ln 3 + 3 \ln 2 + \frac{3}{2} \ln 3 - \frac{3}{2} - 3 \ln 2 + \frac{1}{2} = 2 \ln 2.
\end{aligned}$$

Change of variables

Definition

Let Ω be a compact, Jordan measurable set in $\mathbb{R}^2$ and $F : \Omega \to D \subseteq \mathbb{R}^2$, defined by $F(u, v) = (x(u, v), y(u, v))$, $(u, v) \in \Omega$ a bijective function which can be extended to a C^1-function on an open set $\Omega' \supseteq \Omega$ such that

$$\det(J_F)(u, v) = \frac{D(x, y)}{D(u, v)}(u, v) \neq 0, \forall (u, v) \in \Omega$$

(recall that J_F is the Jacobian matrix of F, while its determinant, $\frac{D(x,y)}{D(u,v)}$ is its Jacobian).Then D is also a compact, Jordan measurable set and F is called is a change of variables (coordinates) from Ω to D.

Proposition

Let $F:\Omega\to D$, $F(u,v)=(x(u,v),y(u,v))$, $(u,v)\in\Omega$ be a change of variables and $f:D\to\mathbb{R}$ a continuous function. Then

$$\iint_D f(x,y)dx\,dy=\iint_\Omega f(x(u,v),y(u,v))\left|\frac{D(x,y)}{D(u,v)}\right|(u,v)dudv.$$

Remarks.

1. We could apply the change of variables method for the example above. Let us set $xy=u$ and $\frac{y}{x}=v$, equivalently $x=\sqrt{\frac{u}{v}}$ and $y=\sqrt{uv}$, with $u\in[1,3]$ and $v\in[1,4]$. Then we obtain

$$\mathrm{area}(D)=\iint_D dxdy=\iint_\Omega\left|\frac{D(x,y)}{D(u,v)}\right|(u,v)dudv,$$

where $\Omega=\{(u,v)\in\mathbb{R}^2\mid 1\le u\le 3,1\le v\le 4\}=[1,3]\times[1,4]$ and

$$\frac{D(x,y)}{D(u,v)}(u,v)=\det\begin{bmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{bmatrix}(u,v)=\det\begin{bmatrix}\frac{1}{2\sqrt{uv}} & -\frac{\sqrt{u}}{2v\sqrt{v}}\\ \frac{\sqrt{v}}{2\sqrt{u}} & \frac{\sqrt{u}}{2\sqrt{v}}\end{bmatrix}=\frac{1}{2v}.$$

Therefore

$$\operatorname{area}(D) = \int_1^3 du \cdot \int_1^4 \left|\frac{1}{2v}\right| dv = \left(u\Big|_1^3\right)\left(\frac{1}{2}\ln v\Big|_1^4\right) = 2\frac{1}{2}\ln 4 = 2\ln 2.$$

2. A common change of variables is given by the transition from the cartesian coordinates (x, y) to *polar coordinates* (r, θ), by the relations

$$\begin{cases} x = r\cos\theta; \\ y = r\sin\theta, \end{cases} \quad \text{with } r \in [r_1, r_2] \subseteq [0, \infty),\ \theta \in [\theta_1, \theta_2] \subseteq [0, 2\pi].$$

The Jacobian of the transformation is
$\frac{D(x,y)}{D(r,\theta)}(r, \theta) = \det \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix} = r(\sin^2\theta + \cos^2\theta) = r.$

3. Sometimes we can use the generalized polar coordinates:

$$\begin{cases} x = ar\cos^\alpha\theta; \\ y = br\sin^\alpha\theta, \end{cases}$$

with $r \in [r_1, r_2] \subseteq [0, \infty)$ and $\theta \in [\theta_1, \theta_2] \subseteq [0, 2\pi]$, while a, b and α are appropriate parameters. When $\alpha = 1$, r and θ are called *elliptic coordinates*, corresponding to the equation of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$.

Example

Let us compute $\iint_D (y - x + 2)dxdy$, where $D = \{(x, y) \in \mathbb{R}^2 \mid \frac{x^2}{4} + \frac{y^2}{9} < 1\}$.
Using the elliptic transformation $(x, y) \to (r, \theta)$ given by $x = 2r\cos\theta$, $y = 3r\sin\theta$, with $0 \le r < 1$ and $0 \le \theta \le 2\pi$, we get

$$\begin{aligned}
\iint_D (y - x + 2)dx\,dy &= \int_0^{2\pi}\left[\int_0^1 (3r\sin\theta - 2r\cos\theta + 2)\left|\frac{D(x, y)}{D(r, \theta)}\right|(r, \theta)dr\right]d\theta \\
&= \int_0^{2\pi}\left[\int_0^1 (3r\sin\theta - 2r\cos\theta + 2)6rdr\right]d\theta \\
&= \int_0^{2\pi}(6\sin\theta - 4\cos\theta + 6)d\theta = 12\pi.
\end{aligned}$$

Mass and center of mass

Another application of the double integral is referring to the computation of the mass of a material plate D, with known mass density ρ, by the formula

$$\mathrm{mass}(D) = \iint_D \rho(x, y)\,dxdy.$$

We can also determine the coordinates of the center of mass (x_G, y_G) of D by the formulae

$$x_G = \frac{\iint_D x\rho(x, y)\,dxdy}{\iint_D \rho(x, y)\,dxdy} \quad \text{and} \quad y_G = \frac{\iint_D y\rho(x, y)\,dxdy}{\iint_D \rho(x, y)\,dxdy}.$$

The triple integral on compact sets

- The triple integral represents the multiple integral in the case $n = 3$.
- It is denoted by

$$\iiint_D f(x, y, z)dxdydz$$

where $f : D \to \mathbb{R}$ and D is a compact, Jordan measurable subset of $\mathbb{R}^3$.

Definition

A subset $D \subseteq \mathbb{R}^3$ is called *simple with respect to the axis Oz* if there exists a compact, Jordan measurable domain $\tilde{D} \subseteq \mathbb{R}^2$ and two continuous functions $\varphi, \psi : \tilde{D} \to \mathbb{R}$ with $\varphi(x, y) < \psi(x, y)$, $\forall (x, y) \in \tilde{D}$, such that

$$D = \{(x, y, z) \in \mathbb{R}^3 \mid \varphi(x, y) \le z \le \psi(x, y),\ \forall (x, y) \in \tilde{D}\}.$$

Such a domain in $\mathbb{R}^3$ has the *volume* (*i.e.*, Jordan measure) given by the formula

$$\mathrm{vol}(D) = \mu_J(D) = \iint_{\tilde{D}} \psi(x, y)dxdy - \iint_{\tilde{D}} \varphi(x, y)dxdy.$$

Proposition

Let $D \subseteq \mathbb{R}^3$ be simple with respect to Oz and let $f : D \to \mathbb{R}$ be a continuous function. Then

$$\iiint_D f(x,y,z)dxdy\,z = \iint_{\tilde{D}} \left(\int_{\varphi(x,y)}^{\psi(x,y)} f(x,y,z)dz \right) dxdy.$$

Example. Let us compute $\iiint_D \sqrt{x^2+y^2}dxdydz$, where D is the domain bounded by the surfaces $z=0$, $z=1$ and $z=\sqrt{x^2+y^2}$. We observe that

$$D = \{(x,y,z) \in \mathbb{R}^3 \mid \sqrt{x^2+y^2} \le z \le 1,\ \forall (x,y) \in \tilde{D}\},$$

where $\tilde{D} = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \le 1\}$. We take $\varphi(x,y) := \sqrt{x^2+y^2}$ and $\psi(x,y) := 1$, so we obtain

$$\begin{aligned}\iiint_D \sqrt{x^2+y^2}dxdydz &= \iint_{\tilde{D}} \left(\int_{\sqrt{x^2+y^2}}^{1} dz \right) \sqrt{x^2+y^2}dxdy \\ &= \iint_{\tilde{D}} \sqrt{x^2+y^2} \left(1 - \sqrt{x^2+y^2} \right) dxdy.\end{aligned}$$

In order to compute this double integral, we use the polar coordinates (r, θ):

$$\iint_{\tilde{D}} \sqrt{x^2+y^2}(1-\sqrt{x^2+y^2})dxdy = \int_0^{2\pi} \left(\int_0^1 r(1-r)rdr \right) d\theta =$$

$$= 2\pi \int_0^1 (r^2 - r^3)dr = 2\pi \left(\frac{r^3}{3} - \frac{r^4}{4} \right) \Big|_0^1 = \frac{\pi}{6}$$

A change of variable formula holds in the case $n = 3$:

Proposition

Let $F : \Omega \to D$, $F(u, v, w) = (x(u, v, w), y(u, v, w), z(u, v, w))$, $(u, v, w) \in \Omega$ be a change of variables between the compact, Jordan measurable domains Ω and D. If $f : D \to \mathbb{R}$ is a continuous function, then

$$\iiint_D f(x, y, z)dxdydz$$
$$= \iiint_\Omega f(x(u, v, w), y(u, v, w), z(u, v, w)) \left| \frac{D(x, y, z)}{D(u, v, w)} \right| (u, v, w)dudvdw.$$

Remarks.

1. The most used change of variables in $\mathbb{R}^3$ is the transition from the cartesian coordinates x, y, z to *spheric coordinates* r, θ, φ, given by

$$\begin{cases} x = r\sin\theta\cos\varphi, & r \in [r_1, r_2] \subseteq [0, +\infty], \\ y = r\sin\theta\sin\varphi, & \theta \in [\theta_1, \theta_2] \subseteq [0, \pi], \\ z = r\cos\theta, & \varphi \in [\varphi_1, \varphi_2] \subseteq [0, 2\pi]. \end{cases}$$

The Jacobian of this transformation is

$$\frac{D(x,y,z,)}{D(r,\theta,\varphi)}(r,\theta,\varphi) = \det \begin{bmatrix} \sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta \\ r\cos\theta\varphi & r\cos\theta\sin\varphi & -r\sin\theta \\ -r\sin\theta\sin\varphi & r\sin\theta\cos\varphi & 0 \end{bmatrix} = r^2\sin\theta.$$

2. Another change of variables for the triple integral is given by *cylindric coordinates*, transformation defined by

$$\begin{cases} x = r\cos\theta, & r \in [r_1, r_2] \subseteq [0, +\infty], \\ y = r\sin\theta, & \theta \in [\theta_1, \theta_2] \subseteq [0, 2\pi], \\ z = z, & z \in [z_1, z_2] \subseteq \mathbb{R}. \end{cases}$$

In this case we have $\frac{D(x,y,z)}{D(r,\theta,z)}(r,\theta,z) = r$.

Previous example:

$$\iiint_D \sqrt{x^2+y^2}dxdydz,$$

where $D = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2 \leq z \leq 1, \forall (x,y) \in \tilde{D}\}$ and $\tilde{D} = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \leq 1\}$. We can use the cylindric coordinates in order to obtain

$$\iiint_D \sqrt{x^2+y^2}dx\,dy\,dz = \int_0^1 \left(\int_0^{2\pi} \left(\int_r^1 rdz\right) d\theta\right) rdr = 2\pi \int_0^1 (1-r)r^2 dr = \frac{\pi}{6}$$

Again, the triple integral can used to the computation of the mass and of the center of mass of a body D, with known mass density ρ, by the formulae

$$\mathrm{mass}(D) = \iiint_D \rho(x,y,z)dxdydz$$

and

$$x_G = \frac{\iiint_D x\rho(x,y,z)dx\,dy\,dz}{\iiint_D \rho(x,y,z)dx\,dy\,dz},\ y_G = \frac{\iiint_D y\rho(x,y,z)dx\,dy\,dz}{\iiint_D \rho(x,y,z)dx\,dy\,dz},$$

$$z_G = \frac{\iiint_D z\rho(x,y,z)dx\,dy\,dz}{\iiint_D \rho(x,y,z)dx\,dy\,dz}.$$

Improper multiple integrals

As in the one-dimensional case, we can extend the notion of integral to the situations where the domain is either not compact or the integrand is not bounded.

Definition

Let D be a subset of $\mathbb{R}^n$ and $f : D \to \mathbb{R}$ a function which is Riemann integrable on any compact, Jordan measurable subset of D. We say that the integral $\int \cdots \int_D f(x_1, \ldots, x_n) dx_1 \ldots dx_n$ is *convergent* if for any sequence of bounded, Jordan measurable sets $(D_k)_{k \in \mathbb{N}^*}$, satisfying

(i) $\overline{D}_k \subset D_{k+1}, \forall k \in \mathbb{N}^*$;

(ii) $\bigcup\limits_{k=1}^{\infty} D_k = D$,

there exists and is finite $\lim\limits_{k \to \infty} \int \cdots \int_{\overline{D}_k} f(x_1, \ldots, x_n) dx_1 \ldots dx_n$, denoted $\int \cdots \int_D f(x_1, \ldots, x_n) dx_1 \ldots dx_n$..

In the case that the above limit does not exist or is infinite, we say that the integral $\int \cdots \int_D f(x_1, \ldots, x_n) dx_1 \ldots dx_n$ is *divergent*.

Examples

1. The integral $\iint_{\mathbb{R}^2} \mathrm{e}^{-x^2-y^2} dxdy$ *is convergent and is equal to* π, *because*

$$\begin{aligned}\iint_{\mathbb{R}^2} \mathrm{e}^{-x^2-y^2} dxdy &= \int_0^{2\pi} \left(\int_0^{\infty} \mathrm{e}^{-r^2} r dr \right) d\theta = (-2\pi) \lim_{a\to\infty} \left(-\frac{1}{2} \mathrm{e}^{-r^2} \Big|_0^a \right) \\ &= \pi \lim_{a\to\infty} (1 - \mathrm{e}^{-a^2}) = \pi.\end{aligned}$$

2. Let us compute the improper (by the singularity in $(0,0)$) integral

$$I = \iint_D \frac{1}{(x^2+y^2)^{\alpha/2}} dxdy,$$

where $D = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 < \rho^2\}$, $\rho > 0$ and $\alpha > 0$: we have

$$I = \lim_{n\to\infty} \iint_{D_n} \frac{1}{(x^2+y^2)^{\alpha/2}} dx\, dy,$$

where $D_n = D \setminus B\left(\mathbf{0}_{\mathbb{R}^2}; \frac{1}{n}\right)$, $n \in \mathbb{N}^*$.

Passing to polar coordinates ($x = r\cos\theta,\ y = r\sin\theta$, with $\frac{1}{n} \le r \le \rho,\ \theta \in [0, 2\pi]$), we get:

$$I = \lim_{n\to\infty} \int_0^{2\pi} \left(\int_{1/n}^{\rho} \frac{r}{r^{\alpha}} dr \right) d\theta = (2\pi) \lim_{n\to\infty} \left(\int_{1/n}^{\rho} r^{1-\alpha} dr \right) =$$

$$= 2\pi \begin{cases} \lim\limits_{n\to\infty} \left(\frac{r^{2-\alpha}}{2-\alpha} \Big|_{1/n}^{\rho} \right), & 0 < \alpha \ne 2; \\ \lim\limits_{n\to\infty} \left(\ln r \big|_{1/n}^{\rho} \right), & \alpha = 2 \end{cases} = \begin{cases} 2\pi\rho^{2-\alpha}, & 0 < \alpha < 2; \\ +\infty, & \alpha \ge 2. \end{cases}$$

As a consequence, the integral is convergent if $\alpha \in (0, 2)$ and divergent if $\alpha \ge 2$.

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