LECTURE 9
LIMITS OF FUNCTIONS. CONTINUOUS FUNCTIONS

We suppose that the reader is familiar with the concept of limit and continuity for real functions of one variable. We would like to extend these notions to functions of several variables (with values in an Euclidean space), or more generally, to functions between metric spaces.

1. Limits of functions

Definition. Let (X, d) and (Y, d') be two metric spaces, $A \subseteq X$ a nonempty set, a function $f : A \to Y$ and $x_0 \in A'$ (i.e., x_0 is a limit point for A). We say that an element $l \in Y$ is the *limit* of f in x_0 if

$$\forall V \in \mathscr{V}_{\mathrm{d}'}(l),\ \exists U \in \mathscr{V}_{\mathrm{d}}(x_0),\ \forall x \in U \smallsetminus \{x_0\} : f(x) \in V,$$

where $\mathscr{V}_{\mathrm{d}}(x_0)$ and $\mathscr{V}_{\mathrm{d}'}(l)$ are the families of neighbourhoods of x_0, respectively l. In this case, we write $\lim\limits_{x\to x_0} f(x) = l$ or $f(x) \xrightarrow{x\to x_0} l$.

As in the case of limits of sequence, one can show that the limit of a function in a point, if existent, is unique. We say that the function f has a limit in the point x_0 if there exists $l \in Y$ such that $\lim\limits_{x\to x_0} f(x) = l$.

Instead of the families of neighbourhoods in the above definition one can use systems of neighbourhoods, in particular the systems of neighbourhoods composed by the balls centered in x_0, respectively l. In this regard, we have the following characterization of the limit of a function:

Proposition 1.1. *Let (X, d) and (Y, d') be two metric spaces, $A \subseteq X$ a nonempty set and a function $f : A \to Y$. For $x_0 \in A'$ and $l \in Y$, the following statements are equivalent:*

(i) $\lim\limits_{x\to x_0} f(x) = l$;

(ii) *if $\mathscr{U}(x_0)$ and $\mathscr{U}'(l)$ are systems of neighbourhoods for x_0, respectively l, then*

$$\forall V \in \mathscr{U}'(l),\ \exists U \in \mathscr{U}(x_0),\ \forall x \in U \smallsetminus \{x_0\} : f(x) \in V;$$

(iii) $\forall \varepsilon > 0,\ \exists \delta > 0,\ \forall x \in B_{\mathrm{d}}(x_0;\delta) \smallsetminus \{x_0\} : \mathrm{d}'(f(x), l) < \varepsilon$, *where $B_{\mathrm{d}}(x_0;\delta)$ is the open ball centered in x_0 of radius δ.*

Relation (iii) can be written

$$\forall \varepsilon > 0,\ \exists \delta > 0 : 0 < \mathrm{d}(x, x_0) < \delta \Rightarrow \mathrm{d}'(f(x), l) < \varepsilon.$$

In the particular case that X and Y are normed spaces, we have:

Proposition 1.2. *Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|')$ be two normed spaces, $A \subseteq X$ a nonempty set and a function $f : A \to Y$. An element $l \in Y$ is the limit of f in some point $x_0 \in A'$ if and only if*

$$\forall \varepsilon > 0,\ \exists \delta > 0 : 0 < \|x - x_0\| < \delta \Rightarrow \|f(x) - l\|' < \varepsilon.$$

With the help of convergent sequences we can give an important characterization for the limit of a function.

Theorem 1.3. *Let (X, d) and (Y, d') be two metric spaces, $A \subseteq X$ a nonempty set and a function $f : A \to Y$. An element $l \in Y$ is the limit of f in some point $x_0 \in A'$ if and only if for every sequence $(x_n)_{n\in\mathbb{N}^*} \subseteq A \smallsetminus \{x_0\}$ such that $\lim\limits_{n\to\infty} x_n = x_0$, we have $\lim\limits_{n\to\infty} f(x_n) = l$.*

Remarks.

1. When someone wants to prove that $\lim\limits_{x\to x_0} f(x) \neq l$, it is enough to give an example of some sequence $(x_n)_{n\in\mathbb{N}^*} \subseteq A \smallsetminus \{x_0\}$ converging to x_0 such that $f(x_n)$ does not converge to l.

2. If, moreover, somebody wants to show that $\lim\limits_{x\to x_0} f(x)$ does not exist, it is sufficient to point out two sequences $(x_n)_{n\in\mathbb{N}^*}$ and $(x'_n)_{n\in\mathbb{N}^*}$ in $A \smallsetminus \{x_0\}$ such that $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} x'_n = x_0$, $\lim\limits_{n\to\infty} f(x_n) = \ell$ and $\lim\limits_{n\to\infty} f(x'_n) = \ell'$, where $\ell \neq \ell'$..

Example. Let the function $f : \mathbb{R}^2 \smallsetminus \{(0,0)\} \to \mathbb{R}$ be defined by

$$f(x,y) := \frac{xy}{x^2 + y^2},\ (x,y) \in \mathbb{R}^2 \smallsetminus \{(0,0)\}.$$

Then $(0,0) \in A'$, where $A := \mathbb{R}^2 \setminus \{(0,0)\}$. If we take the sequence $(x_n, y_n)_{n\in\mathbb{N}^*} \subseteq \mathbb{R}^2 \setminus \{(0,0)\}$, $x_n := \frac{1}{n}$, $y_n := \frac{1}{n}$, $n \in \mathbb{N}^*$, we have $(x_n, y_n) \xrightarrow{n\to\infty} (0,0)$ and

$$f(x_n, y_n) = \frac{1}{2} \xrightarrow{n\to\infty} \frac{1}{2}.$$

On the other hand, if we take the sequence, $(x'_n, y'_n)_{n\in\mathbb{N}^*} \subseteq \mathbb{R}^2 \setminus \{(0,0)\}$, $x'_n := \frac{1}{n}$, $y'_n := \frac{1}{n^2}$, $n \in \mathbb{N}^*$, we have $(x'_n, y'_n) \xrightarrow{n\to\infty} (0,0)$ and

$$f(x'_n, y'_n) = \frac{\frac{1}{n^3}}{\frac{1}{n^2} + \frac{1}{n^4}} = \frac{n}{n^2+1} \xrightarrow{n\to\infty} 0.$$

The conclusion is that the function f does not possess a limit in the point $(0,0)$.

As in the case of limits of sequence, the following series of criteria applies in the case of limits of functions.

Proposition 1.4. *Let (X, d) and (Y, d') be two metric spaces, $A \subseteq X$ a nonempty set, the functions $f : A \to Y$, $g : A \to \mathbb{R}_+$ and $x_0 \in A'$, $l \in Y$. If*

(i) $\mathrm{d}'(f(x), l) \le g(x),\ \forall x \in A$;

(ii) $\lim\limits_{x\to x_0} g(x) = 0$,

then $\lim\limits_{x\to x_0} f(x) = l$.

Theorem 1.5 (Cauchy-Bolzano). *Let (X, d) be a metric space, (Y, d') a complete metric space, $A \subseteq X$ a nonempty set, $x_0 \in A'$ and a function $f : X \to Y$. Then f has a limit in the point x_0 if and only if*

$$\forall \varepsilon > 0,\ \exists \delta > 0,\ \forall x, x' \in B(x_0, \delta) \setminus \{x_0\} : \mathrm{d}'(f(x), f(x')) < \varepsilon.$$

Theorem 1.6. *Let (X, d) be a metric space, $(Y, \|\cdot\|)$ a normed space, $A \subseteq X$ a nonempty set, $x_0 \in A'$ and a function $f : X \to Y$.*

i) *If there exists $\lim\limits_{x\to x_0} f(x) = l$, then $\lim\limits_{x\to x_0} \|f(x)\| = \|l\|$.*

ii) *If $\lim\limits_{x\to x_0} \|f(x)\| = 0$, then $\lim\limits_{x\to x_0} f(x) = \mathbf{0}_Y$.*

iii) *If there exists $\lim\limits_{x\to x_0} \|f(x)\| > 0$, then there exists $\delta > 0$ such that $f(x) \neq \mathbf{0}_Y$, $\forall x \in B(x_0; \delta) \setminus \{x_0\}$.*

For functions having values in a normed space, the addition and the scalar multiplication are closed to the operation of taking limits. More precisely, we have:

Theorem 1.7. *Let (X, d) be a metric space, $(Y, \|\cdot\|)$ a normed space, $A \subseteq X$ a nonempty set and $x_0 \in A'$.*

i) *If $f, g : X \to Y$ are functions such that $\lim\limits_{x\to x_0} f(x) = l_1 \in Y$ and $\lim\limits_{x\to x_0} g(x) = l_2 \in Y$ exist, then we have*

$$\lim_{x\to x_0} (\alpha f + \beta g)(x) = \alpha l_1 + \beta l_2.$$

ii) *If $f : X \to Y$ and $\varphi : X \to \mathbb{R}$ are functions such that $\lim\limits_{x\to x_0} f(x) = l \in Y$ and $\lim\limits_{x\to x_0} \varphi(x) = \alpha \in \mathbb{R}$ exist, then we have*

$$\lim_{x\to x_0} \varphi(x) f(x) = \alpha l.$$

In the case of Euclidean spaces, one can compute the limits of functions on components:

Theorem 1.8. *Let $A \subseteq \mathbb{R}^n$ be a nonempty set, $f : A \to \mathbb{R}^m$ and $\mathbf{x}_0 \in A'$. Then there exists $\lim\limits_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) = \mathbf{l} \in \mathbb{R}^m$ if and only if for every $k \in \{1, \ldots, m\}$ there exists the limit $\lim\limits_{\mathbf{x}\to\mathbf{x}_0} f_k(\mathbf{x}) = l_k \in \mathbb{R}$, where f_k, $1 \le k \le m$ are the m components of the function f. Moreover, in this case, $\mathbf{l} = (l_1, \ldots, l_m)$.*

The next result shows us how we can compute limits for composed functions:

Theorem 1.9 (substitution principle). *Let (X, d), (Y, d') and (Z, d'') be three metric spaces, $A \subseteq X$, $B \subseteq Y$ non-empty sets, the functions $f : A \to B$, $g : B \to Z$ and $x_0 \in A'$, $y_0 \in B'$. If*

(i) $\lim\limits_{x\to x_0} f(x) = y_0$;

(ii) $\lim\limits_{y\to y_0} g(y) = l \in Z$;

(iii) $\exists V \in \mathscr{V}(x_0),\ \forall x \in V \setminus \{x_0\} : f(x) \neq y_0$,

then $\lim\limits_{y\to y_0} g(f(x)) = l$.

A commun mistake when computing limits of functions of several variables is to *iterate* the limit. Let us exemplify that mistake by considering the function $f : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}$ defined by

$$f(x,y) := \frac{x^2y^2}{x^2y^2 + (x-y)^2},\ (x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}.$$

Then, fixing some $y \in \mathbb{R}^*$, we have

$$\lim_{x\to 0} f(x,y) = 0.$$

Letting now $y \to 0$, we obtain the iterated limit

$$\lim_{y\to 0}\lim_{x\to 0} f(x,y) = 0.$$

By symmetry we get the other iterated limit

$$\lim_{x\to 0}\lim_{y\to 0} f(x,y) = 0.$$

However, f does not have a limit in $(0,0)$, since $f(\frac{1}{n},\frac{1}{n}) = 1 \xrightarrow{n\to\infty} 1$ and $f(\frac{1}{n},0) = 0 \xrightarrow{n\to\infty} 0$.

On the other hand, a function f might have a limit in some point, but not iterated limits. For that, let $A := \{(x,y) \in \mathbb{R}^2 \mid xy \neq 0\}$ and $f : A \to \mathbb{R}$ defined by

$$f(x,y) := (x+y)\sin\frac{1}{x}\cdot\sin\frac{1}{y}.$$

Then $|f(x,y)| \le g(x,y) := |x| + |y|$. Since $\lim\limits_{(x,y)\to(0,0)} g(x,y) = 0$, we get, by Proposition 1.4 that $\lim\limits_{(x,y)\to(0,0)} f(x,y) = 0$. On the other hand, if we try to compute the limit $\lim\limits_{x\to 0} f(x,y)$ for some $y \in \mathbb{R}^*$, we obtain that $\lim\limits_{x\to 0} x\sin\frac{1}{x} = 0$ (because $\left|x\sin\frac{1}{x}\right| \le |x|$, $\forall x \in \mathbb{R}^*$), but $x \mapsto \sin\frac{1}{x}$ does not have a limit in 0. Therefore, since

$$f(x,y) = \left(x\sin\frac{1}{x}\right)\sin\frac{1}{y} + \left(\sin\frac{1}{x}\right)\left(y\sin\frac{1}{y}\right),$$

$f(x,y)$ does not have a limit as $x \to 0$ if $\sin\frac{1}{y} \neq 0$, i.e. $y \neq \frac{1}{k\pi}$, $k \in \mathbb{Z}^*$. It is clear now that the problem of existence of the iterated limit $\lim\limits_{y\to 0}\lim\limits_{x\to 0} f(x,y)$ has a negative answer.

The above considerations can be easily extended to the case of more than two variables.

Let us now see what happens if we demand that taking limits to be done over a predefined direction (in a linear space).

Definition. Let $A \subseteq \mathbb{R}^n$ be a nonempty set, $f : A \to \mathbb{R}^m$ and $\mathbf{x}_0 \in \mathbb{R}^n$.

a) We say that the function f has a *limit* in $\mathbf{x}_0$ *along the direction* $\mathbf{u} \in \mathbb{R}^n$ if $0 \in \{t \ge 0 \mid \mathbf{x}_0 + t\mathbf{u} \in A\}'$ (i.e., there exists a sequence $t_n \searrow 0$ such that $\mathbf{x}_0 + t_n\mathbf{u} \in A$, $\forall n \in \mathbb{N}$) and there exists the limit of the function $(0,+\infty) \ni t \mapsto f(\mathbf{x}_0 + t\mathbf{u})$ in $t = 0$, i.e. there exists the limit (to be defined later)

$$l_{\mathbf{u}} := \lim_{t\searrow 0} f(\mathbf{x}_0 + t\mathbf{u}).$$

b) We say that the function f has a (k^{th}-) *partial limit* in $\mathbf{x}_0$ if f has a limit in $\mathbf{x}_0$ along the direction $\mathbf{e}_k$, for $k \in \{1,\dots,n\}$, where $\mathbf{e}_k = (0,\dots,0,\underset{\substack{\uparrow\\ k}}{1},0,\dots,0)$.

The existence of a global limit implies the existence of directional limits. More precisely, we have:

Proposition 1.10. *Let $A \subseteq \mathbb{R}^n$ be a nonempty set, $f : A \to \mathbb{R}^m$ and $\mathbf{x}_0 \in A'$ such that there exists $\lim\limits_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) = \mathbf{l} \in \mathbb{R}^m$. If $\mathbf{u} \in \mathbb{R}^n \setminus \{\mathbf{0}_{\mathbb{R}^n}\}$ is such that $0 \in \{t \ge 0 \mid \mathbf{x}_0 + t\mathbf{u} \in A\}'$, then there exists the limit of f in $\mathbf{x}_0$ along the direction $\mathbf{u}$ and is equal to $\mathbf{l}$.*

The converse of this result is not true; in fact, even if the limits along all the directions exist and are equal, a global limit might not exist, as the following example shows:

Example. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) := \begin{cases} \frac{xy^2}{x^2+y^4}, & (x,y) \neq (0,0);\\ 0, & (x,y) = (0,0).\end{cases}$$

Let (u,v) be a direction in $\mathbb{R}^2 \setminus \{(0,0)\}$. Then, for $t > 0$,

$$f((0,0) + t(u,v)) = f(tu,tv) = \frac{t^3uv^2}{t^2(u^2+t^2v^4)} = \frac{tuv^2}{u^2+t^2v^4}.$$

Hence $\lim\limits_{t\searrow 0} f((0,0)+t(u,v)) = 0$, *i.e.* the limit of f in $(0,0)$ along the direction (u,v) exists and is equal to 0. Of course, this takes place even if $(u,v)=(0,0)$.

However, f does not have a global limit in $(0,0)$ because $f\left(\frac{1}{n^2},\frac{1}{n}\right)=\frac{1}{2}\xrightarrow{n\to\infty}\frac{1}{2}\neq 0$.

When f is a function of one variable, the only possible non-null directions (up to a multiplicative positive scalar) are -1 and 1. In this case, we will speak about *left* and *right* limits.

Definition. Let $A\subseteq\mathbb{R}$ be a nonempty set.

a) We say that $x_0\in\mathbb{R}$ is a *left-limit point* (*right-limit point*) of A if x is a limit point for the set $A\cap(-\infty,x_0)$ $(A\cap(x_0,+\infty))$.

b) If $f:A\to\mathbb{R}^m$ is a function and x_0 is a left-limit point (right-limit point), we say that f has a *left-limit* (*right-limit*) in x_0 if there exists the limit of f in x_0 along the direction -1 (1). In this case, we will denote this limit $\lim\limits_{x\nearrow x_0} f(x)$, $f(x_0-0)$ or $f(x_0^-)$ ($\lim\limits_{x\searrow x_0} f(x)$, $f(x_0+0)$ or $f(x_0^+)$).

In the case $n=1$, the converse of Proposition 1.10 does hold:

Proposition 1.11. *Let $A\subseteq\mathbb{R}^n$ be a nonempty set, $f:A\to\mathbb{R}^m$ and x_0 be both a left-limit and a right-limit point of A. Then the limit $\lim\limits_{x\to x_0} f(x)$ exists if and only if both limits $\lim\limits_{x\nearrow x_0} f(x)$ and $\lim\limits_{x\searrow x_0} f(x)$ exist and are equal. In this case, $\lim\limits_{x\to x_0} f(x)=\lim\limits_{x\nearrow x_0} f(x)=\lim\limits_{x\searrow x_0} f(x)$.*

At the end of this section we recall some of the most usual limits of functions:

- $\lim\limits_{t\to 0}(1+t)^{1/t}=\mathrm{e}$;
- $\lim\limits_{t\to\pm\infty}\left(1+\frac{1}{t}\right)^t=\mathrm{e}$;
- $\lim\limits_{t\to 0}\dfrac{\log_a(1+t)}{t}=\dfrac{1}{\ln a}$ $(a>0,\ a\neq 1)$;
- $\lim\limits_{t\to 0}\dfrac{\ln(1+t)}{t}=1$;
- $\lim\limits_{t\to 0}\dfrac{a^t-1}{t}=\ln a$ $(a>0)$;
- $\lim\limits_{t\to 0}\dfrac{\mathrm{e}^t-1}{t}=1$;
- $\lim\limits_{t\to 0}\dfrac{(1+t)^r-1}{t}=r$ $(r\in\mathbb{R})$;
- $\lim\limits_{t\to 0}\dfrac{\sin t}{t}=1$;
- $\lim\limits_{t\to 0}\dfrac{\operatorname{tg} t}{t}=1$;
- $\lim\limits_{t\to 0}\dfrac{\arcsin t}{t}=1$;
- $\lim\limits_{t\to 0}\dfrac{\operatorname{arctg} t}{t}=1$.

2. Continuous functions

Definition. Let (X,d) and (Y,d') be two metric spaces, $A\subseteq X$ a nonempty set and a function $f:A\to Y$.

a) We say that f is *continuous* in a point $x_0\in A$ if

$$\forall V\in\mathscr{V}_{\mathrm{d}'}(f(x_0)),\ \exists U\in\mathscr{V}_{\mathrm{d}}(x_0),\ \forall x\in U: f(x)\in V.$$

b) We say that f is *discontinuous* in a point $x_0\in A$ if f is not continuous in x_0; in this case, we also say that x_0 is a *discontinuity point* of f.

c) We say that f is *continuous* if f is continuous in x_0, for every $x_0\in A$.

We see that the notion of continuity of a function is very close to that of the limit of a function. In fact, f is continuous in $x_0\in A$ if and only if either x_0 is a limit point for A and $\lim\limits_{x\to x_0} f(x)=f(x_0)$ or x_0 is an isolated point.

As in the case of limits of functions, one can characterize the continuity with systems of neighbourhoods or in the ε-δ language. We present only the latter.

Proposition 2.1. *Let (X,d) and (Y,d') be two metric spaces, $A\subseteq X$ a nonempty set, a function $f:A\to Y$ and $x_0\in A$. Then f is continuous in x_0 if and only if*

$$\forall\varepsilon>0,\ \exists\delta>0:\mathrm{d}(x,x_0)<\delta\Rightarrow\mathrm{d}'(f(x),f(x_0))<\varepsilon.$$

The continuity in some point can be characterized with sequences, too.

Theorem 2.2. *Let (X,d) and (Y,d') be two metric spaces, $A\subseteq X$ a nonempty set, a function $f:A\to Y$ and $x_0\in A$. Then f is continuous in x_0 if and only if for every sequence $(x_n)_{n\in\mathbb{N}^*}\subseteq A$ such that $\lim\limits_{n\to\infty} x_n=x_0$, we have $\lim\limits_{n\to\infty} f(x_n)=f(x_0)$.*

It turns out that the global continuity (*i.e.*, in all points) of a function can be characterized in terms of open or closed sets.

Theorem 2.3. *Let (X, d) and (Y, d') be two metric spaces and a function $f : X \to Y$. Then the following statements are equivalents:*

(i) *f is continuous;*
(ii) *for every open set $D \subseteq Y$, the set $f^{-1}[D]$ is open (with respect to* d*);*
(iii) *for every closed set $F \subseteq Y$, the set $f^{-1}[F]$ is closed;*
(iv) *for every subset $A \subseteq Y$, we have $\overline{f^{-1}[A]} \subseteq f^{-1}[\overline{A}]$.*

Definition. Let (X, d) and (Y, d') be two metric spaces, $A \subseteq X$ a nonempty set, $x_0 \in A'$ and a function $f : A \to Y$. If $\lim\limits_{x \to x_0} f(x) = l \in Y$, then the function $\tilde{f} : A \cup \{x_0\} \to Y$ defined by

$$\tilde{f}(x) := \begin{cases} f(x), & x \in A \setminus \{x_0\}; \\ l, & x = x_0 \end{cases}$$

is continuous in x_0 and is called the *extension by continuity* of f in x_0.

Definition. Let (X, d) and (Y, d') be two metric spaces.

a) We say that a bijective function $f : X \to Y$ is a *homeomorphism* if f and f^{-1} are both continuous.
b) We say that (X, d) and (Y, d') are *homeomorphic* if there exists a homeomorphism between them.

Remark. If f is an isometry between (X, d) and (Y, d'), then f is continuous. If, moreover, f is bijective, then f is a homeomorphism.

Let us introduce some stronger notions of continuity.

Definition. Let (X, d) and (Y, d') be two metric spaces and $f : X \to Y$ a function.

a) The function f is called *uniformly continuous* if

$$\forall \varepsilon > 0,\ \exists \delta > 0,\ \forall x, y \in X : \mathrm{d}(x, y) < \delta \Rightarrow \mathrm{d}'(f(x), f(y)) < \varepsilon.$$

b) The function f is called *Lipschitz-continuous* if there exists a constant $c_1 > 0$, called the *Lipschitz constant* of f, such that

$$\mathrm{d}'(f(x), f(y)) \leq c_1 \mathrm{d}(x, y),\ \forall x, y \in X.$$

c) The function f is called *Hölder-continuous* of order $\alpha \in (0, 1]$ if there exists a constant $c_\alpha > 0$ such that

$$\mathrm{d}'(f(x), f(y)) \leq c_\alpha \left[\mathrm{d}(x, y)\right]^\alpha,\ \forall x, y \in X.$$

Remarks.
1. An uniformly continuous function is continuous.
2. A Lipschitz-continuous is a Hölder-continuous function of order 1 (and vice versa).
3. Any Hölder-continuous function is uniformly continuous (for $\varepsilon > 0$, it is enough to set $\delta := \left(\frac{\varepsilon}{c_\alpha}\right)^{1/\alpha}$).

Theorem 2.4. *Let (X, d), (Y, d') and (Z, d'') be three metric spaces, $A \subseteq X$, $B \subseteq Y$ non-empty sets and the functions $f : A \to B$, $g : B \to Z$.*

i) *If f is continuous in some point $x_0 \in A$ and g is continuous in $y_0 := f(x_0)$, then $g \circ f$ is continuous in x_0.*
ii) *If f and g are continuous, then $g \circ f$ is continuous.*

Theorem 2.5. *Let (X, d) be a metric space, $(Y, \|\cdot\|)$ a normed space, $A \subseteq X$ a nonempty set and $x_0 \in A$.*

i) *If the functions $f, g : X \to Y$ are continuous in x_0, then $\alpha f + \beta g$ is continuous in x_0.*
ii) *If the functions $f : X \to Y$ and $\varphi : X \to \mathbb{R}$ are continuous in x_0, then $\varphi \cdot f$ is continuous in x_0.*

In general, if f is a continuous function between metric spaces, D is an open set and F is a closed set, then $f[D]$ is not necessarely open set, nor $f[F]$ is a closed set. However, there is a property which is preserved by continuity, and that is *compactness*.

Definition. Let (X, d) be a metric space. We say that a subset $K \subseteq X$ is *compact* if every sequence $(x_n)_{n \in \mathbb{N}} \subseteq K$ contains a convergent subsequence.

By Bolzano-Weierstrass theorem, the compact subsets of $\mathbb{R}$ are the closed, bounded subsets.

Theorem 2.6. *Let (X, d), (Y, d') be metric spaces, $K \subseteq X$ a non-empty compact subset and $f : K \to Y$ a continuous function. Then $f[K]$ is compact.*

An immediate consequence is the following result:

Theorem 2.7 (Weierstrass). *Let (X, d) be a metric space, $K \subseteq X$ a non-empty compact subset and $f : K \to \mathbb{R}$ a continuous function. Then the function f is bounded and there exist $x_0, x_1 \in K$ such that $f(x_0) := \min_{x \in K} f(x)$ and $f(x_1) := \max_{x \in K} f(x)$.*

Theorem 2.8 (Cantor). *Let (X, d), (Y, d') be metric spaces, $K \subseteq X$ a non-empty compact subset and $f : K \to Y$ a continuous function. Then f is uniformly continuous.*

Let us now analyze continuity of functions between Euclidean spaces.

Theorem 2.9. *Let $A \subseteq \mathbb{R}^n$ be a nonempty set, $f : A \to \mathbb{R}^m$ and $\mathbf{x}_0 = (x_1^0, \ldots, x_m^0) \in A$. Then f is continuous in $\mathbf{x}_0$ if and only if f_k is continuous in x_k^0 for every $k \in \{1, \ldots, m\}$.*

Proposition 2.10. *If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear mapping, then T is continuous.*

Definition. Let $A \subseteq \mathbb{R}$ be a nonempty set and $f : A \to \mathbb{R}^m$.

- ***a)*** We say that f is *left-continuous* (*right-continuous*) in $x_0 \in A$ if $f|_{A \cap (-\infty, x_0]}$ ($f|_{A \cap [x_0, +\infty)}$) is continuous in x_0.
- ***b)*** We say that f is *left-continuous* (*right-continuous*) if f is left-continuous (right-continuous) in every $x_0 \in A$.

Proposition 2.11. *Let $A \subseteq \mathbb{R}^n$ be a nonempty set, $f : A \to \mathbb{R}^m$ and $x_0 \in A$. Then f is continuous in x_0 if and only if f is both left-continuous and right-continuous in x_0.*

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