LECTURE 11
APPLICATIONS OF DIFFERENTIABILITY. EXTREMA AND CONSTRAINED EXTREMA.

One of the possible applications of differentiability of functions is establishing extremal points for functions involved in optimization problems. More precisely, these problems are targeting the minimization (or the maximization) of a so-called *cost functional* (*profit function*), in presence or absence of some restrictions.

After giving some examples, we will expose the theoretical aspects we need in order to approach such problems.

1. Optimization problems in $\mathbb{R}^n$

Example 1. ***Least squares method***

Suppose that, by experimenting on a certain physical quantity, we have obtained the values $b_1, b_2, \ldots, b_p$, corresponding to the "input" data $a_1, a_2, \ldots, a_p$ ($p \in \mathbb{N}^*$). If we represent the points (a_k, b_k) ($k = \overline{1,p}$) in an orthogonal system of reference in plane, we can (visually) estimate the nature of the unknown function φ satisfying, for any $k = \overline{1,p}$, $\varphi(a_k) = b_k$. More precisely, we estimate that the function φ has a certain type (polynomial, exponential, trigonometric, etc.), whose characteristics (real parameters) $c_1, c_2, \ldots, c_n$ have to be identified. For this purpose we use the method of least squares, by considering the problem of the minimization of the expression

$$\sum_{k=1}^{p} \left(\varphi(a_k; c_1, c_2, \ldots, c_n) - b_k\right)^2,$$

with respect to $(c_1, c_2, \ldots, c_n) \in \mathbb{R}^n$. Solving this problem (of unrestrained extrema), *i.e.* finding $(c_1^0, c_2^0, \ldots, c_n^0) \in \mathbb{R}^n$ such that

$$\min\left\{\sum_{k=1}^{p} \left(\varphi(a_k; c_1, c_2, \ldots, c_n) - b_k\right)^2\right\} = \sum_{k=1}^{p} \left(\varphi(a_k; c_1^0, c_2^0, \ldots, c_n^0) - b_k\right)^2,$$

we can conclude that the physical quantity under study has the law $y = \varphi(x; c_1^0, c_2^0, \ldots, c_n^0)$.

We remark that if the graphic of $\{(a_k, b_k) | k = \overline{1,p}\}$ suggests us that φ is linear, then we can take $n = 2$ and $\varphi(x) := c_1 x + c_2$. In this case, the method of least squares leads us to the problem of minimization of the expression

$$\sum_{k=1}^{p} (c_1 a_k + c_2 - b_k)^2$$

with respect to (c_1, c_2).

Example 2. ***Realization of a maximum profit or a minimum cost***

In an economic theory, the space $\mathbb{R}^n$ is interpreted as the space of *consumer goods output*, where each *consumer good* is characterized by a certain index $i \in \{1, 2, \ldots, n\}$, while the *consumer goods output* is a vector $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, where the component x_i is the quantity in which the consumer good i is produced. In such a context, a *system of prices* is characterized by a function which associates to each consumer goods output a certain value. It is naturally considered that such function is a linear function, hence characterized by a vector $\mathbf{p} = (p_1, p_2, \ldots, p_n) \in \mathbb{R}^n$, where p_i is the "unitary" price of the consumer good i; the system of prices is therefore given by $\sum_{i=1}^{n} p_i x_i$.

When a production company is producing certain consumer goods, its interest is to realize the corresponding production such that the production costs to be minimal and/or the production profit to be maximal. A *cost function* and/or a *profit function* convenably chosen, named objective functions, are used by economical theorist.

In the situation that the set of consumer goods output is $\mathbb{R}^n$, we deal with an unrestained extrema problem. If the set of consumer goods output is $K \subsetneq \mathbb{R}^n$ then we deal with a restricted extrema problem. In both cases, it is a *linear programming* problem if the objective function is linear and K is a linear subspace of $\mathbb{R}^n$. If the objective function is quadratic or convex, then the respective problem is called *quadratic*, respectively *convex optimization problem.*

Example 3. ***Maximal informational entropy***

The *entropy* was introduced as a mathematical quantity by Claude E. Shannon (1947). It is a function corresponding to the quantity of information offered by a certain source, by the means of a certain language, electrical signal or data file. This function, denoted by H is defined on the set of random variables

$$X = \begin{pmatrix} 1 & 2 & \dots & n \\ p_1 & p_2 & \dots & p_n \end{pmatrix}$$

and has the expression

$$H(X) = -\sum_{k=1}^{n} p_k \cdot \log_2 p_k,$$

where p_k is the probability ($p_k \in (0,1)$, $k = \overline{1,n}$ and $\sum_{k=1}^{n} p_k = 1$) that a particular message k is actually transmitted by the above source.

A particular problem is to find an optimal distribution of the random variable X such that the value $H(X)$ is maximal. In other words, we study the problem of maximizing

$$-\sum_{k=1}^{n} p_k \log_2 p_k$$

with respect to $(p_1, p_2, \dots, p_n)$ subject to the restriction $\sum_{k=1}^{n} p_k = 1$, $p_k \in (0,1)$, $k = \overline{1,n}$.

2. Unrestrained extrema

Definition. Let $A \subseteq \mathbb{R}^n$ be a non-empty set, $f : A \to \mathbb{R}$ a function and $\mathbf{x}_0 \in A$. We say that $\mathbf{x}_0$ is:

a) a *minimum point* of f if $f(\mathbf{x}) \geq f(\mathbf{x}_0)$, $\forall \mathbf{x} \in A$;
b) a *maximum point* of f if $f(\mathbf{x}) \leq f(\mathbf{x}_0)$, $\forall \mathbf{x} \in A$;
c) a *strict minimum point* of f if $f(\mathbf{x}) > f(\mathbf{x}_0)$, $\forall \mathbf{x} \in A \setminus \{\mathbf{x}_0\}$;
d) a *strict maximum point* of f if $f(\mathbf{x}) < f(\mathbf{x}_0)$, $\forall \mathbf{x} \in A \setminus \{\mathbf{x}_0\}$;
e) a *local minimum point* of f if there exists $V \in \mathscr{V}(\mathbf{x}_0)$ such that $f(\mathbf{x}) \geq f(\mathbf{x}_0)$, $\forall \mathbf{x} \in A \cap V$;
f) a *local maximum point* of f if there exists $V \in \mathscr{V}(\mathbf{x}_0)$ such that $f(\mathbf{x}) \leq f(\mathbf{x}_0)$, $\forall \mathbf{x} \in A \cap V$;
g) a *local strict minimum point* of f if there exists $V \in \mathscr{V}(\mathbf{x}_0)$ such that $f(\mathbf{x}) > f(\mathbf{x}_0)$, $\forall \mathbf{x} \in A \cap (V \setminus \{\mathbf{x}_0\})$;
h) a *local strict maximum point* of f if there exists $V \in \mathscr{V}(\mathbf{x}_0)$ such that $f(\mathbf{x}) < f(\mathbf{x}_0)$, $\forall \mathbf{x} \in A \cap (V \setminus \{\mathbf{x}_0\})$;
i) an *extremum* (*strict extremum*, *local extremum*, *local strict extremum*) point of f if $\mathbf{x}_0$ is a minimum (strict minimum, local minimum, respectively a local strict minimum) point or a maximum (strict maximum, local maximum, respectively a local strict maximum) point of f.

Remark. Any global extremum point for a function f is always a local extremum point for f. The converse is not true.
1. If $D \subseteq \mathbb{R}^n$ is an open set and $f : D \to \mathbb{R}$ is a function, the problem of determining the global and/or local extremum points (and the associated extreme values) for f is called an *unrestricted extremum problem.*
2. Let $D \subseteq \mathbb{R}^n$ be an open set, $f : D \to \mathbb{R}$ a function and $A \subseteq \mathbb{R}^n$ a set of constraints. The problem of determining the global and/or local extremum points (and the associated extreme values) for $f\big|_{A \cap D}$ is called a *restricted extremum problem.*

Theorem 2.1 (Fermat). *Let $A \subseteq \mathbb{R}^n$ be a non-empty set, $f : A \to \mathbb{R}$ a function and $\mathbf{x}_0 \in \mathring{A}$. If $\mathbf{x}_0$ is a local extremum point of f and f has partial derivatives (of first order) in $\mathbf{x}_0$, then*

$$\frac{\partial f}{\partial x_1}(\mathbf{x}_0) = \cdots = \frac{\partial f}{\partial x_n}(\mathbf{x}_0) = 0.$$

Remark. The conclusion of the theorem can be written as $(\nabla f)(\mathbf{x}_0) = \mathbf{0}_{\mathbb{R}^n}$. If, moreover f is Fréchet differentiable in $\mathbf{x}_0$, this is also equivalent to $\mathrm{d}f(\mathbf{x}_0) = \mathbf{0}_{\mathrm{L}(\mathbb{R}^n;\mathbb{R})}$.

Definition. Let $A \subseteq \mathbb{R}^n$, $\mathbf{x}_0 \in \mathring{A}$ and $f : A \to \mathbb{R}$ a function, Fréchet differentiable in $\mathbf{x}_0$. The element $\mathbf{x}_0$ is called a *critical point* (or *stationary point*) of f if $\mathrm{d}f(\mathbf{x}_0) = \mathbf{0}_{\mathrm{L}(\mathbb{R}^n;\mathbb{R})}$ (*i.e.* $(\nabla f)(\mathbf{x}_0) = \mathbf{0}_{\mathbb{R}^n}$).

Remark. Fermat theorem states that if $\mathbf{x}_0 \in \mathring{A}$ is a local extremum point of a Fréchet differentiable function f in $\mathbf{x}_0$, then $\mathbf{x}_0$ is critical point of f. The converse is not true, as we can see from the following example.

Let $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = x^2 - y^2$, $x, y \in \mathbb{R}$. Then $(0,0)$ is a critical point of f, since $\dfrac{\partial f}{\partial x}(0,0) = 2x\big|_{x=0,\,y=0} = 0$ and $\dfrac{\partial f}{\partial y}(0,0) = -2y\big|_{x=0,\,y=0} = 0$. On the other hand, $f(x,0) = x^2 > f(0,0) = 0 > -y^2 = f(0,y)$ for each $x \neq 0$ and each $y \neq 0$. This implies that $(0,0)$ is not a local minimum point, nor a local maximum point for f.

Definition. Let $A \subseteq \mathbb{R}^n$, $\mathbf{x}_0 \in \mathring{A}$ and $f : A \to \mathbb{R}$ a function, Fréchet differentiable in $\mathbf{x}_0$. If $\mathbf{x}_0$ is a critical point of f, but is not a local extremum for f, we say that $\mathbf{x}_0$ is a *saddle point* of f.

For differentiable functions of order at least 2, there exist criteria (sufficient conditions) for the identification of the extremum points among the critical points. Let us begin with the case $n = 1$:

Theorem 2.2. *Let $A \subseteq \mathbb{R}$ be an interval, $x_0 \in \mathring{A}$ and $f : A \to \mathbb{R}$ a function n-times ($n \geq 2$) derivable in x_0. Suppose that $f'(x_0) = f''(x_0) = \ldots = f^{(n-1)}(x_0) = 0$ and $f^{(n)}(x_0) \neq 0$.*

- ***i)*** *If n is even, then x_0 is a local extremum point of f. More precisely, if $f^{(n)}(x_0) > 0$ then x_0 is a local minimum point of f, while if $f^{(n)}(x_0) < 0$ then x_0 is a local maximum point of f.*
- ***ii)*** *If n is odd, then x_0 is not a local extremum point of f.*

Let now pass to the case $n > 1$; we will characterize extremum points of a function f in $\mathbf{x}_0$ according to the Fréchet derivative of order 2 in $\mathbf{x}_0$. Recall that

$$\left(\mathrm{d}^2 f(\mathbf{x}_0)\right)(\mathbf{v}) = \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{x}_0) v_i v_j, \forall \mathbf{v} = (v_1, v_2, \ldots, v_n) \in \mathbb{R}^n.$$

Theorem 2.3. *Let $A \subseteq \mathbb{R}^n$ a non-empty set and $f : A \to \mathbb{R}$ a function which is C^2 on a neighbourhood of a critical point $\mathbf{x}_0 \in \mathring{A}$.*

- ***i)*** *If $\left(\mathrm{d}^2 f(\mathbf{x}_0)\right)(\mathbf{v}) > 0$, $\forall \mathbf{v} \in \mathbb{R}^n \setminus \{\mathbf{0}_{\mathbb{R}^n}\}$, then $\mathbf{x}_0$ is a local minimum point for f.*
- ***ii)*** *If $\left(\mathrm{d}^2 f(\mathbf{x}_0)\right)(\mathbf{v}) < 0$, $\forall \mathbf{v} \in \mathbb{R}^n \setminus \{\mathbf{0}_{\mathbb{R}^n}\}$, then $\mathbf{x}_0$ is a local maximum point for f.*
- ***iii)*** *If $\mathrm{d}^2 f(\mathbf{x}_0)$ is an undefined quadratic form (i.e. $\exists \mathbf{v}', \mathbf{v}'' \in \mathbb{R}^n$ such that $\left(\mathrm{d}^2 f(\mathbf{x}_0)\right)(\mathbf{v}') < 0$ and $\left(\mathrm{d}^2 f(\mathbf{x}_0)\right)(\mathbf{v}'') > 0$), then $\mathbf{x}_0$ is a saddle point for f.*

Remarks.
1. If $\mathrm{d}^2 f(\mathbf{x}_0)$ is a positive or a negative semidefined quadratic form (*i.e.*, $\left(\mathrm{d}^2 f(\mathbf{x}_0)\right)(\mathbf{v}) \geq 0$, $\forall \mathbf{v} \in \mathbb{R}^n$ or, respectively, $\left(\mathrm{d}^2 f(\mathbf{x}_0)\right)(\mathbf{v}) \leq 0$, $\forall \mathbf{v} \in \mathbb{R}^n$), then we cannot determine the nature of the critical point.
2. By the hypotheses of the above result, we have that $\dfrac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{x}_0) = \dfrac{\partial^2 f}{\partial x_j \partial x_i}(\mathbf{x}_0)$, $\forall i \neq j$, by Schwarz theorem, so $\mathrm{d}^2 f(\mathbf{x}_0)$ is indeed a quadratic form (*i.e.*, coming from a symmetric bilinear form). We call the matrix associated to this quadratic form (relative to the canonical basis) the *Hessian* of f:

$$H_f(\mathbf{x}_0) := \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{x}_0)\right)_{1 \leq i,j \leq n}.$$

By Lecture 8, we have several methods for determining if $H_f(\mathbf{x}_0)$ is positive definite, negative definite or neither.

Proposition 2.4. *Let $A \subseteq \mathbb{R}^n$ a non-empty set and $f : A \to \mathbb{R}$ a function which is C^2 on a neighbourhood of a critical point $\mathbf{x}_0 \in \mathring{A}$.*

- ***i)*** *If all the eigenvalues of $H_f(\mathbf{x}_0)$ are positive, then $\mathbf{x}_0$ is a local minimum point for f.*
- ***ii)*** *If all the eigenvalues of $H_f(\mathbf{x}_0)$ are negative, then $\mathbf{x}_0$ is a local maximum point for f.*
- ***iii)*** *If $H_f(\mathbf{x}_0)$ has at least one positive and one negative eigenvalue, then $\mathbf{x}_0$ is a saddle point for f.*

Of course, we cannot say anything if all the eigenvalues are either all non-negative or all non-positive.

Proposition 2.5. *Let $A \subseteq \mathbb{R}^n$ a non-empty set and $f : A \to \mathbb{R}$ a function having partial derivatives of order 2 which are continuous in a neighbourhood of a critical point $\mathbf{x}_0$ of f. Let $\Delta_k = \det\left(\frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{x}_0)\right)_{1 \leq i,j \leq k}$, $k = \overline{1,n}$, the principal minors of the Hessian $H_f(\mathbf{x}_0)$.*

- ***i)*** *If $\Delta_k > 0$, $\forall k = \overline{1,n}$, then $\mathbf{x}_0$ is a local minimum point for f.*
- ***ii)*** *If $(-1)^{k+1}\Delta_k < 0$, $\forall k = \overline{1,n}$, then $\mathbf{x}_0$ is a local maximum point for f.*
- ***iii)*** *If there exist $j, k = \overline{1,n}$ such that $\Delta_j < 0$ and $(-1)^{k+1}\Delta_k > 0$, then $\mathbf{x}_0$ is a saddle point for f.*

Again, we cannot conclude anything by just knowing that $\Delta_k \geq 0$, $\forall k = \overline{1,n}$ or $(-1)^{k+1}\Delta_k \leq 0$, $\forall k = \overline{1,n}$.
Remark. In the particular case $n = 2$, the above result tells us that if $f : A \subseteq \mathbb{R}^2 \to \mathbb{R}$ is C^2 on a neighbourhood of a critical point $\mathbf{x}_0 \in \mathring{A}$ and we denote $p := \dfrac{\partial^2 f}{\partial x^2}(\mathbf{x}_0)$, $q := \dfrac{\partial^2 f}{\partial x \partial y}(\mathbf{x}_0)$, $r := \dfrac{\partial^2 f}{\partial y^2}(\mathbf{x}_0)$, then:

- i) when $p > 0$ and $pr - q^2 > 0$, $\mathbf{x}_0$ is a local minimum point for f;
- ii) when $p < 0$ and $pr - q^2 > 0$, $\mathbf{x}_0$ is a local maximum point for f;
- iii) when $pr - q^2 < 0$, $\mathbf{x}_0$ is a saddle point for f;
- iv) when $pr - q^2 = 0$, we cannot establish the nature of $\mathbf{x}_0$ by this method.

Coming back to the method of least squares, with φ being linear, we see that the function $f : \mathbb{R}^2 \to \mathbb{R}$, defined by

$$f(c_1, c_2) = \sum_{k=1}^{l} (c_1 a_k + c_2 - b_k)^2$$

is of class C^2 on $\mathbb{R}^2$ and has as critical points the elements $\left(c_1^0, c_2^0\right) \in \mathbb{R}^2$ satisfying

$$\begin{cases} \dfrac{\partial f}{\partial c_1}\left(c_1^0, c_2^0\right) = 2\displaystyle\sum_{k=1}^{l} (c_1^0 a_k + c_2^0 - b_k) a_k = 0; \\ \dfrac{\partial f}{\partial c_2}\left(c_1^0, c_2^0\right) = 2\displaystyle\sum_{k=1}^{l} (c_1^0 a_k + c_2^0 - b_k) = 0. \end{cases}$$

In order to solve this equation, equivalent to the linear system

$$\begin{cases} c_1^0 \displaystyle\sum_{k=1}^{l} a_k^2 + c_2^0 \sum_{k=1}^{l} a_k = \sum_{k=1}^{l} b_k a_k; \\ c_1^0 \displaystyle\sum_{k=1}^{l} a_k^2 + l c_2^0 = \sum_{k=1}^{l} b_k, \end{cases}$$

we compute its determinant

$$\begin{vmatrix} \displaystyle\sum_{k=1}^{l} a_k^2 & \displaystyle\sum_{k=1}^{l} a_k \\ \displaystyle\sum_{k=1}^{l} a_k & l \end{vmatrix} = l \sum_{k=1}^{l} a_k^2 - \left(\sum_{k=1}^{l} a_k\right)^2,$$

which is non-zero when not all $a_1, a_2, \ldots, a_l$ are equal, by Cauchy-Buniakowski-Schwarz inequality. In this case, we have

$$c_1^0 = \frac{l \sum_{k=1}^{l} a_k b_k - \left(\sum_{k=1}^{l} a_k\right)\left(\sum_{k=1}^{l} b_k\right)}{l \sum_{k=1}^{l} a_k^2 - \left(\sum_{k=1}^{l} a_k\right)^2}$$

and

$$c_2^0 = \frac{\left(\sum_{k=1}^{l} b_k\right)\left(\sum_{k=1}^{l} a_k^2\right) - \left(\sum_{k=1}^{l} a_k\right)\left(\sum_{k=1}^{l} a_k b_k\right)}{l \sum_{k=1}^{l} a_k^2 - \left(\sum_{k=1}^{l} a_k\right)^2}.$$

We then have $p = \dfrac{\partial^2 f}{\partial c_1^2}\left(c_1^0, c_2^0\right) = 2\displaystyle\sum_{k=1}^{l} a_k^2$, $q = \dfrac{\partial^2 f}{\partial c_1 \partial c_2}\left(c_1^0, c_2^0\right) = 2\displaystyle\sum_{k=1}^{l} a_k$ and $r = \dfrac{\partial^2 f}{\partial c_2^2}\left(c_1^0, c_2^0\right) = 2l$. Again by Cauchy-Buniakowski-Schwarz inequality, we have $p > 0$ and $pr - q^2 > 0$, *i.e.* $\left(c_1^0, c_2^0\right)$ is a global minimum point for f (not only a local minimum point, because $\left(c_1^0, c_2^0\right)$ is the unique critical point of f).

3. Restricted extrema

As we have already noted, besides the situations we have to solve problems of extremum without constraints, there are also problems where we have to find extrema of functions constrained to some conditions. In this case, the respective extrema are called *restricted extrema*.

Let us specify the setting we will be working in. Let D be an open set (with respect to the Euclidean metric) in $\mathbb{R}^n \times \mathbb{R}^m$ ($n, m \in \mathbb{N}^*$) and $f : D \to \mathbb{R}$, $g : D \to \mathbb{R}^m$ functions of class C^1 on D. We look for the extremum points of f restricted to the supplimentary condition $g(\mathbf{x}, \mathbf{y}) = 0$, where $(\mathbf{x}, \mathbf{y}) \in D$.

Actually, by denoting $A := \{(\mathbf{x}, \mathbf{y}) \in D | g(\mathbf{x}, \mathbf{y}) = \mathbf{0}_{\mathbb{R}^n}\}$ and by $g_1, g_2, \ldots, g_m$ the components of g, we see that the above problem demands the finding of extremum points of $f|_A$. In order to fix the terminology, we say that a point $(\mathbf{x}_0, \mathbf{y}_0) \in A$ is a *local extremum* (*minimum*, *maximum*) or *saddle point* of f, *conditioned to the restrictions* $g_1(\mathbf{x}, \mathbf{y}) = 0$, $g_2(\mathbf{x}, \mathbf{y}) = 0$, $\ldots$, $g_m(\mathbf{x}, \mathbf{y}) = 0$, if it is a local extremum (minimum, maximum) or saddle point of $f|_A$.

The above problem may look as an unrestrained extrema one, but the extremum points $(\mathbf{x}_0, \mathbf{y}_0)$ of $f|_A$ will be elements of $\mathring{A}$ if and only if $g_1 = g_2 = \cdots = g_m = 0$ on a neighbourhood of $(\mathbf{x}_0, \mathbf{y}_0)$, which in fact means that we would not effectively deal with a restricted extrema problem. Hence in gneral we cannot apply the results from the previous section.

If D is bounded, then A is bounded, too; thus, since g is continuous, A is a closed, bounded set, hence compact. By the continuity of f, $f|_A$ will surely have a minimum and a maximum point in A. Therefore, the problem remains to determine these extremum points.

In order to approach this problem, we could try to solve first the system $g_k(\mathbf{x},\mathbf{y}) = 0$, $\forall\, k = \overline{1,m}$, with respect to $\mathbf{y}$. Finding a solution $\mathbf{y} = \varphi(\mathbf{x})$, $\mathbf{x} \in B$, would lead to studying the unrestrained extrema of the function $\mathbf{x} \mapsto f(\mathbf{x}, \varphi(\mathbf{x}))$, with $\mathbf{x} \in B$. The procedure is, however, difficult to apply in practice, because the equation $g(\mathbf{x},\mathbf{y}) = \mathbf{0}_{\mathbb{R}^n}$ is not easily solved.

This is why we will follow the method of *Lagrange multipliers*, described in the sequel.

Theorem 3.1 (existence of Lagrange multipliers). *Let $D \subseteq \mathbb{R}^n \times \mathbb{R}^m$ an open non-empty set, $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}^m$ C^1-functions on D. Let $g_1, g_2, \ldots, g_m$ be the components of g and $(\mathbf{x}_0,\mathbf{y}_0) \in D$ a local extremum point of f, conditioned to the restrictions $g_1(\mathbf{x},\mathbf{y}) = 0$, $g_2(\mathbf{x},\mathbf{y}) = 0, \ldots, g_m(\mathbf{x},\mathbf{y}) = 0$. If $\frac{D(g_1,g_2,\ldots,g_m)}{D(y_1,y_2,\ldots,y_m)}(\mathbf{x}_0,\mathbf{y}_0) \neq 0$, then there exist $\lambda_1, \lambda_2, \ldots, \lambda_m \in \mathbb{R}$ such that $(\mathbf{x}_0,\mathbf{y}_0)$ is a critical point for the C^1-function $L : D \to \mathbb{R}$ defined by*

$$L(\mathbf{x},\mathbf{y}) = f(\mathbf{x},\mathbf{y}) + \lambda_1 g_1(\mathbf{x},\mathbf{y}) + \lambda_2 g_2(\mathbf{x},\mathbf{y}) + \cdots + \lambda_m g_m(\mathbf{x},\mathbf{y}),\ (\mathbf{x},\mathbf{y}) \in D,$$

i.e.

$$(\star)\ \begin{cases} \dfrac{\partial L}{\partial x_k}(\mathbf{x}_0,\mathbf{y}_0) = 0,\ \forall k = \overline{1,n}; \\ \dfrac{\partial L}{\partial y_j}(\mathbf{x}_0,\mathbf{y}_0) = 0,\ \forall j = \overline{1,m}; \\ g_j(\mathbf{x}_0,\mathbf{y}_0) = 0,\ \forall j = \overline{1,m}. \end{cases}$$

Remarks.

1. The numbers $\lambda_1, \lambda_2, \ldots, \lambda_m$ are called *Lagrange multipliers.*

2. The conclusion of the above result can be rephrased as follows: the conditioned local extremum points of f are critical points of the associated function $L = f + \sum_{k=1}^m \lambda_k g_k$.

3. The function $\mathscr{L} : \mathbb{R}^m \times D \to \mathbb{R}$ defined by

$$\mathscr{L}(\boldsymbol{\lambda},\mathbf{x},\mathbf{y}) = f(\mathbf{x},\mathbf{y}) + \lambda_1 g_1(\mathbf{x},\mathbf{y}) + \lambda_2 g_2(\mathbf{x},\mathbf{y}) + \cdots + \lambda_m g_m(\mathbf{x},\mathbf{y}),\ \boldsymbol{\lambda} = (\lambda_1,\ldots,\lambda_m) \in \mathbb{R}^m,\ (\mathbf{x},\mathbf{y}) \in D,$$

is called the *Lagrangian* associated to f and the restriction g (we have added to L the new variable $\boldsymbol{\lambda} \in \mathbb{R}^m$ in order to obtain the function $\mathscr{L}$).

Then the triple $(\boldsymbol{\lambda}_0,\mathbf{x}_0,\mathbf{y}_0)$ satisfies the system $(\star)$ (*i.e.*, $(\mathbf{x}_0,\mathbf{y}_0)$ is a critical point for L, satisfying the restriction $g(\mathbf{x}_0,\mathbf{y}_0) = \mathbf{0}_{\mathbb{R}^m}$) if and only if $(\boldsymbol{\lambda}_0,\mathbf{x}_0,\mathbf{y}_0)$ is a critical point for $\mathscr{L}$, since

$$\frac{\partial \mathscr{L}}{\partial \lambda_j}(\boldsymbol{\lambda}_0,\mathbf{x}_0,\mathbf{y}_0) = g_j(\mathbf{x}_0,\mathbf{y}_0),\ j = \overline{1,m}.$$

4. Suppose now that $(\boldsymbol{\lambda}_0,\mathbf{x}_0,\mathbf{y}_0)$ satisfies the system $(\star)$ and f, g are functions of class C^2 on D (or at least on a neighbourhood of $(\boldsymbol{\lambda}_0,\mathbf{x}_0,\mathbf{y}_0)$). Since

$$L(\mathbf{x},\mathbf{y}) - L(\mathbf{x}_0,\mathbf{y}_0) = f(\mathbf{x},\mathbf{y}) - f(\mathbf{x}_0,\mathbf{y}_0),\ \forall (\mathbf{x},\mathbf{y}) \in A,$$

we see that $(\mathbf{x}_0,\mathbf{y}_0)$ is a conditioned local extremum point of f if and only if $(\mathbf{x}_0,\mathbf{y}_0)$ is a local extremum point for $L\big|_A$. Since $\frac{D(g_1,g_2,\ldots,g_m)}{D(y_1,y_2,\ldots,y_m)}(\mathbf{x}_0,\mathbf{y}_0) \neq 0$, one can solve (at least locally around $(\mathbf{x}_0,\mathbf{y}_0)$) the equation $g(\mathbf{x},\mathbf{y}) = 0$ and find the solution $\mathbf{y} = \varphi(\mathbf{x})$ (this is the implicit function theorem and is beyond the scope of this lecture). Therefore, we can determine sufficient conditions of conditioned local extremality by studying the Hessian of the function $\mathbf{x} \mapsto L(\mathbf{x}, \varphi(\mathbf{x}))$ (still denoted L) in $\mathbf{x}_0$ or, equivalently, the quadratic form associated to the second order Fréchet differential of L in $\mathbf{x}_0$. A practical way to do this is the following:

Let us denote $\mathbf{x} = (\tilde{x}_1,\ldots,\tilde{x}_n) \in \mathbb{R}^n$ and $\mathbf{y} = (\tilde{x}_{n+1},\ldots,\tilde{x}_{n+m}) \in \mathbb{R}^m$; the second order Fréchet differential of L in $(\mathbf{x}_0,\mathbf{y}_0)$ is

$$(\star\star)\ \sum_{i,j=1}^{n+m} \frac{\partial^2 L}{\partial \tilde{x}_i \partial \tilde{x}_j}(\mathbf{x}_0,\mathbf{y}_0)\mathrm{d}\tilde{x}_i \mathrm{d}\tilde{x}_j.$$

By considering the system obtained by differentiating the relations $g_k(\mathbf{x},\mathbf{y}) = 0,\ k = \overline{1,m}$, *i.e.*

$$\sum_{j=1}^{n+m} \frac{\partial g_k}{\partial \tilde{x}_j}(\mathbf{x}_0,\mathbf{y}_0)\mathrm{d}\tilde{x}_j = 0$$

we can retrieve $\mathrm{d}\tilde{x}_{n+1},\ldots,\mathrm{d}\tilde{x}_{n+m}$ as a function of $\mathrm{d}\tilde{x}_1,\ldots,\mathrm{d}\tilde{x}_n$ and replace them in $(\star\star)$. This gives the quadratic form we look for as

$$\mathrm{d}^2 L(\mathbf{x}_0) = \sum_{i,j=1}^{n} a_{ij}\mathrm{d}\tilde{x}_i \mathrm{d}\tilde{x}_j.$$

It is now sufficient to apply Theorem ?? in order to decide if $\mathbf{x}_0$ is a local extremum point or a saddle point for L: if $\mathrm{d}^2L(\mathbf{x}_0)$ is a positive or negative definite form, then $(\mathbf{x}_0, \mathbf{y}_0)$ is a conditioned local minimum, respectively maximum point of f; if $\mathrm{d}^2L(\mathbf{x}_0)$ is undefined, then $(\mathbf{x}_0, \mathbf{y}_0)$ is a conditioned saddle point for f; if $\mathrm{d}^2L(\mathbf{x}_0)$ is positive or negative semi-defined, we cannot decide the nature of $(\mathbf{x}_0, \mathbf{y}_0)$.

The steps described above is the the method of Lagrange multipliers for problems of restricted extrema.

Let now apply this method for the problem of Shannon's entropy, i.e. the extremum problem for

$$\tilde{H}(p_1, p_2, \ldots, p_n) = -\sum_{k=1}^{n} p_k \log_2 p_k,\ p_k \in (0,1),\ \forall k = \overline{1,n},$$

subject to $\sum_{k=1}^{n} p_k = 1$. Since we have only one restriction, we have $m = 1$, $g(p_1, p_2, \ldots, p_n) := \sum_{k=1}^{n} p_k - 1$ and the Lagrangian is

$$\mathscr{L}(\lambda_1, p_1, p_2, \ldots, p_n) = -\sum_{k=1}^{n} p_k \log_2 p_k + \lambda_1 \left(\sum_{k=1}^{n} p_k - 1 \right).$$

Then $(\lambda_1^0, p_1^0, p_2^0, \ldots, p_n^0)$ is a critical point for $\mathscr{L}$ if

$$\begin{cases} \dfrac{\partial \mathscr{L}}{\partial p_k}(\lambda_1^0, p_1^0, p_2^0, \ldots, p_n^0) = -\left(\log_2 p_k^0 + \frac{1}{p_k^0 \ln 2} \right) + \lambda_1^0 = 0,\ \forall k = \overline{1,n} \\ \dfrac{\partial \mathscr{L}}{\partial \lambda_1}(\lambda_1^0, p_1^0, p_2^0, \ldots, p_n^0) = \displaystyle\sum_{k=1}^{n} p_k^0 - 1 = 0. \end{cases}$$

From the first n equations we yield $p_1^0 = p_2^0 = \ldots = p_n^0 = \psi(\lambda_1^0)$, where $\psi(\lambda)$ is the unique solution of the equation in $p \in (0,1)$:

$$\log_2 p + \frac{1}{p \ln 2} = \lambda,$$

while from the last one we retrieve

$$p_1^0 = p_2^0 = \cdots = p_n^0 = \frac{1}{n},\ \lambda_1^0 = \log_2 \frac{1}{n} + \frac{n}{\ln 2}.$$

From the restraint $g(p_1, p_2, \ldots, p_n) = 0$ we get $\mathrm{d}p_1 + \mathrm{d}p_2 + \cdots + \mathrm{d}p_n = 0$; therefore

$$\mathrm{d}^2L(p_1^0, p_2^0, \ldots, p_n^0) = \sum_{k=1}^{n} \frac{1}{p_k^0 \ln 2} \left(\frac{1}{p_k^0} - 1 \right) (\mathrm{d}p_k)^2,$$

which is a negative definite quadratic form. This means that the point $(p_1^0, p_2^0, \ldots, p_n^0) = \left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n} \right)$ is a maximum point for $\tilde{H}$, restricted to $p_1 + p_2 \dotplus \cdots + p_n = 0$.

In consequence, the maximal informational entropy is obtained for the random variable

$$X = \begin{pmatrix} 1 & 2 & \ldots & n \\ \frac{1}{n} & \frac{1}{n} & \ldots & \frac{1}{n} \end{pmatrix}.$$

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