Differentiability

Lecture 10

Mathematics - 1^{st} year, English

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Outline of the lecture

1. Derivatives of functions of a real variable
2. Gâteaux differentiability
3. Fréchet differentiability
4. Higher order derivatives
   - Taylor series

Derivatives of functions of a real variable

The notion of derivative reflects the idea of rate of change of the value of a function with respect to its variable.

Definition

Let $\emptyset \neq A \subseteq \mathbb{R}$ and $f : A \to \mathbb{R}$.

- The *derivative* of f in $x_0 \in A \cap A'$ is the limit (if existent)

$$f'(x_0) = \frac{\mathrm{d}f}{\mathrm{d}x}(x_0) := \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \in \overline{\mathbb{R}}.$$

- We say that f is *derivable* in a point $x_0 \in A \cap A'$ if the derivative of f in x_0 exists and is finite.
- We say that f is *derivable* on a subset $B \subseteq A$ if f is derivable in all points $x_0 \in B$ (this implies that $B \subseteq A \cap A'$).
- We denote f' or $\frac{\mathrm{d}f}{\mathrm{d}x}$ the function $x \mapsto f'(x)$ defined on the subset of A consisting in all points of A where f is derivable.
- If $x_0 \in A$ is a left-limit point (right-limit point) of A, we call the *left-derivative* (*right-derivative*) of f in x_0 the limit (if existent)

$$f'_{\mathrm{l}}(x_0) := \lim_{x \nearrow x_0} \frac{f(x) - f(x_0)}{x - x_0} \in \overline{\mathbb{R}} \quad \left(f'_{\mathrm{r}}(x_0) := \lim_{x \searrow x_0} \frac{f(x) - f(x_0)}{x - x_0} \in \overline{\mathbb{R}} \right).$$

- If $x_0 \in A$ is both a left-limit point and a right-limit point, the derivative $f'(x_0)$ exists if and only if the lateral derivatives $f'_{\mathrm{l}}(x_0)$ and $f'_{\mathrm{r}}(x_0)$ exist and are equal.
- In this case, $f'(x_0) = f'_{\mathrm{l}}(x_0) = f'_{\mathrm{r}}(x_0)$.
- Of course, f is derivable in x_0 if, moreover, the two equal lateral derivatives are finite.

Proposition

Let $\emptyset \neq A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ and $x_0 \in A \cap A'$. If f is derivable in x_0, then f is continuous in x_0.

Definition

Let $\emptyset \neq A \subseteq \mathbb{R}$.

- We say that a function $f : A \to \mathbb{R}$ is of *class* C^1 if f is derivable on A and f' is continuous.
- We denote by $C^1(A)$ the family of all functions $f : A \to \mathbb{R}$ of class C^1.
- We denote by $C(A)$ the family of all continuous functions $f : A \to \mathbb{R}$.

By the above proposition, $C^1(A) \subseteq C(A)$.

Rules of calculus

Theorem

Let $A, B \subseteq \mathbb{R}$ be non-empty sets.

i) *If $f, g : A \to \mathbb{R}$ are derivable in a point $x_0 \in A \cap A'$ and $\alpha, \beta \in \mathbb{R}$, then $\alpha f + \beta g$ and fg are derivable in x_0 and*

$$(\alpha f + \beta g)'(x_0) = \alpha f'(x_0) + \beta g'(x_0);$$
$$(fg)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0) \text{ (Leibniz rule).}$$

If, moreover, $g(x_0) \neq 0$, then $\exists \varepsilon > 0 : g(x) \neq 0, \forall x \in (x_0 - \varepsilon, x_0 + \varepsilon)$ and $1/g$, f/g are derivable in x_0 with

$$\left(\frac{1}{g}\right)'(x_0) = -\frac{g'(x_0)}{g(x_0)^2};$$
$$\left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g(x_0)^2}.$$

Theorem (continuation)

ii) *If the function $f : A \to \mathbb{R}$ is derivable in $x_0 \in A \cap A'$, $f(x_0) \in B \cap B'$ and g is derivable in $f(x_0)$, then $g \circ f$ is derivable in x_0 and*

$$(g \circ f)'(x_0) = g'(f(x_0)) \cdot f'(x_0) \text{ (chain rule).}$$

iii) *If the function $f : A \to B$ is bijective and derivable in $x_0 \in A \cap A'$ such that $f'(x_0) \neq 0$ and $f(x_0) \in B'$, then f^{-1} is derivable in $f(x_0)$ and*

$$(f^{-1})'(f(x_0)) = \frac{1}{f'(x_0)}.$$

If the functions f and g are derivable, the above rules cand be written as:

- $(f+g)' = f' + g'$;
- $(fg)' = f'g + fg'$;
- $\left(\frac{1}{f}\right)' = -\frac{f'}{f}$ (provided that $0 \notin \operatorname{Im} f$);
- $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$ (provided that $0 \notin \operatorname{Im} g$);

- $(g \circ f)' = (g' \circ f) \cdot f'$;
- $\left(f^{-1}\right)' = \dfrac{1}{f' \circ f^{-1}}$ (provided that $0 \notin \operatorname{Im} f'$).

The last two rules can be intuitively remembered if we see f as a *change of variable* $y = f(x)$:

- $\dfrac{\mathrm{d}g}{\mathrm{d}x} = \dfrac{\mathrm{d}g}{\mathrm{d}y} \cdot \dfrac{\mathrm{d}y}{\mathrm{d}x}$;
- $\dfrac{\mathrm{d}x}{\mathrm{d}y} = \dfrac{1}{\frac{\mathrm{d}y}{\mathrm{d}x}}$.

If $f : A \to \mathbb{R}_+^*$ and $g : A \to \mathbb{R}$ are derivable functions, then according to the rules of calculus, we have:

$$(f^g)' = \left(\mathrm{e}^{g \ln f}\right)' = \mathrm{e}^{g \ln f} \left(g' \ln f + \tfrac{gf'}{f}\right) = f^g (\ln f) g' + f^{g-1} f' g.$$

Derivatives of elementary functions

- $c' = \dfrac{\mathrm{d}c}{\mathrm{d}x} = 0$, $x \in \mathbb{R}$, for $c \in \mathbb{R}$;
- $(a^x)' = a^x \ln a$, $x \in \mathbb{R}$, for $a \in \mathbb{R}_+^*$;
- $(\mathrm{e}^x)' = \mathrm{e}^x$, $x \in \mathbb{R}$;
- $(\log_a x)' = \dfrac{1}{x \ln a}$, $x \in \mathbb{R}$, for $a \in \mathbb{R}_+^* \setminus \{1\}$;
- $(\ln x)' = \dfrac{1}{x}$, $x \in \mathbb{R}$;
- $(x^p)' = px^{p-1}$, $x \in \mathcal{D}_p$, for $p \in \mathbb{R}$,

where $\mathcal{D}_p := \mathbb{R}$ if $p \in \mathbb{N}^*$, $\mathcal{D}_p := \mathbb{R}^*$ if $p \in \mathbb{Z} \setminus \mathbb{N}^*$, $\mathcal{D}_p := \mathbb{R}_+$ if $p \in \mathbb{R}_+ \setminus \mathbb{N}$ and $\mathcal{D}_p := \mathbb{R}_+^*$ if $p \in (-\infty, 0) \setminus \mathbb{Z}$;

- $(\sin x)' = \cos x$;
- $(\cos x)' = -\sin x$;
- $(\operatorname{tg} x)' = \dfrac{1}{(\cos x)^2}$, $x \in \mathbb{R} \setminus \left\{(2k+1)\frac{\pi}{2} \mid k \in \mathbb{N}\right\}$;
- $(\operatorname{ctg} x)' = -\dfrac{1}{(\sin x)^2}$, $x \in \mathbb{R} \setminus \{k\pi \mid k \in \mathbb{N}\}$;
- $(\arcsin x)' = \dfrac{1}{\sqrt{1-x^2}}$, $x \in (-1, 1)$;
- $(\arccos x)' = -\dfrac{1}{\sqrt{1-x^2}}$, $x \in (-1, 1)$;
- $(\operatorname{arctg} x)' = \dfrac{1}{1+x^2}$, $x \in \mathbb{R}$;
- $(\operatorname{arcctg} x)' = -\dfrac{1}{1+x^2}$, $x \in \mathbb{R}$.

Derivatives of vector functions of one variable

The definition of derivable real functions can be easily adapted for functions of one variable with values in $\mathbb{R}^m$, because the limit

$$f'(x_0) := \lim_{x \to x_0} \frac{1}{x - x_0} \left(f(x) - f(x_0) \right)$$

makes sense for functions $f : A \to \mathbb{R}^m$ and $x_0 \in A \cap A'$. As in the case of limits of function, the derivatives can be computed on components:

Proposition

Let $\emptyset \neq A \subseteq \mathbb{R}$ and $f : A \to \mathbb{R}^m$ a function with components $f_1, f_2, \ldots, f_m$. If $x_0 \in A \cap A'$, then f is derivable in x_0 if and only if $f_1, f_2, \ldots, f_m$ are derivable in x_0. In this case,

$$f'(x_0) = (f_1'(x_0), \ldots, f_m'(x_0)).$$

Similar rules of calculus apply in this case as well:

Theorem

Let $\emptyset \neq A \subseteq \mathbb{R}$, $x_0 \in A \cap A'$ and $f, g : A \to \mathbb{R}^m$, $\varphi : A \to \mathbb{R}$, derivable in x_0. Then:

i) *$f + g$ is derivable in x_0 and*

$$(f+g)'(x_0) = f'(x_0) + g'(x_0);$$

ii) *$\langle f, g\rangle$ is derivable in x_0 and*

$$(\langle f, g\rangle)'(x_0) = \left\langle f'(x_0), g(x_0)\right\rangle + \left\langle f(x_0), g'(x_0)\right\rangle;$$

iii) *φf is derivable in x_0 and*

$$(\varphi f)'(x_0) = \varphi'(x_0) f(x_0) + \varphi(x_0) f'(x_0).$$

Directional derivatives and Gâteaux differentiability

- For functions of several variables the ratio $\frac{f(\mathbf{x})-f(\mathbf{x}_0)}{\mathbf{x}-\mathbf{x}_0}$ is not defined, so defining derivatives in this way for such functions is not possible.
- There exist several possibilities to circumvent this difficulty: one is to consider *directional derivatives*, which is based on the remark that the derivative of a function $f : A \to \mathbb{R}$ in some point x_0 can be written as
$$f'(x_0) = \lim_{t\to 0} \frac{f(x_0+t) - f(x_0)}{t}.$$

Definition

Let $D \subseteq \mathbb{R}^n$ be a non-empty open set and $f : D \to \mathbb{R}^m$ a function.

- If $\mathbf{x}_0 \in D$ and $\mathbf{u} \in \mathbb{R}^n$, we say that f is *derivable* in $\mathbf{x}_0$ *along the direction* $\mathbf{u}$ if the limit
$$f'(\mathbf{x}_0; \mathbf{u}) := \lim_{t\to 0} \frac{1}{t} \left(f(\mathbf{x}_0 + t\mathbf{u}) - f(\mathbf{x}_0)\right) \in \mathbb{R}^m$$
exists. In this case, $f'(\mathbf{x}_0; \mathbf{u})$ is called the *directional derivative* of f in $\mathbf{x}_0$ in the *direction* $\mathbf{u}$.

Definition (continued)

- If $\mathbf{x}_0 \in D$ and f is derivable in $\mathbf{x}_0$ along each direction $\mathbf{u} \in \mathbb{R}^n$, we say that f is *Gâteaux differentiable* in $\mathbf{x}_0$. The *Gâteaux differential* is then the function $\mathbf{u} \mapsto f'(\mathbf{x}_0; \mathbf{u})$ and is denoted by $\mathrm{D}f(\mathbf{x}_0)$.
- If $\mathbf{x}_0 \in D$ and f is Gâteaux differentiable in $\mathbf{x}_0$, we say that f is *Gâteaux derivable* in $\mathbf{x}_0$ if the Gâteaux differential $\mathrm{D}f(\mathbf{x}_0) : \mathbb{R}^n \to \mathbb{R}^m$ is a linear mapping.
- We say that f is *Gâteaux differentiable* or *derivable* on a subset $D_0 \subseteq D$ if f is Gâteaux differentiable, respectively derivable in any point $\mathbf{x}_0 \in D_0$.

Remark. Since

$$\begin{aligned} f'(\mathbf{x}_0; \alpha\mathbf{u}) &= \lim_{t \to 0} \frac{1}{t} \left(f(\mathbf{x}_0 + t\alpha\mathbf{u}) - f(\mathbf{x}_0) \right) \\ &= \lim_{s \to 0} \frac{\alpha}{s} \left(f(\mathbf{x}_0 + s\mathbf{u}) - f(\mathbf{x}_0) \right) = \alpha f'(\mathbf{x}_0; \mathbf{u}) \end{aligned}$$

for every $\alpha \in \mathbb{R}$ and $\mathbf{u} \in \mathbb{R}^n$, we see that:

- the existence of $f'(\mathbf{x}_0; \alpha\mathbf{u})$ is equivalent to the existence of $f'(\mathbf{x}_0; \mathbf{u})$ in the case $\alpha \in \mathbb{R}^*$, $\mathbf{u} \in \mathbb{R}^n \setminus \{\mathbf{0}_{\mathbb{R}^n}\}$;

- therefore, in the definition of differentiability one can require the existence of $f'(\mathbf{x}_0;\mathbf{u})$ only for versors $\mathbf{u} \in \mathbb{R}^n$;
- if f is differentiable in $\mathbf{x}_0$, the mapping $\mathrm{D}f(\mathbf{x}_0) : \mathbb{R}^n \to \mathbb{R}^m$ is homogeneous; hence for the derivability of f in $\mathbf{x}_0$, it is enough to ask only that $\mathrm{D}f(\mathbf{x}_0)$ is additive.

The constant functions and the linear functions are Gâteaux derivable. Indeed, if $c \in \mathbb{R}$ and $T \in \mathrm{L}(\mathbb{R}^n;\mathbb{R}^m)$, then

$$c'(\mathbf{x}_0;\mathbf{u}) = \lim_{t\to 0} \frac{1}{t}(c-c) = 0$$

and

$$T'(\mathbf{x}_0;\mathbf{u}) = \lim_{t\to 0} \frac{1}{t}\left(T(\mathbf{x}_0 + t\mathbf{u}) - T(\mathbf{x}_0)\right) = T(\mathbf{u}),\ \forall \mathbf{x}_0, \mathbf{u} \in \mathbb{R}^n.$$

In consequence, $\mathrm{D}c(\mathbf{x}_0) = 0$, $\mathrm{D}T(\mathbf{x}_0) = T$, $\forall \mathbf{x}_0 \in \mathbb{R}^n$.

Partial derivatives

Definition

Let $D \subseteq \mathbb{R}^n$ be an open set, $\mathbf{x}_0 \in D$ and $f : A \to \mathbb{R}^m$ a function. Let $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ be the canonical basis in $\mathbb{R}^n$. If f is derivable in $\mathbf{x}_0$ along the direction $\mathbf{e}_k$ for some $k \in \{1, \ldots, n\}$, we say that f admits a *partial derivative* with respect to x_k in $\mathbf{x}_0$ and we denote

$$\frac{\partial f}{\partial x_k}(\mathbf{x}_0) := f'(\mathbf{x}_0; \mathbf{e}_k).$$

We can see that the partial derivative of f with respect to x_k in $\mathbf{x}_0$ is obtained as follows: if $\mathbf{x}_0 = (x_1^0, \ldots, x_n^0) \in D$, then

$$\frac{\partial f}{\partial x_k}(\mathbf{x}_0) = \lim_{t \to 0} \frac{1}{t} \left(f(\mathbf{x}_0 + t\mathbf{e}_k) - f(\mathbf{x}_0) \right)$$

$$= \lim_{x_k \to x_k^0} \frac{1}{x_k - x_k^0} \left(f(x_1^0, \ldots, x_{k-1}^0, x_k, \ldots, x_n^0) - f(x_1^0, \ldots, x_{k-1}^0, x_k^0, \ldots, x_n^0) \right).$$

Of course, $\frac{\partial f}{\partial x_k}(\mathbf{x}_0) = \left(\frac{\partial f_1}{\partial x_k}(\mathbf{x}_0), \ldots, \frac{\partial f_m}{\partial x_k}(\mathbf{x}_0) \right) \in \mathbb{R}^m$.

The existence of partial derivatives of a function of several variables in a point does not imply the existence of all directional derivatives (in other words, the Gâteaux differentiability) in that point, as the following example shows:
Example. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2}, & (x,y) \neq (0,0); \\ 0, & (x,y) = (0,0). \end{cases}$$

Then $\frac{\partial f}{\partial x}(0,0) = 0$ and $\frac{\partial f}{\partial y}(0,0) = 0$, but

$$\frac{f((0,0)+t(u,v)) - f((0,0))}{t} = \frac{\frac{t^2uv}{t^2(u^2+v^2)}}{t} = \frac{1}{t}\frac{uv}{u^2+v^2}.$$

Hence the directional derivative $f'((0,0);(u,v))$ does not exist if $uv \neq 0$.

The Jacobian

Definition

Let $D \subseteq \mathbb{R}^n$ be a non-empty open set, $\mathbf{x}_0 \in D$ and $f : D \to \mathbb{R}^m$ a Gâteaux derivable function in $\mathbf{x}_0$.

- The matrix in $\mathscr{M}_{nm}$ associated to $\mathrm{D}f(\mathbf{x}_0)$ (with respect to the canonical bases in $\mathbb{R}^n$ and $\mathbb{R}^m$) is called the *Jacobian matrix* of f in $\mathbf{x}_0$ and is denoted $\mathrm{J}_f(\mathbf{x}_0)$.
- In the case $m = 1$, the Jacobian matrix of f in $\mathbf{x}_0$ is also called the gradient of f and is denoted by $\nabla f(\mathbf{x}_0)$.
- In the case $m = n$, the determinant of $\mathrm{J}_f(\mathbf{x}_0)$ is called the *Jacobian* of f in $\mathbf{x}_0$ and is denoted by $\frac{\mathrm{D}(f_1,\ldots,f_n)}{\mathrm{D}(x_1,\ldots,x_n)}(\mathbf{x}_0)$, where $f_1, \ldots, f_n$ are the components of f.

Remarks.

1. Let $f : D \to \mathbb{R}^m$ be a Gâteaux derivable function in $\mathbf{x}_0$.It can be easily shown that $\mathrm{J}_f(\mathbf{x}_0) = \begin{bmatrix} \nabla f_1(\mathbf{x}_0) & \ldots & \nabla f_m(\mathbf{x}_0) \end{bmatrix}$, *i.e.* the matrix having as columns the elements of $\nabla f_k(\mathbf{x}_0)$ for $k \in \{1, \ldots, m\}$.

On the other hand,

$$\mathrm{J}_f(\mathbf{x}_0) = \begin{bmatrix} \frac{\partial f}{\partial x_1}(\mathbf{x}_0) \\ \vdots \\ \frac{\partial f}{\partial x_m}(\mathbf{x}_0) \end{bmatrix},$$

i.e. the matrix having as rows the elements of $\frac{\partial f}{\partial x_i}(\mathbf{x}_0)$ for $i \in \{1, \ldots, n\}$:

$$\mathrm{J}_f(\mathbf{x}_0) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(\mathbf{x}_0) & \frac{\partial f_2}{\partial x_1}(\mathbf{x}_0) & \ldots & \frac{\partial f_m}{\partial x_1}(\mathbf{x}_0) \\ \frac{\partial f_1}{\partial x_2}(\mathbf{x}_0) & \frac{\partial f_2}{\partial x_2}(\mathbf{x}_0) & \ldots & \frac{\partial f_m}{\partial x_2}(\mathbf{x}_0) \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_1}{\partial x_n}(\mathbf{x}_0) & \frac{\partial f_2}{\partial x_n}(\mathbf{x}_0) & \ldots & \frac{\partial f_m}{\partial x_n}(\mathbf{x}_0) \end{bmatrix}.$$

Particularizing for the case $m = 1$, we get

$$\nabla f(\mathbf{x}_0) = \begin{bmatrix} \frac{\partial f}{\partial x_1}(\mathbf{x}_0) & \ldots & \frac{\partial f}{\partial x_n}(\mathbf{x}_0) \end{bmatrix}^{\mathrm{T}}.$$

We can see the colum matrix $\nabla f(\mathbf{x}_0)$ as an element of $\mathbb{R}^n$; in this case we can write

$$f'(\mathbf{x}_0; \mathbf{u}) = \langle \nabla f(\mathbf{x}_0), \mathbf{u} \rangle = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\mathbf{x}_0) u_i, \ \forall \mathbf{u} = (u_1, \ldots, u_n) \in \mathbb{R}^n.$$

2. Concerning operations with functions of several variables, we have the following rules, which apply whenever the concerned directional derivatives exist for the functions $f, g : D \subseteq \mathbb{R}^n \to \mathbb{R}^m$ and $\varphi : D \subseteq \mathbb{R}^n \to \mathbb{R}$:

- $(f+g)'(\mathbf{x}_0; \mathbf{u}) = f'(\mathbf{x}_0; \mathbf{u}) + g'(\mathbf{x}_0; \mathbf{u})$;
- $(\varphi f)'(\mathbf{x}_0; \mathbf{u}) = \varphi'(\mathbf{x}_0; \mathbf{u}) f(\mathbf{x}_0; \mathbf{u}) + \varphi(\mathbf{x}_0; \mathbf{u}) f'(\mathbf{x}_0; \mathbf{u})$;
- $\left(\frac{1}{\varphi}\right)'(\mathbf{x}_0; \mathbf{u}) = -\frac{\varphi'(\mathbf{x}_0; \mathbf{u})}{\varphi(\mathbf{x}_0; \mathbf{u})^2}$ if $0 \notin \mathrm{Im}\, \varphi$.

3. If a function $f : D \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is Gâteaux derivable in some point $\mathbf{x}_0 \in D$, we cannot infer the continuity of f in $\mathbf{x}_0$, but only the directional continuity in $\mathbf{x}_0$, *i.e.* the continuity in 0 of the function $t \mapsto f(\mathbf{x}_0 + t\mathbf{u})$ for any $\mathbf{u} \in \mathbb{R}^n$.
However, the situation changes if we require that the partial derivatives exist and are bounded on a neighbourhood of $\mathbf{x}_0$:

Theorem

Let $D \subseteq \mathbb{R}^n$ be an open set, $\mathbf{x}_0 \in D$ and $f : A \to \mathbb{R}^m$. If there exists $V \in \mathscr{V}(\mathbf{x}_0)$ such that the partial derivatives $\frac{\partial f}{\partial x_i}(\mathbf{x})$ exist for any $\mathbf{x} \in V \cap D$ and $\frac{\partial f}{\partial x_i}$ are bounded on $V \cap D$ for every $i = \overline{1, n}$, then f is continuous in $\mathbf{x}_0$.

Remark. A sufficient condition for the functions $\frac{\partial f}{\partial x_i}$ to be bounded on a neighbourhood of $\mathbf{x}_0$ is that they are continuous in $\mathbf{x}_0$.

Definition

Let $A \subseteq \mathbb{R}^n$ be a non-empty set.

- If A is open, we say that a function $f : A \to \mathbb{R}^m$ is of class C^1 if all the partial derivatives of f exist and are continuous.
- If A is open, we denote by $C^1(A; \mathbb{R}^m)$ the family of all functions $f : A \to \mathbb{R}$ of class C^1. If $m = 1$, we simply denote it $C^1(A)$.
- We denote by $C(A; \mathbb{R}^m)$ the family of all continuous functions $f : A \to \mathbb{R}$. If $m = 1$, we simply denote it $C(A)$.

The previous theorem and the remark below allow us to conclude that $C^1(D; \mathbb{R}^m) \subseteq C(D; \mathbb{R}^m)$ for any open subset $D \subseteq \mathbb{R}^n$.

Fréchet differentiability

We remark that a function $f : A \to \mathbb{R}$ is derivable in some point x_0 if there exists $a \in \mathbb{R}$ such that

$$\lim_{x \to x_0} \frac{f(x) - f(x_0) - a(x - x_0)}{|x - x_0|} = 0.$$

In this case, $a = f'(x_0)$. For the case of several variables, another possibility is to replace the real number a by a matrix, or, equivalently, a linear operator.

Definition

Let $D \subseteq \mathbb{R}^n$ be a non-empty open set and $f : D \to \mathbb{R}^m$ a function.

- For $\mathbf{x}_0 \in D$, we say that f is *Fréchet differentiable* in $\mathbf{x}_0$ if there exists a linear operator $T \in \mathrm{L}(\mathbb{R}^n; \mathbb{R}^m)$ such that

$$\lim_{\mathbf{x} \to \mathbf{x}_0} \frac{1}{\|\mathbf{x} - \mathbf{x}_0\|} \left(f(\mathbf{x}) - f(\mathbf{x}_0) - T(\mathbf{x} - \mathbf{x}_0)\right) = \mathbf{0}_{\mathbb{R}^m}.$$

  T is called the *Fréchet differential* of f in $\mathbf{x}_0$ and is denoted by $\mathrm{d}f(\mathbf{x}_0)$.
- We say that f is *Fréchet differentiable* on a subset $D_0 \subseteq D$ if f is Fréchet differentiable in any point $\mathbf{x}_0 \in D_0$.

Remark. Another way to express that f is Fréchet differentiable in $\mathbf{x}_0$ is that there exist $T \in \mathrm{L}(\mathbb{R}^n;\mathbb{R}^m)$ and a continuous function $\alpha : D \to \mathbb{R}^m$ such that $\alpha(\mathbf{x}_0) = \mathbf{0}_{\mathbb{R}^m}$ and

$$f(\mathbf{x}) = f(\mathbf{x}_0) + T(\mathbf{x} - \mathbf{x}_0) + \|\mathbf{x} - \mathbf{x}_0\| \, \alpha(\mathbf{x}), \ \forall \mathbf{x} \in D.$$

In fact, one can define α by

$$\alpha(\mathbf{x}) := \begin{cases} \frac{1}{\|\mathbf{x}-\mathbf{x}_0\|} \left(f(\mathbf{x}) - f(\mathbf{x}_0) - T(\mathbf{x} - \mathbf{x}_0)\right), & \mathbf{x} \in D \setminus \{\mathbf{x}_0\}; \\ \mathbf{0}_{\mathbb{R}^m}, & \mathbf{x} = \mathbf{x}_0. \end{cases}$$

The link between Fréchet differentiability and Gâteaux differentiability is given by the following result:

Theorem

Let $D \subseteq \mathbb{R}^n$ be a non-empty open set and $f : D \to \mathbb{R}^m$ a function. If f is Fréchet differentiable in some point $\mathbf{x}_0 \in D$, then f is Gâteaux derivable in $\mathbf{x}_0$ and $\mathrm{D}f(\mathbf{x}_0) = \mathrm{d}f(\mathbf{x}_0)$.

- A simple consequence of this theorem is that the Fréchet differential is unique, since the Gâteaux derivative is unique, because $\mathrm{D}f(\mathbf{x}_0)(\mathbf{u}) = f'(\mathbf{x}_0;\mathbf{u})$, for every $\mathbf{u} \in \mathbb{R}^n$.

- Another consequence is that if f is Fréchet differentiable in $\mathbf{x}_0 \in D$, then f has partial derivatives in $\mathbf{x}_0$ and

$$\frac{\partial f}{\partial x_i}(\mathbf{x}_0) = \mathrm{d}f(\mathbf{x}_0)(\mathbf{e}_i),\ \forall i \in \{1, \ldots, n\}.$$

- Constant functions and linear mappings are Fréchet differentiable, too. Indeed, if $c \in \mathbb{R}$ and $T \in \mathrm{L}(\mathbb{R}^n; \mathbb{R}^m)$,

$$\lim_{\mathbf{x}\to\mathbf{x}_0} \frac{1}{\|\mathbf{x}-\mathbf{x}_0\|}\Big(c - c - \mathbf{0}_{\mathrm{L}(\mathbb{R}^n;\mathbb{R}^m)}(\mathbf{x}-\mathbf{x}_0)\Big) = \mathbf{0}_{\mathbb{R}^m}$$

and

$$\lim_{\mathbf{x}\to\mathbf{x}_0} \frac{1}{\|\mathbf{x}-\mathbf{x}_0\|}\left(T(\mathbf{x}) - T(\mathbf{x}_0) - T(\mathbf{x}-\mathbf{x}_0)\right) = \mathbf{0}_{\mathbb{R}^m},$$

which clearly prove the claim and moreover, that $\mathrm{d}c(\mathbf{x}_0) = 0$, $\mathrm{d}T(\mathbf{x}_0) = T$, $\forall \mathbf{x}_0 \in \mathbb{R}^n$.

- Let $\mathrm{pr}_i : D \to \mathbb{R}$ be the projection on the i^{th}-component:

$$\mathrm{pr}_i\,(x_1, \ldots, x_n) = x_i,\ (x_1, \ldots, x_n) \in D,\ i = \overline{1, n}.$$

The Fréchet differential of pr_k is traditionally denoted $\mathrm{d}x_k$:

$$\mathrm{d}x_i\,(u_1, \ldots, u_n) = u_i,\ (u_1, \ldots, u_n) \in \mathbb{R}^n,\ i = \overline{1, n}.$$

Since

$$\mathrm{d}f(\mathbf{x}_0)(\mathbf{u}) = f'(\mathbf{x}_0; \mathbf{u}) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\mathbf{x}_0) u_i = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\mathbf{x}_0)\mathrm{d}x_i(\mathbf{u}),$$
$$\forall \mathbf{u} = (u_1, \ldots, u_n) \in \mathbb{R}^n,$$

we have

$$\mathrm{d}f(\mathbf{x}_0) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\mathbf{x}_0)\mathrm{d}x_i.$$

In contrast to Gâteaux derivability, Fréchet differentiability implies continuity:

Theorem

Let $D \subseteq \mathbb{R}^n$ be a non-empty open set and $f : D \to \mathbb{R}^m$ a function. If f is Fréchet differentiable in some point $\mathbf{x}_0 \in D$, then f is continuous in $\mathbf{x}_0$.

A sufficient condition for Fréchet differentiability is given by the following result:

Theorem

Let $D \subseteq \mathbb{R}^n$ be an open set, $\mathbf{x}_0 \in D$ and $f : D \to \mathbb{R}^m$ a function. If there exists $V \in \mathscr{V}(\mathbf{x}_0)$ such that $\frac{\partial f}{\partial x_i}(\mathbf{x})$ exist for any $\mathbf{x} \in V \cap D$ and $\frac{\partial f}{\partial x_i}$ are continuous on $V \cap D$ for every $i = \overline{1, n}$, then f is Fréchet differentiable in $\mathbf{x}_0$.

A consequence of the above result is that if $f \in C^1(D; \mathbb{R}^m)$, then f is Fréchet differentiable.

Theorem

Let $D \subseteq \mathbb{R}^n$ and $E \subseteq \mathbb{R}^m$ be non-empty open sets.

i) *If $f, g : D \to \mathbb{R}^m$ are Fréchet differentiable in $\mathbf{x}_0 \in D$ and $\alpha, \beta \in \mathbb{R}$, then $\alpha f + \beta g$ is Fréchet differentiable in $\mathbf{x}_0$ and*

$$\mathrm{d}(\alpha f + \beta g)(\mathbf{x}_0) = \alpha \mathrm{d}f(\mathbf{x}_0) + \beta \mathrm{d}g(\mathbf{x}_0),\ \forall \alpha, \beta \in \mathbb{R}.$$

ii) *If $f : D \to \mathbb{R}^m$ and $\varphi : D \to \mathbb{R}$ are Fréchet differentiable in $\mathbf{x}_0 \in D$, then φf is Fréchet differentiable in $\mathbf{x}_0$ and*

$$\mathrm{d}(\varphi f)(\mathbf{x}_0) = \mathrm{d}\varphi(\mathbf{x}_0) f(\mathbf{x}_0) + \varphi(\mathbf{x}_0) \mathrm{d}f(\mathbf{x}_0).$$

Theorem (continued)

iii) *If $\varphi : D \to \mathbb{R}$ is Fréchet differentiable in $\mathbf{x}_0 \in D$ and $\varphi(\mathbf{x}_0) \neq 0$, then there exists $D_0 \subseteq D$ an open neighbourhood of $\mathbf{x}_0$ such that $0 \notin \varphi[D_0]$, $\frac{1}{\varphi} : D_0 \to \mathbb{R}$ is Fréchet differentiable in $\mathbf{x}_0$ and*

$$\mathrm{d}\left(\frac{1}{\varphi}\right)(\mathbf{x}_0) = -\frac{1}{\varphi(\mathbf{x}_0)^2}\mathrm{d}\varphi(\mathbf{x}_0).$$

iv) *If E is an open set, $f : D \to E$ is Fréchet differentiable in $\mathbf{x}_0$, $g : E \to \mathbb{R}^p$ is Fréchet differentiable in $f(\mathbf{x}_0)$, then $g \circ f$ is Fréchet differentiable in $\mathbf{x}_0$ and*

$$\mathrm{d}(g \circ f)(\mathbf{x}_0) = \mathrm{d}g(f(\mathbf{x}_0)) \circ \mathrm{d}f(\mathbf{x}_0).$$

The last relation is known as the *chain rule* for Fréchet differentials. In terms of Jacobian matrices, this can be written as

$$\mathrm{J}_{g\circ f}(\mathbf{x}_0) = \mathrm{J}_f(\mathbf{x}_0) \cdot \mathrm{J}_g(f(\mathbf{x}_0))$$

or, in terms of partial derivatives

$$\frac{\partial(g_j \circ f)}{\partial x_i}(\mathbf{x}_0) = \sum_{k=1}^{n} \frac{\partial g_j}{\partial y_k}(f(\mathbf{x}_0))\frac{\partial f_k}{\partial x_i}(\mathbf{x}_0),\ \forall i = \overline{1, n},\ \forall j = \overline{1, p}.$$

In the case $m = n = p$, applying the determinants to the above matrix relation, we get

$$\frac{\mathrm{D}(g_1 \circ f, \ldots, g_n \circ f)}{\mathrm{D}(x_1, \ldots, x_n)}(\mathbf{x}_0) = \frac{\mathrm{D}(g_1, \ldots, g_n)}{\mathrm{D}(y_1, \ldots, y_n)}(f(\mathbf{x}_0)) \cdot \frac{\mathrm{D}(f_1, \ldots, f_n)}{\mathrm{D}(x_1, \ldots, x_n)}(\mathbf{x}_0).$$

Therefore, if $f : D \to E$ is bijective and f^{-1} is also Fréchet differentiable in $f(\mathbf{x}_0)$, then $\mathrm{J}_f(\mathbf{x}_0)$ is non-singular, $\mathrm{J}_{f^{-1}}(f(\mathbf{x}_0)) = (\mathrm{J}_f(\mathbf{x}_0))^{-1}$ and

$$\frac{\mathrm{D}(f_1, \ldots, f_n)}{\mathrm{D}(x_1, \ldots, x_n)}(\mathbf{x}_0) \neq 0;$$
$$\frac{\mathrm{D}(f_1^{-1}, \ldots, f_n^{-1})}{\mathrm{D}(x_1, \ldots, x_n)}(f(\mathbf{x}_0)) = \frac{1}{\frac{\mathrm{D}(f_1, \ldots, f_n)}{\mathrm{D}(x_1, \ldots, x_n)}(\mathbf{x}_0)}.$$

Definition

Let $D, E \subseteq \mathbb{R}^n$ be non-empty open sets. A function $f : D \to E$ is called a *diffeomorphism* if f is bijective, $f \in C^1(D; \mathbb{R}^n)$ and $\mathrm{J}_f(\mathbf{x})$ is non-singular for every $\mathbf{x} \in D$.

It can be shown that if $f : D \to E$ is a diffeomorphism, then $f^{-1} \in C^1(E; \mathbb{R}^n)$.

Higher order derivatives

We consider first the case of real functions of one variable.

- For a function $f : A \to \mathbb{R}$, one can define the derivative function $f' : A_1 \to \mathbb{R}$, where $A_1 \subseteq A \cap A'$ is the set of elements where f is derivable.
- Therefore, we can speak about the derivability of the new function f': the derivative of f' in some point $x_0 \in A_1 \cap A_1'$, if existent, will be denoted $f''(x_0)$ or $\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}(x_0)$ and is called the *second order* derivative of f in x_0.
- Of course, this defines a function, called the *second order derivative* of f: $f'' : A_2 \to \mathbb{R}$, where $A_2 \subseteq A_1 \cap A_1'$ is the set of elements where f' is derivable.
- The process can continue: if $f^{(n-1)} : A_{n-1} \to \mathbb{R}$ is the $(n-1)^{\text{th}}$-order derivative of f (for $n \geq 3$), then $f^{(n)}(x_0)$ or $\frac{\mathrm{d}^n f}{\mathrm{d}x^n}(x_0)$ denotes, if existent, the derivative of $f^{(n-1)}$ in $x_0 \in A_{n-1} \cap A_{n-1}'$ and is called the n^{th}*-order derivative* of f in x_0.
- It defines, in its turn, the function $f^{(n)} : A_n \to \mathbb{R}$, called the n^{th}*-order derivative* of f, where $A_n \subseteq A_{n-1} \cap A_{n-1}'$ is the set of elements where $f^{(n-1)}$ is derivable.

Recursively, one can also define the higher order directional derivatives, higher order Gâteaux differentiability or derivability and higher order Fréchet differentiability. A particular case of higher order directional derivatives is the notion of higher order partial derivatives:
Let $D \subseteq \mathbb{R}^n$ be a non-empty open set and $f : D \to \mathbb{R}^m$ a function. If $i_1, \ldots, i_p \in \{1, \ldots, n\}$ for $p \geq 2$, the *partial derivative of order p of f* with respect to $x_{i_1}, x_{i_2}, \ldots, x_{i_p}$ in a point $\mathbf{x}_0 \in D$ is defined recursively as

$$\frac{\partial^p f}{\partial x_{i_1} \partial x_{i_2} \ldots \partial x_{i_p}}(\mathbf{x}_0) := \frac{\partial \left(\frac{\partial^{p-1} f}{\partial x_{i_2} \ldots \partial x_{i_p}} \right)}{\partial x_1}(\mathbf{x}_0),$$

provided that the partial derivative (of order $p-1$) $\frac{\partial^{p-1} f}{\partial x_2 \ldots \partial x_p}$ exists on some open neighbourhood of $\mathbf{x}_0$ and has a partial derivative with respect to x_1 in $\mathbf{x}_0$.
If $i_1 = \cdots = i_p = i$, instead of $\frac{\partial^p f}{\partial x_1 \ldots \partial x_p}$ we can write $\frac{\partial^p f}{\partial x_i^p}$. If it is not the case, the partial derivative is called a *mixt partial derivative*. The following results gives us sufficient conditions for grouping the indices $i_1, \ldots, i_p$.

Theorem (Schwartz)

Let $D \subseteq \mathbb{R}^n$ be an open set, $\mathbf{x}_0 \in D$, $f : D \to \mathbb{R}^m$ a function and $i, j \in \{1, \dots n\}$ with $i \neq j$. If the mixt partial derivatives $\frac{\partial^2 f}{\partial x_i \partial x_j}$ and $\frac{\partial^2 f}{\partial x_j \partial x_i}$ exist on a neighbourhood of $\mathbf{x}_0$ and they are continuous in $\mathbf{x}_0$, then

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{x}_0) = \frac{\partial^2 f}{\partial x_j \partial x_i}(\mathbf{x}_0).$$

Theorem (Young)

Let $D \subseteq \mathbb{R}^n$ be an open set, $\mathbf{x}_0 \in D$, $f : D \to \mathbb{R}^m$ a function and $i, j \in \{1, \dots n\}$ with $i \neq j$. If the partial derivatives $\frac{\partial f}{\partial x_i}$ and $\frac{\partial f}{\partial x_j}$ exist on an open neighbourhood of $\mathbf{x}_0$ and they are Fréchet differentiable in $\mathbf{x}_0$, then $\frac{\partial^2 f}{\partial x_i \partial x_j}$ and $\frac{\partial^2 f}{\partial x_j \partial x_i}$ exist and are equal.

In the conditions of Schwartz or Young theorems, one can order and group the indices $i_1, \ldots, i_p$ in a mixt partial derivative $\frac{\partial^p f}{\partial x_{i_1} \partial x_{i_2} \ldots \partial x_{i_p}}$ and write it as

$$(\star) \qquad \frac{\partial^p f}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_n^{\alpha_n}},$$

where, for $i = \overline{1, n}$, α_i is the number of i's occuring in the list $i_1, \ldots, i_p$. The vector $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{N}^n$ is called a *multi-index* and we have $p = |\alpha| := \alpha_1 + \cdots + \alpha_n$. In fact, in the expression (**??**), one can omit the terms $\partial x_i^{\alpha_i}$ if $\alpha_i = 0$.

If $D \subseteq \mathbb{R}^n$ is a non-empty open set and $p \geq 2$, $C^p(D; \mathbb{R}^m)$ denotes the set of all functions $f : D \to \mathbb{R}$ such all the partial derivatives of order p exist and are continuous. We also denote by $C^\infty(D; \mathbb{R}^m)$ the set of functions $f : D \to \mathbb{R}$ such that $f \in C^p(D; \mathbb{R}^m)$, for every $p \geq 1$. In the case $m = 1$, we will simply denote $C^p(D)$ instead of $C^p(D; \mathbb{R})$ (for $p \in \mathbb{N}^*$ or $p = \infty$). Of course, we have

$$C^\infty(D; \mathbb{R}^m) \subseteq \cdots \subseteq C^p(D; \mathbb{R}^m) \subseteq \cdots \subseteq C^1(D; \mathbb{R}^m) \subseteq C(D; \mathbb{R}^m).$$

The Fréchet differentiability of higher-order can be introduced as follows:

Definition

Let $D \subseteq \mathbb{R}^n$ be a non-empty open set, $f : D \to \mathbb{R}^m$ a function and $p \in \mathbb{N}^* \setminus \{1\}$.

- We say that f is *Fréchet differentiable* of order p in $\mathbf{x}_0 \in D$ if there exists $D_0 \subseteq D$ an open neighbourhood of $\mathbf{x}_0$ such that all the partial derivatives of order $p-1$ exist and are Fréchet differentiable in $\mathbf{x}_0$.
- We say that f is *Fréchet differentiable* of order p in a subset $D_0 \subseteq D$ if f is Fréchet differentiable of order p in any point $\mathbf{x}_0 \in D_0$.
- If f is Fréchet differentiable of order p in $\mathbf{x}_0 \in D$, then the *Fréchet differential* of order p in $\mathbf{x}_0$ is defined as $\mathrm{d}^p(\mathbf{x}_0) : \mathbb{R}^n \to \mathbb{R}^m$ by

$$\mathrm{d}^p(\mathbf{x}_0)(\mathbf{u}) := \sum_{1 \le i_1, \dots, i_p \le n} \frac{\partial^p f}{\partial x_{i_1} \dots \partial x_{i_p}}(\mathbf{x}_0) \cdot u_{i_1} \cdot \dots \cdot u_{i_n},\ \mathbf{u} = (u_1, \dots, u_n) \in \mathbb{R}^n.$$

Using multi-indexes, the formula defining $\mathrm{d}^p(\mathbf{x}_0)$ is similar to that defining $(u_1 + \dots + u_n)^p$. For instance, if $n = 2$,

$$\mathrm{d}^p(\mathbf{x}_0)(u_1, u_2) = \sum_{j=0}^{p} C_p^j \frac{\partial^p f}{\partial x_1^j \partial x_2^{p-j}}(\mathbf{x}_0) u_1^j u_2^{p-j}.$$

Taylor series

An important application of higher-order derivatives is *Taylor's formula*, which can now be written for functions of several variables.

Theorem (Taylor's formula)

Let $D \subseteq \mathbb{R}^n$ be a non-empty open set, $f : D \to \mathbb{R}^m$ a function Fréchet differentiable of order $p+1$ on some open ball $B(\mathbf{x}_0; r) \subseteq D$, where $p \in \mathbb{N}^*$. Then for every $\mathbf{x} \in B(\mathbf{x}_0; r)$ there exists $t \in (0,1)$ such that

$$\begin{aligned} f(\mathbf{x}) = f(\mathbf{x}_0) + \mathrm{d}f(\mathbf{x}_0)(\mathbf{x}-\mathbf{x}_0) + \frac{1}{2!}\mathrm{d}^2 f(\mathbf{x}_0)(\mathbf{x}-\mathbf{x}_0) + \dots \\ \dots + \frac{1}{p!}\mathrm{d}^p f(\mathbf{x}_0)(\mathbf{x}-\mathbf{x}_0) + \frac{1}{(p+1)!}\mathrm{d}^p f(\boldsymbol{\xi})(\mathbf{x}-\mathbf{x}_0), \end{aligned}$$

where $\boldsymbol{\xi} := \mathbf{x}_0 + t(\mathbf{x}-\mathbf{x}_0)$.

Definition

Let $D \subseteq \mathbb{R}^n$ be a non-empty open set and $f \in C^\infty(D)$.

- The *Taylor series* associated with f around a point $\mathbf{x}_0 \in D$ is the following series
$$f(\mathbf{x}_0) + \sum_{p=1}^{\infty} \frac{1}{p!} \mathrm{d}^p f(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0).$$
- In the case $\mathbf{x}_0 = \mathbf{0}_{\mathbb{R}^n}$ the above Taylor series is called the *Maclaurin series* associated to f.
- We say that a function is *analytic* in a ball $B(\mathbf{x}_0; r) \subseteq D$ if the Taylor series associated with f around $\mathbf{x}_0$ is convergent to $f(\mathbf{x})$ for every $\mathbf{x} \in B(\mathbf{x}_0; r)$.

- In the case $n = 1$, the Taylor series associated with a function is a power series.
- Conversely, if a function f is defined by a power series $\sum_{k=0}^{\infty} a_k (x - x_0)^k$ in its domain of convergence, then f is analytic in $(-r, r)$, where $r \in [0, +\infty]$ is its radius of convergence.

- In fact,

$$f^{(p)}(x) = \sum_{k=0}^{\infty} (k+1) \cdot \dots \cdot (k+p) a_{k+p} (x - x_0)^k, \ \forall x \in (-r, r), \ \forall p \in \mathbb{N}^*.$$

  Hence, $f^{(p)}(x_0) = p! a_p$ and the Taylor series associated to f around x_0 is precisely ${}_{k=0}^{\infty} a_k (x - x_0)^k$ (which proves that f is analytic in $(-r, r)$).
- However, the convergence of Taylor series associated to a function f does not imply that its sum is equal to f. For instance, let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) := \begin{cases} \mathrm{e}^{-\frac{1}{x^2}}, & x > 0; \\ 0, & x \leq 0. \end{cases}$$

  Then $f^{(p)}(0) = f(0) = 0$, $\forall p \in \mathbb{N}^*$, hence its Maclaurin series is the zero series; hence its sum is different from f on any interval centered in 0.

Examples

In the sequel we give some Maclaurin series for some well known analytic functions:

- $\dfrac{1}{1-x} = 1 + x + x^2 + x^3 + \dots,\ x \in (-1, 1);$
- $\ln(1+x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \dots,\ x \in (-1, 1);$
- $\mathrm{e}^x = 1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dots,\ x \in \mathbb{R};$
- $\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \dots,\ x \in \mathbb{R};$
- $\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^3}{3!} - \dfrac{x^5}{5!} + \dots,\ x \in \mathbb{R};$
- $\mathrm{arctg}\, x = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + \dots,\ x \in \mathbb{R}.$

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