LECTURE 1
ELEMENTS OF SET THEORY

1. Sets

In the vision of Georg Cantor, a *set* is a "Many that allows itself to be thought of as a One." More precisely, a set is a collection of elements, in which the order of disposal does not matter. This, however, cannot be taken as a definition of a set, because not every collection of objects can be thought as a set, as the following paradox (due to Bertrand Russell) shows:

Let M be the collection of *all* sets A such that $A \notin A$. Than M *cannot* be a set (exercice!).

Therefore, we will not provide a definition of a set and consider sets as primary objects of the set theory (which lays the foundation of mathematics); instead, some criteria which allow us to tell us which objects of our thought are sets and which are not must be provided. In order to acomplish this task, E. Zermelo and A. Fraenkel introduced a system of *axioms*, bearing now their names. We will adopt a version of their system, which we will present in the sequel. First, let us precise that in the framework of the set theory, all objects are considered to be sets. Between sets, two primary relations are considered: equality (which is a logical relation) and set membership, denoted by $=$, respectively $\in$. The idea of *property* (of a set) is captured by (well formed) *formulas*, which can be written with the aid of *variables* (such as x, y, A, B, etc.), of the above primary relations, of other logical symbols: $\Rightarrow$, $\Leftrightarrow$, $\wedge$, $\vee$, $\neg$, $\exists$, $\forall$ and of auxiliary symbols (*parantheses*).

ZF_1 **(axiom of extensionality)**. Two sets are equal if they have the same elements.

This amounts to say that if A and B are sets, then $\forall x(x \in A \Leftrightarrow x \in B)$ implies $A = B$. Sometimes, we are concerned only with only one side of the biimplication:

Definition. Let A and B be two sets. If all the elements of A are also elements of B, we say that A is a *subset* of B or A is *included* in B and we denote this by $A \subseteq B$. If moreover, $A \neq B$, we say that A is a *proper subset* of B, denoted by $A \subsetneq B$.

Proposition 1.1. *Let A, B and C be sets. Then:*

i) $A \subseteq A$ *(reflexivity);*
ii) $A \subseteq B$ *and* $B \subseteq A$ *implies* $A = B$ *(antisymmetry);*
iii) $A \subseteq B$ *and* $B \subseteq C$ *implies* $A \subseteq C$ *(transitivity).*

ZF_2 **(axiom schema of comprehension)**. For any set A and any property $\mathcal{P}$ (concerning the elements of A), there exists a set consisting only on those elements of A satisfying $\mathcal{P}$.

This set is usually denoted $\{x \in A \mid \mathcal{P}\}$ or $\{x \in A; \mathcal{P}\}$ and it is a subset of A.

We emphasize that specifying the set A is essential. Indeed, having a property $\mathcal{P}$, we can denote $\{x \mid \mathcal{P}\}$ the collection of *all* x satisfying the property $\mathcal{P}$. But such objects, named *classes*, are not sets all the time, so we have to deal carefully with them. For instance, Russell paradox translates into the fact that the class $\{x \mid x \notin x\}$ is not a set. An easy consequence of ZF_2 is that the collection of all sets, $\{x \mid x = x\}$ is not a set.

ZF_3 **(axiom of empty set)**. There exists a set with no elements.

This set (unique, by the axiom of extensionality), is called the *empty set* and is denoted by $\varnothing$. The axiom of empty set is important because it asserts the existence of a least one set.

ZF_4 **(pairing axiom)**. For any sets x and y there exists a set having only x and y as elements.

We will denote $\{x, y\}$ such a set. Of course, $z \in \{x, y\}$ if and only if $z = x$ or $z = y$. For any set x, it is customary to denote $\{x\}$ instead $\{x, x\}$.

Obviously, $\{x, y\} = \{y, x\}$, so the *order* of the elements x and y does not count. This is why the set $\{x, y\}$ is sometimes called an *unordered pair*. In order to introduce a notion of pair in which the order of x and y matters, we simply define

$$(x, y) := \{\{x\}, \{x, y\}\},$$

which we call an *ordered pair*.

Proposition 1.2. *Let x, y, x' and y' be sets. Then $(x, y) = (x', y')$ if and only if $x = x'$ and $y = y'$.*

ZF_5 **(power set axiom)**. For any set A, there exists a set consisting only in the subsets of A.

We will call such a set the *power set* of A and denote it by $\mathscr{P}(A)$. So, $B \in \mathscr{P}(A) \Leftrightarrow B \subseteq A$. Of course, for any set A, $\varnothing \in \mathscr{P}(A)$.

ZF_6 **(union axiom)**. For any set $\mathcal{A}$, there exists a set containing only the elements of the elements of $\mathcal{A}$.

We will call such a set the *union* of $\mathcal{A}$ and denote it by $\bigcup \mathcal{A}$. We have that $x \in \bigcup \mathcal{A}$ if and only if there exists $A \in \mathcal{A}$ such that $x \in A$.

We usually deal with family of sets of the form $\{A_i\}_{i\in I}$, where I is an *index* set; instead of $\bigcup \{A_i\}_{i\in I}$ we will simply write $\bigcup_{i\in I} A_i$. Obviously

$$x \in \bigcup_{i\in I} A_i \Leftrightarrow \exists i \in I : x \in A_i.$$

If A and B are sets, we call the *union* of A and B the set $A \cup B := \bigcup\{A, B\}$. It is clear that $x \in A \cup B$ if and only if $x \in A$ *or* $x \in B$.

A dual notion to the union is that of *intersection*. If $\mathcal{A}$ is a set, the intersection of $\mathcal{A}$ is the class

$$\bigcap \mathcal{A} := \{x \mid x \in A,\ \forall A \in \mathcal{A}\}.$$

If $\mathcal{A}$ is nonempty, then $\bigcap \mathcal{A}$ is a set and $\bigcap \mathcal{A} \subseteq \bigcup \mathcal{A}$. As in the case of unions, if $\{A_i\}_{i\in I}$ is a family of sets, we will write $\bigcap_{i\in I} A_i$ instead of $\bigcap \{A_i\}_{i\in I}$. Also, if A and B are sets, we call the *intersection* of A and B the set $A \cap B := \bigcap\{A, B\}$. Of course, $x \in A \cap B$ if and only if $x \in A$ *and* $x \in B$.

The notions introduced until now can be generalized (by recurrence) for a *finite number* ($n \geq 3$) of sets:

- *unordered n-tuple* with elements $x_1, \ldots, x_n$: $\{x_1, \ldots, x_n\} := \{x_1, \ldots, x_{n-1}\} \cup \{x_n\}$;
- (*ordered*) *n-tuple* with elements $x_1, \ldots, x_n$: $(x_1, \ldots, x_n) := ((x_1, \ldots, x_{n-1}), x_n)$;
- *union* of sets $A_1, \ldots, A_n$: $\bigcup_{k=1}^n A_k = A_1 \cup \cdots \cup A_n := (A_1 \cup \cdots \cup A_{n-1}) \cup A_n = \bigcup \{A_1, \ldots, A_n\}$;
- *intersection* of sets $A_1, \ldots, A_n$: $\bigcap_{k=1}^n A_k = A_1 \cap \cdots \cap A_n := (A_1 \cap \cdots \cap A_{n-1}) \cap A_n = \bigcap \{A_1, \ldots, A_n\}$.

Another set operation is the *set difference*:

Let A and B be sets. The set

$$A \setminus B := \{x \in A \mid x \notin B\}$$

is called the *difference* of sets A and B. The *symmetric difference* of A and B is

$$A \triangle B := (A \setminus B) \cup (B \setminus A).$$

If A is a set, the *absolute complement* of A is the class $A^c := \{x \mid x \notin A\}$. It can be easily proved that A^c is not a set. However, if U is a bigger set than A (meaning that $A \subseteq U$), the *relative complement* of A with respect to U, $\mathrm{C}_A^U := U \setminus A$ is clearly a set. Sometimes, when the set U is implied (and usually, big enough to consider it as a *universe*), we denote C_A instead of C_A^U.

In the following we give a list of the most immediate properties of the set operations just introduced.

Proposition 1.3. *Let A, B, C be sets and U a universe (for the purpose of taking complements). Then:*

i) $A \cup A = A \cap A = A$ *(idempotency);*
ii) $A \cup \varnothing = A$; $A \cap \varnothing = \varnothing$;
iii) $A \cup B = B \cup A$; $A \cap B = B \cap A$ *(commutativity);*
iv) $(A \cup B) \cup C = A \cup (B \cup C)$; $(A \cap B) \cap C = A \cap (B \cap C)$ *(associativity);*
v) $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$; $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$ *(distributivity);*
vi) $A \cup (A \cap B) = A \cap (A \cup B) = A$ *(absorption);*
vii) $\mathrm{C}_{\mathrm{C}_A} = A$;
viii) $\mathrm{C}_{A\cup B} = \mathrm{C}_A \cap \mathrm{C}_B$; $\mathrm{C}_{A\cap B} = \mathrm{C}_A \cup \mathrm{C}_B$ *(De Morgan's laws);*
ix) $C \setminus (A \cup B) = (C \setminus A) \cap (C \setminus B)$; $C \setminus (A \cap B) = (C \setminus A) \cup (C \setminus B)$;
x) $(A \cup B) \setminus C = (A \setminus C) \cup (B \setminus C)$; $(A \cap B) \setminus C = (A \setminus C) \cap (B \setminus C)$;
xi) $A \triangle A = \varnothing$; $A \triangle \varnothing = A$;
xii) $A \triangle B = B \triangle A$;
xiii) $(A \triangle B) \triangle C = A \triangle (B \triangle C)$.

Definition. Let A and B be sets. We call the *cartesian product* of A and B the set

$$A \times B := \{(x, y) \mid x \in A,\ y \in B\}$$ [1]

It is easy to verify that $A \times B$ is indeed a set; indeed, for every $x \in A$ and $y \in B$, $(x, y) \in \mathscr{P}(\mathscr{P}(A \cup B))$ (exercice!).

Proposition 1.4. *Let A, B and C be sets. Then:*

[1]strictly speaking, this is an abreviation for $\{z \mid \exists x \in A,\ \exists y \in B : z = (x, y)\}$

i) $A\times(B\cup C)=(A\times B)\cup(A\times C)$; *iii)* $(A\cup B)\times C=(A\times C)\cup(B\times C)$;
ii) $A\times(B\cap C)=(A\times B)\cap(A\times C)$; *iv)* $(A\cap B)\times C=(A\times C)\cap(B\times C)$.

As in the case of other operations, one can define the *cartesian product* of a *finite number* ($n\geq 3$) of sets:

$$A_1\times\cdots\times A_n := (A_1\times\cdots\times A_{n-1})\times A_n = \{(x_1,\ldots,x_n)\mid x_1\in A_1,\ldots,x_n\in A_n\}.$$

ZF_7 **(axiom schema of replacement)**. For any set A and any property $\mathcal{P}$ such that

$$\forall x(x\in A\Rightarrow \exists! y:\mathcal{P}),$$

there exists a set B such that

$$\forall y(y\in B\Leftrightarrow \exists x\in A:\mathcal{P}).$$

ZF_8 **(axiom of infinity)**. There exists a set C such that:
(i) $\varnothing\in C$;
(ii) $x\in C\Rightarrow x\cup\{x\}\in C$.

With the aid of this axiom we can construct the *set of natural numbers*, as follows. Let C be a set satisfying (i) and (ii); we define

$$\omega := \bigcap\{N\in\mathscr{P}(C)\mid N \text{ satisfies (i) and (ii)}\}.$$

Then ω itself satisfies (i) and (ii). We can define $0:=\varnothing$, $1:=\{0\}=\{\varnothing\}$, $2:=1\cup\{1\}=\{\varnothing,\{\varnothing\}\}$, $3:=2\cup\{2\}=\{\varnothing,\{\varnothing\},\{\varnothing,\{\varnothing\}\}\}$, and so on ... We have that $0,1,2,3,\dots\in\omega$; thus ω becomes a prototype for the *set of natural numbers*, usually denoted by $\mathbb{N}$. Its elements will be called *natural numbers*.

ZF_9 **(axiom of foundation)**. For any nonempty set A, there exists a set $x\in A$ such that $x\cap A=\varnothing$.

This axiom has the rôle of preventing pathologic behaviour for sets. For instance, for any set x, we have $x\notin x$ (it is enough to take $A:=\{x\}$ in ZF_9).

ZF_{10} **(axiom of choice)**. For any set $\mathcal{A}$ consisting of pairwise disjoint nonempty sets, there exists a set C such that for every $A\in\mathcal{A}$, C and A have exactly one commun element.

2. Relations

The following definition attempts to give a mathematical meaning to the concept of relation.

Definition. Let A, B be sets. A (*binary*) *relation from A to B* is a subset of $A\times B$. In the case $A=B$, we say that R is a *relation on A*.

If R is a relation from A to B and $x\in A$, $y\in B$, we often say that x is in the relation R with y if $(x,y)\in R$. Sometimes we will denote xRy instead $(x,y)\in R$.

Definition. Let R be a relation from A to B and S a relation from B to C.

a) The sets

$$\operatorname{Dom}R := \{x\in A\mid \exists y\in B: xRy\},$$
$$\operatorname{Im}R := \{y\in B\mid \exists x\in A: xRy\}$$

are called the *domain*, repectively the *image* of the relation R.

b) The relation from B to A,

$$R^{-1} := \{(y,x)\in B\times A\mid xRy\}$$

is called the *inverse* of the relation R.

c) The relation from A to C

$$S\circ R := \{(x,z)\in A\times C\mid \exists y\in B: xRy\wedge ySz\}$$

is called the *composition* of S with R.

Definition. Let A be a set. The relation

$$1_A := \{(x,x)\mid x\in A\}$$

is called the *identity* on A.

Definition. Let R be a relation on A. We say that the relation R is:

a) *reflexive* if $1_A\subseteq R$, *i.e.* $xRx,\ \forall x\in A$;
b) *irreflexive* if $1_A\nsubseteq R$, *i.e.* $x\neg Rx,\ \forall x\in A$;
c) *symmetric* if $R^{-1}=R$, *i.e.* $xRy\Rightarrow yRx,\ \forall x,y\in A$;
d) *antisymmetric* if $R\cap R^{-1}=1_A$, *i.e.* $xRy\wedge yRx\Rightarrow x=y,\ \forall x,y\in A$;

e) *transitive* if $R \circ R \subseteq R$, *i.e.* $xRy \wedge yRz \Rightarrow xRz,\ \forall x, y, z \in A$;
f) *total* if $R \cup R^{-1} = A \times A$, *i.e.* $xRy \vee yRx,\ \forall x, y \in A$.

A first type of relations we will encounter is that of *equivalence relations*:

Definition. Let R be a relation on A. We say that R is an *equivalence relation* on A if it is reflexive, symmetric and transitive.

Definition. Let R be an equivalence relation on A.

a) For an element $x \in A$, we call the *equivalence class* of x the set
$$\widehat{x}_R := \{y \in A \mid xRy\},$$
(which is denoted also $[x]_R$; sometimes, when it is clear from the context, we renounce to the subscript R).

b) The set of the equivalence classes with respect to R,
$$A/_R := \{\widehat{x}_R \mid x \in A\}$$
is called the *quotient* of A with respect to R.

Proposition 2.1. *Let R be an equivalence relation on A. Then:*

i) $x \in \widehat{x},\ \forall x \in A$;
ii) $y \in \widehat{x} \Leftrightarrow \widehat{x} = \widehat{y} \Leftrightarrow xRy,\ \forall x, y \in A$.

It is customary to denote equivalence relations by symbols as $\sim$, $\simeq$, $\approx$, $\equiv$, $\cong$, etc.

Another frequent type of relations is that of *order relations*:

Definition. Let R be a relation on A. We say that R is:

a) an *order* on A if it is reflexive, antisymmetric and transitive;
b) a *preorder* on A if it is reflexive and transitive;
c) a *total order* on A if it is an order on A and is a total relation.

If A is a set and R is an order (preorder, total order) on A, we say that the pair (A, R) is an *ordered* (a *preordered*, a *totally ordered*) *set*. As in the case of equivalence relations, order relations are usually denoted by symbols as $\leq$, $\preceq$, $\lesssim$, $\leqslant$, etc. The inverses of such relations (which turn out to be still order relations) are denoted by $\geq$, $\succeq$, $\gtrsim$, $\geqslant$, respectively. Also, if $\preceq$ is a preorder on A, $\prec$ will denote the relation $\preceq \setminus 1_A$, *i.e.* $x \prec y \Leftrightarrow (x \preceq y) \wedge (x \neq y)$ for any $x, y \in A$ (similar conventions apply for other symbols).

Definition. Let $(A, \preceq)$ be a preordered set.

a) An element $a \in A$ is called an *upper bound* for the subset $B \subseteq A$ if $x \preceq a,\ \forall x \in B$.
b) An element $a \in A$ is called a *lower bound* for the subset $B \subseteq A$ if $x \succeq a,\ \forall x \in B$.
c) If the subset $B \subseteq A$ admits an upper bound, a lower bound or both, we say that B is *upper bounded*, *lower bounded* or *bounded*, respectively.
d) If $a \in A$ is an upper bound for a subset $B \subseteq A$ and $a \in B$, we say that a is *a maximum* for B.
e) If $a \in A$ is an lower bound for a subset $B \subseteq A$ and $a \in B$, we say that a is *a minimum* for B.

Remark. There can be several maximum points or minimum points for B; however, if $\preceq$ is an order, then the maximum and the minimum, when they exist, are unique. We denote them $\max_{\preceq} B$, respectively $\min_{\preceq} B$. When no confusion is possible, we will use also the notations $\max B$, respectively $\min B$.

Definition. Let $(A, \preceq)$ be an ordered set and B a subset of A.

a) We say that an element $a \in A$ is the *least upper bound* or the *supremum* of B if $a = \min U$, where U is the set of the upper bounds for B. Such an element, if it exists, is denoted $\sup_{\preceq} B$ (or simply $\sup B$).
b) We say that an element $a \in A$ is the *greatest lower bound* or the *infimum* of B if $a = \max L$, where L is the set of the lower bounds for B. Such an element, if it exists, is denoted $\inf_{\preceq} B$ (or simply $\inf B$).

Definition. Let $(A, \preceq)$ be an ordered set.

a) We say that $(A, \preceq)$ is *Dedekind complete* if every non-empty subset of A which is upper bounded admits a supremum.
b) We say that $(A, \preceq)$ is *well-ordered* if every non-empty subset of A admits a minimum.

It can be shown (exercice!) that an ordered set is Dedekind complete if and only if every non-empty subset which is lower bounded admits an infimum.

It is easy to prove (exercice!) that a well-ordered set is a totally ordered *set*. Provided we accept the axiom of choice, it can be proven that every set can be well-ordered (*i.e.*, for every set A there exists an order $\leq$ on A such that $(A, \leq)$ is well-ordered).

3. Functions

Definition. Let A and B be sets. We say that a relation $f \subseteq A \times B$ is a *function from A to B* and we denote $f : A \to B$ if:

(i) $\operatorname{Dom} f = A$;

(ii) $(x, y) \in f,\ (x, z) \in f \Rightarrow y = z,\ \forall x \in A,\ \forall y, z \in B$.

If $f : A \to B$ and $x \in A$, we denote $f(x)$ the unique element $y \in B$ such that $(x, y) \in f$. When we want *to define* a function $f : A \to B$, it is enough *to specify* the value $f(x)$ for every $x \in A$.

For a function $f : A \to B$, $f = \{(x, f(x)) \mid x \in A\}$. Anyway, when we "forget" that f is a relation, we call the set $\{(x, f(x)) \mid x \in A\}$ the *graph* of f.

Of course, the identity on a set A is a function, $1_A : A \to A$ ($1_A(x) = x,\ \forall x \in A$).

If X and Y are sets, we denote $\mathscr{F}(X; Y)$ the set of functions from X to Y.

Definition. Let $f : A \to B$ be a function.

a) If E is a subset of A, the function $f|_E := f \cap (E \times B)$ (i.e., $f|_E(x) := f(x),\ \forall x \in E$) is called the *restriction* of f to E.

b) If E is a subset of A, the set
$$f[E] := \{y \in B \mid \exists x \in E : (x, y) \in f\}$$
is called the *image* of f through E.

c) If F is a subset of B, the set
$$f^{-1}[F] := \{x \in A \mid \exists y \in F : (x, y) \in f\}$$
is called the *preimage* or the *inverse image* of f through F.

For a function $f : A \to B$, $f[\varnothing] = \varnothing$, $f^{-1}[\varnothing] = \varnothing$, $f[A] = \operatorname{Im} f$, $f^{-1}[B] = \operatorname{Dom} f = A$.

Definition. A function $f : A \to B$ is called:

a) *injective* or *one-to-one* if for any $x, y \in A$, $f(x) = f(y) \Rightarrow x = y$ (in other words, the relation f^{-1} is a function from $\operatorname{Im} f$ to A);

b) *surjective* or *onto* if $\operatorname{Im} f = B$;

c) *bijective* if it is both injective and surjective;

d) *invertible* if there exists $g : B \to A$ such that $f \circ g = 1_B$ and $g \circ f = 1_A$.

Proposition 3.1. *A function $f : A \to B$ is bijective if and only if it is invertible. In this case, f^{-1} is a function from B to A and $f \circ f^{-1} = 1_B$, $f^{-1} \circ f = 1_A$.*

Definition. Let $(A, \leq)$, $(B, \leq)$ be ordered sets and $f : A \to B$ a function. We say that f is:

a) *monotone* if for any $x, y \in A$, $x \leq y \Rightarrow f(x) \leq f(y)$;

b) *strictly monotone* if for any $x, y \in A$, $x < y \Rightarrow f(x) < f(y)$.

Let us consider a set U large enough for our purposes (and considered as a universe for taking complements). If A is a subset of U, we call the *characteristic function* of A the function $\chi_A : U \to \{0, 1\}$ defined by
$$\chi_A(x) := \begin{cases} 1, & x \in A; \\ 0, & x \in \complement_A. \end{cases}$$

Proposition 3.2. *Let A and B be two subsets of U. Then:*

i) $A \subseteq B \Leftrightarrow \chi_A \leq \chi_B$;

ii) $\chi_{\complement_A} = 1 - \chi_A$;

iii) $\chi_{A \cap B} = \chi_A \cdot \chi_B$;

iv) $\chi_{A \cup B} = \chi_A + \chi_B - \chi_A \cdot \chi_B$.

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