LECTURE 6
METRIC SPACES

1. Definition. Properties

In the usual physical space, the notion of *distance* or *length* is rather a familiar concept. If P and Q are two points in space having coordinates in a Cartesian system (x_P, y_P, z_P),respectively (x_Q, y_Q, z_Q), the distance between P and Q (or length of the segment PQ) is

$$\mathrm{d}(P,Q) = \sqrt{(x_P - x_Q)^2 + (y_P - y_Q)^2 + (z_P - z_Q)^2}.$$

Our purpose is to extend this concept to more general spaces, known as *metric spaces.* We will see that in such spaces it is possible to define notions of *convergence* and *continuity*, as for the real line. As in the previous lecture, our predilect exemplifications will be on the Euclidean space $\mathbb{R}^n$.

Definition. Let X be a non-empty set. A function $\mathrm{d} : X \times X \to \mathbb{R}_+$ is called a *distance* or a *metric* on X if:

(D_1) $\mathrm{d}(x,y) = 0 \Leftrightarrow x = y,\ \forall x,y \in X$;

(D_2) $\mathrm{d}(x,y) = \mathrm{d}(y,x),\ \forall x,y \in X$ (*symmetry*);

(D_3) $\mathrm{d}(x,z) \le \mathrm{d}(x,y) + \mathrm{d}(y,z),\ \forall x,y,z \in X$ (*triangle property*).

In this case, the couple (X, d) is called a *metric space.*

Proposition 1.1. *Let (X,d) be a metric space. Then:*

i) $\mathrm{d}(x_0,x_n) \le \mathrm{d}(x_0,x_1) + \mathrm{d}(x_1,x_2) + \cdots + \mathrm{d}(x_{n-1},x_n),\ \forall n \in \mathbb{N}^*,\ \forall x_0,x_1,\dots,x_n \in X$;

ii) $|\mathrm{d}(x,z) - \mathrm{d}(y,z)| \le \mathrm{d}(x,y),\ \forall x,y,z \in X$;

iii) $|\mathrm{d}(x,y) - \mathrm{d}(x',y')| \le \mathrm{d}(x,y') + \mathrm{d}(x',y),\ \forall x,y,x',y' \in X$ *(quadrilateral inequality).*

In linear spaces, some distances come from norms, which were introduced in the previous lecture.

Definition. Let $(V, \|\cdot\|)$ be a normed space. Then the mapping $\mathrm{d} : V \times V \to \mathbb{R}_+$ defined by

$$\mathrm{d}(x,y) := \|x - y\|,\ x,y \in \mathbb{R}$$

is a metric, called the *metric induced* by the norm $\|\cdot\|$.

Examples.

1. On $\mathbb{R}$, the application $\mathrm{d} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}_+$ defined by

$$\mathrm{d}(x,y) := |x - y|,\ x,y \in \mathbb{R}$$

is a distance, called the *canonical distance* in $\mathbb{R}$.

2. On $\mathbb{R}^n$, the metric induced by the Euclidean norm is called the *Euclidean metric* on $\mathbb{R}^n$ and is denoted by d_2. We have

$$\mathrm{d}_2(\mathbf{x},\mathbf{y}) := \|\mathbf{x} - \mathbf{y}\|_2 = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2},$$

for $\mathbf{x} = (x_1,\dots,x_n),\ \mathbf{y} = (y_1,\dots,y_n) \in \mathbb{R}^n$.

3. Let, for $p \in [1,+\infty)$, the application $\|\cdot\|_p : \mathbb{R}^n \to \mathbb{R}_+$ be defined by

$$\|\mathbf{x}\|_p := \left(\sum_{i=1}^n |x_i|^p\right)^{1/p},\ \mathbf{x} = (x_1,\dots,x_n) \in \mathbb{R}^n.$$

Then $\|\cdot\|_p$ is a norm (in the case $p = 2$, we already know that, since $\|\cdot\|_p$ coincides with the Euclidean norm, which is induced by the Euclidean scalar product). Indeed, the first properties in the definition of the norm are easy to prove, while the last, the triangle property, being equivalent to

$$\left(\sum_{i=1}^n |x_i + y_i|^p\right)^{1/p} \le \left(\sum_{i=1}^n |x_i|^p\right)^{1/p} + \left(\sum_{i=1}^n |y_i|^p\right)^{1/p},\ \forall x_1,\dots,x_n,y_1\dots,y_n \in \mathbb{R}$$

which is precisely the Minkowski inequality (stated in Lecture 2).

We can also introduce a norm on $\mathbb{R}^n$ even in the case $p = +\infty$, by

$$\|\mathbf{x}\|_\infty := \max_{1\le i\le n} |x_i|\,,\ \mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n.$$

Again, the first properties are easily deduced. In order to show the triangle property, let $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{y} = (y_1, \dots, y_n)$ in $\mathbb{R}^n$. Then

$$\|\mathbf{x}+\mathbf{y}\|_\infty = \max_{1\le i\le n} |x_i + y_i| \le \max_{1\le i\le n} \left(|x_i| + |y_i|\right). \tag{1}$$

Since for each $i \in \{1, \dots, n\}$, $|x_i| \le \|\mathbf{x}\|_\infty$ and $|y_i| \le \|\mathbf{y}\|_\infty$, we have

$$|x_i| + |y_i| \le \|\mathbf{x}\|_\infty + \|\mathbf{y}\|_\infty\,,\ \forall i \in \{1, \dots, n\}.$$

This means that $\max\limits_{1\le i\le n} \left(|x_i| + |y_i|\right) \le \|\mathbf{x}\|_\infty + \|\mathbf{y}\|_\infty$. Combined with (1), we get $\|\mathbf{x}+\mathbf{y}\|_\infty \le \|\mathbf{x}\|_\infty + \|\mathbf{y}\|_\infty$.

The metric induced on $\mathbb{R}^n$ by the p-norm is called *Minkowski distance* and is denoted by d_p. So we have

$$\mathrm{d}_p(\mathbf{x},\mathbf{y}) := \|\mathbf{x}-\mathbf{y}\|_p = \begin{cases} \left(|x_1 - y_1|^p + \dots + |x_n - y_n|^p\right)^{1/p}, & p \in [1, +\infty); \\ \max\left\{|x_1 - y_1|, \dots, |x_n - y_n|\right\}, & p = +\infty, \end{cases}$$

for $\mathbf{x} = (x_1, \dots, x_n)$, $\mathbf{y} = (y_1, \dots, y_n) \in \mathbb{R}^n$. The metric d_1 is sometimes called the *taxi-cab distance* or *Manhattan distance*, while the metric d_∞ is also called *Chebyshev distance*.

If $n = 1$, those distances are the same: $\mathrm{d}_p(x, y) = |x - y|$, $\forall x, y \in \mathbb{R}$, $\forall p \in [1, +\infty]$.

4. The application $\tilde{\mathrm{d}} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+$, defined by

$$\tilde{\mathrm{d}}(\mathbf{x},\mathbf{y}) := \sum_{k=1}^{n} \frac{1}{2^k} \cdot \frac{|x_k - y_k|}{1 + |x_k - y_k|},$$

for $\mathbf{x} = (x_1, \dots, x_n)$, $\mathbf{y} = (y_1, \dots, y_n) \in \mathbb{R}^n$ is a distance on $\mathbb{R}^n$, but is not norm-induced, because the function $\mathbf{x} \mapsto \tilde{\mathrm{d}}(\mathbf{x}, \mathbf{0})$ lacks the homogeneity property.

5. Let X be a non-empty set. The function $\mathrm{d} : X \times X \to \mathbb{R}_+$, defined by

$$\mathrm{d}(x, y) := \begin{cases} 0, & x = y; \\ 1, & x \ne y, \end{cases}$$

for $x, y \in X$, is a metric on X, called the *discrete metric* on X.

6. On the extended real line we can consider the metric $\mathrm{d} : \overline{\mathbb{R}} \times \overline{\mathbb{R}} \to \mathbb{R}_+$, defined by

$$\mathrm{d}(x, y) := |\operatorname{arctg} x - \operatorname{arctg} y|\,,\ x, y \in \overline{\mathbb{R}}$$

(we have extended the function arctg on $\overline{\mathbb{R}}$ by $\operatorname{arctg}(-\infty) := -\pi/2$, $\operatorname{arctg}(+\infty) := \pi/2$).

An important norm on a space of functions is the following:

Definition. Let E be a non-empty set and $\mathscr{B}(E)$ be the space of bounded functions $f : E \to \mathbb{R}$ (*i.e.* $\operatorname{Im} f$ is a bounded set). We set $\|\cdot\|_{\sup} : \mathscr{B}(E) \to \mathbb{R}_+$ defined by

$$\|f\|_{\sup} := \sup_{x\in E} |f(x)|\,.$$

Then $\|\cdot\|_{\sup}$ is a norm on $\mathscr{B}(E)$, called the *uniform norm* or *sup-norm.* The metric induced by $\|\cdot\|_{\sup}$ is called the *uniform distance*, denoted $\mathrm{d}_{\sup}$.

Definition.

a) Let X be a non-empty set. We say that two metrics d and d' on X are *equivalent* if there exist two constants $c_1, c_2 > 0$ such that

$$c_1 \mathrm{d}'(x, y) \le \mathrm{d}(x, y) \le c_2 \mathrm{d}'(x, y),\ \forall x, y \in X.$$

b) Let $(V, +, \cdot)$ be a linear space. We say that two norms $\|\cdot\|$ and $\|\cdot\|'$ are *equivalent* if there exist two constants $c_1, c_2 > 0$ such that

$$c_1 \|\mathbf{x}\|' \le \|\mathbf{x}\| \le c_2 \|\mathbf{x}\|',\ \forall \mathbf{x} \in V.$$

Of course, if two norms on V are equivalent, so are the induced metrics.

Theorem 1.2. *On $\mathbb{R}^n$, all the norms $\|\cdot\|_p$ with $p \in [1, +\infty]$ are equivalent.*

2. Open sets. Closed sets

2.1. **Neighbourhoods.**

Definition. Let (X, d) be a metric space, $x_0 \in X$ an arbitrary point and $r > 0$ a real number.

a) The set
$$B(x_0; r) := \{x \in X \mid \mathrm{d}(x, x_0) < r\}$$
is called the *open ball* of *radius* r and *center* x_0.

b) The set
$$\bar{B}(x_0; r) := \{x \in X \mid \mathrm{d}(x, x_0) \leq r\}$$
is called the *closed ball* of *radius* r and *center* x_0.

c) The set
$$S(x_0; r) := \{x \in X \mid \mathrm{d}(x, x_0) = r\}$$
is called the *sphere* of *radius* r and *center* x_0.

Definition. Let (X, d) be a metric space and $x_0 \in X$ an arbitrary point. We say that a subset V of X is a *neigbourhood* of x_0 if there exists $r > 0$ such that $B(x_0; r) \subseteq V$. We denote the family of all neigbourhoods of x_0 by $\mathscr{V}(x_0)$.

Proposition 2.1. *Let (X, d) be a metric space, $x \in X$ and V, V', U subsets of X.*

i) *If $V \in \mathscr{V}(x)$ and $V \subseteq U$, then $U \in \mathscr{V}(x)$.*
ii) *If $V, V' \in \mathscr{V}(x)$, then $V \cap V' \in \mathscr{V}(x)$.*
iii) *If $V \in \mathscr{V}(x)$, then $x \in V$.*
iv) *If $V \in \mathscr{V}(x)$, then there exists $W \in \mathscr{V}(x)$ such that $V \in \mathscr{V}(y)$, $\forall y \in W$.*

It can be proven that the above four properties fully characterize the family of neigbourhoods $\mathscr{V}(x)$ of all $x \in X$.

Theorem 2.2. *Let (X, d) be a metric space, $x_0 \in X$ an arbitrary point and $r > 0$ a real number. Then*
$$B(x_0; r) \in \mathscr{V}(x), \ \forall x \in B(x_0; r).$$

Definition. Let (X, d) be a metric space and $x_0 \in X$ an arbitrary point. We say that a family $\mathscr{U}(x_0)$ is a *system of neigbourhoods* of x_0 if

(i) $\mathscr{U}(x_0) \subseteq \mathscr{V}(x_0)$;
(ii) for every $V \in \mathscr{V}(x_0)$ there exists $U \in \mathscr{U}(x_0)$ such that $U \subseteq V$.

It is obvious that $\{B(x_0; r)\}_{r \in \mathbb{R}_+^*}$ is a system of neigbourhoods of x_0. In fact, even $\left\{B(x_0; \frac{1}{n})\right\}_{n \in \mathbb{N}^*}$ is a system of neigbourhoods of x_0.

2.2. **Open sets.**

Definition. Let (X, d) be a metric space. A susbset D of X is called an *open set* if it is a neighbourhood for every element of D, *i.e.*
$$\forall x \in D, \ \exists r > 0 : B(x; r) \subseteq D.$$

Examples.

1. By Theorem 2.2, for any $x \in X$ and any $r > 0$, the open ball $B(x; r)$ is an open set. In particular, if X is $\mathbb{R}$ and d is the Euclidean distance, then the interval $(x - \varepsilon, x + \varepsilon)$ is an open set for every $x \in \mathbb{R}$ and $\varepsilon > 0$. This implies that every open interval (a, b) with $a, b \in \mathbb{R}$ with $a < b$ is an open set.

2. The interval $(-1, 2]$ is not open in $\mathbb{R}$. Indeed, $(-1, 2]$ is not a neighbourhood of 2, since $(2 - \varepsilon, 2 + \varepsilon) \nsubseteq (-1, 2]$ for every $\varepsilon > 0$.

3. The sets $(0, 1) \times (2, 4)$ and
$$\left\{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 \neq 4\right\}$$
are open in $\mathbb{R}^2$ (endowed with the Euclidean distance).

Proposition 2.3. *Let X be a non-empty set and two metrics* d, d′ *on X. If* d *and* d′ *are equivalent, then any open set with respect to the metric* d *is open with respect to the metric* d′.

The collection of all open sets of a metric space (X, d) is called the *topology* on X *induced* by d. By the above result, the topologies induced by two equivalent metrics are the same. Therefore, applying Theorem 1.2, the topologies induced by d_p with $p \in [1, +\infty]$ are the same topology, called the *usual topology* on $\mathbb{R}^n$.

Theorem 2.4. *Let (X, d) be a metric space. Then:*

i) an arbitrary union of open sets is an open set;
ii) an intersection of two open sets is an open set;
iii) every open set can be written as an union of open balls;
iv) $\varnothing$ and X are open sets.

2.3. **Closed sets.**

Definition. Let (X, d) be a metric space. A subset F of X is called a *closed set* if $\mathrm{C}_A = X \setminus A$ is an open set.

Proposition 2.5. *Let (X, d) be a metric space. Then:*

i) an arbitrary intersection of closed sets is a closed set;
ii) a union of two closed sets is a closed set;
iii) $\varnothing$ and X are closed sets.

Examples.
1. Every closed interval $[a, b]$ with $a, b \in \mathbb{R}$ with $a \leq b$ is a closed subset of $\mathbb{R}$. In particular, $\{a\}$ is a closed set.

2. There are sets in $\mathbb{R}$ which are neither open nor closed, for instance $(1, 2]$.

3. The sets $[0, 1] \times [2, 4]$ and

$$\left\{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 \geq 4\right\}$$

are closed in $\mathbb{R}^2$ (endowed with the Euclidean distance).

2.4. **Boundedness.**

Definition. Let (X, d) be a metric space, $x_0 \in X$ and A, B two non-empty subsets of X.

a) The *distance* from x_0 to A is defined as

$$\mathrm{d}(x_0, A) := \inf \{\mathrm{d}(x_0, x) \mid x \in A\}.$$

b) The *distance* between A and B is defined as

$$\mathrm{d}(A, B) := \inf \{\mathrm{d}(x, y) \mid x \in A,\ y \in B\}.$$

c) The *diameter* of A is defined as

$$\rho(A) := \sup \{\mathrm{d}(x, y) \mid x, y \in A\}.$$

The diameter of A is an element from $[0, +\infty]$. In the above definition, if we set by convention $\inf \varnothing = +\infty$ and $\sup \varnothing = 0$, we can allow A and B to be the empty set, so we get

$$\mathrm{d}(x_0, \varnothing) = \mathrm{d}(A, \varnothing) = \mathrm{d}(\varnothing, B) = +\infty$$

and

$$\rho(\varnothing) = 0.$$

Proposition 2.6. *Let (X, d) be a metric space and $A, B \subseteq X$.*

i) *If $A \subseteq B$ then $\rho(A) \leq \rho(B)$.*
ii) *$\rho(A) = 0$ if and only if $A = \varnothing$ or $A = \{x\}$ for some $x \in X$.*

Definition. Let (X, d) be a metric space. We say that a subset A of X is *bounded* if $\rho(A) < +\infty$. Otherwise (*i.e.*, $\rho(A) = +\infty$), we say that A is *unbounded*.

Proposition 2.7.

i) *Let (X, d) be a metric space. A subset A of X is bounded if and only if there exist $x_0 \in X$ and $r > 0$ such that $A \subseteq B(x_0; r)$.*
ii) *Let $(V, \|\cdot\|)$ be a normed space. A subset A of V is bounded (with respect to the induced metric) if and only if there exists $r > 0$ such that*

$$\|x\| < r,\ \forall x \in A.$$

This allows us to conclude that there is no danger of confusion when we say that $A \subseteq \mathbb{R}$ is bounded (considering the Euclidean distance on $\mathbb{R}$).

3. The interior and the closure of a set. Limit points

3.1. **The interior of a set.**

Definition. Let (X, d) be a metric space and A a subset of X.

a) We say that an element $x \in X$ is an *interior point* of A if there exists $r > 0$ such that $B(x;r) \subseteq A$ (in other words, $A \in \mathscr{V}(x)$).

b) We call the *interior* of A the set of all interior points of A, denoted $\mathring{A}$ or $\operatorname{int} A$.

Examples. We consider the space $\mathbb{R}$ endowed with the Euclidean metric.

1. If $a, b \in \mathbb{R}$ with $a < b$, then

$$\operatorname{int}(a,b) = \operatorname{int}(a,b] = \operatorname{int}[a,b) = \operatorname{int}[a,b] = (a,b).$$

2. The set $\mathbb{Q}$ of all rational numbers has no interior points ($\mathring{\mathbb{Q}} = \varnothing$), since for every $x \in \mathbb{Q}$ and $\varepsilon > 0$, $(x-\varepsilon, x+\varepsilon) \nsubseteq \mathbb{Q}$, because every non-empty interval contains also irrational numbers.

Theorem 3.1. *Let (X, d) be a metric space. Then:*

i) $\mathring{A} \subseteq A, \ \forall A \subseteq X;$

ii) A *is open* $\Leftrightarrow \mathring{A} = A, \ \forall A \subseteq X;$

iii) $A \subseteq B \Rightarrow \mathring{A} \subseteq \mathring{B}, \ \forall A, B \subseteq X;$

iv) $\overset{\circ}{\widehat{A \cap B}} = \mathring{A} \cap \mathring{B}, \ \forall A, B \subseteq X;$

v) $\mathring{A} \cup \mathring{B} \subseteq \overset{\circ}{\widehat{A \cup B}}, \ \forall A, B \subseteq X.$

Definition. Let (X, d) be a metric space and A a subset of X.

a) We say that an element $x \in X$ is an *exterior point* of A if x is an interior point of $\complement_A = X \setminus A$.

b) We call the *exterior* of A the set of all exterior points of A, denoted $\operatorname{ext} A$.

Proposition 3.2. *Let (X, d) be a metric space, $x \in X$ and A a subset of X. Then $x \in \operatorname{ext} A$ if and only if there exists $V \in \mathscr{V}(x)$ such that $V \cap A = \varnothing$.*

3.2. **The closure of a set.**

Definition. Let (X, d) be a metric space and A a subset of X.

a) We say that an element $x \in X$ is a *closure point* of A if for every neigbourhood V of x we have $V \cap A \neq \varnothing$.

b) We call the *closure* of A the set of all closure points of A, denoted $\overline{A}$ or $\operatorname{cl} A$;

c) We say that A is *dense* (in X) if $\overline{A} = X$.

Proposition 3.3. *Let (X, d) be a metric space, $x \in X$ and A a subset of X. Then $x \in \overline{A}$ if and only if there exists $r > 0$ such that $B(x;r) \cap A \neq \varnothing$.*

Examples. We consider the space $\mathbb{R}$ endowed with the Euclidean metric.

1. If $a, b \in \mathbb{R}$ with $a < b$, then

$$\overline{(a,b)} = \overline{(a,b]} = \overline{[a,b)} = \overline{[a,b]} = [a,b].$$

2. We have $\overline{\mathbb{Q}} = \mathbb{R}$, since for every $x \in \mathbb{Q}$ and $\varepsilon > 0$, $(x-\varepsilon, x+\varepsilon) \cap \mathbb{Q} \neq \varnothing$, because every non-empty interval contains rational numbers. According to the previous definition, $\mathbb{Q}$ is dense in $\mathbb{R}$.

Theorem 3.4. *Let (X, d) be a metric space. Then:*

i) $A \subseteq \overline{A}, \ \forall A \subseteq X;$

ii) A *is closed* $\Leftrightarrow \overline{A} = A, \ \forall A \subseteq X;$

iii) $A \subseteq B \Rightarrow \overline{A} \subseteq \overline{B}, \ \forall A, B \subseteq X;$

iv) $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}, \ \forall A, B \subseteq X;$

v) $\overline{A \cup B} = \overline{A} \cup \overline{B}, \ \forall A, B \subseteq X;$

vi) $\overline{\complement_A} = \complement_{\mathring{A}}, \ \overset{\circ}{\widehat{\complement_A}} = \complement_{\overline{A}}.$

DEFINITION. Let (X, d) be a metric space and A a subset of X. We call the *boundary* of A the set

$$\partial A = \operatorname{Fr} A := \overline{A} \setminus \mathring{A}.$$

Examples. Again, let us consider the Euclidean on $\mathbb{R}$.
1. If $a, b \in \mathbb{R}$ with $a < b$, then

$$\operatorname{Fr}(a,b) = \operatorname{Fr}(a,b] = \operatorname{Fr}[a,b) = \operatorname{Fr}[a,b] = \{a,b\}.$$

2. We have $\operatorname{Fr} \mathbb{Q} = \mathbb{R}$.

3.3. **Limit points.**

DEFINITION. Let (X, d) be a metric space and A a subset of X.

a) We say that an element $x \in X$ is a *limit point* of A if for every neigbourhood V of x we have $V \cap (A \setminus \{x\}) \neq \varnothing$.
b) We call the *derived set* of A the set of all limit points of A, denoted A'.

Theorem 3.5. *Let (X, d) be a metric space. Then:*

i) $A' \subseteq \overline{A} = A \cup A', \ \forall A \subseteq X;$

ii) $\overline{A} = A \Leftrightarrow A' \subseteq A, \ \forall A \subseteq X;$

iii) $A \subseteq B \Rightarrow A' \subseteq B', \ \forall A, B \subseteq X;$

iv) $(A \cup B)' = A' \cup B', \ \forall A, B \subseteq X.$

DEFINITION. Let (X, d) be a metric space and A a subset of X.

a) An element $x \in A \setminus A'$ is called an *isolated point* of A.
b) We call the *discrete part* of A the set of all isolated points of A, *i.e.* $A \setminus A'$.
c) We say that A is *discrete* if every element of A is an isolated point, i.e. $A \cap A' = \varnothing$.

4. SEQUENCES IN METRIC SPACES

Let X be a non-empty set. As in the case of sequences of real numbers, a *sequence* $(x_n)_{n \in \mathbb{N}}$ in X is simply a function $x : \mathbb{N} \to X$. All the terminology related to sequences of real numbers can (and is) translated into this case. If a metric d is defined on X, similar notions associated with sequences can be introduced.

DEFINITION. Let (X, d) be a metric space and $(x_n)_{n \in \mathbb{N}}$ a sequence in X.

a) We say that $(x_n)_{n \in \mathbb{N}}$ is *bounded* if the set $\{x_n\}_{n \in \mathbb{N}}$ is bounded.
b) We say that $(x_n)_{n \in \mathbb{N}}$ is *convergent* if there exists $x \in X$ such that $\mathrm{d}(x_n, x) \underset{n \to \infty}{\longrightarrow} 0$, *i.e.*

$$\forall \varepsilon > 0, \ \exists n_\varepsilon \in \mathbb{N}, \ \forall n \geq n_\varepsilon : \mathrm{d}(x_n, x) < \varepsilon.$$

In this case, we will note $\lim\limits_{n \to \infty} x_n = x$, $x_n \overset{\mathrm{d}}{\to} x$, $x_n \overset{X}{\to} x$ or even $x_n \to x$; the element x will be called the *limit* of $(x_n)_{n \in \mathbb{N}}$.

c) We say that $(x_n)_{n \in \mathbb{N}}$ is *Cauchy* or *fundamental* if

$$\forall \varepsilon > 0, \ \exists n_\varepsilon \in \mathbb{N}, \ \forall m, n \geq n_\varepsilon : \mathrm{d}(x_m, x_n) < \varepsilon$$

or, equivalently,

$$\forall \varepsilon > 0, \ \exists n_\varepsilon \in \mathbb{N}, \ \forall n \geq n_\varepsilon, \ \forall p \in \mathbb{N}^* : \mathrm{d}(x_{n+p}, x_n) < \varepsilon.$$

As in the case of real sequences, one can prove that the limit of a sequence in a metric space is unique.

Proposition 4.1. *Let (X, d) be a metric space and $(x_n)_{n \in \mathbb{N}}$ a convergent sequence in X. Then $(x_n)_{n \in \mathbb{N}}$ is Cauchy.*

The converse of this result is not true, *i.e.* not every Cauchy sequence in an arbitary metric space is convergent. For instance, in the space $X = (0,1)$ endowed with the usual distance ($\mathrm{d}(x,y) := |x - y|$, $x, y \in X$)), the sequence $(1/n)_{n \in \mathbb{N}^*}$ is Cauchy (since it is Cauchy in $\mathbb{R}$), but is not convergent (the usual limit taken in $\mathbb{R}$ is not in $(0,1)$).

The closure points and limit points can be characterized with the help of sequences.

Theorem 4.2. *Let (X, d) be a metric space, $x \in X$ and A a subset of X. Then:*

i) $x \in \overline{A}$ *if and only if there exists a sequence* $(x_n)_{n \in \mathbb{N}} \subseteq A$ *such that* $x_n \overset{\mathrm{d}}{\to} x$;

ii) $x \in A'$ *if and only if there exists a sequence* $(x_n)_{n \in \mathbb{N}} \subseteq A \setminus \{x\}$ *such that* $x_n \overset{\mathrm{d}}{\to} x$.

Let us show that the properties of sequences in Euclidean spaces can be reduced to the properties of the sequences of the coordinates.

Theorem 4.3. *Let $\mathbb{R}^m$, $m \geq 1$ be endowed with the Euclidean metric d_2 and $(\mathbf{x}_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}^m$ with*

$$\mathbf{x}_n = (x_n^1, x_n^2, \ldots, x_n^m),\ \forall n \in \mathbb{N}.$$

- ***i)*** *The sequence $(\mathbf{x}_n)_{n\in\mathbb{N}}$ is bounded if and only if all the sequences $(x_n^i)_{n\in\mathbb{N}}$, $1 \leq i \leq m$, are bounded.*
- ***ii)*** *The sequence $(\mathbf{x}_n)_{n\in\mathbb{N}}$ is convergent if and only if all the sequences $(x_n^i)_{n\in\mathbb{N}}$, $1 \leq i \leq m$, are convergent. In this case, if $\mathbf{x} := \lim\limits_{n\to\infty} \mathbf{x}_n$ and $x^i := \lim\limits_{n\to\infty} x_n^i$, $1 \leq i \leq m$, then $\mathbf{x}_n = (x^1, x^2, \ldots, x^m)$.*
- ***iii)*** *The sequence $(\mathbf{x}_n)_{n\in\mathbb{N}}$ is Cauchy if and only if all the sequences $(x_n^i)_{n\in\mathbb{N}}$, $1 \leq i \leq m$, are Cauchy.*

We can also charcterize the properties of sequences in spaces of functions.

Theorem 4.4. *Let E be a non-empty set and $\mathscr{B}(E)$ be the space of bounded real functions defined on E endowed with the uniform norm. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $\mathscr{B}(E)$.*

- ***i)*** *The sequence $(f_n)_{n\in\mathbb{N}}$ is bounded with respect to the metric $\mathrm{d}_{\sup}$ if and only if it is* uniformly bounded, *i.e.*

$$\exists M > 0, \forall n \in \mathbb{N},\ \forall x \in E : |f(x)| < M.$$

- ***ii)*** *The sequence $(f_n)_{n\in\mathbb{N}}$ is convergent with respect to the metric $\mathrm{d}_{\sup}$ if and only if it is uniformly convergent. Moreover, for a function $f \in \mathscr{B}(E)$, $f_n \overset{\mathrm{d}_{\sup}}{\to} f \Leftrightarrow f_n \overset{\mathrm{u}}{\to} f$.*
- ***iii)*** *The sequence $(f_n)_{n\in\mathbb{N}}$ is Cauchy with respect to the metric $\mathrm{d}_{\sup}$ if and only if it is a uniform Cauchy sequence.*

Definition.

- ***a)*** A metric space (X, d) is called a *complete* metric space if every Cauchy sequence in X is convergent.
- ***b)*** A normed space $(V, \|\cdot\|)$ is called a *Banach space* if V is complete with respect to the induced metric.
- ***c)*** A prehilbertian space $(V, \langle\cdot,\cdot\rangle)$ is called a *Hilbert space* if V is a Banach space with respect to the induced norm.

By Theorem 4.3, $(\mathbb{R}^n, \langle\cdot,\cdot\rangle_2)$ is a Hilbert space. Also, if E is a non-empty set, then $\mathscr{B}(E)$, endowed with the uniform norm, is a Banach space, by results on sequences of functions in Lecture 2.

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