Metric spaces

Lecture 6

Mathematics - 1st year, English

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Distance

If $P(x_P, y_P, z_P)$ and $Q(x_Q, y_Q, z_Q)$ are two points in space, the *distance* between P and Q (or *length* of the segment PQ) is

$$\mathrm{d}(P, Q) = \sqrt{(x_P - x_Q)^2 + (y_P - y_Q)^2 + (z_P - z_Q)^2}.$$

Metric spaces

Definition

Let $X \neq \emptyset$. A function $\mathrm{d} : X \times X \to \mathbb{R}_+$ is called a *distance* or a *metric* on X if:

$(\mathtt{D}_1)$ $\mathrm{d}(x, y) = 0 \Leftrightarrow x = y, \ \forall x, y \in X$;

$(\mathtt{D}_2)$ $\mathrm{d}(x, y) = \mathrm{d}(y, x), \ \forall x, y \in X$ (*symmetry*);

$(\mathtt{D}_3)$ $\mathrm{d}(x, z) \leq \mathrm{d}(x, y) + \mathrm{d}(y, z), \ \forall x, y, z \in X$ (*triangle property*).

In this case, the couple (X, d) is called a *metric space*.

Proposition

Let (X, d) *be a metric space. Then:*

i) $\mathrm{d}(x_0, x_n) \leq \mathrm{d}(x_0, x_1) + \mathrm{d}(x_1, x_2) + \cdots + \mathrm{d}(x_{n-1}, x_n),\ \forall n \in \mathbb{N}^*,\ \forall x_0, x_1, \ldots, x_n \in X$;

ii) $|\mathrm{d}(x, z) - \mathrm{d}(y, z)| \leq \mathrm{d}(x, y),\ \forall x, y, z \in X$;

iii) $|\mathrm{d}(x, y) - \mathrm{d}(x', y')| \leq \mathrm{d}(x, y') + \mathrm{d}(x', y),\ \forall x, y, x', y' \in X$ *(quadrilateral inequality).*

In linear spaces, some distances come from norms.

Definition

Let $(V, \|\cdot\|)$ be a normed space. Then the mapping $\mathrm{d} : V \times V \to \mathbb{R}_+$ defined by

$$\mathrm{d}(x, y) := \|x - y\|,\ x, y \in \mathbb{R}$$

is a metric, called the *metric induced* by the norm $\|\cdot\|$.

Examples

1. On $\mathbb{R}$, the application $\mathrm{d} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}_+$ defined by

$$\mathrm{d}(x, y) := |x - y|, \ x, y \in \mathbb{R}$$

is a distance, called the *canonical distance* in $\mathbb{R}$.

2. On $\mathbb{R}^n$, the metric induced by the Euclidean norm is called the *Euclidean metric* on $\mathbb{R}^n$ and is denoted by d_2. We have

$$\mathrm{d}_2(\mathbf{x}, \mathbf{y}) := \|\mathbf{x} - \mathbf{y}\|_2 = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2},$$

for $\mathbf{x} = (x_1, \ldots, x_n), \ \mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n$.

3. Let, for $p \in [1, +\infty)$, the application $\|\cdot\|_p : \mathbb{R}^n \to \mathbb{R}_+$ be defined by

$$\|\mathbf{x}\|_p := \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}, \ \mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n.$$

Then $\|\cdot\|_p$ is a norm.

Indeed, the triangle property is equivalent to

$$\left(\sum_{i=1}^{n}|x_i+y_i|^p\right)^{1/p} \leq \left(\sum_{i=1}^{n}|x_i|^p\right)^{1/p} + \left(\sum_{i=1}^{n}|y_i|^p\right)^{1/p}$$

which is precisely the Minkowski inequality.
We can also introduce a norm on $\mathbb{R}^n$ even in the case $p=+\infty$, by

$$\|\mathbf{x}\|_\infty := \max_{1\leq i\leq n}|x_i|, \ \mathbf{x}=(x_1,\dots,x_n)\in\mathbb{R}^n.$$

The metric induced on $\mathbb{R}^n$ by the p-norm is called *Minkowski distance* and is denoted by d_p. So we have

$$\mathrm{d}_p(\mathbf{x},\mathbf{y}) := \|\mathbf{x}-\mathbf{y}\|_p = \begin{cases} \left(|x_1-y_1|^p+\cdots+|x_n-y_n|^p\right)^{1/p}, & p\in[1,+\infty); \\ \max\left\{|x_1-y_1|,\dots,|x_n-y_n|\right\}, & p=+\infty, \end{cases}$$

for $\mathbf{x}=(x_1,\dots,x_n),\ \mathbf{y}=(y_1,\dots,y_n)\in\mathbb{R}^n$.

- The metric d_1 is sometimes called the *taxi-cab distance* or *Manhattan distance*.
- The metric d_∞ is also called *Chebyshev distance*.
- If $n=1$: $\mathrm{d}_p(x,y)=|x-y|$, $\forall x,y\in\mathbb{R}$, $\forall p\in[1,+\infty]$.

4. The application $\tilde{\mathrm{d}} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+$, defined by

$$\tilde{\mathrm{d}}(\mathbf{x}, \mathbf{y}) := \sum_{k=1}^{n} \frac{1}{2^k} \cdot \frac{|x_k - y_k|}{1 + |x_k - y_k|},$$

for $\mathbf{x} = (x_1, \dots, x_n)$, $\mathbf{y} = (y_1, \dots, y_n) \in \mathbb{R}^n$ is a distance on $\mathbb{R}^n$, but is not norm-induced, because the function $\mathbf{x} \mapsto \tilde{\mathrm{d}}(\mathbf{x}, \mathbf{0})$ lacks the homogeneity property.

5. Let X be a non-empty set. The function $\mathrm{d} : X \times X \to \mathbb{R}_+$, defined by

$$\mathrm{d}(x, y) := \begin{cases} 0, & x = y; \\ 1, & x \neq y, \end{cases}$$

for $x, y \in X$, is a metric on X, called the *discrete metric* on X.

6. On $\overline{\mathbb{R}}$ we can consider the metric d defined by

$$\mathrm{d}(x, y) := |\operatorname{arctg} x - \operatorname{arctg} y|, \ x, y \in \overline{\mathbb{R}}$$

(we have extended the function arctg on $\overline{\mathbb{R}}$ by $\operatorname{arctg}(-\infty) := -\pi/2$, $\operatorname{arctg}(+\infty) := \pi/2$).

Uniform norm

Definition

Let E be a non-empty set and $\mathscr{B}(E)$ be the space of bounded functions $f : E \to \mathbb{R}$ (*i.e.* $\operatorname{Im} f$ is a bounded set). We set $\|\cdot\|_{\mathsf{sup}} : \mathscr{B}(E) \to \mathbb{R}_+$ defined by

$$\|f\|_{\mathsf{sup}} := \sup_{x \in E} |f(x)| .$$

Then $\|\cdot\|_{\mathsf{sup}}$ is a norm on $\mathscr{B}(E)$, called the *uniform norm* or *sup-norm*. The metric induced by $\|\cdot\|_{\mathsf{sup}}$ is called the *uniform distance*, denoted $\mathrm{d}_{\mathsf{sup}}$.

Definition

- Let X be a non-empty set. We say that two metrics d and d' on X are *equivalent* if there exist two constants $c_1, c_2 > 0$ such that

$$c_1 \mathrm{d}'(x,y) \leq \mathrm{d}(x,y) \leq c_2 \mathrm{d}'(x,y),\ \forall x,y \in X.$$

- Let $(V,+,\cdot)$ be a linear space. We say that two norms $\|\cdot\|$ and $\|\cdot\|'$ are *equivalent* if there exist two constants $c_1, c_2 > 0$ such that

$$c_1 \|\mathbf{x}\|' \leq \|\mathbf{x}\| \leq c_2 \|\mathbf{x}\|',\ \forall \mathbf{x} \in V.$$

Of course, if two norms on V are equivalent, so are the induced metrics.

Theorem

On $\mathbb{R}^n$, all the norms $\|\cdot\|_p$ with $p \in [1,+\infty]$ are equivalent.

In fact, we have, for all $\mathbf{x} \in \mathbb{R}^n$ and $p \in [1,+\infty)$

$$\|\mathbf{x}\|_\infty \leq \|\mathbf{x}\|_p \leq n^{1/p} \|\mathbf{x}\|_\infty .$$

Balls and spheres

Definition

Let (X, d) be a metric space, $x_0 \in X$ an arbitrary point and $r > 0$ a real number.

- The set
$$B(x_0; r) := \{x \in X \mid \mathrm{d}(x, x_0) < r\}$$
is called the *open ball* of *radius* r and *center* x_0.
- The set
$$\bar{B}(x_0; r) := \{x \in X \mid \mathrm{d}(x, x_0) \leq r\}$$
is called the *closed ball* of *radius* r and *center* x_0.
- The set
$$S(x_0; r) := \{x \in X \mid \mathrm{d}(x, x_0) = r\}$$
is called the *sphere* of *radius* r and *center* x_0.

Neighbourhoods

Definition

Let (X, d) be a metric space and $x_0 \in X$ an arbitrary point. We say that a subset V of X is a *neigbourhood* of x_0 if there exists $r > 0$ such that $B(x_0; r) \subseteq V$. We denote the family of all neigbourhoods of x_0 by $\mathscr{V}(x_0)$.

Proposition

Let (X, d) be a metric space, $x \in X$ and V, V', U subsets of X.

i) *If $V \in \mathscr{V}(x)$ and $V \subseteq U$, then $U \in \mathscr{V}(x)$.*

ii) *If $V, V' \in \mathscr{V}(x)$, then $V \cap V' \in \mathscr{V}(x)$.*

iii) *If $V \in \mathscr{V}(x)$, then $x \in V$.*

iv) *If $V \in \mathscr{V}(x)$, then there exists $W \in \mathscr{V}(x)$ such that $V \in \mathscr{V}(y)$, $\forall y \in W$.*

It can be proven that the above four properties fully characterize the family of neigbourhoods $\mathscr{V}(x)$ of all $x \in X$.

Theorem

Let (X, d) *be a metric space,* $x_0 \in X$ *an arbitrary point and* $r > 0$ *a real number. Then*

$$B(x_0; r) \in \mathscr{V}(x), \ \forall x \in B(x_0; r).$$

Definition

Let (X, d) be a metric space and $x_0 \in X$ an arbitrary point. We say that a family $\mathscr{U}(x_0)$ is a *system of neigbourhoods* of x_0 if

- $\mathscr{U}(x_0) \subseteq \mathscr{V}(x_0)$;
- for every $V \in \mathscr{V}(x_0)$ there exists $U \in \mathscr{U}(x_0)$ such that $U \subseteq V$.

It is obvious that $\{B(x_0; r)\}_{r \in \mathbb{R}_+^*}$ is a system of neigbourhoods of x_0. In fact, even $\left\{B(x_0; \frac{1}{n})\right\}_{n \in \mathbb{N}^*}$ is a system of neigbourhoods of x_0.

Open sets

Definition

Let (X, d) be a metric space. A susbset D of X is called an *open set* if it is a neighbourhood for every element of D, *i.e.*

$$\forall x \in D, \ \exists r > 0 : B(x; r) \subseteq D.$$

Examples.

1. For any $x \in X$ and any $r > 0$, the open ball $B(x; r)$ is an open set. In particular, if X is $\mathbb{R}$ and d is the Euclidean distance, then the interval $(x - \varepsilon, x + \varepsilon)$ is an open set for every $x \in \mathbb{R}$ and $\varepsilon > 0$. This implies that every open interval (a, b) with $a, b \in \mathbb{R}$ with $a < b$ is an open set.

2. The interval $(-1, 2]$ is not open in $\mathbb{R}$. Indeed, $(-1, 2]$ is not a neighbourhood of 2, since $(2 - \varepsilon, 2 + \varepsilon) \not\subseteq (-1, 2]$ for every $\varepsilon > 0$.

3. The sets $(0, 1) \times (2, 4)$ and

$$\left\{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 \neq 4\right\}$$

are open in $\mathbb{R}^2$ (endowed with the Euclidean distance).

Topologies

Proposition

Let $X \neq \emptyset$ and two metrics d, d' on X. If d and d' are equivalent, then any open set with respect to the metric d is open with respect to the metric d'.

- The collection of all open sets of a metric space (X, d) is called the *topology* on X *induced* by d.
- The topologies induced by two equivalent metrics are the same.
- Therefore, the topologies induced by d_p with $p \in [1, +\infty]$ are the same topology, called the *usual topology* on $\mathbb{R}^n$.

Theorem

Let (X, d) be a metric space. Then:

i) *an arbitrary union of open sets is an open set;*

ii) *an intersection of two open sets is an open set;*

iii) *every open set can be written as an union of open balls;*

iv) *$\emptyset$ and X are open sets.*

Closed sets

Definition

Let (X, d) be a metric space. A subset F of X is called a *closed set* if $\mathrm{C}_A = X \setminus A$ is an open set.

Proposition

Let (X, d) be a metric space. Then:

i) *an arbitrary intersection of closed sets is a closed set;*

ii) *a union of two closed sets is a closed set;*

iii) *$\emptyset$ and X are closed sets.*

Examples.

1. Every closed interval $[a, b]$ with $a, b \in \mathbb{R}$ with $a \leq b$ is a closed subset of $\mathbb{R}$. In particular, $\{a\}$ is a closed set.

2. There are sets in $\mathbb{R}$ which are neither open nor closed, for instance $(1, 2]$.

3. The sets $[0, 1] \times [2, 4]$ and $\{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 \geq 4\}$ are closed in $\mathbb{R}^2$ (endowed with the Euclidean distance).

Boundedness

Definition

Let (X, d) be a metric space, $x_0 \in X$ and A, B two non-empty subsets of X .

- The *distance* from x_0 to A is defined as

$$\mathrm{d}(x_0, A) := \inf \{\mathrm{d}(x_0, x) \mid x \in A\}.$$

- The *distance* between A and B is defined as

$$\mathrm{d}(A, B) := \inf \{\mathrm{d}(x, y) \mid x \in A, \ y \in B\}.$$

- The *diameter* of A is defined as

$$\rho(A) := \sup \{\mathrm{d}(x, y) \mid x, y \in A\}.$$

- The diameter of A is an element from $[0, +\infty]$.
- If we set by convention $\inf \emptyset = +\infty$ and $\sup \emptyset = 0$, we can allow A and B to be $\emptyset$, so we get
$$\mathrm{d}(x_0, \emptyset) = \mathrm{d}(A, \emptyset) = \mathrm{d}(\emptyset, B) = +\infty \text{ and } \rho(\emptyset) = 0.$$

Proposition

Let (X, d) be a metric space and $A, B \subseteq X$.

i) *If $A \subseteq B$ then $\rho(A) \leq \rho(B)$.*

ii) *$\rho(A) = 0$ if and only if $A = \emptyset$ or $A = \{x\}$ for some $x \in X$.*

Definition

Let (X, d) be a metric space. We say that a subset A of X is *bounded* if $\rho(A) < +\infty$. Otherwise (*i.e.*, $\rho(A) = +\infty$), we say that A is *unbounded*.

Proposition

i) *Let (X, d) be a metric space. A subset A of X is bounded if and only if there exist $x_0 \in X$ and $r > 0$ such that $A \subseteq B(x_0; r)$.*

ii) *Let $(V, \|\cdot\|)$ be a normed space. A subset A of V is bounded (with respect to the induced metric) if and only if there exists $r > 0$ such that*

$$\|x\| < r, \ \forall x \in A.$$

The interior of a set

Definition

Let (X, d) be a metric space and A a subset of X.

i) We say that an element $x \in X$ is an *interior point* of A if there exists $r > 0$ such that $B(x; r) \subseteq A$ (in other words, $A \in \mathscr{V}(x)$).

ii) We call the *interior* of A the set of all interior points of A, denoted $\mathring{A}$ or $\operatorname{int} A$.

Examples. We consider the space $\mathbb{R}$ endowed with the Euclidean metric.
1. If $a, b \in \mathbb{R}$ with $a < b$, then

$$\operatorname{int}(a, b) = \operatorname{int}(a, b] = \operatorname{int}[a, b) = \operatorname{int}[a, b] = (a, b).$$

2. The set $\mathbb{Q}$ of all rational numbers has no interior points ($\mathring{\mathbb{Q}} = \emptyset$), since for every $x \in \mathbb{Q}$ and $\varepsilon > 0$, $(x - \varepsilon, x + \varepsilon) \not\subseteq \mathbb{Q}$, because every non-empty interval contains also irrational numbers.

Theorem

Let (X, d) *be a metric space. Then:*

i) $\mathring{A} \subseteq A, \ \forall A \subseteq X;$

ii) A *is open* $\Leftrightarrow \mathring{A} = A, \ \forall A \subseteq X;$

iii) $A \subseteq B \Rightarrow \mathring{A} \subseteq \mathring{B}, \ \forall A, B \subseteq X;$

iv) $\overset{\circ}{\widehat{A \cap B}} = \mathring{A} \cap \mathring{B}, \ \forall A, B \subseteq X;$

v) $\mathring{A} \cup \mathring{B} \subseteq \overset{\circ}{\widehat{A \cup B}}, \ \forall A, B \subseteq X.$

The exterior of a set

Definition

Let (X, d) be a metric space and A a subset of X.

- We say that an element $x \in X$ is an *exterior point* of A if x is an interior point of $\mathrm{C}_A = X \setminus A$.
- We call the *exterior* of A the set of all exterior points of A, denoted $\operatorname{ext} A$.

Proposition

Let (X, d) be a metric space, $x \in X$ and A a subset of X. Then $x \in \operatorname{ext} A$ if and only if there exists $V \in \mathscr{V}(x)$ such that $V \cap A = \emptyset$.

The closure of a set

Definition

Let (X, d) be a metric space and A a subset of X.

- We say that an element $x \in X$ is a *closure point* of A if for every neighbourhood V of x we have $V \cap A \neq \emptyset$.
- We call the *closure* of A the set of all closure points of A, denoted $\overline{A}$ or $\mathrm{cl}\, A$;
- We say that A is *dense* (in X) if $\overline{A} = X$.

Proposition

Let (X, d) be a metric space, $x \in X$ and A a subset of X. Then $x \in \overline{A}$ if and only if there exists $r > 0$ such that $B(x; r) \cap A \neq \emptyset$.

Examples. We consider the space $\mathbb{R}$ endowed with the Euclidean metric.

1. If $a, b \in \mathbb{R}$ with $a < b$, then

$$\overline{(a, b)} = \overline{(a, b]} = \overline{[a, b)} = \overline{[a, b]} = [a, b].$$

2. We have $\overline{\mathbb{Q}} = \mathbb{R}$, since for every $x \in \mathbb{Q}$ and $\varepsilon > 0$, $(x-\varepsilon, x+\varepsilon) \cap \mathbb{Q} \neq \emptyset$, because every non-empty interval contains rational numbers. According to the previous definition, $\mathbb{Q}$ is dense in $\mathbb{R}$.

Theorem

Let (X, d) be a metric space. Then:

i) $A \subseteq \overline{A}, \ \forall A \subseteq X$;

ii) A *is closed* $\Leftrightarrow \overline{A} = A, \ \forall A \subseteq X$;

iii) $A \subseteq B \Rightarrow \overline{A} \subseteq \overline{B}, \ \forall A, B \subseteq X$;

iv) $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}, \ \forall A, B \subseteq X$;

v) $\overline{A \cup B} = \overline{A} \cup \overline{B}, \ \forall A, B \subseteq X$;

vi) $\overline{\mathrm{C}_A} = \mathrm{C}_{\mathring{A}}, \ \widehat{\mathrm{C}_A}^{\circ} = \mathrm{C}_{\overline{A}}$.

The boundary of a set

Definition

Let (X, d) be a metric space and A a subset of X. We call the *boundary* of A the set

$$\partial A = \operatorname{Fr} A := \overline{A} \setminus \mathring{A}.$$

Examples. Again, let us consider the Euclidean on $\mathbb{R}$.

1. If $a, b \in \mathbb{R}$ with $a < b$, then

$$\operatorname{Fr}(a, b) = \operatorname{Fr}(a, b] = \operatorname{Fr}[a, b) = \operatorname{Fr}[a, b] = \{a, b\}.$$

2. We have $\operatorname{Fr} \mathbb{Q} = \mathbb{R}$.

Limit points

Definition

Let (X, d) be a metric space and A a subset of X.

- We say that an element $x \in X$ is a *limit point* of A if for every neigbourhood V of x we have $V \cap (A \setminus \{x\}) \neq \emptyset$.
- We call the *derived set* of A the set of all limit points of A, denoted A'.

Theorem

Let (X, d) be a metric space. Then:

i) $A' \subseteq \overline{A} = A \cup A', \ \forall A \subseteq X;$

ii) $\overline{A} = A \Leftrightarrow A' \subseteq A, \ \forall A \subseteq X;$

iii) $A \subseteq B \Rightarrow A' \subseteq B', \ \forall A, B \subseteq X;$

iv) $(A \cup B)' = A' \cup B', \ \forall A, B \subseteq X.$

Isolated points

Definition

Let (X, d) be a metric space and A a subset of X.

- An element $x \in A \setminus A'$ is called an *isolated point* of A.
- We call the *discrete part* of A the set of all isolated points of A, *i.e.* $A \setminus A'$.
- We say that A is *discrete* if every element of A is an isolated point, i.e. $A \cap A' = \emptyset$.

Sequences in metric spaces

Let $X \neq \emptyset$. A *sequence* $(x_n)_{n\in\mathbb{N}}$ in X is a function $x : \mathbb{N} \to X$.

Definition

Let (X, d) be a metric space and $(x_n)_{n\in\mathbb{N}}$ a sequence in X.

- We say that $(x_n)_{n\in\mathbb{N}}$ is *bounded* if the set $\{x_n\}_{n\in\mathbb{N}}$ is bounded.
- We say that $(x_n)_{n\in\mathbb{N}}$ is *convergent* if there exists $x \in X$ such that

$$\forall \varepsilon > 0,\ \exists n_\varepsilon \in \mathbb{N},\ \forall n \geq n_\varepsilon : \mathrm{d}(x_n, x) < \varepsilon.$$

  (*i.e.* $\mathrm{d}(x_n, x) \underset{n\to\infty}{\longrightarrow} 0$). In this case, we will note $\lim\limits_{n\to\infty} x_n = x$, $x_n \overset{\mathrm{d}}{\to} x$, $x_n \overset{X}{\to} x$ or even $x_n \to x$; the element x will be called the *limit* of $(x_n)_{n\in\mathbb{N}}$.
- We say that $(x_n)_{n\in\mathbb{N}}$ is *Cauchy* or *fundamental* if

$$\forall \varepsilon > 0,\ \exists n_\varepsilon \in \mathbb{N},\ \forall m, n \geq n_\varepsilon : \mathrm{d}(x_m, x_n) < \varepsilon \text{ or, equivalently,}$$

$$\forall \varepsilon > 0,\ \exists n_\varepsilon \in \mathbb{N},\ \forall n \geq n_\varepsilon,\ \forall p \in \mathbb{N}^* : \mathrm{d}(x_{n+p}, x_n) < \varepsilon.$$

As in the case of real sequences, one can prove that the limit of a sequence in a metric space is unique.

Proposition

Let (X, d) be a metric space and $(x_n)_{n\in\mathbb{N}}$ a convergent sequence in X. Then $(x_n)_{n\in\mathbb{N}}$ is Cauchy.

- The converse of this result is not true, *i.e.* not every Cauchy sequence in an arbitary metric space is convergent.
- For instance, $X = (0,1)$ with the usual distance $(\mathrm{d}(x,y) := |x-y|)$: the sequence $(1/n)_{n\in\mathbb{N}^*}$ is Cauchy, but is not convergent.

Theorem

Let (X, d) be a metric space, $x \in X$ and A a subset of X. Then:

i) *$x \in \overline{A}$ if and only if there exists a sequence $(x_n)_{n\in\mathbb{N}} \subseteq A$ such that $x_n \xrightarrow{\mathrm{d}} x$;*

ii) *$x \in A'$ if and only if there exists a sequence $(x_n)_{n\in\mathbb{N}} \subseteq A \setminus \{x\}$ such that $x_n \xrightarrow{\mathrm{d}} x$.*

Sequences in Euclidean spaces

Theorem

Let $\mathbb{R}^m$, $m \geq 1$ be endowed with the Euclidean metric d_2 and $(\mathbf{x}_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}^m$ with

$$\mathbf{x}_n = (x_n^1, x_n^2, \ldots, x_n^m),\ \forall n \in \mathbb{N}$$

- **i)** *The sequence $(\mathbf{x}_n)_{n\in\mathbb{N}}$ is bounded if and only if all the sequences $(x_n^i)_{n\in\mathbb{N}}$, $1 \leq i \leq m$, are bounded.*
- **ii)** *The sequence $(\mathbf{x}_n)_{n\in\mathbb{N}}$ is convergent if and only if all the sequences $(x_n^i)_{n\in\mathbb{N}}$, $1 \leq i \leq m$, are convergent. In this case, if $\mathbf{x} := \lim\limits_{n\to\infty} \mathbf{x}_n$ and $x^i := \lim\limits_{n\to\infty} x_n^i$, $1 \leq i \leq m$, then $\mathbf{x}_n = (x^1, x^2, \ldots, x^m)$.*
- **iii)** *The sequence $(\mathbf{x}_n)_{n\in\mathbb{N}}$ is Cauchy if and only if all the sequences $(x_n^i)_{n\in\mathbb{N}}$, $1 \leq i \leq m$, are Cauchy.*

Sequences in spaces of functions

Theorem

Let $E \neq \emptyset$. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $\mathscr{B}(E)$.

- **i)** *The sequence $(f_n)_{n\in\mathbb{N}}$ is bounded with respect to the metric $\mathrm{d}_{\mathrm{sup}}$ if and only if it is uniformly bounded, i.e.*

$$\exists M > 0, \forall n \in \mathbb{N},\ \forall x \in E : |f(x)| < M.$$

- **ii)** *The sequence $(f_n)_{n\in\mathbb{N}}$ is convergent with respect to the metric $\mathrm{d}_{\mathrm{sup}}$ if and only if it is uniformly convergent. Moreover, for a function $f \in \mathscr{B}(E)$, $f_n \overset{\mathrm{d}_{\mathrm{sup}}}{\to} f \Leftrightarrow f_n \overset{\mathrm{u}}{\to} f$.*

- **iii)** *The sequence $(f_n)_{n\in\mathbb{N}}$ is Cauchy with respect to the metric $\mathrm{d}_{\mathrm{sup}}$ if and only if it is a uniform Cauchy sequence.*

Complete metric spaces

Definition

- A metric space (X, d) is called a *complete* metric space if every Cauchy sequence in X is convergent.
- A normed space $(V, \|\cdot\|)$ is called a *Banach space* if V is complete with respect to the induced metric.
- A prehilbertian space $(V, \langle\cdot,\cdot\rangle)$ is called a *Hilbert space* if V is a Banach space with respect to the induced norm.

- $(\mathbb{R}^n, \langle\cdot,\cdot\rangle_2)$ is a Hilbert space.
- Also, if E is a non-empty set, then $\mathscr{B}(E)$, endowed with the uniform norm, is a Banach space.

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