LECTURE 13
INTEGRABILITY IN $\mathbb{R}^n$

Multiple integrals are a natural extension of the Riemann integral to the case of functions of several variables. In particular, when the function to be integrated has two variables, we speak about the double integral, while when we deal with three variables, we talk about the triple integral. In this way, we can compute some numeric characteristics of 3D-objects (volume, mass, etc.).

1. Jordan measure

Since the notion of integral (even in $\mathbb{R}$) is strongly related to that of measure of a set (such as length, area or volume), we will start by defining a such a concept in $\mathbb{R}^n$.

Definition.

a) Let $a_1, a_2, \ldots, a_n \in \mathbb{R}$ and $b_1, b_2, \ldots, b_n \in \mathbb{R}$ such that $a_k < b_k$, $\forall k \in \overline{1,n}$. The set

$$I_0 = [a_1, b_1] \times \cdots \times [a_n, b_n] = \left\{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid a_k \le x_k \le b_k,\ \forall k \in \overline{1,n}\right\}$$

is called an *n-dimensional compact interval* (if $n = 2$ or $n = 3$, also called a *rectangle*, respectively a *parallelepiped* with the *edges*, respectively *faces parallel to the coordinates axes*).

Its (*Jordan*) *measure* is the number

$$\mu(I_0) := (b_1^0 - a_1^0)(b_2^0 - a_2^0)\ldots(b_n^0 - a_n^0).$$

(if $n = 2$ or $n = 3$, this is the *area*, respectively, the *volume* of the rectangle, respectively the parallelipiped I_0).

b) We call an *elementary* (*Jordan measurable*) *set* any set in $\mathbb{R}^n$ which can be written as a finite union of compact n-dimensional intervals which have no interior common points, i.e. a set

$$E = \bigcup_{l=1}^{q} I_l$$

such that $I_l = [a_1^l, b_1^l] \times [a_2^l, b_2^l] \times \cdots \times [a_n^l, b_n^l]$, $l = \overline{1,q}$ and such that $\mathring{I}_j \cap \mathring{I}_l = \varnothing$, $\forall j, l \in \{1, 2, \ldots, q\}$, $j \ne l$.

The *Jordan measure* of the set E is defined as

$$\mu(E) := \sum_{l=1}^{q} \mu(I_l),$$

where $\mu(I_l) = \prod_{k=1}^{n}(b_k^l - a_k^l)$.

We will denote by $\mathcal{E}_J^n$ the family of all elementary sets in $\mathbb{R}^n$.

The measure of an elementary set is well-defined because it can be shown that it does not depend (exercice!) on the representation as a finite union of compact intervals with no interior common points (which is not unique).

Definition. Let $A \subseteq \mathbb{R}^n$ be a bounded set.

a) We call the *Jordan interior measure* of the set A the number

$$\mu_*(A) = \sup\left\{\mu(E) \mid E \subseteq A,\ E \in \mathcal{E}_J^n\right\}$$

(if there is no elementary set included in A, $\mu_*(A)$ is then 0).

b) The *Jordan exterior measure* of the set A is the number

$$\mu^*(A) = \inf\left\{\mu(E) \mid E \supseteq A,\ E \in \mathcal{E}_J^n\right\}.$$

c) We say that A is *Jordan measurable* if $\mu_*(A) = \mu^*(A)$. The commun value is called the *Jordan measure* of the set A and is denoted by $\mu_J(A)$ (it is customary to call it *area* if $n = 2$ or *volume* if $n = 3$)

It is obvious that for a bounded set $A \subseteq \mathbb{R}^n$, $\mu_*(A)$ and $\mu^*(A)$ are positive numbers satisfying $\mu_*(A) \le \mu^*(A)$.

Remarks.

1. Any elementary set $E \in \mathcal{E}_J^n$ is Jordan measurable, by the definition.

2. Not every bounded set in $\mathbb{R}^n$ is Jordan measurable. For instance, in $\mathbb{R}^2$ let

$$A_D = \left\{(x, y) \in \mathbb{R}^2 \mid 0 \le x \le 1,\ 0 \le y \le f_D(x)\right\}$$

where $f_D : \mathbb{R} \to \mathbb{R}$ is the Dirichlet function, defined by

$$f_D(x) := \begin{cases} 1, & x \in \mathbb{Q}; \\ 0, & x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Then $\mu_*(A_D) = 0$, since there is no elementary set $E \subseteq A_D$; on the other hand, $\mu^*(A_D) = 1$, since every elementary set $E \supseteq A$ has to include the rectangle $[0,1] \times [0,1]$. Therefore, E is not Jordan measurable.

3. There are non-elementary sets which are Jordan measurable. For instance, the subgraph of a Riemann integrable function $f : [a,b] \to \mathbb{R}_+$, *i.e.* the set

$$\Gamma_f = \left\{(x,y) \in \mathbb{R}^2 \mid a \le x \le b,\ 0 \le y \le f(x)\right\},$$

is Jordan measurable, with $\mu_J(\Gamma_f) = \text{area}(\Gamma_f) = \int\limits_a^b f(x)dx$.

Indeed, if $f \in \mathcal{R}[a,b]$, then $f \in \mathcal{B}[a,b]$. For any partition $\Delta = \{a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b\}$ of the interval $[a,b]$, let the real numbers $m_i := \inf\limits_{x\in[x_{i-1},x_i]} f(x)$ and $M_i = \sup\limits_{x\in[x_{i-1},x_i]} f(x)$, for any $i = \overline{1,n}$.

If we set $E'_\Delta := \bigcup_{i=1}^n [x_{i-1}, x_i] \times [0, m_i]$, w have: $E'_\Delta \in \mathcal{E}^2_J$, $E'_\Delta \subseteq \Gamma_f$ and $\mu(E'_\Delta) = \sum\limits_{i=1}^n m_i(x_i - x_{i-1}) = s_f(\Delta)$ (the Darboux lower sum corresponding to f and Δ). Consequently, it follows that $s_f(\Delta) \le \mu_*(\Gamma_f)$.

Similarly, by setting $E''_\Delta = \bigcup\limits_{i=1}^n [x_{i-1}, x_i] \times [0, M_i]$, we remark that $E''_\Delta \in \mathcal{E}^2_J$, $E''_\Delta \supseteq \Gamma_f$ and $\mu^*(\Gamma_f) \le \mu(E''_\Delta) = \sum\limits_{i=1}^n M_i(x_i - x_{i-1}) = \mathcal{S}_f(\Delta)$ (the Darboux upper sum corresponding to f and Δ).

Therefore,

$$s_f(\Delta) \le \mu_*(\Gamma_f) \le \mu^*(\Gamma_f) \le S_f(\Delta),\ \forall \Delta \in \mathcal{D}[a,b].$$

On the other hand, since f is integrable on $[a,b]$, we have $\underline{I} = \sup_\Delta s_f(\Delta) = \inf_\Delta S_f(\Delta) = \overline{I} = \int_a^b f(x)dx$. Combining the two relations, we get $\mu_*(\Gamma_f) = \mu^*(\Gamma_f) = \int_a^b f(x)dx$, i.e. Γ_f is Jordan measurable and its area is equal to $\int\limits_a^b f(x)\,dx$.

4. More generally, we infer that if $f, g : [a,b] \to \mathbb{R}$ are two Rieman integrable functions on $[a,b]$ such that $f(x) \le g(x)$, $\forall x \in [a,b]$, then the set $\Gamma_{f,g} = \left\{(x,y) \in \mathbb{R}^2 \mid a \le x \le b,\ f(x) \le y \le g(x)\right\}$ is Jordan measurable with

$$\mu_J\left(\Gamma_{f,g}\right) = \int_a^b \left(g(x) - f(x)\right) dx.$$

As an application, we will calculate the area of an ellipse. Let $\tilde{a}, \tilde{b} > 0$ and set $a := -\tilde{a}$, $b := \tilde{a}$. We define the functions $f, g : [a,b] \to \mathbb{R}$ by

$f(x) := -\frac{\tilde{b}}{\tilde{a}}\sqrt{\tilde{a}^2 - x^2}$ şi $g(x) := \frac{\tilde{b}}{\tilde{a}}\sqrt{\tilde{a}^2 - x^2}$, $x \in [a,b] = [-\tilde{a}, \tilde{a}]$. The union of their graphs give the boundary of the ellipse of equation $\frac{x^2}{\tilde{a}^2} + \frac{y^2}{\tilde{b}^2} - 1 = 0$; therefore, the domain bounded by this ellipse is given by

$$\Gamma_{f,g} = \left\{(x,y) \in \mathbb{R}^2 \mid -\tilde{a} \le x \le \tilde{a},\ -\frac{\tilde{b}}{\tilde{a}}\sqrt{\tilde{a}^2 - x^2} \le y \le \frac{\tilde{b}}{\tilde{a}}\sqrt{\tilde{a}^2 - x^2}\right\},$$

By computations, we find $\mu_J(\Gamma_{f,g}) = \dfrac{\tilde{b}}{\tilde{a}}\displaystyle\int_{-\tilde{a}}^{\tilde{a}} \sqrt{\tilde{a}^2 - x^2}dx = \pi\tilde{a}\tilde{b}$. Summarizing, the area of the ellipse with the semiaxes $\tilde{a}$ and $\tilde{b}$ is $\pi\tilde{a}\tilde{b}$.

5. By the definition, a set $B \subseteq \mathbb{R}^n$ is measurable and has null Jordan measure if for every $\varepsilon > 0$, there exists $E_\varepsilon \in \mathcal{E}^n_J$ such that $B \subseteq E_\varepsilon$ şi $\mu_J(E_\varepsilon) < \varepsilon$.

Some necessary and sufficient conditions for Jordan measurability of a set in $\mathbb{R}^n$ are listed in the following result:

Theorem 1.1. *Let $A \subseteq \mathbb{R}^n$ be a bounded set. Then the following statements are equivalent:*

(i) *A is Jordan measurable;*
(ii) *$\forall \varepsilon > 0$, $\exists E'_\varepsilon, E''_\varepsilon \in \mathcal{E}^n_J : E'_\varepsilon \subseteq A \subseteq E''_\varepsilon$ and $\mu_J(E'_\varepsilon) - \mu_J(E''_\varepsilon) < \varepsilon$;*
(iii) *$\partial(A)$ is Jordan measurable and $\mu_J(\partial(A)) = 0$;*
(iv) *there exist sequences $\left(\tilde{E}_m\right)_{m\in\mathbb{N}^*} \subseteq \mathcal{E}^n_J$ and $\left(\hat{E}_m\right)_{m\in\mathbb{N}^*} \subseteq \mathcal{E}^n_J$ such that $\tilde{E}_m \subseteq A \subseteq \hat{E}_m$, $\forall\, m \in \mathbb{N}^*$ and $\lim\limits_{m\to\infty} \mu_J(\tilde{E}_m) = \lim\limits_{m\to\infty} \mu_J(\hat{E}_m)$.*

Remark. For a Jordan measurable set A, $\mu_J(A) \neq 0$ is equivalent to $\mathring{A} \neq \varnothing$.

Let us denote by $\mathcal{M}_J^n$ the family of all subsets of $\mathbb{R}^n$ which are Jordan measurable.

Theorem 1.2 (Properties of the Jordan measure).

i) $\mu_J(A) \geq 0$, $\forall A \in \mathcal{M}_J^n$ (non-negativity).

ii) $\mu_J(A \cup B) = \mu_J(A) + \mu_J(B)$, $\forall A, B \in \mathcal{M}_J^n$ *with* $\mathring{A} \cap \mathring{B} = \varnothing$ (finite additivity).

iii) $\forall A, B \in \mathcal{M}_J^n : B \subseteq A \Rightarrow A \setminus B \in \mathcal{M}_J^n$ *and* $\mu_J(A \setminus B) = \mu_J(A) - \mu_J(B)$ (subtraction).

iv) $\forall A, B \in \mathcal{M}_J^n : B \subseteq A \Rightarrow \mu_J(B) \leq \mu_J(A)$ (monotonicity).

v) $\forall A \in \mathcal{M}_J^n$, $\forall B \subseteq \mathbb{R}^n : \mu_J(A) = 0, B \subseteq A \Longrightarrow B \in \mathcal{M}_J^n$ *and* $\mu_J(B) = 0$ (completion).

Proof.

i) This property is obvious, because for every set $A \subseteq \mathbb{R}^n$, $\mu_*(A) \geq 0$.

ii) Since $A \in \mathcal{M}_J^n$, for every $\varepsilon > 0$, there exist $E'_\varepsilon, E''_\varepsilon \in \mathcal{E}_J^n$ such that $E'_\varepsilon \subset A \subset E''_\varepsilon$ and $\mu_J(E''_\varepsilon) - \mu_J(E'_\varepsilon) < \frac{\varepsilon}{2}$. Also, $B \in \mathcal{M}_J^n$ implies that for every $\varepsilon > 0$, there exist $F'_\varepsilon, F''_\varepsilon \in \mathcal{E}_J^n$ such that $F'_\varepsilon \subset B \subset F''_\varepsilon$ and $\mu_J(F''_\varepsilon) - \mu_J(F'_\varepsilon) < \frac{\varepsilon}{2}$. Then

$$\mu_J(E'_\varepsilon) + \mu_J(F'_\varepsilon) \leq \mu_J(A) + \mu_J(B) \leq \mu_J(E''_\varepsilon) + \mu_J(F''_\varepsilon),$$

$E'_\varepsilon \cup F'_\varepsilon, E''_\varepsilon \cup F''_\varepsilon \in \mathcal{M}_J^n$ (exercice!) and

$$\mu_J(E'_\varepsilon \cup F'_\varepsilon) \leq \mu_*(A \cup B) \leq \mu^*(A \cup B) \leq \mu_J(E''_\varepsilon \cup F''_\varepsilon).$$

But $\mathring{A} \cap \mathring{B} = \varnothing$ implies $\mathring{E}'_\varepsilon \cap \mathring{F}'_\varepsilon = \varnothing$, so $\mu_J(E'_\varepsilon \cup F'_\varepsilon) = \mu_J(E'_\varepsilon) + \mu_J(F'_\varepsilon)$ (exercice!). On the other hand, $\mu_J(E''_\varepsilon \cup F''_\varepsilon) \leq \mu_J(E''_\varepsilon) + \mu_J(F''_\varepsilon)$ (exercice!). Combining the two relations, we get

$$\mu^*(A \cup B) - \varepsilon \leq \mu_J(A) + \mu_J(B) \leq \mu_*(A \cup B) + \varepsilon.$$

Letting $\varepsilon \searrow 0$, this yields

$$\mu^*(A \cup B) = \mu_*(A \cup B) = \mu_J(A) + \mu_J(B),$$

i.e. $A \cup B$ is Jordan measurable and $\mu_J(A \cup B) = \mu_J(A) + \mu_J(B)$.

iii) Again, for every $\varepsilon > 0$, we can find sets $E'_\varepsilon, E''_\varepsilon, F'_\varepsilon, F''_\varepsilon \in \mathcal{E}_J^n$ such that $E'_\varepsilon \subset A \subset E''_\varepsilon$, $F'_\varepsilon \subset B \subset F''_\varepsilon$, $\mu_J(E''_\varepsilon) - \mu_J(E'_\varepsilon) < \frac{\varepsilon}{2}$ and $\mu_J(F''_\varepsilon) - \mu_J(F'_\varepsilon) < \frac{\varepsilon}{2}$. Then $E'_\varepsilon \setminus F''_\varepsilon, E''_\varepsilon \setminus F'_\varepsilon \in \mathcal{M}_J^n$ (exercice!) and

$$\mu_J(E'_\varepsilon) - \mu_J(F''_\varepsilon) \leq \mu_J(E'_\varepsilon \setminus F''_\varepsilon) \leq \mu_*(A \setminus B) \leq \mu^*(A \setminus B) \leq \mu_J(E''_\varepsilon \setminus F'_\varepsilon) = \mu_J(E''_\varepsilon) - \mu_J(F'_\varepsilon).$$

Therefore, by a similar argument as for the previous point,

$$\mu^*(A \setminus B) = \mu_*(A \setminus B) = \mu_J(A) - \mu_J(B),$$

which shows that $A \setminus B$ is Jordan measurable and $\mu_J(A \setminus B) = \mu_J(A) - \mu_J(B)$.

iv) It follows from the first and third properties.

v) Since $\mu_J(A) = 0$, for every $\varepsilon > 0$, we can find $E_\varepsilon \in \mathcal{E}_J^n$ such that $A \subseteq E_\varepsilon$ and $\mu(E_\varepsilon) < \varepsilon$. Therefore, $B \subseteq E_\varepsilon$, $\forall \varepsilon > 0$; together with the inequality $\mu(E_\varepsilon) < \varepsilon$, this implies that $B \in \mathcal{M}_J^n$ and $\mu_J(B) = 0$.

□

Remarks.

1. From the proof of this theorem, we can also infer that if $A, B \in \mathcal{M}_J^n$, then $A \cup B \in \mathcal{M}_J^n$ and $A \setminus B \in \mathcal{M}_J^n$. Moreover, the subadditivity property holds: $\mu_J(A \cup B) \leq \mu_J(A) + \mu_J(B)$.

2. The graph of a continuous function $f : [a, b] \longrightarrow \mathbb{R}_+$ has null area (*i.e.*, has Jordan measure 0). Indeed, since $f \in \mathcal{C}[a, b] \subset \mathcal{R}[a, b]$ and $\Gamma_f = \{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq b,\ 0 \leq y \leq f(x)\}$ is Jordan measurable. Hence, by the equivalence between the first and the third statements of Theorem 1.1, the set $\partial(A)$ has null area. Since $G_f \subseteq \partial(\Gamma_f)$, G_f is also Jordan measurable and has null area.

3. Every set in $\mathbb{R}^2$ whose boundary can be written as a finite union of graphs of continuous functions on compact intervals is Jordan measurable. In particular, any disc in $\mathbb{R}^2$ is Jordan measurable.

2. Riemann multiple integral on compact sets

Let $D \subseteq \mathbb{R}^n$ be a non-empty compact (equivalently, closed and bounded) set such that $D \in \mathcal{M}_J^n$. We will also consider a function $f : D \to \mathbb{R}$ to be integrated.

Definition.

a) We call *partition* of D any finite family $\{D_i\}_{1 \leq i \leq p}$ of subsets of D such that:

i) $D_i \in \mathcal{M}_J^n$, $\forall i \in \overline{1, p}$;

ii) $\mathring{D}_i \cap \mathring{D}_j = \varnothing$, $\forall i, j \in \{1, \ldots, p\}$ with $i \neq j$;

iii) $D = \bigcup_{i=1}^p D_i$.

We denote $\mathcal{D}(D)$ the family of all partitions of D.

b) For a partition Δ we define its *norm* $\|\Delta\|:=\max\limits_{1\le i\le p}\{\operatorname{diam}(D_i)\}$, where $\operatorname{diam}(D_i)$ means the diameter of D_i.

Remark. By the additivity property of the Jordan measure we have $\mu_J(D)=\sum\limits_{i=1}^{p}\mu_J(D_i)$.

Definition. Let $\Delta=\{D_i\}_{1\le i\le p}$ be a partition of D

a) An p-tuple $\xi_\Delta=(\xi^1,\xi^2,\dots,\xi^p)\in(\mathbb{R}^n)^p$ is called an *intermediary point system* of Δ if $\xi^i\in D_i$, $\forall i=\overline{1,n}$. The set of all intermediary point systems of Δ is denoted Ξ_Δ.

b) We call the *Riemann sum* of the function $f:D\to\mathbb{R}$ with respect to Δ and an intermediary point system $\xi_\Delta=(\xi^1,\xi^2,\dots,\xi^n)$ the number

$$\sigma_f(\Delta,\xi_\Delta)=\sum_{i=1}^{n}f(\xi^i)\mu_J(D).$$

Definition. We say that the function $f:D\to\mathbb{R}$ is *Riemann integrable* if there exists $I\in\mathbb{R}$ such that for every $\varepsilon>0$, there exists $\delta_\varepsilon>0$ such that for every partition $\Delta=\{D_i\}_{1\le i\le p}$ of D with $\|\Delta\|<\delta_\varepsilon$ and every intermediary point system $\xi_\Delta=(\xi^1,\xi^2,\dots,\xi^p)$ of Δ we have

$$|\sigma_f(\Delta,\xi_\Delta)-I)|<\varepsilon.$$

The number I is called the *multiple integral* (if $n=2$ or $n=3$, the *double*, respectively *triple integral*) and is denoted by

$$\int\!\cdots\!\int_D f(x_1,x_2,\dots,x_n)dx_1dx_2\dots dx_n.$$

As in the one-dimensional case, it can be shown that a Riemann integrable function on a compact set is bounded. We can also define the *lower* and *upper Darboux sums* of a function $f:D\to\mathbb{R}$ as

$$s_f(\Delta):=\sum_{i=1}^{p}m_i\mu_J(D_i);$$

$$S_f(\Delta):=\sum_{i=1}^{p}M_i\mu_J(D_i),$$

where $\Delta=\{D_i\}_{1\le i\le p}$ is a partition of D and $m_i:=\inf\limits_{x\in D_i}f(x)$, $M_i:=\sup\limits_{x\in D_i}f(x)$, $i=\overline{1,p}$.

It is easy to see that the following relation holds:

$$m\cdot\mu_J(D)\le s_f(\Delta)\le S_f(\Delta)\le M\cdot\mu_J(D),$$

where Δ is an arbitrary partition of D and $m:=\inf\limits_{x\in D}f(x)$, $M:=\sup\limits_{x\in D}f(x)$.

If we denote $\underline{I}:=\sup\limits_{\Delta\in\mathcal{D}(D)}s_f(\Delta)$ and $\overline{I}:=\inf\limits_{\Delta\in\mathcal{D}(D)}s_f(\Delta)$, the *lower*, respectively the *upper Darboux integral* of f, we infer

$$m\cdot\mu_J(D)\le\underline{I}\le\overline{I}\le M\cdot\mu_J(D).$$

As in the case $n=1$, we can show the following result:

Proposition 2.1. *Let $D\subseteq\mathbb{R}^n$ be a non-empty compact set which is Jordan measurable and $f:D\to\mathbb{R}$ a bounded function. Then f is Riemann integrable if and only $\underline{I}=\overline{I}$, condition which is equivalent to*

$$\forall\varepsilon>0,\ \exists\delta_\varepsilon>0,\ \forall\Delta\in\mathcal{D}(D):\|\Delta\|<\varepsilon\Rightarrow S_f(\Delta_\varepsilon)-s_f(\Delta_\varepsilon)<\varepsilon.$$

In this case, $\underline{I}=\overline{I}=\int\!\cdots\!\int_D f(x_1,x_2,\dots,x_n)dx_1dx_2\dots dx_n.$

We can now prove the following result:

Theorem 2.2. *Let $D\subseteq\mathbb{R}^n$ be a non-empty compact set which is Jordan measurable and $f:D\to\mathbb{R}$ a continuous function. Then f is Riemann integrable.*

Proof. Since D is compact, f is uniformly continuous, hence for any $\varepsilon>0$, there exists $\delta_\varepsilon>0$ such that for every $x',x''\in D$ with $\|x'-x''\|<\delta_\varepsilon$, we have $|f(x')-f(x'')|<\dfrac{\varepsilon}{\mu_J(D)}$.

Let $\Delta=\{D_i\}_{1\le i\le p}$ be an arbitrary partition of D with $\|\Delta\|<\delta(\varepsilon)$. Then

$$S_f(\Delta)-s_f(\Delta)=\sum_{i=1}^{p}(M_i-m_i)\mu_J(D_i),$$

where $m_i := \inf\limits_{x\in D_i} f(x)$, $M_i := \sup\limits_{x\in D_i} f(x)$, $i=\overline{1,p}$. By the continuity of f pe D, since D_i are compact subsets of D, it follows that there exist $\xi^i, \eta^i \in D_i$ such that $m_i = f(\xi^i)$ and $M_i = f(\eta^i)$. Therefore,

$$S_f(\Delta) - s_f(\Delta) = \sum_{i=1}^{p} (f(\eta^i) - f(\xi^i))\mu_J(D_i) < \frac{\varepsilon}{\mu_J(D)} \sum_{i=1}^{p} \mu_J(D_i) = \frac{\varepsilon}{\mu_J(D)} \mu_J(D) = \varepsilon.$$

By the previous proposition, f is Riemann integrable. □

A generalization of the above result is the following:

Theorem 2.3. *Let $D \subseteq \mathbb{R}^n$ be a non-empty compact set which is Jordan measurable and $f : D \to \mathbb{R}$ a function which is continuous in every element of D with the exception of a Jordan measurable set having null measure. Then f is Riemann integrable.*

The properties of Riemann integrable functions are similar to the case $n = 1$:

Proposition 2.4. *Let $D \subseteq \mathbb{R}^n$ be a non-empty compact set which is Jordan measurable. Then:*

i) $\int \cdots \int_D 1 \cdot dx_1 dx_2 \dots dx_n = \mu_J(D)$;

ii) for every Riemann integrable functions $f, g : D \to \mathbb{R}$ and $\alpha, \beta \in \mathbb{R}$, $\alpha f + \beta g$ is Riemann integrable and

$$\int \cdots \int_D (\alpha f(x_1, \dots, x_n) + \beta g(x_1, \dots, x_n)) dx_1 \dots dx_n = \alpha \int \cdots \int_D f(x_1, \dots, x_n) dx_1 \dots dx_n + \beta \int \cdots \int_D g(x_1, \dots, x_n) dx_1 \dots dx_n;$$

iii) for every Riemann integrable functions $f, g : D \to \mathbb{R}$ with $f(x) \le g(x)$, $\forall x \in D$, we have:

$$\int \cdots \int_D f(x_1, \dots, x_n) dx_1 \dots dx_n \le \int \cdots \int_D g(x_1, \dots, x_n) dx_1 \dots dx_n;$$

iv) for every Riemann integrable function $f : D \to \mathbb{R}$, $|f|$ is also Riemann integrable and

$$\left| \int \cdots \int_D f(x_1, \dots, x_n) dx_1 \dots dx_n \right| \le \int \cdots \int_D \left| f(x_1, \dots, x_n) \right| dx_1 \dots dx_n;$$

v) for every Riemann integrable function $f : D \to \mathbb{R}$, there exists $\lambda \in \left[\inf\limits_{x\in D} f(x), \sup\limits_{x\in D} f(x) \right]$ such that:

$$\int \cdots \int_D f(x_1, \dots, x_n) dx_1 \dots dx_n = \lambda \mu_J(D).$$

If, moreover, $f \in C(D)$ and D is connected (i.e., it cannot be divided into two disjoint nonempty closed sets), then there exists $\xi \in D$ such that

$$\int \cdots \int_D f(x_1, \dots, x_n) dx_1 \dots dx_n = f(\xi) \mu_J(D);$$

vi) if D is the union of two non-empty compact, Jordan-measurable sets D_1 and D_2, with $\mathring{D}_1 \cap \mathring{D}_2 = \emptyset$, and f is Riemann integrable on both D_1 and D_2, then f is Riemann integrable on D and

$$\int \cdots \int_D f(x_1, \dots, x_n) dx_1 \dots dx_n = \int \cdots \int_{D_1} f(x_1, \dots, x_n) dx_1 \dots dx_n + \int \cdots \int_{D_2} f(x_1, \dots, x_n) dx_1 \dots dx_n;$$

vii) for every $f, g \in C(D)$ with $g(x) \ge 0$, $\forall x \in D$, there exists $\eta \in D$ such that

$$\int \cdots \int_D f(x_1, \dots, x_n) g(x_1, \dots, x_n) dx_1 \dots dx_n = f(\eta) \int \cdots \int_D g(x_1, \dots, x_n) dx_1 \dots dx_n.$$

2.1. **The double integral on compact sets.**

As already mentionned, in the particular case $n = 2$, the multiple integral is called *double integral.* If $f : D \to \mathbb{R}$ is a Riemann integrable function on a non-empty compact, Jordan measurable set $D \subseteq \mathbb{R}^2$, we will denote its double integral $\iint_D f(x, y) dx\, dy$. In the sequel we will present a few ways to compute it.

Proposition 2.5 (rectangle case). *Let $a, b, c, d \in \mathbb{R}$ with $a < b$, $c < d$, $D := [a, b] \times [c, d]$ and $f : D \to \mathbb{R}$ a Riemann integrable function. If, for every $x \in [a, b]$, $f(x, \cdot)$ is Riemann integrable and the function $x \mapsto \int_c^d f(x, y) dy$ is also Riemann integrable on $[a, b]$, then*

$$\iint_{[a,b]\times[c,d]} f(x, y) dxdy = \int_a^b \Big(\int_c^d f(x, y) dy \Big) dx.$$

Moreover, if $f(x,y) = f_1(x)f_2(y)$, $\forall (x,y) \in [a,b]\times[c,d]$ and $f_1 \in \mathcal{R}[a,b]$, $f_2 \in \mathcal{R}[c,d]$, then we have

$$\iint_{[a,b]\times[c,d]} f_1(x)f_2(y)dx\,dy = \int_a^b f_1(x)dx \cdot \int_c^d f_2(y)dy.$$

Remarks.

1. We get a similar result by inversing the roles of x and y, obtaining, as a conclusion, the equality

$$\iint_{[a,b]\times[c,d]} f(x,y)dxdy = \int_c^d \Big(\int_a^b f(x,y)dx\Big)dy.$$

2. A sufficient condition for the conditions of the above result to be fulfilled is $f \in C([a,b]\times[c,d])$.

Definition.

a) A subset $D \subseteq \mathbb{R}^2$ is called *simple with respect to the axis Oy* if there exist continuous functions $\varphi, \psi : [a,b] \to \mathbb{R}$ with $\varphi(x) < \psi(x)$, $\forall x \in [a,b]$, such that

$$D = \{(x,y) \in \mathbb{R}^2 \mid a \le x \le b,\ \varphi(x) \le y \le \psi(x)\}.$$

b) A subset $D \subseteq \mathbb{R}^2$ is called *simple with respect to the axis Ox* if there exist continuous functions $\gamma, \omega : [c,d] \to \mathbb{R}$ with $\gamma(y) < \omega(y)$, $\forall y \in [c,d]$, such that

$$D = \{(x,y) \in \mathbb{R}^2 \mid \gamma(y) \le x \le \omega(y),\ c \le y \le d\}.$$

Theorem 2.6. *Let $D \subseteq \mathbb{R}^2$ be a simple domain with respect to the axis Oy and $f \in C(D)$. Then*

$$\iint_D f(x,y)dxdy = \int_a^b \Big(\int_{\varphi(x)}^{\psi(x)} f(x,y)dy\Big)dx,$$

where the functions $\varphi, \psi : [a,b] \to \mathbb{R}$ with $\varphi(x) < \psi(x)$ are such that $D = \{(x,y) \in \mathbb{R}^2 \mid a \le x \le b,\ \varphi(x) \le y \le \psi(x)\}$.

Remark.

1. If $f \in C(D)$, with D being simple with respect to the axis Ox, *i.e.*

$$D = \{(x,y) \in \mathbb{R}^2 \mid \gamma(y) \le x \le \omega(y),\ c \le y \le d\},$$

then the corresponding formula is the following

$$\iint_D f(x,y)dxdy = \int_c^d \left(\int_{\gamma(y)}^{\omega(y)} f(x,y)dx\right)dy.$$

Example. Let $D = \{(x,y) \in \mathbb{R}_+^2 | 1 \le xy \le 3,\ 1 \le \frac{y}{x} \le 4\}$. We will compute the area of D.

By applying *i)* of Proposition 2.4, we have

$$\text{area}(D) = \mu_J(D) = \iint_D dx\,dy.$$

Since $D = D_1 \cup D_2 \cup D_3$, with $\overset{\circ}{D}_i \cap \overset{\circ}{D}_j = \varnothing$, $\forall i,j \in \{1,2,3\}$, $i \neq j$, where $D_1 = \{(x,y) \in \mathbb{R}^2 \mid \gamma_1(y) = \frac{1}{y} \le x \le \omega_1(y) = y, 1 \le y \le \sqrt{3}\}$, $D_2 = \{(x,y) \in \mathbb{R}^2 \mid \gamma_2(y) = \frac{1}{y} \le x \le \omega_2(y) = \frac{3}{y}, \sqrt{3} \le y \le 2\}$ and $D_3 = \{(x,y) \in \mathbb{R}^2 \mid \gamma_3(y) = \frac{y}{4} \le x \le \omega_3(y) = \frac{3}{y}, 2 \le y \le 2\sqrt{3}\}$, we get, since D_1, D_2, D_3 are simple domains with respect to the axis Ox:

$$\begin{aligned}
\text{area}(D) &= \iint_D dx\,dy = \iint_{D_1} dx\,dy + \iint_{D_2} dx\,dy + \iint_{D_3} dx\,dy = \\
&= \int_1^{\sqrt{3}} \left(\int_{1/y}^{y} dx\right) dy + \int_{\sqrt{3}}^{2} \left(\int_{1/y}^{3/y} dx\right) dy + \int_2^{2\sqrt{3}} \left(\int_{y/4}^{3/y} dx\right) dy = \\
&= \int_1^{\sqrt{3}} \left(y - \frac{1}{y}\right) dy + \int_{\sqrt{3}}^{2} \frac{2}{y} dy + \int_2^{2\sqrt{3}} (\frac{3}{y} - \frac{y}{4}) dy = \\
&= \left(\frac{y^2}{2} - \ln y\right)\Bigg|_1^{\sqrt{3}} + 2\ln y\Bigg|_{\sqrt{3}}^{2} + \left(3\ln y - \frac{y^2}{8}\right)\Bigg|_2^{2\sqrt{3}} = \\
&= \frac{3}{2} - \frac{1}{2}\ln 3 - \frac{1}{2} + 2\ln 2 - \ln 3 + 3\ln 2 + \frac{3}{2}\ln 3 - \frac{3}{2} - 3\ln 2 + \frac{1}{2} = 2\ln 2.
\end{aligned}$$

In some conditions, a double integral on a compact, Jordan measurable can be computed by an appropriate change of variables, the goal being mainly the transformation of the domain and/or the integrand such that it would ease the calculations.

Definition. Let Ω be a compact, Jordan measurable set in $\mathbb{R}^2$ and $F : \Omega \to D \subseteq \mathbb{R}^2$, defined by $F(u,v) = (x(u,v), y(u,v))$, $(u,v) \in \Omega$ a bijective function which can be extended to a C^1-function on an open set $\Omega' \supseteq \Omega$ such that

$$\det(J_F)(u,v) = \frac{D(x,y)}{D(u,v)}(u,v) \neq 0, \forall (u,v) \in \Omega$$

(recall that J_F is the Jacobian matrix of F, while its determinant, $\frac{D(x,y)}{D(u,v)}$ is its Jacobian).Then D is also a compact, Jordan measurable set and F is called is a change of variables (coordinates) from Ω to D.

The next result states how can an integral over D can be transformed into one over Ω by a change of variables.

Proposition 2.7. *Let $F : \Omega \to D$, $F(u,v) = (x(u,v), y(u,v))$, $(u,v) \in \Omega$ be a change of variables and $f : D \to \mathbb{R}$ a continuous function. Then*

$$\iint_D f(x,y)dx\,dy = \iint_\Omega f(x(u,v), y(u,v)) \left| \frac{D(x,y)}{D(u,v)} \right| (u,v)dudv.$$

Remarks.
1. We could apply the change of variables method for the example above. Let us set $xy = u$ and $\frac{y}{x} = v$, equivalently $x = \sqrt{\frac{u}{v}}$ and $y = \sqrt{uv}$, with $u \in [1,3]$ and $v \in [1,4]$. Then we obtain

$$\text{area}(D) = \iint_D dxdy = \iint_\Omega \left| \frac{D(x,y)}{D(u,v)} \right| (u,v)dudv,$$

where $\Omega = \{(u,v) \in \mathbb{R}^2 \mid 1 \leq u \leq 3, 1 \leq v \leq 4\} = [1,3] \times [1,4]$ and

$$\frac{D(x,y)}{D(u,v)}(u,v) = \det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} (u,v) = \det \begin{bmatrix} \frac{1}{2\sqrt{uv}} & -\frac{\sqrt{u}}{2v\sqrt{v}} \\ \frac{\sqrt{v}}{2\sqrt{u}} & \frac{\sqrt{u}}{2\sqrt{v}} \end{bmatrix} = \frac{1}{2v}.$$

Therefore

$$\text{area}(D) = \int_1^3 du \cdot \int_1^4 \left| \frac{1}{2v} \right| dv = \left(u\Big|_1^3 \right) \left(\frac{1}{2} \ln v \Big|_1^4 \right) = 2\frac{1}{2} \ln 4 = 2 \ln 2,$$

which confirms the value founded above.

2. A common change of variables is given by the transition from the cartesian coordinates (x,y) to *polar coordinates* (r,θ), by the relations

$$\begin{cases} x = r\cos\theta; \\ y = r\sin\theta, \end{cases} \quad \text{with } r \in [r_1, r_2] \subseteq [0,\infty),\ \theta \in [\theta_1, \theta_2] \subseteq [0, 2\pi].$$

The Jacobian of the transformation is $\frac{D(x,y)}{D(r,\theta)}(r,\theta) = \det \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix} = r(\sin^2\theta + \cos^2\theta) = r.$

3. Sometimes we can use the generalized polar coordinates:

$$\begin{cases} x = ar\cos^\alpha\theta; \\ y = br\sin^\alpha\theta, \end{cases}$$

with $r \in [r_1, r_2] \subseteq [0,\infty)$ and $\theta \in [\theta_1, \theta_2] \subseteq [0, 2\pi]$, while a, b and α are appropriate parameters. When $\alpha = 1$, r and θ are called *elliptic coordinates*, corresponding to the equation of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$.

Example. Let us compute $\iint_D (y - x + 2)dxdy$, where $D = \{(x,y) \in \mathbb{R}^2 \mid \frac{x^2}{4} + \frac{y^2}{9} < 1\}$.

Using the elliptic transformation $(x,y) \to (r,\ \theta)$ given by $x = 2r\cos\theta$, $y = 3r\sin\theta$, with $0 \leq r < 1$ and $0 \leq \theta \leq 2\pi$, we get

$$\begin{aligned} \iint_D (y - x + 2)dx\,dy &= \int_0^{2\pi} \left[\int_0^1 (3r\sin\theta - 2r\cos\theta + 2) \left| \frac{D(x,y)}{D(r,\theta)} \right| (r,\theta)dr \right] d\theta = \\ &= \int_0^{2\pi} \left[\int_0^1 (3r\sin\theta - 2r\cos\theta + 2)6rdr \right] d\theta = \int_0^{2\pi} (6\sin\theta - 4\cos\theta + 6)d\theta = \\ &= (-6\cos\theta - 4\sin\theta + 6\theta) \Big|_0^{2\pi} = 12\pi. \end{aligned}$$

Another application of the double integral is referring to the computation of the mass of a material plate D, with known mass density ρ, by the formula

$$\text{mass}(D) = \iint_D \rho(x,y)dxdy.$$

We can also determine the coordinates of the center of mass (x_G, y_G) of D, cu densitatea de masă ρ, by the formulae

$$x_G = \frac{\iint_D x\rho(x,y)dxdy}{\iint_D \rho(x,y)dxdy} \quad \text{and} \quad y_G = \frac{\iint_D y\rho(x,y)dxdy}{\iint_D \rho(x,y)dxdy}.$$

2.2. **The triple integral on compact sets.**

The triple integral represents the multiple integral in the case $n = 3$. It is denoted by

$$\iiint_D f(x,y,z)dxdydz$$

where $f : D \to \mathbb{R}$ and D is a compact, Jordan measurable subset of $\mathbb{R}^3$. By analogy with the case $n = 2$, the methods of computing the triple integral are similar.

Definition. A subset $D \subseteq \mathbb{R}^3$ is called *simple with respect to the axis Oz* if there exists a compact, Jordan measurable domain $\tilde{D} \subseteq \mathbb{R}^2$ and two continuous functions $\varphi, \psi : \tilde{D} \to \mathbb{R}$ with $\varphi(x,y) < \psi(x,y)$, $\forall (x,y) \in \tilde{D}$, such that

$$D = \{(x,y,z) \in \mathbb{R}^3 \mid \varphi(x,y) \le z \le \psi(x,y),\ \forall (x,y) \in \tilde{D}\}.$$

Such a domain in $\mathbb{R}^3$ has the *volume* (*i.e.*, Jordan measure) given by the formula

$$\text{vol}(D) = \mu_J(D) = \iint_{\tilde{D}} \psi(x,y)dxdy - \iint_{\tilde{D}} \varphi(x,y)dxdy.$$

More generally, we can formulate a similar result to Theorem 2.6 for the 3D-case:

Proposition 2.8. *Let $D \subseteq \mathbb{R}^3$ be simple with respect to Oz and let $f : D \to \mathbb{R}$ be a continuous function. Then*

$$\iiint_D f(x,y,z)dxdy\,z = \iint_{\tilde{D}} \left(\int_{\varphi(x,y)}^{\psi(x,y)} f(x,y,z)dz \right) dxdy.$$

Example. Let us compute $\iiint_D \sqrt{x^2+y^2}dxdydz$, where D is the domain bounded by the surfaces $z = 0$, $z = 1$ and $z = \sqrt{x^2+y^2}$. We observe that

$$D = \{(x,y,z) \in \mathbb{R}^3 \mid \sqrt{x^2+y^2} \le z \le 1,\ \forall (x,y) \in \tilde{D}\},$$

where $\tilde{D} = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \le 1\}$. We take $\varphi(x,y) := \sqrt{x^2+y^2}$ and $\psi(x,y) := 1$, so we obtain

$$\iiint_D \sqrt{x^2+y^2}dxdydz = \iint_{\tilde{D}} \left(\int_{\sqrt{x^2+y^2}}^{1} dz \right) \sqrt{x^2+y^2}dxdy = \iint_{\tilde{D}} \sqrt{x^2+y^2} \left(1 - \sqrt{x^2+y^2}\right) dxdy.$$

In order to compute this double integral, we use the polar coordinates (r, θ):

$$\iint_{\tilde{D}} \sqrt{x^2+y^2}(1 - \sqrt{x^2+y^2})dxdy = \int_0^{2\pi} \left(\int_0^1 r(1-r)rdr \right) d\theta =$$

$$= 2\pi \int_0^1 (r^2 - r^3)dr = 2\pi \left(\frac{r^3}{3} - \frac{r^4}{4} \right)\Bigg|_0^1 = \frac{\pi}{6}.$$

A change of variable formula holds in the case $n = 3$:

Proposition 2.9. *Let $F : \Omega \to D$, $F(u,v,w) = (x(u,v,w), y(u,v,w), z(u,v,w))$, $(u,v,w) \in \Omega$ be a change of variables between the compact, Jordan measurable domains Ω and D. If $f : D \to \mathbb{R}$ is a continuous function, then*

$$\iiint_D f(x,y,z)dxdydz = \iiint_\Omega f(x(u,v,w), y(u,v,w), z(u,v,w)) \left| \frac{D(x,y,z)}{D(u,v,w)} \right| (u,v,w)dudvdw.$$

Remarks.

1. The most used change of variables in $\mathbb{R}^3$ is the transition from the cartesian coordinates x, y, z to *spheric coordinates* r, θ, φ, given by

$$\begin{cases} x = r\sin\theta\cos\varphi, & r \in [r_1, r_2] \subseteq [0, +\infty], \\ y = r\sin\theta\sin\varphi, & \theta \in [\theta_1, \theta_2] \subseteq [0, \pi], \\ z = r\cos\theta, & \varphi \in [\varphi_1, \varphi_2] \subseteq [0, 2\pi]. \end{cases}$$

The Jacobian of this transformation is

$$\frac{D(x,y,z,)}{D(r,\theta,\varphi)}(r,\theta,\varphi)=\det\begin{bmatrix}\sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta\\ r\cos\theta\varphi & r\cos\theta\sin\varphi & -r\sin\theta\\ -r\sin\theta\sin\varphi & r\sin\theta\cos\varphi & 0\end{bmatrix}=r^2\sin\theta.$$

2. Another change of variables for the triple integral is given by *cylindric coordinates*, transformation defined by

$$\begin{cases}x=r\cos\theta, & r\in[r_1,r_2]\subseteq[0,+\infty],\\ y=r\sin\theta, & \theta\in[\theta_1,\theta_2]\subseteq[0,2\pi],\\ z=z, & z\in[z_1,z_2]\subseteq\mathbb{R}.\end{cases}$$

In this case we have $\frac{D(x,y,z)}{D(r,\theta,z)}(r,\theta,z)=r$.

Resuming the example above,

$$\iiint_D\sqrt{x^2+y^2}dxdydz,$$

where $D=\{(x,y,z)\in\mathbb{R}^3\mid x^2+y^2\le z\le 1,\ \forall(x,y)\in\tilde{D}\}$ and $\tilde{D}=\{(x,y)\in\mathbb{R}^2\mid x^2+y^2\le 1\}$, we can use this last change of variables in order to obtain

$$\iiint_D\sqrt{x^2+y^2}dx\,dy\,dz=\int_0^1\left(\int_0^{2\pi}\left(\int_r^1 rdz\right)d\theta\right)rdr=2\pi\int_0^1(1-r)r^2dr=\frac{\pi}{6}.$$

Again, the triple integral can used to the computation of the mass and of the center of mass of a body D, with known mass density ρ, by the formulae

$$\text{mass}(D)=\iiint_D\rho(x,y,z)dxdydz$$

and

$$x_G=\frac{\iiint_D x\rho(x,y,z)dx\,dy\,dz}{\iiint_D\rho(x,y,z)dx\,dy\,dz},\ y_G=\frac{\iiint_D y\rho(x,y,z)dx\,dy\,dz}{\iiint_D\rho(x,y,z)dx\,dy\,dz},\ z_G=\frac{\iiint_D z\rho(x,y,z)dx\,dy\,dz}{\iiint_D\rho(x,y,z)dx\,dy\,dz}.$$

2.3. Coming back to the general case of a multiple integral on a compact, Jordan measurable domain, its computation can usually be done by one of the following two formulae:

$$\int\cdots\int_D f(x_1,\ldots,x_n)dx_1\ldots dx_n=\int\cdots\int_{\tilde{D}}\left(\int_{\varphi(x_1,\ldots,x_{n-1})}^{\psi(x_1,\ldots,x_{n-1})}f(x_1,x_2,\ldots,x_n)dx_n\right)dx_1\ldots dx_n$$

(when D is simple with respect to Ox_n, *i.e.* $D=\{(x_1,\ldots,x_{n-1},x_n)\in\mathbb{R}^n\mid\varphi(x_1,\ldots,x_{n-1})\le x_n\le\psi(x,y),\ \forall(x_1,\ldots,x_{n-1})\in\tilde{D}\}$) and

$$\int\cdots\int_D f(x_1,\ldots,x_n)dx_1\ldots dx_n=$$

$$=\int\cdots\int_\Omega f(x_1(y_1,\ldots,y_n),\ldots,x_n(y_1,\ldots,y_n))\left|\frac{D(x_1,x_2,\ldots,x_n)}{D(y_1,y_2,\ldots,y_n)}\right|(y_1,y_2,\ldots,y_n)dy_1\ldots dy_n$$

(for a change of variables from $(x_1,\ldots,x_n)\in D$ to the coordinates $(y_1,\ldots,y_n)\in\Omega$).

Example. Let us compute $\int\cdots\int_D dx_1\ldots dx_n$, where D is the set

$$D=\{(x_1,\ldots,x_n)\in\mathbb{R}^n\mid x_1\ge 0,\ x_2\ge 0,\ldots x_n\ge 0,\ x_1+x_2+\ldots+x_n\le 1\}.$$

By using the first formula, we get

$$\int\cdots\int_D dx_1\ldots dx_n=\int_0^1\left(\int_0^{1-x_1}\ldots\left(\int_0^{1-x_1-\ldots-x_{n-1}}dx_n\right)\ldots dx_2\right)dx_1=$$

$$=\int_0^1\left(\int_0^{1-x_1}\ldots\left(\int_0^{1-x_1-\ldots-x_{n-2}}(1-x_1-\ldots-x_{n-1})\,dx_{n-1}\right)\ldots dx_2\right)dx_1=$$

$$=\int_0^1\left(\int_0^{1-x_1}\ldots\left(\int_0^{1-x_1-\ldots-x_{n-3}}\frac{(1-x_1-\ldots-x_{n-2})^2}{2!}dx_{n-2}\right)\ldots dx_2\right)dx_1=\ldots=\frac{1}{n!}.$$

3. Improper multiple integrals

As in the one-dimensional case, we can extend the notion of integral to the situations where the domain is either not compact or the integrand is not bounded.

Definition. Let D be a subset of $\mathbb{R}^n$ and $f : D \to \mathbb{R}$ a function which is Riemann integrable on any compact, Jordan measurable subset of D. We say that the integral $\int\cdots\int_D f(x_1,\dots,x_n)dx_1\dots dx_n$ is *convergent* if for any sequence of bounded, Jordan measurable sets $(D_k)_{k\in\mathbb{N}^*}$, satisfying

(i) $\overline{D}_k \subset D_{k+1}$, $\forall k \in \mathbb{N}^*$;

(ii) $\bigcup\limits_{k=1}^{\infty} D_k = D$,

there exists and is finite $\lim\limits_{k\to\infty}\int\cdots\int_{D_k} f(x_1,\dots,x_n)dx_1\dots dx_n$, denoted $\int\cdots\int_D f(x_1,\dots,x_n)dx_1\dots dx_n$, its value being independent of the choice of $(D_k)_{k\in\mathbb{N}^*}$.

In the case that the above limit does not exist or is infinite, we say that the integral $\int\cdots\int_D f(x_1,\dots,x_n)dx_1\dots dx_n$ is *divergent.*

As in the case $n = 1$, we can establish various convergence/divergence criteria.

Examples.

1. The integral $\iint_{\mathbb{R}^2} \mathrm{e}^{-x^2-y^2}dxdy$ *is convergent and is equal to* π, *because*

$$\iint_{\mathbb{R}^2} \mathrm{e}^{-x^2-y^2}dxdy = \int_0^{2\pi}\left(\int_0^{\infty}\mathrm{e}^{-r^2}rdr\right)d\theta = (-2\pi)\lim_{a\to\infty}\left(-\frac{1}{2}\mathrm{e}^{-r^2}\bigg|_0^a\right) = \pi\lim_{a\to\infty}(1-\mathrm{e}^{-a^2}) = \pi.$$

2. Let us compute the improper (by the singularity in $(0,0)$) integral

$$I = \iint_D \frac{1}{(x^2+y^2)^{\alpha/2}}dxdy,$$

where $D = \{(x,y)\in\mathbb{R}^2 \mid x^2+y^2<\rho^2\}$, $\rho>0$ and $\alpha>0$: we have

$$I = \lim_{n\to\infty}\iint_{D_n}\frac{1}{(x^2+y^2)^{\alpha/2}}dx\,dy,$$

where $D_n = D\setminus B\left(\mathbf{0}_{\mathbb{R}^2};\frac{1}{n}\right)$, $n\in\mathbb{N}^*$.

Passing to polar coordinates ($x = r\cos\theta,\ y = r\sin\theta$, with $\frac{1}{n}\le r\le\rho,\ \theta\in[0,2\pi]$), we get:

$$I = \lim_{n\to\infty}\int_0^{2\pi}\left(\int_{1/n}^{\rho}\frac{r}{r^\alpha}dr\right)d\theta = (2\pi)\lim_{n\to\infty}\left(\int_{1/n}^{\rho}r^{1-\alpha}dr\right) =$$

$$= 2\pi\begin{cases}\lim\limits_{n\to\infty}\left(\frac{r^{2-\alpha}}{2-\alpha}\bigg|_{1/n}^{\rho}\right), & 0<\alpha\neq 2;\\ \lim\limits_{n\to\infty}\left(\ln r\big|_{1/n}^{\rho}\right), & \alpha=2\end{cases} = \begin{cases}2\pi\rho^{2-\alpha}, & 0<\alpha<2;\\ +\infty, & \alpha\ge 2.\end{cases}$$

As a consequence, the integral is convergent if $\alpha\in(0,2)$ and divergent if $\alpha\ge 2$.

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