$\underbrace{\{\varphi_1, \varphi_2, \dots \varphi_n\}}_{\Gamma} \models \varphi$ ddacă pt orice $\tau : A \to B$ a.î. $\hat{\tau}(\varphi_1) = \hat{\tau}(\varphi_2) = \dots = \hat{\tau}(\varphi_n) = 1$

avem ~~si~~ $\hat{\tau}(\varphi) = 1$

$\Gamma = \{p, \neg p\}$ → nu există niciun $\tau : A \to B$ a.î. $\hat{\tau}(p) = \hat{\tau}(\neg p) = 1$

→ Γ - inconsistentă

Def $\Gamma = \{\varphi_1, \dots \varphi_n\}$ este consistentă ddacă există $\tau : A \to B$ a.î.

$\hat{\tau}(\varphi_1) = \hat{\tau}(\varphi_2) = \dots = \hat{\tau}(\varphi_n) = 1$.

Lema $\Gamma = \{\varphi_1, \dots \varphi_n\}$ - consistentă ddacă $(((\varphi_1 \wedge \varphi_2) \wedge \varphi_3) \wedge \dots \wedge \varphi_n)$ este satisfiabilă.

$\Rightarrow$

$\Leftarrow$

Ex: pt orice $\varphi \in LP$, dacă $\Gamma = \{\varphi_1, \dots \varphi_n\} \subseteq 2^{LP}$ este inconsistentă

atunci $\Gamma \models \varphi$ → poate fi inclusiv $(p \wedge \neg p)$

Ex: dacă $\Gamma = \{\varphi_1, \dots \varphi_n\} \models \underbrace{(p \wedge \neg p)}_{\equiv \bot}$ atunci Γ - inconsistentă.

Secvențele : $\underbrace{\{\varphi_1, \dots \varphi_n\}}_{\Gamma} \vdash \varphi$ (φ consecință sintactică din Γ)

$\varphi, \varphi_1, \varphi_2, \dots \varphi_n \in LP_{\neg, \wedge, \vee, \Rightarrow, \bot}$

Regulă de inferență : un tuplu format din :

- o mulțime de secvențe $S_1, S_2, \dots S_n$ → ipotezele
- o secvență S → concluzia
- o condiție (opțională)
- un nume

$$\text{nume}\ \frac{S_1 \quad S_2 \quad \dots \quad S_n}{S}\ \text{condiție}$$

$$\wedge_i \frac{\overset{S_1}{\Gamma \vdash \varphi} \qquad \overset{S_2}{\Gamma \vdash \varphi'}}{\underset{S}{\Gamma \vdash (\varphi \wedge \varphi')}}$$

nume

schema de secvente

instanță: $\wedge_i \frac{\overbrace{\overbrace{\{p, q\}}^{\Gamma} \vdash \overset{\varphi}{p}}^{S_1} \qquad \{p, q\} \vdash \overset{\varphi'}{q}}{\{p, q\} \vdash (p \wedge q)}$

ipoteză $\frac{}{\Gamma \vdash \varphi}$ $\varphi \in \Gamma$

ipoteză $\frac{}{\underbrace{\{(p \wedge q), r\}}_{\Gamma} \vdash \underbrace{(p \wedge q)}_{\varphi}}$ $\varphi \in \Gamma$ ✓

Sistem deductiv = o mulțime de reguli de inferență.

Demonstrație formală = o listă de secvente.

1. S_1
2. S_2
3. S_3

⋮

n. S_n

a.î. pt orice $1 \le i \le n$, S_i: concluzia unei instanțe a unei reguli având ca ipoteze doar secvente dintre $S_1 \ldots S_{i-1}$

1. $\overbrace{\{p, q\} \vdash p}^{S_1}$ (ipoteză)   ipoteză $\frac{}{\{p, q\} \vdash p}$ $p \in \{p, q\}$

2. $\{p, q\} \vdash q$ (ipoteză)

3. $\{p, q\} \vdash (p \wedge q)$ $(\wedge_i, 1, 2)$ $\wedge_i \frac{\{p, q\} \vdash p \qquad \{p, q\} \vdash q}{\{p, q\} \vdash (p \wedge q)}$

4. $\{p, q\} \vdash ((p \wedge q) \wedge q)$ $(\wedge_i, 3, 2)$!!! $\wedge_i \frac{\{p, q\} \vdash \overset{\varphi}{(p \wedge q)} \qquad \{p, q\} \vdash \overset{\varphi'}{q}}{\{p, q\} \vdash ((p \wedge q) \wedge q)}$

Secventă validă în D : $\Gamma \vdash \varphi$ ddacă există o dem formală $S_1, \dots S_n$

a.î. $S_n = \Gamma \vdash \varphi$

Ex : $\{p, q\} \vdash ((p \wedge q) \wedge q)$ - este secv. validă.

<u>Deductia Naturală</u> (DN) - un sistem deductiv pt $LP_{\neg, \wedge, \vee, \rightarrow, \perp}$

① conjunctie

$$\wedge_i \frac{\Gamma \vdash \varphi \qquad \Gamma \vdash \varphi'}{\Gamma \vdash (\varphi \wedge \varphi')} \qquad \wedge_{e_1} \frac{\Gamma \vdash (\varphi \wedge \varphi')}{\Gamma \vdash \varphi} \qquad \wedge_{e_2} \frac{\Gamma \vdash (\varphi \wedge \varphi')}{\Gamma \vdash \varphi'}$$

$\{(p \wedge q), r\} \vdash (p \wedge r)$ - secv. validă

1. $\{(p \wedge q), r\} \vdash (p \wedge q)$ (ipoteza)

2. ~~i.~~ $\{(p \wedge q), r\} \vdash p$ $(\wedge_{e_1}, 1)$ → ipoteza : $\{(p \wedge q), r\} \vdash (p \wedge$ ~~—~~ $)$

3. ~~j.~~ $\{(p \wedge q), r\} \vdash \underbrace{r}_{\varphi}$ (ipoteză) $(r \in \Gamma)$

4. ~~n.~~ $\underbrace{\{(p \wedge q), r\}}_{\Gamma} \vdash (\overset{\varphi}{p} \wedge \overset{\varphi'}{r})$ $(\wedge_i, \overset{2}{\cancel{i}}, \overset{3}{\cancel{j}})$

② implicatie

$$\rightarrow_e \frac{\Gamma \vdash (\varphi \rightarrow \varphi') \qquad \Gamma \vdash \varphi}{\Gamma \vdash \varphi'}$$

1. $\{\overbrace{(p\to r), (p\wedge q)}^{\Gamma}\} \vdash (\overset{\varphi}{p} \to \overset{\varphi'}{r})$ (ipoteză)

2 ~~j~~. $\{(p\to r), (p\wedge q)\} \vdash (p\wedge q)$ (ipoteză) $(p\wedge q)\in\Gamma$

3 ~~i~~. $\{(p\to r), (p\wedge q)\} \vdash p$ $(\wedge e_1, \overset{2}{\cancel{j}})$

4 ~~n~~. $\{\underbrace{(p\to r), (p\wedge q)}_{\Gamma}\} \vdash \underset{\varphi'}{r}$ $(\to e, 1, \overset{3}{\cancel{i}})$

$$\to i \; \frac{\Gamma, \varphi \vdash \varphi'}{\Gamma \vdash (\varphi\to\varphi')} \qquad\qquad \to i \; \frac{\overbrace{\Gamma\cup\{\varphi\}}^{\Gamma'} \vdash \varphi'}{\Gamma' \vdash (\varphi\to\varphi')}$$

1. $\{(p\to q),(q\to r), p\} \vdash (p\to q)$ (ipoteze)

2. $\{p\to q,\ q\to r,\ p\} \vdash p$ (ipoteze)

3 ~~m~~. $\{(p\to q),(q\to r), p\} \vdash q$ $(\to e, 1, 2)$

4 ~~k~~. $\{(p\to q),(q\to r), p\} \vdash (q\to r)$ (ipoteză)

$$\to e \; \frac{\Gamma' \vdash (q\to r) \quad \Gamma' \vdash q}{\Gamma' \vdash r}$$

5 ~~j~~. $\{\overbrace{\underbrace{(p\to q),(q\to r)}_{\Gamma}, p}^{\Gamma'}\} \vdash r$ $(\to e, \overset{4}{\cancel{k}}, \overset{3}{\cancel{m}})$

6 ~~n~~. $\{\underbrace{(p\to q),(q\to r)}_{\Gamma}\} \vdash (\overset{\varphi}{p}\to \overset{\varphi'}{r})$ $(\to i, \overset{5}{\cancel{j}})$

③ disjuncția

$$\vee i_1 \; \frac{\Gamma\vdash\varphi_1}{\Gamma\vdash(\varphi_1\vee \underbrace{\varphi_2}_{\text{orice}})} \qquad\qquad \vee i_2 \; \frac{\Gamma\vdash\varphi_2}{\Gamma\vdash(\underbrace{\varphi_1}_{\text{orice}}\vee\varphi_2)}$$

$1\ \cancel{k}\quad \{(p \wedge q)\} \vdash (p \wedge q) \quad (\text{ipoteză})$

$$\wedge e_1 \frac{\Gamma \vdash (\varphi_1 \wedge \varphi_2)}{\Gamma \vdash \varphi_1}$$

$2\ \cancel{i}\quad \{(p \wedge q)\} \vdash p \quad (\wedge e_1, \cancel{k})$

$3\ \cancel{r}\quad \underbrace{\{(p \wedge q)\}}_{\Gamma} \vdash (\overset{\varphi_1}{p} \vee \overset{\varphi_2}{r}) \quad (\vee i_1, \cancel{i})$

caz 1 φ_1 "A" într-o atribuire — caz 2

$$\vee e \frac{\Gamma \vdash (\varphi_1 \vee \varphi_2) \quad \Gamma, \varphi_1 \vdash \varphi' \quad \Gamma, \varphi_2 \vdash \varphi'}{\Gamma \vdash \varphi'}$$

$1\quad \{(p \vee q)\} \vdash (\underset{\varphi_1}{p} \vee \underset{\varphi_2}{q}) \quad (\text{ipoteză})$

$2.\quad \{(p \vee q), p\} \vdash p \quad (\text{ipoteză})$

$3\ \cancel{i}\quad \underbrace{\{(p \vee q), p\}}_{\Gamma'} \vdash (q \vee p) \quad (\vee i_2, 2)$

$\cancel{4}\quad \{(p \vee q), q\} \vdash q \quad (\text{ipoteză})$

$5\ \cancel{j}\quad \underbrace{\{(p \vee q), q\}}_{\Gamma''} \vdash (q \vee p) \quad (\vee i_1, \cancel{4})$

$6\ \cancel{r}\quad \underbrace{\{(p \vee q)\}}_{\Gamma} \vdash \underbrace{(q \vee p)}_{\varphi'} \quad (\vee e, 1, \cancel{i}, \cancel{j})$

④ negație

inconsistentă

$$\neg i \frac{\Gamma, \varphi \vdash \bot}{\Gamma \vdash \neg\varphi} \qquad \neg e \frac{\Gamma \vdash \varphi \quad \Gamma \vdash \neg\varphi}{\Gamma \vdash \bot} \qquad \bot e \frac{\Gamma \vdash \bot}{\Gamma \vdash \varphi}$$

orice

1. $\{p, \neg p\} \vdash p$ (ip)

2. $\{p, \neg p\} \vdash \neg p$ (ip)

j. $\overbrace{\{p, \neg p\}}^{\Gamma'} \vdash \bot$ $(\neg e, 1, 2)$

n. $\underbrace{\{p\}}_{\Gamma} \vdash \underbrace{\neg\neg p}_{\varphi}$ $(\neg i, j)$

---

1. $\{p, \neg p\} \vdash p$ (ip)
2. $\{p, \neg p\} \vdash \neg p$ (ip)
j. $\{p, \neg p\} \vdash \bot$ $(\neg e, 1, 2)$
n. $\{p, \neg p\} \vdash \underline{r}$ $(\bot e, j$

5. extindere $$\frac{\Gamma \vdash \varphi}{\Gamma, \varphi' \vdash \varphi}$$

1. $\{p, q\} \vdash p$ (ip)

2. $\{p, q, \underline{r \vee p}\} \vdash p$ (extindere, 1) (ip)

$$\neg\neg e \; \frac{\Gamma \vdash \neg\neg\varphi}{\Gamma \vdash \varphi}$$

1. $\{(\neg p \to q), \neg q, \neg p\} \vdash (\neg p \to q)$ (ip)

2. $\{(\neg p \to q), \neg q, \neg p\} \vdash \neg p$ (ip)

3 $\cancel{k_1}$ $\{(\neg p \to q), \neg q, \neg p\} \vdash q$ $(\to e, 1, 2)$

4 $\cancel{k_2}$ $\{(\neg p \to q), \neg q, \neg p\} \vdash \neg q$ (ip)

5 $\cancel{i}$ $\{(\neg p \to q), \neg q, \underbrace{\neg p}_{\varphi}\} \vdash \bot$ $(\neg e, \overset{3}{\cancel{k_1}}, \overset{4}{\cancel{k_2}})$

6 $\cancel{j}$ $\{(\neg p \to q), \neg q\} \vdash \neg\neg p$ $(\neg i, \overset{5}{\cancel{i}})$

7 $\cancel{n}$ $\underbrace{\{(\neg p \to q), \neg q\}}_{\Gamma} \vdash p$ $(\neg\neg e, \overset{6}{j})$

$$\neg\neg i \ \frac{\Gamma \vdash \varphi}{\Gamma \vdash \neg\neg\varphi}$$

Th 109 (Corectitudine) pt orice $\Gamma \subseteq \mathbb{LP}_{\neg,\wedge,\vee,\to,\bot}$, și orice φ

dacă $\Gamma \vdash \varphi$ atunci $\Gamma \models \varphi$

Th 110 (Completitudine) pt orice Γ și orice φ

dacă $\Gamma \models \varphi$ atunci $\Gamma \vdash \varphi$

7.6 Deducția naturală

Deducția naturală este sistemul deductiv alcătuit din toate regulile din secțiunile precedente. Iată aici toate regulile:

$$\wedge i \ \frac{\Gamma \vdash \varphi \qquad \Gamma \vdash \varphi'}{\Gamma \vdash (\varphi \wedge \varphi')}, \qquad \wedge e_1 \ \frac{\Gamma \vdash (\varphi \wedge \varphi')}{\Gamma \vdash \varphi},$$

$$\wedge e_2 \ \frac{\Gamma \vdash (\varphi \wedge \varphi')}{\Gamma \vdash \varphi'}, \qquad \to e \ \frac{\Gamma \vdash (\varphi \to \varphi') \qquad \Gamma \vdash \varphi}{\Gamma \vdash \varphi'}, \qquad \to i \ \frac{\Gamma, \varphi \vdash \varphi'}{\Gamma \vdash (\varphi \to \varphi')},$$

$$\vee i_1 \ \frac{\Gamma \vdash \varphi_1}{\Gamma \vdash (\varphi_1 \vee \varphi_2)}, \qquad \vee i_2 \ \frac{\Gamma \vdash \varphi_2}{\Gamma \vdash (\varphi_1 \vee \varphi_2)},$$

$$\vee e \ \frac{\Gamma \vdash (\varphi_1 \vee \varphi_2) \qquad \Gamma, \varphi_1 \vdash \varphi' \qquad \Gamma, \varphi_2 \vdash \varphi'}{\Gamma \vdash \varphi'},$$

$$\neg e \ \frac{\Gamma \vdash \varphi \qquad \Gamma \vdash \neg\varphi}{\Gamma \vdash \bot}, \qquad \neg i \ \frac{\Gamma, \varphi \vdash \bot}{\Gamma \vdash \neg\varphi}, \qquad \bot e \ \frac{\Gamma \vdash \bot}{\Gamma \vdash \varphi},$$

$$\text{IPOTEZĂ} \ \frac{}{\Gamma \vdash \varphi} \ \varphi \in \Gamma, \qquad \text{EXTINDERE} \ \frac{\Gamma \vdash \varphi}{\Gamma, \varphi' \vdash \varphi}, \qquad \neg\neg e \ \frac{\Gamma \vdash \neg\neg\varphi}{\Gamma \vdash \varphi}.$$