Integrability

Lecture 12

Mathematics - 1^{st} year, English

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January 8, 2019

Integral

The notion of *integral* is central not only in mathematics, serving to:

- the determination of the state of a dynamical process whose speed of evolution is known;
- the computation of: numeric characteristics of geometric shapes (length, area, volume, position coordinates, center of mass);
- of physical quantities (momentum, potential or work);
- numeric characteristics of random variables in probability (distribution function, mean and variance).

Outline of the lecture

1. Antiderivatives
2. Riemann Integral
3. Improper integrals
4. Integrals with parameters
   - Gamma and Beta functions

Antiderivatives

Definition

Let $I \subseteq \mathbb{R}$ with $\mathring{I} \neq \emptyset$ and $f : I \to \mathbb{R}$.

- A function $F : I \to \mathbb{R}$ is called an *antiderivative* of f if F is derivable on I and $F'(x) = f(x)$, $\forall x \in I$.
- If f has at least an antiderivative on I, then the set of all antiderivatives of f is called the *indefinite integral* of f and is denoted $\int f(x)dx$.

Remarks.

1. If $F : I \to \mathbb{R}$ is antiderivative of a function $f : I \to \mathbb{R}$, then any other antiderivative of f has the form $F + c$, where c is a real constant.

- By denoting $\mathcal{C}$ the set of all constant functions on I, we have $\int f(x)dx = F + \mathcal{C}$.
- By language abuse, we can write $\int f(x)dx = F(x) + c$, $\forall x \in I$.

2. If $f : I \to \mathbb{R}$ is a derivable function on I, then f is an antiderivative of f'.

3. Any antiderivative of a function $f : I \to \mathbb{R}$ is continuous, because any derivable function is continuous.
4. The space $\mathcal{P}(I)$ of all functions $f : I \to \mathbb{R}$ which admit antiderivatives is a linear space (subspace of $\mathscr{F}(I;\mathbb{R})$), because

$$\int (\alpha f(x) + \beta g(x))dx = \alpha \int f(x)dx + \beta \int g(x)dx, \ \forall \alpha, \beta \in \mathbb{R}.$$

5. Any function $f : I \to \mathbb{R}$ admiting antiderivatives has the so called *Darboux property*: for any $x_1, x_2 \in I$ and any λ between $f(x_1)$ and $f(x_2)$, there exists $\tilde{x}$ between x_1 and x_2 such that $f(\tilde{x}) = \lambda$.

List of usual indefinite integrals:

- $\int x^\alpha dx = c + \begin{cases} \dfrac{x^{\alpha+1}}{\alpha+1}, & \alpha \in \mathbb{R} \setminus \{-1\}; \\ \ln|x|, & \alpha = -1; \end{cases}$
- $\int \dfrac{dx}{x^2-a^2} = \dfrac{1}{2a} \ln\left|\dfrac{x-a}{x+a}\right| + c;\quad \int \dfrac{dx}{x^2+a^2} = \dfrac{1}{a}\,\mathrm{arctg}\,\dfrac{x}{a} + c,\ a \in \mathbb{R}^*;$
- $\int \dfrac{dx}{\sqrt{x^2+a^2}} = \ln\left(x + \sqrt{x^2+a^2}\right) + c;\quad \int \dfrac{dx}{\sqrt{a^2-x^2}} = \arcsin \dfrac{x}{|a|} + c,\ a \in \mathbb{R}^*;$
- $\int a^x dx = \dfrac{1}{\ln a} a^x + c,\ a \in \mathbb{R}_+^* \setminus \{1\};$
- $\int \sin x dx = -\cos x + c;\quad \int \cos x dx = \sin x + c;$
- $\int \mathrm{sh}\, x\, dx = \int \dfrac{\mathrm{e}^x - \mathrm{e}^{-x}}{2} dx = \mathrm{ch}\, x + c;\quad \int \mathrm{ch}\, x\, dx = \int \dfrac{\mathrm{e}^x + \mathrm{e}^{-x}}{2} dx = \mathrm{sh}\, x + c,$

where $c \in \mathbb{R}$.

Integration by parts

Let $f, g : I \to \mathbb{R}$ two derivabile functions, with f' and g' continuous on I. Then

$$\int f'(x)g(x)dx = f(x)g(x) - \int f(x)g'(x)dx,\ x \in I,$$

We can apply this formula in order two complete the list of indefinite integrals:

- $\int \sqrt{a^2 - x^2}dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\arcsin\frac{x}{|a|} + c,\ a \in \mathbb{R}_+^*;$
- $\int \sqrt{x^2 \pm a^2}dx = \frac{x}{2}\sqrt{x^2 \pm a^2} \pm \frac{a^2}{2}\ln\left|x + \sqrt{x^2 \pm a^2}\right| + c,\ a \in \mathbb{R}^*;$
- $\int \ln x dx = x(\ln x - 1) + c.$

Integration by parts is recommended for integrals of the form

$$\int P(x)f(x)dx,$$

where $P \in \mathbb{R}[X]$ and f is an elementary function: e^x, $\ln x$, $\arcsin x$, $\arccos x$, $\mathrm{arctg}\, x$, $\mathrm{arcctg}\, x$, a^x, $\log_a x$, etc. By applying this method, one can reduce by one unit the degree of the polynomial function P.

Method of algebraic transformations

- It is mostly used for computing the antiderivatives of *rational functions* of the form $f(x) = \dfrac{P(x)}{Q(x)}$, where $P, Q \in \mathbb{R}[X]$, defined on an interval $I \subseteq \mathbb{R}$ such that $\mathring{I} \neq \emptyset$ and $Q(x) \neq 0$ on I.
- It is well-known (from algebra) that f can be uniquely decomposed as a sum of "simple" rational functions

$$f(x) = G(x) + \frac{H(x)}{Q(x)} = G(x) + \sum\nolimits_1 \frac{A_{k,m}}{(x - x_k)^m} + \sum\nolimits_2 \frac{B_{k,m}x + C_{k,m}}{(x^2 + p_k x + q_k)^m},$$

  where:

  - G is a polynomial function (equal to 0 when $\deg P < \deg Q$),
  - H still a polynomial function with $\deg H < \deg Q$,
  - $\sum_1$ is a finite sum with respect to all real roots x_k of Q and
  - $\sum_2$ is a finite sum with respect to all complex roots of Q (with $p_k, q_k \in \mathbb{R}$ such that $p_k^2 - 4q_k < 0$).
- The integration of f is then reduced to computing the antiderivatives of all components of the above decomposition.

If Q has multiple roots, computing the antiderivative of $\dfrac{P(x)}{Q(x)}$ can be also done by *Gauss-Ostrogradski method*, based on the formula

$$(*) \qquad \int \frac{P(x)}{Q(x)}dx = \frac{P_1(x)}{Q_1(x)} + \int \frac{P_2(x)}{Q_2(x)}dx,\ x \in I,$$

where

- $Q_1 \in \mathbb{R}[X]$ is the greatest common divisor of Q and Q',
- $Q_2 = \dfrac{Q}{Q_1}$ and
- P_1, P_2 are polynomials having the degree one unit smaller than $\deg Q_1$, respectively $\deg Q_2$.

Finding P_1 and P_2 can be realized by derivating relation $(*)$, *i.e.*

$$\frac{P(x)}{Q(x)} = \frac{P_1'(x)Q_1(x) - P_1(x)Q_1'(x)}{Q_1^2(x)} + \frac{P_2(x)}{Q_2(x)},\ x \in I.$$

Method of trigonometric transformations

It is often combined with the *substitution method* and is used for computing the antiderivatives of functions expressed with the help of trigonometric functions.

- For *trigonometric integrals* of the form
$$\int E(\sin x, \cos x)dx, \ x \in I = (-\pi, \pi),$$
where E is a rational function of two variabiles: substitution $\operatorname{tg} \frac{x}{2} = t$.
- Since $\sin x = \dfrac{2t}{1+t^2}$, $\cos x = \dfrac{1-t^2}{1+t^2}$, $x = 2 \operatorname{arctg} t$, $dx = \dfrac{2dt}{1+t^2}$: transformation into a rational function in the new variable t.
- There are some cases in which the computations can be simplified, by avoiding the standard substitution $\operatorname{tg} \frac{x}{2} = t$:
    i) if $E(-\sin x, \cos x) = -E(\sin x, \cos x)$, *i.e.* E is odd in $\sin x$, then the substitution $\cos x = t$ is recommended;
    ii) if $E(\sin x, -\cos x) = -E(\sin x, \cos x)$, *i.e.* E is odd in $\cos x$, then the substitution $\sin x = t$ is recommended;
    iii) if $E(-\sin x, -\cos x) = E(\sin x, \cos x)$, *i.e.* E is even in $\sin x$ and $\cos x$, then the substitution $\operatorname{tg} x = t$ is recommended.

Irrational integrals

We still apply the substitution method for computing the so-called *irrational integrals*, in order to reduce them to integrals of rational functions.
We use the *Euler substitutions* for integrals of the form

$$\int E\left(x, \sqrt{ax^2+bx+c}\right) dx, \; x \in I,$$

with $a, b, c \in \mathbb{R}$ and E a rational function of two variabiles. The change of variable is done according to each of the following case:

i) $\sqrt{ax^2+bx+c} = \pm x\sqrt{a} \pm t$, when $a > 0$;

ii) $\sqrt{ax^2+bx+c} = \pm tx \pm \sqrt{c}$, when $c > 0$;

iii) $\sqrt{ax^2+bx+c} = t(x - x_0)$, when $b^2 - 4ac > 0$, where x_0 is a real root of the equation $ax^2 + bx + c = 0$.

For irrational integrals of the form

$$\int E\left(x, \left(\frac{ax+b}{cx+d}\right)^{p_1/q_1}, \dots, \left(\frac{ax+b}{cx+d}\right)^{p_k/q_k}\right) dx,\ x \in I,$$

where E is a rational function of $k+1$ ($k \in \mathbb{N}^*$) real variables, $a, b, c, d \in \mathbb{R}$, $a^2+b^2+c^2+d^2 \neq 0$, $cx+d \neq 0$, $\forall x \in I$, $\dfrac{ax+b}{cx+d} > 0$, $\forall x \in I$, $p_i \in \mathbb{Z}$, $q_i \in \mathbb{N}^*$, $\forall i = \overline{1, k}$, we use the substitution $\dfrac{ax+b}{cx+d} = t^{q_0}$, where q_0 is the least common multiple of $q_1, q_2, \dots, q_k$.

Chebyshev substitutions are used for the calculus of *bynomial integrals*, having the form

$$\int x^p (ax^q + b)^r \, dx, \; x \in I,$$

where $a \in \mathbb{R}^*$, $b \in \mathbb{R}$ and $p, q, r \in \mathbb{Q}$. The computation of such integrals is reduced to that of the antiderivatives of irrational functions only in the following three cases:

i) $r \in \mathbb{Z}$: the substitution $x = t^m$, with m being the least common multiple of p and q;

ii) $\frac{p+1}{q} \in \mathbb{Z}$: the substitution $ax^q + b = t^\ell$, where ℓ is the denominator of r.

iii) $\frac{p+1}{q} + r \in \mathbb{Z}$: the substitution $a + bx^{-q} = t^\ell$, ℓ being the denominator r.

Computing integrals of the form

$$\int E\left(a^{r_1 x}, a^{r_2 x}, \dots, a^{r_n x}\right) \, dx,$$

where $a \in \mathbb{R}_+^* \setminus \{1\}$, $r_1, r_2, \dots, r_n \in \mathbb{Q}$ and E is a rational functions of n ($n \in \mathbb{N}^*$) real variables can be done by the substitution $a^x = t^\nu$, where $t > 0$ and ν is the least common multiple of the denominators of $r_1, r_2, \dots, r_n$.

Elementary functions that do not possess elementary antiderivatives:

- *elliptic integrals*

$$\int \sqrt{(1 - a^2 \sin^2 x)^{\pm 1}}dx, \ \ a \in (0,1),$$

- $\int \frac{\sin x}{x} dx, \quad \int \frac{\cos x}{x} dx;$
- $\int \frac{dx}{\ln x}, \ \int \frac{e^x}{x}\, dx;$
- $\int e^{-x^2}\, dx$ (Poisson antiderivative);
- $\int \cos(x^2) dx$, $\int \sin(x^2) dx$ (Fresnel antiderivatives).

Riemann Integral

Let $a, b \in \mathbb{R}$, $a < b$ and $f : [a, b] \to \mathbb{R}$.

Definition

- We call a *partition* of the interval $[a, b]$ a finite set $\Delta = \{x_0, x_1, \ldots, x_n\}$ such that $a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b$. The intervals $[x_i, x_{i+1}]$ $(i = \overline{0, n-1})$ are called *subintervals* of the partition Δ.
- The number
$$\|\Delta\| = \max_{1 \le i \le n} \{x_i - x_{i-1}\}$$
(denoted also by $\nu(\Delta)$) is called the *mesh* or *norm* of the partition Δ.
- A partition Δ of the interval $[a, b]$ is called equidistant if $x_i - x_{i-1} = \dfrac{b-a}{n}$, $\forall i = \overline{1, n}$; in this case we have $\|\Delta\| = \dfrac{b-a}{n}$ and $x_i = a + i\dfrac{b-a}{n}$, $\forall i = \overline{0, n}$.

- We will denote by $\mathcal{D}[a, b]$ the set of all partitions of a compact interval $[a, b]$.
- If $\Delta_1, \Delta_2 \in \mathcal{D}[a, b]$ and $\Delta_1 \subseteq \Delta_2$, we say that Δ_2 is *finer* than Δ_1 and we denote $\Delta_1 \preceq \Delta_2$.

Definition

Let $\Delta = \{x_0, x_1, \ldots, x_n\}$ with $a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b$ be a partition of $[a, b]$.

- An n-uple $\xi_\Delta = (\xi_1, \xi_2, \ldots, \xi_n) \in \mathbb{R}^n$ is called an *intermediary point system* of Δ if $\xi_i \in [x_{i-1}, x_i]$, $\forall i = \overline{1, n}$. The set of all intermediary point systems of Δ is denoted Ξ_Δ.
- We call the *Riemann sum* of the function $f : [a, b] \to \mathbb{R}$ with respect to Δ and an intermediary point system $\xi_\Delta = (\xi_1, \xi_2, \ldots, \xi_n)$ the number

$$\sigma_f(\Delta, \xi_\Delta) = \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}).$$

Definition

The function $f : [a, b] \to \mathbb{R}$ is called *Riemann integrable* (or *$\mathcal{R}$-integrable*) if there exists a real number I, called the *Riemann integral* of f, such that for any $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$ such that for any partition $\Delta \in \mathcal{D}[a, b]$ with $\|\Delta\| < \delta_\varepsilon$ and any $\xi_\Delta \in \Xi_\Delta$ we have $|\sigma_f(\Delta, \xi_\Delta) - I| < \varepsilon$.
The Riemann integral (which is unique) is denoted by

$$\int_a^b f(x)dx \quad \text{or} \quad (\mathcal{R}) \int_{[a,b]} f(x)dx.$$

The set of all $\mathcal{R}$-integrable functions on $[a, b]$ is denoted $\mathcal{R}[a, b]$.

Proposition

If a function $f : [a, b] \to \mathbb{R}$ is Riemann integrable, then it is bounded.

Remark. If we denote $\mathcal{B}([a,b])$ the set of all bounded functions $f:[a,b]\to\mathbb{R}$, then $\mathcal{R}[a,b]\subseteq\mathcal{B}([a,b])$.

- The inclusion is strict, because there exist bounded functions which are not $\mathcal{R}$-integrable.
- An example is the Dirichlet function, $f:[a,b]\to\mathcal{R}$, defined by
  $f(x)=\begin{cases}1, & x\in[a,b]\cap\mathbb{Q};\\ 0, & x\in[a,b]\setminus\mathbb{Q}.\end{cases}$

Theorem (Cauchy criterion of Riemann integrability)

The function $f:[a,b]\to\mathbb{R}$ is $\mathcal{R}$-integrable if and only if

$$\forall\varepsilon>0,\ \exists\delta_\varepsilon>0,\ \forall\Delta\in\mathcal{D}[a,b],\ \forall\xi'_\Delta,\xi''_\Delta\in\Xi_\Delta:\|\Delta\|<\delta_\varepsilon \Rightarrow \left|\sigma_f(\Delta,\xi'_\Delta)-\sigma_f(\Delta,\xi''_\Delta)\right|<\varepsilon.$$

Properties

Proposition

i) *If* $f \in \mathcal{R}[a,b]$*, then* $f|_{[c,d]} \in \mathcal{R}[c,d]$*, for any interval* $[c,d] \subseteq [a,b]$*.*

ii) *Let* $f : [a,b] \to \mathbb{R}$ *and* $c \in (a,b)$*. If* $f|_{[a,c]} \in \mathcal{R}[a,c]$ *and* $f|_{[c,b]} \in \mathcal{R}[c,b]$*, then* $f \in \mathcal{R}[a,b]$ *and*

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

iii) *If* $f \in \mathcal{R}[a,b]$*, then* $|f| \in \mathcal{R}[a,b]$ *and*

$$\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|\, dx.$$

Proposition (continuation)

iv) *If $f, g \in \mathcal{R}[a,b]$, then $f \cdot g \in \mathcal{R}[a,b]$ and the following Cauchy-Schwarz inequality for $\mathcal{R}$-integrable functions holds:*

$$\left(\int_a^b f(x)g(x)dx\right)^2 \leq \left(\int_a^b f^2(x)\,dx\right)\left(\int_a^b g^2(x)\,dx\right).$$

v) *If $f \in \mathcal{R}[a,b]$ and $|f(x)| \geq \mu > 0$, $\forall x \in [a,b]$, then $\frac{1}{f} \in \mathcal{R}[a,b]$.*

vi) *If $f, g \in \mathcal{R}[a,b]$ and $\alpha, \beta \in \mathbb{R}$, then $\alpha f + \beta g \in \mathcal{R}[a,b]$ and*

$$\int_a^b (\alpha f(x) + \beta g(x))\,dx = \alpha \int_a^b f(x)dx + \beta \int_a^b g(x)dx$$

(in other words, $\mathcal{R}[a,b]$ is a linear subspace of $\mathscr{F}([a,b];\mathbb{R})$).

vii) *If $f \in \mathcal{R}[a,b]$ and $f(x) \geq 0$, $\forall x \in [a,b]$, then $\int_a^b f(x)dx \geq 0$.*

Remarks.

1. A generalization of Cauchy-Schwarz inequality is, similar to finite sums of real numbers, *Hölder's inequality* pentru for $\mathcal{R}$-integrable functions:

$$\left|\int_a^b f(x)g(x)dx\right| \leq \left(\int_a^b |f(x)|^p dx\right)^{\frac{1}{p}} \left(\int_a^b |g(x)|^q dx\right)^{\frac{1}{q}},$$

where $f, g \in \mathcal{R}[a, b]$, $p, q \in (1, +\infty)$, with $\frac{1}{p} + \frac{1}{q} = 1$.

2. The Riemann integral is a monotone functional, *i.e.* if $f, g \in \mathbb{R}[a, b]$ such that $f(x) \leq g(x)$, $\forall x \in [a, b]$, then

$$\int_a^b f(x)dx \leq \int_a^b g(x)dx.$$

3. If $f \in \mathcal{R}[a, b]$, we define $\int_b^a f(x)dx := -\int_a^b f(x)dx$ and $\int_a^a f(x)dx := 0$.

4. Let $f \in \mathcal{R}[a,b]$ and $m = \inf\limits_{x\in[a,b]} f(x) \in \mathbb{R}$, $M = \sup\limits_{x\in[a,b]} f(x) \in \mathbb{R}$. By the monotonicity of the Riemann integral, we have

$$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a).$$

Moreover, if $f \in C([a,b])$ (*i.e.*, f is continuous on $[a,b]$), then there exists $x_1, x_2 \in [a,b]$ such that $f(x_1) = m$, $f(x_2) = M$; it follows that

$$f(x_1) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq f(x_2)$$

Since f has the Darboux property (implied by the continuity of f), there exists c between x_1 and x_2 (with possibility of equality) such that $f(c) = \dfrac{1}{b-a}\int_a^b f(x)dx$, *i.e.* the following *mean equality* holds:

$$\int_a^b f(x)dx = f(c)(b-a).$$

Darboux sums

If $f : [a, b] \to \mathbb{R}$ is a bounded function and $\Delta = \{x_0, x_1, \ldots, x_n\}$ with $a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b$ is a partition of $[a, b]$, we can define the *lower* and *upper Darboux sums* associated with Δ by

$$s_f(\Delta) := \sum_{i=1}^{n} m_i(x_i - x_{i-1});$$
$$S_f(\Delta) := \sum_{i=1}^{n} M_i(x_i - x_{i-1}),$$

where $m_i := \inf_{x \in [x_{i-1}, x_i]} f(x)$ and $M_i := \sup_{x \in [x_{i-1}, x_i]} f(x)$, $\forall i = \overline{1, n}$.

Definition

The number $\underline{I} := \sup_{\Delta \in \mathcal{D}[a,b]} s_f(\Delta)$ is called the *lower Darboux integral*, while the number $\overline{I} := \inf_{\Delta \in \mathcal{D}[a,b]} S_f(\Delta)$ is called the *upper Darboux integral*.

We always have $\underline{I} \le \overline{I}$.

Theorem (Darboux criterion of Riemann integrability)

Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Then f is Riemann integrable if and only if $\underline{I} = \overline{I}$, condition which is equivalent to

$$\forall \varepsilon > 0, \ \exists \Delta_\varepsilon \in \mathcal{D}[a, b] : S_f(\Delta_\varepsilon) - s_f(\Delta_\varepsilon) < \varepsilon.$$

In this case, $\underline{I} = \overline{I} = \int_a^b f(x)dx$.

Using the Cauchy or Darboux criteria, one can highlight some important categories of functions which are Riemann integrable.

Theorem

Let $f : [a, b] \to \mathbb{R}$ be a function.

i) *If $f \in C([a, b])$, then $f \in \mathcal{R}[a, b]$.*

ii) *If f is monotone on $[a, b]$ (or, more generally, piecewise monotone on $[a, b]$, i.e., $f\big|_{[c_{i-1}, c_i]}$ is monotone for each $i = \overline{1, n}$, where $a = c_0 < c_1 < \ldots < c_{n-1} < c_n = b$), then $f \in \mathcal{R}[a, b]$.*

Leibniz-Newton formula

Theorem

Let $f : [a,b] \to \mathbb{R}$ be a Riemann integrable function. We define $F : [a,b] \to \mathbb{R}$ by

$$F(x) = \int_a^x f(t)\,dt, \ \ x \in [a,b].$$

Then:

i) $F \in C([a,b])$*; moreover, there exists* $L > 0$ *such that*

$$|F(x) - F(\tilde{x})| \le L|x - \tilde{x}|, \ \ \forall x, \tilde{x} \in [a,b];$$

ii) *if f is continuous in some $x_0 \in [a,b]$, then F is derivable in x_0 and $F'(x_0) = f(x_0)$.*

- if $f \in C([a,b])$, then F is an antiderivative of f;
- if $f \in C([a,b])$ and $F' = f$, then the *Leibniz-Newton* formula holds:

$$\int_a^b f(x)\,dx = F(x)|_a^b := F(b) - F(a).$$

- In order to compute the Riemann integral of a function $f \in C([a, b])$, we can use the *change of variables*, by the formula

$$\int_{\alpha}^{\beta} (f \circ \varphi)(t)\varphi'(t)dt = \int_{\varphi(\alpha)}^{\varphi(\beta)} f(x)dx,$$

  if $\varphi : [\alpha, \beta] \to [a, b]$ is a C^1-function.
- A second change of variables formula, equivalent to the first one, is

$$\int_{a}^{b} f(x)dx = \int_{\psi^{-1}(a)}^{\psi^{-1}(b)} (f \circ \psi)(t)\psi'(t)dt,$$

  $\psi : [a, b] \to [\alpha, \beta]$ is a bijective, C^1-function.
- Another way of computing Riemann integrals is the *integration by parts* method, given by the formula

$$\int_{a}^{b} f(x)g'(x)\, dx = f(x)g(x)|_{a}^{b} - \int_{a}^{b} f'(x)g(x)\, dx,$$

  for $f, g : [a, b] \to \mathbb{R}$ derivable on $[a, b]$ with $f', g' \in \mathcal{R}[a, b]$ (in particular, $f, g \in C^1[a, b]$).

The uniform convergence of functions preserves the Riemann integrablity, as the following result asserts:

Proposition

Let $(f_n)_{n\in\mathbb{N}^*} \subseteq \mathcal{R}[a,b]$ be a uniformly convergent sequence of functions to $f:[a,b]\to\mathbb{R}$. Then $f\in\mathcal{R}[a,b]$ and

$$\int_a^b f(x)dx \left(=\int_a^b \lim_{n\to\infty} f_n(x)dx\right) = \lim_{n\to\infty}\int_a^b f_n(x)dx.$$

Improper integrals

A natural extension of the Riemann integral:

- the function to be integrated is unbounded or
- the interval of integration is unbounded.

Both cases can be reduced to the case where the interval of integration is not compact, when deal with the so-called *improper integrals*.
We will give the definition of improper integrals only on intervals of the form $[a, b)$ with $a \in \mathbb{R}$, $b \in \overline{\mathbb{R}}$, $a < b$, the case of intervals $(a, b]$ or (a, b) (or even the case $(a, b) \setminus \{\gamma_1, \ldots, \gamma_n\}$ with $\gamma_1, \ldots, \gamma_n \in (a, b)$) being treated in a similar manner.

Definition

Let $f : [a, b) \to \mathbb{R}$ such that f is *locally Riemann integrable* on $[a, b)$, *i.e.* $f \in \mathcal{R}[a, c]$ for any $c \in (a, b)$.

- If there exists the limit
$$I := \lim_{c \nearrow b} \int_a^c f(x)dx \in \overline{\mathbb{R}},$$
we call I the (*generalized*) *Riemann integral* of f on $[a, b)$, denoted $\int_a^{b-0} f(x)dx$ (or $(\mathcal{R})\ \int_{[a,b)} f(x)dx$). If $b = +\infty$, I can be simply denoted $\int_a^{+\infty} f(x)dx$.
- If $I \in \mathbb{R}$, we say that f is *improperly Riemann integrable* on $[a, b)$ or the integral $\int_a^b f(x)dx$ *is convergent* (shortly, $\int_a^b f(x)dx$ (C)).
- If $I \in \{-\infty, +\infty\}$ or the limit $\lim\limits_{c \nearrow b} \int_a^c f(x)dx$ does not exist, we say that the integral $\int_a^b f(x)dx$ *is divergent* (shortly, $\int_a^b f(x)dx$ (D)).

Similar notations can be established in the case of intervals $(a, b]$ or (a, b): $\int_{a-0}^{b} f(x)dx$, (or $(\mathcal{R})\ \int_{(a,b]} f(x)dx$), $\int_{-\infty}^{b} f(x)dx$, respectively $\int_{a+0}^{b-0} f(x)dx$ (or $(\mathcal{R})\ \int f(x)dx$), $\int^{b-0} f(x)dx$, $\int^{+\infty} f(x)dx$ or $\int^{+\infty} f(x)dx$

Principal value

Suppose that $f : [a, b] \setminus \{c\} \to \mathbb{R}$, $c \in (a, b)$ such that f is locally Riemann integrable function on $[a, b] \setminus \{c\}$. If the limit

$$\lim_{\varepsilon \to 0} \left(\int_a^{c-\varepsilon} f(x)dx + \int_{c+\varepsilon}^b f(x)dx \right)$$

exists, we call it the *principal value* of the integral $\int_a^b f(x)dx$. If, moreover, this limit is finite, we say that f is integrable on $[a, b]$ *in the sense of the principal value*.

This is a weaker notion than the improper integrability, since $\int_a^b f(x)dx$ (C) is equivalent to the existence of both limits $\lim_{\varepsilon \searrow 0} \int_a^{c-\varepsilon} f(x)dx$ and $\lim_{\varepsilon \searrow 0} \int_{c+\varepsilon}^b f(x)dx$.

Example

Let $p \in \mathbb{R}$. Then we have:

- $\int_{0+0}^{1} x^p dx = \lim\limits_{c \searrow 0} \int_c^1 x^p dx = \begin{cases} \lim\limits_{c \searrow 0} \left. \dfrac{x^{p+1}}{p+1} \right|_c^1, & p \neq -1; \\ \lim\limits_{c \searrow 0} \ln x|_c^1, & p = -1 \end{cases} =$

  $\begin{cases} \lim\limits_{c \searrow 0} \dfrac{1}{p+1}(1 - c^{p+1}), & p \neq -1; \\ \lim\limits_{c \searrow 0} -\ln c, & p = -1 \end{cases} = \begin{cases} \dfrac{1}{p+1}, & p > -1; \\ +\infty, & p \leq -1; \end{cases}$

- $\int_1^{+\infty} x^p dx = \lim\limits_{c \nearrow +\infty} \int_1^c x^p dx = \begin{cases} \lim\limits_{c \nearrow +\infty} \left. \dfrac{x^{p+1}}{p+1} \right|_1^c, & p \neq -1; \\ \lim\limits_{c \nearrow +\infty} \ln x|_1^c, & p = -1 \end{cases} =$

  $\begin{cases} \lim\limits_{c \nearrow +\infty} \dfrac{1}{p+1}(c^{p+1} - 1), & p \neq -1; \\ \lim\limits_{c \nearrow +\infty} \ln c, & p = -1 \end{cases} = \begin{cases} -\dfrac{1}{p+1}, & p < -1; \\ +\infty, & p \geq -1. \end{cases}$

- $\int_0^1 x^p dx$ (C) if and only if $p > -1$; $\int_1^{+\infty} x^p dx$ (C) if and only if $p < -1$.

Proposition (Cauchy's criterion of convergence)

The integral $\int_a^{b-0} f(x)dx$ *is convergent if and only if for every* $\varepsilon > 0$ *there exists* $a_\varepsilon \in (a, b)$ *such that for any* $a', a'' \in (a_\varepsilon, b)$ *we have* $\left|\int_{a'}^{a''} f(x)dx\right| < \varepsilon$.

Definition

Let $f : [a, b) \to \mathbb{R}$ such that f is locally Riemann integrable on $[a, b)$.

- If the integral $\int_a^b |f(x)|dx$ is convergent, we say that the integral $\int_a^b f(x)dx$ is *absolutely convergent*, denoting $\int_a^b f(x)dx$ (AC)
- If the integral $\int_a^b f(x)dx$ is convergent, but $\int_a^b |f(x)|dx$ is divergent, we say that $\int_a^b f(x)dx$ is *semiconvergent*.

Cauchy's criterion of convergence: if $\int_a^b f(x)dx$ (AC), then $\int_a^b f(x)dx$ (C).

Similar with the criteria of convergence for series, we have the following *comparison criterion*:

Proposition

Let $f, g : [a, b) \to \mathbb{R}$ be locally Riemann integrable functions on $[a, b)$. If $|f(x)| \leq g(x)$, $\forall x \in [a, b)$ and $\int_a^b g(x)dx$ (C), then $\int_a^b f(x)dx$ (AC).

Improper integrals on unbounded intervals.

We will consider only integrals of the form $\int_a^{+\infty} f(x)dx$ with $a \in \mathbb{R}$, since the cases $\int_{-\infty}^a f(x)dx$ and $\int_{-\infty}^{+\infty} f(x)dx$ can be reduced to this one.

Necessary criterion of integrability: Suppose that the limit $\ell = \lim\limits_{x \nearrow +\infty} f(x)$ exists.

If $\int_a^{+\infty} f(x)dx$ (C), then $\ell = 0$.

Theorem (β-criterion)

Let $\beta \in \mathbb{R}$. Suppose that there exists $\ell = \lim\limits_{x \to +\infty} x^{\beta} |f(x)|$. Then:

i) $\int_a^{+\infty} f(x) dx$ *(AC) if* $\beta > 1$ *and* $\ell < +\infty$*;*

ii) $\int_a^{+\infty} |f(x)| \, dx$ *(D) if* $\beta \leq 1$ *and* $0 < \ell$*.*

Proposition (Integral criterion of Cauchy)

If the function $f : [1, +\infty) \to \mathbb{R}_+$ is decreasing, then the improper integral $\int_1^{+\infty} f(x) \, dx$ has the same nature with the series ${}_{n=1}^{\infty} f(n)$.

Integrals of unbounded functions on bounded intervals.

Let $a, b \in \mathbb{R}$ with $a < b$.

Theorem (α-criterion)

Let $\alpha \in \mathbb{R}$ and $f : [a, b) \to \mathbb{R}$ (respectively $f : (a, b] \to \mathbb{R}$) a locally Riemann integrable function. Suppose that there exists the limit $L = \lim\limits_{x \nearrow b} \left[(b - x)^{\alpha} |f(x)|\right]$ (respectively $L = \lim\limits_{x \searrow a} \left[(x - a)^{\alpha} |f(x)|\right]$). Then:

i) $\int_a^b f(x)dx$ *(AC) if $\alpha < 1$ and $L < +\infty$;*

ii) $\int_a^b f(x)dx$ *(D) if $\alpha \geq 1$ and $L > 0$.*

Integrals with parameters

- Let $A \subseteq \mathbb{R}^k$ be a non-empty set, $a, b \in \mathbb{R}$ such that $a < b$ and $f : [a, b] \times A \to \mathbb{R}$ such that for every $\mathbf{y} \in A$, arbitrarily fixed, the function $f(\cdot, \mathbf{y})$ is Riemann integrable on $[a, b]$. We can define then $F : A \to \mathbb{R}$ by

$$F(\mathbf{y}) = \int_a^b f(x, \mathbf{y})dx, \ \mathbf{y} = (y_1, y_2, \ldots, y_k) \in A,$$

  called the *Riemann integral* of f on $[a, b]$, of *parameters* $y_1, y_2, \ldots, y_k$.
- More generally, the limits of integration can also depend on parameters: if the functions $p, q : A \to [a, b]$ are given, we can define the function $G : A \to \mathbb{R}$ by

$$G(y) = \int_{p(\mathbf{y})}^{q(\mathbf{y})} f(x, \mathbf{y})dx, \ \mathbf{y} \in A.$$

Transfer of properties by integrability

It is clear that only the existence of the limit $\lim_{\mathbf{y}\to\mathbf{y}_0} f(x,\mathbf{y})$, for every $x \in [a,b]$ in some point $\mathbf{y}_0 \in A'$ is not enough to infer the existence of $\lim_{\mathbf{y}\to\mathbf{y}_0} F(\mathbf{y})$ or, in the affirmative case, the equality

$$\lim_{\mathbf{y}\to\mathbf{y}_0} F(\mathbf{y}) \left(= \lim_{\mathbf{y}\to\mathbf{y}_0} \int_a^b f(x,\mathbf{y})dx \right) = \int_a^b \lim_{\mathbf{y}\to\mathbf{y}_0} f(x,\mathbf{y})dx.$$

A solution to this issue, again by similarity, is to demand that the limit is *uniform*.

Definition

For $\mathbf{y}_0 \in A'$, we say that the function $f : [a,b] \times A \to \mathbb{R}$ *converges* to $g : [a,b] \to \mathbb{R}$ as $\mathbf{y} \to \mathbf{y}_0$ (*i.e.*, $g(x) = \lim_{\mathbf{y}\to\mathbf{y}_0} f(x,\mathbf{y})$), *uniformly* with respect to $x \in [a,b]$, if

$$\forall \varepsilon > 0,\ \exists V_\varepsilon \in \mathcal{V}(\mathbf{y}_0),\ \forall x \in [a,b],\ \forall \mathbf{y} \in V_\varepsilon \setminus \{\mathbf{y}_0\} : |f(x,\mathbf{y}) - g(x)| < \varepsilon.$$

Proposition

If $f : [a, b] \times A \to \mathbb{R}$ is Riemann integrable on $[a, b]$ for every $\mathbf{y} \in A$ and for $\mathbf{y}_0 \in A'$, we have $\lim_{\mathbf{y} \to \mathbf{y}_0} f(x, \mathbf{y}) = g(x)$, uniformly with respect to $x \in [a, b]$, then g is Riemann integrable on $[a, b]$ and

$$\lim_{\mathbf{y} \to \mathbf{y}_0} \int_a^b f(x, \mathbf{y}) dx = \int_a^b g(x) dx \left(= \int_a^b \lim_{\mathbf{y} \to \mathbf{y}_0} f(x, \mathbf{y}) dx \right).$$

The following result concerns the transfer of continuity for the function G (the most general case):

Proposition

Suppose that $A \subseteq \mathbb{R}^k$ is a compact set, $f \in C([a, b] \times A)$ and $p, q : A \to [a, b]$ are continuous functions such that $p \leq q$. Then $G \in C(A)$. In particular, if $p \equiv a$ and $q \equiv b$, we obtain $F \in C(A)$.

In applications, the most useful transfer property is that with respect to the derivability:

Proposition

Suppose that $A = [a_1, b_1] \times \cdots \times [a_k, b_k]$ *is a compact parallelipiped in* $\mathbb{R}^k$, $f : [a, b] \times A \to \mathbb{R}$ *a continuous function on* $[a, b] \times A$ *admiting partial derivatives* $\dfrac{\partial f}{\partial y_i}$, $i = \overline{1, k}$, *continuous on* $[a, b] \times A$, *and* $p, q : A \to [a, b]$ *such that* $p \leq q$ *admit partial derivatives on* A, $\dfrac{\partial p}{\partial y_i}, \dfrac{\partial q}{\partial y_i}$, $i = \overline{1, k}$. *Then* G *(and therefore* F, *for the particular case where* p *and* q *are constants) has partial derivatives on* A *and the Leibniz formula takes place:*

$$\frac{\partial G}{\partial y_i}(\mathbf{y}) = f\left(q(\mathbf{y}), \mathbf{y}\right) \frac{\partial q}{\partial y_i}(\mathbf{y}) - f\left(p(\mathbf{y}), \mathbf{y}\right) \frac{\partial p}{\partial y_i}(\mathbf{y}) + \int_{p(\mathbf{y})}^{q(\mathbf{y})} \frac{\partial f}{\partial y_i}(x, \mathbf{y}) dx, \ \forall \mathbf{y} \in A.$$

Concerning the $\mathcal{R}$-integrability of parameter integrals, we mention the following result:

Proposition

If $A = [c, d] \subseteq \mathbb{R}$ with $c < d$ and $f \in C([a, b] \times [c, d])$, then the function $F : [c, d] \to \mathbb{R}$ (given by $F(y) = \int_a^b f(x, y)dx$, $y \in [c, d]$) is $\mathcal{R}$-integrable $[c, d]$ and

$$\int_c^d F(y)dy \left(= \int_c^d \left(\int_a^b f(x, y)\, dx \right) dy \right) = \int_a^b \left(\int_c^d f(x, y)dy \right) dx.$$

Improper integrals with parameters

When the compact domain $[a, b]$ or $[p(\mathbf{y}), q(\mathbf{y})]$ in the definition of F, respectively G, is replaced by a non-compact set, we talk about *improper integrals with parameters*.

Definition

Let $a \in \mathbb{R}$, $b \in \overline{\mathbb{R}}$ with $a < b$, $A \subseteq \mathbb{R}^k$ a non-empty set and $f : [a, b) \times A \to \mathbb{R}$ a function such that $f(\cdot, \mathbf{y})$ is Riemann integrable on each compact interval $[a, c]$ with $c < b$ for each $\mathbf{y} \in A$.

- The improper integral $\int_a^b f(x, \mathbf{y})dx$, $\mathbf{y} \in A$, is called *pointwise convergent* on A to $F : A \to \mathbb{R}$ if $\lim\limits_{c \nearrow b} \int_a^c f(x, \mathbf{y})dx = F(\mathbf{y})$, $\forall \mathbf{y} \in A$.
- We say that $\int_a^b f(x, \mathbf{y})dx$ is uniformly convergent on A to $F : A \to \mathbb{R}$ if $\lim\limits_{c \nearrow b} \int_a^c f(x, \mathbf{y})dx = F(\mathbf{y})$, uniformly with respect to $\mathbf{y} \in A$.

Let us state the result concerning the derivability transfer with respect to integrability.

Proposition

Let $a \in \mathbb{R}$, $b \in \overline{\mathbb{R}}$ with $a < b$, $c, d \in \mathbb{R}$ with $c < d$ and $f : [a, b) \times [c, d] \to \mathbb{R}$ a continuous function such that $\dfrac{\partial f}{\partial y}$ exists and is continuous on $[a, b) \times [c, d]$. Suppose that:

(i) *the improper integral $\int_a^b f(x, y)dx$ is pointwise convergent to $F(y)$, for $y \in [c, d]$;*

(ii) *the improper integral $\int_a^b \dfrac{\partial f}{\partial y}(x, y)\, dx$ is uniformly convergent with respect to $y \in [c, d]$.*

Then the function F is derivable for each $y \in [c, d]$ and

$$F'(y) = \int_a^{b-0} \frac{\partial f}{\partial y}(x, y)dx, \ \forall y \in [c, d].$$

Gamma and Beta functions

Among the improper integrals with parameters worth to be mentionned, we recall:

- the *Dirichlet integral* $\int_0^{+\infty} \frac{\sin x}{x^\alpha}$, $\alpha > 0$,
- *Euler-Poisson integral* $\int_0^{+\infty} e^{-ax^2}\, dx$, $a \in \mathbb{R}$ and
- *Euler integrals (functions)*: the *Gamma function* and the *Beta function*:

Gamma function

This function is defined as the improper integral

$$\Gamma(p) := \int_0^{+\infty} x^{p-1} \mathrm{e}^{-x} dx, \;\; p \in \mathbb{R}_+^*.$$

It is convergent for any $p \in (0, +\infty)$, from the application of β and α-criteria.

Properties of the Gamma function

1. $\Gamma(p+1) = p\Gamma(p),\ \forall p > 0;$
2. $\Gamma(1) = 1;$
3. $\Gamma(n+1) = n!,\ \forall n \in \mathbb{N};$
4. $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi};$
5. $\Gamma(p)\Gamma(1-p) = \dfrac{\pi}{\sin(p\pi)},\ \forall p \in (0,1);$
6. $\Gamma(p) = \lim\limits_{n\to\infty} \dfrac{n!n^p}{p(p+1)(p+2)\cdots(p+n)},\ \forall p > 0;$
7. $(\Gamma(p))^{-1} = pe^{\gamma p}{}_{n=1}^{\infty}\left(1+\dfrac{p}{n}\right)\mathrm{e}^{-p/n},\ \forall\, p > 0$ (*Weierstrass*), where $\gamma = 0,5772...$ is Euler's constant.

Beta function

It is defined by

$$B(p,q) := \int_0^1 x^{p-1}(1-x)^{q-1}dx,\ \ p>0,\ \ q>0$$

and satisfies the relations:

1. $B(p,q) = \int_0^{+\infty} \dfrac{t^{p-1}}{(1+t)^{p+q}}dt,\ \ \forall p,q>0;$
2. $B(p,q) = 2\int_0^{\pi/2} \sin^{2p-1}\theta \cos^{2q-1}\theta d\theta,\ \ \forall p,q>0;$
3. $B(p,q) = B(q,p),\ \ \forall p,q>0;$
4. $B(p,q) = \dfrac{\Gamma(p)\cdot\Gamma(q)}{\Gamma(p+q)},\ \ \forall p,q>0;$
5. $B(p,q+1) = \dfrac{q}{p+q}B(p,q) = \dfrac{q}{p}B(p+1,q),\ \ \forall p,q>0;$
6. $B(p,q) = B(p+1,q) + B(p,q+1),\ \ \forall p,q>0;$
7. $B(p,n+1) = \dfrac{n!}{p(p+1)\cdots(p+n)},\ \ \forall p>0,\ \ \forall n\in\mathbb{N}.$

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