LECTURE 12
INTEGRABILITY

The notion of *integral* is central to many areas of mathematics, but also to other domains that rely on mathematics. For instance, it serves to: the determination of the state of a dynamical process whose speed of evolution is known, the computation of numeric characteristics of geometric shapes (length, area, volume, position coordinates, center of mass), of physical quantities (momentum, potential or work) or numeric characteristics of random variables in probability (distribution function, mean and variance).

1. Antiderivatives

Definition. Let $I \subseteq \mathbb{R}$ be an interval with non-empty interior and $f : I \to \mathbb{R}$.

a) A function $F : I \to \mathbb{R}$ is called an *antiderivative* of f if F is derivable on I and $F'(x) = f(x)$, $\forall x \in I$.

b) If f has at least an antiderivative on I, then the set of all antiderivatives of f is called the *indefinite integral* of f and is denoted $\int f(x)dx$.

Remarks.

1. If $F : I \to \mathbb{R}$ is antiderivative of a function $f : I \to \mathbb{R}$, then any other antiderivative of f has the form $F + c$, where c is a real constant. By denoting $\mathcal{C}$ the set of all constant functions on I, we have $\int f(x)dx = F + \mathcal{C}$. By language abuse, we can write $\int f(x)dx = F(x) + c,\ \forall x \in I$.

2. If $f : I \to \mathbb{R}$ is a derivable function on I, then f is an antiderivative of f'.

3. Any antiderivative of a function $f : I \to \mathbb{R}$ is continuous, because any derivable function is continuous.

4. The space $\mathcal{P}(I)$ of all functions $f : I \to \mathbb{R}$ which admit antiderivatives is a linear space (subspace of $\mathscr{F}(I;\mathbb{R})$), because

$$\int (\alpha f(x) + \beta g(x))dx = \alpha \int f(x)dx + \beta \int g(x)dx,\ \forall \alpha, \beta \in \mathbb{R}.$$

5. Any function $f : I \to \mathbb{R}$ admiting antiderivatives has the so called *Darboux property*: for any $x_1, x_2 \in I$ and any λ between $f(x_1)$ and $f(x_2)$, there exists $\tilde{x}$ between x_1 and x_2 such that $f(\tilde{x}) = \lambda$.

In the sequel we analyse some computation methods for antiderivatives, starting with a list of usual indefinite integrals:

- $\displaystyle\int x^\alpha dx = c + \begin{cases} \dfrac{x^{\alpha+1}}{\alpha+1}, & \alpha \in \mathbb{R} \setminus \{-1\}; \\ \ln|x|, & \alpha = -1; \end{cases}$
- $\displaystyle\int \frac{dx}{x^2 - a^2} = \tfrac{1}{2a} \ln\left|\frac{x-a}{x+a}\right| + c;\quad \int \frac{dx}{x^2 + a^2} = \frac{1}{a}\operatorname{arctg}\frac{x}{a} + c,\ a \in \mathbb{R}^*;$
- $\displaystyle\int \frac{dx}{\sqrt{x^2 + a^2}} = \ln\left(x + \sqrt{x^2 + a^2}\right) + c;\quad \int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin\frac{x}{|a|} + c,\ a \in \mathbb{R}^*;$
- $\displaystyle\int a^x dx = \frac{1}{\ln a} a^x + c,\ a \in \mathbb{R}_+^* \setminus \{1\};$
- $\displaystyle\int \sin x dx = -\cos x + c;\quad \int \cos x dx = \sin x + c;$
- $\displaystyle\int \operatorname{sh} x\, dx = \int \frac{e^x - e^{-x}}{2} dx = \operatorname{ch} x + c;\quad \int \operatorname{ch} x\, dx = \int \frac{e^x + e^{-x}}{2} dx = \operatorname{sh} x + c,$

where $c \in \mathbb{R}$ and $I \subseteq \mathbb{R}$ is an interval with non-empty interior such that the functions under the integral sign are defined on I.

Integration by parts. Let $f, g : I \to \mathbb{R}$ two derivabile functions, with f' and g' continuous on I. Then

$$\int f'(x)g(x)dx = f(x)g(x) - \int f(x)g'(x)dx,\ x \in I,$$

We can apply this formula in order two complete the list of indefinite integrals:

- $\int \sqrt{a^2 - x^2} dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2} \arcsin \frac{x}{|a|} + c,\ a \in \mathbb{R}_+^*$;
- $\int \sqrt{x^2 \pm a^2} dx = \frac{x}{2}\sqrt{x^2 \pm a^2} \pm \frac{a^2}{2} \ln \left| x + \sqrt{x^2 \pm a^2} \right| + c,\ a \in \mathbb{R}^*$;
- $\int \ln x dx = x(\ln x - 1) + c$.

Integration by parts is recommended for integrals of the form

$$\int P(x) f(x) dx,$$

where $P \in \mathbb{R}[X]$ and f is an elementary function: e^x, $\ln x$, $\arcsin x$, $\arccos x$, $\operatorname{arctg} x$, $\operatorname{arcctg} x$, a^x, $\log_a x$, etc. By applying this method, one can reduce by one unit the degree of the polynomial function P.

Method of algebraic transformations. It is mostly used for computing the antiderivatives of *rational functions* of the form $f(x) = \frac{P(x)}{Q(x)}$, where $P, Q \in \mathbb{R}[X]$, defined on an interval $I \subseteq \mathbb{R}$ such that $\mathring{I} \neq \varnothing$ and $Q(x) \neq 0$ on I. It is well-known (from algebra) that f can be uniquely decomposed as a sum of "simple" rational functions

$$f(x) = \frac{P(x)}{Q(x)} = G(x) + \frac{H(x)}{Q(x)} = G(x) + \sum\nolimits_1 \frac{A_{k,m}}{(x - x_k)^m} + \sum\nolimits_2 \frac{B_{k,m} x + C_{k,m}}{(x^2 + p_k x + q_k)^m},\ x \in I,$$

where G is a polynomial function (equal to 0 when $\deg P < \deg Q$), H still a polynomial function with $\deg H < \deg Q$, $\sum_1$ is a finite sum with respect to all real roots x_k of Q and $\sum_2$ is a finite sum with respect to all complex roots of Q (with $p_k, q_k \in \mathbb{R}$ such that $p_k^2 - 4q_k < 0$). The integration of f is then reduced to computing the antiderivatives of all components of the above decomposition.

If Q has multiple roots, computing the antiderivative of $\frac{P(x)}{Q(x)}$ can be also done by *Gauss-Ostrogradski method*, based on the formula

$$(*) \qquad \int \frac{P(x)}{Q(x)} dx = \frac{P_1(x)}{Q_1(x)} + \int \frac{P_2(x)}{Q_2(x)} dx,\ x \in I,$$

where $Q_1 \in \mathbb{R}[X]$ is the greatest common divisor of Q and Q' (the derivative of Q), $Q_2 = \frac{Q}{Q_1}$, and P_1, P_2 are polynomials having the degree one unit smaller than $\deg Q_1$, respectively $\deg Q_2$. Finding P_1 and P_2 can be realized by derivating relation $(*)$, *i.e.*

$$\frac{P(x)}{Q(x)} = \frac{P_1'(x) Q_1(x) - P_1(x) Q_1'(x)}{Q_1^2(x)} + \frac{P_2(x)}{Q_2(x)},\ x \in I.$$

Method of trigonometric transformations. It is often combined with the *substitution method* and is used for computing the antiderivatives of functions expressed with the help of trigonometric functions.

For *trigonometric integrals* of the form

$$\int E(\sin x, \cos x) dx,\ x \in I = (-\pi, \pi),$$

where E is a rational function of two variabiles one can use the substitution $\operatorname{tg} \frac{x}{2} = t$. Since $\sin x = \frac{2t}{1+t^2}$, $\cos x = \frac{1-t^2}{1+t^2}$, $x = 2 \operatorname{arctg} t$, $dx = \frac{2dt}{1+t^2}$, computing the above integral is reduced to the computation of the antiderivative of a rational function in the new variable t. There are some cases in which the computations can be simplified, by avoiding the standard substitution $\operatorname{tg} \frac{x}{2} = t$:

i) if $E(-\sin x, \cos x) = -E(\sin x, \cos x)$, *i.e.* E is odd in $\sin x$, then the substitution $\cos x = t$ is recommended;
ii) if $E(\sin x, -\cos x) = -E(\sin x, \cos x)$, *i.e.* E is odd in $\cos x$, then the substitution $\sin x = t$ is recommended;
iii) if $E(-\sin x, -\cos x) = E(\sin x, \cos x)$, *i.e.* E is even in $\sin x$ and $\cos x$, then the substitution $\operatorname{tg} x = t$ is recommended.

We still apply the substitution method for computing the so-called *irrational integrals,* in order to reduce them to integrals of rational functions. We use the *Euler substitutions* for integrals of the form

$$\int E\left(x, \sqrt{ax^2 + bx + c}\right) dx,\ x \in I,$$

with $a, b, c \in \mathbb{R}$ and E a rational function of two variabiles. The change of variable is done according to each of the following case:

i) $\sqrt{ax^2+bx+c} = \pm x\sqrt{a} \pm t$, when $a > 0$;
ii) $\sqrt{ax^2+bx+c} = \pm tx \pm \sqrt{c}$, when $c > 0$;
iii) $\sqrt{ax^2+bx+c} = t(x-x_0)$, when $b^2-4ac > 0$, where x_0 is a real root of the equation $ax^2+bx+c=0$.

For irrational integrals of the form

$$\int E\left(x, \left(\frac{ax+b}{cx+d}\right)^{p_1/q_1}, \ldots, \left(\frac{ax+b}{cx+d}\right)^{p_k/q_k}\right) dx,\ x \in I,$$

where E is a rational function of $k+1$ ($k \in \mathbb{N}^*$) real variables, $a,b,c,d \in \mathbb{R}$, $a^2+b^2+c^2+d^2 \neq 0$, $cx+d \neq 0$, $\forall x \in I$, $\dfrac{ax+b}{cx+d} > 0$, $\forall x \in I, p_i \in \mathbb{Z}$, $q_i \in \mathbb{N}^*$, $\forall i = \overline{1,k}$, we use the substitution $\dfrac{ax+b}{cx+d} = t^{q_0}$, where q_0 is the least common multiple of $q_1, q_2, \ldots, q_k$.

Chebyshev substitutions are used for the calculus of *bynomial integrals*, having the form

$$\int x^p(ax^q+b)^r\, dx,\ x \in I,$$

where $a \in \mathbb{R}^*$, $b \in \mathbb{R}$ and $p,q,r \in \mathbb{Q}$. The computation of such integrals is reduced to that of the antiderivatives of irrational functions only in the following three cases:

i) $r \in \mathbb{Z}$: the substitution $x = t^m$, with m being the least common multiple of p and q;
ii) $\frac{p+1}{q} \in \mathbb{Z}$: the substitution $ax^q+b = t^l$, where l is the denominator of r.
iii) $\frac{p+1}{q} + r \in \mathbb{Z}$: the substitution $a + bx^{-q} = t^l$, l being the denominator r.

Computing integrals of the form

$$\int E\left(a^{r_1x}, a^{r_2x}, \ldots, a^{r_nx}\right)\, dx,$$

where $a \in \mathbb{R}_+^* \setminus \{1\}$, $r_1, r_2, \ldots, r_n \in \mathbb{Q}$ and E is a rational functions of n ($n \in \mathbb{N}^*$) real variables can be done by the substitution $a^x = t^\nu$, where $t > 0$ and ν is the least common multiple of the denominators of $r_1, r_2, \ldots, r_n$.

Finally, we want to emphasize that there are elementary functions that do not possess elementary antiderivatives. It is the case of *elliptic integrals*

$$\int \sqrt{(1-a^2\sin^2 x)^{\pm 1}} dx,\ a \in (0,1),$$

but also of the following integrals:

- $\displaystyle\int \frac{\sin x}{x} dx$, $\displaystyle\int \frac{\cos x}{x} dx$;
- $\displaystyle\int \frac{dx}{\ln x}$, $\displaystyle\int \frac{e^x}{x}\, dx$;
- $\displaystyle\int e^{-x^2}\, dx$ (Poisson antiderivative), $\displaystyle\int \cos(x^2)dx$, $\displaystyle\int \sin(x^2)dx$ (Fresnel antiderivatives).

2. Riemann Integral

Let $a,b \in \mathbb{R}$, $a < b$ and $f : [a,b] \to \mathbb{R}$.

Definition.

a) We call a *partition* of the interval $[a,b]$ a finite set $\Delta = \{x_0, x_1, \ldots, x_n\}$ such that $a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b$. The intervals $[x_i, x_{i+1}]$ ($i = \overline{0, n-1}$) are called *subintervals* of the partition Δ.

b) The number

$$\|\Delta\| = \max_{1 \leq i \leq n} \{x_i - x_{i-1}\}$$

(denoted also by $\nu(\Delta)$) is called the *mesh* or *norm* of the partition Δ.

c) A partition Δ of the interval $[a,b]$ is called equidistant if $x_i - x_{i-1} = \dfrac{b-a}{n}$, $\forall i = \overline{1,n}$; in this case we have $\|\Delta\| = \dfrac{b-a}{n}$ and $x_i = a + i\dfrac{b-a}{n}$, $\forall i = \overline{0,n}$.

We will denote by $\mathcal{D}[a,b]$ the set of all partitions of a compact interval $[a,b]$. Let $\Delta_1, \Delta_2 \in \mathcal{D}[a,b]$. We say that Δ_2 is *finer* than Δ_1 and we denote $\Delta_1 \preceq \Delta_2$ if $\Delta_1 \subseteq \Delta_2$.

Definition. Let $\Delta = \{x_0, x_1, \ldots, x_n\}$ with $a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b$ be a partition of $[a,b]$.

a) An n-uple $\xi_\Delta = (\xi_1, \xi_2, \ldots, \xi_n) \in \mathbb{R}^n$ is called an *intermediary point system* of Δ if $\xi_i \in [x_{i-1}, x_i]$, $\forall i = \overline{1,n}$. The set of all intermediary point systems of Δ is denoted Ξ_Δ.

b) We call the *Riemann sum* of the function $f : [a, b] \to \mathbb{R}$ with respect to Δ and an intermediary point system $\xi_\Delta = (\xi_1, \xi_2, \ldots, \xi_n)$ the number

$$\sigma_f(\Delta, \xi_\Delta) = \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}).$$

Definition. The function $f : [a, b] \to \mathbb{R}$ is called *Riemann integrable* (or *$\mathcal{R}$-integrable*) if there exists a real number I, called the *Riemann integral* of f, such that for any $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$ such that for any partition $\Delta \in \mathcal{D}[a, b]$ with $\|\Delta\| < \delta_\varepsilon$ and any $\xi_\Delta \in \Xi_\Delta$ we have $\left|\sigma_f(\Delta, \xi_\Delta) - I\right| < \varepsilon$.

The Riemann integral (which is unique) is denoted by $\displaystyle\int_a^b f(x)dx$ (or $(\mathcal{R}) \displaystyle\int_{[a,b]} f(x)dx$). The set of all $\mathcal{R}$-integrable functions on $[a, b]$ is denoted $\mathcal{R}[a, b]$.

Proposition 2.1. *If a function $f : [a, b] \to \mathbb{R}$ is Riemann integrable, then it is bounded.*

Remark. If we denote $\mathcal{B}([a, b])$ the set of all bounded functions $f : [a, b] \to \mathbb{R}$, then, by the above result, $\mathcal{R}[a, b] \subseteq \mathcal{B}([a, b])$. The inclusion is strict, however, because there exist bounded functions which are not $\mathcal{R}$-integrable. An example is the Dirichlet function, $f : [a, b] \to \mathcal{R}$, defined by $f(x) = \begin{cases} 1, & x \in [a, b] \cap \mathbb{Q}; \\ 0, & x \in [a, b] \setminus \mathbb{Q}. \end{cases}$

A necessary and sufficient condition of Riemann integrability is given by the following Cauchy condition:

Theorem 2.2 (Cauchy criterion of Riemann integrability). *The function $f : [a, b] \to \mathbb{R}$ is $\mathcal{R}$-integrable if and only if*

$$\forall \varepsilon > 0,\ \exists \delta_\varepsilon > 0,\ \forall \Delta \in \mathcal{D}[a, b],\ \forall \xi'_\Delta, \xi''_\Delta \in \Xi_\Delta : \|\Delta\| < \delta_\varepsilon \Rightarrow \left|\sigma_f(\Delta, \xi'_\Delta) - \sigma_f(\Delta, \xi''_\Delta)\right| < \varepsilon.$$

The following result displays some useful properties of $\mathcal{R}$-integrabile functions.

Proposition 2.3.

i) *If $f \in \mathcal{R}[a, b]$, then $f\big|_{[c,d]} \in \mathcal{R}[c, d]$, for any interval $[c, d] \subseteq [a, b]$.*

ii) *Let $f : [a, b] \to \mathbb{R}$ and $c \in (a, b)$. If $f\big|_{[a,c]} \in \mathcal{R}[a, c]$ and $f\big|_{[c,b]} \in \mathcal{R}[c, b]$, then $f \in \mathcal{R}[a, b]$ and*

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

iii) *If $f \in \mathcal{R}[a, b]$, then $|f| \in \mathcal{R}[a, b]$ and*

$$\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|\, dx.$$

iv) *If $f, g \in \mathcal{R}[a, b]$, then $f \cdot g \in \mathcal{R}[a, b]$ and the following* Cauchy-Schwarz inequality *for $\mathcal{R}$-integrable functions holds:*

$$\left(\int_a^b f(x)g(x)dx\right)^2 \leq \left(\int_a^b f^2(x)\, dx\right)\left(\int_a^b g^2(x)\, dx\right).$$

v) *If $f \in \mathcal{R}[a, b]$ and $|f(x)| \geq \mu > 0$, $\forall x \in [a, b]$, then $\dfrac{1}{f} \in \mathcal{R}[a, b]$.*

vi) *If $f, g \in \mathcal{R}[a, b]$ and $\alpha, \beta \in \mathbb{R}$, then $\alpha f + \beta g \in \mathcal{R}[a, b]$ and*

$$\int_a^b (\alpha f(x) + \beta g(x))\, dx = \alpha \int_a^b f(x)dx + \beta \int_a^b g(x)dx$$

(in other words, $\mathcal{R}[a, b]$ is a linear subspace of $\mathscr{F}([a, b]; \mathbb{R})$).

vii) *If $f \in \mathcal{R}[a, b]$ and $f(x) \geq 0$, $\forall x \in [a, b]$, then $\displaystyle\int_a^b f(x)dx \geq 0$.*

Remarks.

1. A generalization of Cauchy-Schwarz inequality is, similar to finite sums of real numbers, *Hölder's inequality* pentru for $\mathcal{R}$-integrable functions:

$$\left|\int_a^b f(x)g(x)dx\right| \leq \left(\int_a^b |f(x)|^p dx\right)^{\frac{1}{p}} \left(\int_a^b |g(x)|^q dx\right)^{\frac{1}{q}},$$

where $f, g \in \mathcal{R}[a, b]$, $p, q \in (1, +\infty)$, with $\frac{1}{p} + \frac{1}{q} = 1$.

2. The Riemann integral is a monotone functional, *i.e.* if $f, g \in \mathbb{R}[a,b]$ such that $f(x) \le g(x)$, $\forall x \in [a,b]$, then

$$\int_a^b f(x)dx \le \int_a^b g(x)dx.$$

3. If $f \in \mathcal{R}[a,b]$, we define $\int_b^a f(x)dx := -\int_a^b f(x)dx$ and $\int_a^a f(x)dx := 0$.

4. Let $f \in \mathcal{R}[a,b]$ and $m = \inf\limits_{x\in[a,b]} f(x) \in \mathbb{R}$, $M = \sup\limits_{x\in[a,b]} f(x) \in \mathbb{R}$. By the monotonicity of the Riemann integral, we have

$$m(b-a) \le \int_a^b f(x)dx \le M(b-a).$$

Moreover, if $f \in C([a,b])$ (*i.e.*, f is continuous on $[a,b]$), then there exists $x_1, x_2 \in [a,b]$ such that $f(x_1) = m$, $f(x_2) = M$; it follows that

$$f(x_1) \le \frac{1}{b-a}\int_a^b f(x)dx \le f(x_2)$$

Since f has the Darboux property (implied by the continuity of f), there exists c between x_1 and x_2 (with possibility of equality) such that $f(c) = \frac{1}{b-a}\int_a^b f(x)dx$, *i.e.* the following *mean equality* holds:

$$\int_a^b f(x)dx = f(c)(b-a).$$

Darboux sums.

If $f : [a,b] \to \mathbb{R}$ is a bounded function and $\Delta = \{x_0, x_1, \ldots, x_n\}$ with $a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b$ is a partition of $[a,b]$, we can define the *lower* and *upper Darboux sums* associated with Δ by

$$s_f(\Delta) := \sum_{i=1}^n m_i(x_i - x_{i-1});$$
$$S_f(\Delta) := \sum_{i=1}^n M_i(x_i - x_{i-1}),$$

where $m_i := \inf\limits_{x\in[x_{i-1},x_i]} f(x)$ and $M_i := \sup\limits_{x\in[x_{i-1},x_i]} f(x)$, $\forall i = \overline{1,n}$.

Definition. The number $\underline{I} := \sup\limits_{\Delta\in\mathcal{D}[a,b]} s_f(\Delta)$ is called the *lower Darboux integral*, while the number $\overline{I} := \inf\limits_{\Delta\in\mathcal{D}[a,b]} S_f(\Delta)$ is called the *upper Darboux integral.*

We always have $\underline{I} \le \overline{I}$. The following result sets another necessary and sufficient condition of $\mathcal{R}$-integrability:

Theorem 2.4 (Darboux criterion of Riemann integrability). *Let $f : [a,b] \to \mathbb{R}$ be a bounded function. Then f is Riemann integrable if and only if $\underline{I} = \overline{I}$, condition which is equivalent to*

$$\forall \varepsilon > 0,\ \exists \Delta_\varepsilon \in \mathcal{D}[a,b] : S_f(\Delta_\varepsilon) - s_f(\Delta_\varepsilon) < \varepsilon.$$

In this case, $\underline{I} = \overline{I} = \int_a^b f(x)dx.$

Using the Cauchy or Darboux criteria, one can highlight some important categories of functions which are Riemann integrable.

Theorem 2.5. *Let $f : [a,b] \to \mathbb{R}$ be a function.*

i) *If $f \in C([a,b])$, then $f \in \mathcal{R}[a,b]$.*

ii) *If f is monotone on $[a,b]$ (or, more generally,* piecewise monotone *on $[a,b]$, i.e., $f|_{[c_{i-1},c_i]}$ is monotone for each $i = \overline{1,n}$, where $a = c_0 < c_1 < \ldots < c_{n-1} < c_n = b$), then $f \in \mathcal{R}[a,b]$.*

Theorem 2.6. *Let $f : [a,b] \to \mathbb{R}$ be a Riemann integrable function. We define $F : [a,b] \to \mathbb{R}$ by*

$$F(x) = \int_a^x f(t)\,dt,\ x \in [a,b].$$

Then:

i) $F \in C([a,b])$; *moreover, there exists* $L > 0$ *such that*

$$|F(x) - F(\tilde{x})| \le L|x - \tilde{x}|, \ \forall x, \tilde{x} \in [a,b]$$

(i.e. F *is Lipschitz-continuous);*

ii) if f *is continuous in some* $x_0 \in [a,b]$*, then* F *is derivable in* x_0 *and* $F'(x_0) = f(x_0)$.

Remarks.

1. As a consequence of the second assertion of the theorem, if $f \in C([a,b])$, then F is an antiderivative of f.
2. Another consequence is that if $f \in C([a,b])$ and f has an antiderivative F, then the *Leibniz-Newton* formula holds:

$$\int_a^b f(x)\,dx = F(x)\big|_a^b := F(b) - F(a).$$

In order to compute the Riemann integral of a function $f \in C([a,b])$, we can use the *change of variables*, by the formula

$$\int_\alpha^\beta (f \circ \varphi)(t)\varphi'(t)dt = \int_{\varphi(\alpha)}^{\varphi(\beta)} f(x)dx,$$

if $\varphi : [\alpha, \beta] \to [a,b]$ is a C^1-function. A second change of variables formula, equivalent to the first one, is

$$\int_a^b f(x)dx = \int_{\psi^{-1}(a)}^{\psi^{-1}(b)} (f \circ \psi)(t)\psi'(t)dt,$$

$\psi : [a,b] \to [\alpha, \beta]$ is a bijective, C^1-function.

Another way of computing Riemann integrals is the *integration by parts* method, given by the formula

$$\int_a^b f(x)g'(x)\,dx = f(x)g(x)\big|_a^b - \int_a^b f'(x)g(x)\,dx,$$

for $f, g : [a,b] \to \mathbb{R}$ derivable on $[a,b]$ with $f', g' \in \mathcal{R}[a,b]$ (in particular, $f, g \in C^1[a,b]$).

The uniform convergence of functions preserves the Riemann integrablity, as the following result asserts:

Proposition 2.7. *Let* $(f_n)_{n \in \mathbb{N}^*} \subseteq \mathcal{R}[a,b]$ *be a uniformly convergent sequence of functions to* $f : [a,b] \to \mathbb{R}$*. Then* $f \in \mathcal{R}[a,b]$ *and*

$$\int_a^b f(x)dx \left(= \int_a^b \lim_{n\to\infty} f_n(x)dx \right) = \lim_{n\to\infty} \int_a^b f_n(x)dx.$$

3. Improper integrals

A natural extension of the Riemann integral is given by the cases where the function to be integrated is unbounded or the interval of integration is unbounded. Both cases can be reduced to the case where the interval of integration is not compact, when deal with the so-called *improper integrals*.

We will give the definition of improper integrals only on intervals of the form $[a,b)$ with $a \in \mathbb{R}$, $b \in \overline{\mathbb{R}}$, $a < b$, the case of intervals $(a,b]$ or (a,b) (or even the case $(a,b) \setminus \{\gamma_1, \dots, \gamma_n\}$ with $\gamma_1, \dots, \gamma_n \in (a,b)$) being treated in a similar manner.

Definition. Let $f : [a,b) \to \mathbb{R}$ such that f is *locally Riemann integrable* on $[a,b)$, *i.e.* $f \in \mathcal{R}[a,c]$ for any $c \in (a,b)$.

a) If there exists the limit

$$I := \lim_{c \nearrow b} \int_c^b f(x)dx \in \overline{\mathbb{R}},$$

we call I the (*generalized*) *Riemann integral* of f on $[a,b)$, denoted $\int_a^{b-0} f(x)dx$ (or $(\mathcal{R}) \int_{[a,b)} f(x)dx$). If $b = +\infty$, I can be simply denoted $\int_a^{+\infty} f(x)dx$.

b) If $I \in \mathbb{R}$, we say that f is *improperly Riemann integrable* on $[a,b)$ or the integral $\int_a^b f(x)dx$ *is convergent* (shortly, $\int_a^b f(x)dx$ (C)).

c) If $I \in \{-\infty, +\infty\}$ or the limit $\lim_{c \nearrow b} \int_c^b f(x)dx$ does not exist, we say that the integral $\int_a^b f(x)dx$ *is divergent* (shortly, $\int_a^b f(x)dx$ (D)).

Similar notations can be established in the case of intervals $(a,b]$ or (a,b): $\int_{a-0}^{b} f(x)dx$, (or $(\mathcal{R})$ $\int_{(a,b]} f(x)dx$), $\int_{-\infty}^{b} f(x)dx$, respectively $\int_{a+0}^{b-0} f(x)dx$ (or $(\mathcal{R})$ $\int_{(a,b)} f(x)dx$), $\int_{-\infty}^{b-0} f(x)dx$, $\int_{a+0}^{+\infty} f(x)dx$ or $\int_{-\infty}^{+\infty} f(x)dx$.

Remark. Suppose that $f : [a,b] \setminus \{c\} \to \mathbb{R}$, $c \in (a,b)$ such that f is locally Riemann integrable function on $[a,b] \setminus \{c\}$. If the limit

$$\lim_{\varepsilon \to 0} \left(\int_{a}^{c-\varepsilon} f(x)dx + \int_{c+\varepsilon}^{b} f(x)dx \right)$$

exists, we call it the *principal value* of the integral $\int_{a}^{b} f(x)dx$. If, moreover, this limit is finite, we say that f is integrable on $[a,b]$ *in the sense of the principal value.*

This is a weaker notion than the improper integrability, since $\int_{a}^{b} f(x)dx$ (C) is equivalent to the existence of both limits $\lim_{\varepsilon \searrow 0} \int_{a}^{c-\varepsilon} f(x)dx$ and $\lim_{\varepsilon \searrow 0} \int_{c+\varepsilon}^{b} f(x)dx$.

Example. Let $p \in \mathbb{R}$. Then we have:

- $$\int_{0+0}^{1} x^p dx = \lim_{c \searrow 0} \int_{c}^{1} x^p dx = \begin{cases} \lim\limits_{c \searrow 0} \dfrac{x^{p+1}}{p+1}\Big|_{c}^{1}, & p \neq -1; \\ \lim\limits_{c \searrow 0} \ln x|_{c}^{1}, & p = -1 \end{cases} = \begin{cases} \lim\limits_{c \searrow 0} \dfrac{1}{p+1}(1 - c^{p+1}), & p \neq -1; \\ \lim\limits_{c \searrow 0} -\ln c, & p = -1 \end{cases} = \begin{cases} \dfrac{1}{p+1}, & p > -1; \\ +\infty, & p \leq -1; \end{cases}$$
- $$\int_{1}^{+\infty} x^p dx = \lim_{c \nearrow +\infty} \int_{1}^{c} x^p dx = \begin{cases} \lim\limits_{c \nearrow +\infty} \dfrac{x^{p+1}}{p+1}\Big|_{1}^{c}, & p \neq -1; \\ \lim\limits_{c \nearrow +\infty} \ln x|_{1}^{c}, & p = -1 \end{cases} = \begin{cases} \lim\limits_{c \nearrow +\infty} \dfrac{1}{p+1}(c^{p+1} - 1), & p \neq -1; \\ \lim\limits_{c \nearrow +\infty} \ln c, & p = -1 \end{cases}$$
  $$= \begin{cases} -\dfrac{1}{p+1}, & p < -1; \\ +\infty, & p \geq -1. \end{cases}$$

Therefore, $\int_{0}^{1} x^p dx$ (C) if and only if $p > -1$, while $\int_{1}^{+\infty} x^p dx$ (C) if and only if $p < -1$.

Proposition 3.1 (Cauchy's criterion of convergence). *Let $f : [a,b) \to \mathbb{R}$ be a locally Riemann integrable function on $[a,b)$. Then $\int_{a}^{b-0} f(x)dx$ is convergent if and only if for every $\varepsilon > 0$ there exists $a_\varepsilon \in (a,b)$ such that for any $a', a'' \in (a_\varepsilon, b)$ we have*

$$\left| \int_{a'}^{a''} f(x)dx \right| < \varepsilon.$$

The result can be obtained by characterizing the existence (with a Cauchy condition) of the finite limit $\lim\limits_{c \nearrow b} \int_{a}^{c} f(x)dx$.

Definition. Let $f : [a,b) \to \mathbb{R}$ such that f is locally Riemann integrable on $[a,b)$.

a) If the integral $\int_{a}^{b} |f(x)|dx$ is convergent, we say that the integral $\int_{a}^{b} f(x)dx$ is *absolutely convergent*, denoting $\int_{a}^{b} f(x)dx$ (AC)

b) If the integral $\int_{a}^{b} f(x)dx$ is convergent, but $\int_{a}^{b} |f(x)|dx$ is divergent, we say that $\int_{a}^{b} f(x)dx$ is *semiconvergent.*

A consequence of Cauchy's criterion of convergence is that if $\int_{a}^{b} f(x)dx$ is absolutely convergent, then $\int_{a}^{b} f(x)dx$ is convergent.

Similar with the criteria of convergence for series, we have the following *comparison criterion*:

Proposition 3.2. *Let $f, g : [a,b) \to \mathbb{R}$ be locally Riemann integrable functions on $[a,b)$. If $|f(x)| \leq g(x)$, $\forall x \in [a,b)$ and $\int_{a}^{b} g(x)dx$ (C), then $\int_{a}^{b} f(x)dx$ (AC).*

Improper integrals on unbounded intervals.

We will consider only integrals of the form $\int_a^{+\infty} f(x)dx$ with $a \in \mathbb{R}$, since the cases $\int_{-\infty}^{a} f(x)dx$ and $\int_{-\infty}^{+\infty} f(x)dx$ can be reduced to this one.

Supposing that the limit $l = \lim_{x \nearrow +\infty} f(x)$ exists, we can infer that if $\int_a^{+\infty} f(x)dx$ (C), then for every $\varepsilon > 0$ and $\delta > 0$, there exists $a_\varepsilon, a_\delta > a$ such that for any $a' > a_\varepsilon$ we have $-\varepsilon < \int_{a'}^{a'+1} f(x)dx < \varepsilon$ (by letting $a'' = a' + 1$ in the Cauchy criterion of convergence) and $f(x) \in (l-\delta, l+\delta)$, $\forall x > a_\delta$. We obtain that a necessary condition of Riemann integrability (*attention:* only when the limit $\lim_{x\to\infty} f(x)$ exists) is that $\lim_{x\to+\infty} f(x) = 0$.

Theorem 3.3 (β-criterion). *Let $\beta \in \mathbb{R}$. Suppose that there exists $l = \lim_{x\to+\infty} x^\beta |f(x)|$. Then:*

i) $\int_a^{+\infty} f(x)dx$ *(AC) if $\beta > 1$ and $l < +\infty$;*
ii) $\int_a^{+\infty} |f(x)|\, dx$ *(D) if $\beta \leq 1$ and $0 < l$.*

Proposition 3.4 (Integral criterion of Cauchy). *If the function $f : [1, +\infty) \to \mathbb{R}_+$ is decreasing, then the improper integral* $\int_1^{+\infty} f(x)\, dx$ *has the same nature with the series* $\sum_{n=1}^{\infty} f(n)$.

Integrals of unbounded functions on bounded intervals.

Let $a, b \in \mathbb{R}$ with $a < b$.

Theorem 3.5 (α-criterion). *Let $\alpha \in \mathbb{R}$ and $f : [a, b) \to \mathbb{R}$ (respectively $f : (a, b] \to \mathbb{R}$) a locally Riemann integrable function. Suppose that there exists the limit $L = \lim_{x\nearrow b} [(b-x)^\alpha |f(x)|]$ (respectively $L = \lim_{x\searrow a} [(x-a)^\alpha |f(x)|]$). Then:*

i) $\int_a^b f(x)dx$ *(AC) if $\alpha < 1$ and $L < +\infty$;*
ii) $\int_a^b f(x)dx$ *(D) if $\alpha \geq 1$ and $L > 0$.*

4. Integrals with parameters

Let $A \subseteq \mathbb{R}^k$ be a non-empty set, a,$b \in \mathbb{R}$ such that $a < b$ and $f : [a, b] \times A \to \mathbb{R}$ such that for every $\mathbf{y} \in A$, arbitrarily fixed, the function $f(\cdot, \mathbf{y})$ is Riemann integrable on $[a, b]$. We can define then $F : A \to \mathbb{R}$ by

$$F(\mathbf{y}) = \int_a^b f(x, \mathbf{y})dx, \ \mathbf{y} = (y_1, y_2, \dots, y_k) \in A,$$

called the *Riemann integral* of f on $[a, b]$, of *parameters* $y_1, y_2, \dots, y_k$.

More generally, the limits of integration can also depend on parameters: if the functions $p, q : A \to [a, b]$ are given, we can define the function $G : A \to \mathbb{R}$ by

$$G(y) = \int_{p(\mathbf{y})}^{q(\mathbf{y})} f(x, \mathbf{y})dx, \ \mathbf{y} \in A.$$

As in the case of sequences of functions, we are interested in the transfer of integrability with respect to the parameter $\mathbf{y}$. By analogy, it is clear that only the existence of the limit $\lim_{\mathbf{y}\to\mathbf{y}_0} f(x, \mathbf{y})$, for every $x \in [a, b]$ in some point $\mathbf{y}_0 \in A'$ is not enough to infer the existence of $\lim_{\mathbf{y}\to\mathbf{y}_0} F(\mathbf{y})$ or, in the affirmative case, the equality $\lim_{\mathbf{y}\to\mathbf{y}_0} F(\mathbf{y}) \left(= \lim_{\mathbf{y}\to\mathbf{y}_0} \int_a^b f(x, \mathbf{y})dx \right) = \int_a^b \lim_{\mathbf{y}\to\mathbf{y}_0} f(x, \mathbf{y})dx$. A solution to this issue, again by similarity, is to demand that the limit is *uniform.*

Definition. For $\mathbf{y}_0 \in A'$, we say that the function $f : [a, b] \times A \to \mathbb{R}$ has the *limit* $g : [a, b] \to \mathbb{R}$ as $\mathbf{y} \to \mathbf{y}_0$ (*i.e.*, $g(x) = \lim_{\mathbf{y}\to\mathbf{y}_0} f(x, \mathbf{y})$), *uniformly* with respect to $x \in [a, b]$, if

$$\forall \varepsilon > 0, \ \exists V_\varepsilon \in \mathcal{V}(\mathbf{y}_0), \ \forall x \in [a, b], \ \forall \mathbf{y} \in V_\varepsilon \setminus \{\mathbf{y}_0\} : |f(x, \mathbf{y}) - g(x)| < \varepsilon.$$

Proposition 4.1. *If $f : [a,b] \times A \to \mathbb{R}$ is Riemann integrable on $[a,b]$ for every $\mathbf{y} \in A$ and for $\mathbf{y}_0 \in A'$, we have $\lim\limits_{\mathbf{y}\to\mathbf{y}_0} f(x,\mathbf{y}) = g(x)$, uniformly with respect to $x \in [a,b]$, then g is Riemann integrable on $[a,b]$ and*

$$\lim_{\mathbf{y}\to\mathbf{y}_0} \int_a^b f(x,\mathbf{y})dx = \int_a^b g(x)dx \left(= \int_a^b \lim_{\mathbf{y}\to\mathbf{y}_0} f(x,\mathbf{y})dx \right).$$

The following result concerns the transfer of continuity for the function G (the most general case):

Proposition 4.2. *Suppose that $A \subseteq \mathbb{R}^k$ is a compact set, $f \in C([a,b] \times A)$ and $p, q : A \to [a,b]$ are continuous functions such that $p \leq q$. Then $G \in C(A)$. In particular, if $p \equiv a$ and $q \equiv b$, we obtain $F \in C(A)$.*

In applications, the most useful transfer property is that with respect to the derivability:

Proposition 4.3. *Suppose that $A = [a_1,b_1] \times \cdots \times [a_k,b_k]$ is a compact parallelipiped in $\mathbb{R}^k$, $f : [a,b] \times A \to \mathbb{R}$ a continuous function on $[a,b] \times A$ admiting partial derivatives $\dfrac{\partial f}{\partial y_i}$, $i = \overline{1,k}$, continuous on $[a,b] \times A$, and $p, q : A \to [a,b]$ such that $p \leq q$ admit partial derivatives on A, $\dfrac{\partial p}{\partial y_i}$, $\dfrac{\partial q}{\partial y_i}$, $i = \overline{1,k}$. Then G (and therefore F, for the partciular case where p and q are constants) has partial derivatives on A and the* Leibniz formula *takes place:*

$$\frac{\partial G}{\partial y_i}(\mathbf{y}) = f\left(q(\mathbf{y}),\mathbf{y}\right) \frac{\partial q}{\partial y_i}(\mathbf{y}) - f\left(p(\mathbf{y}),\mathbf{y}\right) \frac{\partial p}{\partial y_i}(\mathbf{y}) + \int_{p(\mathbf{y})}^{q(\mathbf{y})} \frac{\partial f}{\partial y_i}(x,\mathbf{y})dx, \ \forall \mathbf{y} \in A.$$

Concerning the $\mathcal{R}$-integrability of parameter integrals, we mention the following result:

Proposition 4.4. *If $A = [c,d] \subseteq \mathbb{R}$ with $c < d$ and $f \in C\left([a,b] \times [c,d]\right)$, then the function $F : [c,d] \to \mathbb{R}$ (given by $F(y) = \displaystyle\int_a^b f(x,y)dx$, $y \in [c,d]$) is $\mathcal{R}$-integrable $[c,d]$ and*

$$\int_c^d F(y)dy \left(= \int_c^d \left(\int_a^b f(x,y)\, dx \right) dy \right) = \int_a^b \left(\int_c^d f(x,y)dy \right) dx.$$

When the compact domain $[a,b]$ or $[p(\mathbf{y}),q(\mathbf{y})]$ in the definition of F, respectively G, is replaced by a non-compact set, we talk about *improper integrals with parameters.*

Definition. Let $a \in \mathbb{R}$, $b \in \overline{\mathbb{R}}$ with $a < b$, $A \subseteq \mathbb{R}^k$ a non-empty set and $f : [a,b) \times A \to \mathbb{R}$ a function such that $f(\cdot,\mathbf{y})$ is Riemann integrable on each compact interval $[a,c]$ with $c < b$ for each $\mathbf{y} \in A$.

(1) The improper integral $\displaystyle\int_a^b f(x,\mathbf{y})dx$, $\mathbf{y} \in A$, is called *pointwise convergent* on A to $F : A \to \mathbb{R}$ if $\displaystyle\lim_{c \nearrow b} \int_a^c f(x,\mathbf{y})dx = F(\mathbf{y})$, $\forall \mathbf{y} \in A$.

(2) We say that $\displaystyle\int_a^b f(x,\mathbf{y})dx$ is uniformly convergent on A to $F : A \to \mathbb{R}$ if $\displaystyle\lim_{c \nearrow b} \int_a^c f(x,\mathbf{y})dx = F(\mathbf{y})$, uniformly with respect to $\mathbf{y} \in A$.

Let us state the result concerning the derivability transfer with respect to integrability.

Proposition 4.5. *Let $a \in \mathbb{R}$, $b \in \overline{\mathbb{R}}$ with $a < b$, $c, d \in \mathbb{R}$ with $c < d$ and $f : [a,b) \times [c,d] \to \mathbb{R}$ a continuous function such that $\dfrac{\partial f}{\partial y}$ exists and is continuous on $[a,b) \times [c,d]$. Suppose that:*

(i) *the improper integral $\displaystyle\int_a^b f(x,y)dx$ is pointwise convergent to $F(y)$, for $y \in [c,d]$;*

(ii) *the improper integral $\displaystyle\int_a^b \frac{\partial f}{\partial y}(x,y)\, dx$ is uniformly convergent with respect to $y \in [c,d]$.*

Then the function F is derivable for each $y \in [c,d]$ and

$$F'(y) = \int_a^{b-0} \frac{\partial f}{\partial y}(x,y)dx, \ \forall y \in [c,d].$$

4.1. **Gamma and Beta functions.**

Among the improper integrals with parameters worth to be mentionned, we recall the *Dirichlet integral* $\displaystyle\int_0^{+\infty} \frac{\sin x}{x^\alpha}$, $\alpha > 0$, *Euler-Poisson integral* $\displaystyle\int_0^{+\infty} e^{-ax^2}\, dx$, $a \in \mathbb{R}$ and *Euler integrals (functions)*, which we will talk about in what follows.

Gamma function

This function is defined as the improper integral

$$\Gamma(p) := \int_0^{+\infty} x^{p-1} \mathrm{e}^{-x} dx,\ p \in \mathbb{R}_+^*.$$

It is convergent for any $p \in (0, +\infty)$, from the application of β and α-criteria. We present next some properties of the Gamma function:

1. $\Gamma(p+1) = p\Gamma(p),\ \forall p > 0;$
2. $\Gamma(1) = 1;$
3. $\Gamma(n+1) = n!,\ \forall n \in \mathbb{N};$
4. $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi};$
5. $\Gamma(p)\Gamma(1-p) = \dfrac{\pi}{\sin(p\pi)},\ \forall p \in (0,1);$
6. $\Gamma(p) = \lim\limits_{n\to\infty} \dfrac{n!n^p}{p(p+1)(p+2)\cdots(p+n)},\ \forall p > 0;$
7. $(\Gamma(p))^{-1} = pe^{\gamma p} \prod\limits_{n=1}^{\infty} \left(1 + \dfrac{p}{n}\right) \mathrm{e}^{-p/n},\ \forall\, p > 0$ (*Weierstrass*), where $\gamma = 0,5772\ldots$ is Euler's constant.

Beta function

It is defined by

$$B(p,q) := \int_0^1 x^{p-1}(1-x)^{q-1} dx,\ p > 0,\ q > 0$$

and satisfies the relations:

1. $B(p,q) = \displaystyle\int_0^{+\infty} \frac{t^{p-1}}{(1+t)^{p+q}} dt,\ \forall p,q > 0;$
2. $B(p,q) = 2\displaystyle\int_0^{\pi/2} \sin^{2p-1}\theta \cos^{2q-1}\theta d\theta,\ \forall p,q > 0;$
3. $B(p,q) = B(q,p),\ \forall p,q > 0;$
4. $B(p,q) = \dfrac{\Gamma(p)\cdot\Gamma(q)}{\Gamma(p+q)},\ \forall p,q > 0;$
5. $B(p,q+1) = \dfrac{q}{p+q} B(p,q) = \dfrac{q}{p} B(p+1,q),\ \forall p,q > 0;$
6. $B(p,q) = B(p+1,q) + B(p,q+1),\ \forall p,q > 0;$
7. $B(p,n+1) = \dfrac{n!}{p(p+1)\cdots(p+n)},\ \forall p > 0,\ \forall n \in \mathbb{N}.$

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