Limits of functions. Continuous functions

Lecture 9

Mathematics - 1^{st} year, English

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Outline of the lecture

1. Limits of functions
2. Continuous functions

Limits of functions

Definition

Let (X, d), (Y, d') be metric spaces, $\emptyset \neq A \subseteq X$, $f : A \to Y$ and $x_0 \in A'$. We say that an element $\ell \in Y$ is the *limit* of f in x_0 if

$$\forall V \in \mathscr{V}_{\mathrm{d}'}(\ell),\ \exists U \in \mathscr{V}_{\mathrm{d}}(x_0),\ \forall x \in (A \cap U) \setminus \{x_0\} : f(x) \in V,$$

In this case, we write $\lim\limits_{x \to x_0} f(x) = \ell$ or $f(x) \xrightarrow{x \to x_0} \ell$.

- As in the case of limits of sequence, one can show that the limit of a function in a point, if existent, is unique.
- We say that the function f has a limit in the point x_0 if there exists $\ell \in Y$ such that $\lim\limits_{x \to x_0} f(x) = \ell$.

Characterizations

Proposition

Let (X, d), (Y, d') *be metric spaces,* $\emptyset \neq A \subseteq X$, $f : A \to Y$, $x_0 \in A'$ *and* $\ell \in Y$. *The following statements are equivalent:*

1. $\lim\limits_{x \to x_0} f(x) = \ell$;
2. *if* $\mathscr{U}(x_0)$ *and* $\mathscr{U}'(\ell)$ *are systems of neighbourhoods for* x_0, *respectively* ℓ, *then*
$$\forall V \in \mathscr{U}'(\ell),\ \exists U \in \mathscr{U}(x_0),\ \forall x \in (A \cap U) \setminus \{x_0\} : f(x) \in V;$$
3. $$\forall \varepsilon > 0,\ \exists \delta > 0,\ \forall x \in (A \cap B_{\mathrm{d}}(x_0; \delta)) \setminus \{x_0\} : \mathrm{d}'(f(x), \ell) < \varepsilon.$$

The last relation can be written

$$\forall \varepsilon > 0,\ \exists \delta > 0, \forall x \in A : 0 < \mathrm{d}(x, x_0) < \delta \Rightarrow \mathrm{d}'(f(x), \ell) < \varepsilon.$$

In the particular case that X and Y are normed spaces, we have:

Proposition

Let $(X, \|\cdot\|)$, $(Y, \|\cdot\|')$ be normed spaces, $\emptyset \neq A \subseteq X$, $f : A \to Y$. An element $\ell \in Y$ is the limit of f in $x_0 \in A'$ if and only if

$$\forall \varepsilon > 0,\ \exists \delta > 0,\ \forall x \in A : 0 < \|x - x_0\| < \delta \Rightarrow \|f(x) - \ell\|' < \varepsilon.$$

Theorem (Characterization with sequences)

Let (X, d), (Y, d') be metric spaces, $\emptyset \neq A \subseteq X$, $f : A \to Y$. An element $\ell \in Y$ is the limit of f in some point $x_0 \in A'$ if and only if for every sequence $(x_n)_{n\in\mathbb{N}^*} \subseteq A \setminus \{x_0\}$ such that $\lim\limits_{n\to\infty} x_n = x_0$, we have $\lim\limits_{n\to\infty} f(x_n) = \ell$.

Remarks.

1. For proving that $\lim\limits_{x\to x_0} f(x) \neq \ell$, it is enough to provide some sequence $(x_n)_{n\in\mathbb{N}^*} \subseteq A \setminus \{x_0\}$ converging to x_0 such that $f(x_n)$ does not converge to ℓ.

2. If, moreover, we want to show that $\lim\limits_{x\to x_0} f(x)$ doesn't exist, it is sufficient to point out two sequences $(x_n)_{n\in\mathbb{N}^*}, (x'_n)_{n\in\mathbb{N}^*} \subseteq A \setminus \{x_0\}$ such that $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} x'_n = x_0$, $\lim\limits_{n\to\infty} f(x_n) = \ell$ and $\lim\limits_{n\to\infty} f(x'_n) = \ell'$, where $\ell \neq \ell'$.

Example

Let the function $f : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}$ be defined by

$$f(x,y) := \frac{xy}{x^2+y^2},\ (x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}.$$

Then $(0,0) \in A'$, where $A := \mathbb{R}^2 \setminus \{(0,0)\}$. If we take the sequence $(x_n, y_n)_{n\in\mathbb{N}^*} \subseteq \mathbb{R}^2 \setminus \{(0,0)\}$, $x_n := \dfrac{1}{n}$, $y_n := \dfrac{1}{n}$, $n \in \mathbb{N}^*$, we have $(x_n, y_n) \overset{n\to\infty}{\longrightarrow} (0,0)$ and

$$f(x_n, y_n) = \frac{1}{2} \overset{n\to\infty}{\longrightarrow} \frac{1}{2}.$$

On the other hand, if we take the sequence, $(x_n', y_n')_{n\in\mathbb{N}^*} \subseteq \mathbb{R}^2 \setminus \{(0,0)\}$, $x_n' := \frac{1}{n}$, $y_n' := \frac{1}{n^2}$, $n \in \mathbb{N}^*$, we have $(x_n', y_n') \overset{n\to\infty}{\longrightarrow} (0,0)$ and

$$f(x_n', y_n') = \frac{\frac{1}{n^3}}{\frac{1}{n^2}+\frac{1}{n^4}} = \frac{n}{n^2+1} \overset{n\to\infty}{\longrightarrow} 0.$$

The conclusion is that the function f does not possess a limit in the point $(0,0)$.

As in the case of limits of sequence, the following series of criteria applies in the case of limits of functions.

Proposition

Let (X, d), (Y, d') be metric spaces, $\emptyset \neq A \subseteq X$, $f : A \to Y$, $g : A \to \mathbb{R}_+$ and $x_0 \in A'$, $\ell \in Y$. If

1. $\mathrm{d}'(f(x), \ell) \leq g(x), \ \forall x \in A$;
2. $\lim\limits_{x \to x_0} g(x) = 0$,

then $\lim\limits_{x \to x_0} f(x) = \ell$.

Theorem (Cauchy-Bolzano)

Let (X, d) be a metric space, (Y, d') a complete metric space, $\emptyset \neq A \subseteq X$, $x_0 \in A'$ and $f : X \to Y$. Then f has a limit in the point x_0 if and only if

$$\forall \varepsilon > 0, \ \exists \delta > 0, \ \forall x, x' \in (A \cap B(x_0, \delta)) \setminus \{x_0\} : \mathrm{d}'(f(x), f(x')) < \varepsilon.$$

Theorem

(X, d): metric space, $(Y, \|\cdot\|)$: normed space, $\emptyset \neq A \subseteq X$, $x_0 \in A'$, $f : X \to Y$.

- **i)** *If* $\lim\limits_{x \to x_0} f(x) = \ell$, *then* $\lim\limits_{x \to x_0} \|f(x)\| = \|\ell\|$.
- **ii)** *If* $\lim\limits_{x \to x_0} \|f(x)\| = 0$, *then* $\lim\limits_{x \to x_0} f(x) = \mathbf{0}_Y$.
- **iii)** *If* $\lim\limits_{x \to x_0} \|f(x)\| > 0$, *then* $\exists \delta > 0$, $\forall x \in (A \cap B(x_0, \delta)) \setminus \{x_0\} : f(x) \neq \mathbf{0}_Y$.

Theorem

Let (X, d) be a metric space, $(Y, \|\cdot\|)$ a normed space, $\emptyset \neq A \subseteq X$, $x_0 \in A'$, $f, g : X \to Y$ and $\varphi : X \to \mathbb{R}$.

- **i)** *If* $\lim\limits_{x \to x_0} f(x) = \ell_1 \in Y$ *and* $\lim\limits_{x \to x_0} g(x) = \ell_2 \in Y$, *then we have*

$$\lim_{x \to x_0} (\alpha f + \beta g)(x) = \alpha \ell_1 + \beta \ell_2, \ \forall \alpha, \beta \in \mathbb{R}.$$

- **ii)** *If* $\lim\limits_{x \to x_0} f(x) = \ell \in Y$ *and* $\lim\limits_{x \to x_0} \varphi(x) = \alpha \in \mathbb{R}$, *then we have*

$$\lim_{x \to x_0} \varphi(x) f(x) = \alpha \ell.$$

In the case of Euclidean spaces, one can compute the limits of functions on components:

Theorem

Let $\emptyset \neq A \subseteq \mathbb{R}^n$, $f : A \to \mathbb{R}^m$ *and* $\mathbf{x}_0 \in A'$. *Let* f_k, $1 \leq k \leq m$ *be the* m *components of the function* f. *Then there exists* $\lim\limits_{\mathbf{x} \to \mathbf{x}_0} f(\mathbf{x}) = \boldsymbol{\ell} \in \mathbb{R}^m$ *if and only if for every* $k = \overline{1, m}$ *there exists the limit* $\lim\limits_{\mathbf{x} \to \mathbf{x}_0} f_k(\mathbf{x}) = \ell_k \in \mathbb{R}$. *In this case,* $\boldsymbol{\ell} = (\ell_1, \ldots, \ell_m)$.

The next result shows us how we can compute limits for composed functions:

Theorem (substitution principle)

Let (X, d), (Y, d') *and* (Z, d'') *be metric spaces,* $\emptyset \neq A \subseteq X$, $\emptyset \neq B \subseteq Y$, $f : A \to B$, $g : B \to Z$ *and* $x_0 \in A'$, $y_0 \in B'$. *If*

1. $\lim\limits_{x \to x_0} f(x) = y_0$;
2. $\lim\limits_{y \to y_0} g(y) = \ell \in Z$;
3. $\exists V \in \mathscr{V}(x_0),\ \forall x \in (A \cap V) \setminus \{x_0\} : f(x) \neq y_0$,

then $\lim\limits_{y \to y_0} g(f(x)) = \ell$.

Iterate limits

A commun mistake when computing limits of functions of several variables is to *iterate* the limit:

Let $f : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}$ be defined by

$$f(x,y) := \frac{x^2y^2}{x^2y^2 + (x-y)^2}, \ (x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}.$$

Then, fixing some $y \in \mathbb{R}^*$, we have

$$\lim_{x\to 0} f(x,y) = 0.$$

Letting now $y \to 0$, we obtain the *iterated* limit

$$\lim_{y\to 0}\lim_{x\to 0} f(x,y) = 0.$$

By symmetry we get the other iterated limit

$$\lim_{x\to 0}\lim_{y\to 0} f(x,y) = 0.$$

However, f does not have a limit in $(0,0)$, since $f(\frac{1}{n}, \frac{1}{n}) = 1 \stackrel{n\to\infty}{\longrightarrow} 1$ and $f(\frac{1}{n}, 0) = 0 \stackrel{n\to\infty}{\longrightarrow} 0$.

On the other hand, a function could have a limit in some point, but not iterated limits.
Let $A := \left\{(x, y) \in \mathbb{R}^2 \mid xy \neq 0\right\}$ and $f : A \to \mathbb{R}$ defined by

$$f(x, y) := (x + y) \sin \frac{1}{x} \cdot \sin \frac{1}{y}.$$

Then $|f(x, y)| \leq g(x, y) := |x| + |y|$. Since $\lim\limits_{(x,y)\to(0,0)} g(x, y) = 0$, we have $\lim\limits_{(x,y)\to(0,0)} f(x, y) = 0$.
If we try to compute the limit $\lim\limits_{x\to 0} f(x, y)$ for some $y \in \mathbb{R}^*$, we obtain that $\lim\limits_{x\to 0} x \sin \frac{1}{x} = 0$ (because $\left|x \sin \frac{1}{x}\right| \leq |x|$, $\forall x \in \mathbb{R}^*$), but $x \mapsto \sin \frac{1}{x}$ does not have a limit in 0. Therefore, since

$$f(x, y) = \left(x \sin \frac{1}{x}\right) \sin \frac{1}{y} + \left(\sin \frac{1}{x}\right) \left(y \sin \frac{1}{y}\right),$$

$f(x, y)$ does not have a limit as $x \to 0$ if $\sin \frac{1}{y} \neq 0$, i.e. $y \neq \frac{1}{k\pi}$, $k \in \mathbb{Z}^*$.
It is clear now that the problem of existence of the iterated limit $\lim\limits_{y\to 0} \lim\limits_{x\to 0} f(x, y)$ has a negative answer.

Directional limits

Definition

Let $\emptyset \neq A \subseteq \mathbb{R}^n$, $f : A \to \mathbb{R}^m$ and $\mathbf{x}_0 \in \mathbb{R}^n$.

- We say that the function f has a *limit* in $\mathbf{x}_0$ *along the direction* $\mathbf{u} \in \mathbb{R}^n$ if

  $$0 \in \{t \geq 0 \mid \mathbf{x}_0 + t\mathbf{u} \in A\}'$$

  (i.e., there exists a sequence $t_n \searrow 0$ such that $\mathbf{x}_0 + t_n\mathbf{u} \in A$, $\forall n \in \mathbb{N}$) and there exists the limit of the function

  $$(0, +\infty) \ni t \mapsto f(\mathbf{x}_0 + t\mathbf{u}) \text{ in } t = 0,$$

  i.e. there exists the limit (to be defined later)

  $$\ell_{\mathbf{u}} := \lim_{t \searrow 0} f(\mathbf{x}_0 + t\mathbf{u}).$$

- We say that the function f has a $(k^{th}$-) *partial limit* in $\mathbf{x}_0$ if f has a limit in $\mathbf{x}_0$ along the direction $\mathbf{e}_k$, for $k \in \{1, \ldots, n\}$, where $\mathbf{e}_k = (0, \ldots, 0, \underset{\substack{\uparrow \\ k}}{1}, 0, \ldots, 0)$.

The existence of a global limit implies the existence of directional limits:

Proposition

Let $\emptyset \neq A \subseteq \mathbb{R}^n$, $f : A \to \mathbb{R}^m$ and $\mathbf{x}_0 \in A'$ such that $\lim\limits_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) = \boldsymbol{\ell} \in \mathbb{R}^m$. If $\mathbf{u} \in \mathbb{R}^n \setminus \{\mathbf{0}_{\mathbb{R}^n}\}$ is such that $0 \in \{t \geq 0 \mid \mathbf{x}_0 + t\mathbf{u} \in A\}'$, then there exists the limit of f in $\mathbf{x}_0$ along the direction $\mathbf{u}$ and is equal to $\boldsymbol{\ell}$.

- The converse of this result is not true;
- in fact, even if the limits along all the directions exist and are equal, a global limit might not exist:

Example. Let $f : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}$ be defined by $f(x,y) := \dfrac{xy^2}{x^2+y^4}$.

Let (u,v) be a direction in $\mathbb{R}^2 \setminus \{(0,0)\}$. Then, for $t > 0$,

$$f\left((0,0) + t(u,v)\right) = f(tu,tv) = \frac{t^3uv^2}{t^2(u^2+t^2v^4)} = \frac{tuv^2}{u^2+t^2v^4}.$$

Hence $\lim\limits_{t\searrow 0} f\left((0,0)+t(u,v)\right) = 0$, *i.e.* the limit of f in $(0,0)$ along the direction (u,v) exists and is equal to 0. However, f does not have a global limit in $(0,0)$ because $f\left(\frac{1}{n^2},\frac{1}{n}\right) = \frac{1}{2} \xrightarrow{n\to\infty} \frac{1}{2} \neq 0$.

Left and right limits

When f is a function of one variable, we will speak about *left* and *right* limits.

Definition

Let $A \subseteq \mathbb{R}$ be a nonempty set.

- We say that $x_0 \in \mathbb{R}$ is a *left-limit point* (*right-limit point*) of A if x is a limit point for the set $A \cap (-\infty, x_0)$ $(A \cap (x_0, +\infty))$.
- If $f : A \to \mathbb{R}^m$ is a function and x_0 is a left-limit point (right-limit point), we say that f has a *left-limit* (*right-limit*) in x_0 if there exists the limit of f in x_0 along the direction -1 (1). In this case, we will denote this limit $\lim\limits_{x \nearrow x_0} f(x)$, $f(x_0 - 0)$ or $f(x_0^-)$ $(\lim\limits_{x \searrow x_0} f(x)$, $f(x_0 + 0)$ or $f(x_0^+))$.

Proposition

Let $\emptyset \neq A \subseteq \mathbb{R}^n$, $f : A \to \mathbb{R}^m$ and x_0 be both a left-limit and a right-limit point of A. Then the limit $\lim\limits_{x \to x_0} f(x)$ exists if and only if both limits $\lim\limits_{x \nearrow x_0} f(x)$ and $\lim\limits_{x \searrow x_0} f(x)$ exist and are equal. In this case, $\lim\limits_{x \to x_0} f(x) = \lim\limits_{x \nearrow x_0} f(x) = \lim\limits_{x \searrow x_0} f(x)$.

Usual limits

$$\lim_{t\to 0}(1+t)^{1/t} = \mathrm{e}; \qquad \lim_{t\to\pm\infty}\left(1+\tfrac{1}{t}\right)^t = \mathrm{e};$$

$$\lim_{t\to 0}\frac{\log_a(1+t)}{t} = \frac{1}{\ln a}\ (a>0,\ a\neq 1); \qquad \lim_{t\to 0}\frac{\ln(1+t)}{t} = 1;$$

$$\lim_{t\to 0}\frac{a^t-1}{t} = \ln a\ (a>0); \qquad \lim_{t\to 0}\frac{\mathrm{e}^t-1}{t} = 1;$$

$$\lim_{t\to 0}\frac{(1+t)^r-1}{t} = r\ (r\in\mathbb{R});$$

$$\lim_{t\to 0}\frac{\sin t}{t} = 1; \qquad \lim_{t\to 0}\frac{\operatorname{tg} t}{t} = 1;$$

$$\lim_{t\to 0}\frac{\arcsin t}{t} = 1; \qquad \lim_{t\to 0}\frac{\operatorname{arctg} t}{t} = 1.$$

Continuous functions

Definition

Let (X, d), (Y, d') be metric spaces, $\emptyset \neq A \subseteq X$ and $f : A \to Y$.

- We say that f is *continuous* in a point $x_0 \in A$ if

$$\forall V \in \mathscr{V}_{\mathrm{d}'}(f(x_0)),\ \exists U \in \mathscr{V}_{\mathrm{d}}(x_0),\ \forall x \in U \cap A : f(x) \in V.$$

- We say that f is *discontinuous* in a point $x_0 \in A$ if f is not continuous in x_0; in this case, we also say that x_0 is a *discontinuity point* of f.
- We say that f is *continuous* if f is continuous in x_0, for every $x_0 \in A$.

Relation with limits: f is continuous in $x_0 \in A$ if and only if either x_0 is a limit point for A and $\lim_{x \to x_0} f(x) = f(x_0)$ or x_0 is an isolated point.

Characterizations

Proposition

Let (X, d), (Y, d') be metric spaces, $\emptyset \neq A \subseteq X$, $f : A \to Y$ and $x_0 \in A$. Then f is continuous in x_0 if and only if

$$\forall \varepsilon > 0,\ \exists \delta > 0,\ \forall x \in A : \mathrm{d}(x, x_0) < \delta \Rightarrow \mathrm{d}'(f(x), f(x_0)) < \varepsilon.$$

The continuity in some point can be characterized with sequences, too.

Theorem

Let (X, d), (Y, d') be metric spaces, $\emptyset \neq A \subseteq X$, $f : A \to Y$ and $x_0 \in A$. Then f is continuous in x_0 if and only if for every sequence $(x_n)_{n\in\mathbb{N}^*} \subseteq A$ such that $\lim\limits_{n\to\infty} x_n = x_0$, we have $\lim\limits_{n\to\infty} f(x_n) = f(x_0)$.

It turns out that the global continuity (*i.e.*, in all points) of a function can be characterized in terms of open or closed sets.

Theorem

Let (X, d) and (Y, d') be two metric spaces and a function $f : X \to Y$. Then the following statements are equivalents:

1. *f is continuous;*
2. *for every open set $D \subseteq Y$, the set $f^{-1}[D]$ is open (with respect to d);*
3. *for every closed set $F \subseteq Y$, the set $f^{-1}[F]$ is closed;*
4. *for every subset $A \subseteq Y$, we have $\overline{f^{-1}[A]} \subseteq f^{-1}[\overline{A}]$.*

Definition

Let (X, d), (Y, d') be metric spaces, $\emptyset \neq A \subseteq X$, $x_0 \in A'$ and $f : A \to Y$. If $\lim\limits_{x \to x_0} f(x) = \ell \in Y$, then the function $\tilde{f} : A \cup \{x_0\} \to Y$ defined by

$$\tilde{f}(x) := \begin{cases} f(x), & x \in A \setminus \{x_0\}; \\ \ell, & x = x_0 \end{cases}$$

is continuous in x_0 and is called the *extension by continuity* of f in x_0.

Definition

Let (X, d) and (Y, d') be two metric spaces.

- We say that a bijective function $f : X \to Y$ is a *homeomorphism* if f and f^{-1} are both continuous.
- We say that (X, d) and (Y, d') are *homeomorphic* if there exists a homeomorphism between them.

Remark. If f is an isometry between (X, d) and (Y, d'), then f is continuous. If, moreover, f is bijective, then f is a homeomorphism.

Stronger notions of continuity

Definition

Let (X, d) and (Y, d') be two metric spaces and $f : X \to Y$ a function.

- The function f is called *uniformly continuous* if

$$\forall \varepsilon > 0,\ \exists \delta > 0,\ \forall x, y \in X : \mathrm{d}(x, y) < \delta \Rightarrow \mathrm{d}'(f(x), f(y)) < \varepsilon.$$

- The function f is called *Lipschitz-continuous* if there exists a constant $c_1 > 0$, called the *Lipschitz constant* of f, such that

$$\mathrm{d}'(f(x), f(y)) \leq c_1 \mathrm{d}(x, y),\ \forall x, y \in X.$$

- The function f is called *Hölder-continuous* of order $\alpha \in (0, 1]$ if $\exists c_\alpha > 0$:

$$\mathrm{d}'(f(x), f(y)) \leq c_\alpha \left[\mathrm{d}(x, y)\right]^\alpha,\ \forall x, y \in X.$$

- An uniformly continuous function is continuous.
- Any Hölder-continuous function is uniformly continuous (for $\varepsilon > 0$, it is enough to set $\delta := (\varepsilon / c_\alpha)^{1/\alpha}$).

Operations with continuous functions

Theorem

Let (X, d), (Y, d') and (Z, d'') be metric spaces, $A \subseteq X$, $B \subseteq Y$ non-empty sets and the functions $f : A \to B$, $g : B \to Z$.

- **i)** *If f is continuous in some point $x_0 \in A$ and g is continuous in $y_0 := f(x_0)$, then $g \circ f$ is continuous in x_0.*
- **ii)** *If f and g are continuous, then $g \circ f$ is continuous.*

Theorem

Let (X, d) be a metric space, $(Y, \|\cdot\|)$ a normed space, $A \subseteq X$ a nonempty set and $x_0 \in A$.

- **i)** *If the functions $f, g : X \to Y$ are continuous in x_0, then $\alpha f + \beta g$ is continuous in x_0.*
- **ii)** *If the functions $f : X \to Y$ and $\varphi : X \to \mathbb{R}$ are continuous in x_0, then $\varphi \cdot f$ is continuous in x_0.*

Compactness

If f is a continuous function between metric spaces, D is an open set and F is a closed set, then $f[D]$ is not necessarely open set, nor $f[F]$ is a closed set. However, there is a property, named *compactness*, which is preserved by continuity.

Definition

Let (X, d) be a metric space. We say that a subset $K \subseteq X$ is *compact* if every sequence $(x_n)_{n\in\mathbb{N}} \subseteq K$ contains a convergent subsequence to an element of K.

By Bolzano-Weierstrass theorem, the compact subsets of $\mathbb{R}$ are the closed, bounded subsets.

Theorem

Let (X, d), (Y, d') be metric spaces, $\emptyset \neq K \subseteq X$ a compact subset and $f : K \to Y$ a continuous function. Then $f[K]$ is compact.

An immediate consequence is the following result:

Theorem (Weierstrass)

Let (X, d) be a metric space, $\emptyset \neq K \subseteq X$ a compact subset and $f : K \to \mathbb{R}$ a continuous function. Then the function f is bounded and there exist $x_0, x_1 \in K$ such that $f(x_0) := \min_{x \in K} f(x)$ and $f(x_1) := \max_{x \in K} f(x)$.

Theorem (Cantor)

Let (X, d), (Y, d') be metric spaces, $K \subseteq X$ a non-empty compact subset and $f : K \to Y$ a continuous function. Then f is uniformly continuous.

Continuous functions in Euclidean spaces

Theorem

Let $A \subseteq \mathbb{R}^n$ be a nonempty set, $f : A \to \mathbb{R}^m$ and $\mathbf{x}_0 = (x_1^0, \ldots, x_m^0) \in A$. Then f is continuous in $\mathbf{x}_0$ if and only if f_k is continuous in x_k^0 for every $k \in \{1, \ldots, m\}$.

Proposition

If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear mapping, then T is continuous.

Definition

Let $\emptyset \neq A \subseteq \mathbb{R}$ and $f : A \to \mathbb{R}^m$. We say that f is *left-continuous* (*right-continuous*) in $x_0 \in A$ if $f|_{A\cap(-\infty,x_0]}$ ($f|_{A\cap[x_0,+\infty)}$) is continuous in x_0.

Proposition

Let $A \subseteq \mathbb{R}^n$ be a nonempty set, $f : A \to \mathbb{R}^m$ and $x_0 \in A$. Then f is continuous in x_0 if and only if f is both left-continuous and right-continuous in x_0.

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