LECTURE 2
SEQUENCES OF REAL NUMBERS AND REAL FUNCTIONS. REMARKABLE INEQUALITIES IN $\mathbb{R}$

1. The set of real numbers

We will suppose that the reader is acquainted with the algebraic properties of the sets $\mathbb{N}$ (*natural numbers*), $\mathbb{Z}$ (*integers*), $\mathbb{Q}$ (*rationals*) and $\mathbb{R}$ (*real numbers*). For his sake, we remind the definition of $\mathbb{R}$:

Definition. A *set of real numbers* is a *Dedekind-complete ordered field*, *i.e.* a quadruplet $(\mathbb{R}, +, \cdot, \leq)$ where $\mathbb{R}$ is a set with at least two elements, $+$ (*addition*) and $\cdot$ (*multiplication*) are two *algebraic operations* on $\mathbb{R}$ and $\leq$ is a total order on $\mathbb{R}$ such that:

- (F_1) $x + (y + z) = (x + y) + z, \ \forall x, y, z \in \mathbb{R}$;
- (F_2) $\exists 0 \in \mathbb{R}, \ \forall x \in \mathbb{R} : x + 0 = 0 + x = x$;
- (F_3) $\forall x \in \mathbb{R}, \ \exists (-x) \in \mathbb{R} : x + (-x) = (-x) + x = 0$;
- (F_4) $x + y = y + x, \ \forall x, y \in \mathbb{R}$;
- (F_5) $(x \cdot y) \cdot z = x \cdot (y \cdot z), \ \forall x, y, z \in \mathbb{R}$;
- (F_6) $\exists 1 \in R : x \cdot 1 = 1 \cdot x = x, \ \forall x \in \mathbb{R}$;
- (F_7) $\forall x \in \mathbb{R}^*, \ \exists x^{-1} \in \mathbb{R} : x \cdot x^{-1} = x^{-1} \cdot x = 1$;[1]
- (F_8) $x \cdot y = y \cdot x, \ \forall x, y \in \mathbb{R}$;
- (F_9) $x \cdot (y + z) = x \cdot y + x \cdot z, \ \forall x, y, z \in \mathbb{R}$;
- (O_1) $x \leq y \Rightarrow x + z \leq y + z, \ \forall x, y, z \in \mathbb{R}$;
- (O_2) $(x \leq y) \wedge (0 \leq z) \Rightarrow x \cdot z \leq y \cdot z$;
- (C) the ordered set $(\mathbb{R}, \leq)$ is Dedekind-complete.

Every property of real numbers can be proved starting from the above "axioms". As usual, *substraction* and *division* can be introduced as:

$$x - y := x + (-y), \ x, y \in \mathbb{R};$$
$$\frac{x}{y} = x/y := x \cdot (y^{-1}), \ x, y \in \mathbb{R}.$$

It can be proven (but this is beyond the scope of this presentation) that there exists a unique (up to a *homeomorphism* of ordered fields) set of real numbers, so we will call $\mathbb{R}$ *the* set of real numbers (or the *real line*) and its elements *real numbers*. In fact, one can construct $\mathbb{R}$ starting with $\mathbb{N}$ (constructed in the previous lecture as ω), then continuing with $\mathbb{Z}$, $\mathbb{Q}$ and finally $\mathbb{R}$.

Reciprocally, if we already have set $\mathbb{R}$, we can define $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$ as follows:

- $\mathbb{N} := \bigcap \{N \in \mathscr{P}(\mathbb{R}) \mid 0 \in N, \ n \in N \Rightarrow n + 1 \in N, \ \forall n \in N\} = \{0, 1, 1 + 1, (1 + 1) + 1, \dots\} = \{0, 1, 2, 3, \dots\}$;
- $\mathbb{Z} := \mathbb{N} \cup \{-n \mid n \in \mathbb{N}\}$;
- $\mathbb{Q} := \{m/n \mid m \in \mathbb{Z}, \ n \in \mathbb{N}^*\}$.

We will adopt this point of view from now on.

The *absolute value* of a number x is defined as

$$|x| := \begin{cases} x, & x \geq 0; \\ -x, & x < 0. \end{cases}$$

Proposition 1.1. *We have:*

i) $|x| \geq 0, \ \forall x \in \mathbb{R}$;

ii) $|x| = 0 \Leftrightarrow x = 0, \ \forall x \in \mathbb{R}$;

iii) $|xy| = |x| \cdot |y|, \ \forall x, y \in \mathbb{R}$;

iv) $|x + y| \leq |x| + |y|, \ \forall x, y \in \mathbb{R}$.

Concerning the supremum and infimum of a non-empty subset of $\mathbb{R}$, they can be characterized as follows:

Proposition 1.2. *Let A be a non-empty subset of $\mathbb{R}$.*

i) *An element $\alpha \in \mathbb{R}$ is the supremum of A if and only if:*

(a) $x \leq \alpha, \ \forall x \in A$;

(b) $\forall \varepsilon > 0, \ \exists x_\varepsilon \in A : \alpha - \varepsilon < x_\varepsilon$.

[1] $\mathbb{R}^*$ denotes $\mathbb{R} \setminus \{0\}$; similar notations for $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$

ii) *An element $\beta \in \mathbb{R}$ is the infimum of A if and only if:*
  (a) $x \geq \beta,\ \forall x \in A;$
  (b) $\forall \varepsilon > 0,\ \exists x_\varepsilon \in A : \beta + \varepsilon > x_\varepsilon.$

Proof. We will show just the *supremum* property, the proof of the other being analoguous.

$\Rightarrow$ Since $\alpha = \sup A$, α is an upper bound for A; therefore, $x \leq \alpha,\ \forall x \in A$. On the other hand, it is the smallest upper bound; hence for some $\varepsilon > 0$, $\alpha - \varepsilon$ cannot be an upper bound for A. Consequently, the order $\leq$ being total, there exists $x_\varepsilon \in A$ such that $\alpha - \varepsilon < x_\varepsilon$.

$\Leftarrow$ Let $\alpha \in \mathbb{R}$ satisfying (a) and (b). Then, from (a), α is an upper bound for A. Suppose now that there exists another upper bound for A, $\eta \in \mathbb{R}$, such that $\alpha > \eta$. Then, taking $\varepsilon := \alpha - \eta > 0$, by (b) there exists $x_\varepsilon \in A$ such that $\alpha - \varepsilon < x_\varepsilon$, *i.e.* $\eta < x_\varepsilon$. But this contradicts the fact that η is an upper bound for A.

In conclusion, α is the least upper bound of A.

□

It is now easy to check that, for $a, b \in \mathbb{R}$ with $a < b$,
- $\inf[a, b] = \inf[a, b) = \inf(a, b] = \inf(a, b) = a$;
- $\sup[a, b] = \sup[a, b) = \sup(a, b] = \sup(a, b) = b$.

Having recalled a few algebraic and set theoretic properties of $\mathbb{R}$, let us focus on the so called "topological" properties of the *real line*.

2. Sequences of real numbers

A *sequence* of real numbers is simply a function $x : \mathbb{N} \to \mathbb{R}$. It is customary to denote x_n instead of $x(n)$, for $n \in \mathbb{N}$; also, the function x itself will be denoted $(x_n)_{n\in\mathbb{N}}$, $(x_n)_{n\geq 0}$ or even (x_n). If $p \in \mathbb{N}^*$, by $(x_n)_{n\geq p}$ we understand the sequence $(x_{n+p})_{n\in\mathbb{N}}$ or even a function $x : \{n \in \mathbb{N} \mid n \geq p\} \to \mathbb{R}$. By $\{x_n\}_{n\in\mathbb{N}}$ we denote the set $\{x_n \mid n \in \mathbb{N}\}$. Sometimes, if $A \subseteq \mathbb{R}$ is a set and $(x_n)_{n\in\mathbb{N}}$ is a sequence, we denote $(x_n)_{n\in\mathbb{N}} \subseteq A$ (abuse of language) instead of $\{x_n\}_{n\in\mathbb{N}} \subseteq A$, *i.e.* $x_n \in A$, $\forall n \in \mathbb{N}$.

Definition. We say that a sequence $(x_n)_{n\in\mathbb{N}}$ is:
a) *upper bounded* if $\{x_n\}_{n\in\mathbb{N}}$ is upper bounded, *i.e.* there exists $M \in \mathbb{R}$ such that $x_n \leq M$, $\forall n \in \mathbb{N}$;
b) *lower bounded* if $\{x_n\}_{n\in\mathbb{N}}$ is lower bounded, *i.e.* there exists $m \in \mathbb{R}$ such that $x_n \geq m$, $\forall n \in \mathbb{N}$;
c) *bounded* if $\{x_n\}_{n\in\mathbb{N}}$ is bounded, *i.e.* there exists $m, M \in \mathbb{R}$ such that $m \leq x_n \leq M$, $\forall n \in \mathbb{N}$;
d) *unbounded* if $\{x_n\}_{n\in\mathbb{N}}$ is not bounded.

For instance, the sequence $((-1)^n)_{n\geq 1}$ is bounded (since $\{(-1)^n\}_{n\geq 1} = \{-1, 1\}$), but $(2^n)_{n\in\mathbb{N}}$ is not bounded (it is lower bounded, but not upper bounded).

Definition. We say that a sequence $(x_n)_{n\in\mathbb{N}}$ is:
a) *increasing* if $x_n \leq x_{n+1},\ \forall n \in \mathbb{N}$;
b) *decreasing* if $x_n \geq x_{n+1},\ \forall n \in \mathbb{N}$;
c) *monotone* if it is increasing or decreasing;
d) *strictly increasing* if $x_n < x_{n+1},\ \forall n \in \mathbb{N}$;
e) *strictly decreasing* if $x_n > x_{n+1},\ \forall n \in \mathbb{N}$;
f) *strictly monotone* if it is strictly increasing or strictly decreasing.

Again, the sequence $((-1)^n)_{n\geq 1}$ is not monotone, but $(2^n)_{n\in\mathbb{N}}$ and $\left(\frac{1}{n}\right)_{n\geq 1}$ are (the first is strictly increasing, the second strictly decreasing).

Definition. We say that a sequence $(x_n)_{n\in\mathbb{N}}$ is *convergent* if there exists an element $x \in \mathbb{R}$, called the *limit* of the sequence $(x_n)_{n\in\mathbb{N}}$, such that for any $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that:

$$|x_n - x| < \varepsilon,\ \forall n \geq n_\varepsilon.$$

In this case we say that $(x_n)_{n\in\mathbb{N}}$ *converges* to x and we write this $x_n \underset{n\to\infty}{\longrightarrow} x$ (more simple, $x_n \to x$) or $\lim\limits_{n\to\infty} x_n = x$.

This last notation is legitimated by the following result:

Proposition 2.1. *The limit of a sequence of real numbers is unique.*

Proof. Suppose that the sequence $(x_n)_{n\in\mathbb{N}}$ has two distinct limits x and y. Let us take $\varepsilon := |x - y|/2 > 0$. Then, by the definition, there exist some $n_\varepsilon, m_\varepsilon \in \mathbb{N}$ such that:

$$|x_n - x| < \varepsilon,\ \forall n \geq n_\varepsilon;$$
$$|x_n - y| < \varepsilon,\ \forall n \geq m_\varepsilon.$$

If $n \geq \max\{n_\varepsilon, m_\varepsilon\}$, then

$$|x-y| \leq |x-x_n| + |x_n - y| < 2\varepsilon = |x-y|,$$

which is, of course, absurd. Therefore we cannot have $x \neq y$. □

Proposition 2.2. *Any convergent sequence is bounded.*

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence converging to $x \in \mathbb{R}$. Then, letting $\varepsilon := 1$, there exists $n_1 \in \mathbb{N}$ such that $|x_n - x| < 1$, $\forall n \geq n_1$. This implies that

$$|x_n| \leq |x_n - x| + |x| < 1 + |x|, \ \forall n \geq n_1.$$

Letting $M := \max\{|x_1|, \dots, |x_{n_1-1}|, 1+|x|\}$, we can see that

$$|x_n| \leq M, \ \forall n \in \mathbb{N}.$$

The sequence is therefore bounded. □

Examples.

- Perhaps the most used example is the sequence $\left(\frac{1}{n}\right)_{n\geq 1}$. We have

  $$\lim_{n\to\infty} \frac{1}{n} = 0,$$

  because for every $\varepsilon > 0$ there exists $n_\varepsilon := \left\lfloor \frac{1}{\varepsilon} \right\rfloor + 1$ such[2] that $n \geq n_\varepsilon$ implies

  $$-\varepsilon < 0 < \frac{1}{n} \leq \frac{1}{n_\varepsilon} < \varepsilon.$$

- The constant sequence $(c)_{n\in\mathbb{N}}$ (for $c \in \mathbb{R}$) is increasing and decreasing in the same time; it is convergent to c.
- The sequence $(a^n)_{n\geq 1}$ is convergent for $a \in (-1, 1]$ and divergent for $a \in \mathbb{R} \setminus (-1,1]$; we have

  $$\lim_{n\to\infty} a^n = \begin{cases} 0, & -1 < a < 1; \\ 1, & a = 1. \end{cases}$$

- Another well known limit is

  $$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = \mathrm{e},$$

  where e is *Euler*'s number (in fact, the sequence $\left(\left(1+\frac{1}{n}\right)^n\right)_{n\geq 1}$ is increasing and bounded, so e can be defined as the limit of this sequence – see Theorem ?? below).

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of real numbers. A *subsequence* of $(x_n)_{n\in\mathbb{N}}$ is a sequence $(x_{n_k})_{k\in\mathbb{N}}$ where $(n_k)_{k\in\mathbb{N}} \subseteq \mathbb{N}$ is a strictly increasing sequence of natural numbers.

Proposition 2.3. *Let $(x_n)_{n\in\mathbb{N}}$ be a sequence converging to $x \in \mathbb{R}$. If $(x_{n_k})_{k\in\mathbb{N}}$ is a subsequence of $(x_n)_{n\in\mathbb{N}}$, then $(x_{n_k})_{k\in\mathbb{N}}$ converges to x.*

Proof. Let $\varepsilon > 0$. Since $x_n \to x$, there exists $n_\varepsilon \in \mathbb{N}$ such that $|x_n - x| < \varepsilon$, $\forall n \geq n_\varepsilon$. On the other hand, the sequence $(n_k)_{k\in\mathbb{N}}$ is strictly increasing, so there exists k_ε such that

$$n_k \geq n_\varepsilon, \ \forall k \geq k_\varepsilon$$

(otherwise, we would have $\{n_k\}_{k\in\mathbb{N}} \subseteq \{0, 1, \dots, n_\varepsilon - 1\}$). Therefore,

$$|x_{n_k} - x| < \varepsilon, \ \forall k \geq k_\varepsilon.$$

This proves that $\lim_{k\to\infty} x_{n_k} = x$. □

Therefore, a simple way for proving that a sequence $(x_n)_{n\in\mathbb{N}}$ does *not* converge is to put our hands on two subsequences $(x_{n_k})_{k\in\mathbb{N}}$ and $(x_{m_k})_{k\in\mathbb{N}}$ *converging* to different limits. For instance, the sequence $((-1)^n)_{n\geq 1}$ is not convergent, because the subsequence $\left((-1)^{2k}\right)_{k\geq 1}$ is the constant sequence converging to 1, while the subsequence $\left((-1)^{2k+1}\right)_{k\geq 0}$ is the constant sequence converging to -1.

Concerning how operations on $\mathbb{R}$ and sequences relate, we have the following properties:

Proposition 2.4. *Let (x_n) and (y_n) be sequences of real numbers, convergent to $x \in \mathbb{R}$, repectively to $y \in \mathbb{R}$. Then:*

i) $\lim_{n\to\infty}(x_n + y_n) = x + y$;
ii) $\lim_{n\to\infty}(x_n y_n) = xy$;
iii) $\lim_{n\to\infty}(x_n - y_n) = x - y$;

[2] for $x \in \mathbb{R}$, $\lfloor x \rfloor := \sup\{n \in \mathbb{Z} \mid n \leq x\}$ is called the *floor* or the *integer part* of x

iv) *if* $y \neq 0$, *then there exists* $n_0 \in \mathbb{N}$ *such that* $y_n \neq 0,\ \forall n \geq n_0$ *and* $\lim_{n\to\infty}\left(\frac{x_n}{y_n}\right) = \frac{x}{y}$;

v) *if* $x_n \leq y_n,\ \forall n \in \mathbb{N}$, *then* $x \leq y$;

vi) (squeeze theorem) *if the sequence* (z_n) *is such that* $x_n \leq z_n \leq y_n,\ \forall n \in \mathbb{N}$ *and* $x = y$, *then* (z_n) *is convergent and* $\lim_{n\to\infty} z_n = x$.

A simple consequence of the last property is the following criterium of convergence:

Proposition 2.5. *Let* $(x_n)_{n\in\mathbb{N}}$ *be a sequence of real numbers and* $x \in \mathbb{R}$. *If there exists a sequence* $(\alpha_n)_{n\in\mathbb{N}}$ *converging to* 0 *such that*

$$|x_n - x| \leq \alpha_n,\ \forall n \in \mathbb{N},$$

then (x_n) *converges to* x.

The following result is also known as the *Monotone convergence theorem.*

Theorem 2.6 (Weierstrass). *Let* $(x_n)_{n\in\mathbb{N}}$ *be a sequence of real numbers.*

i) *If* $(x_n)_{n\in\mathbb{N}}$ *is increasing and upper bounded, then it converges to* $\sup\{x_n\}_{n\in\mathbb{N}}$.

ii) *If* $(x_n)_{n\in\mathbb{N}}$ *is decreasing and lower bounded, then it converges to* $\inf\{x_n\}_{n\in\mathbb{N}}$.

Proof. We will prove only the first part; we can treat the second in the same manner.

Since the non-empty set $\{x_n\}_{n\in\mathbb{N}}$ is upper bounded, it has a least upper bound; let $\alpha := \sup\{x_n\}_{n\in\mathbb{N}}$. By Proposition ??,

$$x_n \leq \alpha,\ \forall n \in \mathbb{N}$$

and

$$\forall \varepsilon > 0,\ \exists n_\varepsilon \in \mathbb{N} : \alpha - \varepsilon < x_{n_\varepsilon}.$$

Since (x_n) is increasing, we have that $x_n \geq x_{n_\varepsilon},\ \forall n \geq n_\varepsilon$. Combining the two inequalities, we get for $\varepsilon > 0$,

$$\alpha - \varepsilon < x_n,\ \forall n \geq n_\varepsilon.$$

Therefore,

$$|x_n - \alpha| = \alpha - x_n < \varepsilon,\ \forall n \geq n_\varepsilon.$$

This relation, ε being choosen in an arbitrary way, expresses the fact that (x_n) is convergent to α. □

Lema 2.7. *Any sequence* $(x_n)_{n\in\mathbb{N}}$ *of real numbers has a monotone subsequence.*

Proof. For $n \in \mathbb{N}$, let the set $A_n \subseteq \mathbb{N}$ be defined by

$$A_n := \{k > n \mid x_k \geq x_n\}.$$

I. Suppose first that there exists some $n_0 \in \mathbb{N}$ such that A_n is an infinite set for every $n \geq n_0$. Then, starting with $n_0 := 0$, one can construct a strictly increasing sequence $(n_k)_{k\in\mathbb{N}}$ such that $n_{k+1} \in A_{n_k},\ \forall k \in \mathbb{N}$. This clearly implies that the sequence $(x_{n_k})_{k\in\mathbb{N}}$ is increasing.

II. The other possibility is that the set

$$A := \{n \in \mathbb{N} \mid A_n \text{ is finite}\}$$

is infinite. Let $(n_k)_{k\in\mathbb{N}}$ be a strictly increasing sequence such that $A = \{n_k\}_{k\in\mathbb{N}}$ (such a sequence can be constructed based on the existence of a bijective function between $\mathbb{N}$ and A).

Therefore, for every $k \in \mathbb{N}$, A_{n_k} is finite, which implies that for any $k \in \mathbb{N}$ we can choose $k' > k$ such that $x_{n_{k'}} < x_{n_k}$ (because $A \cap \{n \geq n_k \mid x_n < x_{n_k}\}$ is infinite, hence non-empty).

With the aid of this property we can easily construct a strictly decreasing subsequence $\left(x_{n_{k_l}}\right)_{l\in\mathbb{N}}$ of $(x_{n_k})_{k\in\mathbb{N}}$. But $\left(x_{n_{k_l}}\right)_{l\in\mathbb{N}}$ is also a subsequence of $(x_n)_{n\in\mathbb{N}}$; this ends the proof. □

The following result is now an easy consequence of Theorem ?? and of the above Lemma.

Theorem 2.8 (Bolzano–Weierstrass). *If a sequence* $(x_n)_{n\in\mathbb{N}}$ *is bounded, then it possesses a convergent subsequence* $(x_{n_k})_{k\in\mathbb{N}}$.

Definition. We say that a sequence $(x_n)_{n\in\mathbb{N}}$ is *Cauchy* (or *fundamental*) if for any $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that:

$$|x_n - x_m| < \varepsilon,\ \forall n, m \geq n_\varepsilon.$$

The following result is an important criterium for determining if a sequence is convergent, without even guessing its supposed limit.

Theorem 2.9. *A sequence of real numbers is convergent if and only if it is Cauchy.*

The "if" part of this result, applied to general spaces where convergence can be defined, is the definition of a *complete space*; thus this theorem tells us that $\mathbb{R}$ is a complete space.

Proof.

- $\Rightarrow$ If $(x_n)_{n\in\mathbb{N}}$ is a sequence convergent to some $x \in \mathbb{R}$, then, according to the definition of convergence, for any $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that
$$|x_n - x| < \varepsilon/2, \ \forall n \geq n_\varepsilon.$$
Hence, if $m, n \geq n_\varepsilon$, we have
$$|x_m - x_n| \leq |x_m - x| + |x - x_n| < \varepsilon.$$
This proves that (x_n) is a Cauchy sequence.
- $\Leftarrow$ Suppose that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Then $(x_n)_{n\in\mathbb{N}}$ is bounded (the proof goes approximatively the same as for ??: if $\varepsilon := 1$, then $|x_n| \leq |x_{n_1}| + |x_{n_1} - x_n| < 1 + |x_{n_1}|$, $\forall n \geq n_1$). By Bolzano-Weierstrass theorem, it has a convergent subsequence $(x_{n_k})_{k\in\mathbb{N}}$. Let $x \in \mathbb{R}$ be the limit of this subsequence. Our aim is to show that the whole sequence $(x_n)_{n\in\mathbb{N}}$ converges to x.
Let now $\varepsilon > 0$. Since $x_{n_k} \to x$, there exists some $k_\varepsilon \in \mathbb{N}$ such that
$$|x_{n_k} - x| < \varepsilon/2, \ \forall k \geq k_\varepsilon.$$
On the other hand, since (x_n) is Cauchy, there exists some $n_\varepsilon \in \mathbb{N}$ such that
$$|x_m - x_n| < \varepsilon/2, \ \forall m, n \geq n_\varepsilon.$$
Let $n'_\varepsilon := \max\{n_\varepsilon, n_{k_\varepsilon}\}$ and k'_ε such that $n_{k'_\varepsilon} \geq n'_\varepsilon$. If $n \geq n_{k'_\varepsilon}$, then
$$|x_n - x| \leq \left|x_n - x_{n_{k'_\varepsilon}}\right| + \left|x_{n_{k'_\varepsilon}} - x\right| < \varepsilon,$$
because $n \geq n_{k'_\varepsilon} \geq n_\varepsilon$ and $k'_\varepsilon \geq k_\varepsilon$.

□

3. The extended real line. Limit points

Obviously, not every subset of $\mathbb{R}$ posses a supremum or an infimum (for instance, $\mathbb{R}$ itself); a necessary condition is to be bounded. If we want to get rid of this restriction, we should add the "bounds" of $\mathbb{R}$: an upper bound and a lower bound.

Let $+\infty$ and $-\infty$ be two distinct points, neither elements of $\mathbb{R}$ ($-\infty, +\infty \notin \mathbb{R}$), which we call "plus infinity", respectively "minus infinity". We denote $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$, the *extended real line*. We extend the natural order $\leq$ on $\mathbb{R}$ to $\overline{\mathbb{R}}$ in the following manner:

- $-\infty \leq +\infty$;
- $-\infty \leq x, \ x \leq +\infty, \ \forall x \in \mathbb{R}$ (in fact, the inequalities are strict, since the related elements are distinct).

With this extended order, every subset of $\mathbb{R}$ has a supremum and an infimum. In fact, for $A \subseteq \mathbb{R}$, we have:

- $\sup A = +\infty$ if and only if A is not upper bounded;
- $\inf A = -\infty$ if and only if A is not lower bounded;
- $\sup A = -\infty \Leftrightarrow \inf A = +\infty \Leftrightarrow A = \varnothing$.

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of real numbers. We write $\lim\limits_{n\to\infty} x_n = +\infty$ ($x_n \to +\infty$) or $\lim\limits_{n\to\infty} x_n = -\infty$ ($x_n \to -\infty$) if for every $a \in \mathbb{R}$, there exists $n_a \in \mathbb{N}$ such that:
$$x_n > a, \ \forall n \geq n_a,$$
respectively
$$x_n < a, \ \forall n \geq n_a.$$

Proposition 3.1. *Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of real numbers.*

i) *If $(x_n)_{n\in\mathbb{N}}$ is increasing and unbounded, then $\lim\limits_{n\to\infty} x_n = +\infty$.*

ii) *If $(x_n)_{n\in\mathbb{N}}$ is decreasing and unbounded, then $\lim\limits_{n\to\infty} x_n = -\infty$.*

Therefore, any monotone sequence in $\mathbb{R}$ has a limit in $\overline{\mathbb{R}}$.

Concerning the usual operations, they can be extended partially to $\overline{\mathbb{R}}$ in order to obtain an analoguous result to Proposition ?? (exercice: state and prove the result!). In this regard, we introduce:

- $(-\infty) + a = a + (-\infty) := -\infty$, for $-\infty \leq a < +\infty$; $(+\infty) + a = a + (+\infty) := +\infty$, for $-\infty < a \leq +\infty$;
- $(-\infty) \cdot a = a \cdot (-\infty) := -\infty$, $(+\infty) \cdot a = a \cdot (+\infty) := +\infty$, for $0 < a \leq +\infty$;
  $(-\infty) \cdot a = a \cdot (-\infty) := +\infty$, $(+\infty) \cdot a = a \cdot (+\infty) := -\infty$, for $-\infty \leq a < 0$;
- $-(-\infty) := +\infty$, $-(+\infty) := -\infty$, $1/(-\infty) = 1/(+\infty) = 0$.

The operations $(-\infty) + (+\infty)$, $(+\infty) + (-\infty)$, $(-\infty) - (+\infty)$, $(+\infty) - (-\infty)$, $0 \cdot (-\infty)$, $0 \cdot (+\infty)$, $\dfrac{\pm\infty}{\pm\infty}$, $\dfrac{\pm\infty}{0}$ (partially) remain not defined.

Definition. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of real numbers.

a) We call $x \in \overline{\mathbb{R}}$ a *limit point* of the sequence $(x_n)_{n\in\mathbb{N}}$ if there exists a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that $x_{n_k} \to x$.

b) The set of the limit points of the sequence $(x_n)_{n\in\mathbb{N}}$ is denoted $\mathrm{L}_{(x_n)}$.

An immediate application of Lemma ?? shows that $\mathrm{L}_{(x_n)} \neq \varnothing$ for any sequence $(x_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$.

Definition. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of real numbers.

a) We call the *inferior limit* of $(x_n)_{n\in\mathbb{N}}$ the number (in $\overline{\mathbb{R}}$):

$$\liminf_{n\to\infty} x_n = \varliminf_{n\to\infty} x_n := \inf \mathrm{L}_{(x_n)} .$$

b) We call the *superior limit* of $(x_n)_{n\in\mathbb{N}}$ the number (in $\overline{\mathbb{R}}$):

$$\limsup_{n\to\infty} x_n = \varlimsup_{n\to\infty} x_n := \sup \mathrm{L}_{(x_n)} .$$

Remarks. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of real numbers.

- We have
$$\liminf_{n\to\infty} x_n \leqslant \limsup_{n\to\infty} x_n .$$
- If $x \in \overline{\mathbb{R}}$, then $\lim\limits_{n\to\infty} x_n = x$ if and only if $\mathrm{L}_{(x_n)} = \{x\}$, *i.e.*
$$\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n = x.$$
- It can be shown (see the proof of Lemma ??) that there exists a monotone subsequence of $(x_n)_{n\in\mathbb{N}}$ which has the limit $\varliminf\limits_{n\to\infty} x_n$ and there exists a monotone subsequence of $(x_n)_{n\in\mathbb{N}}$ which has the limit $\varlimsup\limits_{n\to\infty} x_n$.

4. Sequences of functions

Sometimes, the terms of a sequence depend on a *parameter*, as for instance, the sequence $\left(\left(1+\frac{x}{n}\right)^n\right)_{n\geq 1}$, where $x \in \mathbb{R}$ (we already know that this sequence converges to e^x). We can regard such a sequence as a sequence of functions (in this example, the functions would be $x \mapsto \left(1+\frac{x}{n}\right)^n$).

In general, let E be a set and, for every $n \in \mathbb{N}$, $f_n : E \to \mathbb{R}$ be a function. By analogy with sequences of real numbers, we say that the function $F : \mathbb{N} \to \mathscr{F}(E;\mathbb{R})$ defined by $F(n) := f_n$, is a *sequence of functions.* Also, we denote $(f_n)_{n\in\mathbb{N}}$ instead F.

In most applications, E will be a subset of $\mathbb{R}$ (or, later during the course, of $\mathbb{R}^m$).

In contrast with sequences of real numbers, we will have here several types of convergence:

Definition. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions from a set E to $\mathbb{R}$ and $f : E \to \mathbb{R}$. We say that:

a) $(f_n)_{n\in\mathbb{N}}$ *converges pointwise* to f if $f_n(x) \to f(x)$, $\forall x \in E$ (we note $f_n \xrightarrow{\mathrm{p}} f$ or $f_n \xrightarrow[E]{\mathrm{p}} f$, if we want to specify on which set the pointwise convergence holds);

b) $(f_n)_{n\in\mathbb{N}}$ *converges uniformly* to f if for any $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that:

$$|f_n(x) - f(x)| < \varepsilon,\ \forall n \geq n_\varepsilon,\ \forall x \in E.$$

In this case we note $f_n \xrightarrow{\mathrm{u}} f$ or $f_n \xrightarrow[E]{\mathrm{u}} f$.

Of course, if $(f_n)_{n\in\mathbb{N}}$ converges uniformly to f, it will also converge pointwise to f.

Also, $f_n \xrightarrow{\mathrm{u}} f$ if and only if $\sup_{x\in E} |f_n(x) - f(x)| \in \mathbb{R}$ for n sufficiently large and

$$\lim_{n\to\infty} \sup_{x\in E} |f_n(x) - f(x)| = 0. \tag{*}$$

Example. Let $f_n : [0,1] \to \mathbb{R}$ be defined as $f_n(x) := x^n$, for $n \geq 1$. It is clear that

$$\lim_{n\to\infty} x^n = \begin{cases} 0, & x \in [0,1); \\ 1, & x = 1. \end{cases}$$

Therefore, $(f_n)_{n\in\mathbb{N}}$ converges pointwise to f, where $f : [0,1] \to \mathbb{R}$ is defined as $f(x) := \begin{cases} 0, & x \in [0,1); \\ 1, & x = 1. \end{cases}$

On the other hand,

$$\sup_{x\in[0,1]} |f_n(x) - f(x)| = \sup_{x\in[0,1)} x^n = 1,$$

hence $\sup_{x\in[0,1]} |f_n(x) - f(x)| \not\to 0$. By (*), this means suppose that $(f_n)_{n\in\mathbb{N}}$ doesn't converge uniformly to f.

The following results assert some criteria for the uniform convergence of functions:

Proposition 4.1. *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions from a set E to $\mathbb{R}$ and $f : E \to \mathbb{R}$. If there exists a sequence $(\alpha_n)_{n\in\mathbb{N}}$ converging to* 0 *such that*

$$|f_n(x) - f(x)| \leq \alpha_n,\ \forall n \in \mathbb{N},\ \forall x \in E$$

then (f_n) converges uniformly to f.

Theorem 4.2. *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions from a set E to $\mathbb{R}$. Then there exists a function $f : E \to \mathbb{R}$ such that $f_n \xrightarrow{\text{u}} f$ if and only if $(f_n)_{n\in\mathbb{N}}$ is a uniform Cauchy sequence, i.e. for any $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that:*

$$|f_m(x) - f_n(x)| < \varepsilon,\ \forall m, n \geq n_\varepsilon,\ \forall x \in E.$$

As we will see later, uniform convergence is closed to (keeps) properties as boundedness, continuity or integrability.

5. Remarkable inequalities

Proposition 5.1 (Hölder inequality). *Let $n \in \mathbb{N}^*$, $a_1, \ldots, a_n$, $b_1, \ldots, b_n \in \mathbb{R}_+$ and $p, q \in \mathbb{R}_+^*$ [3]such that $\frac{1}{p} + \frac{1}{q} = 1$. Then:*

$$\sum_{i=1}^{n} a_i b_i \leq \left(\sum_{i=0}^{n} a_i^p\right)^{\frac{1}{p}} \cdot \left(\sum_{i=0}^{n} b_i^q\right)^{\frac{1}{q}}.$$

It is easy to prove (exercice!) a variant of this inequality, the *weighted Hölder inequality*:

$$\sum_{i=1}^{n} \lambda_i a_i b_i \leq \left(\sum_{i=1}^{n} \lambda_i a_i^p\right)^{\frac{1}{p}} \cdot \left(\sum_{i=1}^{n} \lambda_i b_i^q\right)^{\frac{1}{q}},$$

where $n \in \mathbb{N}^*$, $\lambda_1, \ldots \lambda_n$, $a_1, \ldots, a_n$, $b_1, \ldots, b_n \in \mathbb{R}_+$ and $p, q \in \mathbb{R}_+^*$ such that $\frac{1}{p} + \frac{1}{q} = 1$.

In the case $p = q = 2$ we obtain the *Cauchy-Buniakowski-Schwarz inequality*:

$$\sum_{i=1}^{n} a_i b_i \leq \left(\sum_{i=1}^{n} a_i^2\right)^{\frac{1}{2}} \cdot \left(\sum_{i=1}^{n} b_i^2\right)^{\frac{1}{2}}.$$

The equality holds if and only if there exist $u, v \in \mathbb{R}$ with $u^2 + v^2 \neq 0$, such that $ua_i + vb_i = 0$, $\forall\ i \in \{1, 2, \ldots n\}$.

Proposition 5.2 (Minkowski inequality). *Let $n \in \mathbb{N}^*$, $a_1, \ldots, a_n$, $b_1, \ldots, b_n \in \mathbb{R}_+^*$ and $p \in \mathbb{R}_+^*$.*

i) *If $p \geq 1$, then*

$$\left(\sum_{i=1}^{n} (a_i + b_i)^p\right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} b_i^p\right)^{\frac{1}{p}}.$$

ii) *If $0 < p < 1$, then*

$$\left(\sum_{i=1}^{n} (a_i + b_i)^p\right)^{\frac{1}{p}} \geq \left(\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} b_i^p\right)^{\frac{1}{p}}.$$

In both cases, if $p \neq 1$, the equality holds if the n-tuples $(a_0, a_1, \ldots, a_n)$ and $(b_0, b_1, \ldots, b_n)$ are proportional.

Proposition 5.3 (Carleman inequality). *For any $n \in \mathbb{N}^*$ and $a_1, a_2, \ldots, a_n \in \mathbb{R}_+$ it holds*

$$\sum_{k=1}^{n} (a_1 a_2 \ldots a_k)^{\frac{1}{k}} \leq \mathrm{e} \sum_{k=1}^{n} a_k.$$

The equality holds if and only if $a_1 = a_2 = \ldots = a_n = 0$.

Selective bibliography

[1] D. Buşneag, D. Piciu, *Lecţii de algebră*, Ed. Universitaria, Craiova, 2002.
[2] C. G. Denlinger, *Elements of Real Analysis*, International Series in Mathematics, Jones and Bartlett Publishers International, London, 2012.
[3] M. O. Drâmbe, *Inegalităţi. Idei şi metode.*, Ed. GIL, Zalău, 2003.
[4] R. Luca-Tudorache, *Analiză matematică*, Editura Tehnopress, Iaşi, 2005.
[5] G. Păltineanu, *Analiză matematică. Calcul diferenţial*, Editura AGIR, Bucureşti, 2002.
[6] M. Postolache, *Analiză matematică (teorie şi aplicaţii)*, Ed. Fair Partners, Bucureşti, 2011.
[7] A. Precupanu, *Bazele analizei matematice*, Editura Polirom, Iaşi, 1998.

[3] recall that $\mathbb{R}_+ := [0, +\infty)$ and $\mathbb{R}_+^* := \mathbb{R}_+ \setminus \{0\} = (0, +\infty)$