Outline of the lecture

1. Definition and properties
   - Examples
   - Cauchy test
   - Operations with series

2. Series with positive terms
   - Comparison criteria
   - Cauchy's criterion of condensation
   - Root test of Cauchy
   - Kummer criterion
   - Gauss criterion

Series of real numbers

What is an *infinite sum*

$$x_1 + \cdots + x_n + \dots \quad ?$$

A natural approach is to study the behaviour of the partial sums $x_1 + \cdots + x_n$ for n very large (going to $+\infty$).

Definition

Let $(x_n)_{n\geq 1} \subseteq \mathbb{R}$. The *series* with *general terms* x_n, $x_n \in \mathbb{N}$ is the sequence

$$S_n := \sum_{k=1}^{n} x_k = x_1 + \cdots + x_n, \;\; n \in \mathbb{N}^*.$$

We will denote this series by $\sum\limits_{n=1}^{\infty} x_n$, $\sum\limits_{n\geq 1} x_n$ or $x_1 + \cdots + x_n + \dots$.

- For $n \in \mathbb{N}^*$, the term $S_n := x_1 + \cdots + x_n$ is called the *partial sum* of order n of the series.
- If (S_n) is convergent, we say that the series is *convergent*; we denote this

$$\sum_{n=1}^{\infty} x_n \text{ (C)}.$$

- If (S_n) is divergent (it has no limit or has infinite limit), we say that the series is *divergent*; we denote this

$$\sum_{n=1}^{\infty} x_n \text{ (D)}.$$

- If $S_n \to S \in \overline{\mathbb{R}}$, we write

$$\sum_{n=1}^{\infty} x_n = S.$$

Remainder of a series

If $p \in \mathbb{N}^*$, the series $\sum\limits_{n=1}^{\infty} x_n$ and $\sum\limits_{n=p+1}^{\infty} x_n$ have the *same nature*, i.e. they are either *both convergent* or *both divergent*.

Definition

If $p \in \mathbb{N}$, we call the *remainder* of order p of the series $\sum\limits_{n=1}^{\infty} x_n$ the series $\sum\limits_{n=p+1}^{\infty} x_n$.

Proposition

Let $\sum\limits_{n=1}^{\infty} x_n$ be a convergent series of real numbers. Then, for any $p \in \mathbb{N}$, the remainder of order p is convergent. Moreover, if we denote

$$R_p := \sum_{n=p+1}^{\infty} x_n, \ \ p \in \mathbb{N},$$

then $\lim\limits_{p \to +\infty} R_p = 0$.

Examples

1. The *geometric series* of *ratio* $r \in \mathbb{R}$: $\sum_{n=0}^{\infty} r^n$.

Partial sums: $$S_n = 1 + r + \cdots + r^n = \begin{cases} \dfrac{1-r^n}{1-r}, & r \neq 1, \\ n+1, & r = 1. \end{cases}$$

- $\sum_{n=0}^{\infty} r^n$ (C) for $r \in (-1, 1)$;
- $\sum_{n=0}^{\infty} r^n$ (D) for $r \in (-\infty, -1] \cup [1, +\infty)$.
- We have also:

$$\sum_{n=0}^{\infty} r^n = \begin{cases} \dfrac{1}{1-r}, & r \in (-1, 1), \\ +\infty, & r \geq 1. \end{cases}$$

In the case $r = 1$, the series $\sum_{n=0}^{\infty} (-1)^n = 1 - 1 + 1 - 1 + \ldots$ is also known as *Grandi* series; it is a divergent series.

2. The series $\sum_{n=1}^{\infty} \ln\left(1+\frac{1}{n}\right)$ is divergent, because its partial sums go to $+\infty$:

$$\begin{aligned} S_n &= \sum_{k=1}^{n} \ln\left(1+\frac{1}{k}\right) = \sum_{k=1}^{n} \ln\frac{k+1}{k} \\ &= \underbrace{\sum_{k=1}^{n} [\ln(k+1) - \ln k]}_{\text{telescopic sum}} = \ln(n+1) - \ln 1 = \ln(n+1), \end{aligned}$$

3. The series $\sum_{n=2}^{\infty} \frac{n-\sqrt{n^2-1}}{\sqrt{n^2-n}}$ is convergent, since we have

$$S_n := \sum_{k=2}^{n} \frac{k-\sqrt{k^2-1}}{\sqrt{k^2-k}} = \sum_{k=2}^{n} \left(\frac{k}{\sqrt{k^2-k}} - \frac{\sqrt{k^2-1}}{\sqrt{k^2-k}} \right)$$

$$= \sum_{k=2}^{n} \left(\sqrt{\frac{k^2}{k^2-k}} - \sqrt{\frac{k^2-1}{k^2-k}} \right) = \underbrace{\sum_{k=2}^{n} \left(\sqrt{\frac{k}{k-1}} - \sqrt{\frac{k+1}{k}} \right)}_{\text{telescopic sum}}$$

$$= \sqrt{2} - \sqrt{\frac{n+1}{n}} \underset{n\to+\infty}{\longrightarrow} \sqrt{2}-1.$$

Of course, we will have $\sum_{n=2}^{\infty} \frac{n-\sqrt{n^2-1}}{\sqrt{n^2-n}} = \sqrt{2}-1$.

Necessary condition of convergence

Theorem

Let $\sum_{n=1}^{\infty} x_n$ *be a convergent series. Then* $\lim_{n\to\infty} x_n = 0$.

Proof.

Let $S_n := \sum_{k=1}^{n} x_k$, $n \in \mathbb{N}^*$ and $S := \lim_{n\to\infty} S_n \in \mathbb{R}$. Then

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} (S_n - S_{n-1}) = S - S = 0.$$

□

Remarks.

- If $x_n \not\to 0$, the series $\sum_{n=1}^{\infty} x_n$ is clearly divergent;
- $\lim_{n\to\infty} x_n = 0$ is just a **necessary** condition for the convergence of $\sum_{n=1}^{\infty} x_n$, but **not** a **sufficient** one.

Cauchy criterion

Theorem

The series $\sum_{n=1}^{\infty} x_n$ *is convergent if and only if*

$$\forall \varepsilon > 0,\ \exists n_\varepsilon \in \mathbb{N}^*,\ \forall n \geq n_\varepsilon,\ \forall p \in \mathbb{N}^* : \left| x_{n+1} + \cdots + x_{n+p} \right| < \varepsilon.$$

Proof.

Let $S_n := \sum_{k=1}^{n} x_k,\ n \in \mathbb{N}^*$. By *Cauchy's criterion for the convergence of the sequences*, the sequence (S_n) is convergent if and only if

$$\forall \varepsilon > 0,\ \exists n_\varepsilon \in \mathbb{N}^*,\ \forall n \geq n_\varepsilon,\ \forall p \in \mathbb{N}^* : \left| S_{n+p} - S_n \right| < \varepsilon.$$

But $S_{n+p} - S_n = x_{n+1} + \cdots + x_{n+p}$ for any $n, p \in \mathbb{N}^*$, which proves the assertion. $\square$

By negating the Cauchy condition of convergence for a series, we obtain:

Proposition

The series $\sum_{n=1}^{\infty} x_n$ *is divergent if and only if*

$$\exists \varepsilon > 0,\ \forall n \in \mathbb{N}^*,\ \exists k_n \geq n,\ \forall p_n \in \mathbb{N}^* : \left| x_{k_n+1} + \cdots + x_{k_n+p_n} \right| \geq \varepsilon.$$

The harmonic series

The series $\sum_{n=1}^{\infty} \frac{1}{n}$ is called the *harmonic* series. It is divergent, by the previous result. Indeed,

$$\frac{1}{n+1} + \cdots + \frac{1}{n+p} \geq \frac{p}{n+p}, \ \forall n, p \in \mathbb{N}^*.$$

For $\varepsilon := 1/2$ and any $n \in \mathbb{N}^*$, we can set $k_n := n$ and $p_n := n$; we will have

$$\left| \frac{1}{k_n+1} + \cdots + \frac{1}{k_n+p_n} \right| \geq \frac{p_n}{k_n+p_n} = \frac{1}{2} \geq \varepsilon.$$

Therefore $\sum_{n=1}^{\infty} \frac{1}{n}$ (D).

Operations with series

Let now $\lambda \in \mathbb{R}$ and $\sum_{n=1}^{\infty} x_n$, $\sum_{n=1}^{\infty} y_n$ two series of real numbers.

- The series $\sum_{n=1}^{\infty} (x_n + y_n)$ is called the *sum* of the series $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$.
- The series $\sum_{n=1}^{\infty} (\lambda x_n)$ is called the *product* of the series $\sum_{n=1}^{\infty} x_n$ with the number (scalar) λ.

Theorem

Let $S := \sum_{n=1}^{\infty} x_n$ and $S' := \sum_{n=1}^{\infty} y_n$. If $S, S' \in \mathbb{R}$, then:

i) *if $x_n \leq y_n$, $\forall n \in \mathbb{N}^*$, then $S \leq S'$;*

ii) *the series $\sum_{n=0}^{\infty} (x_n + y_n)$ is convergent and $\sum_{n=0}^{\infty} (x_n + y_n) = S + S'$;*

iii) *the series $\sum_{n=1}^{\infty} (\lambda x_n)$ is convergent and $\sum_{n=1}^{\infty} (\lambda x_n) = \lambda S$.*

Theorem

If we associate the terms of a convergent series in finite groups, by keeping the order of the terms, we still obtain a convergent series, with the same sum.

Associating the terms of $\sum_{n=1}^{\infty} x_n$ means constructing a new series

$$\sum_{k=1}^{\infty} (x_{n_k} + \cdots + x_{n_{k+1}-1}),$$

where $(n_k)_{k\in\mathbb{N}^*} \subseteq \mathbb{N}^*$ is a strictly increasing sequence with $n_1 = 1$.

Remark.

- Associating terms of a divergent series sometimes gives a convergent series.
- For instance, if we associate every two terms in *Grandi* series $\sum_{n=0}^{\infty} (-1)^n$ we obtain the convergent series

$$\begin{aligned}\sum_{k=0}^{\infty} \left((-1)^{2k} + (-1)^{2k+1}\right) &= (-1+1) + (-1+1) + \cdots + (-1+1) + \ldots \\ &= 0 + 0 + \cdots + 0 + \ldots.\end{aligned}$$

Series with positive terms

We say that a series $\sum_{n=1}^{\infty} x_n$ has *positive terms* if $x_n \geq 0$, $\forall n \in \mathbb{N}^*$.
A series with positive terms *always* has a sum (which may be finite or infinite).

Proposition

A series with positive terms is convergent if and only if the sequence of its partial terms is bounded.

This means $\sum_{n=1}^{\infty} x_n$ is convergent if and only if $(S_n)_{n\geq 1}$ is bounded, where

$$S_n := x_1 + \cdots + x_n, \ \ n \in \mathbb{N}^*.$$

Comparison criterion I

We will study convergence and divergence criteria for series with positive sums. The first we state are called *comparison criteria*; they specify the nature of a series (*i.e.*, it is convergent or divergent) by comparing it with another series whose nature is already known.

Let $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$ be two series with positive terms.

Theorem (CC1)

Suppose that $x_n \leq y_n$, $\forall n \in \mathbb{N}^*$.

- **i)** *If* $\sum_{n=1}^{\infty} y_n$ *(C) then* $\sum_{n=1}^{\infty} x_n$ *(C).*
- **ii)** *If* $\sum_{n=1}^{\infty} x_n$ *(D) then* $\sum_{n=1}^{\infty} y_n$ *(D).*

Comparison criterion II

Theorem (CC2)

Suppose that $x_n > 0$, $y_n > 0$ and $\dfrac{x_{n+1}}{x_n} \leq \dfrac{y_{n+1}}{y_n}$, for every $n \in \mathbb{N}^*$.

i) *If $\sum\limits_{n=1}^{\infty} y_n$ (C) then $\sum\limits_{n=1}^{\infty} x_n$ (C).*

ii) *If $\sum\limits_{n=1}^{\infty} x_n$ (D) then $\sum\limits_{n=1}^{\infty} y_n$ (D).*

Comparison criterion III

Theorem (CC3)

Suppose that $y_n > 0$, $\forall n \in \mathbb{N}^*$ *and there exists* $\lim\limits_{n\to+\infty} \frac{x_n}{y_n} = \ell \in [0, +\infty]$.

1. *If* $\ell \in (0, +\infty)$, *the series* $\sum\limits_{n=1}^{\infty} x_n$ *and* $\sum\limits_{n=1}^{\infty} y_n$ *have the same nature.*
2. *If* $\ell = 0$, *we have:*
   - **i)** *if* $\sum\limits_{n=1}^{\infty} y_n$ (C) *then* $\sum\limits_{n=1}^{\infty} x_n$ (C);
   - **ii)** *if* $\sum\limits_{n=1}^{\infty} x_n$ (D) *then* $\sum\limits_{n=1}^{\infty} y_n$ (D).
3. *If* $\ell = +\infty$, *we have:*
   - **i)** *if* $\sum\limits_{n=1}^{\infty} x_n$ (C) *then* $\sum\limits_{n=1}^{\infty} y_n$ (C);
   - **ii)** *if* $\sum\limits_{n=1}^{\infty} y_n$ (D) *then* $\sum\limits_{n=1}^{\infty} x_n$ (D).

Cauchy's criterion of condensation

Theorem

Let $(x_n)_{n\geq 1} \subseteq \mathbb{R}_+$ *a decreasing sequence. Then the series* $\sum\limits_{n=1}^{\infty} x_n$ *has the same nature as the series* $\sum\limits_{n=1}^{\infty} (2^n x_{2^n})$.

- The above result holds also if we require only that $(x_n)_{n\geq 1} \subseteq \mathbb{R}_+$ is a monotone sequence.
- Indeed, if (x_n) is not decreasing, we have that both sequences (x_n) and $(2^n x_{2^n})$ do *not* converge to 0, which implies that the series $\sum\limits_{n=1}^{\infty} x_n$ and $\sum\limits_{n=1}^{\infty} (2^n x_{2^n})$ are divergent.

Generalized harmonic series

Example. The series $\sum_{n=1}^{\infty} \frac{1}{n^\alpha}$ is called the *generalized harmonic series* (*of parameter* $\alpha \in \mathbb{R}$).

- In the case $\alpha = 1$, we recover the *harmonic* series.
- The series $\sum_{n=1}^{\infty} \frac{1}{n^\alpha}$ has the same nature with the series

$$\sum_{n=1}^{\infty} \frac{2^n}{(2^n)^\alpha} = \sum_{n=1}^{\infty} \frac{2^n}{2^{n\alpha}} = \sum_{n=1}^{\infty} \left(2^{1-\alpha}\right)^n.$$

- This series is the *geometric series* with ratio $2^{1-\alpha}$ which converges if and only if $2^{1-\alpha} \in (-1, 1)$, *i.e.* $\alpha > 1$.
- As a conclusion,

$$\sum_{n=1}^{\infty} \frac{1}{n^\alpha} \text{ (C), if } \alpha > 1;$$

$$\sum_{n=1}^{\infty} \frac{1}{n^\alpha} \text{ (D), if } \alpha \leq 1.$$

Root test of Cauchy

Theorem

Let $\sum_{n=1}^{\infty} x_n$ be a series with positive terms such that there exists

$$\lim_{n\to\infty} \sqrt[n]{x_n} = \ell \in [0, +\infty].$$

i) *If $\ell < 1$, then $\sum_{n=1}^{\infty} x_n$ (C).*

ii) *If $\ell > 1$, then $\sum_{n=1}^{\infty} x_n$ (D).*

- In the case $\ell = 1$, we cannot say anything about the nature of the series $\sum_{n=1}^{\infty} x_n$ (take for example the generalized harmonic function).
- In that case, we should apply further tests (criteria).

Kummer criterion

Theorem

Let $\sum\limits_{n=1}^{\infty} x_n$ *be a series with* $x_n > 0$, $\forall n \in \mathbb{N}^*$ *and a sequence* $(a_n)_{n\geq 1} \subseteq \mathbb{R}_+^*$.
Suppose that there exists

$$\lim_{n\to\infty}\left(a_n \cdot \frac{x_n}{x_{n+1}} - a_{n+1}\right) = \ell \in \overline{\mathbb{R}}.$$

i) *If* $\ell > 0$, *then* $\sum\limits_{n=1}^{\infty} x_n$ *(C).*

ii) *If* $\ell < 0$ *and* $\sum\limits_{n=1}^{\infty} \frac{1}{a_n}$ *(D), then* $\sum\limits_{n=1}^{\infty} x_n$ *(D).*

Again, in the case $\ell = 0$, we cannot say anything about the nature of the series.
The following criteria are applications of Kummer criterion for $a_n = 1$, n or $n \ln n$.

Ratio test (d'Alembert criterion)

Corollary

Let $\sum_{n=1}^{\infty} x_n$ *be a series with* $x_n > 0$, $\forall n \in \mathbb{N}^*$. *Suppose that there exists*

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = L \in [0, +\infty].$$

i) *If* $L < 1$, *then* $\sum_{n=1}^{\infty} x_n$ *(C).*

ii) *If* $L > 1$, *then* $\sum_{n=1}^{\infty} x_n$ *(D).*

Raabe-Duhamel criterion

Corollary

Let $\sum\limits_{n=1}^{\infty} x_n$ *be a series with* $x_n > 0$, $\forall n \in \mathbb{N}^*$. *Suppose that there exists*

$$\lim_{n\to\infty} n\left(\frac{x_n}{x_{n+1}} - 1\right) = \rho \in \overline{\mathbb{R}}.$$

i) *If* $\rho > 1$, *then* $\sum\limits_{n=1}^{\infty} x_n$ *(C).*

ii) *If* $\rho < 1$, *then* $\sum\limits_{n=1}^{\infty} x_n$ *(D).*

- In order to show the second part of this criterion we use the divergence of the *harmonic series* $\sum\limits_{n=1}^{\infty} \frac{1}{n}$.
- We usually try to apply this criterion when the ratio test fails.

Bertrand criterion

Corollary

Let $\sum_{n=1}^{\infty} x_n$ *be a series with* $x_n > 0$, $\forall n \in \mathbb{N}^*$. *Suppose that there exists*

$$\lim_{n\to\infty}\left(n\ln n\cdot\frac{x_n}{x_{n+1}}-(n+1)\ln(n+1)\right)=\mu\in\overline{\mathbb{R}}.$$

i) *If* $\mu > 0$, *then* $\sum_{n=1}^{\infty} x_n$ *(C).*

ii) *If* $\mu < 0$, *then* $\sum_{n=1}^{\infty} x_n$ *(D).*

- We use the fact (for the 2nd part) that the series $\sum_{n=1}^{\infty}\frac{1}{n\ln n}$ is divergent.
- Indeed, by Cauchy's criterion of condensation, it has the same nature with

$$\sum_{n=1}^{\infty}\frac{2^n}{2^n\ln(2^n)}=\sum_{n=1}^{\infty}\frac{1}{n\ln 2}=\frac{1}{\ln 2}\sum_{n=1}^{\infty}\frac{1}{n}=+\infty.$$

Gauss criterion

Theorem

Let $\sum_{n=1}^{\infty} x_n$ *be a series with* $x_n > 0$, $\forall n \in \mathbb{N}^*$. *Suppose that there exists* $\alpha, \beta \in \mathbb{R}$, $\gamma \in \mathbb{R}_+^*$ *and a* **bounded** *sequence* $(y_n)_{n\geq 1}$ *such that*

$$\frac{x_n}{x_{n+1}} = \alpha + \frac{\beta}{n} + \frac{y_n}{n^{1+\gamma}},\ \forall n \in \mathbb{N}^*.$$

i) *If* $\alpha > 1$, *then* $\sum_{n=1}^{\infty} x_n$ *(C).*

ii) *If* $\alpha < 1$, *then* $\sum_{n=1}^{\infty} x_n$ *(D).*

iii) *If* $\alpha = 1$ *and* $\beta > 1$, *then* $\sum_{n=1}^{\infty} x_n$ *(C).*

iv) *If* $\alpha = 1$ *and* $\beta \leq 1$, *then* $\sum x_n$ *(D).*

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