Linear spaces

Lecture 5

Mathematics - 1^{st} year, English

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Outline of the lecture

1. Definition. Properties
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   - Linear dependence
   - Algebraic bases
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Linear spaces

A *vector space* (also called a *linear space*) is a collection of objects called *vectors*, which may be added together and multiplied ("scaled") by numbers, called *scalars*.
Euclidian spaces: the real line, the real plane, the real space and the real hyperspace (a n-dimensional space with $n \geq 4$).

Definition

Let $V \neq \emptyset$, $+ : V \times V \to V$ (operation) and $\cdot : \mathbb{R} \times V \to V$ (external operation). We say that $(V, +, \cdot)$ is a *linear space* or a *vectorial space* if:

- $\mathbf{x} + (\mathbf{y} + \mathbf{z}) = (\mathbf{x} + \mathbf{y}) + \mathbf{z},\ \forall \mathbf{x}, \mathbf{y}, \mathbf{z} \in V$;
- $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x},\ \forall \mathbf{x}, \mathbf{y} \in V$;
- $\exists \mathbf{0} \in V,\ \forall \mathbf{x} \in V : \mathbf{x} + \mathbf{0} = \mathbf{0} + \mathbf{x} = \mathbf{x}$;
- $\forall \mathbf{x} \in V,\ \exists (-\mathbf{x}) \in V : \mathbf{x} + (-\mathbf{x}) = (-\mathbf{x}) + \mathbf{x} = \mathbf{0}$;
- $\alpha \cdot (\mathbf{x} + \mathbf{y}) = \alpha \cdot \mathbf{x} + \alpha \cdot \mathbf{y}$, $\forall \alpha \in \mathbb{R}$, $\forall \mathbf{x}, \mathbf{y} \in V$;
- $(\alpha + \beta) \cdot \mathbf{x} = \alpha \cdot \mathbf{x} + \beta \cdot \mathbf{x}$, $\forall \alpha, \beta \in \mathbb{R}$, $\forall \mathbf{x} \in V$;
- $\alpha \cdot (\beta \cdot \mathbf{x}) = (\alpha\beta) \cdot \mathbf{x}$, $\forall \alpha, \beta \in \mathbb{R}$, $\forall \mathbf{x} \in V$;
- $1 \cdot \mathbf{x} = \mathbf{x}$, $\forall \mathbf{x} \in V$.

- The elements of V are usually called *vectors*;
- the elements of $\mathbb{R}$ are called *scalars*;
- the operation $+$ is called the *addition of vectors*;
- the external operation $\cdot$ is called the *multiplication with scalars*;
- the element $\mathbf{0}$ is called the *null-vector*;
- the vector $-\mathbf{x}$ is called the *opposite* of the vector $\mathbf{x} \in V$.

The Euclidean space

Theorem

Let $n \in \mathbb{N}^*$ and $\mathbb{R}^n := \underbrace{\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}}_{n \text{ times}}$. We define the operations $+ : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and $\cdot : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ by:

$$(x_1, x_2, \ldots, x_n) + (y_1, y_2, \ldots, y_n) := (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n);$$

$$\alpha \cdot (x_1, x_2, \ldots, x_n) := (\alpha x_1, a x_2, \ldots, \alpha x_n).$$

Then $(\mathbb{R}^n, +, \cdot)$ is a linear space, with $\mathbf{0} = (0, \ldots, 0)$.

- The above two operations are named the *canonical operations* on $\mathbb{R}^n$.
- If $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, we will call the numbers $x_1, x_2, \ldots, x_n$ the *coordinates* of $\mathbf{x}$.

Examples

1. Let, for $m, n \in \mathbb{N}^*$, $\mathscr{M}_{m,n}$ be the set of all real $m \times n$-matrices.

- $+$ is the usual addition between matrices;
- $\cdot$ is the multiplication of matrices with reals.

Then $(\mathscr{M}_{m,n}, +, \cdot)$ is a linear space.

2. Let $\mathbb{R}[X]$ be the set of all *polynomials* with real coefficients.

- $+$ is the usual addition between polynomials;
- $\cdot$ is the multiplication of polynomials with reals.

Then $(\mathbb{R}[X], +, \cdot)$ is a linear space.

3. Let E be a set, $(V, +, \cdot)$ a linear space and $\mathscr{F}(E; V)$ the collection of all functions $f : E \to V$. The operations

$$\begin{aligned} + &: \mathscr{F}(E;V) \times \mathscr{F}(E;V) \to \mathscr{F}(E;V); \\ \cdot &: \mathbb{R} \times \mathscr{F}(E;V) \to \mathscr{F}(E;V) \end{aligned}$$

are defined by

- $(f+g)(x) := f(x) + g(x)$, $f, g \in \mathscr{F}(E;V)$, $x \in E$;
- $(\alpha \cdot f)(x) := \alpha \cdot f(x)$, $\alpha \in \mathbb{R}$, $f \in \mathscr{F}(E;V)$, $x \in E$.

Then $(\mathscr{F}(E;V), +, \cdot)$ is a linear space.

Particularizing E and $(V, +, \cdot)$ we get other or already known examples.

- For instance, if we take $E := \{1, \ldots, m\} \times \{1, \ldots, n\}$ and $V := \mathbb{R}$, we obtain once again the linear space $(\mathscr{M}_{m,n}, +, \cdot)$, since $\mathscr{M}_{m,n}$ is precisely $\mathscr{F}(\{1, \ldots, m\} \times \{1, \ldots, n\}; \mathbb{R})$.
- If $m, n \in \mathbb{N}^*$, $E \subseteq \mathbb{R}^n$ and $V := \mathbb{R}^m$, then $(\mathscr{F}(E; \mathbb{R}^m), +, \cdot)$ is a vectorial space of functions of n variables with values in $\mathbb{R}^m$.
- If $E := \mathbb{N}$ and $V := \mathbb{R}$, then $\mathscr{F}(E; V)$ is the space of real sequences $(x_n)_{n\in\mathbb{N}}$.

Properties

Theorem

Let $(V, +, \cdot)$ *be a linear space. Then, for any* $\alpha \in \mathbb{R}$ *and* $\mathbf{x} \in V$ *we have:*

i) $\alpha \cdot \mathbf{0} = 0 \cdot \mathbf{x} = \mathbf{0}$;

ii) $(-\alpha) \cdot \mathbf{x} = \alpha \cdot (-\mathbf{x}) = -(\alpha \cdot \mathbf{x})$;

iii) $(-\alpha) \cdot (-\mathbf{x}) = \alpha \cdot \mathbf{x}$;

iv) $\alpha \cdot \mathbf{x} = \mathbf{0} \Rightarrow \alpha = 0$ *or* $\mathbf{x} = \mathbf{0}$.

Linear subspaces

Definition

Let $(V, +, \cdot)$ be a linear space and $\emptyset \neq W \subseteq V$. We say that W is a *linear subspace* of V if for any $\alpha \in \mathbb{R}$, $\mathbf{x}, \mathbf{y} \in W$ we have that $\mathbf{x} + \mathbf{y} \in W$ and $\alpha \cdot \mathbf{x} \in W$.

Examples.

1. If $m, n \in \mathbb{N}^*$ and $m \leq n$, the set

$$W_m := \{(x_1, \ldots, x_m, 0, \ldots, 0) \in \mathbb{R}^n \mid (x_1, \ldots, x_m) \in \mathbb{R}^m\}$$

is a linear subspace of $\mathbb{R}^n$. Since we can identify W_m with $\mathbb{R}^m$, we often consider $\mathbb{R}^m$ as a subset of $\mathbb{R}^n$ (as we consider $\mathbb{R}$ a subset of $\mathbb{C}$).

2. Let $n \in \mathbb{N}^*$ and $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ such that not all $\alpha_1, \ldots, \alpha_n$ are 0 (*i.e.*, $(\alpha_1, \ldots, \alpha_n) \neq \mathbf{0}$). The set

$$H := \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid \alpha_1 x_1 + \cdots + \alpha_n x_n = 0\}$$

is a linear subspace of $\mathbb{R}^n$, called a *hyperplane*.

3. The set of *even* real functions,

$$\{f \in \mathscr{F}(\mathbb{R}; \mathbb{R}) \mid f(-x) = f(x),\ \forall x \in \mathbb{R}\}$$

is a linear subspace of $\mathscr{F}(\mathbb{R}; \mathbb{R})$.

Proposition

Let W_1 and W_2 be two linear subspaces of a linear space $(V, +, \cdot)$. Then $W_1 \cap W_2$ is again a linear subspace of V.

Proof.

Let $\alpha \in \mathbb{R}$ and $\mathbf{x}, \mathbf{y} \in W_1 \cap W_2$. Then

$$\begin{aligned} &\mathbf{x}, \mathbf{y} \in W_1 \Rightarrow \mathbf{x} + \mathbf{y} \in W_1 \text{ and} \\ &\mathbf{x}, \mathbf{y} \in W_2 \Rightarrow \mathbf{x} + \mathbf{y} \in W_2 \end{aligned}$$

so $\mathbf{x} + \mathbf{y} \in W_1 \cap W_2$.

Also,

$$\begin{aligned} &\alpha \in \mathbb{R},\ \mathbf{x} \in W_1 \Rightarrow \alpha \cdot \mathbf{x} \in W_1 \text{ and} \\ &\alpha \in \mathbb{R}, \mathbf{x} \in W_2 \Rightarrow \alpha \cdot \mathbf{x} \in W_2, \end{aligned}$$

hence $\alpha \cdot \mathbf{x} \in W_1 \cap W_2$. ∎

- In contrast to the intersection, the union of two linear subspaces of V is *not* a linear subspace of V, in general.
- The above result can be extended to an arbitrary number of intersections.

Linear combinations

Definition

Let $(V, +, \cdot)$ be a linear space. A *linear combination* of the vectors $\mathbf{x}_1, \ldots, \mathbf{x}_n \in V$ is a vector $\mathbf{y} \in V$ which can be written as

$$\mathbf{y} = \alpha_1 \mathbf{x}_1 + \cdots + \alpha_n \mathbf{x}_n,$$

where $n \in \mathbb{N}^*$ and $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$.

Remark. If W is a linear subspace of V, any linear combination of the vectors $\mathbf{x}_1, \ldots, \mathbf{x}_n \in W$ is again an element of W.

Definition

Let $(V, +, \cdot)$ be a linear space and U be a non-empty subset of V. The set of *all* linear combinations of elements of U,

$$\{\alpha_1 \mathbf{x}_1 + \cdots + \alpha_n \mathbf{x}_n \mid n \in \mathbb{N}^*,\ \alpha_1, \ldots, \alpha_n \in \mathbb{R},\ \mathbf{x}_1, \ldots, \mathbf{x}_n \in V\}$$

is called the *linear subspace generated* by U, denoted $\mathrm{Lin}(U)$.

- It is easy to prove that $U \subseteq \mathrm{Lin}(U)$ and $\mathrm{Lin}(U)$ is a linear subspace of V (hence the name).
- Moreover, it can be shown that $\mathrm{Lin}(U)$ is the smallest linear subspace of V which contains U.

Example. If $V := \mathbb{R}^3$, the linear subspace generated by $U := \{(1, 3, 2)\}$ is the line $\{(\alpha, 3\alpha, 2\alpha) \mid \alpha \in \mathbb{R}\}$.

Linear dependence

Definition

Let $(V, +, \cdot)$ be a linear space.

- For $n \in \mathbb{N}^*$, the vectors $\mathbf{x}_1, \dots, \mathbf{x}_n \in V$ are called *linearly dependent* if there exist $\alpha_1, \dots, \alpha_n \in \mathbb{R}$, not all 0, such that
$$\alpha_1 \mathbf{x}_1 + \dots + \alpha_n \mathbf{x}_n = 0.$$
Otherwise, $\mathbf{x}_1, \dots, \mathbf{x}_n$ are called *linearly independent*.
- A subset U of V is called *linearly independent* if for any *distinct* vectors $\mathbf{x}_1, \dots, \mathbf{x}_n \in U$, $\mathbf{x}_1, \dots, \mathbf{x}_n$ are *linearly independent*.

Remarks.

- By the above definition, $\mathbf{x}_1, \dots, \mathbf{x}_n \in V$ are linearly independent if and only if the equation
$$\alpha_1 \mathbf{x}_1 + \dots + \alpha_n \mathbf{x}_n = \mathbf{0}$$
has as unique solution $\alpha_1 = \dots = \alpha_n = 0$.

- If $\mathbf{0} \in \{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$, then clearly $\mathbf{x}_1, \ldots, \mathbf{x}_n$ are linearly dependent (we take all α_k, $1 \leq k \leq n$, to be 0, except the α_k corresponding to the $\mathbf{x}_k$ which is $\mathbf{0}$). Hence, if $U \subseteq V$ is linearly independent, $0 \notin U$.
- The necessary and sufficient condition for the vectors $\mathbf{x}_1, \ldots, \mathbf{x}_n \in V$ to be linearly dependent is that we can write a vector among $\mathbf{x}_1, \ldots, \mathbf{x}_n$ as a linear combination of the others. Indeed, if

$$\mathbf{x}_k = \alpha_1 \mathbf{x}_1 + \cdots + \alpha_{k-1}\mathbf{x}_{k-1} + \alpha_{k+1}\mathbf{x}_{k+1} + \cdots + \alpha_n \mathbf{x}_n,$$

then

$$\alpha_1 \mathbf{x}_1 + \cdots + \alpha_{k-1}\mathbf{x}_{k-1} + \alpha_k \mathbf{x}_k + \alpha_{k+1}\mathbf{x}_{k+1} + \cdots + \alpha_n \mathbf{x}_n = \mathbf{0},$$

where $\alpha_k = -1 \neq 0$. Conversely, if

$$\alpha_1 \mathbf{x}_1 + \cdots + \alpha_n \mathbf{x}_n = \mathbf{0}$$

for some $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$, not all 0, let $k \in \{1, \ldots, n\}$ such that $\alpha_k \neq 0$. Then

$$\mathbf{x}_k = \left(-\frac{\alpha_1}{\alpha_k}\right)\mathbf{x}_1 + \cdots + \left(-\frac{\alpha_{k-1}}{\alpha_k}\right)\mathbf{x}_{k-1} + \left(-\frac{\alpha_{k+1}}{\alpha_k}\right)\mathbf{x}_{k+1} + \cdots + \left(-\frac{\alpha_n}{\alpha_k}\right)\mathbf{x}_n.$$

Algebraic bases

Definition

Let $(V, +, \cdot)$ be a linear space. A subset $B \subseteq V$ is called an *algebraic basis* or *Hamel basis* (or simply, a *basis*) of V if B is linearly independent and $\mathrm{Lin}(B) = V$.

Theorem

Let $n \in \mathbb{N}^*$. Then the set $\{\mathbf{e}_1, \dots, \mathbf{e}_n\} \subseteq \mathbb{R}^n$, where

$$\mathbf{e}_k := (\underbrace{0, \dots, 0}_{k-1 \text{ times}}, 1, 0, \dots, 0), \ 1 \le k \le n,$$

is a basis of $\mathbb{R}^n$, called the canonical basis of $\mathbb{R}^n$.

Dimension

Definition

Let $(V, +, \cdot)$ be a linear space. We say that V is *finite-dimensional* if there exists a finite basis of V. Otherwise, V is called *infinite-dimensional*.

Theorem

Let $(V, +, \cdot)$ be a finite-dimensional linear space, $n \in \mathbb{N}^*$ and $\{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ a basis of V. Let $X = (\alpha_{i,j})_{\substack{1 \le i \le n \\ 1 \le j \le m}}$ be a matrix in $\mathscr{M}_{n,m}$. Then the m vectors

$$\mathbf{x}_k := \alpha_{1,k}\mathbf{b}_1 + \cdots + \alpha_{n,k}\mathbf{b}_n, \ 1 \le k \le m$$

are linearly independent if and only if the rank of the matrix X is m.

Proof.

The vectors $\mathbf{x}_1, \ldots, \mathbf{x}_m$ are linearly independent if and only if the equation

$$(*) \qquad \xi_1 \mathbf{x}_1 + \cdots + \xi_m \mathbf{x}_m = \mathbf{0}$$

has only the trivial solution $\xi_1 = \cdots = \xi_m = 0$. Writting down the expression for each $\mathbf{x}_k$, $1 \leq k \leq m$, we get that (*) is equivalent to

$$\xi_1(\alpha_{1,1}\mathbf{b}_1 + \cdots + \alpha_{n,1}\mathbf{b}_n) + \cdots + \xi_m(\alpha_{1,m}\mathbf{b}_1 + \cdots + \alpha_{n,m}\mathbf{b}_n) = \mathbf{0}, \ \textit{i.e.}$$

$$(\alpha_{1,1}\xi_1 + \cdots + \alpha_{1,m}\xi_m)\mathbf{b}_1 + \cdots + (\alpha_{n,1}\xi_1 + \cdots + \alpha_{n,m}\xi_m)\mathbf{b}_n = \mathbf{0}.$$

Therefore, $\mathbf{x}_1, \ldots, \mathbf{x}_m$ are linearly independent if and only if the homogeneous system with n equations and m unknowns

$$\begin{cases} \alpha_{1,1}\xi_1 + \cdots + \alpha_{1,m}\xi_m & = & 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots & & \vdots \\ \alpha_{n,1}\xi_1 + \cdots + \alpha_{n,m}\xi_m & = & 0 \end{cases}$$

has only the trivial solution. By the theory of linear systems in $\mathbb{R}$, this is equivalent to the fact that the matrix X has the rank m. ∎

Corollary

Let $(V, +, \cdot)$ be a finite-dimensional linear space. If B is a basis of V with n elements and $\mathbf{x}_1, \ldots, \mathbf{x}_m \in V$ are linearly independent vectors, then $m \leq n$.

Theorem

Let $(V, +, \cdot)$ be a finite-dimensional linear space. Then there exists a unique $n \in \mathbb{N}$, called the dimension of V and denoted $\dim V$, such that every basis of V has precisely n elements.

Remark. The linear space $\mathbb{R}^n$ is finite dimensional and has dimension n.

Theorem

Let W be a linear subspace of a finite-dimensional linear space $(V, +, \cdot)$. Then W is finite-dimensional and $\dim W \leq \dim V$.

Proposition

Let $(V, +, \cdot)$ be a finite-dimensional linear space and $n := \dim V$. If $m \leq n$ and $\mathbf{x}_1, \ldots, \mathbf{x}_m \in V$ are linearly independent vectors, then there exist vectors $\mathbf{y}_{m+1}, \ldots, \mathbf{y}_n \in V$ such that $\{\mathbf{x}_1, \ldots, \mathbf{x}_m, \mathbf{y}_{m+1}, \ldots, \mathbf{y}_n\}$ forms a basis of V.

Coordinates

Proposition

Let $(V, +, \cdot)$ be a finite-dimensional linear space with dimension $n \in \mathbb{N}^*$. If $\{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ is a basis of V, then for every $\mathbf{x} \in V$ there exist and are unique $\alpha_1, \dots, \alpha_n \in \mathbb{R}$ such that

$$\mathbf{x} = \alpha_1 \mathbf{b}_1 + \cdots + \alpha_n \mathbf{b}_n.$$

The scalars $\alpha_1, \dots, \alpha_n$ are called the coordinates of $\mathbf{x}$ with respect to $\mathbf{b}_1, \dots, \mathbf{b}_n$.

Remarks.

1. In $\mathbb{R}^n$, the coordinates of a vector $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$ with respect to the elements of the canonical basis $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ are precisely $x_1, \dots, x_n$ (*i.e.*, the coordinates of $\mathbf{x}$).

2. Let $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a basis of $\mathbb{R}^n$ (not necessarily the canonical one), $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $\alpha_1, \ldots, \alpha_n$ the coordinates of $\mathbf{x}$ with respect to $\mathbf{b}_1, \ldots, \mathbf{b}_n$. Then the relation

$$\mathbf{x} = \alpha_1 \mathbf{b}_1 + \cdots + \alpha_n \mathbf{b}_n$$

can be written in a matrix-way as

$$\mathbf{x}^T = \mathbf{B} \cdot X_B,$$

where

$$\mathbf{B} = [\mathbf{b}_1^T \ldots \mathbf{b}_n^T] \in \mathscr{M}_n$$

is the matrix having on the k-th column the coordinates of $\mathbf{b}_k$, while

$$\mathbf{X_B} = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} \in \mathscr{M}_{n,1}$$

is the column-matrix of the coordinates of of $\mathbf{x}$ with respect to $\mathbf{b}_1, \ldots, \mathbf{b}_n$.

Definition

Let $(V, +, \cdot)$ be a finite-dimensional linear space with dimension $n \in \mathbb{N}^*$, $B = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ a basis of V and $B' = \{\mathbf{b}'_1, \dots, \mathbf{b}'_n\}$ a set of m vectors in V. We call the *transition matrix* from B to B' the matrix

$$\mathrm{S} = \begin{bmatrix} s_{1,1} & \dots & s_{1,m} \\ \vdots & & \vdots \\ s_{n,1} & \dots & s_{n,m} \end{bmatrix} \in \mathscr{M}_{n,m},$$

where, for $1 \leq k \leq m$, $s_{1,k}, \dots, s_{n,k}$ are the coordinates of the vector $\mathbf{b}'_k$ with respect to $\mathbf{b}_1, \dots, \mathbf{b}_n$.

Formally, we can write

$$\mathbf{B}' = \mathbf{B} \cdot S,$$

where B and B' are the row-matrix formed with the elements of B and respectively B':

$$\mathbf{B} = [\mathbf{b}_1 \dots \mathbf{b}_n], \mathbf{B} = [\mathbf{b}'_1 \dots \mathbf{b}'_m]$$

Theorem

Let $(V, +, \cdot)$ be a finite-dimensional linear space with dimension $n \in \mathbb{N}^*$. If $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$, $B' = \{\mathbf{b}'_1, \ldots, \mathbf{b}'_n\}$ are two bases of V and S is the transition matrix from B to B', then the matrix S is non-singular and S^{-1} is the transition matrix from B' to B.
Moreover, if $\mathbf{x} \in V$ and $\alpha_1, \ldots, \alpha_n$, $\alpha'_1, \ldots, \alpha'_n$ are the coordinates of $\mathbf{x}$ with respect to $\mathbf{b}_1, \ldots, \mathbf{b}_n$, respectively $\mathbf{b}'_1, \ldots, \mathbf{b}'_n$, then

$$X_{B'} = \mathrm{S}^{-1} \cdot X_B,$$

where

$$\mathbf{X_B} = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} \in \mathscr{M}_{n,1}, \mathbf{X'_B} = \begin{bmatrix} \alpha'_1 \\ \vdots \\ \alpha'_n \end{bmatrix} \in \mathscr{M}_{n,1}$$

Definition

Let $(V, +, \cdot)$ be a finite-dimensional linear space with dimension $n \in \mathbb{N}^*$. Let $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$, $B' = \{\mathbf{b}'_1, \ldots, \mathbf{b}'_n\}$ be two bases of V and S the transition matrix from B to B'. We say that B and B' have the *same orientation* if $\det \mathrm{S} > 0$.

Scalar products

Definition

Let $(V,+,\cdot)$ be a linear space. We say that an application $\langle\cdot,\cdot\rangle : V\times V\to\mathbb{R}$ is a *scalar product* on V if:

- $\langle\mathbf{x},\mathbf{x}\rangle\geq 0,\ \forall\mathbf{x}\in V$ (*positive definitness*);
- $\langle\mathbf{x},\mathbf{x}\rangle = 0\Rightarrow\mathbf{x}=0,\ \forall\mathbf{x}\in V$;
- $\langle\mathbf{x},\mathbf{y}\rangle = \langle\mathbf{y},\mathbf{x}\rangle,\ \forall\mathbf{x},\mathbf{y}\in V$ (*symmetry*);
- $\langle\alpha\cdot\mathbf{x}+\beta\cdot\mathbf{y},\mathbf{z}\rangle = \alpha\langle\mathbf{x},\mathbf{z}\rangle+\beta\langle\mathbf{y},\mathbf{z}\rangle$,
  $\langle\mathbf{x},\alpha\cdot\mathbf{y}+\beta\cdot\mathbf{z}\rangle = \alpha\langle\mathbf{x},\mathbf{y}\rangle+\beta\langle\mathbf{x},\mathbf{z}\rangle,\ \forall\alpha,\beta\in\mathbb{R},\ \forall\mathbf{x},\mathbf{y},\mathbf{z}\in V$ (*bilinearity*).

In this case, the quadruple $(V,+,\cdot,\langle\cdot,\cdot\rangle)$ is called a *prehilbertian space*.

Proposition

Let $n\in\mathbb{N}^*$ and $\langle\cdot,\cdot\rangle:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$ be defined as

$$\langle(x_1,\dots,x_n),(y_1,\dots,y_n)\rangle := x_1y_1+\cdots+x_ny_n.$$

Then $\langle\cdot,\cdot\rangle$ is a scalar product on $\mathbb{R}^n$, called the Euclidian scalar product.

Orthogonality

Definition

Let $(V, \langle \cdot, \cdot \rangle)$ be a prehilbertian space.

- We say that two vectors $\mathbf{x} \in V$ and $\mathbf{y} \in V$ are *orthogonal* and we denote $\mathbf{x} \perp \mathbf{y}$ if $\langle \mathbf{x}, \mathbf{y} \rangle = 0$.
- Let $\mathbf{x} \in V$ and U a non-empty subset of V. We say that x is orthogonal on U and we denote $\mathbf{x} \perp U$ if $\mathbf{x} \perp \mathbf{y}$ for every $\mathbf{y} \in U$.
- If U is non-empty subset of V, we call U an *orthogonal system* if $\mathbf{x} \perp \mathbf{y}$ for any distinct $\mathbf{x}, \mathbf{y} \in U$.
- Let $U \subseteq V$. The *orthogonal complement* of U is the set

$$U^{\perp} := \{\mathbf{x} \in V \mid \mathbf{x} \perp U\}.$$

Remark. Let $\emptyset \neq U \subseteq V$.

- One can show that if $\mathbf{x} \in V$, then $\mathbf{x} \perp U$ if and only if $\mathbf{x} \perp \mathrm{Lin}(U)$.
- Therefore, $U^{\perp} = \mathrm{Lin}(U)^{\perp}$.
- It is also easy to prove that $U \cap U^{\perp} = \{\mathbf{0}\}$.

Angle between vectors

Definition

Let $(V, \langle\cdot,\cdot\rangle)$ be a prehilbertian space. For $\mathbf{x},\mathbf{y} \in V \setminus \{\mathbf{0}\}$, we call the *angle* between $\mathbf{x}$ and $\mathbf{y}$ the number

$$\widehat{(\mathbf{x},\mathbf{y})} = \sphericalangle(\mathbf{x},\mathbf{y}) := \arccos \frac{\langle \mathbf{x},\mathbf{y}\rangle}{\sqrt{\langle \mathbf{x},\mathbf{x}\rangle}\sqrt{\langle \mathbf{y},\mathbf{y}\rangle}}.$$

It is clear that $\widehat{(\mathbf{x},\mathbf{y})} = \widehat{(\mathbf{y},\mathbf{x})} \in [0,\pi]$, $\forall \mathbf{x},\mathbf{y} \in V \setminus \{\mathbf{0}\}$. Moreover, if $\mathbf{x},\mathbf{y} \in V \setminus \{\mathbf{0}\}$, $\widehat{(\mathbf{x},\mathbf{y})} = \pi/2$ if and only if $\mathbf{x} \perp \mathbf{y}$.

Norms

Definition

Let $(V, +, \cdot)$ be a linear space. We say that an application $\|\cdot\| : V \to \mathbb{R}$ is a *norm* on V if:

- $\|\mathbf{x}\| \geq 0, \ \forall \mathbf{x} \in V$;
- $\|\mathbf{x}\| = 0 \Rightarrow \mathbf{x} = 0, \ \forall \mathbf{x} \in V$;
- $\|\lambda \mathbf{x}\| = |\lambda| \, \|\mathbf{x}\|, \ \forall \lambda \in \mathbb{R}, \ \forall \mathbf{x} \in V$ (*homogeneity*);
- $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|, \ \forall \mathbf{x}, \mathbf{y} \in V$ (*triangle property*).

In this case, the quadruple $(V, +, \cdot, \|\cdot\|)$ is called a *normed space*.

Proposition

Let $(V, \langle\cdot,\cdot\rangle)$ be a prehilbertian space. Then the mapping $\|\cdot\| : V \to \mathbb{R}$ defined by

$$\|\mathbf{x}\| := \sqrt{\langle \mathbf{x}, \mathbf{x}\rangle},\ \mathbf{x} \in V$$

is a norm on V, called the norm induced by the scalar product $\langle\cdot,\cdot\rangle$.

Definition

Let $n \in \mathbb{N}^*$. The norm induced by the Euclidean scalar product on $\mathbb{R}^n$ is called the *Euclidean norm* and is denoted $\|\cdot\|_2$.

If $(x_1, \ldots, x_n) \in \mathbb{R}^n$, then $\|(x_1, \ldots, x_n)\|_2 = \sqrt{x_1^2 + \cdots + x_n^2}$.

Definition

Let $(V, \|\cdot\|)$ be a normed space. A vector $\mathbf{x} \in V$ such that $\|\mathbf{x}\| = 1$ is called a *versor*.

Definition

Let $(V, \langle\cdot,\cdot\rangle)$ be a prehilbertian space.

- A non-empty subset $U \subseteq V$ is called an *orthonormal system* if U is an orthogonal system and every element of U is a versor.
- If B is a basis of V and B is an orthogonal system, then B is called an *orthogonal basis*.
- If B is a basis of V and B is an orthonormal system, then B is called an *orthonormal basis*.

In other words, U is an orthonormal system if and only if for any $\mathbf{x}, \mathbf{y} \in U$ we have

$$\langle \mathbf{x}, \mathbf{y} \rangle = \begin{cases} 0, & \mathbf{x} \neq \mathbf{y}; \\ 1, & \mathbf{x} = \mathbf{y}. \end{cases}$$

Of course, the canonical basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ in $\mathbb{R}^n$ is an orthonormal basis.

Definition

Let $(V, \langle\cdot,\cdot\rangle)$ be a finite-dimensional prehilbertian space with dimension $n \in \mathbb{N}^*$ and $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ a basis of V. We call the *Gram determinant* associated with the basis B the number $\det \mathrm{G} \in \mathbb{R}$, where

$$\mathrm{G} := \begin{bmatrix} \langle \mathbf{b}_1, \mathbf{b}_1\rangle & \langle \mathbf{b}_1, \mathbf{b}_2\rangle & \ldots & \langle \mathbf{b}_1, \mathbf{b}_n\rangle \\ \langle \mathbf{b}_2, \mathbf{b}_1\rangle & \langle \mathbf{b}_2, \mathbf{b}_2\rangle & \ldots & \langle \mathbf{b}_2, \mathbf{b}_n\rangle \\ \vdots & \vdots & & \vdots \\ \langle \mathbf{b}_n, \mathbf{b}_1\rangle & \langle \mathbf{b}_n, \mathbf{b}_2\rangle & \ldots & \langle \mathbf{b}_n, \mathbf{b}_n\rangle \end{bmatrix} \in \mathscr{M}_n$$

- G is a symmetric and non-singular matrix.
- The basis B is orthogonal or orthonormal if and only if G is a *diagonal matrix*, respectively $\mathrm{G} = \mathrm{I}_n$.
- If $\mathbf{x}, \mathbf{y} \in V$ and $\alpha_1, \ldots, \alpha_n$, $\beta_1, \ldots, \beta_n$ are the coordinates of $\mathbf{x}$, respectively $\mathbf{y}$, with respect to $\mathbf{b}_1, \ldots, \mathbf{b}_n$, then

$$\langle \mathbf{x}, \mathbf{y}\rangle = \sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i\beta_j \langle \mathbf{b}_i, \mathbf{b}_j\rangle = X_B^{\mathrm{T}} \cdot \mathrm{G} \cdot Y_B,$$

where $X_B = [\alpha_1, \ldots, \alpha_n]^T \in \mathscr{M}_{n,1}$ and $Y_B = [\beta_1, \ldots, \beta_n]^T \in \mathscr{M}_{n,1}$.

Theorem (Gram-Schmidt orthonormalization procedure)

Let $(V, \langle\cdot,\cdot\rangle)$ be a finite-dimensional prehilbertian space with dimension $n \in \mathbb{N}^*$. If $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ a basis of V, there exists an orthonormal basis $B' = \{\mathbf{b}'_1, \ldots, \mathbf{b}'_n\}$ such that

$$\mathrm{Lin}(\{\mathbf{b}'_1, \ldots, \mathbf{b}'_k\}) = \mathrm{Lin}(\{\mathbf{b}_1, \ldots, \mathbf{b}_k\})$$

for every $k \in \{1, \ldots, n\}$.

- One important aspect of this result is that every finite-dimensional prehilbertian space has an orthonormal basis.
- It is enough to prove that the basis B' should be only orthogonal (if $\{\mathbf{b}'_1, \ldots, \mathbf{b}'_n\}$ is an orthogonal basis, then $\left\{\frac{\mathbf{b}'_1}{\|\mathbf{b}'_1\|}, \ldots, \frac{\mathbf{b}'_n}{\|\mathbf{b}'_n\|}\right\}$ is an orthonormal basis).

Proof.

Step 1. We take $\mathbf{b}'_1 = \mathbf{b}_1$.
Step 2. Suppose that, for $k < n$, we have already found $\mathbf{b}'_1, \ldots, \mathbf{b}'_k$ with $\{\mathbf{b}'_1, \ldots, \mathbf{b}'_k\}$ an orthogonal system such that

$$\mathrm{Lin}(\{\mathbf{b}'_1, \ldots, \mathbf{b}'_k\}) = \mathrm{Lin}(\{\mathbf{b}_1, \ldots, \mathbf{b}_k\}).$$

We determine $\mathbf{b}'_{k+1} = \lambda_1 \mathbf{b}'_1 + \cdots + \lambda_k \mathbf{b}'_k + \mathbf{b}_{k+1}$ such that $\mathbf{b}'_{k+1} \perp \mathbf{b}'_j$, $\forall j \in \{1, \ldots, k\}$. This means that

$$\lambda_j \left\|\mathbf{b}'_j\right\|^2 + \left\langle \mathbf{b}_{k+1}, \mathbf{b}'_j \right\rangle = 0, \ \forall j \in \{1, \ldots, k\},$$

i.e. $\lambda_j = -\frac{\left\langle \mathbf{b}_{k+1}, \mathbf{b}'_j \right\rangle}{\left\|\mathbf{b}'_j\right\|^2}$ for $j \in \{1, \ldots, k\}$. In conclusion, we have found

$$\mathbf{b}'_{k+1} = \mathbf{b}_{k+1} - \frac{\left\langle \mathbf{b}_{k+1}, \mathbf{b}'_1 \right\rangle}{\left\|\mathbf{b}'_1\right\|^2}\mathbf{b}'_1 - \cdots - \frac{\left\langle \mathbf{b}_{k+1}, \mathbf{b}'_k \right\rangle}{\left\|\mathbf{b}'_k\right\|^2}\mathbf{b}'_k.$$

Step 3. We repeat **Step 2** until we arrive to $k + 1 = n$. ∎

The above algorithm is called the *Gram-Schmidt orthonormalization procedure*.

- V. Borcea, C. Davideanu, Corina Forăscu, *Probleme de Algebră Liniară*, Gh. Asachi Publishing House, Iași, 2000.
- S. Corbu, *Algebră liniară.Geometrie analitică (elemente de teorie și aplicații)*, Editura "Fair Partners", București, 2011.
- N. Cotfas, *Elemente de algebră liniară*, Editura Univ. București, 2009.
- M. Gorunescu, F. Gorunescu, A. Prodan, *Matematici superioare*, Biostatistică și Informatică, Editura Albastră, Cluj-Napoca, 2002.
- J. Hefferon, *Linear Algebra*, Saint Michael's College, 2014.
- Ion D. Ion, R. Nicolae, *Algebră*, Editura Didactică și Pedagogică, București, 1981.
- V. Postolică, *Eficiență prin matematica aplicată*, Editura Matrix Rom, București, 2006.
- A. Precupanu, *Bazele analizei matematice*, Editura Polirom, Iași, 1998.
- R. Singh, M. Bhatt, *Linear Algebra and Vector Calculus*, Mc Graw Hill Ed., 2016.
- S. Treil, *Linear Algebra Done Wrong*, Brown University, 2013.