LECTURE 4
SERIES OF GENERAL REAL NUMBERS. POWER SERIES

1. Series of real numbers – the general case

In this section we will analyse series with terms which are not necessarely positive. Before giving some criteria of convergence, let us study a simple example.

Example. The series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$ is called the *alternate harmonic series.* It is a convergent series. Indeed, by noting $x_n := (-1)^{n+1}\frac{1}{n}$, $n \in \mathbb{N}^*$, we have for $n, p \in \mathbb{N}^*$

$$\begin{aligned}\left|x_{n+1} + \cdots + x_{n+p}\right| &= \left|(-1)^{n+2}\frac{1}{n+1} + (-1)^{n+3}\frac{1}{n+2} + \cdots + (-1)^{n+p+1}\frac{1}{n+p}\right| \\ &= \frac{1}{n+1} - \frac{1}{n+2} + \cdots + (-1)^{p-1}\frac{1}{n+p} \le \frac{1}{n+1}.\end{aligned}$$

Since $\lim_{n\to\infty}\frac{1}{n+1} = 0$, for every $\varepsilon > 0$, we can find n_ε (for instance $n_\varepsilon := \left\lfloor \frac{1}{\varepsilon}\right\rfloor$) such that $\frac{1}{n+1} < \varepsilon$, for every $n \ge n_\varepsilon$. Therefore, $\left|x_{n+1} + \cdots + x_{n+p}\right| < \varepsilon$ for every $n \ge n_\varepsilon$ and $p \in \mathbb{N}^*$. This implies that the series satisfies the Cauchy test of convergence, so it is convergent.

Moreover, one can prove (exercice!) that its sum is $\ln 2$.

A series $\sum_{n=1}^{\infty} x_n$ such that $x_n \cdot x_{n+1} \le 0$, $\forall n \in \mathbb{N}^*$ is called an *alternate series.* The name "alternate harmonic series" for the series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$ is therefore consistent with this nomenclature.

1.1. **Convergence criteria**.

Theorem 1.1 (Dirichlet criterion). *Let $(x_n)_{n\ge1}$ and $(y_n)_{n\ge1}$ be sequences of real numbers. Let $S_n := x_1 + \cdots + x_n$, $n \in \mathbb{N}^*$. If*

(i) *the sequence $(S_n)_{n\ge1}$ is bounded;*
(ii) *the sequence $(y_n)_{n\ge1}$ is monotone and* $\lim_{n\to\infty} y_n = 0$,

then the series $\sum_{n=1}^{\infty} x_n y_n$ is convergent.

Proof. Let $m := \inf_{n\in\mathbb{N}} S_n$ şi $M := \sup_{n\in\mathbb{N}} S_n$. Since the sequence (S_n) is bounded, we have that $m, M \in \mathbb{R}$. We can suppose, without loss of generality, that (y_n) is decreasing.

We will apply Cauchy's convergence test to $\sum_{n=1}^{\infty} x_n y_n$. Let $\varepsilon > 0$. There exists $n_\varepsilon \in \mathbb{N}^*$ such that $0 < y_n < \frac{\varepsilon}{M-m}$, $\forall n \ge n_\varepsilon$. For each $n \ge n_\varepsilon$ and $p \in \mathbb{N}^*$ we have

$$\sum_{k=n+1}^{n+p} x_k y_k = \sum_{k=n+1}^{n+p}(S_k - S_{k-1})y_k = \sum_{k=n+1}^{n+p} S_k y_k - \sum_{k=n}^{n+p-1} S_k y_{k+1} = \sum_{k=n+1}^{n+p-1} S_k(y_k - y_{k+1}) + S_{n+p}y_{n+p} - S_n y_{n+1}.$$

Therefore, since $\sum_{k=n+1}^{n+p-1}(y_k - y_{k+1}) = y_{n+1} - y_{n+p}$,

$$m(y_{n+1} - y_{n+p}) + m y_{n+p} - M y_{n+1} \le \sum_{k=n+1}^{n+p} x_k y_k \le M(y_{n+1} - y_{n+p}) + M y_{n+p} - m y_{n+1}.$$

Hence

$$\left|\sum_{k=n+1}^{n+p} x_k y_k\right| < (M-m)\frac{\varepsilon}{M-m} = \varepsilon.$$

Since ε arbitrarely choosen, the series $\sum_{n=0}^{\infty} x_n y_n$ is convergent. □

Example. Let us consider the series $\sum_{n=1}^{\infty}\frac{\cos n}{\sqrt{n}}$. In order to apply Dirichlet criterion, it is enough to study the sequence of partial sums of the series $\sum_{n=1}^{\infty}\cos n$. Let

$$S_n := \cos 1 + \cos 2 + \cdots + \cos n.$$

One can explicitely calculate S_n, by multiplying it with $2\sin\frac{1}{2}$:

$$\begin{aligned}2\sin\frac{1}{2}\cdot S_n &= 2\cos 1\cdot\sin\frac{1}{2} + 2\cos 2\cdot\sin\frac{1}{2} + \cdots + 2\cos n\cdot\sin\frac{1}{2}\\ &= \left[\sin\left(1+\frac{1}{2}\right) - \sin\left(1-\frac{1}{2}\right)\right] + \left[\sin\left(2+\frac{1}{2}\right) - \sin\left(2-\frac{1}{2}\right)\right] + \cdots + \left[\sin\left(n+\frac{1}{2}\right) - \sin\left(n-\frac{1}{2}\right)\right]\\ &= \sin\left(n+\frac{1}{2}\right) - \sin\left(\frac{1}{2}\right) = 2\sin\frac{n}{2}\cdot\cos\frac{n+1}{2}.\end{aligned}$$

We have then

$$|S_n| = \left|\frac{\sin\frac{n}{2}\cdot\cos\frac{n+1}{2}}{\sin\frac{1}{2}}\right| \le \frac{1}{\left|\sin\frac{1}{2}\right|},\ \forall n\in\mathbb{N}^*,$$

so the sequence $(S_n)_{n\ge1}$ is bounded. The sequence $\left(\frac{1}{\sqrt{n}}\right)_{n\ge1}$ is decreasing and convergent to 0; by Dirichlet criterion it follows that the series $\sum_{n=1}^{\infty}\frac{\cos n}{\sqrt{n}}$ is convergent.

Corollary (Leibniz criterion). *Let $(x_n)_{n\ge1}\subseteq\mathbb{R}$ be a monotone sequence with $\lim_{n\to\infty}x_n = 0$. Then the alternate series $\sum_{n=1}^{\infty}(-1)^n x_n$ is convergent.*

Proof. In order to apply Dirichlet criterion for the series $\sum_{n=1}^{\infty}(-1)^n x_n$, it is enough to see that the sequence of the partial sums of Grandi series, $\sum_{n=1}^{\infty}(-1)^n$, is bounded. □

It is easy now to see, by applying Leibniz criterion, that the alternate harmonic series is convergent.

Theorem 1.2 (Abel criterion). *Let $(x_n)_{n\ge1}$ and $(y_n)_{n\ge1}$ be sequences of real numbers. If*

(i) *the series $\sum_{n=1}^{\infty}x_n$ is convergent;*

(ii) *the sequence $(y_n)_{n\ge1}$ is monotone and bounded,*

then the series $\sum_{n=1}^{\infty}x_n y_n$ is convergent.

Proof. Since $(y_n)_{n\ge1}$ is monotone and bounded, it is convergent. Let $y\in\mathbb{R}$ be its limit and $\tilde{y}_n := y_n - y$. Then the sequence $(\tilde{y}_n)_{n\ge1}$ is monotone with $\lim_{n\to\infty}\tilde{y}_n = 0$.

By Dirichlet criterion, the series $\sum_{n=1}^{\infty}x_n\tilde{y}_n$ is convergent ($\sum_{n=1}^{\infty}x_n$ (C) implies, of course, that the sequence of its partial sums is bounded). On the other hand, the series $\sum_{n=1}^{\infty}x_n y$ is also convergent (because $\sum_{n=1}^{\infty}x_n$ is convergent). Summing the two convergent series, we obtain that the series $\sum_{n=1}^{\infty}x_n(\tilde{y}_n + y)$ is convergent, *i.e.* $\sum_{n=1}^{\infty}x_n y_n$ (C). □

1.2. **Absolute convergent series**.

Definition. We say that a series of real numbers $\sum_{n=1}^{\infty}x_n$ is:

a) *absolute convergent*, if $\sum_{n=1}^{\infty}|x_n|$ is convergent;

b) *semiconvergent*, if $\sum_{n=1}^{\infty}x_n$ is convergent, but $\sum_{n=1}^{\infty}|x_n|$ is divergent.

We sometimes express that $\sum_{n=1}^{\infty}x_n$ is absolute convergent by $\sum_{n=1}^{\infty}x_n$ (AC) and that $\sum_{n=1}^{\infty}x_n$ is semiconvergent by $\sum_{n=1}^{\infty}x_n$ (SC).

Remarks. For series with positive terms, absolute convergence is equivalent with convergence.

The alternate harmonic series is semiconvergent, because it is convergent and $\sum_{n=1}^{\infty}\left|(-1)^{n+1}\frac{1}{n}\right| = \sum_{n=1}^{\infty}\frac{1}{n}$ is divergent (it is the harmonic series).

Proposition 1.3. *If a series of real numbers is absolute convergent, then it is convergent.*

Proof. We will simply apply Cauchy's convergence test. Let $\sum_{n=1}^{\infty} x_n$ be an absolute convergent series. Let $\varepsilon > 0$; since $\sum_{n=1}^{\infty} |x_n|$ (C), we can find $n_\varepsilon \in \mathbb{N}^*$ such that

$$|x_{n+1}| + \cdots + |x_{n+p}| < \varepsilon, \ \forall n \geq n_\varepsilon, \ \forall p \in \mathbb{N}^*.$$

But $|x_{n+1} + \cdots + x_{n+p}| \leq |x_{n+1}| + \cdots + |x_{n+p}|$, so

$$|x_{n+1} + \cdots + x_{n+p}| < \varepsilon, \ \forall n \geq n_\varepsilon, \ \forall p \in \mathbb{N}^*.$$

Applying again Cauchy's test, we deduce that $\sum_{n=1}^{\infty} x_n$ is convergent. □

We can now say something about another operation with series, called the *Cauchy product.*

Definition. Let $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$ be two series of real numbers. The series $\sum_{n=1}^{\infty} c_n$, where

$$c_n := x_1 y_n + x_2 y_{n-1} + \cdots + x_n y_1,$$

is called the *Cauchy product* of the series $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$.

Of course, this operation between series is commutative. We give, without proof, a criterion of convergence for the Cauchy product of two series.

Theorem 1.4 (Mertens). *Let $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$ be two series of real numbers. If $\sum_{n=1}^{\infty} x_n$ (AC) and $\sum_{n=1}^{\infty} y_n$ (C), then the Cauchy product of $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$ is convergent. Moreover, its sum is equal to the product of the sums of the two series.*

This result has a simple consequence, regarding the Cauchy product of two absolute convergent series.

Corollary (Cauchy Theorem). *The Cauchy product of two absolute convergent series is absolute convergent.*

Remark. The Cauchy product of two convergent series is not necessarely convergent. Let, for $n \in \mathbb{N}$, $x_n := (-1)^n \frac{1}{\sqrt{n+1}}$ and $y_n := x_n$.

By Leibniz criterion, the alternate series $\sum_{n=0}^{\infty} x_n$ and $\sum_{n=0}^{\infty} y_n$ are convergent. We define, for $n \in \mathbb{N}$,

$$c_n := \sum_{k=0}^{n} x_k y_{n-k} = \sum_{k=0}^{n} (-1)^n \frac{1}{\sqrt{(k+1)(n-k+1)}} = (-1)^n \sum_{k=0}^{n} \frac{1}{\sqrt{(k+1)(n-k+1)}}.$$

Since, by mean inequality,

$$\sqrt{(k+1)(n-k+1)} \leq \frac{1}{2}\left[(k+1) + (n-k+1)\right] = \frac{n+2}{2}, \ \forall k = 0, 1 \ldots, n,$$

we have

$$|c_n| = \sum_{k=0}^{n} \frac{1}{\sqrt{(k+1)(n-k+1)}} \geq \sum_{k=0}^{n} \frac{2}{n+2} = \frac{2(n+1)}{n+2}.$$

Therefore $c_n \not\to 0$, hence $\sum_{n=0}^{\infty} c_n$ is clearly not convergent.

1.3. **Unconditionally convergent series.**

We will now see, still without proofs, under which conditions the nature of a series does not change when we permute its terms.

Theorem 1.5 (Riemann). *Let $\sum_{n=1}^{\infty} x_n$ be semiconvergent series. Then, for any $S \in \overline{\mathbb{R}}$ there exists a bijective function (a permutation) $\varphi : \mathbb{N}^* \to \mathbb{N}^*$ such that $\sum_{n=1}^{\infty} x_{\varphi(n)} = S$.*

An immediate consequence of the above result, by letting $S = +\infty$ or $S = -\infty$, is that we can permute the terms of a semiconvergent series in order to obtain a divergent one.

Definition. We say that a series of real numbers $\sum_{n=1}^{\infty} x_n$ is *unconditionally convergent* if for any bijective function $\varphi : \mathbb{N}^* \to \mathbb{N}^*$, the series $\sum_{n=1}^{\infty} x_{\varphi(n)}$ is convergent.

Obviously, by taking $\varphi := \mathbf{1}_{\mathbb{N}^*}$, an unconditionally convergent series is convergent. One can say more, by applying Riemann theorem:

Theorem 1.6. *A series $\sum_{n=1}^{\infty} x_n$ is unconditionally convergent if and only if it is absolute convergent. In this case, for every bijective function $\varphi : \mathbb{N}^* \to \mathbb{N}^*$ we have*

$$\sum_{n=1}^{\infty} x_{\varphi(n)} = \sum_{n=1}^{\infty} x_n.$$

1.4. p**-adic representation of real numbers.**

Proposition 1.7. *Let $p \in \mathbb{N}^* \setminus \{1\}$ and a sequence $(a_n)_{n\geq 1}$ of natural numbers such that $a_n \leq p-1$, $\forall n \in \mathbb{N}^*$. Then the series*

$$\sum_{n=1}^{\infty} \frac{a_n}{p^n}$$

is convergent and its sum is a real number between 0 *and* 1.

The converse also holds and the result is known as the *p-adic representation of real numbers.*

Theorem 1.8. *Let $a \in (0,1]$ and $p \in \mathbb{N}^* \setminus \{1\}$. Then there exists a unique sequence $(a_n)_{n\geq 1}$ of natural numbers satisfying $a_n \leq p-1$, $\forall n \in \mathbb{N}^*$ and $\{n \in \mathbb{N}^* \mid a_n \neq 0\}$ is infinite such that*

$$\sum_{n=1}^{\infty} \frac{a_n}{p^n} = a.$$

1.5. **Approximation of convergent series.**

For an alternate series with its general term converging to 0, we have the following estimate of its sum:

Theorem 1.9. *Let $(x_n)_{n\geq 1} \subseteq \mathbb{R}$ be a monotone sequence with $\lim_{n\to\infty} x_n = 0$. If we denote by S the sum of the alternate series $\sum_{n=1}^{\infty} (-1)^n x_n$ and by $S_n := x_1 + \cdots + x_n$, $n \in \mathbb{N}^*$, its partial sums, then we have*

$$|S_n - S| \leq |x_{n+1}|, \ \forall n \in \mathbb{N}^*.$$

We can also estimate the rate of convergence for a series whose general term decrease exponentially:

Theorem 1.10. *Let $\sum_{n=1}^{\infty} x_n$ be a series of real numbers. Let $S_n := x_1 + \cdots + x_n$, $n \in \mathbb{N}^*$.*

i) *If there exists $n_0 \in \mathbb{N}^*$ and $\lambda < 1$ such that $\sqrt[n]{|x_n|} < \lambda$, $\forall n \geq n_0$, then the series is absolute convergent and, if we denote $S := \sum_{n=1}^{\infty} x_n$,*

$$|S_n - S| \leq \frac{\lambda^{n+1}}{1-\lambda}, \ \forall n \in \mathbb{N}^*.$$

ii) *If there exists $n_0 \in \mathbb{N}^*$ and $\lambda < 1$ such that $\frac{|x_{n+1}|}{|x_n|} < \lambda$, $\forall n \geq n_0$, then the series is absolute convergent and, if we denote $S := \sum_{n=1}^{\infty} x_n$,*

$$|S_n - S| \leq \frac{|x_{n+1}|}{1-\lambda}, \ \forall n \in \mathbb{N}^*.$$

2. Power series

2.1. **Uniform convergence**.

Let us introduce first, by analogy with sequences of functions, the notion of *uniform convergence* for *series of functions.*

If $(f_n)_{n\geq 1}$ is a sequence of functions from a set E to $\mathbb{R}$, by the *series of functions* $\sum_{n=1}^{\infty} f_n$ we understand the sequence of functions $(S_n)_{n\geq 1}$, where the functions $S_n : E \to \mathbb{R}$, $n \in \mathbb{N}^*$ are the *partial sums* of the series $\sum_{n=1}^{\infty} f_n$, defined by

$$S_n(x) := f_1(x) + \cdots + f_n(x),\ x \in E.$$

Definition. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions from a set E to $\mathbb{R}$ and $D \subseteq E$. Let $(S_n)_{n\geq 1}$ be the sequence of the partial sums of $\sum_{n=1}^{\infty} f_n$.

a) We say that $\sum_{n=1}^{\infty} f_n$ *converges pointwise* on D if $\sum_{n=1}^{\infty} f_n(x)$ is convergent for every $x \in D$, *i.e.* there exists a function $S : D \to \mathbb{R}$ such that $S_n \xrightarrow[D]{\text{p}} S$. In this case we will write

$$\sum_{n=1}^{\infty} f_n = S \text{ on } D.$$

b) We say that $\sum_{n=1}^{\infty} f_n$ *converges uniformly* on D if there exists a function $S : D \to \mathbb{R}$ such that $S_n \xrightarrow[D]{\text{u}} S$, *i.e.*

$$\forall \varepsilon > 0,\ \exists n_\varepsilon \in \mathbb{N}^*,\ \forall n \geq n_\varepsilon,\ \forall x \in D : \left| \sum_{k=1}^{n} f_k(x) - S(x) \right| < \varepsilon.$$

In the case we will write

$$\sum_{n=1}^{\infty} f_n(x) = S(x) \text{ (UC)},\ x \in D.$$

As in the case of numeric series, we have a Cauchy test for uniform convergence:

Theorem 2.1 (Cauchy test of uniform convergence). *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions from a set E to $\mathbb{R}$ and $D \subseteq E$. Then $\sum_{n=1}^{\infty} f_n$* converges uniformly *on D if and only if*

$$\forall \varepsilon > 0,\ \exists n_\varepsilon \in \mathbb{N},\ \forall n \geq n_\varepsilon,\ \forall p \in \mathbb{N}^*,\ \forall x \in D : \left| f_{n+1}(x) + \cdots + f_{n+p}(x) \right| < \varepsilon.$$

2.2. **Power series**.

Definition. Let $(a_n) \subseteq \mathbb{R}$ be a sequence and $x_0 \in \mathbb{R}$. The series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ with parameter $x \in \mathbb{R}$ is called the *power series* centered in x_0, with coefficients a_n, $n \in \mathbb{N}$. The set of those $x \in \mathbb{R}$ for which the series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ is convergent (absolute convergent) is called the *domain of convergence* (*domain of absolute convergence*) of the power series.

It is clear that the power series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ can be seen as a series of functions $\sum_{n=0}^{\infty} f_n$, where the functions $f_n : \mathbb{R} \to \mathbb{R}$, $n \in \mathbb{N}$ are defined by

$$f_n(x) := a_n(x-x_0)^n,\ x \in \mathbb{R}.$$

We will sometimes denote $\mathrm{D_c}$ and $\mathrm{D_{ac}}$ the domain of convergence, respectively the domain of absolute convergence of the power series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$.

For the sequel, we will suppose that $x_0 = 0$. This is not a restriction of the generality, since any power series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ can be brought to the form $\sum_{n=0}^{\infty} \tilde{a}_n x^n$, with different coefficients $\tilde{a}_n$, $n \in \mathbb{N}$.

Theorem 2.2. *Let* $\sum_{n=0}^{\infty} a_n x^n$ *be a power series. Then there exists a unique* $r \in [0, +\infty]$, *called* radius of convergence *of* $\sum_{n=0}^{\infty} a_n x^n$, *such that*

$$(-r, r) \subseteq \mathrm{D_{ac}} \subseteq \mathrm{D_c} \subseteq [-r, +r] \tag{*}$$

Moreover, we have:

i) $r = \dfrac{1}{\limsup_{n\to\infty} \sqrt[n]{|a_n|}}$;

ii) the series $\sum_{n=0}^{\infty} a_n x^n$ *is uniformly convergent with respect to* $x \in [a, b]$, *for any* $a, b \in \mathrm{D_{ac}}$ *with* $a \le b$;

iii) the function $S : \mathrm{D_c} \to \mathbb{R}$, *defined as*

$$S(x) := \sum_{n=0}^{\infty} a_n x^n, \ x \in \mathrm{D_c}$$

is continuous.

The last part of this result is also called *Abel theorem.*

Let us discuss some aspects concerning this theorem. Relation (*) tells us that the series $\sum_{n=0}^{\infty} a_n x^n$ is absolutely convergent for any $x \in (-r, r)$ and it is divergent for any $x \in \mathbb{R} \setminus [-r, r]$. When $r = 0$, the only point of (absolute) convergence for $\sum_{n=0}^{\infty} a_n x^n$ is $x = 0$ (in this case, $\mathrm{D_c} = \{0\}$). If $r = +\infty$, the series is absolutely convergent (and, therefore, convergent) for any $x \in \mathbb{R}$.

Also, the sets $\mathrm{D_{ac}}$ and $\mathrm{D_c}$ are intervals in $\mathbb{R}$, with $\mathrm{D_{ac}}$ being symmetric ($x \in \mathrm{D_{ac}} \Rightarrow -x \in \mathrm{D_{ac}}$).

When the limit $l := \lim_{n\to+\infty} \sqrt[n]{|a_n|}$ exists, the radius of convergence is precisely $r := \dfrac{1}{l}$. In particular, if $a_n \neq 0$, $\forall n \in \mathbb{N}$, the existence of $\lambda := \lim_{n\to+\infty} \dfrac{|a_{n+1}|}{|a_n|}$ implies that $\lim_{n\to+\infty} \sqrt[n]{|a_n|} = \lambda$ and, consequently, $r := \dfrac{1}{\lambda}$.

The continuity on $(-r, r)$ of the function S defined in *iii)* can be easily retrived from *ii)*, by the transfer of the continuity carried by the uniform convergence. Abel theorem insures that S is also continuous in the extremities of the interval $[-r, r]$ (of course, in the case $r < +\infty$) for which the power series might converge.

Examples. In the following we present some examples of power series of the form $\sum_{n=0}^{\infty} a_n x^n$.

1. The *null series*: $a_n := 0$, $n \in \mathbb{N}$. We have $r = +\infty$, $\mathrm{D_{ac}} = \mathrm{D_c} = \mathbb{R}$.
2. The *geometric series*, $\sum_{n=0}^{\infty} x^n$. We have $r = 1$, $\mathrm{D_{ac}} = \mathrm{D_c} = (-1, 1)$.
3. The series $\sum_{n=0}^{\infty} n! x^n$: $r = 0$, $\mathrm{D_{ac}} = \mathrm{D_c} = \{0\}$.
4. The series $\sum_{n=1}^{\infty} \dfrac{1}{n^\alpha} x^n$, with $\alpha \in \mathbb{R}$. We have $r = 1$ and
   - $\mathrm{D_{ac}} = \begin{cases} (-1, 1), & \alpha \le 1; \\ [-1, 1], & \alpha > 1; \end{cases}$
   - $\mathrm{D_c} = \begin{cases} (-1, 1), & \alpha \le 0; \\ [-1, 1), & \alpha \in (0, 1]; \\ [-1, 1], & \alpha > 1. \end{cases}$
5. The *exponential series*, $\sum_{n=0}^{\infty} \dfrac{x^n}{n!}$. We have $r = +\infty$, $\mathrm{D_{ac}} = \mathrm{D_c} = \mathbb{R}$. Also,

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = \mathrm{e}^x, \ \forall x \in \mathbb{R}.$$

6. The *trigonometric series*, $\sum_{n=0}^{\infty} \dfrac{(-1)^n}{(2n+1)!} x^{2n+1}$ and $\sum_{n=0}^{\infty} \dfrac{(-1)^n}{(2n)!} x^{2n}$. Again we have $r = +\infty$, $\mathrm{D_{ac}} = \mathrm{D_c} = \mathbb{R}$. Also,

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = \sin x, \ \forall x \in \mathbb{R};$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = \cos x, \ \forall x \in \mathbb{R}.$$

7. The *binomial series.* For $n \in \mathbb{N}$ and $\alpha \in \mathbb{R} \setminus \{n + p \mid p \in \mathbb{N}\}$, we define

$$\mathrm{C}_\alpha^n := \begin{cases} \dfrac{\alpha \cdot \dots \cdot (\alpha - n + 1)}{n!}, & n > 0; \\ 1, & n = 0 \end{cases}$$

(therefore, C_α^n is now defined for any $n \in \mathbb{N}$ and $\alpha \in \mathbb{R}$). The series $\sum_{n=0}^{\infty} \mathrm{C}_\alpha^n x^n$, where $\alpha \in \mathbb{R}$, is called the *binomial series* (*of parameter* α). We have:

- if $\alpha \in \mathbb{N}$: $r = +\infty$, $\mathrm{D_{ac}} = \mathrm{D_c} = \mathbb{R}$;
- if $\alpha \le -1$: $r = 1$, $\mathrm{D_{ac}} = \mathrm{D_c} = (-1, 1)$;
- if $\alpha \in (-1, 0)$: $r = 1$, $\mathrm{D_{ac}} = (-1, 1)$, $\mathrm{D_c} = (-1, 1]$;
- if $\alpha \in \mathbb{R} \setminus \mathbb{N}$, $\alpha > 0$, then $r = 1$, $\mathrm{D_{ac}} = \mathrm{D_c} = [-1, 1]$.

Moreover, for any $\alpha \in \mathbb{R}$,

$$\sum_{n=0}^{\infty} \mathrm{C}_\alpha^n x^n = (1 + x)^\alpha, \ \forall x \in \mathrm{D_c}.$$

This generalizes the binomial formula, already know for $\alpha \in \mathbb{N}$, hence the name.

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