Substitutie este o functie $\sigma : X \to T$ cu proprietatea că

$\sigma(x) \neq x$ pt un nr finit de var $x \in X$.

Ex $\sigma_1 : X \to T$

- $\sigma_1(x) = i(y)$
- $\sigma_1(y) = x$
- $\sigma_1(x_1) = y_1$
- $\sigma_1(z) = z$ pt orice $z \in X \setminus \{x, y, x_1\}$

$dom(\sigma) = \{ x \in X \mid \sigma(x) \neq x \}$ – domeniul substitutiei σ

multime finită

Ex $dom(\sigma_1) = \{x, y, x_1\}$

$t = f(x, i(y)) \in T \longrightarrow \sigma_1^{\#}(t) = f(i(y), i(x))$

$\sigma^{\#} : T \to T$ – extinderea lui σ peste T

$$\sigma^{\#}(t) = \begin{cases} \sigma(x) & , t = x \in X \\ c & , t = c \in F_0 \\ f(\sigma^{\#}(t_1), \ldots \sigma^{\#}(t_n)) & , t = f(t_1, t_2, \ldots t_n) \quad n \geq 1. \end{cases}$$

$$\sigma_1^{\#}(f(x, i(y))) = f(\sigma_1^{\#}(x), \sigma_1^{\#}(i(y))) =$$

$$= f(\sigma_1(x), i(\sigma_1^{\#}(y))) =$$

$$= f\Big(i(y),\ i\big(\sigma_1(y)\big)\Big) =$$

$$= f\big(i(y),\ i(x)\big)$$

dacă $dom(\sigma) = \{x_1, \dots x_n\}$ atunci scriem $\sigma = \{x_1 \mapsto \sigma(x_1), \dots\ x_n \mapsto \sigma(x_n)\}$

Ex: $\sigma_1 = \{x \mapsto i(y),\ y \mapsto x,\ x_1 \mapsto y_1\}$

$\sigma/_V : X \to T$ — restricția lui σ la mulțimea V

$dom(\sigma) = \{x_1, \dots x_n\}$

$dom(\sigma/_V) = dom(\sigma) \cap V$

$\sigma/_V(x) = \sigma(x)$ pt orice $x \in V$

$\sigma/_V(x) = x$ pt orice $x \in X \setminus V$

Ex: $V = \{x, x_2, y\}$

$\sigma_1/_V = \{x \mapsto i(y),\ y \mapsto x\}$

$\sigma^b : LP1 \to LP1$ — extinderea lui σ peste $LP1$.

↳ aparițiile libere sunt înlocuite conform cu σ

ap. lib. ap. leg.

$$\varphi = \big(\forall x.\ P(x_1, x) \wedge P(x, y)\big) \vee Q(x)$$

$$\sigma_1^b(\varphi) = \Big(\forall x.\ P(y_1, x) \wedge P(x, x)\Big) \vee Q(i(y))$$

$\sigma^\flat : LP_1 \to LP1.$

1. $\sigma^\flat(P(t_1, \dots t_n)) = P(\sigma^\#(t_1), \dots, \sigma^\#(t_n))$

2. $\sigma^\flat(\neg \varphi) = \neg \sigma^\flat(\varphi)$

3. $\sigma^\flat((\varphi_1 \land \varphi_2)) = (\sigma^\flat(\varphi_1) \land \sigma^\flat(\varphi_2))$

4. $\sigma^\flat((\varphi_1 \lor \varphi_2)) = (\sigma^\flat(\varphi_1) \lor \sigma^\flat(\varphi_2))$

5. $\sigma^\flat((\varphi_1 \to \varphi_2)) = (\sigma^\flat(\varphi_1) \to \sigma^\flat(\varphi_2))$

6. $\sigma^\flat((\varphi_1 \leftrightarrow \varphi_2)) = (\sigma^\flat(\varphi_1) \leftrightarrow \sigma^\flat(\varphi_2))$

7. $\sigma^\flat((\forall x. \varphi)) = (\forall x. \rho^\flat(\varphi))$ $\quad \rho = \sigma|_{dom(\sigma) \setminus \{x\}}$

toate aparițiile lui x sunt legate.

$\varphi = P(x, y)$

8. $\sigma^\flat((\exists x. \varphi)) = (\exists x. \rho^\flat(\varphi))$ $\quad \rho = \sigma|_{dom(\sigma) \setminus \{x\}}$

ap. lib. $\quad$ ap. leg.

$\varphi = ((\forall x. (P(x_1, x) \land P(x, y))) \lor Q(x))$

$\sigma_1^\flat(\varphi) = (\sigma_1^\flat((\forall x. (P(x_1, x) \land P(x, y)))) \lor \sigma_1^\flat(Q(x)))$

$= ((\forall x. \rho_1^\flat((P(x_1, x) \land P(x, y)))) \lor Q(\sigma_1^\#(x)))$

$\sigma_1 = \{ x \mapsto i(y), y \mapsto x, x_1 \mapsto y_1 \}$

$\rho_1 = \sigma_1|_{\{x, y, x_1\} \setminus \{x\}} = \sigma_1|_{\{y, x_1\}} = \{ y \mapsto x, x_1 \mapsto y_1 \}$

$$= \left(\left(\forall x. \left(\rho_1^b(P(x_1, x)) \wedge \rho_1^b(P(x, y)) \right) \right) \vee Q\left(\sigma_1(x)\right) \right)$$

$$= \left(\left(\forall x. \left(P\left(\rho_1^\#(x_1), \rho_1^\#(x)\right) \wedge P\left(\rho_1^\#(x), \rho_1^\#(y)\right) \right) \right) \vee Q\left(i(y)\right) \right)$$

$$= \left(\left(\forall x. \left(P\left(y_1, x\right) \wedge P\left(x, x\right) \right) \right) \vee Q\left(i(y)\right) \right)$$

Secventa : $\underbrace{\{\varphi_1, \ldots \varphi_n\}}_{\Gamma} \vdash \varphi$ $\quad \varphi$ cons sintactică din Γ

$\Sigma = (\{P, Q\}, \{f, g, a, b\})$

$ar(P) = ar(Q) = 1$

$ar(f) = ar(g) = 1$

$ar(a) = ar(b) = 0$

Ex secvente : $\{P(a), Q(x)\} \vdash (P(a) \wedge Q(x))$

1. $\{(\varphi_1 \vee \varphi_2), \neg\varphi_2\} \vdash (\varphi_1 \vee \varphi_2)$ (ip)
2. $\{(\varphi_1 \vee \varphi_2), \neg\varphi_2\} \vdash \neg\varphi_2$ (ip)

caz 1 3 $\{(\varphi_1 \vee \varphi_2), \neg\varphi_2, \varphi_1\} \vdash \varphi_1$ (ip)

caz 2
4. $\{(\varphi_1 \vee \varphi_2), \neg\varphi_2, \varphi_2\} \vdash \varphi_2$ (ip)
5. —— // —— $\vdash \neg\varphi_2$ (ip)
6. —— // —— $\vdash \bot$ $(\neg e, 4, 5)$

7 ~~k~~. $\{(\varphi_1 \vee \varphi_2), \neg\varphi_2, \varphi_2\} \vdash \varphi_1$ $(\bot e, 6)$

n. $\{(\varphi_1 \vee \varphi_2), \neg\varphi_2\} \vdash \varphi_1$ $(\vee e, 1, 3, \overset{7}{\cancel{k}})$

$$\neg e \; \frac{\Gamma' \vdash \varphi \qquad \Gamma' \vdash \neg\varphi}{\Gamma' \vdash \bot}$$

$$\bot e \; \frac{\Gamma' \vdash \bot}{\Gamma' \vdash \varphi}$$

$$\sigma = \{x \mapsto \overset{i(y)}{t}\} \qquad \sigma^{\flat}(\varphi)$$

$$\{x \mapsto t\} \qquad \varphi[x \mapsto t]$$

$$\forall_e \quad \frac{\Gamma \vdash (\forall x.\ \varphi)}{\Gamma \vdash \varphi[x \mapsto t]}$$

1. $\{\forall x.(Om(x) \to Muritor(x)),\ Om(s)\} \vdash \forall x.(Om(x) \to Muritor(x))$ (ip)

2. $\{\forall x.(Om(x) \to Muritor(x)),\ Om(s)\} \vdash Om(s) \to Muritor(s)$ $(\forall_e, 1, x \mapsto s)$

3. $\{\forall x.(Om(x) \to Muritor(x)),\ Om(s)\} \vdash Om(s)$ (ip)

4. $\underbrace{\{\forall x.(Om(x) \to Muritor(x)),\ Om(s)\}}_{\Gamma} \vdash Muritor(s)$ $(\to_e, 2, 3)$

$$\exists_i \quad \frac{\Gamma \vdash \varphi[x \mapsto t]}{\Gamma \vdash \exists x.\ \varphi}$$

1. $\{P(a)\} \vdash P(\overset{x}{a})$ (ip) $\qquad P(a) = \underbrace{P(x)}_{\varphi}\,[x \mapsto a]$

2. $\{P(a)\} \vdash \exists x.\ P(x)$ $(\exists_i, 1)$

$P(t)$

$$\forall_i \quad \frac{\Gamma \vdash \varphi[x \mapsto x_0]}{\Gamma \vdash \forall x.\ \varphi} \quad x_0 \notin var(\Gamma, \varphi)$$

x_0 arbitrar.

1. $\{\forall x. (P(x) \to Q(x)), \forall x. P(x)\} \vdash \forall x. (P(x) \to Q(x))$ (ip)

2. $\{\forall x. (P(x) \to Q(x)), \forall x. P(x)\} \vdash P(x_0) \to Q(x_0)$ $(\forall_e, 1, x \mapsto x_0)$

3. $\{\forall x. (P(x) \to Q(x)), \forall x. P(x)\} \vdash \forall x. P(x)$ (ip)

4 $\cancel{j}$. $\{\forall x. (P(x) \to Q(x)), \forall x. P(x)\} \vdash P(x_0)$ $(\forall_e, 3, x \mapsto x_0)$

5 $\cancel{k}$. $\{\forall x. (P(x) \to Q(x)), \forall x. P(x)\} \vdash Q(x_0)$ $(\to_e, 2, \overset{4}{\cancel{j}})$

6 $\cancel{n}$. $\{\forall x. (P(x) \to Q(x)), \forall x. P(x)\} \vdash \forall x. Q(x)$ $(\forall_i, \overset{5}{\cancel{k}})$

acoperă toate cazurile pt x.

$$\exists_e \frac{\Gamma \vdash (\exists x. \varphi) \qquad \Gamma, \varphi[x \mapsto x_0] \vdash \psi}{\Gamma \vdash \psi} \quad x_0 \notin var(\Gamma, \varphi, \psi)$$