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$T: \mathbb{R}_2[t] \to \mathbb{R}_2[t]$

$T(a_0 + a_1 t + a_2 t^2) = a_0 + a_1 + (a_1 + a_2)t + (a_2 - a_0)t^2$

$\mathbb{R}_2[t]$ – mulț. pol. de grad cel mult t

$B_c = \{t^0, t^1, t^2\} = \{1, t, t^2\}$ bază canonică din $\mathbb{R}_2[t]$

$t^2 - t + 1 = 1 \cdot 1 + (-1) \cdot t + 1 \cdot t^2$

a) $T(\underbrace{1}_{a_0} \underbrace{-2}_{a_1} t + \underbrace{3}_{a_2} t^2) = 1 + (-2) + (-2+3)t + (3-1)t^2$
$= -1 + t + 2t^2$

$\mathbb{R}_2[t] = \{a_0 + a_1 t + a_2 t^2, \ a_0, a_1, a_2 \in \mathbb{R}\}$

b) A_{B_c} $\quad B_c = \{1, t, t^2\}$

$T(1) = \alpha_{11} \cdot 1 + \alpha_{12} \cdot t + \alpha_{13} t^2$
$T(t) = \alpha_{21} \cdot 1 + \alpha_{22} t + \alpha_{23} t^2$
$T(t^2) = \alpha_{31} \cdot 1 + \alpha_{32} t + \alpha_{33} t^2$

$A_{B_c} = \begin{pmatrix} \alpha_{11} & \alpha_{21} & \alpha_{31} \\ \alpha_{12} & \alpha_{22} & \alpha_{32} \\ \alpha_{13} & \alpha_{23} & \alpha_{33} \end{pmatrix}$

$T(1) = T(\underbrace{1}_{a_0} \cdot 1 + \underbrace{0}_{a_1} \cdot t + \underbrace{0}_{a_2} \cdot t^2) = 1 - t^2 = 1 \cdot 1 + 0 \cdot t - 1 \cdot t^2$
$a_0 = 1$
$a_1 = 0 = a_2$
$T(t) = T(0 \cdot 1 + \underbrace{1}_{a_1} \cdot t + 0 t^2) = 1 + t$
$T(t^2) = T(\underbrace{0}_{a_0} \cdot 1 + \underbrace{0}_{a_1} \cdot t + \underbrace{1}_{a_2} \cdot t^2) = 0 + 1 \cdot t + 1 \cdot t^2 = t + t^2$

$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -1 & 0 & 1 \end{pmatrix}$
$\quad \downarrow \quad \downarrow \quad \downarrow$
$\quad T(1) \ T(t) \ T(t^2)$

$T(a_0 + a_1 t + a_2 t^2) = a_0 T(1) + a_1 T(t) + a_2 T(t^2)$
$= a_0(1 - t^2) + a_1(1 + t) + a_2(t + t^2)$
$= a_0 + a_1 + (a_1 + a_2)t + (-a_0 + a_2)t^2$

rang $A = \text{rang} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -1 & 0 & 1 \end{pmatrix}$

det $A \neq 0 \Rightarrow$ rang $A = 3$
det $A = 0 \Rightarrow$ rang $A < 3$
Caut minor de ord 2: $\begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix} = 1 \neq 0 \Rightarrow$ rang $A = 2$

c) rang T = rang A = 2

$\text{Im}\,T = \{y \in \mathbb{R}_2[t] \mid \exists x \in \mathbb{R}_2[t] \text{ a.î. } Tx = y\}$
$= \{y \in \mathbb{R}_2[t] \mid \exists \bar{x} = a_0 + a_1 t + a_2 t^2 \text{ a.î. } T(a_0 + a_1 t + a_2 t^2) = y\}$
$= \{y \in \mathbb{R}_2[t] \mid \exists \bar{x} \text{ a.î. } a_0 T(1) + a_1 T(t) + a_2 T(t^2) = y\}$
$= \text{Lin}(T(1), T(t^2), T(t))$

Studiez liniara ind.
$\{T(1), T(t), T(t^2)\}$

$\alpha_1 T(1) + \alpha_2 T(t) + \alpha_3 T(t^2) = 0 + 0t + 0t^2 \Rightarrow \alpha_i = 0, i = \overline{1,3}$

$\alpha_1(1 - t^2) + \alpha_2(1 + t) + \alpha_3(t + t^2) = \bar{0}_{\mathbb{R}_2[t]}$

$\begin{cases} \alpha_1 + \alpha_2 = 0 \\ \alpha_2 + \alpha_3 = 0 \\ -\alpha_1 + \alpha_3 = 0 \end{cases}$ $\quad \det \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -1 & 0 & 1 \end{pmatrix} = 0 \Rightarrow \{T(1), T(t), T(t^2)\}$ liniar dependenți

$\begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix} = 1 \neq 0$ nec. princ α_1, α_2
sec : α_3
ec. prim 2, 3

$\begin{cases} \alpha_2 + \alpha_3 = 0 \\ -\alpha_1 + \alpha_3 = 0 \end{cases} \Rightarrow \begin{cases} \alpha_2 = -\alpha_3 \\ \alpha_1 = \alpha_3 \end{cases}$

$\Rightarrow \boxed{\alpha_1 = \alpha_3}$ $\quad \boxed{\alpha_2 = -\alpha_3}$

Rel. de dependență:

$\alpha_1 T(1) + \alpha_2 T(t) + \alpha_3 T(t^2) = 0$
$\alpha_3 T(1) - \alpha_3 T(t) + \alpha_3 T(t^2) = 0 \quad | : \alpha_3$
$\boxed{T(1) - T(t) + T(t^2) = 0}$
$T(t) = T(1) + T(t^2)$

$\text{Im}\,T = \{y \in \mathbb{R}_2[t] \mid \exists x \text{ a.î. } \underbrace{a_0 T(1) + a_1(T(1) + T(t^2)) + a_2 T(t^2)}_{(a_0 + a_1)T(1) + (a_1 + a_2)T(t^2)} = y\}$
$= \text{Lin}(T(1), T(t^2))$

$\{T(1), T(t^2)\}$ lin indep

$\alpha T(1) + \beta T(t^2) = 0 \Rightarrow \alpha = \beta = 0$
$\alpha(1 - t^2) + \beta(t + t^2) = 0 \cdot 1 + 0t + 0t^2$

$\alpha = 0$
$\beta = 0$
$-\alpha + \beta = 0$
$\Rightarrow \alpha = \beta = 0$ $\{T(1), T(t^2)\}$ – lin ind.

$\Rightarrow \{T(1), T(t^2)\}$ bază pt $\text{Im}\,T$

$\dim(\text{Im}\,T) = 2 = \text{rang}\,T$

$\text{Ker}\,T = \{\bar{x} \in \mathbb{R}_2[t] \text{ a.î. } T(\bar{x}) = 0\}$
$= \{x = a_0 + a_1 t + a_2 t^2 \in \mathbb{R}_2[t] \text{ a.î. } T(a_0 + a_1 t + a_2 t^2) = 0 + 0t + 0t^2\}$

$T(x) = 0 \iff a_0 + a_1 + (a_1 + a_2)t + (a_2 - a_0)t^2 = 0 \cdot 1 + 0t + 0t^2$

$\begin{cases} a_0 + a_1 = 0 \\ a_1 + a_2 = 0 \\ a_2 - a_0 = 0 \end{cases}$ $\quad \det \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -1 & 0 & 1 \end{pmatrix} = 0$

rang $A = 2$. nec. princ a_0, a_1

$\begin{cases} a_1 = -a_2 \\ a_0 = a_2 \end{cases}$

$\text{Ker}\,T = \{\bar{x} = a_2 + (-a_2)t + a_2 t^2, a_2 \in \mathbb{R}\}$
$= \{\bar{x} = a_2(1 - t + t^2), a_2 \in \mathbb{R}\} = \text{Lin}(1 - t + t^2)$

$\text{def}\,T = \dim(\text{Ker}\,T) = 1$

$(1 - t + t^2)$ lin ind? $\quad \alpha(1 - t + t^2) = 0 + 0 \cdot t + 0t^2$
$\alpha = 0 \quad -\alpha = 0 \quad \alpha = 0 \Rightarrow \alpha = 0$

$\dim \mathbb{R}_2[t] = 3 = \text{def}\,T + \text{rang}\,T$

$B = \{1, t, t^2\}$ $\quad a_0 + a_1 t + a_2 t^2 = a_0 \cdot 1 + a_1 t + a_2 t^2$

Matricea lui T în baza:
$B' = \{-1 + t + 3t^2, -t + 2t^2, 2 + 4t - t^2\}$

B' bază? $\quad \begin{vmatrix} -1 & 0 & 2 \\ 1 & -1 & 4 \\ +3 & 2 & -1 \end{vmatrix} = -1 + 4 + 6 + 8 \neq 0$

$\Rightarrow B'$ lin indep
card $B' = \dim \mathbb{R}_2[t]$ $\Big| \Rightarrow B'$ bază

$A_{B'B'}$

① $T(-1 + t + 3t^2) = a_{11}(-1 + t + 3t^2) + a_{12}(-t + 2t^2) + a_{13}(2 + 4t - t^2)$ $\quad = 2 + 4t + 4t^2$
$T(-t + 2t^2) = a_{21} b_1' + a_{22} b_2' + a_{23} b_3'$
$T(2 + 4t - t^2) = a_{31} b_1' + a_{32} b_2' + a_{33} b_3'$

$A_{B'} = \begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}$

① $\begin{cases} -a_{11} + 2a_{13} = 0 \\ a_{11} - a_{12} + 4a_{13} = 4 \\ 3a_{11} + 2a_{12} - a_{13} = 4 \end{cases}$

$B' = B_c \cdot S_{B_cB'}$

$A_{B'B'} = (S_{B_cB'})^{-1} A_{B_c} (S_{B_cB'})$

$A_{B'} = (S_{B_cB'})^{-1} A_{B_c} (S_{B_cB'})$

$S_{B_cB'} = \begin{pmatrix} -1 & 0 & 2 \\ 1 & -1 & 4 \\ 3 & 2 & -1 \end{pmatrix}$ $\quad B' = \{-1 + t + 3t^2, -t + 2t^2, 2 + 4t - t^2\}$

$A_{B'} = S_{B_cB'}^{-1} \cdot \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 2 \\ 1 & -1 & 4 \\ 3 & 2 & -1 \end{pmatrix}$

$S_{B_cB'}^T = \begin{pmatrix} -1 & 1 & 3 \\ 0 & -1 & 2 \\ 2 & 4 & -1 \end{pmatrix}$ $\quad \det S_{B_cB'} = -1 + 4 + 6 + 8 = 17$

$S_{B_cB'}^* = \begin{pmatrix} -7 & 4 & 2 \\ 13 & -5 & 6 \\ 5 & 2 & 1 \end{pmatrix}$

$A_{B'} = \frac{1}{17} \begin{pmatrix} -7 & 4 & 2 \\ 13 & -5 & 6 \\ 5 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 2 \\ 1 & -1 & 4 \\ 3 & 2 & -1 \end{pmatrix}$

$= \frac{1}{17} \begin{pmatrix} -9 & -3 & 6 \\ 7 & 8 & 1 \\ 4 & 7 & 3 \end{pmatrix} \begin{pmatrix} -1 & 0 & 2 \\ 1 & -1 & 4 \\ 3 & 2 & -1 \end{pmatrix}$

$= \frac{1}{17} \begin{pmatrix} 24 & 15 & -36 \\ 4 & -6 & 45 \\ 12 & -1 & 33 \end{pmatrix}$