Outline of the lecture

1. Series of real numbers – the general case
   - Convergence criteria
   - Absolute convergent series
   - Unconditionally convergent series

2. Power series
   - Series of functions. Uniform convergence
   - Power series

Series of real numbers – the general case

The alternate harmonic series. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$:

By noting $x_n := (-1)^{n+1} \frac{1}{n}$, $n \in \mathbb{N}^*$, we have:

$$\begin{aligned}|x_{n+1} + \cdots + x_{n+p}| &= \left|(-1)^{n+2} \frac{1}{n+1} + (-1)^{n+3} \frac{1}{n+2} + \cdots + (-1)^{n+p+1} \frac{1}{n+p}\right| \\ &= \frac{1}{n+1} - \frac{1}{n+2} + \cdots + (-1)^{p-1} \frac{1}{n+p} \leq \frac{1}{n+1}.\end{aligned}$$

Since $\lim_{n\to\infty} \frac{1}{n+1} = 0$,

$$\forall \varepsilon > 0, \ \exists n_\varepsilon \in \mathbb{N}, \ n \geq n_\varepsilon, p \in \mathbb{N}^* : |x_{n+1} + \cdots + x_{n+p}| < \varepsilon.$$

By Cauchy's convergence test, $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$ is convergent.

- A series $\sum_{n=1}^{\infty} x_n$ such that $x_n \cdot x_{n+1} \leq 0$, $\forall n \in \mathbb{N}^*$ is called an *alternate series*.

Dirichlet criterion

Theorem

Let $(x_n)_{n\geq 1}$ *and* $(y_n)_{n\geq 1}$ *be sequences of real numbers. Let* $S_n := x_1 + \cdots + x_n$, $n \in \mathbb{N}^*$. *If*

1. *the sequence* $(S_n)_{n\geq 1}$ *is bounded;*
2. *the sequence* $(y_n)_{n\geq 1}$ *is monotone and* $\lim\limits_{n\to\infty} y_n = 0$,

then the series $\sum\limits_{n=1}^{\infty} x_n y_n$ *is convergent.*

Example. Let us consider the series $\sum\limits_{n=1}^{\infty} \dfrac{\cos n}{\sqrt{n}}$. Let

$$S_n := \cos 1 + \cos 2 + \cdots + \cos n.$$

We have:

$$\begin{aligned}
2\sin\tfrac{1}{2}\cdot S_n &= 2\cos 1\cdot \sin\tfrac{1}{2} + 2\cos 2\cdot\sin\tfrac{1}{2} + \cdots + 2\cos n\cdot\sin\tfrac{1}{2}\\
&= \left[\sin\left(1+\tfrac{1}{2}\right) - \sin\left(1-\tfrac{1}{2}\right)\right] + \left[\sin\left(2+\tfrac{1}{2}\right) - \sin\left(2-\tfrac{1}{2}\right)\right] + \dots\\
&\cdots + \left[\sin\left(n-\tfrac{1}{2}\right) - \sin\left(n-\tfrac{3}{2}\right)\right] + \left[\sin\left(n+\tfrac{1}{2}\right) - \sin\left(n-\tfrac{1}{2}\right)\right]\\
&= \sin\left(n+\tfrac{1}{2}\right) - \sin\left(\frac{1}{2}\right) = 2\sin\tfrac{n}{2}\cdot\cos\tfrac{n+1}{2}.
\end{aligned}$$

Then the sequence $(S_n)_{n\geq 1}$ is bounded, because

$$|S_n| = \left|\frac{\sin\frac{n}{2}\cdot\cos\frac{n+1}{2}}{\sin\frac{1}{2}}\right| \leq \frac{1}{\left|\sin\frac{1}{2}\right|},\ \forall n\in\mathbb{N}^*.$$

The sequence $\left(\frac{1}{\sqrt{n}}\right)_{n\geq 1}$ is decreasing and convergent to 0; by Dirichlet criterion it follows that the series $\sum\limits_{n=1}^{\infty}\frac{\cos n}{\sqrt{n}}$ is convergent.

Leibniz criterion

Corollary

Let $(x_n)_{n\geq 1} \subseteq \mathbb{R}$ be a monotone sequence with $\lim\limits_{n\to\infty} x_n = 0$. Then the alternate series $\sum\limits_{n=1}^{\infty} (-1)^n x_n$ is convergent.

Proof.

In order to apply Dirichlet criterion for the series $\sum\limits_{n=1}^{\infty} (-1)^n x_n$, it is enough to see that the sequence of the partial sums, $\sum\limits_{n=1}^{\infty} (-1)^n$, is bounded. $\square$

Abel criterion

Theorem

Let $(x_n)_{n\geq 1}$ and $(y_n)_{n\geq 1}$ be sequences of real numbers. If

1. *the series $\sum_{n=1}^{\infty} x_n$ is convergent;*
2. *the sequence $(y_n)_{n\geq 1}$ is monotone and bounded,*

then the series $\sum_{n=1}^{\infty} x_n y_n$ is convergent.

Proof.

Since (y_n) is monotone and bounded, it is convergent. Let $y \in \mathbb{R}$ be its limit and $\tilde{y}_n := y_n - y$. Then $(\tilde{y}_n)$ is monotone with $\lim_{n\to\infty} \tilde{y}_n = 0$.

By Dirichlet criterion, $\sum_{n=1}^{\infty} x_n \tilde{y}_n$ is convergent. Also, $\sum_{n=1}^{\infty} x_n y$ is also convergent.

We then obtain that $\sum_{n=1}^{\infty} x_n(\tilde{y}_n + y)$ is convergent, *i.e.* $\sum_{n=1}^{\infty} x_n y_n$ (C). $\square$

Absolute convergent series

Definition

We say that a series of real numbers $\sum_{n=1}^{\infty} x_n$ is:

- *absolute convergent*, if $\sum_{n=1}^{\infty} |x_n|$ is convergent – we note: $\sum_{n=1}^{\infty} x_n$ (AC);
- *semiconvergent*, if $\sum_{n=1}^{\infty} x_n$ is convergent, but $\sum_{n=1}^{\infty} |x_n|$ is divergent – we note: $\sum_{n=1}^{\infty} x_n$ (SC).

- For series with positive terms, absolute convergence is equivalent with convergence.
- The alternate harmonic series is semiconvergent: it is convergent and $\sum_{n=1}^{\infty} \left|(-1)^{n+1}\frac{1}{n}\right|$ is divergent.

Theorem

If a series of real numbers is absolute convergent, then it is convergent.

Proof.

Let $\sum_{n=1}^{\infty} x_n$ be an absolute convergent series.

Let $\varepsilon > 0$; since $\sum_{n=1}^{\infty} |x_n|$ (C), by Cauchy's convergence test, we can find $n_\varepsilon \in \mathbb{N}^*$ such that

$$|x_{n+1}| + \cdots + |x_{n+p}| < \varepsilon, \ \forall n \geq n_\varepsilon, \ \forall p \in \mathbb{N}^*.$$

But $|x_{n+1} + \cdots + x_{n+p}| \leq |x_{n+1}| + \cdots + |x_{n+p}|$, so

$$|x_{n+1} + \cdots + x_{n+p}| < \varepsilon, \ \forall n \geq n_\varepsilon, \ \forall p \in \mathbb{N}^*.$$

Applying again Cauchy's test, we deduce that $\sum_{n=1}^{\infty} x_n$ is convergent. $\square$

The Cauchy product

Definition

Let $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$ be two series of real numbers. The series $\sum_{n=1}^{\infty} c_n$, where

$$c_n := x_1 y_n + x_2 y_{n-1} + \cdots + x_n y_1,$$

is called the *Cauchy product* of the series $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$.

This operation between series is commutative.

Theorem (Mertens)

Let $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$ be two series. If $\sum_{n=1}^{\infty} x_n$ (AC) and $\sum_{n=1}^{\infty} y_n$ (C), then the Cauchy product of $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$ is convergent. Moreover, its sum is equal to the product of the sums of the two series.

Cauchy Theorem

Mertens theorem has a simple consequence:

Theorem

The Cauchy product of two absolute convergent series is absolute convergent.

Remark. The Cauchy product of two convergent series is not necessarely convergent.

For example, set $x_n := (-1)^n \frac{1}{\sqrt{n+1}}$ and $y_n := x_n$ for $n \in \mathbb{N}$. By Leibniz criterion, the alternate series $\sum_{n=0}^{\infty} x_n$ and $\sum_{n=0}^{\infty} y_n$ are convergent. We define, for $n \in \mathbb{N}$,

$$c_n := \sum_{k=0}^{n} x_k y_{n-k} = \sum_{k=0}^{n} (-1)^n \frac{1}{\sqrt{(k+1)(n-k+1)}} = (-1)^n \sum_{k=0}^{n} \frac{1}{\sqrt{(k+1)(n-k+1)}}.$$

Since

$$|c_n| = \sum_{k=0}^{n} \frac{1}{\sqrt{(k+1)(n-k+1)}} \geq \sum_{k=0}^{n} \frac{2}{n+2} = \frac{2(n+1)}{n+2},$$

we have $c_n \not\to 0$. Hence $\sum_{n=0}^{\infty} c_n$ is not convergent.

Unconditionally convergent series

Theorem (Riemann)

Let $\sum_{n=1}^{\infty} x_n$ *be semiconvergent series. Then, for any* $S \in \overline{\mathbb{R}}$ *there exists a bijective function (a permutation)* $\varphi : \mathbb{N}^* \to \mathbb{N}^*$ *such that* $\sum_{n=1}^{\infty} x_{\varphi(n)} = S$.

Letting $S = +\infty$ or $S = -\infty$: we can permute the terms of a semiconvergent series in order to obtain a divergent one.

Definition

We say that a series of real numbers $\sum_{n=1}^{\infty} x_n$ is *unconditionally convergent* if for any bijective function $\varphi : \mathbb{N}^* \to \mathbb{N}^*$, the series $\sum_{n=1}^{\infty} x_{\varphi(n)}$ is convergent.

Obviously, an unconditionally convergent series is convergent.

Theorem

A series $\sum_{n=1}^{\infty} x_n$ *is unconditionally convergent if and only if it is absolute convergent. In this case, for every bijective function* $\varphi : \mathbb{N}^* \to \mathbb{N}^*$ *we have*

$$\sum_{n=1}^{\infty} x_{\varphi(n)} = \sum_{n=1}^{\infty} x_n.$$

Series of functions

Let $(f_n)_{n\geq 1}$ is a sequence of functions from a set E to $\mathbb{R}$.

By the *series of functions* $\sum_{n=1}^{\infty} f_n$ we understand the sequence of functions $(S_n)_{n\geq 1}$, where the functions $S_n : E \to \mathbb{R}$, $n \in \mathbb{N}^*$ are the *partial sums* of the series $\sum_{n=1}^{\infty} f_n$, defined by

$$S_n(x) := f_1(x) + \cdots + f_n(x), \ x \in E.$$

Uniform convergence

Definition

Let $f_n : E \to \mathbb{R}$, $n \in \mathbb{N}^*$ and $D \subseteq E$. Let S_n, $n \in \mathbb{N}^*$ be the partial sums of $\sum\limits_{n=1}^{\infty} f_n$.

- We say that $\sum\limits_{n=1}^{\infty} f_n$ *converges pointwise* on D if $\sum\limits_{n=1}^{\infty} f_n(x)$ is convergent for every $x \in D$, *i.e.* $\exists S : D \to \mathbb{R}$ such that $S_n \xrightarrow[D]{\mathrm{p}} S$. In this case we will write
$$\sum_{n=1}^{\infty} f_n = S \text{ on } D.$$
- We say that $\sum\limits_{n=1}^{\infty} f_n$ *converges uniformly* on D if $\exists S : D \to \mathbb{R}$ such that $S_n \xrightarrow[D]{\mathrm{u}} S$. In the case we will write
$$\sum_{n=1}^{\infty} f_n(x) = S(x) \text{ (UC)},\ x \in D.$$

Cauchy test of uniform convergence

As in the case of numeric series, we have a Cauchy test for uniform convergence:

Theorem

Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions from a set E to $\mathbb{R}$ and $D \subseteq E$. Then $\sum_{n=1}^{\infty} f_n$ converges uniformly on D if and only if

$$\forall \varepsilon > 0,\ \exists n_\varepsilon \in \mathbb{N},\ \forall n \geq n_\varepsilon,\ \forall p \in \mathbb{N}^*,\ \forall x \in D : |f_{n+1}(x) + \cdots + f_{n+p}(x)| < \varepsilon.$$

Power series

Definition

Let $(a_n) \subseteq \mathbb{R}$ be a sequence and $x_0 \in \mathbb{R}$.

- The series with parameter $x \in \mathbb{R}$,

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n$$

  is called the *power series* centered in x_0, with coefficients a_n, $n \in \mathbb{N}$.

- The set of those $x \in \mathbb{R}$ for which the series $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ is convergent (absolute convergent) is called the *domain of convergence* (*domain of absolute convergence*) of the power series, denoted $\mathrm{D_c}$ ($\mathrm{D_{ac}}$).

Theorem (Abel)

Let $\sum\limits_{n=0}^{\infty} a_n x^n$ be a power series. Then there exists a unique $r \in [0, +\infty]$, called radius of convergence of $\sum\limits_{n=0}^{\infty} a_n x^n$, such that

$$(-r, r) \subseteq \mathrm{D}_{\mathrm{ac}} \subseteq \mathrm{D}_{\mathrm{c}} \subseteq [-r, +r]$$

Moreover, we have:

1. $r = \dfrac{1}{\ell}$, *where* $\ell := \limsup\limits_{n\to\infty} \sqrt[n]{|a_n|}$ *(or* $\lim\limits_{n\to+\infty} \sqrt[n]{|a_n|}$, $\lim\limits_{n\to+\infty} \dfrac{|a_{n+1}|}{|a_n|}$, *whenever they exist);*
2. *the series* $\sum\limits_{n=0}^{\infty} a_n x^n$ *is uniformly convergent with respect to* $x \in [a, b] \subseteq \mathrm{D}_{\mathrm{ac}}$*;*
3. *the function* $S : \mathrm{D}_{\mathrm{c}} \to \mathbb{R}$, *defined as*

$$S(x) := \sum_{n=0}^{\infty} a_n x^n,\ x \in \mathrm{D}_{\mathrm{c}}$$

is continuous.

Examples

- The *null series*: $a_n := 0,\ n \in \mathbb{N}$. We have $r = +\infty$, $\mathrm{D_{ac}} = \mathrm{D_c} = \mathbb{R}$.
- The *geometric series*, $\sum_{n=0}^{\infty} x^n$. We have $r = 1$, $\mathrm{D_{ac}} = \mathrm{D_c} = (-1, 1)$.
- The series $\sum_{n=0}^{\infty} n! x^n$: $r = 0$, $\mathrm{D_{ac}} = \mathrm{D_c} = \{0\}$.
- The series $\sum_{n=1}^{\infty} \frac{1}{n^\alpha} x^n$, with $\alpha \in \mathbb{R}$. We have $r = 1$ and
  - $\mathrm{D_{ac}} = \begin{cases} (-1,1), & \alpha \le 1; \\ [-1,1], & \alpha > 1; \end{cases}$
  - $\mathrm{D_c} = \begin{cases} (-1,1), & \alpha \le 0; \\ [-1,1), & \alpha \in (0,1]; \\ [-1,1], & \alpha > 1. \end{cases}$
- The *exponential series*, $\sum_{n=0}^{\infty} \frac{x^n}{n!}$. We have $r = +\infty$, $\mathrm{D_{ac}} = \mathrm{D_c} = \mathbb{R}$. Also,

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = \mathrm{e}^x, \ \forall x \in \mathbb{R}.$$

- The *trigonometric series*, $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$ and $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$. Again we have $r = +\infty$, $\mathrm{D}_{\mathrm{ac}} = \mathrm{D}_{\mathrm{c}} = \mathbb{R}$. Also,

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = \sin x, \ \forall x \in \mathbb{R};$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = \cos x, \ \forall x \in \mathbb{R}.$$

- The *binomial series*. For $n \in \mathbb{N}$ and $\alpha \in \mathbb{R} \setminus \{n+p \mid p \in \mathbb{N}\}$, we define

$$\mathrm{C}_\alpha^n := \begin{cases} \dfrac{\alpha \cdot \dots \cdot (\alpha - n + 1)}{n!}, & n > 0; \\ 1, & n = 0 \end{cases}$$

(therefore, C_α^n is now defined for any $n \in \mathbb{N}$ and $\alpha \in \mathbb{R}$).

The series $\sum_{n=0}^{\infty} \mathrm{C}_\alpha^n x^n$ is called the *binomial series* (*of parameter* $\alpha \in \mathbb{R}$). We have:

  - if $\alpha \in \mathbb{N}$: $r = +\infty$, $\mathrm{D_{ac}} = \mathrm{D_c} = \mathbb{R}$;
  - if $\alpha \leq -1$: $r = 1$, $\mathrm{D_{ac}} = \mathrm{D_c} = (-1, 1)$;
  - if $\alpha \in (-1, 0)$: $r = 1$, $\mathrm{D_{ac}} = (-1, 1)$, $\mathrm{D_c} = (-1, 1]$;
  - if $\alpha \in \mathbb{R} \setminus \mathbb{N}$, $\alpha > 0$, then $r = 1$, $\mathrm{D_{ac}} = \mathrm{D_c} = [-1, 1]$.

Moreover, for any $\alpha \in \mathbb{R}$,

$$\sum_{n=0}^{\infty} \mathrm{C}_\alpha^n x^n = (1+x)^\alpha, \ \forall x \in \mathrm{D_c}.$$

This generalizes the binomial Newton formula, already known for $\alpha \in \mathbb{N}$.