Applications of differentiability. Extrema of functions
Lecture 11

Mathematics - 1$^{\text{st}}$ year, English

Faculty of Computer Science
Alexandru Ioan Cuza University of Iasi

e-mail: corina.forascu@gmail.com

facebook: Corina Forăscu

December 18, 2018

Outline of the lecture

1. Optimization problems in Euclidean spaces

2. Unrestrained extrema

3. Restricted extrema

Optimization problems in Euclidean spaces

- One of the possible applications of differentiability: establishing extremal points for functions involved in optimization problems.
- More precisely, these problems are targeting the minimization (or the maximization) of a so-called *cost functional* (*profit function*), in presence or absence of some restrictions.

Example 1. Least squares method

Suppose that we are experimenting on a certain physical quantity:

- "input" data: $a_1, a_2, \ldots, a_p$;
- we obtain the "output" values $b_1, b_2, \ldots, b_p$;
- we represent the points (a_k, b_k) $(k = \overline{1, p})$ in the Euclidean plane;
- we can (visually) estimate the nature of the unknown function φ satisfying $\varphi(a_k) = b_k$.
- More precisely, we estimate that the function φ has a certain type (polynomial, exponential, trigonometric, etc.), whose characteristics (real parameters) $c_1, c_2, \ldots, c_n$ have to be identified.

- For this purpose we use *the method of least squares*, by considering the problem of the minimization of the expression

$$\sum_{k=1}^{p} \left(\varphi(a_k; c_1, c_2, \ldots, c_n) - b_k \right)^2,$$

with respect to $(c_1, c_2, \ldots, c_n) \in \mathbb{R}^n$.

- Solving this problem (of unrestrained extrema) means finding $(c_1^0, c_2^0, \ldots, c_n^0) \in \mathbb{R}^n$ such that

$$\min \left\{ \sum_{k=1}^{p} \left(\varphi(a_k; c_1, c_2, \ldots, c_n) - b_k \right)^2 \right\} = \sum_{k=1}^{p} \left(\varphi(a_k; c_1^0, c_2^0, \ldots, c_n^0) - b_k \right)^2.$$

- We can conclude that the physical quantity under study has the law $y = \varphi(x; c_1^0, c_2^0, \ldots, c_n^0)$.

- We remark that if the graphic of $\{(a_k, b_k) | k = \overline{1, p}\}$ suggests us that φ is linear, then we can take $n = 2$ and $\varphi(x) := c_1 x + c_2$.

- In this case, the method of least squares leads us to the problem of minimization of the expression

$$\sum_{k=1}^{p} (c_1 a_k + c_2 - b_k)^2.$$

Example 2. Realization of a maximum profit/minimum cost

- In an economic theory, the space $\mathbb{R}^n$ is interpreted as the space of *consumer goods output*:
- each *consumer good* is characterized by a certain index $i \in \{1, 2, \ldots, n\}$;
- the *consumer goods output* is a vector $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, where the component x_i is the quantity in which the consumer good i is produced.
- In such a context, a *system of prices* is characterized by a function which associates to each consumer goods output a certain value.
- It is naturally considered that such function is a linear function, hence characterized by a vector $\mathbf{p} = (p_1, p_2, \ldots, p_n) \in \mathbb{R}^n$, where p_i is the "unitary" price of the consumer good i; the system of prices is therefore given by $\sum\limits_{i=1}^{n} p_i x_i$.

- When a production company is producing certain consumer goods, its interest is to realize the corresponding production such that the production costs to be minimal and/or the production profit to be maximal.
- A *cost function* and/or a *profit function* convenably chosen, named *objective functions*, are used by economical theorists.
- In the situation that the set of consumer goods output is $\mathbb{R}^n$, we deal with an unrestained extrema problem.
- If the set of consumer goods output is $K \subsetneq \mathbb{R}^n$ then we deal with a restricted extrema problem.
- In both cases, it is a *linear programming* problem if the objective function is linear and K is a linear subspace of $\mathbb{R}^n$.
- If the objective function is quadratic or convex, then the respective problem is called *quadratic*, respectively *convex optimization problem*.

Example 3. Maximal informational entropy

- The *entropy* was introduced as a mathematical quantity by Claude E. Shannon (1947);
- it is a function corresponding to the quantity of information offered by a certain source, by the means of a certain language, electrical signal or data file.
- This function, denoted by H is defined on the set of random variables

$$X = \left(\begin{array}{cccc} 1 & 2 & \ldots & n \\ p_1 & p_2 & \ldots & p_n \end{array} \right)$$

and has the expression

$$H(X) = -\sum_{k=1}^{n} p_k \cdot \log_2 p_k,$$

where p_k is the probability ($p_k \in (0, 1)$, $k = \overline{1, n}$ and $\sum_{k=1}^{n} p_k = 1$) that a particular message k is actually transmitted by the above source.

- A particular problem is to find an optimal distribution of the random variable X such that the value $H(X)$ is maximal.
- In other words, we study the problem of maximizing

$$- \sum_{k=1}^{n} p_k \log_2 p_k$$

with respect to $(p_1, p_2, \ldots, p_n)$ subject to the restriction $\sum_{k=1}^{n} p_k = 1$, $p_k \in (0, 1)$, $k = \overline{1, n}$.

Definition

Let $\emptyset \neq A \subseteq \mathbb{R}^n$, $f : A \to \mathbb{R}$ a function and $\mathbf{x}_0 \in A$. We say that $\mathbf{x}_0$ is:

- a *minimum point* of f if $f(\mathbf{x}) \geq f(\mathbf{x}_0)$, $\forall \mathbf{x} \in A$;
- a *maximum point* of f if $f(\mathbf{x}) \leq f(\mathbf{x}_0)$, $\forall \mathbf{x} \in A$;
- a *strict minimum point* of f if $f(\mathbf{x}) > f(\mathbf{x}_0)$, $\forall \mathbf{x} \in A \setminus \{\mathbf{x}_0\}$;
- a *strict maximum point* of f if $f(\mathbf{x}) < f(\mathbf{x}_0)$, $\forall \mathbf{x} \in A \setminus \{\mathbf{x}_0\}$;
- a *local minimum point* of f if there exists $V \in \mathscr{V}(\mathbf{x}_0)$ such that $f(\mathbf{x}) \geq f(\mathbf{x}_0)$, $\forall \mathbf{x} \in A \cap V$;
- a *local maximum point* of f if there exists $V \in \mathscr{V}(\mathbf{x}_0)$ such that $f(\mathbf{x}) \leq f(\mathbf{x}_0)$, $\forall \mathbf{x} \in A \cap V$;
- a *local strict minimum point* of f if there exists $V \in \mathscr{V}(\mathbf{x}_0)$ such that $f(\mathbf{x}) > f(\mathbf{x}_0)$, $\forall \mathbf{x} \in A \cap (V \setminus \{\mathbf{x}_0\})$;
- a *local strict maximum point* of f if there exists $V \in \mathscr{V}(\mathbf{x}_0)$ such that $f(\mathbf{x}) < f(\mathbf{x}_0)$, $\forall \mathbf{x} \in A \cap (V \setminus \{\mathbf{x}_0\})$;
- an *extremum* (*strict extremum*, *local extremum*, *local strict extremum*) point of f if $\mathbf{x}_0$ is a minimum (strict minimum, local minimum, respectively a local strict minimum) point or a maximum (strict maximum, local maximum, respectively a local strict maximum) point of f.

Remarks. Any global extremum point for a function f is always a local extremum point for f. The converse is not true.

1. If $D \subseteq \mathbb{R}^n$ is an open set and $f : D \to \mathbb{R}$ is a function, the problem of determining the global and/or local extremum points (and the associated extreme values) for f is called an *unrestricted extremum problem*.

2. Let $D \subseteq \mathbb{R}^n$ be an open set, $f : D \to \mathbb{R}$ a function and $A \subseteq \mathbb{R}^n$ a set of constraints. The problem of determining the global and/or local extremum points (and the associated extreme values) for $f|_{A \cap D}$ is called a *restricted extremum problem*.

Fermat theorem

Theorem (Fermat)

Let $A \subseteq \mathbb{R}^n$ be a non-empty set, $f : A \to \mathbb{R}$ a function and $\mathbf{x}_0 \in \overset{\circ}{A}$. If $\mathbf{x}_0$ is a local extremum point of f and f has partial derivatives (of first order) in $\mathbf{x}_0$, then

$$\frac{\partial f}{\partial x_1}(\mathbf{x}_0) = \cdots = \frac{\partial f}{\partial x_n}(\mathbf{x}_0) = 0.$$

Remark. The conclusion of the theorem can be written as $(\nabla f)(\mathbf{x}_0) = \mathbf{0}_{\mathbb{R}^n}$. If, moreover f is Fréchet differentiable in $\mathbf{x}_0$, this is also equivalent to

$$\mathrm{d}f(\mathbf{x}_0) = \mathbf{0}_{\mathrm{L}(\mathbb{R}^n;\mathbb{R})}.$$

Definition

Let $A \subseteq \mathbb{R}^n$, $\mathbf{x}_0 \in \overset{\circ}{A}$ and $f : A \to \mathbb{R}$ a function, Fréchet differentiable in $\mathbf{x}_0$. The element $\mathbf{x}_0$ is called a *critical point* (or *stationary point*) of f if $\mathrm{d}f(\mathbf{x}_0) = \mathbf{0}_{\mathrm{L}(\mathbb{R}^n;\mathbb{R})}$ (*i.e.* $(\nabla f)(\mathbf{x}_0) = \mathbf{0}_{\mathbb{R}^n}$).

Remark. Fermat theorem states that if $\mathbf{x}_0 \in \mathring{A}$ is a local extremum point of a Fréchet differentiable function f in $\mathbf{x}_0$, then $\mathbf{x}_0$ is critical point of f. The converse is not true, as we can see from the following example.

Example. Let $f : \mathbb{R}^2 \to \mathbb{R}$,

$$f(x, y) = x^2 - y^2, x, y \in \mathbb{R}.$$

- Then $(0, 0)$ is a critical point of f, since $\dfrac{\partial f}{\partial x}(0, 0) = 2x\big|_{x=0, y=0} = 0$ and $\dfrac{\partial f}{\partial y}(0, 0) = -2y\big|_{x=0, y=0} = 0$.

- On the other hand, $f(x, 0) = x^2 > f(0, 0) = 0 > -y^2 = f(0, y)$ for each $x \neq 0$ and each $y \neq 0$. This implies that $(0, 0)$ is not a local minimum point, nor a local maximum point for f.

Definition

Let $A \subseteq \mathbb{R}^n$, $\mathbf{x}_0 \in \mathring{A}$ and $f : A \to \mathbb{R}$ a function, Fréchet differentiable in $\mathbf{x}_0$. If $\mathbf{x}_0$ is a critical point of f, but is not a local extremum for f, we say that $\mathbf{x}_0$ is a *saddle point* of f.

Sufficent conditions of extremum
c

ase n=1

Theorem

Let $A \subseteq \mathbb{R}$ be an interval, $x_0 \in \mathring{A}$ and $f : A \to \mathbb{R}$ a function n-times ($n \geq 2$) derivable in x_0. Suppose that $f'(x_0) = f''(x_0) = \ldots = f^{(n-1)}(x_0) = 0$ and $f^{(n)}(x_0) \neq 0$.

   i) *If n is even, then x_0 is a local extremum point of f. More precisely, if $f^{(n)}(x_0) > 0$ then x_0 is a local minimum point of f, while if $f^{(n)}(x_0) < 0$ then x_0 is a local maximum point of f.*

   ii) *If n is odd, then x_0 is not a local extremum point of f.*

Case n>1

We will characterize extremum points of a function f in $\mathbf{x}_0$ according to the Fréchet derivative of order 2 in $\mathbf{x}_0$. Recall that

$$\left(\mathrm{d}^2 f(\mathbf{x}_0)\right)(\mathbf{v}) = \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{x}_0) v_i v_j, \forall \mathbf{v} = (v_1, v_2, \ldots, v_n) \in \mathbb{R}^n.$$

Theorem

Let $A \subseteq \mathbb{R}^n$ a non-empty set and $f : A \to \mathbb{R}$ a function which is C^2 on a neighbourhood of a critical point $\mathbf{x}_0 \in \mathring{A}$.

  i) *If $\left(\mathrm{d}^2 f(\mathbf{x}_0)\right)(\mathbf{v}) > 0$, $\forall \mathbf{v} \in \mathbb{R}^n \smallsetminus \{\mathbf{0}_{\mathbb{R}^n}\}$, then $\mathbf{x}_0$ is a local minimum point for f.*

  ii) *If $\left(\mathrm{d}^2 f(\mathbf{x}_0)\right)(\mathbf{v}) < 0$, $\forall \mathbf{v} \in \mathbb{R}^n \smallsetminus \{\mathbf{0}_{\mathbb{R}^n}\}$, then $\mathbf{x}_0$ is a local maximum point for f.*

  iii) *If $\mathrm{d}^2 f(\mathbf{x}_0)$ is an undefined quadratic form (i.e. $\exists \mathbf{v}', \mathbf{v}'' \in \mathbb{R}^n$ such that $\left(\mathrm{d}^2 f(\mathbf{x}_0)\right)(\mathbf{v}') < 0$ and $\left(\mathrm{d}^2 f(\mathbf{x}_0)\right)(\mathbf{v}'') > 0$), then $\mathbf{x}_0$ is a saddle point for f.*

Remarks.

1. If $\mathrm{d}^2 f(\mathbf{x}_0)$ is a positive or a negative semidefined quadratic form (*i.e.*, $\left(\mathrm{d}^2 f(\mathbf{x}_0)\right)(\mathbf{v}) \geq 0$, $\forall \mathbf{v} \in \mathbb{R}^n$ or, respectively, $\left(\mathrm{d}^2 f(\mathbf{x}_0)\right)(\mathbf{v}) \leq 0$, $\forall \mathbf{v} \in \mathbb{R}^n$), then we cannot determine the nature of the critical point.

2. By the hypotheses of the above result, we have that $\dfrac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{x}_0) = \dfrac{\partial^2 f}{\partial x_j \partial x_i}(\mathbf{x}_0)$, $\forall i \neq j$, by Schwarz theorem, so $\mathrm{d}^2 f(\mathbf{x}_0)$ is indeed a quadratic form (*i.e.*, coming from a symmetric bilinear form). We call the matrix associated to this quadratic form (relative to the canonical basis) the *Hessian* of f:

$$H_f(\mathbf{x}_0) := \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{x}_0) \right)_{1 \leq i, j \leq n}.$$

By Lecture 8, we have several methods for determining if $H_f(\mathbf{x}_0)$ is positive definite, negative definite or neither.

Proposition

Let $\emptyset \neq A \subseteq \mathbb{R}^n$ and $f : A \to \mathbb{R}$ a function which is C^2 on a neighbourhood of a critical point $\mathbf{x}_0 \in \mathring{A}$.

   i) *If all the eigenvalues of $H_f(\mathbf{x}_0)$ are positive, then $\mathbf{x}_0$ is a local minimum point for f.*
   ii) *If all the eigenvalues of $H_f(\mathbf{x}_0)$ are negative, then $\mathbf{x}_0$ is a local maximum point for f.*
   iii) *If $H_f(\mathbf{x}_0)$ has at least one positive and one negative eigenvalue, then $\mathbf{x}_0$ is a saddle point for f.*

Of course, we cannot say anything if all the eigenvalues are either all non-negative or all non-positive.

Proposition

Let $\varnothing \neq A \subseteq \mathbb{R}^n$ and $f : A \to \mathbb{R}$ a function having partial derivatives of order 2 which are continuous in a neighbourhood of a critical point $\mathbf{x}_0$ of f. Let $\Delta_k = \det \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{x}_0) \right)_{1 \leq i,j \leq k}$, $k = \overline{1, n}$, the principal minors of the Hessian $H_f(\mathbf{x}_0)$.

i) If $\Delta_k > 0$, $\forall k = \overline{1, n}$, then $\mathbf{x}_0$ is a local minimum point for f.

ii) If $(-1)^{k+1}\Delta_k < 0$, $\forall k = \overline{1, n}$, then $\mathbf{x}_0$ is a local maximum point for f.

iii) If there exist $j, k = \overline{1, n}$ such that $\Delta_j < 0$ and $(-1)^{k+1}\Delta_k > 0$, then $\mathbf{x}_0$ is a saddle point for f.

Again, we cannot conclude anything by just knowing that $\Delta_k \geq 0$, $\forall k = \overline{1, n}$ or $(-1)^{k+1}\Delta_k \leq 0$, $\forall k = \overline{1, n}$.

Case n=2

Remark. In the particular case $n = 2$, the above result tells us that if $f : A \subseteq \mathbb{R}^2 \to \mathbb{R}$ is C^2 on a neighbourhood of a critical point $\mathbf{x}_0 \in \mathring{A}$ and we denote $p := \dfrac{\partial^2 f}{\partial x^2}(\mathbf{x}_0)$, $q := \dfrac{\partial^2 f}{\partial x \partial y}(\mathbf{x}_0)$, $r := \dfrac{\partial^2 f}{\partial y^2}(\mathbf{x}_0)$, then:

  i) when $p > 0$ and $pr - q^2 > 0$, $\mathbf{x}_0$ is a local minimum point for f;

 ii) when $p < 0$ and $pr - q^2 > 0$, $\mathbf{x}_0$ is a local maximum point for f;

iii) when $pr - q^2 < 0$, $\mathbf{x}_0$ is a saddle point for f;

 iv) when $pr - q^2 = 0$, we cannot establish the nature of $\mathbf{x}_0$ by this method.

Method of least squares revisited

Coming back to the method of least squares, with φ being linear, we see that the function $f : \mathbb{R}^2 \to \mathbb{R}$, defined by

$$f(c_1, c_2) = \sum_{k=1}^{\ell} (c_1 a_k + c_2 - b_k)^2$$

is of class C^2 on $\mathbb{R}^2$ and has as critical points the elements $(c_1^0, c_2^0) \in \mathbb{R}^2$ satisfying

$$\begin{cases} \dfrac{\partial f}{\partial c_1}\left(c_1^0, c_2^0\right) = 2_{k=1}^{\ell}(c_1^0 a_k + c_2^0 - b_k)a_k = 0; \\ \dfrac{\partial f}{\partial c_2}\left(c_1^0, c_2^0\right) = 2_{k=1}^{\ell}(c_1^0 a_k + c_2^0 - b_k) = 0. \end{cases}$$

In order to solve this equation, equivalent to the linear system

$$\begin{cases} c_1^0{}_{k=1}^{\ell} a_k^2 + c_2^0{}_{k=1}^{\ell} a_k = {}_{k=1}^{\ell} b_k a_k; \\ c_1^0{}_{k=1}^{\ell} a_k^2 + \ell c_2^0 = {}_{k=1}^{\ell} b_k, \end{cases}$$

we compute its determinant

$$\begin{vmatrix} \sum_{k=1}^{\ell} a_k^2 & \sum_{k=1}^{\ell} a_k \\ \sum_{k=1}^{\ell} a_k & \ell \end{vmatrix} = \ell \sum_{k=1}^{\ell} a_k^2 - \left(\sum_{k=1}^{\ell} a_k \right)^2,$$

which is non-zero when not all $a_1, a_2, \ldots, a_\ell$ are equal, by Cauchy-Buniakowski-Schwarz inequality.
In this case, we have

$$c_1^0 = \frac{\ell \sum_{k=1}^{\ell} a_k b_k - \left(\sum_{k=1}^{\ell} a_k \right) \left(\sum_{k=1}^{\ell} b_k \right)}{\ell \sum_{k=1}^{\ell} a_k^2 - \left(\sum_{k=1}^{\ell} a_k \right)^2}$$

and

$$c_2^0 = \frac{\left(\sum_{k=1}^{\ell} b_k \right) \left(\sum_{k=1}^{\ell} a_k^2 \right) - \left(\sum_{k=1}^{\ell} a_k \right) \left(\sum_{k=1}^{\ell} a_k b_k \right)}{\ell \sum_{k=1}^{\ell} a_k^2 - \left(\sum_{k=1}^{\ell} a_k \right)^2}.$$

We then have

- $p = \dfrac{\partial^2 f}{\partial c_1^2} \left(c_1^0, c_2^0 \right) = 2 \sum_{k=1}^{\ell} a_k^2,$

- $q = \dfrac{\partial^2 f}{\partial c_1 \partial c_2} \left(c_1^0, c_2^0 \right) = 2 \sum_{k=1}^{\ell} a_k$ and

- $r = \dfrac{\partial^2 f}{\partial c_2^2} \left(c_1^0, c_2^0 \right) = 2\ell.$

- Again by Cauchy-Buniakowski-Schwarz inequality, we have $p > 0$ and $pr - q^2 > 0$, *i.e.* $\left(c_1^0, c_2^0 \right)$ is a global minimum point for f (not only a local minimum point, because $\left(c_1^0, c_2^0 \right)$ is the unique critical point of f).

Restricted extrema

Besides the situations we have to solve problems of extremum without constraints, there are also problems where we have to find extrema of functions constrained to some conditions. In this case, the respective extrema are called *restricted extrema*.

Setting. Let D be an open set (with respect to the Euclidean metric) in $\mathbb{R}^n \times \mathbb{R}^m$ ($n, m \in \mathbb{N}^*$) and $f : D \to \mathbb{R}$, $g : D \to \mathbb{R}^m$ functions of class C^1 on D.

- We look for the extremum points of f restricted to the supplimentary condition $g(\mathbf{x}, \mathbf{y}) = 0$, where $(\mathbf{x}, \mathbf{y}) \in D$.
- by denoting $A := \{(\mathbf{x}, \mathbf{y}) \in D \,|\, g(\mathbf{x}, \mathbf{y}) = \mathbf{0}_{\mathbb{R}^n}\}$ and by $g_1, g_2, \ldots, g_m$ the components of g, we see that the above problem demands the finding of extremum points of $f\big|_A$.
- *Terminology:* we say that a point $(\mathbf{x}_0, \mathbf{y}_0) \in A$ is a *local extremum* (*minimum*, *maximum*) or *saddle point* of f, *conditioned to the restrictions* $g_1(\mathbf{x}, \mathbf{y}) = 0$, $g_2(\mathbf{x}, \mathbf{y}) = 0$, ..., $g_m(\mathbf{x}, \mathbf{y}) = 0$, if it is a local extremum (minimum, maximum) or saddle point of $f\big|_A$.

- The above problem may look as an unrestrained extrema one, but the extremum points $(\mathbf{x}_0, \mathbf{y}_0)$ of $f|_A$ will be elements of $\mathring{A}$ if and only if $g_1 = g_2 = \cdots = g_m = 0$ on a neighbourhood of $(\mathbf{x}_0, \mathbf{y}_0)$, which in fact means that we would not effectively deal with a restricted extrema problem. Hence in general we cannot apply the results from the previous section.

- If D is bounded, then A is bounded, too; thus, since g is continuous, A is a closed, bounded set, hence compact.

- By the continuity of f, $f|_A$ will surely have a minimum and a maximum point in A. Therefore, the problem remains to determine these extremum points.

- In order to approach this problem, we could try to solve first the system $g_k(\mathbf{x}, \mathbf{y}) = 0$, $\forall\, k = \overline{1, m}$, with respect to $\mathbf{y}$. Finding a solution $\mathbf{y} = \varphi(\mathbf{x})$, $\mathbf{x} \in B$, would lead to studying the unrestrained extrema of the function $\mathbf{x} \mapsto f(\mathbf{x}, \varphi(\mathbf{x}))$, with $\mathbf{x} \in B$. The procedure is, however, difficult to apply in practice, because the equation $g(\mathbf{x}, \mathbf{y}) = \mathbf{0}_{\mathbb{R}^n}$ is not easily solved.

- This is why we will follow the method of *Lagrange multipliers*, described in the sequel.

Method of Lagrange multipliers

Theorem (existence of Lagrange multipliers)

Let $\varnothing \neq D \subseteq \mathbb{R}^n \times \mathbb{R}^m$ an open set, $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}^m$ C^1-functions on D. Let $g_1, g_2, \ldots, g_m$ be the components of g and $(\mathbf{x}_0, \mathbf{y}_0) \in D$ a local extremum point of f, conditioned to the restrictions $g_1(\mathbf{x}, \mathbf{y}) = 0$, $g_2(\mathbf{x}, \mathbf{y}) = 0$, $\ldots$, $g_m(\mathbf{x}, \mathbf{y}) = 0$. If $\frac{D(g_1, g_2, \ldots, g_m)}{D(y_1, y_2, \ldots, y_m)}(\mathbf{x}_0, \mathbf{y}_0) \neq 0$, then there exist $\lambda_1, \lambda_2, \ldots, \lambda_m \in \mathbb{R}$ such that $(\mathbf{x}_0, \mathbf{y}_0)$ is a critical point for the C^1-function $L : D \to \mathbb{R}$ defined by

$$L(\mathbf{x}, \mathbf{y}) = f(\mathbf{x}, \mathbf{y}) + \lambda_1 g_1(\mathbf{x}, \mathbf{y}) + \lambda_2 g_2(\mathbf{x}, \mathbf{y}) + \cdots + \lambda_m g_m(\mathbf{x}, \mathbf{y}), \ (\mathbf{x}, \mathbf{y}) \in D,$$

i.e.

$$(\star) \begin{cases} \dfrac{\partial L}{\partial x_k}(\mathbf{x}_0, \mathbf{y}_0) = 0, \ \forall k = \overline{1, n}; \\[2mm] \dfrac{\partial L}{\partial y_j}(\mathbf{x}_0, \mathbf{y}_0) = 0, \ \forall j = \overline{1, m}; \\[2mm] g_j(\mathbf{x}_0, \mathbf{y}_0) = 0, \ \forall j = \overline{1, m}. \end{cases}$$

Remarks.

1. The numbers $\lambda_1, \lambda_2, \ldots, \lambda_m$ are called *Lagrange multipliers*.

2. The conclusion of the above result can be rephrased as follows: the conditioned local extremum points of f are critical points of the associated function $L = f + \sum_{k=1}^{m} \lambda_k g_k$.

3. The function $\mathscr{L} : \mathbb{R}^m \times D \to \mathbb{R}$ defined by

$$\mathscr{L}(\boldsymbol{\lambda}, \mathbf{x}, \mathbf{y}) = f(\mathbf{x}, \mathbf{y}) + \lambda_1 g_1(\mathbf{x}, \mathbf{y}) + \lambda_2 g_2(\mathbf{x}, \mathbf{y}) + \cdots + \lambda_m g_m(\mathbf{x}, \mathbf{y}),$$

for $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_m) \in \mathbb{R}^m$, is called the *Lagrangian* associated to f and the restriction g.

Then the triple $(\boldsymbol{\lambda}_0, \mathbf{x}_0, \mathbf{y}_0)$ satisfies the system $(\star)$ (*i.e.*, $(\mathbf{x}_0, \mathbf{y}_0)$ is a critical point for L, satisfying the restriction $g(\mathbf{x}_0, \mathbf{y}_0) = \mathbf{0}_{\mathbb{R}^m}$) if and only if $(\boldsymbol{\lambda}_0, \mathbf{x}_0, \mathbf{y}_0)$ is a critical point for $\mathscr{L}$, since

$$\frac{\partial \mathscr{L}}{\partial \lambda_j}(\boldsymbol{\lambda}_0, \mathbf{x}_0, \mathbf{y}_0) = g_j(\mathbf{x}_0, \mathbf{y}_0), \ j = \overline{1, m}.$$

4. Suppose now that $(\lambda_0, \mathbf{x}_0, \mathbf{y}_0)$ satisfies the system $(\star)$ and f, g are functions of class C^2 on D (or at least on a neighbourhood of $(\lambda_0, \mathbf{x}_0, \mathbf{y}_0)$). Since

$$L(\mathbf{x}, \mathbf{y}) - L(\mathbf{x}_0, \mathbf{y}_0) = f(\mathbf{x}, \mathbf{y}) - f(\mathbf{x}_0, \mathbf{y}_0), \ \forall (\mathbf{x}, \mathbf{y}) \in A,$$

we see that $(\mathbf{x}_0, \mathbf{y}_0)$ is a conditioned local extremum point of f if and only if $(\mathbf{x}_0, \mathbf{y}_0)$ is a local extremum point for $L|_A$.

A practical way to determine sufficient conditions of conditioned local extremality is the following:

Let us denote $\mathbf{x} = (\tilde{x}_1, \ldots, \tilde{x}_n) \in \mathbb{R}^n$ and $\mathbf{y} = (\tilde{x}_{n+1}, \ldots, \tilde{x}_{n+m}) \in \mathbb{R}^m$; the second order Fréchet differential of L in $(\mathbf{x}_0, \mathbf{y}_0)$ is

$$(\star\star) \quad \sum_{i,j=1}^{n+m} \frac{\partial^2 L}{\partial \tilde{x}_i \partial \tilde{x}_j}(\mathbf{x}_0, \mathbf{y}_0) \mathrm{d}\tilde{x}_i \mathrm{d}\tilde{x}_j.$$

By considering the system obtained by differentiating the relations $g_k(\mathbf{x}, \mathbf{y}) = 0$, $k = \overline{1, m}$, i.e.

$$\sum_{j=1}^{n+m} \frac{\partial g_k}{\partial \tilde{x}_j}(\mathbf{x}_0, \mathbf{y}_0) \mathrm{d}\tilde{x}_j = 0$$

we can retrieve $\mathrm{d}\tilde{x}_{n+1}, \ldots, \mathrm{d}\tilde{x}_{n+m}$ as a function of $\mathrm{d}\tilde{x}_1, \ldots, \mathrm{d}\tilde{x}_n$ and replace them in $(\star\star)$.

This gives the quadratic form we look for as

$$\mathrm{d}^2 L(\mathbf{x}_0) = \sum_{i,j=1}^{n} a_{ij} \mathrm{d}\tilde{x}_i \mathrm{d}\tilde{x}_j.$$

Now we can decide if $\mathbf{x}_0$ is a local extremum point or a saddle point for L:

- if $\mathrm{d}^2 L(\mathbf{x}_0)$ is a positive or negative definite form, then $(\mathbf{x}_0, \mathbf{y}_0)$ is a conditioned local minimum, respectively maximum point of f;
- if $\mathrm{d}^2 L(\mathbf{x}_0)$ is undefined, then $(\mathbf{x}_0, \mathbf{y}_0)$ is a conditioned saddle point for f;
- if $\mathrm{d}^2 L(\mathbf{x}_0)$ is positive or negative semi-defined, we cannot decide the nature of $(\mathbf{x}_0, \mathbf{y}_0)$.

Shannon's entropy revisited

Let now apply this method for finding the extremum points for

$$\tilde{H}(p_1, p_2, \ldots, p_n) = - \sum_{k=1}^{n} p_k \log_2 p_k, \ p_k \in (0,1), \ \forall k = \overline{1,n},$$

subject to $\sum_{k=1}^{n} p_k = 1$. Since we have only one restriction, we have $m = 1$, $g(p_1, p_2, \ldots, p_n) := \sum_{k=1}^{n} p_k - 1$ and the Lagrangian is

$$\mathscr{L}(\lambda_1, p_1, p_2, \ldots, p_n) = - \sum_{k=1}^{n} p_k \log_2 p_k + \lambda_1 \left(\sum_{k=1}^{n} p_k - 1 \right).$$

Then $(\lambda_1^0, p_1^0, p_2^0, \ldots, p_n^0)$ is a critical point for $\mathscr{L}$ if

$$\begin{cases} \dfrac{\partial \mathscr{L}}{\partial p_k}(\lambda_1^0, p_1^0, p_2^0, \ldots, p_n^0) = - \left(\log_2 p_k^0 + \dfrac{1}{p_k^0 \ln 2} \right) + \lambda_1^0 = 0, \ \forall k = \overline{1,n} \\ \dfrac{\partial \mathscr{L}}{\partial \lambda_1}(\lambda_1^0, p_1^0, p_2^0, \ldots, p_n^0) = \sum_{k=1}^{n} p_k^0 - 1 = 0. \end{cases}$$

From the first n equations we yield $p_1^0 = p_2^0 = \ldots = p_n^0 = \psi(\lambda_1^0)$, where $\psi(\lambda)$ is the unique solution of the equation in $p \in (0,1)$: $\log_2 p + \dfrac{1}{p \ln 2} = \lambda$,

Lecture 11

while from the last one we retrieve

$$p_1^0 = p_2^0 = \cdots = p_n^0 = \frac{1}{n}, \ \lambda_1^0 = \log_2 \frac{1}{n} + \frac{n}{\ln 2}.$$

From the restraint $g(p_1, p_2, \ldots, p_n) = 0$ we get $\mathrm{d}p_1 + \mathrm{d}p_2 + \cdots + \mathrm{d}p_n = 0$; therefore

$$\mathrm{d}^2 L(p_1^0, p_2^0, \ldots, p_n^0) = \sum_{k=1}^{n} \frac{1}{p_k^0 \ln 2} \left(\frac{1}{p_k^0} - 1 \right) (\mathrm{d}p_k)^2,$$

which is a negative definite quadratic form in $\mathrm{d}p_1, \ldots, \mathrm{d}p_{n-1}$.

This means that the point $(p_1^0, p_2^0, \ldots, p_n^0) = \left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n} \right)$ is a maximum point for $\tilde{H}$, restricted to $p_1 + p_2 + \cdots + p_n = 0$.

In consequence, the maximal informational entropy is obtained for the random variable

$$X = \left(\begin{array}{cccc} 1 & 2 & \ldots & n \\ \frac{1}{n} & \frac{1}{n} & \ldots & \frac{1}{n} \end{array} \right).$$

📚 C. Canuto, A. Tabacco, *Mathematical Analysis II,* Springer, 2015.

📚 B. Edwards, R. Larson, *Calculus (9th Edition)*, Brooks/Cole-Cengage Learning, 2014.

📚 K. P. Evans, N. Jacob, *A Course in Analysis*, World Scientific, 2016.

📚 M. Gorunescu, *Lecții de analiză matematică pentru informaticieni*, Reprografia Universității Craiova, 2000.

📚 F. Iacob, *Matematică pentru anul II - ID*, seria 2004-2005.

📚 R. Luca-Tudorache, *Analiză matematică*, Editura Tehnopress, Iași, 2005.

📚 G. Păltineanu, *Analiză matematică. Calcul diferențial*, Editura AGIR, București, 2002.

📚 E. Popescu, *Analiză matematică. Calcul diferențial*, Editura Matrix Rom, București, 2006.

📚 M. Postolache, *Analiză matematică (teorie și aplicații)*, Ed. Fair Partners, București, 2011.

📚 V. Postolică, *Eficiență prin matematică aplicată. Analiză matematică*, Editura Matrix Rom, București, 2006.