LECTURE 3
SERIES OF REAL NUMBERS. SERIES WITH POSITIVE TERMS

1. Definition and properties

The purpose of this lecture is to familiarize the reader with *infinite sums*, known as (*infinite*) *series*. As opposite to *finite sum*, which is an algebraic concept, the series are a topological one, *i.e.* it is more related to the notion of limit or convergence.

Definition. Let $(x_n)_{n\geq 1}$ a sequence of real numbers. The *series* with *general terms* x_n, $x_n \in \mathbb{N}$ is the sequence

$$S_n := \sum_{k=1}^{n} x_k = x_1 + \cdots + x_n,\ n \in \mathbb{N}^*.$$

We will denote this series by $\sum\limits_{n=1}^{\infty} x_n$, $\sum\limits_{n\geq 1} x_n$ or even $x_1 + \cdots + x_n + \ldots$

a) For $n \in \mathbb{N}^*$, the term $S_n \in \mathbb{R}$ is called the *partial sum* of order n of the series.

b) If the sequence (S_n) is convergent, we say that the series is *convergent*; we denote this $\sum\limits_{n=1}^{\infty} x_n$ (C).

c) If the sequence (S_n) is divergent (it has no limit or has infinite limit), we say that the series is *divergent*; we denote this $\sum\limits_{n=1}^{\infty} x_n$ (D).

d) If $S_n \to S \in \overline{\mathbb{R}}$, we call S the *sum* of the series $\sum\limits_{n=1}^{\infty} x_n$ and write $\sum\limits_{n=1}^{\infty} x_n = S$.

Sometimes, we slightly adapt the definition for series of the form $\sum\limits_{n=p}^{\infty} x_n$ (also denoted $\sum\limits_{n\geq p} x_n$ or $x_p + \cdots + x_n + \ldots$) starting from $p \in \mathbb{N}$ with $p \neq 1$.

When we want to determine the nature of a series $\sum\limits_{n=1}^{\infty} x_n$ (*i.e.*, if it is convergent or divergent), it doesn't matter if we eliminate a finite number of terms from the series. In other words, $\sum\limits_{n=1}^{\infty} x_n$ has the same nature with the series $\sum\limits_{n=p}^{\infty} x_n$, where $p > 1$. However, its sum can change.

Definition. Let $\sum\limits_{n=1}^{\infty} x_n$ be a series of real numbers. For $p \in \mathbb{N}$, we call the *remainder* of order p of $\sum\limits_{n=1}^{\infty} x_n$ the series $\sum\limits_{n=p+1}^{\infty} x_n$.

Proposition 1.1. *Let $\sum\limits_{n=1}^{\infty} x_n$ be a convergent series of real numbers. Then, for any $p \in \mathbb{N}$, the remainder of order p is convergent. Moreover, if we denote*

$$R_p := \sum_{n=p+1}^{\infty} x_n,\ p \in \mathbb{N},$$

then $\lim_{p\to+\infty} R_p = 0$.

Proof. Let $S_n := \sum_{k=1}^{n} x_k$, $n \in \mathbb{N}^*$ and, for $p \in \mathbb{N}$ and $n \geq p+1$, $S_{n,p} := \sum_{k=p+1}^{n} x_n$. Then, for $p \in \mathbb{N}$,

$$S_{n,p} = S_n - S_p,\ \forall n \geq p+1.$$

Since the sequence $(S_n)_{n\geq 1}$ is convergent, it immediately follows that $(S_{n,p})_{n\geq p+1}$ is convergent, which means that $\sum_{n=p+1}^{\infty} x_n$ is convergent.

Moreover, if R_p is its sum, we have

$$R_p = S - S_p,$$

where $S := \sum_{n=1}^{\infty} x_n$, *i.e.* $S = \lim_{n\to+\infty} S_n$. Letting $p \to +\infty$ in the above relation, we get

$$\lim_{p\to+\infty} R_p = S - \lim_{p\to+\infty} S_p = S - S = 0.$$

□

Examples.

1. The series $\sum_{n=0}^{\infty} r^n$, where $r \in \mathbb{R}$ (the general term is $x_n = r^n$), is called the *geometric series* (of *ratio* r). The sequence of its partial sums is given by

$$S_n = 1 + r + \cdots + r^n = \begin{cases} \dfrac{1-r^n}{1-r}, & r \neq 1, \\ n+1, & r = 1. \end{cases}$$

Therefore, (S_n) converges for $r \in (-1,1)$ and (S_n) diverges for $r \in \mathbb{R} \setminus (-1,1)$, i.e. $\sum_{n=0}^{\infty} r^n$ (C) for $r \in (-1,1)$ and $\sum_{n=0}^{\infty} r^n$ (D) for $r \in \mathbb{R} \setminus (-1,1)$. We have also:

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r},\ r \in (-1,1);$$

$$\sum_{n=0}^{\infty} r^n = +\infty,\ r \geq 1.$$

In the case $r = 1$, the series $\sum_{n=0}^{\infty} (-1)^n = 1 - 1 + 1 - 1 + \ldots$ is also known as *Grandi* series; it is a divergent series.

2. The series $\sum_{n=1}^{\infty} \ln\left(1 + \frac{1}{n}\right)$ is divergent, because we can write its partial sums as

$$S_n = \sum_{k=1}^{n} \ln\left(1 + \frac{1}{k}\right) = \sum_{k=1}^{n} \ln \frac{k+1}{k} = \sum_{k=1}^{n} [\ln(k+1) - \ln k] = \ln(n+1) - \ln 1 = \ln(n+1),$$

which has $+\infty$ as its limit.

3. The series $\sum_{n=2}^{\infty} \frac{n-\sqrt{n^2-1}}{\sqrt{n^2-n}}$ is convergent, since we have

$$S_n := \sum_{k=2}^{n} \frac{k-\sqrt{k^2-1}}{\sqrt{k^2-k}} = \sum_{k=2}^{n} \left(\frac{k}{\sqrt{k^2-k}} - \frac{\sqrt{k^2-1}}{\sqrt{k^2-k}} \right) = \sum_{k=2}^{n} \left(\sqrt{\frac{k^2}{k^2-k}} - \sqrt{\frac{k^2-1}{k^2-k}} \right) = \sum_{k=2}^{n} \left(\sqrt{\frac{k}{k-1}} - \sqrt{\frac{k+1}{k}} \right)$$

$$= \sqrt{2} - \sqrt{\frac{n+1}{n}} \underset{n\to+\infty}{\longrightarrow} \sqrt{2} - 1.$$

Of course, we will have $\sum_{n=2}^{\infty} \frac{n-\sqrt{n^2-1}}{\sqrt{n^2-n}} = \sqrt{2} - 1$.

Remark. In examples **2** and **3**, we could write the partial sums as *telescopic sums*, which facilitates the finding of the sum.

Theorem 1.2. *Let* $\sum_{n=1}^{\infty} x_n$ *be a convergent series. Then* $\lim_{n\to\infty} x_n = 0$.

Proof. Let $S_n := \sum_{k=1}^{n} x_k$, $n \in \mathbb{N}^*$. Since $(S_n)_{n\geq 1}$ is convergent (because $\sum_{n=1}^{\infty} x_n$ is convergent) to some real $S \in \mathbb{R}$, we have that

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} (S_n - S_{n-1}) = S - S = 0.$$

□

Remark. When someone wants to find out if a series $\sum_{n=1}^{\infty} x_n$ is convergent, the first thing to check is that $\lim_{n\to\infty} x_n = 0$. If not, the series is clearly divergent. Attention, though: $\lim_{n\to\infty} x_n = 0$ does not necessarily imply that $\sum_{n=1}^{\infty} x_n$ is convergent!

A useful general criterion for the convergence of a series is the following:

Theorem 1.3 (Cauchy criterium). *The series* $\sum_{n=1}^{\infty} x_n$ *is convergent if and only if*

$$\forall \varepsilon > 0,\ \exists n_\varepsilon \in \mathbb{N}^*,\ \forall n \geq n_\varepsilon,\ \forall p \in \mathbb{N}^* : \left| x_{n+1} + \cdots + x_{n+p} \right| < \varepsilon.$$

Proof. Let $S_n := \sum_{k=1}^{n} x_k$, $n \in \mathbb{N}^*$. By Cauchy's criterion for the convergence of the sequences (see Lecture 2), the sequence (S_n) is convergent if and only if

$$\forall \varepsilon > 0,\ \exists n_\varepsilon \in \mathbb{N}^*,\ \forall n \geq n_\varepsilon,\ \forall p \in \mathbb{N}^* : \left| S_{n+p} - S_n \right| < \varepsilon.$$

But $S_{n+p} - S_n = x_{n+1} + \cdots + x_{n+p}$ for any $n, p \in \mathbb{N}^*$, which proves the assertion. □

By negating the Cauchy condition of convergence for a series, we obtain:

Proposition 1.4. *The series* $\sum\limits_{n=1}^{\infty} x_n$ *is divergent if and only if*

$$\exists \varepsilon > 0,\ \forall n \in \mathbb{N}^*,\ \exists k_n \geq n,\ \forall p_n \in \mathbb{N}^* : \left| x_{k_n+1} + \cdots + x_{k_n+p_n} \right| \geq \varepsilon.$$

Example. The series $\sum\limits_{n=1}^{\infty} \frac{1}{n}$ is called the *harmonic* series. With the help of the above result, it is easy to prove that it diverges. Indeed,

$$\frac{1}{n+1} + \cdots + \frac{1}{n+p} \geq \frac{p}{n+p},\ \forall n, p \in \mathbb{N}^*.$$

For $\varepsilon := 1/2$ (but it works for any other $\varepsilon \in (0,1)$) and any $n \in \mathbb{N}^*$, we can set $k_n := n$ and $p_n := n$; we will have

$$\left| \frac{1}{k_n+1} + \cdots + \frac{1}{k_n+p_n} \right| \geq \frac{p_n}{k_n+p_n} = \frac{1}{2} \geq \varepsilon.$$

Therefore $\sum\limits_{n=1}^{\infty} \frac{1}{n}$ (D).

Let now $\lambda \in \mathbb{R}$ and $\sum\limits_{n=1}^{\infty} x_n$, $\sum\limits_{n=1}^{\infty} y_n$ two series of real numbers. The series $\sum\limits_{n=1}^{\infty} (x_n + y_n)$ is called the *sum* of the series $\sum\limits_{n=1}^{\infty} x_n$ and $\sum\limits_{n=1}^{\infty} y_n$; the series $\sum\limits_{n=1}^{\infty} (\lambda x_n)$ is called the *product* of the series $\sum\limits_{n=1}^{\infty} x_n$ with the number (scalar) λ.

Theorem 1.5. *Let* $\lambda \in \mathbb{R}$ *and* $\sum\limits_{n=1}^{\infty} x_n$, $\sum\limits_{n=1}^{\infty} y_n$ *two convergent series, with* $S := \sum\limits_{n=1}^{\infty} x_n$ *and* $S' := \sum\limits_{n=1}^{\infty} y_n$. *Then:*

(i) *if* $x_n \leq y_n,\ \forall n \in \mathbb{N}^*$, *then* $S \leq S'$;

(ii) *the series* $\sum\limits_{n=0}^{\infty} (x_n + y_n)$ *is convergent and* $\sum\limits_{n=0}^{\infty} (x_n + y_n) = S + S'$;

(iii) *the series* $\sum\limits_{n=1}^{\infty} (\lambda x_n)$ *is convergent and* $\sum\limits_{n=1}^{\infty} (\lambda x_n) = \lambda S$.

Remark. The conclusions relative to S and S' remain true if we allow S and/or S' to be infinite, with the condition that the operations concerning them to be defined.

Theorem 1.6. *If we associate the terms of a convergent series in finite groups, by keeping the order of the terms, we still obtain a convergent series, with the same sum.*

Proof. Let $\sum\limits_{n=1}^{\infty} x_n$ be a convergent series and $(n_k)_{k \in \mathbb{N}^*} \subseteq \mathbb{N}^*$ a strictly increasing sequence of natural numbers with $n_1 = 1$. Let S be the sum of the series and S_n, $n \in \mathbb{N}^*$ its partial sums. By associating the terms from n_k to $n_{k+1} - 1$, for every $k \in \mathbb{N}^*$, we form a new series with the general term

$$y_k := x_{n_k} + \cdots + x_{n_{k+1}-1},\ k \in \mathbb{N}^*.$$

Let us calculate the partial sum of order k of the new series:

$$\begin{aligned} T_k &= y_1 + \cdots + y_k = (x_{n_1} + \cdots + x_{n_2-1}) + (x_{n_2} + \cdots + x_{n_3-1}) + \cdots + (x_{n_k} + \cdots + x_{n_{k+1}-1}) \\ &= x_1 + \cdots + x_{n_{k+1}-1} = S_{n_{k+1}-1}, \end{aligned}$$

by the associativity of the addition. Since $S_n \xrightarrow[n \to +\infty]{} S$, we also have that its subsequence $\left(S_{n_{k+1}-1}\right)_{k \in \mathbb{N}^*}$ also converges to S, which means that $\lim_{k \to +\infty} T_k = S$.

This proves that $\sum\limits_{k=1}^{\infty} y_k$ is convergent and its sum equals S. □

Remark. Sometimes, associating the terms of a divergent series provides a convergent series. For instance, if we associate every two terms in *Grandi* series $\sum\limits_{n=0}^{\infty} (-1)^n$ we obtain the series

$$\sum_{k=0}^{\infty} \left((-1)^{2k} + (-1)^{2k+1} \right) = (-1+1) + (-1+1) + \cdots + (-1+1) + \ldots,$$

which is obviously convergent (with sum 0).

2. Series with positive terms

We say that a series $\sum_{n=1}^{\infty} x_n$ has *positive terms* if $x_n \geq 0$, $\forall n \in \mathbb{N}^*$. For such a series, it is clear that the sequence of its partial terms is increasing, so it has a limit, which can be a positive real or $+\infty$. Therefore, a series with positive terms *always* has a sum (which may be finite or infinite). The following immediate result specifies when this sum is finite.

Proposition 2.1. *A series with positive terms is convergent if and only if the sequence of its partial terms is bounded.*

In this section we will mostly study convergence and divergence criteria for series with positive sums. The first we state are called *comparison criteria*; they specify the nature of a series (*i.e.*, it is convergent or divergent) by comparing it with another series whose nature is already known.

Let, for this purpose, $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$ be two series with positive terms.

Theorem 2.2 (Comparison criterion I – CC1). *Suppose that $x_n \leq y_n$, $\forall n \in \mathbb{N}^*$.*

- ***i)*** *If $\sum_{n=1}^{\infty} y_n$ (C) then $\sum_{n=1}^{\infty} x_n$ (C).*
- ***ii)*** *If $\sum_{n=1}^{\infty} x_n$ (D) then $\sum_{n=1}^{\infty} y_n$ (D).*

Proof. It is sufficient to prove the first assertion, the other being its reciproc.

Let $S_n := \sum_{k=1}^{n} x_k$, $T_n := \sum_{k=1}^{n} y_k$, for $n \in \mathbb{N}^*$. Since $\sum_{n=1}^{\infty} y_n$ is convergent, the sequence (T_n) is bounded, by Proposition 2.1. By the hypotheses,

$$S_n = \sum_{k=1}^{n} x_k \leq \sum_{k=1}^{n} y_k = T_n, \forall n \in \mathbb{N}^*,$$

hence (S_n) is also bounded. By the same Proposition, $\sum_{n=1}^{\infty} x_n$ is convergent. □

Theorem 2.3 (Comparison criterion II – CC2). *Suppose that $x_n > 0$, $y_n > 0$ and $\frac{x_{n+1}}{x_n} \leq \frac{y_{n+1}}{y_n}$, for every $n \in \mathbb{N}^*$.*

- ***i)*** *If $\sum_{n=1}^{\infty} y_n$ (C) then $\sum_{n=1}^{\infty} x_n$ (C).*
- ***ii)*** *If $\sum_{n=1}^{\infty} x_n$ (D) then $\sum_{n=1}^{\infty} y_n$ (D).*

Proof. Again, we will prove only the first assertion. For every $n \in \mathbb{N}^*$ we have that

$$x_{n+1} = x_1 \cdot \frac{x_2}{x_1} \cdot \dots \cdot \frac{x_{n+1}}{x_n} \leq x_1 \cdot \frac{y_2}{y_1} \cdot \dots \cdot \frac{y_{n+1}}{y_n} = \frac{x_1}{y_1} \cdot y_{n+1},$$

i.e. $x_n \leq \frac{x_1}{y_1} \cdot y_n$, $\forall n \in \mathbb{N}^*$. Since $\sum_{n=1}^{\infty} y_n$ is convergent, the series $\sum_{n=1}^{\infty} \left(\frac{x_1}{y_1} \cdot y_n\right)$ is also convergent. By CC1, $\sum_{n=1}^{\infty} x_n$ (C). □

Theorem 2.4 (Comparison criterion III – CC3). *Suppose that $y_n > 0$, $\forall n \in \mathbb{N}^*$ and there exists $\lim_{n\to+\infty} \frac{x_n}{y_n} = l \in [0, +\infty]$.*

- ***i)*** *If $l \in (0, +\infty)$, the series $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$ have the same nature.*
- ***ii)*** *If $l = 0$, we have:*
  - (a) *if $\sum_{n=1}^{\infty} y_n$ (C) then $\sum_{n=1}^{\infty} x_n$ (C);*
  - (b) *if $\sum_{n=1}^{\infty} x_n$ (D) then $\sum_{n=1}^{\infty} y_n$ (D).*
- ***iii)*** *If $l = +\infty$, we have:*
  - (a) *if $\sum_{n=1}^{\infty} x_n$ (C) then $\sum_{n=1}^{\infty} y_n$ (C);*
  - (b) *if $\sum_{n=1}^{\infty} y_n$ (D) then $\sum_{n=1}^{\infty} x_n$ (D).*

Proof.

i) We use the fact that $\lim_{n\to+\infty} \frac{x_n}{y_n} = l$; taking $\varepsilon := \frac{l}{2} > 0$, we get the existence of some $n_0 \in \mathbb{N}^*$ such that

$$\left|\frac{x_n}{y_n} - l\right| < \frac{l}{2},\ \forall n \geq n_0,$$

i.e.

$$\frac{l}{2} \cdot y_n < x_n < \frac{3l}{2} \cdot y_n,\ \forall n \geq n_0.$$

Since the series $\sum\limits_{n=1}^{\infty} x_n$ has the same nature with the series $\sum\limits_{n=n_0}^{\infty} x_n$ and the series $\sum\limits_{n=1}^{\infty} y_n$ has the same nature with each of series $\sum\limits_{n=n_0}^{\infty} \left(\frac{l}{2} \cdot y_n\right)$ and $\sum\limits_{n=n_0}^{\infty} \left(\frac{3l}{2} \cdot y_n\right)$, by CC1 it follows that $\sum\limits_{n=1}^{\infty} x_n$ and $\sum\limits_{n=1}^{\infty} y_n$ have the same nature.

ii) By using $\lim_{n\to+\infty} \frac{x_n}{y_n} = 0$ and taking $\varepsilon := 1$, there exists $n_0 \in \mathbb{N}^*$ such that

$$\frac{x_n}{y_n} < 1,\ \forall n \geq n_0,$$

i.e.

$$x_n < y_n,\ \forall n \geq n_0.$$

Now, since the series $\sum\limits_{n=1}^{\infty} x_n$ has the same nature with the series $\sum\limits_{n=n_0}^{\infty} x_n$ and the series $\sum\limits_{n=1}^{\infty} y_n$ has the same nature with $\sum\limits_{n=n_0}^{\infty} y_n$, the conclusion holds from CC1.

iii) The proof is very similar to that of the second point, with the difference that the reverse inequality is used. Indeed, since $\lim_{n\to+\infty} \frac{x_n}{y_n} = +\infty$, there exists $n_0 \in \mathbb{N}^*$ such that

$$\frac{x_n}{y_n} > 1,\ \forall n \geq n_0,$$

i.e.

$$x_n > y_n,\ \forall n \geq n_0.$$

The conclusion now follows from CC1, by swaping the roles of $\sum\limits_{n=1}^{\infty} x_n$ and $\sum\limits_{n=1}^{\infty} y_n$.

□

The next criterion applies to series $\sum\limits_{n=1}^{\infty} x_n$ where the sequence $(x_n)_{n\geq 1} \subseteq \mathbb{R}_+$ is monotone.

Theorem 2.5 (Cauchy's criterion of condensation). *Let $(x_n)_{n\geq 1} \subseteq \mathbb{R}_+$ a decreasing sequence. Then the series $\sum\limits_{n=1}^{\infty} x_n$ has the same nature as the series $\sum\limits_{n=1}^{\infty} (2^n x_{2^n})$.*

The above result holds also if we require only that $(x_n)_{n\geq 1} \subseteq \mathbb{R}_+$ is a monotone sequence. Indeed, if (x_n) is not decreasing, we have that both sequences (x_n) and $(2^n x_{2^n})$ do *not* converge to 0, which implies that the series $\sum\limits_{n=1}^{\infty} x_n$ and $\sum\limits_{n=1}^{\infty} (2^n x_{2^n})$ are divergent.

Proof. Let, for $n \in \mathbb{N}^*$, $S_n := \sum\limits_{k=1}^{n} x_k$ and $T_n := \sum_{k=1}^{n} 2^k x_{2^k}$. We have, by the fact that $(x_n)_{n\geq 1}$ is decreasing,

$$T_n = \sum_{k=1}^{n} 2^k x_{2^k} \leq \sum_{k=1}^{n} 2(x_{2^{k-1}} + \cdots + x_{2^k - 1}) = 2(x_1 + \cdots + x_{2^n - 1}) = 2S_{2^n - 1},\ \forall n \in \mathbb{N}^*,$$

that the convergence of $\sum\limits_{n=1}^{\infty} x_n$ implies the convergence of $\sum\limits_{n=1}^{\infty} (2^n x_{2^n})$. Conversely, by the same property of $(x_n)_{n\geq 1}$,

$$S_{2^n - 1} = \sum_{k=1}^{n} (x_{2^{k-1}} + \cdots + x_{2^k - 1}) \leq \sum_{k=1}^{n} 2^{k-1} x_{2^{k-1}} = \sum_{k=0}^{n-1} 2^k x_{2^k} = x_1 + T_{n-1},\ \forall n \in \mathbb{N}^*,$$

so the convergence of $\sum\limits_{n=1}^{\infty} (2^n x_{2^n})$ implies the convergence of $\sum\limits_{n=1}^{\infty} x_n$, by Proposition 2.1.

Therefore, the nature of the two series is the same. □

Example. The series $\sum\limits_{n=1}^{\infty} \frac{1}{n^\alpha}$ is called the *generalized harmonic series* (*of parameter $\alpha \in \mathbb{R}$*). In the case $\alpha = 1$, we recover the harmonic series.

Applying Cauchy's criterion of condensation, the series $\sum_{n=1}^{\infty} \frac{1}{n^{\alpha}}$ has the same nature with the series $\sum_{n=1}^{\infty} \frac{2^n}{(2^n)^{\alpha}} = \sum_{n=1}^{\infty} \frac{2^n}{2^{n\alpha}} = \sum_{n=1}^{\infty} \left(2^{1-\alpha}\right)^n$. But the last series is the geometric series with ratio $2^{1-\alpha}$ and we know that it converges if and only if $2^{1-\alpha} \in (-1,1)$, *i.e.* $\alpha > 1$. As a conclusion, $\sum_{n=1}^{\infty} \frac{1}{n^{\alpha}}$ (C) for $\alpha > 1$ and $\sum_{n=1}^{\infty} \frac{1}{n^{\alpha}}$ (D) if $\alpha \leq 1$ (in particular, we retrive that the harmonic series is divergent).

Theorem 2.6 (root test of Cauchy). *Let $\sum_{n=1}^{\infty} x_n$ be a series with positive terms such that there exists $\lim_{n\to\infty} \sqrt[n]{x_n} = l \in [0,+\infty]$.*

i) *If $l < 1$, then $\sum_{n=1}^{\infty} x_n$ (C).*

ii) *If $l > 1$, then $\sum_{n=1}^{\infty} x_n$ (D).*

In the case $l = 1$, we cannot say anything about the nature of the series $\sum_{n=1}^{\infty} x_n$ (take for example the generalized harmonic function). In that case, we should apply further tests (criteria).

Proof.

i) If $\alpha \in \mathbb{R}$ is such that $l < \alpha < 1$, we can find $n_0 \in \mathbb{N}^*$ such that

$$\sqrt[n]{x_n} < \alpha, \ \forall n \geq n_0,$$

i.e. $x_n < \alpha^n$ for any $n \geq n_0$. By CC1, since the geometric series $\sum_{n=n_0}^{\infty} \alpha^n$ is convergent, we have that $\sum_{n=1}^{\infty} x_n$ is convergent, too.

ii) Since $l > 1$, we can find $n_0 \in \mathbb{N}^*$ such that

$$\sqrt[n]{x_n} > 1, \ \forall n \geq n_0,$$

i.e. $x_n > 1$ for any $n \geq n_0$. By CC1, since the (geometric) series $\sum_{n=n_0}^{\infty} 1$ is divergent, we have that $\sum_{n=1}^{\infty} x_n$ is divergent, too.

□

Remark. In the case that $\sqrt[n]{x_n}$ does not necessarely possess a limit, we can replace $\lim_{n\to\infty} \sqrt[n]{x_n}$ by $\limsup_{n\to\infty} \sqrt[n]{x_n}$ in the statement of the result.

Theorem 2.7 (Kummer criterion). *Let $\sum_{n=1}^{\infty} x_n$ be a series with $x_n > 0$, $\forall n \in \mathbb{N}^*$ and a sequence $(a_n)_{n\geq 1} \subseteq \mathbb{R}_+^*$. Suppose that there exists $\lim_{n\to\infty} \left(a_n \cdot \frac{x_n}{x_{n+1}} - a_{n+1}\right) = l \in \overline{\mathbb{R}}$.*

i) *If $l > 0$, then $\sum_{n=1}^{\infty} x_n$ (C).*

ii) *If $l < 0$ and $\sum_{n=1}^{\infty} \frac{1}{a_n}$ (D), then $\sum_{n=1}^{\infty} x_n$ (D).*

Again, in the case $l = 0$, we cannot say anything about the nature of the series $\sum_{n=1}^{\infty} x_n$.

Proof.

i) Let $\varepsilon \in (0,l)$; we can find $n_0 \in \mathbb{N}^*$ such that

$$a_n \cdot \frac{x_n}{x_{n+1}} - a_{n+1} > \varepsilon, \ \forall n \geq n_0,$$

i.e.

$$a_n x_n - a_{n+1} x_{n+1} > \varepsilon x_{n+1}, \ \forall n \geq n_0.$$

By summing these inequalities from n_0 to $n-1$ we obtain

$$a_{n_0} x_{n_0} - a_n x_n > \varepsilon (x_{n_0} + \cdots + x_n), \ \forall n \geq n_0.$$

Therefore,

$$x_{n_0} + \cdots + x_n < \frac{a_{n_0} x_{n_0} - a_n x_n}{\varepsilon} \leq \frac{a_{n_0} x_{n_0}}{\varepsilon}.$$

This implies that the sequence of the partial sums of the series $\sum_{n=n_0}^{\infty} x_n$ is bounded, so $\sum_{n=1}^{\infty} x_n$ is convergent.

ii) Since $l < 0$, there exists $n_0 \in \mathbb{N}^*$ such that

$$a_n \cdot \frac{x_n}{x_{n+1}} - a_{n+1} < 0, \ \forall n \geq n_0,$$

i.e.

$$\frac{x_{n+1}}{x_n} > \frac{a_n}{a_{n+1}} = \frac{\frac{1}{a_{n+1}}}{\frac{1}{a_n}}, \ \forall n \geq n_0.$$

By CC2, since $\sum_{n=n_0}^{\infty} \frac{1}{a_n}$ is divergent, $\sum_{n=1}^{\infty} x_n$ is divergent, too.

□

By choosing particular sequences $(a_n)_{n\geq 1}$ – $(1)_{n\geq 1}$, $(n)_{n\geq 1}$, or $(n \ln n)_{n\geq 1}$ – we get three different criteria of convergence for series with positive terms:

Corollary (ratio test or d'Alembert criterion). *Let* $\sum_{n=1}^{\infty} x_n$ *be a series with* $x_n > 0$, $\forall n \in \mathbb{N}^*$. *Suppose that there exists* $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = l \in [0, +\infty]$.

i) *If* $l < 1$, *then* $\sum_{n=1}^{\infty} x_n$ *(C).*

ii) *If* $l > 1$, *then* $\sum_{n=1}^{\infty} x_n$ *(D).*

Corollary (Raabe-Duhamel criterion). *Let* $\sum_{n=1}^{\infty} x_n$ *be a series with* $x_n > 0$, $\forall n \in \mathbb{N}^*$. *Suppose that there exists*

$$\lim_{n\to\infty} n\left(\frac{x_n}{x_{n+1}} - 1\right) = \rho \in \overline{\mathbb{R}}.$$

i) *If* $\rho > 1$, *then* $\sum_{n=1}^{\infty} x_n$ *(C).*

ii) *If* $\rho < 1$, *then* $\sum_{n=1}^{\infty} x_n$ *(D).*

In order to show the second part of this criterion we use the divergence of the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$. We usually try to apply this criterion when the ratio test fails.

Corollary (Bertrand criterion). *Let* $\sum_{n=1}^{\infty} x_n$ *be a series with* $x_n > 0$, $\forall n \in \mathbb{N}^*$. *Suppose that there exists*

$$\lim_{n\to\infty} \left(n \ln n \cdot \frac{x_n}{x_{n+1}} - (n+1)\ln(n+1)\right) = \mu \in \overline{\mathbb{R}}.$$

i) *If* $\mu > 0$, *then* $\sum_{n=1}^{\infty} x_n$ *(C).*

ii) *If* $\mu < 0$, *then* $\sum_{n=1}^{\infty} x_n$ *(D).*

For proving the second part of this criterion we use the fact that the series $\sum_{n=1}^{\infty} \frac{1}{n \ln n}$ is divergent. Indeed, by Cauchy's criterion of condensation, this series has the same nature with $\sum_{n=1}^{\infty} \frac{2^n}{2^n \ln(2^n)} = \sum_{n=1}^{\infty} \frac{1}{n \ln 2} = \frac{1}{\ln 2} \sum_{n=1}^{\infty} \frac{1}{n} = +\infty$.

The last criterion we give here is also the most general; it is usually applied when Raabe-Duhamel criterion fails.

Theorem 2.8 (Gauss criterion). *Let* $\sum_{n=1}^{\infty} x_n$ *be a series with* $x_n > 0$, $\forall n \in \mathbb{N}^*$. *Suppose that there exists* $\alpha, \beta \in \mathbb{R}$, $\gamma \in \mathbb{R}_+^*$ *and a bounded sequence* $(y_n)_{n\geq 1}$ *such that*

$$\frac{x_n}{x_{n+1}} = \alpha + \frac{\beta}{n} + \frac{y_n}{n^{1+\gamma}}, \ \forall n \in \mathbb{N}^*.$$

i) *If* $\alpha > 1$, *then* $\sum_{n=1}^{\infty} x_n$ *(C).*

ii) *If* $\alpha < 1$, *then* $\sum_{n=1}^{\infty} x_n$ *(D).*

iii) *If* $\alpha = 1$ *and* $\beta > 1$, *then* $\sum_{n=1}^{\infty} x_n$ *(C).*

iv) *If* $\alpha = 1$ *and* $\beta \le 1$, *then* $\sum_{n=1}^{\infty} x_n$ *(D).*

Proof. Since $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \frac{1}{\alpha}$ and $\lim_{n\to\infty} n\left(\frac{x_n}{x_{n+1}} - \alpha\right) = \beta$, we have to study only the case $\alpha = 1$ and $\beta = 1$; indeed, in the case $\alpha \neq 1$ we simply apply the ratio test, while in the case $\alpha = 1$, $\beta \neq 1$ we apply Raabe-Duhamel criterion.

Let us apply Bertrand criterion. We have

$$\begin{aligned} n\ln n \cdot \frac{x_n}{x_{n+1}} - (n+1)\ln(n+1) &= n\ln n\left(1 + \frac{1}{n} + \frac{y_n}{n^{1+\gamma}}\right) - (n+1)\ln(n+1) \\ &= (n+1)\left[\ln n - \ln(n+1)\right] + \frac{y_n \ln n}{n^{\gamma}} \\ &= \ln\left[\left(1+\frac{1}{n}\right)^{-(n+1)}\right] + \frac{y_n \ln n}{n^{\gamma}}. \end{aligned}$$

Since $\lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^{-(n+1)} = \mathrm{e}^{-1}$, $\lim_{n\to+\infty} \frac{\ln n}{n^{\gamma}} = 0$ and the sequence $(y_n)_{n\ge 1}$ is bounded, we get

$$\lim_{n\to\infty}\left(n\ln n \cdot \frac{x_n}{x_{n+1}} - (n+1)\ln(n+1)\right) = -1.$$

Therefore, in the case $\alpha = 1$, $\beta = 1$, the series $\sum_{n=1}^{\infty} x_n$ is divergent, by Bertrand criterion. □

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