# Mathematics
# course notes – 1st chapter

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# Agenda

1. Sets
2. Relations
3. Functions

# Sets

Slides based on those of Johnnie Baker, Univ. of Kent

# Set Theory - Definitions and notation

A ***set*** is an unordered collection of objects referred to as elements.

A set is said to contain its elements.

Different ways of describing a set.

**1 – Explicitly: listing the elements of a set**

$\{1, 2, 3\}$ is the set containing "1" and "2" and "3." list the members between braces.
$\{1, 1, 2, 3, 3\} = \{1, 2, 3\}$ since repetition is irrelevant.
$\{1, 2, 3\} = \{3, 2, 1\}$ since sets are unordered.
$\{1,2,3, \ldots, 99\}$ is the set of positive integers less than 100; use ellipses when the general pattern of the elements is obvious.
$\{1, 2, 3, \ldots\}$ is a way we denote an infinite set (in this case, the natural numbers).
$\varnothing = \{\}$ is the empty set, or the set containing no elements.

Note: $\varnothing \neq \{\varnothing\}$

# Set Theory - Definitions and notation

**2 – Implicitly: by using a set builder notations, stating the property or properties of the elements of the set.**

$S = \{m \mid 2 \leq m \leq 100, m \text{ is integer}\}$

S is
the set of
all m
such that
m is between 2 and 100
and
m is integer.

# Set Theory - Ways to define sets

: and | are read "such that" or "where"

Explicitly: {John, Paul, George, Ringo}

Implicitly: {1,2,3,…}, or {2,3,5,7,11,13,17,…}

Set builder: { x : x is prime }, { x | x is odd }. In general { x : P(x) is true }, where P(x) is some description of the set.

Let D(x,y) denote "x is divisible by y."

Give another name for

$$\{ x : \forall y \, ((y > 1) \land (y < x)) \rightarrow \neg D(x,y) \}.$$

Primes

Can we use **any** predicate P to define a set S = { x : P(x) }?

Reveals contradiction in Frege's naïve set theory.

"the set of all sets that do not contain themselves as members“ ☺

# Set Theory - Russell's Paradox

Can we use **any** predicate P to define a set $S = \{ x : P(x) \}$?

No!

Define $S = \{ x : x \text{ is a set where } x \notin x \}$

Then, if $S \in S$, then by defn of S, $S \notin S$.

So S must not be in S, right?

But, if $S \notin S$, then by defn of S, $S \in S$.

ARRRGH!

There is a town with a barber who shaves all the people (and only the people) who don't shave themselves.

Who shaves the barber?

Aside: this layman's version of Russell's paradox has some drawbacks.

# Set Theory - Russell's Paradox

There is a town with a barber who shaves all the people (and only the people) who don't shave themselves.

Who shaves the barber?

Does the barber shave himself?
If the barber does not shave himself, he must abide by the rule and shave himself.
If he does shave himself, according to the rule he will not shave himself.

$$(\exists x)\,(barber(x) \wedge (\forall y)(\neg shaves(y,y) \leftrightarrow shaves(x,y))$$

This sentence is unsatisfiable (a contradiction) because of the universal quantifier.
The universal quantifier y will include every single element in the domain,
including our infamous barber x. So when the value x is assigned to y,
the sentence can be rewritten to:

$$\{\neg shaves(x,x) \leftrightarrow shaves(x,x)\} \equiv$$

$$\{(shaves(x,x) \vee shaves(x,x)) \wedge (\neg shaves(x,x) \vee \neg shaves(x,x))\} \equiv$$

$$\{shaves(x,x) \wedge (\neg shaves(x,x)\}$$

Contradiction!

# Set Theory - Definitions and notation

Important Sets

**N** = {0,1,2,3,…}, the set of **natural numbers**, non negative integers, (occasionally IN)

**Z** = { …, -2, -1, 0, 1, 2,3, …), the set of **integers**

$\mathbf{Z}^+$ = {1,2,3,…} set of **positive integers**

**Q** = {p/q | p $\in$ Z, q $\in$Z, and q$\neq$0}, set of **rational numbers**

**R**, the set of real numbers

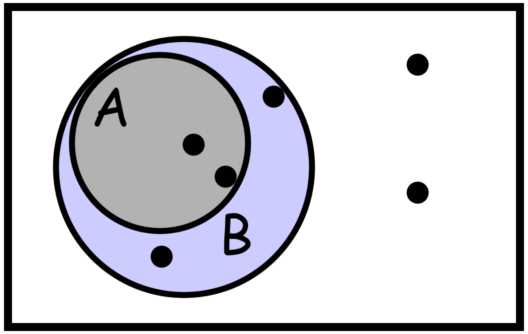
Note: Real number are the numbers that can be represented by an infinite decimal representation, such as 3.4871773339…. The real numbers include both **rationa**l, and **irrational** numbers such as $\pi$ and the $\sqrt{2}$ and can be represented as points along an infinitely long number line.

# Set Theory - Definitions and notation

$x \in S$ means "x is an element of set S."
$x \notin S$ means "x is not an element of set S."

$A \subseteq B$ means "A is a subset of B."

or, "B contains A."
or, "every element of A is also in B."
or, $\forall x\, ((x \in A) \rightarrow (x \in B))$.




Venn Diagram

# Set Theory - Definitions and notation

$A \subseteq B$ means "A is a subset of B."
$A \supseteq B$ means "A is a superset of B."

$A = B$ if and only if A and B have exactly the same elements.

iff, $A \subseteq B$ and $B \subseteq A$
iff, $A \subseteq B$ and $A \supseteq B$
iff, $\forall x\, ((x \in A) \leftrightarrow (x \in B))$.

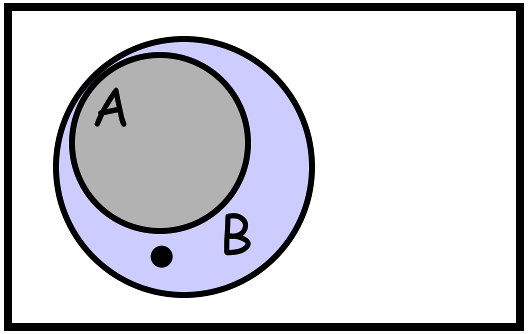
So to show equality of sets A and B, show:

- $A \subseteq B$
- $B \subseteq A$

# Set Theory - Definitions and notation

$A \subset B$ means "A is a proper subset of B."

- $A \subseteq B$, and $A \neq B$.
- $\forall x\ ((x \in A) \rightarrow (x \in B)) \wedge \neg\forall x\ ((x \in B) \rightarrow (x \in A))$
- $\forall x\ ((x \in A) \rightarrow (x \in B)) \wedge \exists x\ \neg(\neg(x \in B) \vee (x \in A))$
- $\forall x\ ((x \in A) \rightarrow (x \in B)) \wedge \exists x\ ((x \in B) \wedge \neg(x \in A))$
- $\forall x\ ((x \in A) \rightarrow (x \in B)) \wedge \exists x\ ((x \in B) \wedge (x \notin A))$



# Set Theory - Definitions and notation

Quick examples:

$\{1,2,3\} \subseteq \{1,2,3,4,5\}$

$\{1,2,3\} \subset \{1,2,3,4,5\}$

Is $\emptyset \subseteq \{1,2,3\}$? Yes! $\forall x\ (x \in \emptyset) \rightarrow (x \in \{1,2,3\})$ holds (for all over empty domain)

Is $\emptyset \in \{1,2,3\}$? No!

Is $\emptyset \subseteq \{\emptyset,1,2,3\}$? Yes!

Is $\emptyset \in \{\emptyset,1,2,3\}$? Yes!

# Set Theory - Definitions and notation

A few more:

Is $\{a\} \subseteq \{a\}$? Yes

Is $\{a\} \in \{a,\{a\}\}$? Yes

Is $\{a\} \subseteq \{a,\{a\}\}$? Yes

Is $\{a\} \in \{a\}$? No

# Set Theory - Cardinality

If S is finite, then the *cardinality* of S, $|S|$, is the number of distinct elements in S.

If $S = \{1,2,3\}$, $|S| = 3$.

If $S = \{3,3,3,3,3\}$, $|S| = 1$.

If $S = \varnothing$, $|S| = 0$.

If $S = \{ \varnothing, \{\varnothing\}, \{\varnothing,\{\varnothing\}\} \}$, $|S| = 3$.

If $S = \{0,1,2,3,\ldots\}$, $|S|$ is infinite.

# Set Theory - Power sets

If S is a set, then the ***power set*** of S is

$$2^S = \{ x : x \subseteq S \}.$$

aka P(S)

We say, "P(S) is the set of all subsets of S."

If $S = \{a\}$, $2^S = \{\emptyset, \{a\}\}$.

If $S = \{a,b\}$, $2^S = \{\emptyset, \{a\}, \{b\}, \{a,b\}\}$.

If $S = \emptyset$, $2^S = \{\emptyset\}$.

If $S = \{\emptyset,\{\emptyset\}\}$, $2^S = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset,\{\emptyset\}\}\}$.

Fact: if S is finite, $|2^S| = 2^{|S|}$. (if $|S| = n$, $|2^S| = 2^n$)

Why?

# Set Theory – Ordered Tuples
## Cartesian Product

**When order matters**, we use ordered n-tuples

The ***Cartesian Product*** of two sets A and B is:

$$A \times B = \{ \langle a,b \rangle : a \in A \wedge b \in B \}$$

If A = {Charlie, Lucy, Linus}, and B = {Brown, VanPelt}, then

A x B = {<Charlie, Brown>, <Lucy, Brown>, <Linus, Brown>, <Charlie, VanPelt>, <Lucy, VanPelt>, <Linus, VanPelt>}

We'll use these special sets soon!

$$A_1 \times A_2 \times \ldots \times A_n = \{\langle a_1, a_2, \ldots, a_n \rangle : a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n\}$$

Size?

$n^n$ if $(\forall i)\ |A_i| = n$

A,B finite → |AxB| = ?

a) AxB
b) |A|+|B|
c) |A+B|
d) |A||B|

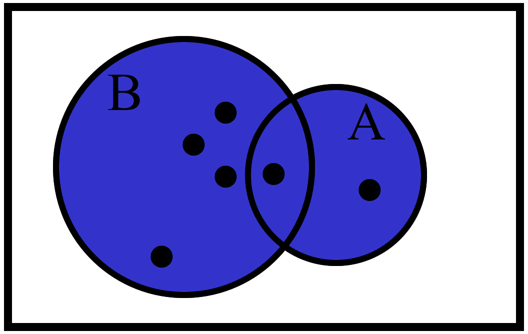
# Set Theory - Operators

The *union* of two sets A and B is:

$$A \cup B = \{ x : x \in A \vee x \in B \}$$

If A = {Charlie, Lucy, Linus}, and B = {Lucy, Desi}, then

$A \cup B$ = {Charlie, Lucy, Linus, Desi}



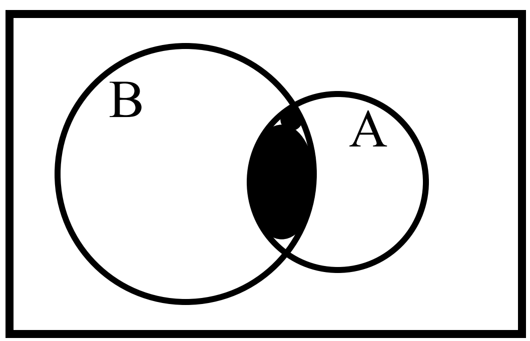
# Set Theory - Operators

The *intersection* of two sets A and B is:

$$A \cap B = \{ x : x \in A \wedge x \in B\}$$

If A = {Charlie, Lucy, Linus}, and B = {Lucy, Desi}, then

$A \cap B = \{\text{Lucy}\}$



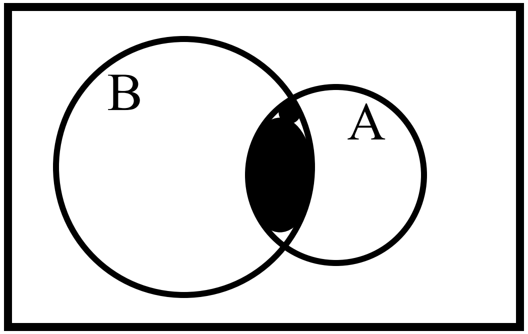
# Set Theory - Operators

The *intersection* of two sets A and B is:

$$A \cap B = \{ x : x \in A \wedge x \in B \}$$

If A = {x : x is a US president}, and B = {x : x is deceased}, then

$A \cap B$ = {x : x is a deceased US president}



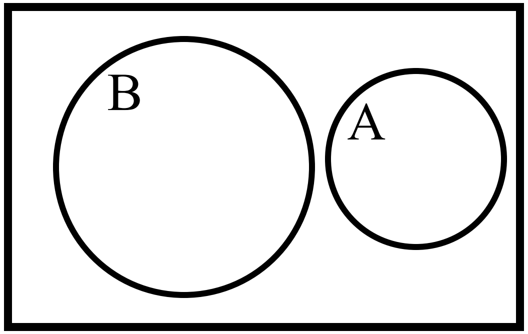
# Set Theory - Operators

The *intersection* of two sets A and B is:

$$A \cap B = \{ x : x \in A \wedge x \in B \}$$

If $A = \{x : x \text{ is a US president}\}$, and $B = \{x : x \text{ is in this room}\}$, then

$A \cap B = \{x : x \text{ is a US president in this room}\} = \varnothing$




Sets whose intersection is empty are called *disjoint* sets

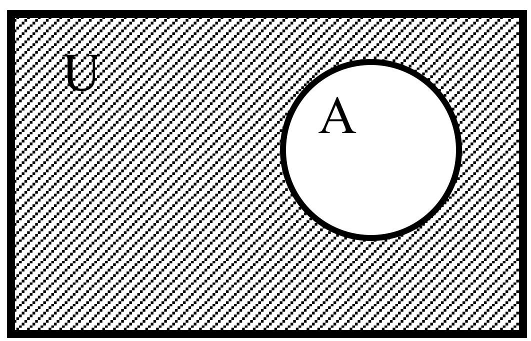
# Set Theory - Operators

The *complement* of a set A is:

$$C_A = \{ x : x \notin A\}$$

If A = {x : x is bored}, then

$C_A$ = {x : x is not bored}

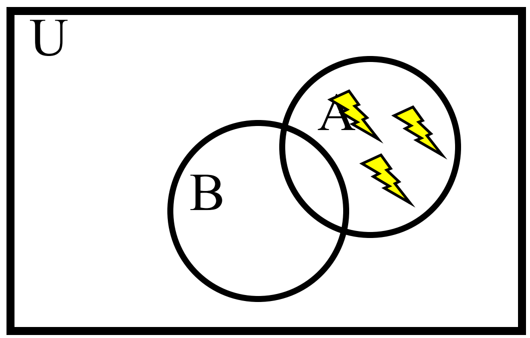



$C_\varnothing = U$
and
$C_U = \varnothing$

I.e., $C_A = U - A$, where U is the universal set.
"A set fixed within the framework of a theory and consisting of all objects considered in the theory. "

# Set Theory - Operators

The *set difference*, A - B, is:




$$A - B = \{ x : x \in A \wedge x \notin B \}$$

$$A - B = A \cap C_B$$

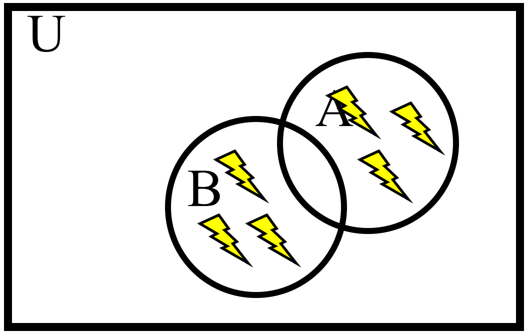
# Set Theory - Operators

The *symmetric difference*, $A \Delta B$, is:

$$A \Delta B = \{ x : (x \in A \wedge x \notin B) \vee (x \in B \wedge x \notin A)\}$$

$$= (A - B) \cup (B - A)$$

like "exclusive or"



Theorem: $A \Delta B = (A - B) \cup (B - A)$

Proof: $A \Delta B = \{ x : (x \in A \wedge x \notin B) \vee (x \in B \wedge x \notin A)\}$

$= \{ x : (x \in A - B) \vee (x \in B - A)\}$

$= \{ x : x \in ((A - B) \cup (B - A))\}$

$= (A - B) \cup (B - A)$

Q.E.D.

Directly from defns.
Semantically clear.

# Set Theory - Identities

*Identity*

$$A \cap U = A$$
$$A \cup \emptyset = A$$

*Domination*

$$A \cup U = U$$
$$A \cap \emptyset = A$$

*Idempotent*

$$A \cup A = A$$
$$A \cap A = A$$

# Set Theory – Identities, cont.

*Complement Laws*

$$A \cup C_A = U$$

$$A \cap C_A = \varnothing$$

*Double complement*

$$C_{C_A} = A$$

# Set Theory - Identities, cont.

*Commutativity*

$$A \cup B = B \cup A$$

$$A \cap B = B \cap A$$

*Associativity*

$$(A \cup B) \cup C = A \cup (B \cup C)$$

$$(A \cap B) \cap C = A \cap (B \cap C)$$

*Distributivity*

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

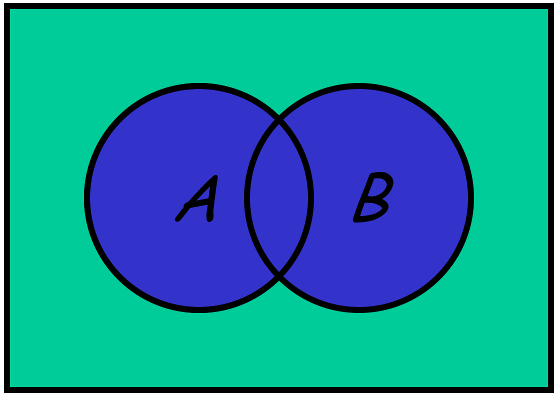
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

*DeMorgan's I*

$$C(A \cup B) = C(A) \cap C(B)$$

*DeMorgan's II*

$$C(A \cap B) = C(A) \cup C(B)$$



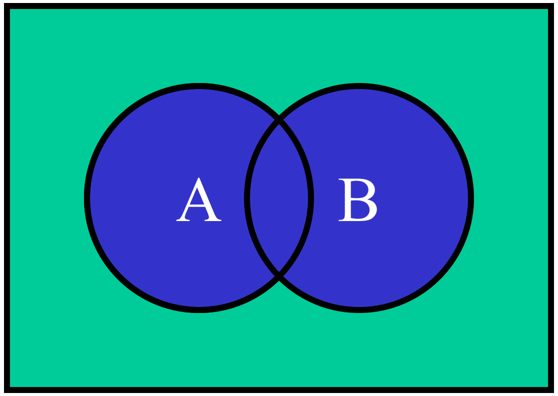

Proof by "diagram" (useful!), but we aim for a more formal proof.

# Proving identities

Prove that $\overline{(A \cup B)} = \overline{A} \cap \overline{B}$ (De Morgan)

1. ($\subseteq$) $(x \in \overline{A \cup B})$
   $\Rightarrow (x \notin (A \cup B))$
   $\Rightarrow (x \notin A \text{ and } x \notin B)$
   $\Rightarrow (x \in \overline{A} \cap \overline{B})$

2. ($\supseteq$) $(x \in (\overline{A} \cap \overline{B}))$
   $\Rightarrow (x \notin A \text{ and } x \notin B)$
   $\Rightarrow (x \notin A \cup B)$
   $\Rightarrow (x \in \overline{A \cup B})$




# Alt. proof

Prove that $\overline{(A \cup B)} = \overline{A} \cap \overline{B}$ using a membership table.

0 : x is not in the specified set

1 : otherwise

| $A$ | $B$ | $\overline{A}$ | $\overline{B}$ | $\overline{A} \cap \overline{B}$ | $A \cup B$ | $\overline{A \cup B}$ |
|---|---|---|---|---|---|---|
| 1 | 1 | 0 | 0 | 0 | 1 | 0 |
| 1 | 0 | 0 | 1 | 0 | 1 | 0 |
| 0 | 1 | 1 | 0 | 0 | 1 | 0 |
| 0 | 0 | 1 | 1 | 1 | 0 | 1 |

Haven't we seen this before?

General connection via Boolean algebras (Rosen chapt. 11)

Proof using logically equivalent set definitions.

$$\overline{(A \cup B)} = \overline{A} \cap \overline{B}$$

$$\overline{(A \cup B)} = \{x : \neg(x \in A \vee x \in B)\}$$

$$= \{x : \neg(x \in A) \wedge \neg(x \in B)\}$$

$$= \{x : (x \in \overline{A}) \wedge (x \in \overline{B})\}$$

$$= \overline{A} \cap \overline{B}$$

**TABLE 1 Set Identities.**

| *Identity* | *Name* |
|---|---|
| $A \cup \varnothing = A$<br>$A \cap U = A$ | Identity laws |
| $A \cup U = U$<br>$A \cap \varnothing = \varnothing$ | Domination laws |
| $A \cup A = A$<br>$A \cap A = A$ | Idempotent laws |
| $\overline{(\overline{A})} = A$ | Complementation law |
| $A \cup B = B \cup A$<br>$A \cap B = B \cap A$ | Commutative laws |
| $A \cup (B \cup C) = (A \cup B) \cup C$<br>$A \cap (B \cap C) = (A \cap B) \cap C$ | Associative laws |
| $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$<br>$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ | Distributive laws |
| $\overline{A \cup B} = \overline{A} \cap \overline{B}$<br>$\overline{A \cap B} = \overline{A} \cup \overline{B}$ | De Morgan's laws |
| $A \cup (A \cap B) = A$<br>$A \cap (A \cup B) = A$ | Absorption laws |
| $A \cup \overline{A} = U$<br>$A \cap \overline{A} = \varnothing$ | Complement laws |

# Relations

# Relations - definitions

If we want to describe a relationship between elements of two sets A and B, we can use **ordered pairs** *(a, b)* with their first element taken $a \in$ A and their second element $b \in$ B.

Since this is a relation between **two sets**, it is called a **binary relation** (or **correspondence**, **dyadic relation** or **2-place relation**).

**Definition:** Let A and B be sets. A **binary relation from A to B** is a subset of A×B.

For a binary relation R: $R \subseteq A \times B$.

We use the notation $a$ R $b$ to denote that $(a, b) \in R$ and we say that ***a* is related to *b* by R**.

$D(R) = \{x \in A \mid \exists\, y \in B \text{ s.t. } (x,y) \in R\}$ is called the **domain of the relation** R.

$Im(R) = \{y \in B \mid \exists\, x \in A \text{ s.t. } (x,y) \in R\}$ is called the **co-domain (range) of the relation** R.

In particular, a subset of A×A is called a **relation on A** or a **homogeneous relation**.

# Relations - definitions

For $R \subseteq A\times B$, we define $R^{-1}$ – the **inverse relation** of R

$$R^{-1} = \{(y, x) \mid (x,y) \in R\}$$

For $R \subseteq A\times B$ and $S \subseteq B\times C$ s.t. $Im(R) \cap D(S) \neq \emptyset$, the **composite of R and S** is the relation consisting of ordered pairs $(a, c)$, where $a\in A$, $c\in C$, and for which there exists an element $b\in B$ such that $(a, b)\in R$ and $(b, c)\in S$. We denote the composite of R and S by $S\circ R$.

$$S\circ R = \{(a, c) \subseteq A\times C \mid \exists\, b \in B \text{ s.t. } (a, b)\in R \text{ and } (b, c)\in S\}$$

$\mathbf{1_A} = \{(a,a) \mid a \in A\}$ – **identity relation on A**.

On a set A with $n$ elements we can define $2^{n^2}$ different relations!!

# Relations - properties

A relation R on a non-empty set A is called:

- **reflexive** if $(a, a) \in R$ for every element $a \in A$, or $1_A \subseteq R$.
- **irreflexive** if $(a, a) \notin R$ for every element $a \in A$.
- **coreflexive** if $a=b$ whenever $(a, b) \in R$
- **symmetric** if $(b, a) \in R$ whenever $(a, b) \in R$ for all $a, b \in A$; or $R^{-1} = R$
- **antisymmetric** if $a = b$ whenever $(a, b) \in R$ and $(b, a) \in R$; or $R \cap R^{-1} = 1_A$
- **asymmetric** if $(a, b) \in R$ implies that $(b, a) \notin R$ for all $a, b \in A$
- **transitive** if whenever $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$ for $a, b, c \in A$.
- **total** if for all $a, b \in A$ it holds that aRb or bRa (or both).

# Relations - properties

A relation that is reflexive, symmetric, and transitive is called an **equivalence relation**.

A relation that is only symmetric and transitive (without necessarily being reflexive) is called a **partial equivalence relation**.

A relation that is reflexive, antisymmetric, and transitive is called a **partial order**.

A partial order that is total is called a **total order, simple order, linear order**, or a **chain**.

A linear order where every nonempty subset has a least element is called a **well-order**.

# Relations - properties

For R – a equivalence relation on a set A and $a \in A$

$[a]_R = [a] = \hat{a}_R = \{a \in A \mid (a, b) \in R\}$ = the **equivalence class** of the element a.

$A_{|R} = \{[a] \mid a \in A\}$ - the set of all equivalence classes in *A* with respect to an equivalence relation *R* is denoted as *A*/*R* and called *A* **modulo** *R* (or the **quotient set** of *A* by *R*)

The surjective map a → [a] from A onto A/R, which maps each element to its equivalence class, is called the **canonical surjection** or the **canonical projection map**.

Every element a of A is a member of the equivalence class [a].

Every two equivalence classes [a] and [b] are either equal or disjoint.

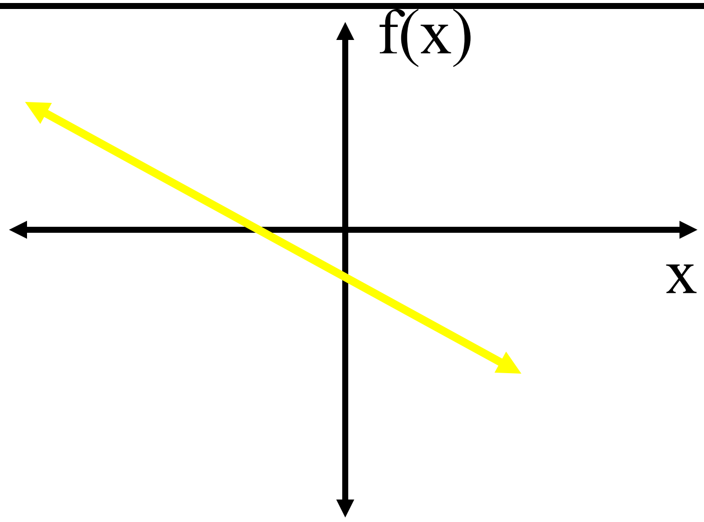
Therefore, the set of all equivalence classes of A forms a partition of A: every element of A belongs to one and only one equivalence class.

a R b if and only if [a] = [b].

# Functions

# Functions

Suppose we have:



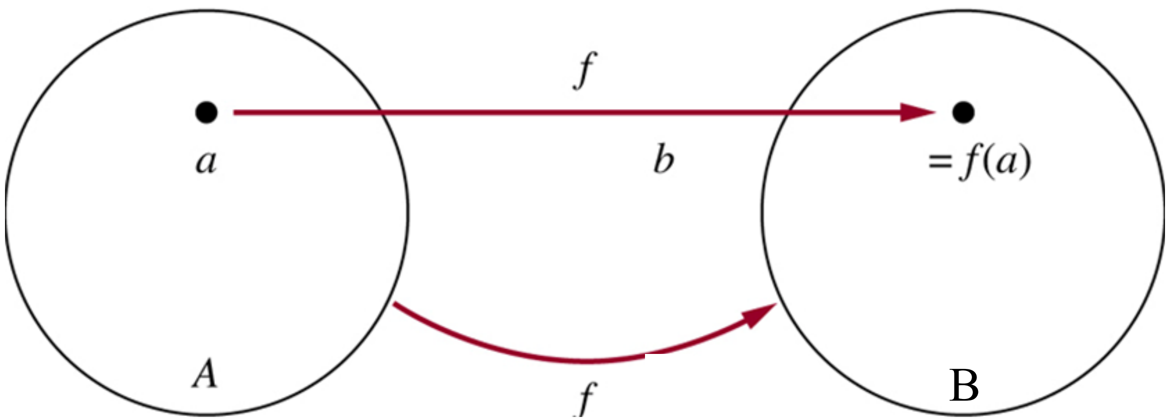

How do you describe the yellow function?

What's a function ?

$f(x) = -(1/2)x - 1/2$

# Functions

More generally:



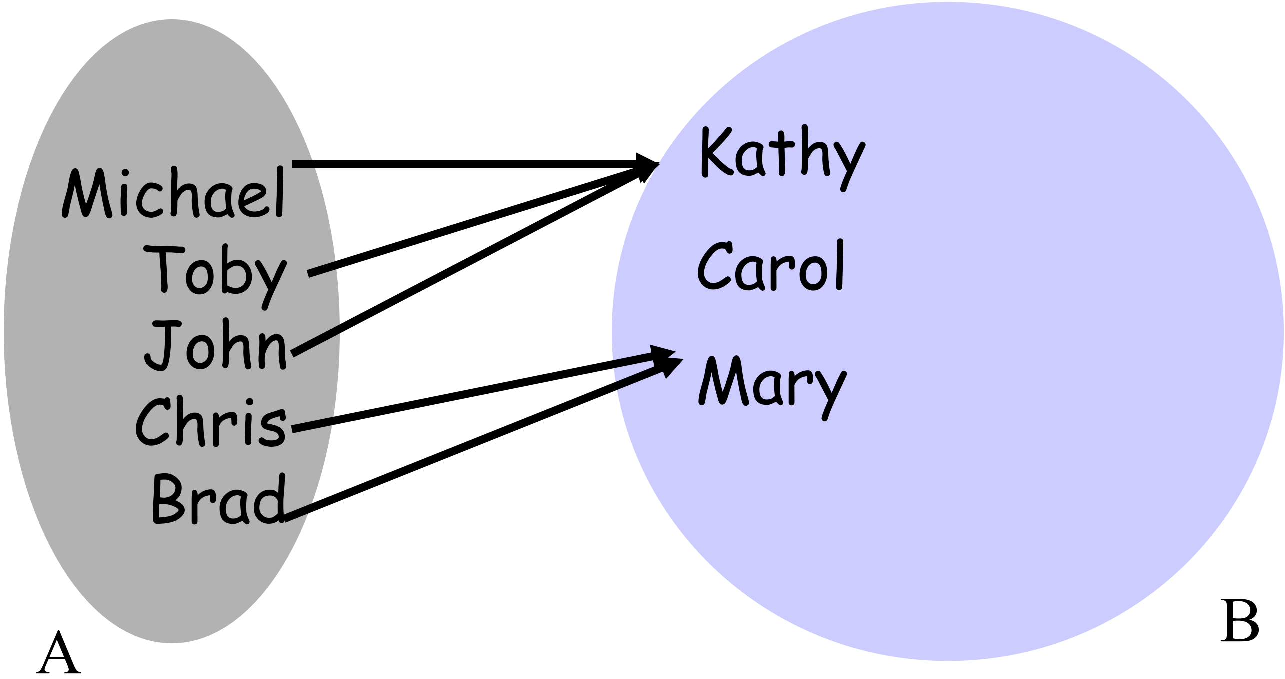

**Definition:**
Given A and B, nonempty sets, a **function** f from A to B is an assignment of exactly one element of B to each element of A. We write f(a)=b if b is the element of B assigned by function f to the element a of A.
If f is a function from A to B, we write f : A→B.

Note: Functions are also called **mappings** or **transformations.**

# Functions

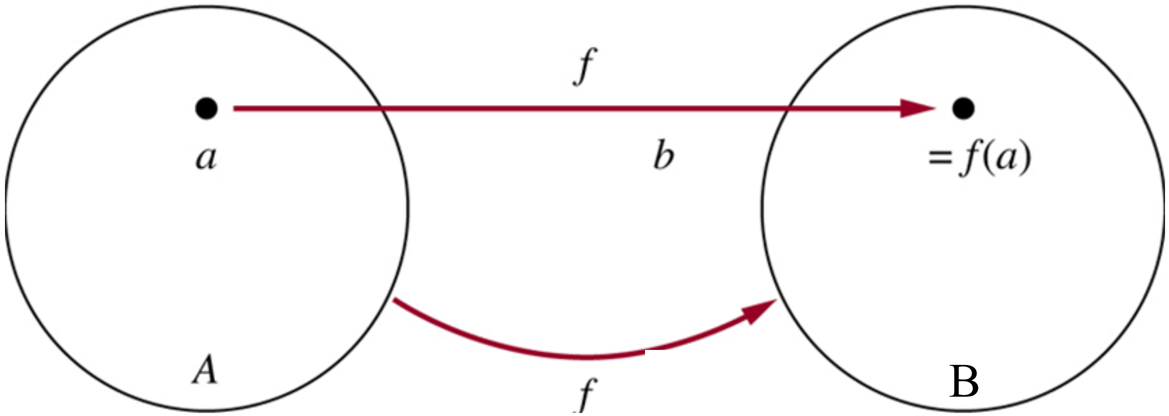
A = {Michael, Toby , John , Chris , Brad }
B = { Kathy, Carla, Mary}

Let f: A $\rightarrow$ B be defined as f(a) = mother(a).



# Functions

More generally:




A - Domain of f

B- Co-Domain of f

f: R→R, f(x) = -(1/2)x - 1/2

domain

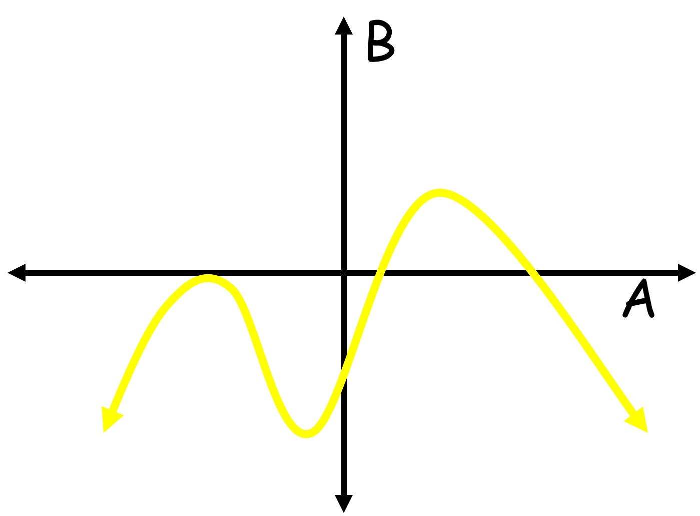
co-domain

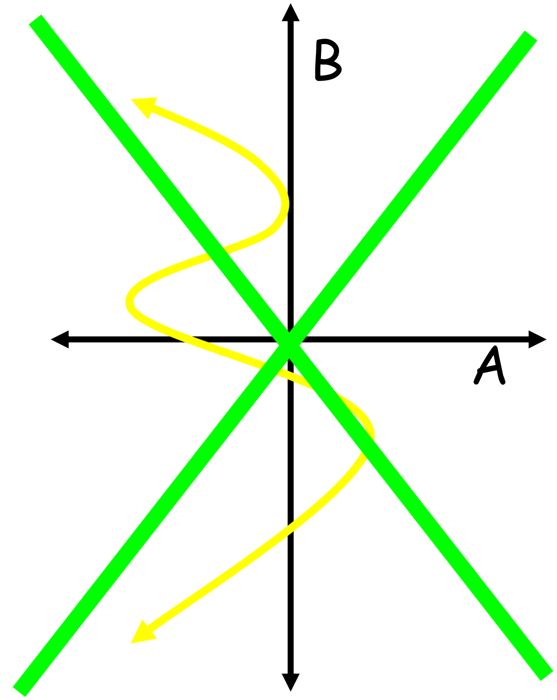
a collection of points!

# unctions

More formally: a function $f: A \rightarrow B$ is a subset of $A \times B$ where
$\forall\, a \in A, \exists!\, b \in B$ and $\langle a,b \rangle \in f$.

a point!




Why not?

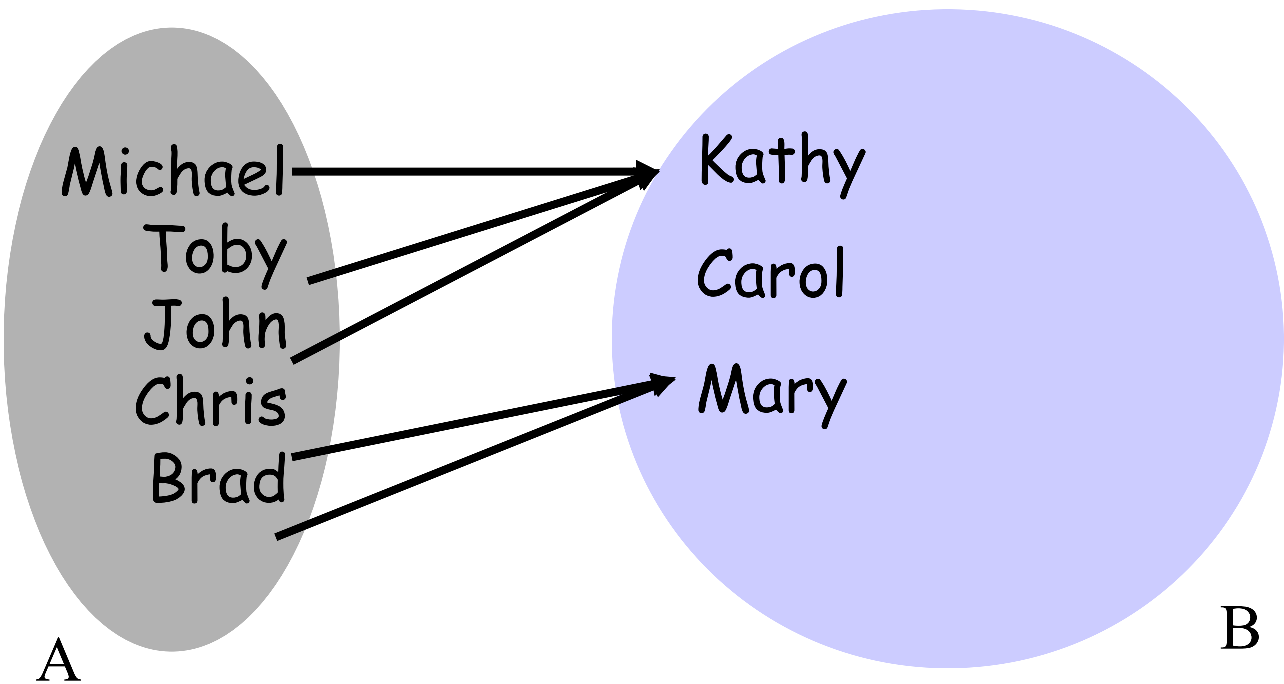
# Functions - image & preimage

image(S)

For any set $S \subseteq A$, $\text{image}(S) = \{b : \exists a \in S, f(a) = b\}$

So, image({Michael, Toby}) = {Kathy} image(A) = B - {Carol}

range of f
image(A)

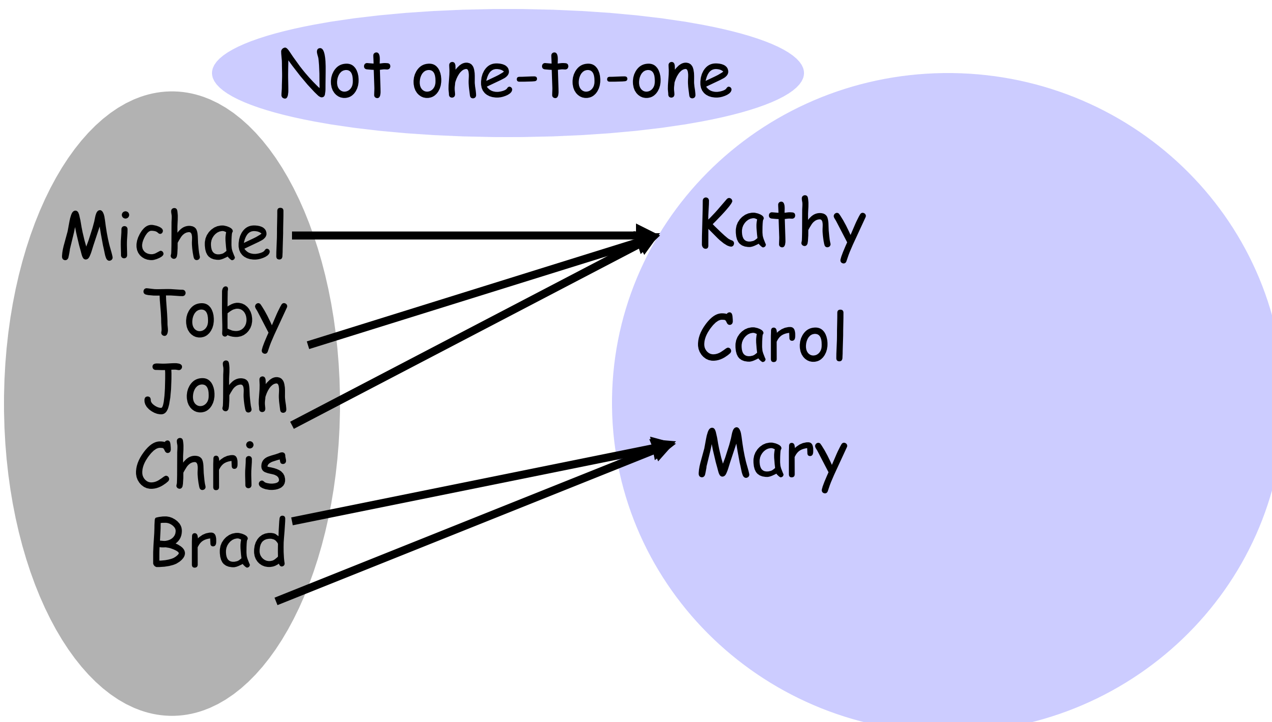



image(John) = {Kathy}        pre-image(Kathy) = {John, Toby, Michael}

Every $b \in B$ has at most 1 preimage.

# Functions - injection

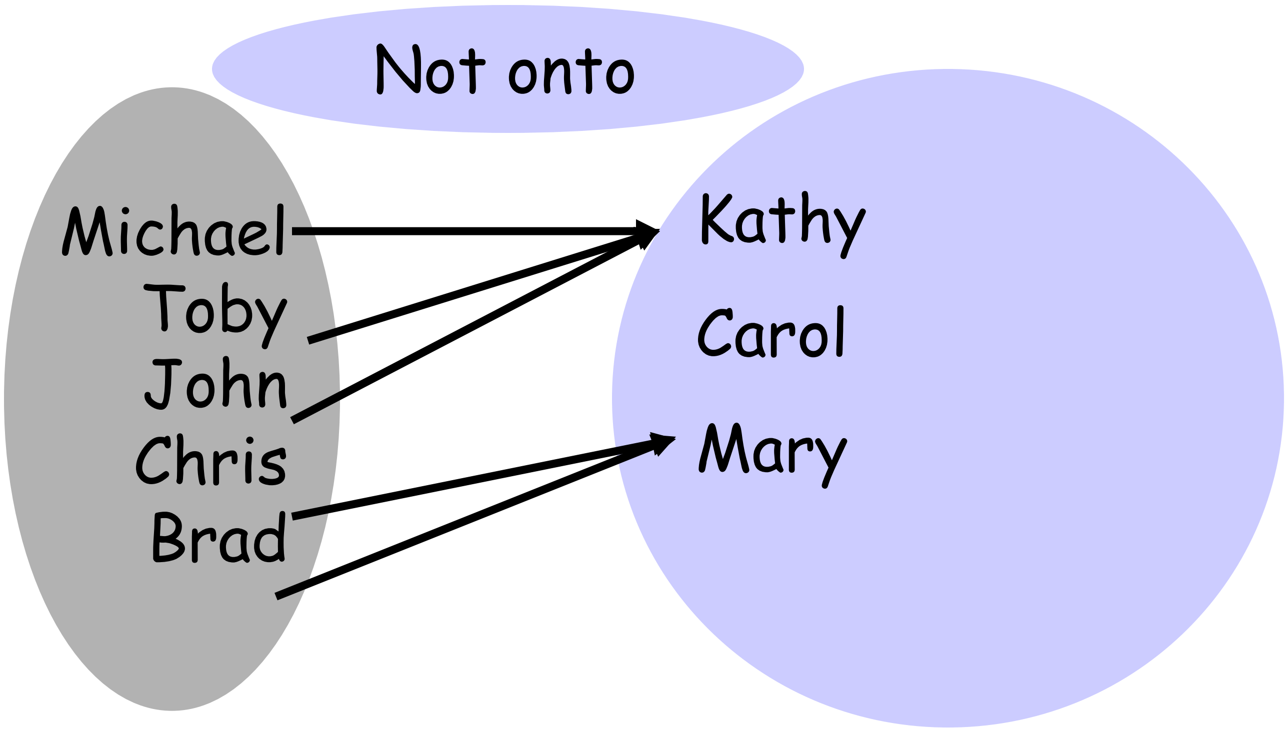
A function $f: A \rightarrow B$ is one-to-one (injective, an injection) if
$\forall a,b,c,\ (f(a) = b \wedge f(c) = b) \rightarrow a = c$



# Functions - surjection

Every $b \in B$ has at least 1 preimage.

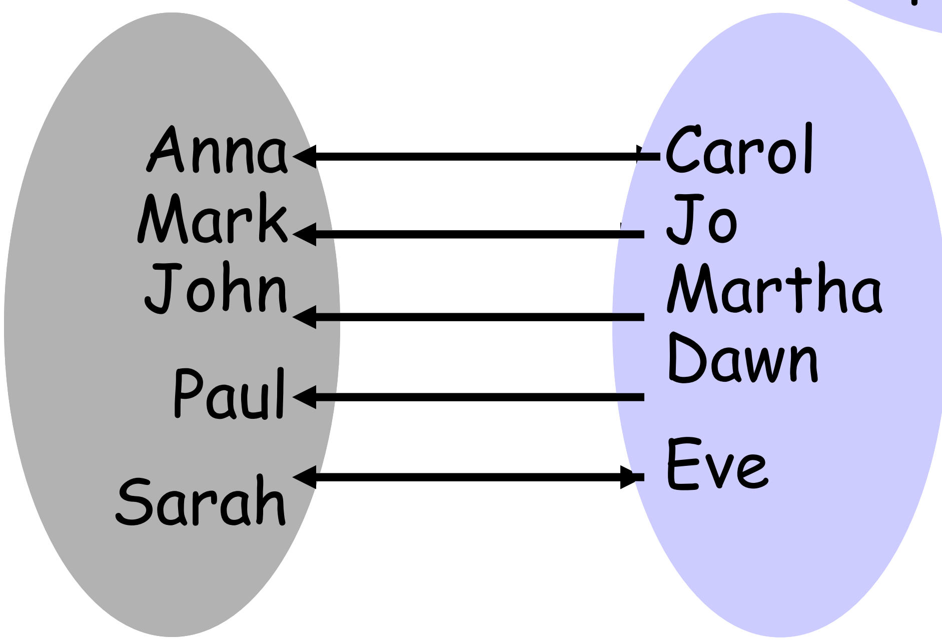
A function $f: A \to B$ is onto (surjective, a surjection) if $\forall b \in B, \exists a \in A\ f(a) = b$



# Functions – one-to-one-correspondence or bijection

A function f: A $\rightarrow$ B is bijective if it is one-to-one and onto.

Every b $\in$ B has exactly 1 preimage.

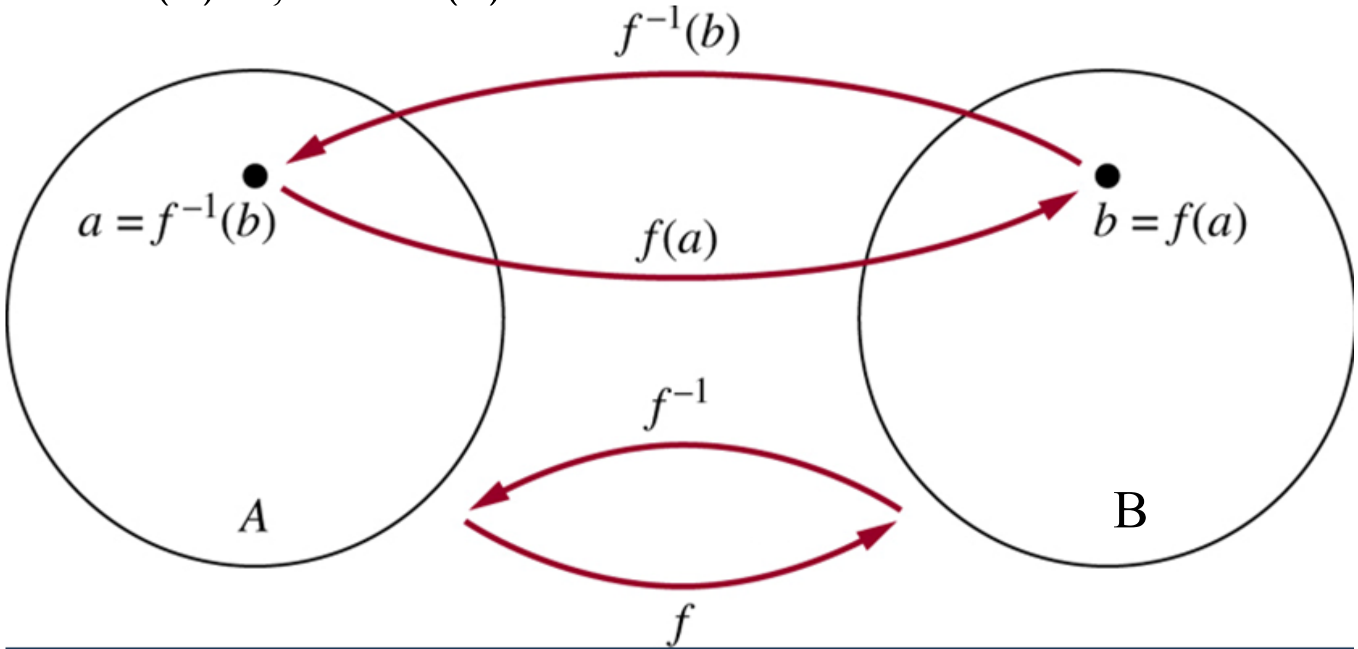



An important implication of this characteristic: The preimage ($f^{-1}$) is a function! They are **invertible**.

# Functions: inverse function

Definition:

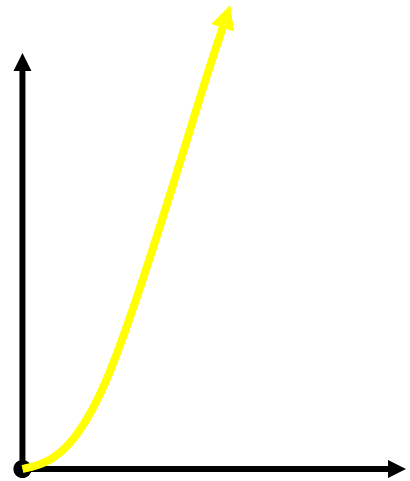
Given f, a one-to-one correspondence from set A to set B, the **inverse function of f** is the function that assigns to an element b belonging to B the unique element a in A such that f(a)=b. The inverse function is denoted $f^{-1}$ . $f^{-1}$ (b)=a, when f(a)=b.



# Functions - examples

Suppose f: $R^+ \rightarrow R^+$, $f(x) = x^2$.

Is f one-to-one? yes
Is f onto? yes
Is f bijective? yes



This function is invertible.

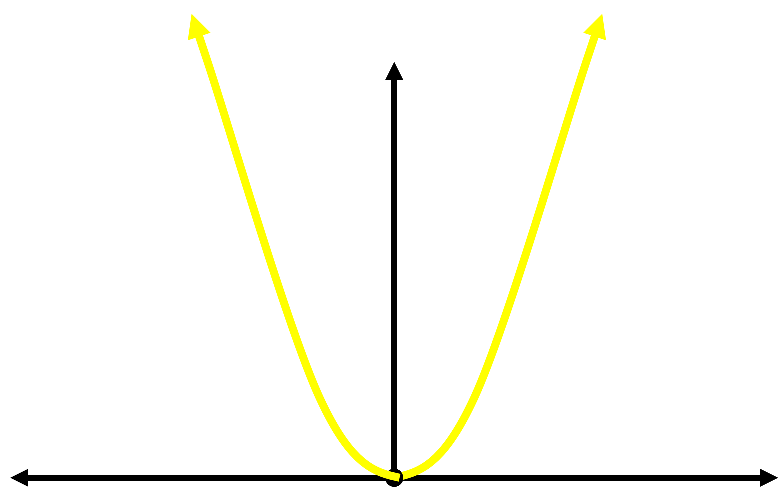
# Functions - examples

Suppose f: $R \rightarrow R^{+}$, $f(x) = x^2$.

Is f one-to-one? no

Is f onto? yes

Is f bijective? no



This function is not invertible.

# Functions - examples

Suppose f: R $\rightarrow$ R, $f(x) = x^2$.

Is f one-to-one?

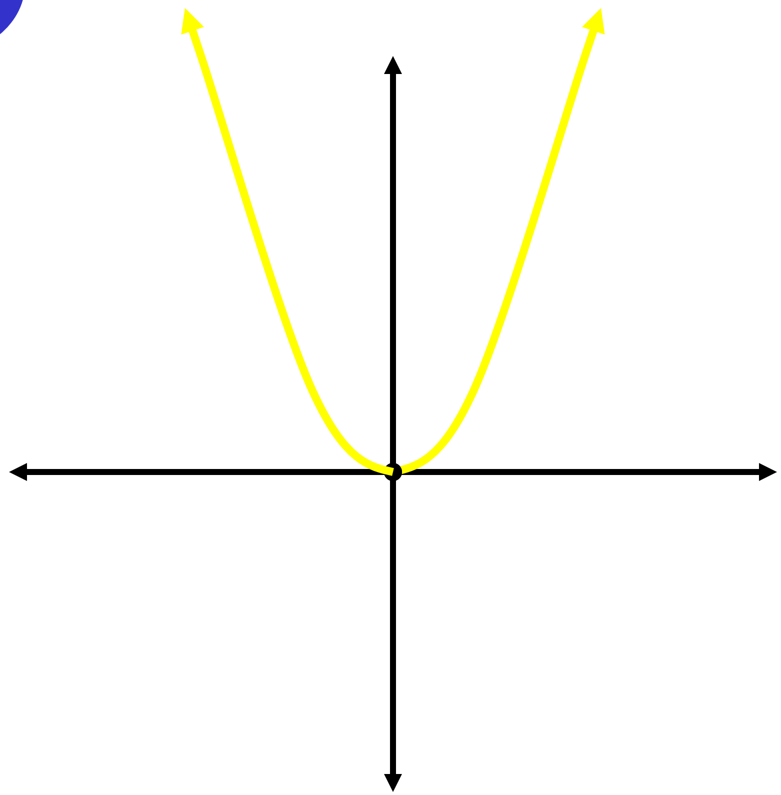
no

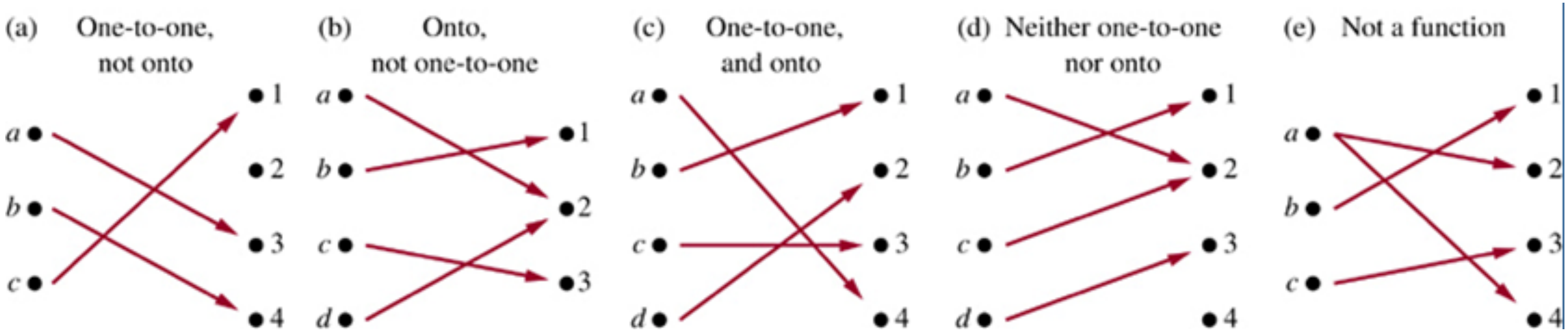
Is f onto?

no

Is f bijective?

no



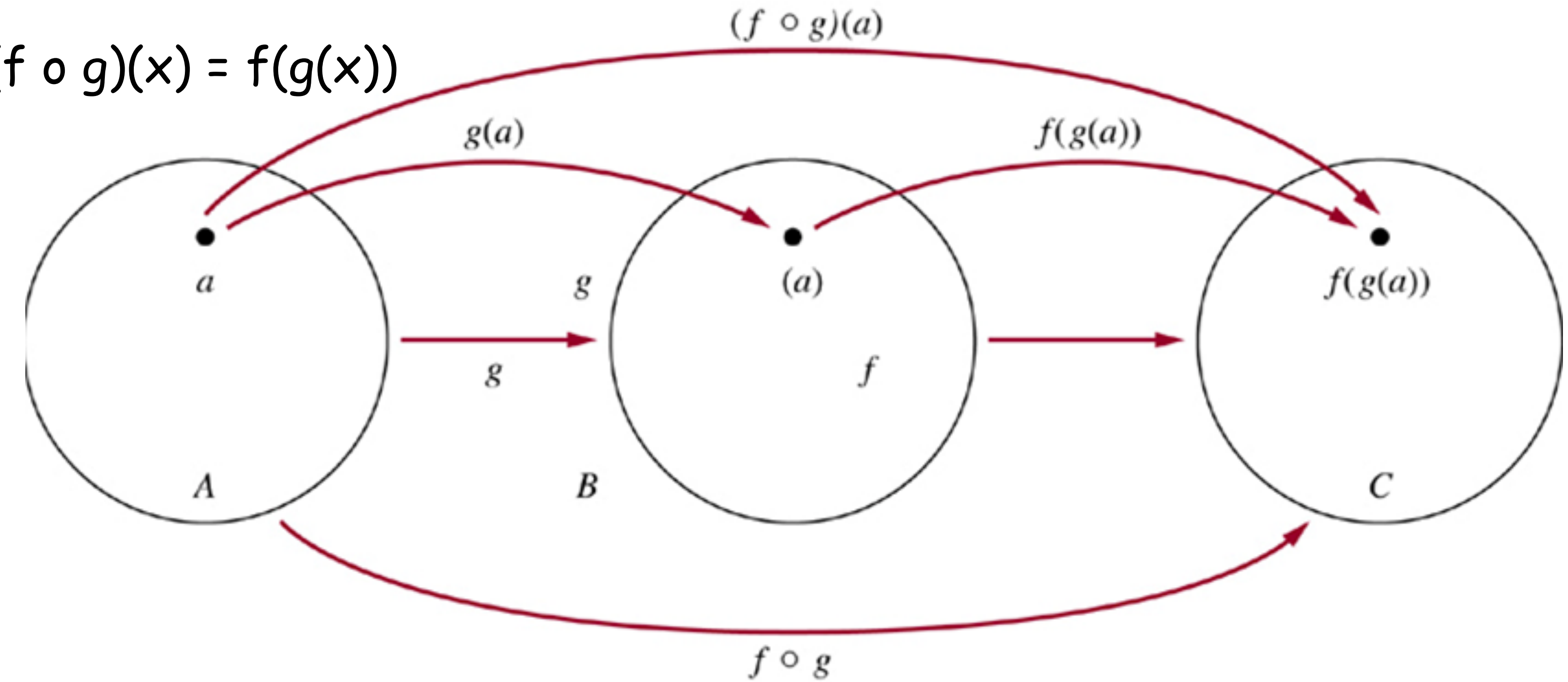

(a) One-to-one, not onto
(b) Onto, not one-to-one
(c) One-to-one, and onto
(d) Neither one-to-one nor onto
(e) Not a function
a
b
c
d
1
2
3
4

“f composed with g”

# Functions - composition

Let f: A→B, and g: B→C be functions. Then the composition of f and g is:

(f o g)(x) = f(g(x))




Note: (f o g) cannot be defined unless the range of g is a subset of the domain of f.

Example:

Let $f(x) = 2x + 3$; $g(x) = 3x + 2$;

$(f \circ g)(x) = f(3x + 2) = 2(3x + 2) + 3 = 6x + 7.$

$(g \circ f)(x) = g(2x + 3) = 3(2x + 3) + 2 = 6x + 11.$

As this example shows, $(f \circ g)$ and $(g \circ f)$ are not necessarily equal – i.e, the composition of functions is not commutative.

Note:

$(f^{-1} \circ f)(a) = f^{-1}(f(a)) = f^{-1}(b) = a.$

$(f \circ f^{-1})(b) = f(f^{-1}(b)) = f^{-}(a) = b.$

Therefore $(f^{-1} \circ f) = I_A$ and $(f \circ f^{-1}) = I_B$ where $I_A$ and $I_B$ are the identity function on the sets A and B. $(f^{-1})^{-1} = f$

# Some important functions

**Absolute value**:

Domain R; Co-Domain = $\{0\} \cup R^+$

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Ex: $|-3| = 3$; $|3| = 3$

**Floor function (or greatest integer function):**

Domain = R; Co-Domain = Z

$\lfloor x \rfloor$ = largest integer not greater than x

Ex: $\lfloor 3.2 \rfloor = 3$; $\lfloor -2.5 \rfloor = -3$

# Some important functions

**Ceiling function:**

Domain = R;

Co-Domain = Z

$\lceil x \rceil$ = smallest integer greater than x

Ex: $\lceil 3.2 \rceil = 4$; $\lceil -2.5 \rceil = -2$

**TABLE 1 Useful Properties of the Floor and Ceiling Functions.**
($n$ is an integer)

| | |
|---|---|
| (1a) | $\lfloor x \rfloor = n$ if and only if $n \le x < n + 1$ |
| (1b) | $\lceil x \rceil = n$ if and only if $n - 1 < x \le n$ |
| (1c) | $\lfloor x \rfloor = n$ if and only if $x - 1 < n \le x$ |
| (1d) | $\lceil x \rceil = n$ if and only if $x \le n < x + 1$ |
| (2) | $x - 1 < \lfloor x \rfloor \le x \le \lceil x \rceil < x + 1$ |
| (3a) | $\lfloor -x \rfloor = -\lceil x \rceil$ |
| (3b) | $\lceil -x \rceil = -\lfloor x \rfloor$ |
| (4a) | $\lfloor x + n \rfloor = \lfloor x \rfloor + n$ |
| (4b) | $\lceil x + n \rceil = \lceil x \rceil + n$ |

# Some important functions

**Factorial function:** Domain = Range = N **Error on range**

n! = n (n-1)(n-2) …, 3 x 2 x 1

Ex: 5! = 5 x 4 x 3 x 2 x 1 = 120

Note: 0! = 1 by convention.

# Some important functions

**Mod (or remainder):**

Domain = $N \times N^+ = \{(m,n) | \ m \in N, \ n \in N+ \}$
Co-domain Range = N

$m \ \mathit{mod} \ n = m - \lfloor m/n \rfloor n$

Ex: $8 \bmod 3 = 8 - \lfloor 8/3 \rfloor 3 = 2$
$57 \bmod 12 = 9$;

Note: This function computes the remainder when m is divided by n.
The name of this function is an abbreviation of m modulo n, where modulus means with respect to a modulus (size) of n, which is defined to be the remainder when m is divided by n. Note also that this function is an example in which the domain of the function is a 2-tuple.

# Some important functions: Exponential Function

**Exponential function:**

Domain = $R^+ \times R$ = $\{(a,x) | a \in R+, x \in R\}$

Co-domain Range = $R^+$

$f(x) = a^x$

Note: *a* is a **positive** constant; x varies.

Ex: $f(n) = a^n = a \times a \ldots, \times a$ (n times)

How do we define f(x) if x is not a positive integer?

# Some important functions: Exponential function

**Exponential function:**
How do we define f(x) if x is not a positive integer?
Important properties of exponential functions:

(1) $a^{(x+y)} = a^x a^y$; (2) $a^1 = a$ (3) $a^0 = 1$

See:

$$a^2 = a^{1+1} = a^1 a^1 = a \times a;$$

$$a^3 = a^{2+1} = a^2 a^1 = a \times a \times a;$$

$$\ldots$$

$$a^n = a \times \cdots \times a \quad (n\ times)$$

We get:

$$a = a^1 = a^{1+0} = a \times a^0 \quad therefore \quad a^0 = 1$$

$$1 = a^0 = a^{b+(-b)} = a^b \times a^{-b} \quad therefore \quad a^{-b} = 1/a^b$$

$$a = a^1 = a^{\frac{1}{2}+\frac{1}{2}} = a^{\frac{1}{2}} \times a^{\frac{1}{2}} = (a^{\frac{1}{2}})^2 \, therefore \quad a^{\frac{1}{2}} = \sqrt{a}$$

By similar arguments:

$$a^{\frac{1}{k}} = \sqrt[k]{a}$$

$$a^{mx} = a^x \times \cdots a^x \ (m \quad times) = (a^x)^m, \quad therefore \quad a^{\frac{m}{n}} = (a^{\frac{1}{n}})^m = (\sqrt[n]{a})^m$$

Note: This determines $a^x$ for all x rational. x is irrational by continuity (we'll skip "details").

# Some important functions: Logarithm Function

**Logarithm base a:**

Domain = $R^+ \times R$ = $\{(a,x) \mid a \in R+, a>1, x \in R\}$

Co-domain Range = R

$y = \log_a(x) \Leftrightarrow a^y = x$

Ex: $\log_2(8) = 3$; $\log_2(16) = 3$; $3 < \log_2(15) < 4$.

Key properties of the log function (they follow from those for exponential):

1. $\log_a(1)=0$ (because $a^0 = 1$)
2. $\log_a(a)=1$ (because $a^1 = a$)
3. $\log_a(xy) = \log_a(x) + \log_a(x)$ (similar arguments)
4. $\log_a(x^r) = r \log_a(x)$
5. $\log_a(1/x) = -\log_a(x)$ (note $1/x = x^{-1}$)
6. $\log_b(x) = \log_a(x) / \log_a(b)$

# Logarithm Functions

Examples:

$\log_2 (1/4) = -\log_2 (4) = -2.$

$\log_2 (-4)$ undefined

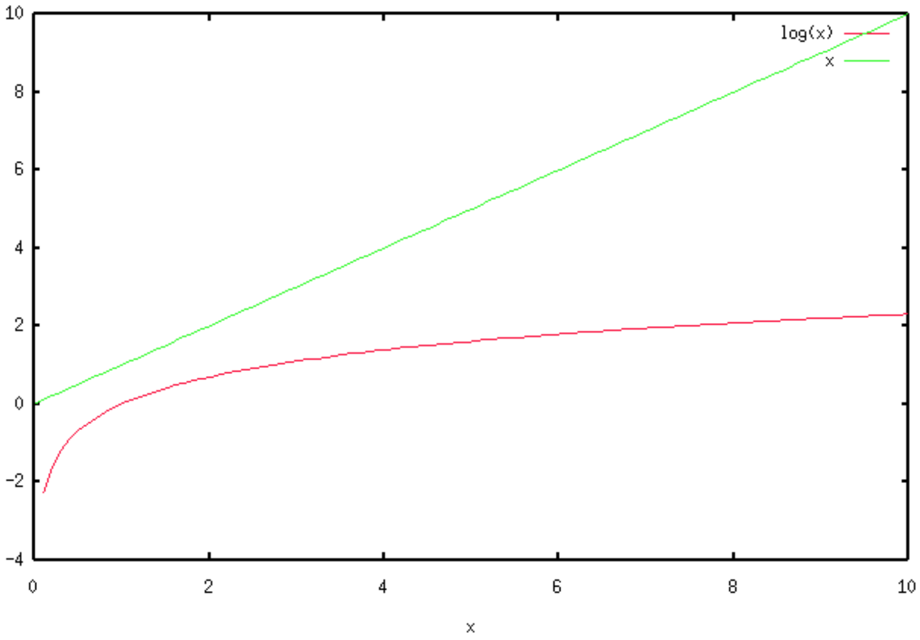
$\log_2 (2^{10}\, 3^5) = \log_2 (2^{10}) + \log_2 (3^5) = 10 \log_2 (2) + 5 \log_2 (3) =$

$= 10 + 5 \log_2 (3)$

# Limit Properties of Log Function

$$\lim_{x \to \infty} \log(x) = \infty$$

$$\lim_{x \to \infty} \frac{\log(x)}{x} = 0$$




As x gets large, *log(x)* grows without bound.
But $x$ grows **MUCH** faster than *log*($x$)…more soon on growth rates.

# Some important functions: Polynomials

**Polynomial function:**

Domain = usually R

Co-domain Range = usually R

$P_n(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x^1 + a_0$

n, a nonnegative integer is the degree of the polynomial;

$a_n \neq 0$ (so that the term $a_n x^n$ actually appears)

$(a_n, a_{n-1}, \ldots, a_1, a_0)$ are the coefficients of the polynomial.

Ex:

$y = P_1(x) = a_1 x^1 + a_0$ linear function

$y = P_2(x) = a_2 x^2 + a_1 x^1 + a_0$ quadratic polynomial or function

# Further readings

chapter 1 = pages 1-19 from

S. Pugachev, I. N. Sinitsyn , Lectures on Functional Analysis and Applications, World Scientific, 1999 - http://bookzz.org/book/719253/3326cc

(try to watch: https://www.youtube.com/watch?v=SqRY1Bm8EVs&feature=share

With more details at http://www.gresham.ac.uk/lectures-and-events/cantors-infinities