"There are basically two types of people. People who accomplish things, and people who claim to have accomplished things. The first group is less crowded." – Mark Twain

LECTURE 8

MORE GROUPS EXAMPLES

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3.2. Definitions and Examples

Definition 3.2.9. A **Cayley table** of a group G is a table where rows and columns are marked by elements of a group and the interior of the table in row g and column h is the product gh.

Example 3.2.10. The Cayley table for $(\mathbb{Z}_5, +)$ is given to the right.

$+$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

The identity in a group can be identified as the unique element with a row or column which matches with the group indicators perfectly. Every other row or column will be a rearrangement of the group elements.

Also, the inverse of g can be found on the Cayley table by finding the column which contains the identity in the row of g. That column corresponds to g^{-1}. For example, in row 2, the identity, 0, is in row 3. Thus, $-2 \equiv 3 \pmod 5$. ∎

Example 3.2.11. The Cayley table for $(\mathbb{Z}_8^*, \cdot)$ is given to the right.

$\cdot$	1	3	5	7
1	1	3	5	7
3	3	1	7	5
5	5	7	1	3
7	7	5	3	1

Commutativity can be identified from the Cayley table, because the table of an Abelian group will be symmetric across the main diagonal. This is apparent in this and the last example.

In any row or column of a Cayley table, every element of the group appears once and only once. Such a table is called a **latin square**, like in a game of Sudoku. ∎

Example 3.2.12. The set $M_n(\mathbb{R})$ of all $n \times n$ matrices with real scalars is not a group, because not every matrix has an inverse. On the other hand, let $\mathrm{GL}_n(\mathbb{R})$ be the set of nonsingular, real, $n \times n$ matrices. This does form a group, known as the **general linear group**. The identity of this group is I_n, the $n \times n$ identity matrix. ∎

Example 3.2.13. There are a variety of ways of building groups out of subsets of the general linear group. For example, let

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad I = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

$$J = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad K = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

be matrices in $\mathrm{GL}_2(\mathbb{C})$. Then the set $Q_8 = \{1, -1, I, -I, J, -J, K, -K\}$ forms a group under matrix multiplication, known as the **quaternion group**. Its Cayley table is displayed.

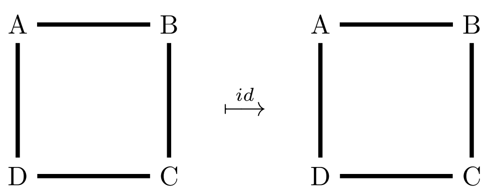
	1	-1	I	$-I$	J	$-J$	K	$-K$
1	1	-1	I	$-I$	J	$-J$	K	$-K$
-1	-1	1	$-I$	I	$-J$	J	$-K$	K
I	I	$-I$	-1	1	K	$-K$	$-J$	J
$-I$	$-I$	I	1	-1	$-K$	K	J	$-J$
J	J	$-J$	$-K$	K	-1	1	I	$-I$
$-J$	$-J$	J	K	$-K$	1	-1	$-I$	I
K	K	$-K$	J	$-J$	$-I$	I	-1	1
$-K$	$-K$	K	$-J$	J	I	$-I$	1	-1

Note that Q_8 is a non-Abelian group since $IJ = K \neq -K = JI$. ∎

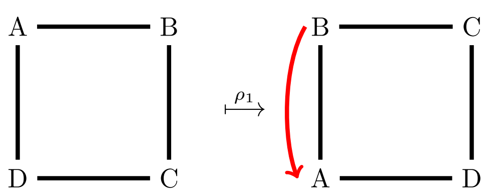
Definition 3.2.14. A **symmetry** of a geometric figure is a rearrangement of the figure preserving the arrangement of its sides and vertices as well as its distances and angles. The set of all symmetries of the regular n-gon is denoted D_n, called the **dihedral group**.

Note that a symmetry of a polygon is just a special type of permutation of the vertices. As such, multiplication of symmetries is function composition and is associative. The identity function is a trivial symmetry, and the inverse of a symmetry is again a symmetry. Symmetry groups are typically noncommutative.

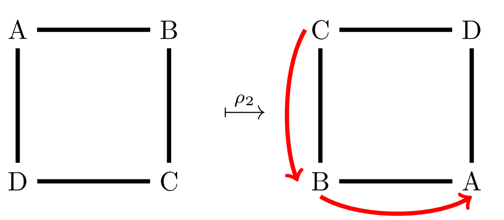
Example 3.2.15. In this example, we will explore D_4, the eight symmetries of a square.



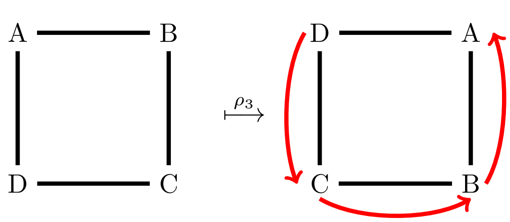
$$id = \begin{pmatrix} A & B & C & D \\ A & B & C & D \end{pmatrix}$$



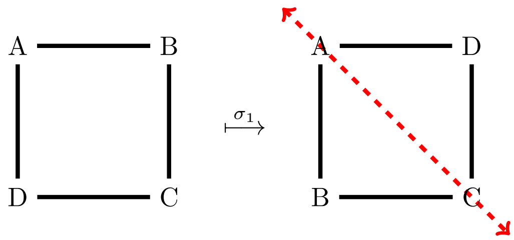
$$\rho_1 = \begin{pmatrix} A & B & C & D \\ D & A & B & C \end{pmatrix}$$



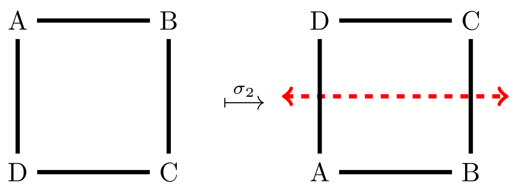
$$\rho_2 = \begin{pmatrix} A & B & C & D \\ C & D & A & B \end{pmatrix}$$



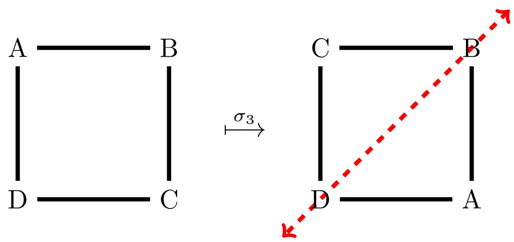
$$\rho_3 = \begin{pmatrix} A & B & C & D \\ B & C & D & A \end{pmatrix}$$



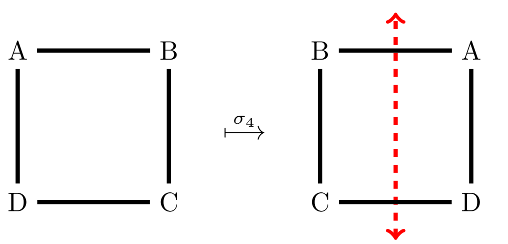
$$\sigma_1 = \begin{pmatrix} A & B & C & D \\ A & D & C & B \end{pmatrix}$$



$$\sigma_2 = \begin{pmatrix} A & B & C & D \\ D & C & B & A \end{pmatrix}$$



$$\sigma_3 = \begin{pmatrix} A & B & C & D \\ C & B & A & D \end{pmatrix}$$



$$\sigma_4 = \begin{pmatrix} A & B & C & D \\ B & A & D & C \end{pmatrix}$$

Multiplication of symmetries is the same permutation multiplication that we introduced earlier. For example,

$$\rho_1\sigma_2 = \begin{pmatrix} A & B & C & D \\ D & A & B & C \end{pmatrix}\begin{pmatrix} A & B & C & D \\ D & C & B & A \end{pmatrix} = \begin{pmatrix} A & B & C & D \\ D & C & B & A \\ C & B & A & D \end{pmatrix} = \sigma_3$$

and

$$\sigma_4\sigma_1 = \begin{pmatrix} A & B & C & D \\ B & A & D & C \end{pmatrix}\begin{pmatrix} A & B & C & D \\ A & D & C & B \end{pmatrix} = \begin{pmatrix} A & B & C & D \\ A & D & C & B \\ B & C & D & A \end{pmatrix} = \rho_3$$

In this group, id is the identity element, and $\rho_1^{-1} = \rho_3$ while every other element is its own inverse. This group is non-Abelian. The Cayley table for D_4 is provided in Table 3.2.1.

■

Homework:
Judson 3.5: 2, 3, 4, 6, Pick 1 (15, 17)

Table 3.2.1. Cayley Table for D_4

	id	ρ_1	ρ_2	ρ_3	σ_1	σ_2	σ_3	σ_4
id	id	ρ_1	ρ_2	ρ_3	σ_1	σ_2	σ_3	σ_4
ρ_1	ρ_1	ρ_2	ρ_3	id	σ_2	σ_3	σ_4	σ_1
ρ_2	ρ_2	ρ_3	id	ρ_1	σ_3	σ_4	σ_1	σ_2
ρ_3	ρ_3	id	ρ_1	ρ_2	σ_4	σ_1	σ_2	σ_3
σ_1	σ_1	σ_4	σ_3	σ_2	id	ρ_3	ρ_2	ρ_1
σ_2	σ_2	σ_1	σ_4	σ_3	ρ_1	id	ρ_3	ρ_2
σ_3	σ_3	σ_2	σ_1	σ_4	ρ_2	ρ_1	id	ρ_3
σ_4	σ_4	σ_3	σ_2	σ_1	ρ_3	ρ_2	ρ_1	id