"My happiness grows in direct proportion to my acceptance, and in inverse proportion to my expectations."
– Michael J. Fox

LECTURE 3

INVERSE FUNCTIONS AND PERMUTATIONS

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1.2. Sets and Equivalence Relations

Example 1.2.22. Consider the mapping $f : A \to A$, with $A \neq \emptyset$ defined by the rule $f(a) = a$ for all $a \in A$. This mapping is known as the **identity function**, is denoted id_A, and is always bijective. ■

Definition 1.2.23. Let $f : A \to B$ be a function. We say that f is **invertible** if there exists a function $f^{-1} : B \to A$, called the **inverse** of f, such that

$$f^{-1} \circ f = id_A \qquad \text{and} \qquad f \circ f^{-1} = id_B.$$

When an inverse exists, it is unique. We will see that this is a property true for all groups.

Example 1.2.24. The functions $f(x) = x^3$ and $g(x) = e^x$ are invertible, real-valued functions since $f^{-1}(x) = \sqrt[3]{x}$ and $g^{-1}(x) = \ln x$. Note that $f^{-1} : \mathbb{R} \to \mathbb{R}$ and $g^{-1} : (0, \infty) \to \mathbb{R}$.

On the other hand, the function $h_1 : \mathbb{R} \to [0, \infty)$ given by $h_1(x) = x^2$ is NOT invertible. It is tempting to believe that $h_1^{-1}(x) = \sqrt{x}$, but this is where the domain/codomain conventions of calculus can be ambiguous. Note that although $\sqrt{x}^2 = x$ for all $x \geq 0$, we have $\sqrt{x^2} = |x|$, which does not equal x for $x < 0$. On the other hand, the mapping $h_2 : [0, \infty) \to [0, \infty)$ given by $h_2(x) = x^2$ is invertible with $h_2^{-1}(x) = \sqrt{x}$. It is important to recognize that as functions $h_1 \neq h_2$ even though they are defined by the same mathematical formula. ■

Theorem 1.2.25. *A mapping is invertible if and only if it is both one-to-one and onto.*

Proof. Let $f : A \to B$. If f is bijective, then for each $b \in B$ there is a unique $a \in A$ such that $f(a) = b$. Define a mapping $g : B \to A$ such that $g(b) = a$ exactly when $f(a) = b$. By the unique statement mentioned before, g is well-defined. Next, note that $f \circ g = id_B$ and $g \circ f = id_A$. Therefore, f is invertible.

If f is invertible, then $f \circ f^{-1} = id_B$, which is a surjective map. By exercise 1.3.22 (b), f is surjective. Likewise, $f^{-1} \circ f = id_A$, which is an injective map. By exercise 1.3.22 (c), f is injective. Therefore, f is bijective. □

Example 1.2.26. Let A be an $m \times n$ matrix. Then this matrix naturally produces a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ using the rule

$$T(\boldsymbol{x}) = A\boldsymbol{x}.$$

As was learned in Linear Algebra class, an $n \times n$ matrix A is invertible, it has an inverse A^{-1} if and only if A is nonsingular, that is, $A\boldsymbol{x} = \boldsymbol{b}$ has a unique solution. Notice that this statement is a special case of the previous theorem. We note that A being nonsingular is the same thing as T being bijective. Note that $A\boldsymbol{x} = \boldsymbol{b}$ having a solution means that $\boldsymbol{b}$ is the image of some vector $\boldsymbol{x}$ via T, that is, $T(\boldsymbol{x}) = \boldsymbol{b}$. This is surjectivity. Likewise, since $A\boldsymbol{x} = \boldsymbol{b}$ has a unique solution means there is exactly one vector $\boldsymbol{x}$ such that $A\boldsymbol{x} = \boldsymbol{b}$. Thus, there is only one vector $\boldsymbol{x}$ such that $T(\boldsymbol{x}) = \boldsymbol{b}$. This is injectivity. ■

Definition 1.2.27. For any set X, a bijective mapping $\pi : X \to X$ is called a **permutation** of X. The set of permutations on X is denoted S_X. When the set has the form $X = \{1, 2, \ldots, n\} \subseteq \mathbb{N}$, we denote S_X as S_n.

When working with permutations on a finite set X, it can be convenient to use a two–row tableaux to express the permutation, where the first row is an arrangement of the elements of X and the second row is the images of the first row with respect to π. Thus, the second row will be a rearrangement of the elements of X, which is why these maps are called permutations. For example, the permutation $\pi : X \to X$ where $X = \{1, \ldots, n\}$ might have the form

$$\pi = \begin{pmatrix} 1 & 2 & \cdots & n \\ \pi(1) & \pi(2) & \cdots & \pi(n) \end{pmatrix}.$$

Theorem 1.2.28. *Let X be a finite, nonempty set. Then $|S_X| = |X|!$.*

Proof. Without the loss of generality, we may assume that $X = \{1, \ldots, n\}$, where $n = |X|$. Let $\pi : X \to X$ be a permutation. Then there are n choices for the assignment $\pi(1)$ since 1 can be mapped to any element of X. Likewise, there are $n-1$ choices for the assignment $\pi(2)$ since 2 can be mapped to any element of X other than $\pi(1)$. This is to guarantee that π is injective. Likewise, there are $n-3$ choices for $\pi(3)$, since it can be any element of X except $\pi(1)$ and $\pi(2)$. Continuing in this pattern, there will always be $n-k+1$ choices for $\pi(k)$. Finally, the product of these choices will give us the correct number of permutation which is $n \cdot (n-1) \cdot (n-2) \cdots 1 = n!$. □

Example 1.2.29. Suppose that $X = \{1, 2, 3\}$. Define a map $\pi : X \to X$ by

$$\pi(1) = 2, \quad \pi(2) = 1, \quad \pi(3) = 3$$

This is a bijective map and hence a permutation. In particular,

$$\pi = \begin{pmatrix} 1 & 2 & 3 \\ \pi(1) & \pi(2) & \pi(3) \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}.$$

There are five other elements of S_3:

$$id_X = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \quad \pi = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

It is always possible to compose two permutations since they have the same domains as codomains, and this composite will itself be a permutation. A three-row tableaux can be used to compute their composite. The right two rows will be the same as the permutation on the right, but the third row will be the image of this second row by the permutation on the left. For example,

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 2 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}.$$

Permutations always have inverses since they are bijective. They can be easily construct by reflecting the above permutation tableaux (and reordering the elements, if necessary). For example, $\pi^{-1} : S \to S$ is given as

$$\pi^{-1} = \begin{pmatrix} \pi(1) & \pi(2) & \pi(3) \\ 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = \pi.$$

Interesting enough, this permutation is equal to its own inverse, although this is not always the case. ■

Homework:
Judson 1.4 : 19, 23, 24