LECTURE 1

SETS – THE FUNDAMENTAL BUILDING BLOCKS OF MATHEMATICS

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1.2. Sets and Equivalence Relations

Definition 1.2.1. A **set** is a well-defined collection of distinct objects. The objects of a set are called its **elements**. By **well-defined**, we mean that there is a rule that enables one to determine whether a given object is an element of the set or not. If a set has no elements, it is called the **empty set** and is denoted $\emptyset$ or $\{\}$.

A set is usually specified either by listing all of its elements inside a pair of braces or by stating the property that determines whether or not an object x belongs to the set. We might write

$$X = \{x_1, x_2, \ldots, x_n\}$$

for a set containing elements $x_1, x_2, \ldots, x_n$ or

$$X = \{x \mid x \text{ satisfies } \mathcal{P}\}$$

if each x in X satisfies a certain property $\mathcal{P}$.

Because elements in a set are distinct, we do not let repeats to occur in a set. In other words, the sets $\{1, 3, 2, 2\}$ and $\{1, 3, 2\}$ are the same, that is,

$$\{1, 3, 2, 2\} = \{1, 3, 2\}.$$

Also, the order or arrangement of the elements in a set is irrelevant. So,

$$\{1, 2, 3\} = \{1, 3, 2\} = \{3, 2, 1\}.$$

Definition 1.2.2. When considering if an object is an element of a set or not, the symbol $\in$ means "is an element of" and $\notin$ means "is not an element of."

Example 1.2.3. Let $A = \{1, 2, 3, 4\}$. To denote that 2 is an element of the set A, we write $2 \in A$. To denote that 5 is not an element of A, we write $5 \notin A$. ■

Some famous sets include:

$$\begin{aligned}
\mathbb{N} &= \{n \mid n \text{ is a natural number}\} = \{0, 1, 2, 3, 4, \ldots\}; \\
\mathbb{Z} &= \{n \mid n \text{ is an integer}\} = \{\ldots, -2, -1, 0, 1, 2, 3, 4, \ldots\}; \\
\mathbb{Q} &= \{r \mid r \text{ is a rational number}\} = \{p/q \mid p, q \in \mathbb{Z} \text{ where } q \neq 0\}; \\
\mathbb{R} &= \{x \mid x \text{ is a real number}\}; \quad \mathbb{C} = \{z \mid z \text{ is a complex number}\}.
\end{aligned}$$

One can compare real numbers by determining whether one is bigger than the other or if they are the same. A similar comparison exists for sets.

Definition 1.2.4. Let A and B be two sets. If A and B have exactly the same elements as each other, then $A = B$.

If every element of A is an element of B, then A is a **subset** of B and we denote this as $A \subseteq B$. For example,

$$\{1, 2\} \subseteq \{1, 2, 3\}.$$

Note that it is always true that $\emptyset \subseteq A$ for any set A. The reason this holds is that there exists no counterexamples, that is, there does not exist an element of $\emptyset$ which is not in A.

If $A \subseteq B$ and $A \neq B$, then $A \subset B$ and A is a **proper subset** of B.

Example 1.2.5. Consider the three sets $A =$ the set of all even numbers, $B = \{2, 4, 6\}$, and $C = \{2, 3, 4, 6\}$.

Here, $B \subseteq A$ since every element of B is also an even number, so is an element of A. Of course, $B \neq A$ since $8 \in A$ but $8 \notin B$. So, $B \subset A$.

It is also true that $B \subset C$. On the other hand, $C \not\subseteq A$ since $3 \in C$ but $3 \notin A$. ∎

Definition 1.2.6. If A and B are sets, then the **intersection** of A and B, denoted $A \cap B$, is the set consisting of elements that belong to both A *and* B. The **union** of A and B, denoted $A \cup B$, is the set consisting of elements that belong to A *or* B (or both).

For a list of sets $A_1, A_2, \ldots, A_n$, then

$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \ldots \cap A_n \quad \text{and} \quad \bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \ldots \cup A_n.$$

Two sets are called **disjoint** if their intersection is empty.

Example 1.2.7. Let $A = \{1, 3, 5, 8\}$, $B = \{3, 5, 7\}$, and $C = \{2, 4, 6, 8\}$.

(a) $A \cap B = \boxed{\{3, 5\}}$

(b) $A \cup B = \boxed{\{1, 3, 5, 7, 8\}}$

(c) To calculate $B \cap (A \cup C)$, we first calculate $A \cup C$.

$$A \cup C = \{1, 2, 3, 4, 5, 6, 8\}.$$

Then

$$B \cap (A \cup C) = \boxed{\{3, 5\}}.$$

∎

Usually, we will work with subsets of a bigger set, which we call the **universal set** U.

Definition 1.2.8. If A is a set, the **complement** of A, denoted A', is the set of all elements in the universal set U that are not in A. Other texts sometimes denote this as $\overline{A}$, A^c, or $\sim A$.

Example 1.2.9. If the universal set $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, $A = \{1, 2, 5, 7, 9\}$, and $B = \{1, 2, 4\}$, then

$$A' = \boxed{\{3, 4, 6, 8\}} \qquad \text{and} \qquad B' = \boxed{\{3, 5, 6, 7, 8, 9\}}.$$

∎

Example 1.2.10. Consider the sets $A = \{\text{red, green, blue}\}$ and $B = \{\text{red, orange, yellow, green, blue, purple}\}$ inside the universal set of all colors. Then

$$B \cap A' = \boxed{\{\text{orange, yellow, purple}\}}.$$

∎

Note that in the previous example, the set $B \cap A'$ is the set of all elements of B not contained in A. This is called the **set difference** and is often denoted $B \setminus A$ or $B - A$. In this context, the set difference only depends on B and A and not on the universal set U even though A' depends on U.

Still considering the previous example, note that $A \smallsetminus B = A \cap B' = \emptyset$ since every element of A is contained in B. In fact, $A \subseteq B$. In general, if $A \subseteq B$ then $A \smallsetminus B = \emptyset$.

Proposition 1.2.11. *For any set $A, B, C \subseteq U$,*

1) *(Idempotency) $A \cup A = A$ and $A \cap A = A$;*
2) *(Identity) $A \cup \emptyset = A$ and $A \cap U = A$;*
3) *(Absorption) $A \cup U = U$, $A \cap \emptyset = \emptyset$, and $A \smallsetminus \emptyset = A$;*
4) *(Complements) $A \cup A' = U$, $A \cap A' = \emptyset$, and $A \smallsetminus A = \emptyset$;*
5) *(Commutativity) $A \cup B = B \cup A$ and $A \cap B = B \cap A$;*
6) *(Associativity) $A \cup (B \cup C) = (A \cup B) \cup C$ and $A \cap (B \cap C) = (A \cap B) \cap C$;*
7) *(Distributivity) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ and $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$;*
8) *(De Morgan's Laws) $(A \cap B)' = A' \cup B'$ and $(A \cup B)' = A' \cap B'$.*

Proof. Many of these identities are straight forward to prove and many will be proven in the homework. To illustrate the basic template for proving two sets are equal we prove the first distributive law $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$. To prove that two sets are equal, we will show that the two sets are subsets of each other.

We begin by showing that $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$. Let $x \in A \cup (B \cap C)$. So, $x \in A$ or $x \in B \cap C$. Suppose first that $x \in A$. Now, $A \subseteq A \cup B, A \cup C$. So, $x \in A \cup B, A \cup C$. Thus, $x \in (A \cup B) \cap (A \cup C)$. Next, suppose that $x \in B \cap C$. So, $x \in B$ and $x \in C$. Now, $B \subseteq A \cup B$ and $C \subseteq A \cup C$. Thus, $x \in (A \cup B) \cap (A \cup C)$. Either way, we get $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$.

Next, we show that $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$. Let $x \in (A \cup B) \cap (A \cup C)$. So, $x \in A \cup B$ and $x \in A \cup C$. If $x \in A$, then $x \in A \cup (B \cap C)$ and we're done. Suppose then that $x \notin A$. That implies that $x \in B$ and $x \in C$. Thus, $x \in B \cap C$. Therefore, $x \in A \cup (B \cap C)$, which implies that $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$.

Combining the two statements, this shows that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$. □

Homework:
Judson 1.4 : 1, Pick 2 from (4-10), Pick 2 from (12-16)