"The function of education is to teach one to think intensively and to think critically. Intelligence plus character – that is the goal of true education." – Martin Luther King, Jr.

LECTURE 2

FUNCTIONS

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1.2. Sets and Equivalence Relations

Definition 1.2.12. Given sets A and B, we can define a new set $A \times B$, called the **Cartesian product** of A and B, as a set of ordered pairs. That is,

$$A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}.$$

We define the Cartesian product of n sets to be

$$\prod_{i=1}^{n} A_i = A_1 \times \ldots \times A_n = \{(a_1, \ldots, a_n) \mid a_i \in A_i \text{ for } i = 1, \ldots, n\}.$$

If $A = A_1 = A_2 = \ldots = A_n$, we often write A^n for $A \times \ldots \times A$ (where A would be written n times). For example, the set $\mathbb{R}^3$ consists of all of 3-tuples of real numbers.

Example 1.2.13. If $A = \{x, y\}$, $B = \{1, 2, 3\}$, and $C = \emptyset$, then $A \times B$ is the set

$$A \times B = \{(x, 1), (x, 2), (x, 3), (y, 1), (y, 2), (y, 3)\} \quad \text{and} \quad A \times C = \emptyset.$$ ∎

Definition 1.2.14. A **relation** R between the sets A and B is a subset of their Cartesian product, that is, $R \subseteq A \times B$. For elements $a \in A$ and $b \in B$, we say that a is **related to** b with respect to R, denoted $a\ R\ b$, if and only if $(a, b) \in R$.

We say that a relation $f \subseteq A \times B$ is a **function** or **mapping** from set A to set B, denoted $f : A \to B$ or $A \xrightarrow{f} B$, if each element $a \in A$ occurs in exactly one ordered pair of the form (a, b) in f, that is, each $a \in A$ is related to exactly one $b \in B$. This unique element b is called the **image** of a with respect to f, and this unique relationship is expressed as $f(a) = b$ or $f : a \mapsto b$. The set A is called the **domain** of f, the set B is called the **codomain** of f, the set

$$f(A) = \{f(a) \mid a \in A\}$$

is called the **image** (or **range**) of f, and, any subset $X \subseteq B$, the set

$$f^{-1}(X) = \{a \in A \mid f(a) \in X\}$$

is called the **pre-image** of X with respect to f.

Let B^A denote the set of all functions of the form $f : A \to B$. This notation is used because if $|A| = a$ and $|B| = b$, then $|B^A| = b^a$, where $|S|$ denotes the cardinality of the set S.

Example 1.2.15. In calculus, functions between real numbers are often defined using formulas such as $f(x) = x^3$ or $g(x) = e^x$. Such function expressions are to be understood as meaning the mapping $f : x \mapsto f(x)$ is given by substitution, for example $f(2) = (2)^3 = 8$ and $g(0) = e^0 = 1$.

In this context, the domain and codomain is usually not stated, but the convention is to set them as the maximal subsets of $\mathbb{R}$ such that $f(x) \in \mathbb{R}$ for each $x \in \operatorname{dom} f$. For example, if $f_1(x) = \dfrac{1}{x}$ and $f_2(x) = \sqrt{x}$, then $f_1(0)$ and $f_2(-1)$ do not evaluate to be real numbers. Therefore, the convention is to restrict the sets so that

$$f_1 : (\infty, 0) \cup (0, \infty) \to (\infty, 0) \cup (0, \infty) \quad \text{and} \quad f_2 : [0, \infty) \to [0, \infty).$$

On the other hand, the functions above have the form

$$f : \mathbb{R} \to \mathbb{R} \qquad \text{and} \qquad g : \mathbb{R} \to (0, \infty). \qquad \blacksquare$$

Example 1.2.16. Consider the relation $f : \mathbb{Q} \to \mathbb{Z}$ given by $f(p/q) = p$. We know that $\dfrac{1}{2} = \dfrac{2}{4}$, but is $f\left(\dfrac{1}{2}\right) = 1$ or 2? This relation cannot be a mapping because it is not *well-defined*. A relation is **well-defined** if each element in the domain is assigned to a unique element in the range. This is particularly a problem when "mappings" are defined on sets of equivalence classes[†] according to representatives from the classes. If not carefully defined, such "mappings" might not be functions. ■

Definition 1.2.17. Given two functions $f : B \to C$ and $g : A \to B$[‡] the **composition** of f and g, denoted $f \circ g : A \to C$, is defined by

$$(f \circ g)(x) = f(g(x)).$$

Example 1.2.18. If $f(x) = x^3$ and $g(x) = e^x$, then $(f \circ g)(x) = (e^x)^3 = e^{3x}$. ■

Definition 1.2.19. Let $f : A \to B$ be a function. Let $a_1, a_2 \in A$. We say that f is **injective** or **one-to-one** if $f(a_1) = f(a_2)$ implies that $a_1 = a_2$, or in other words, for each $b \in f(A)$ there is a unique $a \in A$ such that $f(a) = b$.

We say that f is **surjective** or **onto** if the image of f equals the codomain of f $(f(A) = B)$, or in other words, for all $b \in B$ there exists an $a \in A$ such that $f(a) = b$.

If f is both injective and surjective, then we say that f is **bijective**.

Example 1.2.20. If $f : \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$, and $h : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^3$, $g(x) = e^x$, and $h(x) = x^2$, then f is bijective, g is injective but not surjective, and h is neither injective nor surjective. On the other hand, the mapping $g_1 : \mathbb{R} \to (0, \infty)$ and $h_1 : \mathbb{R} \to [0, \infty)$ given by $g_1(x) = e^x$ and $h_2(x) = x^2$ are both surjective maps but only g_1 is injective. If we consider $h_2(x) : [0, \infty) \to [0, \infty)$ given by $h_2(x) = x^2$, then h_2 is bijective. For these reasons, the convention for defining functions in calculus is very imprecise for higher mathematics. ■

Theorem 1.2.21. *Let $f : A \to B$, $g : B \to C$, and $h : C \to D$. Then*

(a) The composition of mappings is associative, that is, $(h \circ g) \circ f = h \circ (g \circ f)$;

(b) If f and g are both surjective, then $g \circ f$ is surjective;

(c) If f and g are both injective, then $g \circ f$ is injective;

(d) If f and g are both bijective, then $g \circ f$ is bijective.

Proof. Let $a \in A$. Then

$$[(h \circ g) \circ f](a) = (h \circ g)(f(a)) = h(g(f(a))) = h((g \circ f)(a)) = [h \circ (g \circ f)](a),$$

[†]Equivalence relations will be reviewed in the next lecture.

[‡]It is not necessary that the codomain of g and domain of f be equal. Instead function composition can be defined when the domain of f is a subset of the codomain of g.

which proves (a).

Let $c \in C$. Since g is surjective, there exists some $b \in B$ such that $g(b) = c$. Likewise, since f is surjective, there exists some $a \in A$ such that $f(a) = b$. Thus,

$$(g \circ f)(a) = g(f(a)) = g(b) = c,$$

which proves (b).

Part (c) is left as an exercise to the student. Part (d) follows immediately from (b) and (c). $\square$

Homework:
Judson 1.4 : 2, 17, 18, 20, 22