"The worst form of inequality is to try to make unequal things equal." – *Aristotle*

LECTURE 4

EQUIVALENCE CLASSES

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1.2. Sets and Equivalence Relations

Definition 1.2.30. An **equivalence relation** $\sim$ on a set X is a relation $\sim\subseteq X \times X$ such that

1) (Reflexive property) $x \sim x$ for all $x \in X$;

2) (Symmetric property) $x \sim y$ implies $y \sim x$;

3) (Transitive property) If $x \sim y$ and $y \sim z$, then $x \sim z$.

Let $x \in X$. Then define the set $[x] = \{y \in X \mid x \sim y\}$. This set $[x]$ is called the **equivalence class** of x and each element of this set is called a **representative** of the equivalence class.

Example 1.2.31. Let $\sim$ be a relation on $X = \{(a, b) \mid a, b \in \mathbb{Z}, b \neq 0\}$ given by

$$(a, b) \sim (c, d) \quad \text{if and only if} \quad ad = bc.$$

This forms an equivalence relation on X.

1) (Reflexive property) Note that $ab = ab$. So, $(a, b) \sim (a, b)$.

2) (Symmetry property) If $(a, b) \sim (c, d)$ then $ad = bc$. This implies that $cb = da$. Thus, $(c, d) \sim (a, b)$.

3) (Transitive property) Let $(a, b) \sim (c, d)$ and $(c, d) \sim (e, f)$. Then $ad = bc$ and $cf = de$. Then $adf = bcf = bde$. Since $d \neq 0$, we can cancel it from both sides and get $af = be$. Thus, $(a, b) \sim (e, f)$.

Therefore, $\sim$ is an equivalence relation. Note that $\mathbb{Q} = \{[x] \mid x \in X\}$, the set of equivalence classes of ordered pairs of integers. The ratio in lowest terms is typically chosen for the representative of these equivalence classes. ∎

Example 1.2.32. Let A, B be $n \times n$ matrices. We say that A is **similar** to B, denoted $A \sim B$, if there is a nonsingular $n \times n$ matrix P such that

$$A = PBP^{-1}.$$

Similarity is an equivalence relation on $n \times n$ matrices.

1) (Reflexive property) Note that $A = I_n A I_n^{-1}$, where I_n is the $n \times n$ identity matrix. So, $A \sim A$.

2) (Symmetry property) If $A \sim B$ then $A = PBP^{-1}$. Then $B = P^{-1}AP = P^{-1}A(P^{-1})^{-1})$, where P^{-1} is an invertible matrix. Thus, $B \sim A$.

3) (Transitive property) Let $A \sim B$ and $B \sim C$. Then there exists invertible $n \times n$ matrices P and Q such that $A = PBP^{-1}$ and $B = QCQ^{-1}$. Then $A = PBP^{-1} = P(QCQ^{-1})P^{-1} = (PQ)C(PQ)^{-1}$, where PQ is a nonsingular matrix. Thus, $A \sim C$.

Therefore, $\sim$ is an equivalence relation as claimed. Similar matrices share many linear algebraic properties, for example, they have the same determinants, traces, and eigenvalues. The Jordan Canonical Form is often chosen to represent these similarity classes. ∎

Example 1.2.33. Let $f, g : \mathbb{R} \to \mathbb{R}$ be two differentiable functions. Then say that $f \cong g$ if and only if $f' = g'$. This is an equivalence relation on the set of differentiable real-valued functions. From calculus, we know that $f(x) = g(x) + C$ and when integrating the antiderivative with $C = 0$ is usually chosen to represent this class. ■

Definition 1.2.34. A **partition** $\mathcal{P}$ of a set X is a collection of nonempty subsets $X_1, X_2, \ldots$ such that $X_i \cap X_j = \emptyset$ for $i \neq j$ and $\bigcup_k X_k = X$.

Theorem 1.2.35. *Given an equivalence relation $\sim$ on a set X, the equivalence classes of X form a partition of X. Conversely, if $\mathcal{P} = \{X_k\}$ is a partition of a set X, then there is an equivalence relation on X with equivalence classes X_k.*

Proof. Suppose that $\sim$ is an equivalence relation on X. Then each class $[x] \neq \emptyset$ since $x \in [x]$, which follows from the reflexive property. Similarly, $\bigcup_x [x] = X$ since every element $x \in X$ is contained in $[x]$. Finally, if $[x] \cap [y] \neq \emptyset$, then let $z \in [x] \cap [y]$. Then $z \sim x$ and $z \sim y$. By the symmetric property, $x \sim z$. Then $x \sim y$ by the transitive property. This shows that $x \in [y]$, which implies that $[x] \subseteq [y]$. Similarly, we have that $[y] \subseteq [x]$, which implies that $[x] = [y]$. Therefore, the equivalence classes of $\sim$ form a partition of X.

Suppose that $\mathcal{P}$ is a partition of X. We define a relation on X by the rule $x \sim y$ if and only if x and y are contained in the same partition class. We will call $[x]$ the class containing x. Thus, $x \sim y$ means $x \in [y]$. Furthermore, $x \sim y$ if and only if $[x] = [y]$. Therefore, $\sim$ is an equivalence relation on X since $=$ is an equivalence relation on the classes of $\mathcal{P}$. □

Corollary 1.2.36. *Two equivalence classes of an equivalence relation are either disjoint or equal.*

Definition 1.2.37. Let r and s be two integers and suppose that n be a positive integer. We say that r is **congruent to s modulo n**, denoted $r \equiv s \pmod{n}$, if and only if $r - s = kn$ for some $k \in \mathbb{Z}$.

Example 1.2.38. Compute the following residues.

(a) $10 \equiv 1 \pmod{3}$ since $10 - 1 = 9 = 3 \cdot 3$.

(b) $15 \equiv 0 \pmod{5}$ since $15 - 0 = 15 = 3 \cdot 5$.

(c) $15 \not\equiv 0 \pmod{7}$ since $15 = 15 - 0$ is not a multiple of 7.

(d) $30 \equiv 48 \pmod{9}$ since $30 - 48 = -18 = -2 \cdot 9$.

(e) $23 \equiv 2 \pmod{7}$ since $23 - 2 = 21 = 3 \cdot 7$. ■

Proposition 1.2.39. *Congruence modulo n is an equivalence relation on $\mathbb{Z}$ for any positive integer n.*

Proof. Let $r, s, t \in \mathbb{Z}$.

1) (Reflexive property) Note that $r - r = 0 = 0n$. So, $r \equiv r \pmod{n}$.

2) (Symmetry property) If $r \equiv s \pmod{n}$, then $r - s = kn$ for some $k \in \mathbb{Z}$. Then $s - r = (-k)n$, where $-k \in \mathbb{Z}$. Then $s \equiv r \pmod{n}$.

3) (Transitive property) Let $r \equiv s \pmod{n}$ and $s \equiv t \pmod{n}$. Then $r - s = kn$ and $s - t = \ell n$ for some $k, \ell \in \mathbb{Z}$. Adding these together, we get $r - t = (r - s) + (s - t) = kn + \ell n = (k + \ell)n$. Since $k + \ell \in \mathbb{Z}$, we conclude that $r \equiv t \pmod{n}$, which finishes the proof. □

If we consider the equivalence relation established by the integers modulo 3, then

$$[0] = \{\ldots, -3, 0, 3, 6, \ldots\}$$

$$
\begin{aligned}
[1] &= \{\ldots, -2, 1, 4, 7, \ldots\} \\
[2] &= \{\ldots, -1, 2, 5, 8, \ldots\}
\end{aligned}
$$

Notice that $[0] \cup [1] \cup [2] = \mathbb{Z}$ and also that the sets are disjoint. The sets $[0]$, $[1]$, and $[2]$ form a partition of the integers. In particular, there will always be n congruence classes modulo n, and each class can be represented by a unique natural number x such that $0 \leq x < n$. In fact, this representative x is just the remainder of the integer when divided by n, a concept to be explored in the next chapter. The set of all congruence classes modulo n is denoted $\mathbb{Z}_n$. This is a set containing n elements.

Homework:
Judson 1.4 : 21, 25, 26, 28, 29