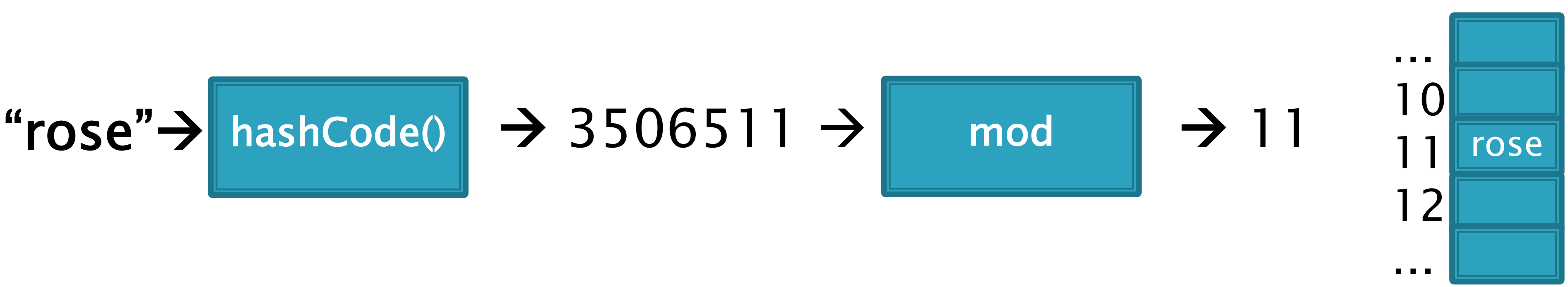
CSSE 230

Hash table basics

How can hash tables perform both **contains()** in O(1) time and **add()** in amortized O(1) time, given enough space?



Hashing

Efficiently putting 5 pounds of data in a 20 pound bag

Reminder: sets hold unique items

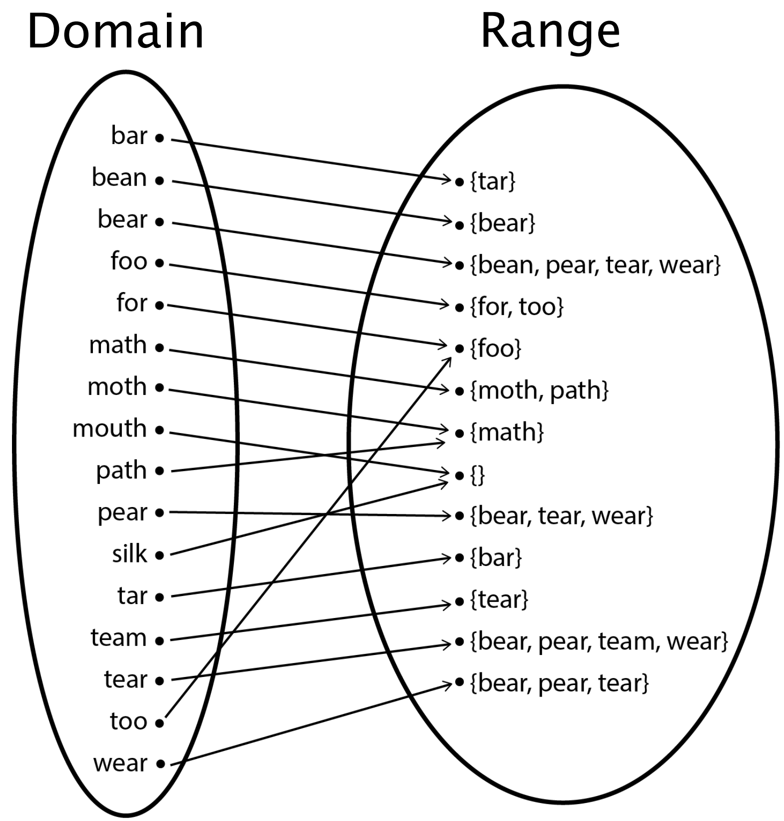
- **Implementation choices:**
  - **TreeSet** (and TreeMap) uses a balanced tree: O(log n)
    - Uses a red-black tree
  - **HashSet** (and HashMap) uses a hash table: amortized O(1) time

HashSet<E> - Method Summary (some, not all)

```
void clear()
boolean contains(Object o)
boolean isEmpty()
boolean add(E e)
boolean remove(Object o)
int size()
```

Related: map concept

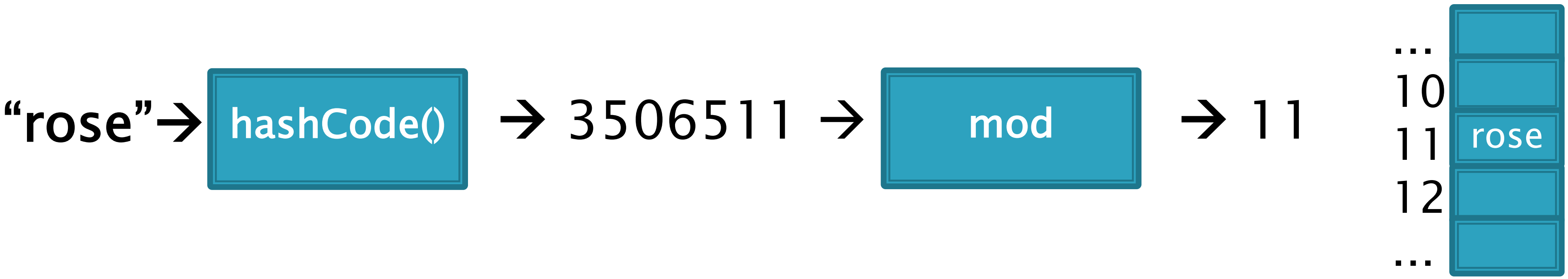
- Allows us to keep track of the mapping from domain to range values
- Map is a container:
  - insert(key, value)
  - delete(key)
  - lookUp(key)
- Domain contains *key* values
- This is *function*, so key values must be unique



HashMap<K,V> – Method Summary (some, not all)

```
void clear()
boolean containsKey(Object key)
V get(Object key)
boolean isEmpty()
V put(K key, V value)
V remove(Object key)
V replace(K key, V value)
int size()
```

Big ideas of hash tables

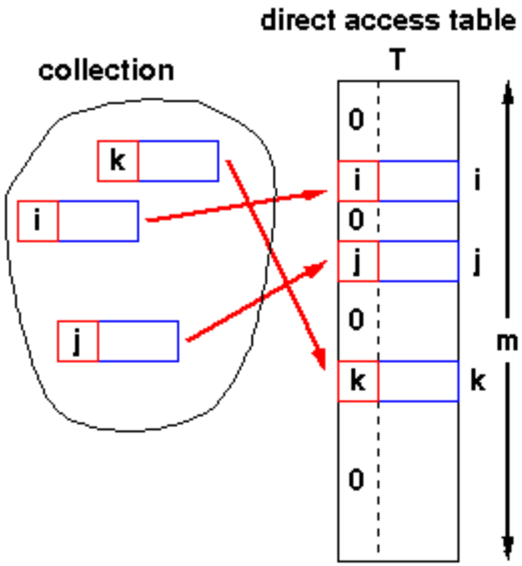


1. The underlying storage?
   Growable array

2. Calculate the index to store an item **from the item itself.** How?
   Hashcode. Fast but un-ordered.

3. What if that location is already occupied with another item?
   Collision. Two methods to resolve

Direct Address Tables



- Array of size **m**
- **n** elements with unique keys
- If **n** $\leq$ m, then use the key as an array index.
  - Clearly O(1) lookup of keys

- Issues?
  - Keys must be unique.
  - Often the range of potential keys is much larger than the storage we want for an array
    - Example: RHIT student IDs vs. # Rose students

Diagram from John Morris, University of Western Australia

When Direct Address Tables are not feasible …

Three step process used for accessing hash tables:

1. Transform *key* into an integer X
2. Use a calculation on X to generate a natural number Y in the range $[0..m-1]$
3. Use Y to index into the hash table array, i.e., `hTable[Y]`

- Step #1 is handled by Java's hashCode() method
- Step #2's m is the size of the hash table array
- Step #2 is often implemented by: $Y = X \bmod m$
  - Using *mod* operation is called the 'Division Method'
  - 'Multiplication Methods' also exist

Javadoc prototype for Object's hashCode() method:

```
int hashCode()
```
Returns a hash code value for the object

We attempt to create unique keys by applying a .hashCode() function …

key → hashCode() → integer

Required property of Java's hashCode() method:

- Given x.equals(y), i.e., x is equal to y,
  then x.hashCode() = y.hashCode()

Desirable properties:

- Should be **fast** to calculate
- Should produce integers that have a nice uniform distribution

hashCode("rose")= 3506511
hashCode("hulman")= -1206158341 (can be negative if overflows)
hashCode("institute") = 36682261

…and then take it mod the table size (m) to get an index into the array.

- Example: if $m = 100$:

hashCode("rose") = 3506511

hashCode("hulman") = -1206158341

hashCode("institute") = 36682261

mod m

→11

→07*

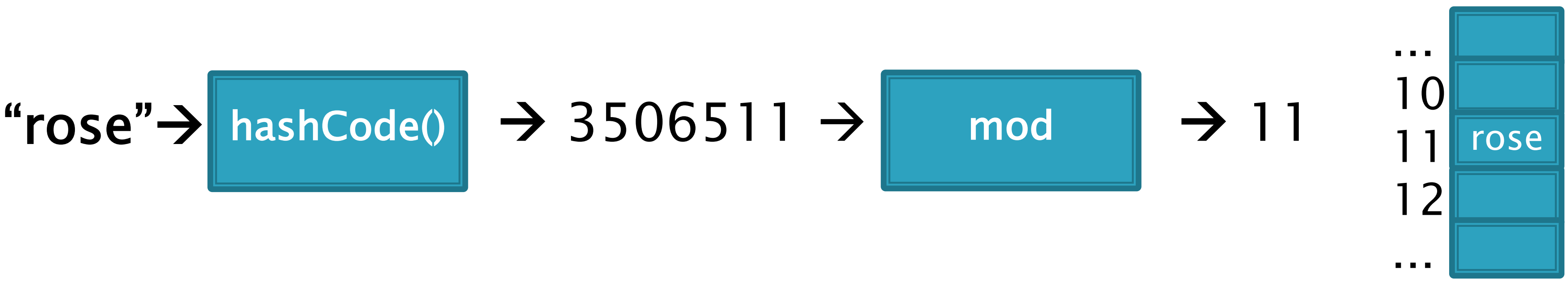
→61

* Note: since the hashCode is an integer, it might be negative…

- If it is negative, add Integer.MAX_VALUE + 1 to make it positive before you mod. (Same as ANDing with 0x7fffffff, or removing sign bit from two's complement)
- This mimics what's actually done in practice: when m is a power of 2, say 2^k, we can just truncate, keeping the last k bits (instead of taking mod m). Sign bit is lost.

Index calculated from the object itself, not from a comparison with other objects

- How Java's **hashCode() is used:**



  - Unless this position is already occupied

a "collision"

Some hashCode() implementations

- Default if you inherit `Object`'s: memory location (platform-specific, actually)
- Many JDK classes override **hashCode()**
  - Integer: the value itself
  - Double: XOR first 32 bits with last 32 bits
  - String: we'll see shortly!
  - Date, URL, ...
- Custom classes should override hashCode()
  - Use a combination of **final** fields.
  - If key is based on mutable field, then the hashcode will change and you will lose it!
  - Developers often use strings when feasible

A simple hashCode function for Strings is a function of every character

```
// This could be in the String class
public static int hash(String s) {
  int total = 0;
  for (int i = 0; i < s.length(); i++)
    total = total + s.charAt(i);
  return total;
}
```

- Advantages?

- Disadvantages?

A better hashCode function for Strings uses place value

```
// This could be in the String class
public static int hash(String s) {
 int total = 0;
 for (int i = 0; i < s.length(); i++)
   total = total*256 + s.charAt(i);
 return total;
}
```

- Spreads out the values more, and anagrams not an issue.
- What about overflow during computation?
  - What happens to first characters?

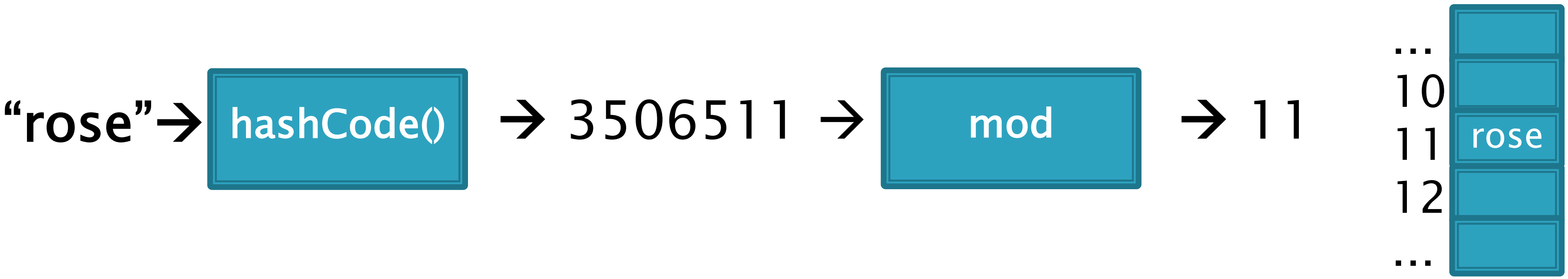
A better hashCode function for Strings uses place value with a base that's prime


```
// This could be in the String class
public static int hash(String s) {
 int total = 0;
 for (int i = 0; i < s.length(); i++)
   total = total*31 + s.charAt(i);
 return total;
}
```

- Spread out, anagrams OK, overflow OK.
- This is **String**'s `hashCode()` method.
- The ($x = 31x + y$) pattern is a good one to follow.

- See https://docs.oracle.com/javase/8/docs/api/java/lang/String.html#hashCode--

Collisions are inevitable

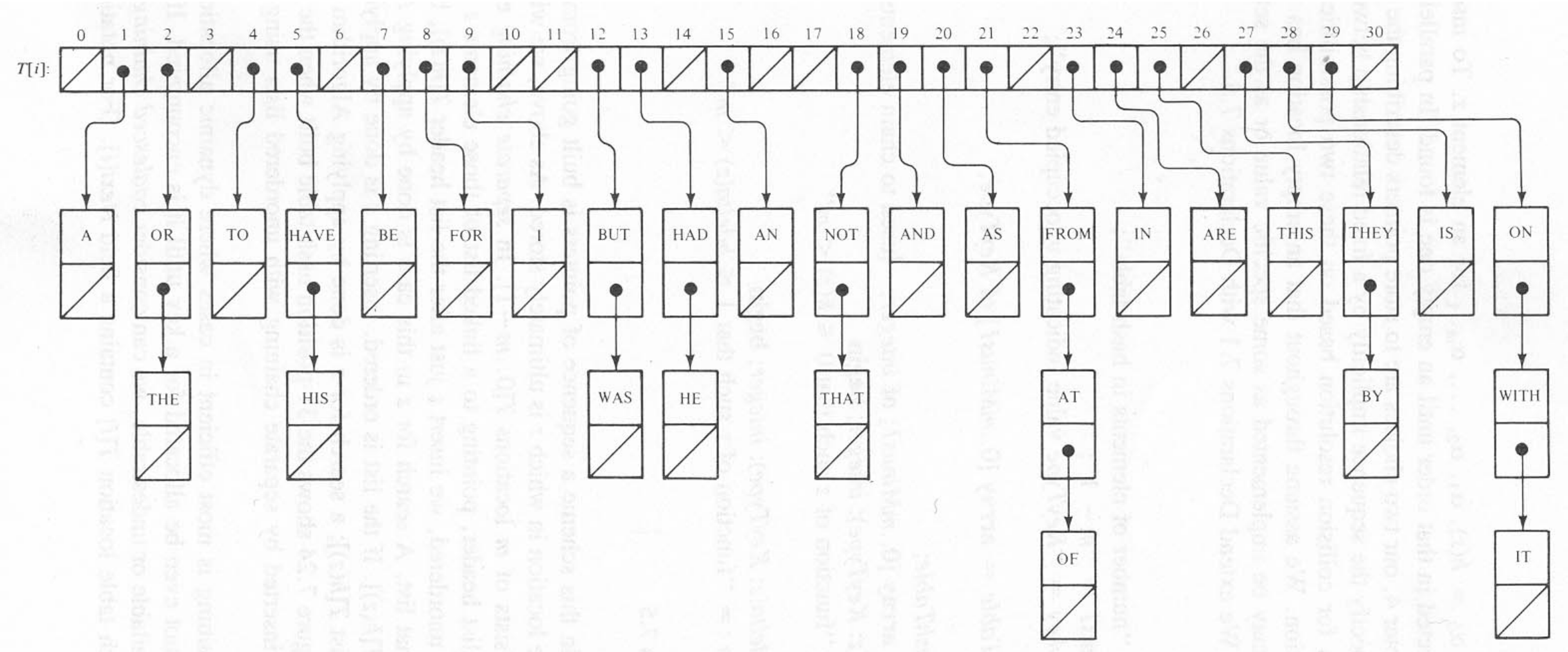


- A good hashCode operation distributes keys uniformly, but collisions will still happen
- hashCode() are ints → only ~4 billion unique values.
  - How many 16 character ASCII strings are possible? "aaaaaaaaaaaaaaaa" <- Here's one
- If n is small (n = # of keys), tables should be much smaller
  - mod will cause collisions too!
- Solutions:
  - Chaining
  - Probing (Linear, Quadratic)

Separate chaining: an array of linked lists

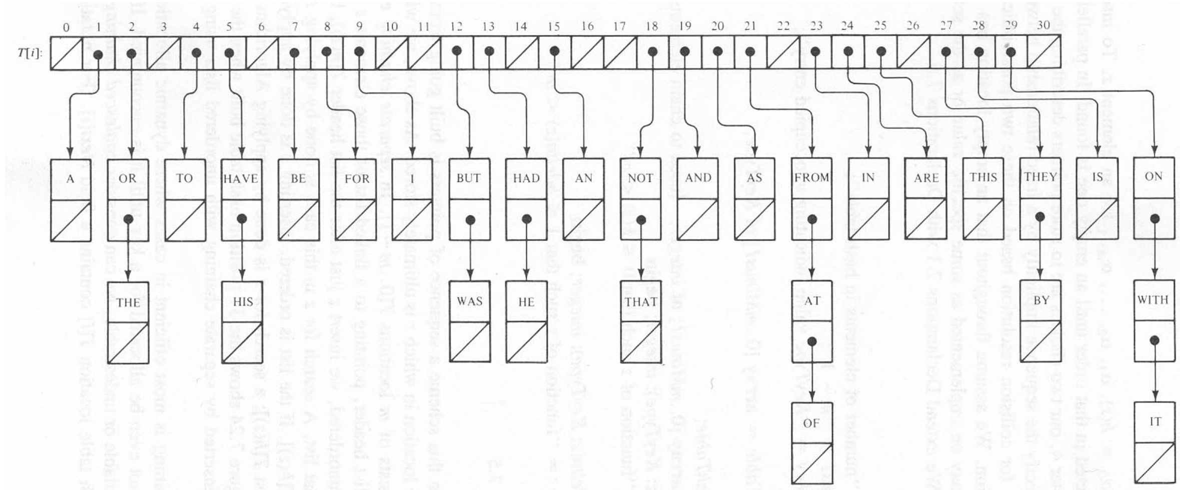
Grow in another direction

Examples: .get(“at”), .get(“him), (hashcode=18), .add(“him”), .delete(“with”)



Java’s **HashMap** uses chaining and a table size that is a power of 2.

Runtime of hashing with chaining depends on the load factor



m array slots,
n items.
Load factor, $\lambda = n/m$.

Runtime = $O(\lambda)$

Space-time trade-off

1. If m constant, then this is O(n). Why?

2. If keep m~0.5n (by doubling), then this is amortized O(1). Why?

Alternative: Store collisions in other array slots.

- No need to grow in second direction

- No memory required for pointers
  - Historically, this was important!
  - Still is for some data…

- Will still need to keep load factor ($\lambda=n/m$) low or else collisions degrade performance
  - We'll grow the array again

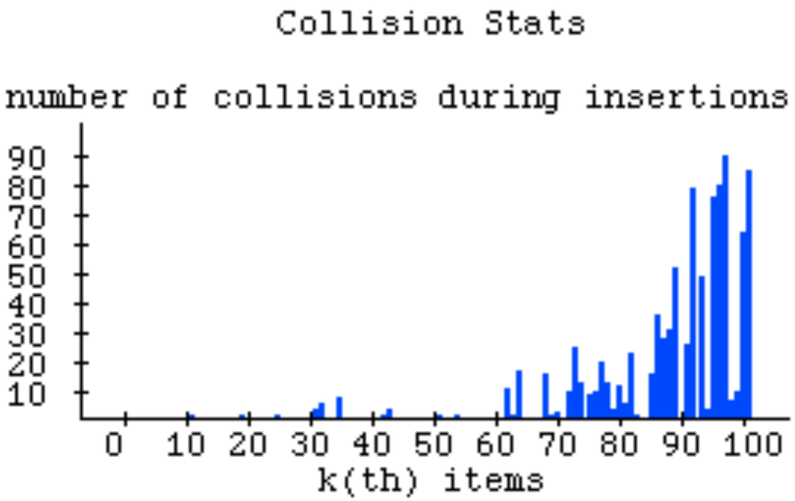
Collision Resolution: Linear Probing

- Probe H (see if it causes a collision)
- Collision? Also probe the next available space:
  - Try H, H+1, H+2, H+3, …
  - Wraparound at the end of the array
- Example on board: .add() and .get()

- Problem: Clustering

- Animation:
  - http://www.cs.auckland.ac.nz/software/AlgAnim/hash_tables.html
  - Applet deprecated on most browsers.
  - See next slide for a few freeze-frames.

Clustering Example



Linear probing efficiency also depends on load factor, $\lambda = n/m$

- For probing to work, $0 \leq \lambda \leq 1$.

- For a given λ, what is the expected number of probes before an empty location is found?

Rough Analysis of Linear Probing

- Assume all locations are equally likely to be occupied, and equally likely to be the next one we look at
- λ = probability that a given cell is full
  $(1-\lambda)$ = probability a given cell is empty
- What's the expected number of probes to find an empty cell?

$$\sum_{p=1}^{\infty} \lambda^{p-1}(1-\lambda)p = \frac{1}{1-\lambda}$$

If $\lambda = 0.5$
Then $\frac{1}{1-0.5} = 2$

From https://en.wikipedia.org/wiki/List_of_mathematical_series:

$$\sum_{k=1}^{n} kz^k = z\frac{1-(n+1)z^n + nz^{n+1}}{(1-z)^2}$$

Rough Analysis of Linear Probing

- $\lambda = 0.5$
- p = # of probes
- prob(1) = $\lambda^{(p-1)} * (1- \lambda)p$ = (1−0.75) = 0.25
- prob(2) = $\lambda^{(2-1)} * (1- \lambda)2$ = 0.375
  1 st probe found full cell, 2nd probe found empty
- prob(3) = $\lambda^{(3-1)} * (1- \lambda)3$ = ~0.422

$$\sum_{p=1}^{\infty} \lambda^{p-1}(1-\lambda)p = \frac{1}{1-\lambda}$$

If $\lambda = 0.5$
Then $\frac{1}{1-0.5} = 2$

From https://en.wikipedia.org/wiki/List_of_mathematical_series:

$$\sum_{k=1}^{n} kz^k = z\frac{1-(n+1)z^n + nz^{n+1}}{(1-z)^2}$$

Start Here for 2nd Day on Hashing

Better Analysis of Linear Probing

- Clustering!
  - Blocks of occupied cells are formed
  - Any collision in a block makes the block bigger
- Two sources of collisions:
  - Identical hash values
  - Hash values that hit a cluster
- Actual average number of probes for large λ:

$$\frac{1}{2}\left(1+\frac{1}{(1-\lambda)^2}\right)$$

For a proof, see Knuth, The Art of Computer Programming, Vol 3: Searching Sorting, 2nd ed, Addision-Wesley, Reading, MA, 1998. (1st edition = 1968)

Why consider linear probing?

- Easy to implement
- Works well when load factor is low
  - In practice, once $\lambda > 0.5$, we usually **double the size of the array** and rehash
  - This is more efficient than letting the load factor get high
- Works well with caching

To reduce clustering, probe farther apart

- Reminder: Linear probing:
  - Collision at H? Try H, H+1, H+2, H+3,...
- New: **Quadratic** probing:
  - Collision at H? Try H, $H+1^2$. $H+2^2$, $H+3^2$, ...
  - Eliminates primary clustering. “Secondary clustering” isn’t as problematic

```
                                          H
hash ( 89, 10 ) = 9
hash ( 18, 10 ) = 8
hash ( 49, 10 ) = 9
hash ( 58, 10 ) = 8
hash (  9, 10 ) = 9
```

49 inserted at $H + 1^2$

58 inserted at $H + 2^2$

9 inserted at $H + 2^2$

	After insert 89	*After insert 18*	*After insert 49*	*After insert 58*	*After insert 9*
0			49	49	49
1					
2				58	58
3					9
4					
5					
6					
7					
8		18	18	18	18
9	89	89	89	89	89

figure 20.7

A quadratic probing hash table after each insertion (note that the table size was poorly chosen because it is not a prime number).

Try:

H

$H + 1^2$

$H + 2^2$

$H + 3^2$

.

.

Quadratic Probing works best with low λ and prime m

- Choose a prime number for the array size, m
- Then if $\lambda \leq 0.5$:
  - Guaranteed insertion
    - If there is a "hole", we'll find it
  - **So no cell is probed twice**

- **Can show with m=17, H=6.**

For a proof, see Theorem 20.4:

Suppose the table size is prime, and that we repeat a probe before trying more than half the slots in the table

See that this leads to a contradiction

Note: a ≡ b (mod m) is true
If and only if a mod m = b mod m

Theorem 20.4

If quadratic probing is used and the table size is prime, then a new element can always be inserted if the table is at least half empty. Furthermore, in the course of the insertion, no cell is probed twice.

Proof

Let M be the size of the table. Assume that M is an odd prime greater than 3. We show that the first $\lceil M/2 \rceil$ alternative locations (including the original) are distinct. Two of these locations are $H + i^2 (\text{mod } M)$ and $H + j^2 (\text{mod } M)$, where $0 \le i, j \le \lfloor M/2 \rfloor$. Suppose, for the sake of contradiction, that these two locations are the same but that $i \neq j$. Then

$$\begin{aligned} H + i^2 &\equiv H + j^2 (\text{mod } M) \\ i^2 &\equiv j^2 (\text{mod } M) \\ i^2 - j^2 &\equiv 0 (\text{mod } M) \\ (i - j)(i + j) &\equiv 0 (\text{mod } M) \end{aligned}$$

Because M is prime, it follows that either $i - j$ or $i + j$ is divisible by M. As i and j are distinct and their sum is smaller than M, neither of these possibilities can occur. Thus we obtain a contradiction. It follows that the first $\lceil M/2 \rceil$ alternatives (including the original location) are all distinct and guarantee that an insertion must succeed if the table is at least half empty.

Quadratic Probing runs quickly if we implement it correctly

Use an algebraic technique to calculate next index

- Difference between successive probes yields:
  - Probe i location, $H_i = (H_{i-1} + 2i - 1) \% M$

1. Just use bit shift to multiply i by 2
   - probeLoc= probeLoc + (i << 1) - 1;

   ...faster than multiplication

2. Since i is at most M/2, can just check:
   - if (probeLoc >= M)
     probeLoc -= M;

   ...faster than mod

When growing array, can't double!

- Can use, e.g., BigInteger.nextProbablePrime()

Quadratic probing analysis

- No one has been able to analyze it!
- Experimental data shows that it works well
  - Provided that the array size is prime, and $\lambda < 0.5$

Summary:
Hash tables are fast for some operations

Data structure used to implement a Map	*Insert* Map's put(k,v)	*Find value* Map's get(k)	*Find max value* Find key in Map with max value
Unsorted array			
Sorted array			
Balanced BST			
Hash table *Chaining, and low load factor, and uniform distribution*			

- Finish the quiz.
- Then check your answers with the next slide

Answers:

Data structure used to implement a Map	*Insert* Map's put(k,v)	*Find value* Map's get(k)	*Find max value* Find key in Map with max value
Unsorted array	Amortized $\theta(1)$ *Insert at end of array*	$\theta(n)$ *Brute force linear search*	$\theta(n)$ *Brute force search*
Sorted array	$\theta(n)$ *Making room for inserted (k,v) requires moving* n *items in worst case*	$\theta(\log n)$ *Binary search*	$\theta(1)$ *Max key will be at end of array*
Balanced BST	$\theta(\log n)$ *Traverse to a leaf*	$\theta(\log n)$ *Traverse to a leaf*	$\theta(\log n)$ *Traverse to a leaf*
Hash table *Chaining, and low load factor, and uniform distribution*	Amortized $\theta(1)$	$\theta(1)$	$\theta(n)$ *Brute force search*

In practice

- Constants matter!

- 727MB data, ~190M elements
  - Many inserts, followed by many finds
  - Microsoft's C++ STL

Structure	build (seconds)	Size (MB)	100k finds (seconds)
Hash map	22	6,150	24
Tree map	114	3,500	127
Sorted array	17	727	25

- Why?
- Sorted arrays are nice **if** they don’t have to be updated frequently!
- Trees still nice when interleaved insert/find

Review: discuss with a partner

- Why use 31 and not 256 as a base in the String hash function?
- Consider chaining, linear probing, and quadratic probing.
  - What is the purpose of all of these?
  - For which can the load factor go over 1?
  - For which should the table size be prime to avoid probing the same cell twice?
  - For which is the table size a power of 2?
  - For which is clustering a major problem?
  - For which must we grow the array and rehash every element when the load factor is high?

Today’s worktime

…Next week‘s Small Programming HW 4 is StringHashSet – it will be posted by tonight – good idea to work on it after Milestone 2 is completed

…is acceptable to use for EditorTrees Milestone 2 group worktime, especially if you have questions for me