Day 27 QS Average Case Analysis

P1

$T(N) = T(L) + T(R) + N$: Recurrence for QS

If $T(N)$ represents the average cost to QS N items the ave cost of a recursive call is obtained by averaging the costs of all possible subproblem sizes

$0, 1, 2, \ldots, N-1$: all possible subproblem sizes

$T(0), T(1), \ldots, T(N-1)$: costs of all subproblem sizes

Now:

To get average, add up costs + divide by N

Get: $$\frac{T(0) + T(1) + \ldots + T(N-1)}{N}$$

Now $$T(N) = \underbrace{\frac{T(0) + \ldots + T(N-1)}{N}}_{T(L)} + \underbrace{\frac{T(0) + \ldots + T(N-1)}{N}}_{T(R)} + N$$

Get: $$T(N) = \frac{2}{N}\left(T(0) + \ldots + T(N-1)\right) + N$$

Now: use variable X Get: $$T(X) = \frac{2}{X}\left(T(0) + \ldots + T(X-1)\right) + X$$

+ Remember N = constant, size of problem

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continued

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$$T(x) = \frac{2}{x}\left(T(0) + \dots + T(x-1)\right) + x$$

Now: Experience tells us that we need to eliminate that pesky $2/x$, i.e., the division by x on RHS

So, Multiply both sides by x

Get: $xT(x) = 2\left(T(0) + \dots + T(x-1)\right) + x^2$

Now: Generate 2 equations from this formula plug in N for x to get 1st
$N-1$ for x to get other

Get:

(1) $$NT(N) = 2\left(T(0) + \dots + T(N-2) + T(N-1)\right) + N^2$$

(2) $$(N-1)T(N-1) = 2\left(T(0) + \dots + T(N-2)\right) + (N-1)^2$$

Now: subtract (2) from (1), but 1st note: $(N-1)^2 = N^2 - 2N + 1$

get: $NT(N) - (N-1)T(N-1) = 2T(N-1) + 2N - 1$

Now: recall that we are looking for upper bound & so "-1" is insignificant, so drop it

get: $NT(N) - (N-1)T(N-1) = 2T(N-1) + 2N$

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continued

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$$NT(N) - (N-1)T(N-1) = 2T(N-1) + 2N$$

Now: rearrange by algebra

Get: $NT(N) = 2T(N-1) + 2N + (N-1)T(N-1)$

$NT(N) = (N+1)T(N-1) + 2N$ Note: $N-1+2 = N+1$

We now have $T(N)$ in terms of $T(N-1)$ and are almost ready to telescope

Now: use experience to get an equation we can telescope, i.e., divide both sides by $N(N+1)$

So:
$$\frac{NT(N)}{N(N+1)} = \frac{(N+1)T(N-1) + 2N}{N(N+1)}$$

Get:
$$\frac{T(N)}{N+1} = \frac{T(N-1)}{N} + \frac{2}{N+1}$$

Now: change back to variable X

$$\frac{T(X)}{X+1} = \frac{T(X-1)}{X} + \frac{2}{X+1}$$

$\longrightarrow$

continued

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$$\frac{T(X)}{X+1} = \frac{T(X-1)}{X} + \frac{2}{X+1}$$

Now: Telescope, plug in N, N-1, N-2, ... for X

Get: N : $\frac{T(N)}{N+1} = \frac{T(N-1)}{N} + \frac{2}{N+1}$

N-1 : $\frac{T(N-1)}{N} = \frac{T(N-2)}{N-1} + \frac{2}{N}$

N-2 : $\frac{T(N-2)}{N-1} = \frac{T(N-3)}{N-2} + \frac{2}{N-1}$

$\vdots \qquad \vdots \qquad \vdots$

N-(N-2) : $\frac{T(2)}{3} = \frac{T(1)}{2} + \frac{2}{3}$

Now: Add equations

Get:

$$\frac{T(N)}{N+1} + \frac{T(N-1)}{N} + \frac{T(N-2)}{N-1} + \ldots + \frac{T(2)}{3} = \overbrace{\frac{T(N-1)}{N} + \frac{2}{N+1}} + \overbrace{\frac{T(N-2)}{N-1} + \frac{2}{N}} + \overbrace{\frac{T(N-3)}{N-2} + \frac{2}{N-1}} + \ldots + \overbrace{\frac{T(1)}{2} + \frac{2}{3}}$$

Now: Simplify by cancellation

get: $\frac{T(N)}{N+1} = \frac{2}{N+1} + \frac{2}{N} + \frac{2}{N-1} + \ldots + \frac{T(1)}{2} + \frac{2}{3}$

$= 2\left(\frac{1}{N+1} + \frac{1}{N} + \frac{1}{N-1} + \ldots + \frac{1}{3} + \frac{1}{1}\right)$

$=$

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continued

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$$\frac{T(N)}{N+1} = 2\left(\frac{1}{3} + \dots + \frac{1}{N-1} + \frac{1}{N} + \frac{1}{N+1}\right) + \frac{T(1)}{2}$$

Now: Recall that $\sum_{k=1}^{N} 1/k = \ln(N) + 0.577$

Theorem 5.5

which is $\Theta(\log_2(N))$

So: we need to get RHS (above) in the form of the summation above

existing RHS: $2\left(\frac{1}{3} + \dots + \frac{1}{N-1} + \frac{1}{N} + \frac{1}{N+1}\right) + \frac{T(1)}{2}$

New RHS: $2\left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{N} + \frac{1}{N+1}\right) - \frac{5}{2}$

add in these

$\boxed{\text{This came from } \frac{T(1)}{2}}$

Subtract this to offset added terms at front end

Apply Theorem 5.5

get: $\frac{T(N)}{N+1} = \Theta(\log_2 N)$

Now: multiply both sides by $N+1$

get $T(N) = N\log N + \log N$

$= \Theta(N \log N)$

Done: So QS Ave performance is $\Theta(N \cdot \log N)$