1 Up in the air (40 pts)

Some potentially useful integral relationships:

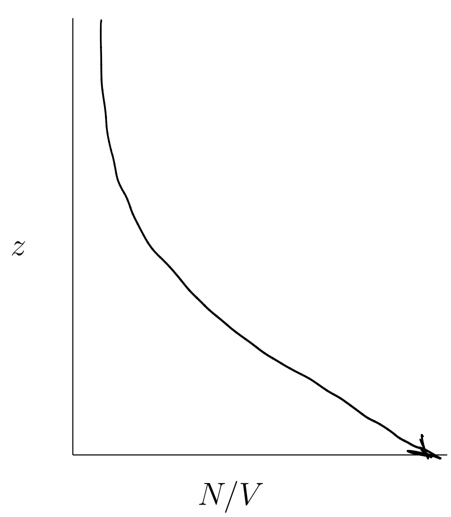
$$\int_0^\infty e^{-ax}dx = 1/a \qquad \int_0^\infty xe^{-ax}dx = 1/a^2$$

The Boltzmann distribution describes how energy is distributed amongst available energy states at thermal equilibrium. For example, if the potential energy of a gas molecule a distance z above the surface of the earth is $U(z) = mgz$, where m is the object's mass and $g = 9.8\,\mathrm{m\,s^{-2}}$ is the acceleration due to gravity, then the Boltzmann distribution tells us the vertical distribution of those molecules in the atmosphere at a given temperature.

1.1 (10 pts) Assuming the temperature of a column of air in the atmosphere is a constant T, write an expression for the relative probability of a molecule of mass m to be a distance z above the surface of the earth. No need to normalize.

$$P(z) \propto e^{-mgz/k_BT}$$

1.2 (10 pts) Based on your answer, sketch the expected number density N/V of gas molecules of mass m vs distance z (altitude) above the earth's surface.



1.3 (8 pts) Calculate the ratio of the number density of CO_2 molecules (mass 44 amu, or $0.044\,\mathrm{kg\,mol^{-1}}$) at an altitude of 11 km (about the altitude that a commercial airliner flies) to that at the earth's surface, assuming a constant $T = 25\,°\mathrm{C}$.

$$\frac{\frac{N}{V}(11000\,m)}{\frac{N}{V}(0\,m)} = \frac{P(11000\,m)}{P(0\,m)} = \frac{e^{-mg(11000\,m)/k_BT}}{e^0}$$

$$\frac{mg}{RT} = \frac{0.044\,kg\,mol^{-1}(9.8\,m/s^2)}{8.314\,JK^{-1}\,mol^{-1}\,(298\,K)} = 0.000174\,m^{-1}$$

$$EXP(-0.000174\,m^{-1} \cdot 11000\,m) = 0.147$$

1.4 (12 pts) Calculate the expectation value of the altitude of a CO_2 molecule at $25\,°\mathrm{C}$.

$$\langle z\rangle = \frac{\int_0^\infty z\,e^{-mgz/k_BT}\,dz}{\int_0^\infty e^{-mgz/k_BT}\,dz}$$

$$= \frac{1/(mg/k_BT)^2}{1/(mg/k_BT)} = \frac{k_BT}{mg}$$

$$= 5747\,m$$

2 Separating the big ones from the little ones (30 pts)

Uranium comes in primarily two isotopes, ^{238}U, natural abundance about 99.3%, and ^{235}U, natural abundance about 0.7%. The ^{235}U isotope has the shorter half-life and is the useful one for nuclear reactors (and bombs!). Uranium is "enriched" by selectively increasing the proportion of the ^{235}U isotope.

2.1 (10 pts) One way to enrich a mixture of uranium isotopes is by taking advantage of the different rates of effusion of gaseous $^{235}UF_6$ and $^{238}UF_6$. If a vessel is filled with a mixture of these two gases in their natural proportions at 50 °C, what is the ratio $^{235}UF_6/^{238}UF_6$ of gases exiting the vessel?

	Atomic mass (amu)
^{235}U	235
^{238}U	238
^{19}F (only isotope)	19

Graham's effusion law $\frac{dN}{dt} \propto 1/m^{1/2}$

$$MW\ {}^{235}UF_6 = 235 + 6(19) = 349 \text{ amu}$$

$$MW\ {}^{238}UF_6 = 238 + 6(19) = 352 \text{ amu}$$

$$\frac{^{235}UF_6}{^{238}UF_6} = \frac{\frac{1}{\sqrt{349}}}{\frac{1}{\sqrt{352}}} = \sqrt{\frac{352}{349}} \boxed{= 1.0043}$$

$P = P_0 e^{-t/\tau}$

2.2 (5 pts) How would increasing the temperature of the vessel at constant volume affect the **proportion** of exiting gases?

mass is not changing

$\tau = \frac{V}{A}\left(\frac{2\pi m}{k_B T}\right)^{1/2}$

Increase ______ Decrease ______ No change __X__

2.3 (5 pts) How would increasing the temperature of the vessel at constant volume affect the **rate** gases exit the vessel?

$T \uparrow$, $\therefore \tau \downarrow$, so rate $\uparrow$

Increase __X__ Decrease ______ No change ______

2.4 (10 pts) How many such effusion devices would have to be connected in series to increase the fraction of ^{235}U to 3%, about that used in a commercial nuclear power plant?

enrichment factor from 2.1 = 1.0043

$$\frac{^{235}F_{out}}{^{238}F_{out}} = (1.0043)^n \cdot \frac{^{235}F_{in}}{^{238}F_{\cancel{out}\,in}}$$

$$\left(\frac{0.03}{0.97}\right) = (1.0043)^n \cdot \left(\frac{.007}{.993}\right)$$

$$\ln(4.387) = \ln\left((1.0043)^n\right)$$

$$1.479 = n \cdot \ln(0.00429)$$

can also take the log base 1.0043

$\boxed{n \approx 345 \text{ devices}}$

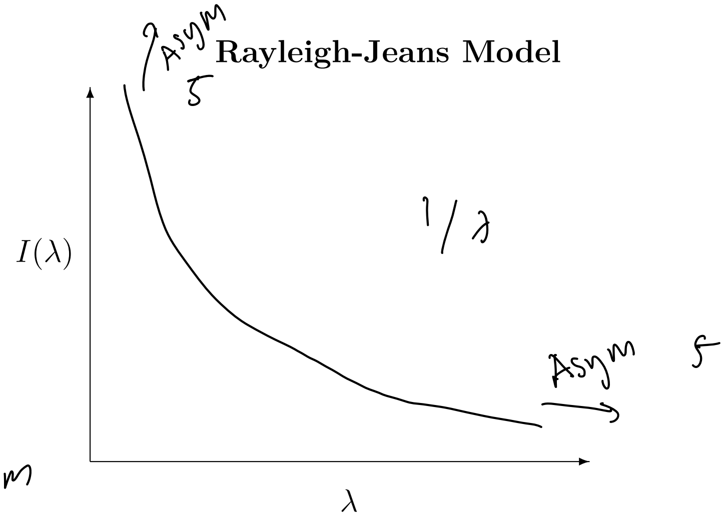
3 It's raining photons (30 pts)

3.1 (10 pts) On the graph on the left below, provide a rough sketch of the spectrum of an ideal blackbody radiator. On the right, provide a rough sketch of what Rayleigh-Jeans models says it should look like.

10pts

Observation

Rayleigh-Jeans Model



3.2 (6 pts) What two things about the light in the blackbody radiator did Planck have to assume to explain the blackbody spectrum?

3pts 1. Thing 1: energy $\propto h\nu$

3pts 2. Thing 2: $E(\nu) = nh\nu$ Discrete quantities of energy

3.3 (10 pts) The earth's surface receives about 340 W m^{-2} of energy from the sun, averaged over the planet's rotation and orbit. What would the equilibrium temperature of the earth's surface be if it behaved as a perfect blackbody radiator and re-emitted all this incoming energy?

$I = \sigma_{SB} T^4$ $\sigma_{SB} = 5.6704 \times 10^{-8}\ J/sm^2K^4$

$340\ W/m^2 = 3.67 \times 10^{-8}\ J/sm^4 K^4 \cdot T^4$

$$T = \sqrt[4]{\frac{340\ W/m^2}{5.6704 \times 10^{-8}\ J/sm^2K^4}}$$

$= 278.3\ K$

3.4 (4 pts) Very briefly, why is the earth warmer than your answer?

Earth is warmer because of the greenhouse effect. Atmosphere is transparent in energy region that the earth emits.

