Problem #1 $V(x) = \frac{1}{2} kx^2$ $\quad -\infty < x < \infty$

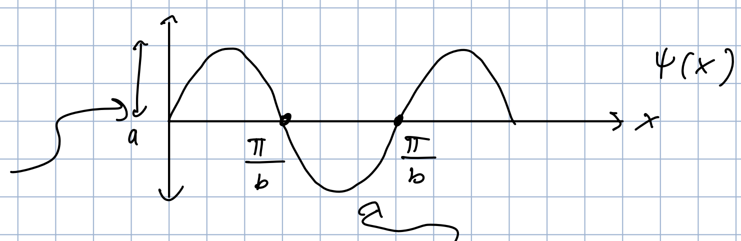
we will write the 1-D Schrodinger equation for Harmonic Oscillator

$$\underbrace{-\frac{\hbar^2}{2m_e}\frac{\partial^2 \psi(x)}{\partial x^2}}_{KE} + \underbrace{\left(\tfrac{1}{2}kx^2\right)\psi(x)}_{PE} = \underbrace{E\,\psi(x)}_{\text{total energy}}$$

Problem #2 The Schrodinger equation must have the following Properties:

1) continuous
2) Single value
3) twice differentiable
4) Square integrable $\rightarrow \int_{-\infty}^{\infty} |f(x)|^2\, dx < \infty$

a) $\psi(x) = a \sin(bx)$



continuous ? ✓

differentiable ? ✓

single value? ✓

$\psi(x) = a \sin(bx)$

$\frac{\partial \psi(x)}{\partial x} = ab \cos(bx)$

square integrable ? No. not bounded by anything

$\int \psi(x)\,\psi(x)\, dx$ $\rightarrow$ wolfram alpha

b) How about $\psi(x) = a \exp\left(-\frac{x^2}{2b^2}\right)$?

continuous? yes.

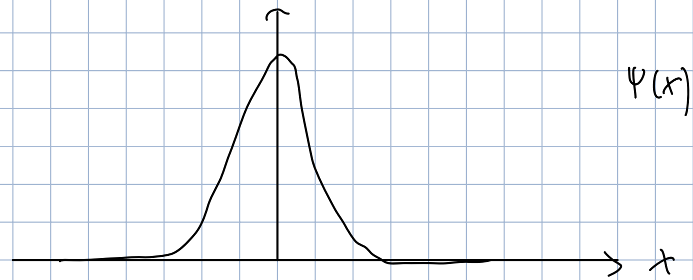
differentiable? yes, can differentiate

single value? yes

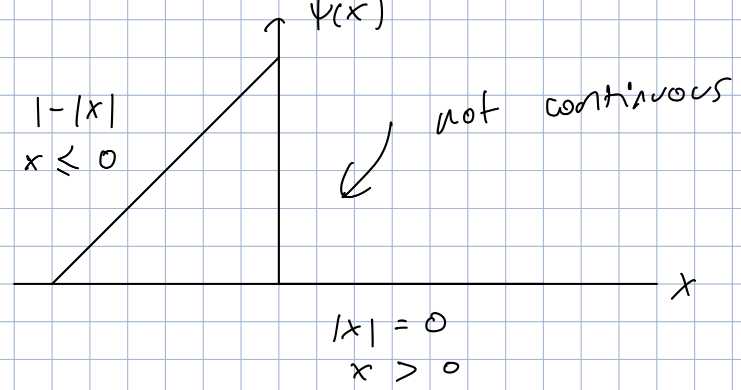
square integrable? yes

e.g. $\int_{-\infty}^{\infty} \exp(-x^2)\, dx = \sqrt{\pi}$

treating a and b as const.



c) it's not continuous ; it has a discontinuity



Problem # 3 Normalize

$$\rho(x) = \psi^*(x)\,\psi(x)$$

$$= \left(a \exp\left(-\frac{x^2}{2b}\right)\right)^2$$

$$\int_{-\infty}^{\infty} a^2 \left(\exp\left(\frac{-x^2}{2b}\right)\right)^2 dx = \sqrt{\pi}\, a^2\, b$$

we normalize s.t.

$$\sqrt{\pi}\, a^2\, b = 1 \Rightarrow \tilde{\psi}(x) = \sqrt{\frac{1}{\sqrt{\pi}\, b}} \exp\left(-\frac{x^2}{2b^2}\right)$$

$$a = \sqrt{\frac{1}{\sqrt{\pi}\, b}}$$

Problem # 4

we will non-dimensionalize x

$$\hat{x} = \frac{x}{b}$$

$$\hat{V} = \frac{V}{(\kappa b^2)}$$

$$V(x) = 1/2 \, \kappa x^2$$

$$\hat{V} \kappa b^2 = 1/2 \, \kappa (b\hat{x})^2$$

$$\hat{V} = 1/2 \, K (\hat{x})^2$$

$$\tilde{\psi}(\hat{x}) = \sqrt{\frac{1}{\sqrt{\pi}\, b}} \exp\left(\frac{-(\cancel{b}\hat{x})^2}{2\cancel{b^2}} \right)$$

$$\tilde{\psi}(\hat{x})^2 = \frac{1}{\sqrt{\pi}\, b} \left(\exp\left(-\frac{\hat{x}}{2} \right) \right)^2$$

Problem #5

$$P(0 < x < b) = \int_0^b \tilde{\psi}(x)^2 \, dx =$$

$$= \int_0^b \frac{1}{\sqrt{\pi}\, b} \left(\exp\left(-\frac{x^2}{2b^2} \right) \right) dx$$

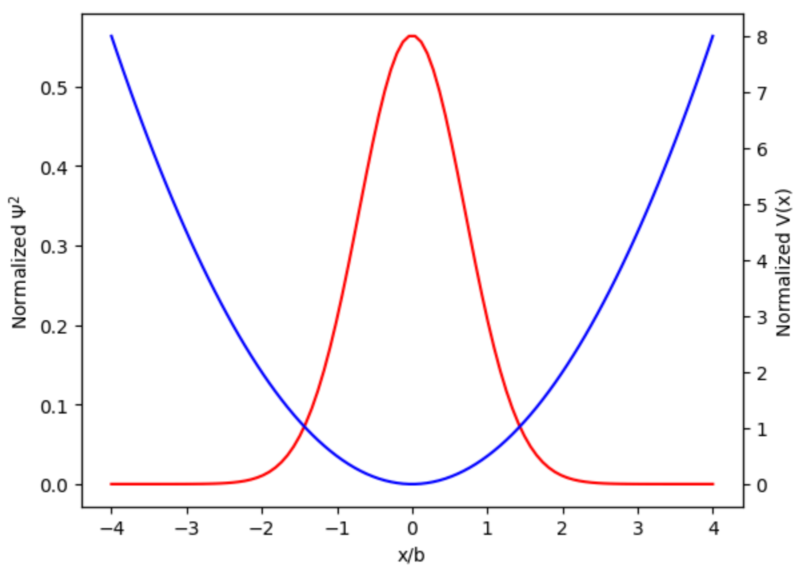
let $u = \frac{x}{b}$ $\quad dx = b \, du$

$$P(0 < u < 1) = \int_0^1 \frac{1}{\pi^{1/2}} e^{-u^2} du$$ } use wolfram

$$= 0.421$$

```
import numpy as np
import matplotlib.pyplot as plt
import math

# Define potential function V(x) (simple harmonic oscillator)
def V(x):
    return (x**2)/2

# Define normalized wavefunction squared (psi(x)^2)
def psi_squared(x,b):
    return  1/(((math.pi)**(1/2)) * b)* np.exp(-x**2)

# Defining:
b = 1  # Unit of length
hbar = 1.0545e-34

# Generate x (normalized) values
x_values = np.linspace(-4, 4, 100)

# Calculate potential values
V_values = V(x_values)

# Calculate wavefunction squared normalized values
psi_squared_values_normalized = psi_squared(x_values,b)

# Calculate normalized wavefunction squared values with the updated normalization constant
#psi_squared_values_normalized = psi_squared(x_values, a, b)
fig, ax_1 = plt.subplots()

# Plot potential and wavefunction squared
ax_1.set_xlabel('x/b')
ax_1.set_ylabel(r'$\mathrm{Normalized\ }\Psi^2$')
ax_1.plot(x_values,psi_squared_values_normalized, color = 'r')

ax_2 = ax_1.twinx()
ax_2.set_ylabel('Normalized V(x)')
ax_2.plot(x_values,V_values, color = 'b')
```

```
[<matplotlib.lines.Line2D at 0x7bad157af910>]
```



Problem #6

want to show

$$\left(-\frac{\hbar^2}{2m_e}\frac{\partial^2}{\partial x^2} + \frac{1}{2}\varkappa x^2\right)\psi \overset{?}{=} E\psi$$

normalized wavefunction is in the form

$$\tilde{\psi}(x) = Ae^{-ax^2}$$

$$\tilde{\psi}(x) = \frac{1}{\pi^{1/4}\sqrt{b}}\exp\left(-\frac{x^2}{2b^2}\right)$$

evaluate the first term $\left(-\frac{\hbar^2}{2m_e}\frac{\partial^2}{\partial x^2}\right)$

$$\frac{\partial^2}{\partial x^2}\tilde{\psi}(x) = \frac{1}{\pi^{1/4}b^{1/2}}\left(2\cdot\frac{1}{2b^2}\exp\left(\frac{-x^2}{2b^2}\right)\overbrace{\left(2\cdot\frac{1}{b^2}x^2 - 1\right)}^{\text{from the hint)}}\right.$$

$$= \frac{1}{b^2}\tilde{\psi}(x)\left(\frac{x^2}{b^2}-1\right)$$

Plug back into Schrodingers

$$-\frac{\hbar^2}{2m_e}\left(\frac{1}{b^2}\tilde{\psi}(x)\left(\frac{x^2}{b^2}-1\right)\right) + \frac{1}{2}\varkappa x^2\tilde{\psi}(x) = E\tilde{\psi}(x)$$

$$\left(-\frac{\hbar^2}{2m_e}\left(\frac{1}{b^2}\left(\frac{x^2}{b^2}-1\right)\right) + \frac{1}{2}\varkappa x^2\right)\tilde{\psi}(x) = E\tilde{\psi}(x)$$

$$\left(-\frac{\hbar^2}{2m_e}\left(\frac{x^2}{b^4}-\frac{1}{b^2}\right) + \frac{1}{2}\varkappa x^2\right)\cancel{\tilde{\psi}(x)} = E\cancel{\tilde{\psi}(x)}$$

$$\left(-\frac{\hbar^2}{2m_e}\left(\frac{x^2}{\frac{\hbar^2}{m_e\varkappa}} - \frac{1}{\left(\frac{\hbar^2}{m_e\varkappa}\right)^{1/2}}\right)\right) + \frac{1}{2}\varkappa x^2 = E$$

$$\cancel{\frac{\varkappa x^2}{2}} + \frac{\hbar\sqrt{\varkappa}}{2\sqrt{m_e}} + \cancel{\frac{1}{2}\varkappa x^2} = E$$

$$E = \frac{1}{2}\hbar\sqrt{\frac{\varkappa}{m}}$$ } eigenfunction to shrodingers

Problem #7

$$\hat{p} = -i\hbar \frac{\partial}{\partial x}$$

$$\hat{p}\,\hat{\Psi}(x) = -i\hbar \frac{\partial}{\partial x}\left(\frac{1}{\pi^{1/4}\, b^{1/2}} \cdot \exp\left(\frac{-x^2}{2b^2}\right)\right)$$

$$= \frac{i\hbar}{\pi^{1/4} b^{1/2}} \cdot \exp\left(\frac{-x^2}{2b}\right) \cdot \frac{x}{b^2}$$

$$= \frac{-i\hbar}{\pi^{1/4} b^{5/2}} \cdot \underline{x} \cdot \exp\left(-\frac{x^2}{2b}\right)$$

because there is an x,

$\tilde{\Psi}(x) \neq$ eigenfunction of linear momentum

Problem #8

From Q7, $\tilde{\Psi}(x) \neq$ eigenfunction of linear momentum

therefore, if we measure the linear momentum of many electrons, we will not get the same answer every time.

9. What is the average value for the linear momentum you measure?

Expectation value

$\langle \Psi | \hat{p} | \tilde{\Psi} \rangle$

$\hat{p} = -i\hbar \frac{d}{dx}$

$$\hat{p} | \tilde{\Psi} \rangle = -i\hbar \frac{d}{dx} \left(\tilde{\Psi}(x) \right)$$

$$= \frac{i\hbar}{\pi^{1/4} b^{5/2}} x e^{-x^2/2b^2}$$

$$\frac{i\hbar}{(\pi^{1/4})^2 b^{\frac{1}{2}+\frac{5}{2}}} = c$$

$$\langle \tilde{\Psi} | \hat{p} | \tilde{\Psi} \rangle = \int_{-\infty}^{\infty} \left(\frac{1}{\pi^{1/4} b^{1/2}} e^{-x^2/2b^2} \right) \left(\frac{i\hbar}{\pi^{1/4} b^{5/2}} x e^{-x^2/2b^2} \right) dx$$

$$= c \int_{-\infty}^{\infty} x e^{-x^2/b^2} dx = \boxed{0}$$

can also use symmetry to argue this has to be zero

10. What is the uncertainty in the momentum?

$\Delta p = \sqrt{\langle p^2\rangle - \cancel{\langle p\rangle^2}}$

$\hookrightarrow \langle p\rangle = 0, \therefore \langle p\rangle^2 = 0$

$\Delta p = \sqrt{\langle p^2\rangle}$

$$\hat{p}^2 = \hat{p}\cdot\hat{p} = -i^2\hbar^2\frac{d^2}{dx^2} = -\hbar^2\frac{d^2}{dx^2}$$

$$\hat{p}^2|\tilde{\psi}\rangle = -\hbar^2\frac{d^2}{dx^2}\left(\frac{1}{\pi^{1/4}b^{1/2}}e^{-x^2/2b^2}\right) \qquad \frac{d}{dx}(\psi) \text{ from \#9}$$

$$= -\hbar^2\frac{d}{dx}\left(\frac{1}{\pi^{1/4}b^{5/2}}xe^{-x^2/2b^2}\right)$$

$$= \frac{-\hbar^2}{\pi^{1/4}b^{9/2}}(x-b^2)e^{-x^2/2b^2} \qquad \text{Using Wolfram Alpha}$$

$$\langle\tilde{\psi}|p^2|\tilde{\psi}\rangle = \int_{-\infty}^{\infty}\left(\frac{1}{\pi^{1/4}b^{1/2}}e^{-x^2/2b^2}\right)\left(\frac{-\hbar^2}{\pi^{1/4}b^{9/2}}(x^2-b^2)e^{-x^2/2b^2}\right)dx$$

exponents add w/ multiplication

$$\frac{1}{\pi^{1/4}b^{1/2}}\cdot\frac{-\hbar^2}{\pi^{1/4}b^{9/2}}\int_{-\infty}^{\infty}(x^2-b^2)e^{-x^2/2b^2 + -x^2/2b^2}dx$$

$\frac{1}{2}+\frac{9}{2} = \frac{10}{2} = 5$

$$\frac{-\hbar^2}{\pi^{1/2}b^5}\underbrace{\int_{-\infty}^{\infty}(x^2-b^2)e^{\frac{-2x^2}{2b^2}}dx}_{\frac{-\sqrt{\pi}b^3}{2}} \quad \text{using Wolfram Alpha}$$

$$\langle p^2\rangle = \frac{-\hbar^2}{\cancel{\pi^{1/2}}b^5}\cdot\frac{-\cancel{\pi^{1/2}}b^3}{2} = \frac{\hbar^2}{2b^2}$$

$$\boxed{\Delta p = \sqrt{\langle p^2\rangle} = \frac{\hbar}{\sqrt{2}b}}$$

11. Maximum precision of the position?

Heisenberg Uncertainty : $\Delta x \Delta p \geq \frac{\hbar}{2}$

$$\Delta x \cdot \frac{\hbar}{\sqrt{2}b} \geq \frac{\hbar}{2} \qquad \frac{\hbar}{2} \div \frac{\hbar}{\sqrt{2}b} = \frac{\hbar}{2} \cdot \frac{\sqrt{2}b}{\hbar}$$

$$\boxed{\Delta x \geq \frac{b}{\sqrt{2}}}$$

12. $\tilde{\Psi}(x)$ is nonzero for all values on the domain