For $F : \mathbb{R} \to \mathbb{R}$ a cumulative distribution function, we can define the pseudo-inverse by

$$Q(p) = F^{-1}(p) := \inf \{x \in \mathbb{R} \,;\, p \leq F(x)\} \,, \quad p \in \mathbb{R}$$

1) The set $A = \{x \in \mathbb{R} \,;\, p \leq F(x)\}$ can be empty or not, bounded or not, depending on $p \in \mathbb{R}$.

Indeed,

i) $p = 0$:

$$\{x \in \mathbb{R} \,;\, 0 \leq F(x)\} = \mathbb{R}$$

ii) $p > 1$

$$\{x \in \mathbb{R} \,,\, p \leq F(x)\} = \emptyset$$

iii) $p < 0$

$$\{x \in \mathbb{R} \,;\, p \leq F(x)\} = \mathbb{R}$$

In cases i) and iii), the $\inf \mathbb{R}$ is defined to be $-\infty$. Then

$$Q(p) = -\infty \quad \text{if } p \leq 0.$$

In case $p > 1$, the $\inf \emptyset$ is defined to be $+\infty$. Then, $Q(p) = +\infty$.

When $0 < p < 1$, the set is always bounded and non-empty. Indeed, by definition

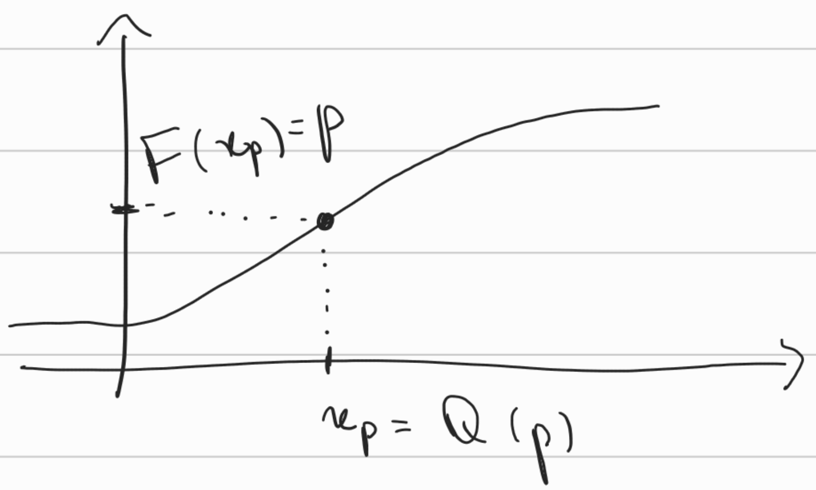
$$F(x) = \mathbb{P}(X \leq x) \in [0, 1]$$

Then, the inf is a true min:

$$Q(p) = \min\{x \in \mathbb{R};\ p \leq F(x)\},\ p \in (0,1)$$

(2) When F is strictly increasing, the pseudo-inverse is a true inverse:

$$Q(p) = x_p \Leftrightarrow F(x_p) = p$$



It means that there is only one $x_p \in \mathbb{R}$ s.t. $F(x_p) = p$. Indeed, for all $x \le x_p$, we have $F(x) < F(x_p)$ since F is strictly increasing and since $F(x_p) = p$,

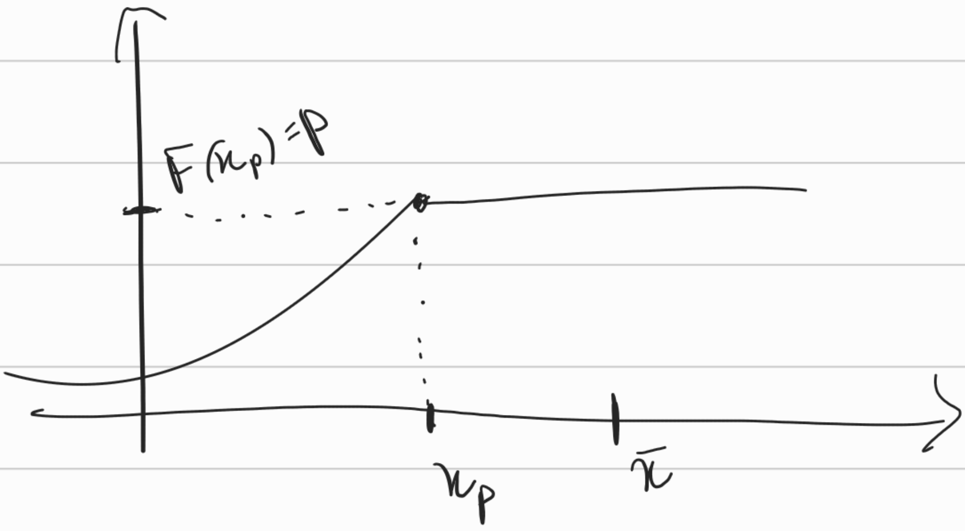
$p \le F(x_p)$.

Then $\inf \{x \in \mathbb{R} ; \ p \le F(x)\} = x_p$

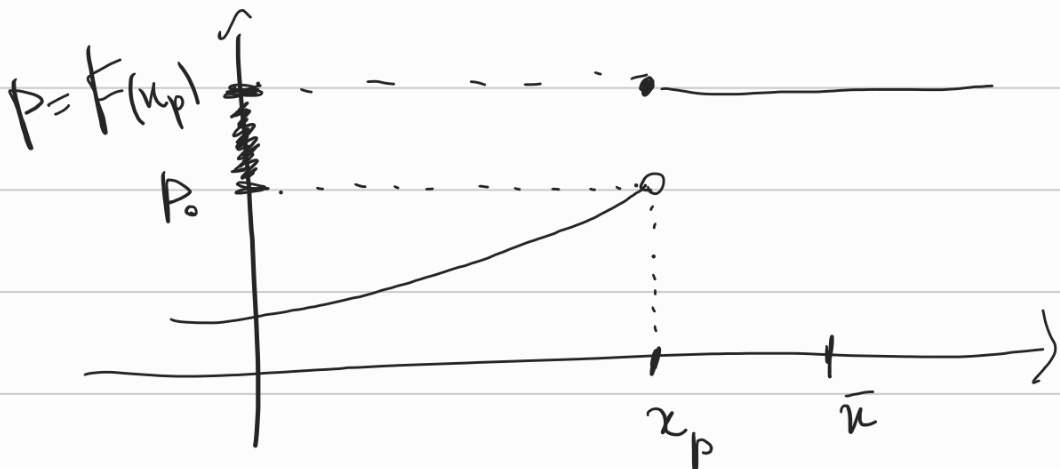
and

$$\begin{cases} F(Q(p)) = F(x_p) = p \\ Q(F(x_p)) = Q(p) = x_p. \end{cases}$$

③ When F is only non-decresing, for instance,



or



then $Q(p) = x_p$ is not a true inverse. Indeed, for any $\bar{x} > x_p$ we have $F(\bar{x}) = F(x_p) = p$

then $F^{-1}(p) := \{x \in \mathbb{R};\ F(x) = p\}$
$= [x_p, +\infty)$.

In that case, $Q(p) = x_p$ and also $Q(\bar{p}) = x_p$ for all $\bar{p} \in [p_0, p]$

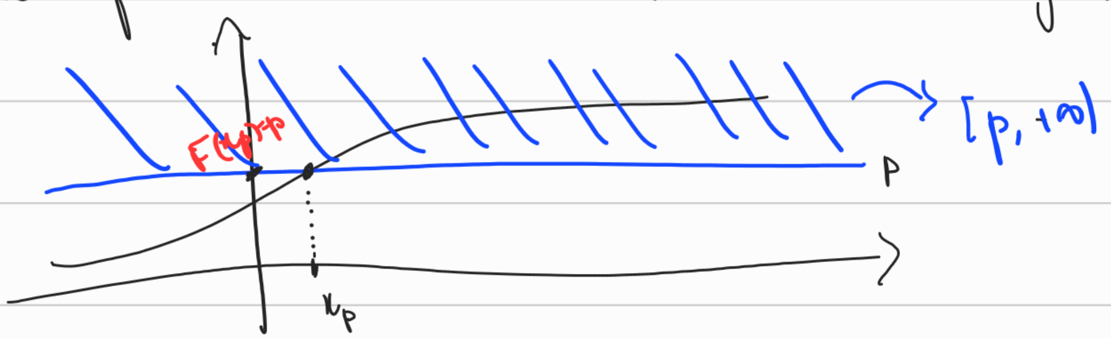
OBS:
We can rewrite the set

$$\{x \in \mathbb{R};\ p \leq F(x)\} = F^{-1}(p)$$

as the pre-image of $[p, +\infty)$ by F:

$$F^{-1}([p, +\infty)) = \{x \in \mathbb{R};\ F(x) \in [p, +\infty)\}$$

In that is easier to interpret the quantile $Q(p)$ as the "first" $x \in \mathbb{R}$ such that $F(x)$ belongs to $[p, +\infty)$



$\{x \in \mathbb{R} ;\ p \leq F(x)\} = F^{-1}([p, +\infty))$

x_0 tq

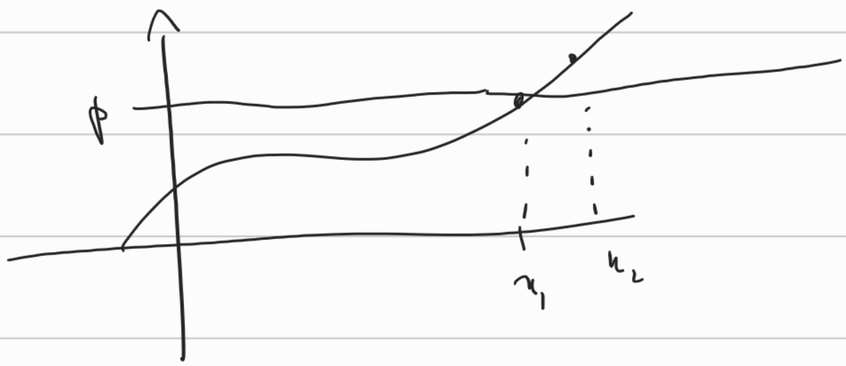
$p \leq F(x_0)$ intão $x_0 \in F^{-1}([p, +\infty))$

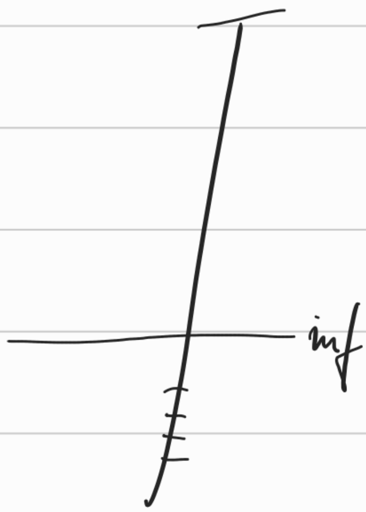
$\Rightarrow F(x_0) \in [p, +\infty)$

$\Rightarrow F(x_0) \geq p$

$a \in F^{-1}([p, +\infty)) \Rightarrow F(a) \geq p$

$\inf F^{-1}([p, +\infty))$





$F(x_2) \geq F(x_1) = p$

$\Rightarrow x_2 \in F^{-1}[p, +\infty))$

but $x_2 \geq x_1$