① $(X_i)_{i=1}^{n} \overset{iid}{\sim} \text{Pois}(\lambda)$, $\hat{\lambda} = \frac{1}{n}\sum_{i=1}^{n} X_i$

Find bias, se and MSE.

$$\text{bias}\,\hat{\lambda} = \mathbb{E}\hat{\lambda} - \lambda$$

$$= \mathbb{E}\frac{1}{n}\sum_{i=1}^{n} X_i - \lambda = \frac{n}{n}\mathbb{E}X_i - \lambda = \lambda - \lambda = 0$$

$\Leftrightarrow$ unbiased

$$\text{Var}\,\hat{\lambda} = \text{Var}\left(\frac{1}{n}\sum X_i\right) = \frac{1}{n^2}\text{Var}\sum_{i=1}^{n} X_i$$

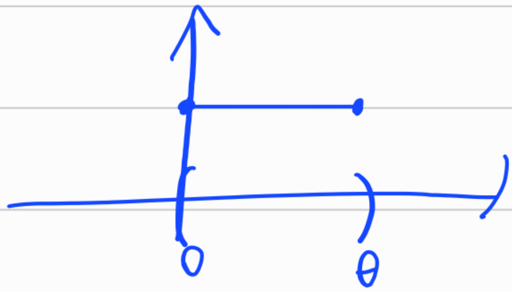
$$= \frac{1}{n^2}\, n\, \text{Var}X_i = \frac{1}{n}\text{Var}X_i = \frac{\lambda}{n}$$

$$se = \sqrt{\text{Var}\,\hat{\lambda}} = \sqrt{\frac{\lambda}{n}}\,;$$

$$\text{MSE}(\hat{\lambda}) = \text{bias}^2(\hat{\lambda}) + \text{Var}\,\hat{\lambda}$$

$$= 0 + \frac{\lambda}{n} = \frac{\lambda}{n}$$

② $(X_i)_{i=1}^{n} \overset{iid}{\sim} \text{Unif}(0,\theta)$



$\hat{\theta} = \max_{1\le i\le n} X_i$

$F_X(t) = \int_0^t \mathbb{1}_{[0,\theta)}(y) \frac{1}{\theta} dy$

$\text{bias } \hat{\theta} = \mathbb{E} \max_{1\le i\le n} X_i - \theta$

$= \int_{[0,t]\cap[0,\theta]} 1\, dy$

$Y = \max_{1\le i\le n} X_i \Rightarrow Y \sim ?$

$= \frac{t}{\theta},\ t<\theta$

$$\mathbb{P}(Y \le t) = \mathbb{P}\left(\max_{1\le i\le n} X_i \le t\right)$$

$$= \mathbb{P}\left(\bigcap_{i=1}^{n} \{X_i \le t\}\right) \overset{ind}{=} \prod_{i=1}^{n} \mathbb{P}(X_i \le t)$$

$$\overset{i.d.}{=} \mathbb{P}(X_i \le t)^n$$

$$= \begin{cases} (t/\theta)^n & \text{se } t<\theta. \\ 0, & \text{c.c.} \end{cases}$$

$\boxed{\sim \text{Beta}(n,1)\cdot\theta}$ $\quad F' = f$

$f_Y(t) = \frac{n}{\theta}(t/\theta)^{n-1},\ t<\theta$

$= \frac{n\,t^{n-1}}{\theta\ \theta^{n-1}} = \frac{n\,t^{n-1}}{\theta^n}$

$$\mathbb{E}Y = \int_0^\theta t\, f_Y(t)\,dt$$

$$= \int_0^\theta t \frac{n}{\theta^n} t^{n-1}\, dt = \frac{n}{\theta^n}\int_0^\theta t^n dt = \frac{n}{\theta^n} \frac{t^{n+1}}{n+1}\Big|_0^\theta$$

$$= \frac{n}{\theta^n}\left[\frac{\theta^{n+1}}{n+1} - \frac{0^{n+1}}{n+1}\right] = \frac{n\theta}{n+1}$$

$$\text{bias}\,\hat{\theta} = \frac{n\theta}{n+1} - \theta = \frac{n\theta - n\theta - \theta}{n+1} = \frac{-\theta}{n+1}$$

$\Rightarrow$ biased!

$$se = \sqrt{\operatorname{Var}\hat{\theta}}$$

$$\operatorname{Var}\hat{\theta} = \mathbb{E}\hat{\theta}^2 - (\mathbb{E}\hat{\theta})^2$$

$$\mathbb{E}\hat{\theta}^2 = \int_0^\theta t^2 f_Y(t)\,dt = \int_0^\theta t^2 \frac{n t^{n-1}}{\theta^n}\,dt = \frac{n}{\theta^n}\int_0^\theta t^{n+1}\,dt$$

$$= \frac{n}{\theta^n}\frac{t^{n+2}}{n+2}\Big|_0^\theta = \frac{n}{\theta^n}\frac{\theta^{n+2}}{n+2} = \frac{n\theta^2}{n+2}$$

$$\operatorname{Var}\hat{\theta} = \frac{n\theta^2}{n+2} - \left(\frac{n\theta}{n+1}\right)^2 = \frac{n\theta^2}{n+2} - \frac{n^2\theta^2}{(n+1)^2}$$

$$= \theta^2\left(\frac{n}{n+2} - \frac{n^2}{(n+1)^2}\right) = \theta^2\left[\frac{(n+1)^2 n - n^2(n+2)}{(n+2)(n+1)^2}\right]$$

$$= \theta^2\frac{\left[(\cancel{n^3} + \cancel{2n^2} + n) - \cancel{n^3} - \cancel{2n^2}\right]}{(n+1)^2(n+2)}$$

$$= \theta^2\frac{n}{(n+1)^2(n+2)}$$

$$se = \sqrt{Var\,\hat{\theta}} = \frac{\theta}{(n+1)}\sqrt{\frac{n}{n+2}}$$

$$MSE(\hat{\theta}) = \frac{\theta^2}{(n+1)^2} + \frac{\theta^2 n}{(n+1)^2(n+2)}$$

③ $(X_i)_{i=1}^{n} \overset{iid}{\sim} \text{Unif}(0,\theta)$, $\hat{\theta} = 2\bar{X}$

bias, se, MSE

$$\mathbb{E}\hat{\theta} = \mathbb{E}2\bar{X} = 2\,\mathbb{E}\frac{1}{n}\sum_{i=1}^{n} X_i = 2\,\mathbb{E}X_i = 2\frac{\theta}{2} = \theta$$

$$\therefore \text{ bias } \hat{\theta} = \mathbb{E}\hat{\theta} - \theta = \theta - \theta = 0$$

$\Rightarrow$ unbiased

$$Var\,\hat{\theta} = Var\,2\bar{X} = 2^2\,Var\frac{1}{n}\sum X_i = \frac{4}{n^2}\,n\,Var X_i = \frac{4}{n}Var X_i$$

$$\mathrm{Var}\,X_i = \int_0^\theta x^2 \frac{1}{\theta}\,dx - \left(\frac{\theta}{2}\right)^2 = \frac{1}{\theta}\frac{x^3}{3}\Big|_0^\theta - \frac{\theta^2}{4} = \frac{1}{\theta}\frac{\theta^3}{3} - \frac{\theta^2}{4}$$

$$= \frac{4\theta^2 - 3\theta^2}{12}$$

$$= \frac{\theta^2}{12}$$

$$\therefore \mathrm{Var}\,\hat{\theta} = \frac{4}{n}\frac{\theta^2}{12} = \frac{\theta^2}{3n}$$

$$\Rightarrow se = \sqrt{\frac{\theta^2}{3n}} = \frac{\theta}{\sqrt{3n}}$$

$$\mathrm{MSE}\,\hat{\theta} = \mathrm{bias}^2\,\hat{\theta} + \mathrm{Var}\,\hat{\theta}$$

$$= 0 + \frac{\theta^2}{3n}$$