② $X_i \sim Pois(\lambda_i)$, $(\lambda_i)_{i\in\mathbb{N}} \nearrow +\infty$, $Z_n = \frac{X_n - \mathbb{E}X_n}{\sqrt{Var\, X_n}}$

a) $M_{X_n}(t) = ?$, $M_{Z_n}(t) = ?$

b) Expansão de Taylor em $M_{Z_n}(t)$ para mostrar que

$$M_{Z_n}(t) \longrightarrow \exp(t^2/2)$$

a) $M_{X_n}(t) = \mathbb{E} \exp(t X_n)$

$$= \sum_{k=0}^{\infty} \exp(t k) \frac{\lambda^k \exp(-\lambda)}{k!}$$

$$= \exp(-\lambda) \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \exp(t k) = \exp(-\lambda) \sum_{k=0}^{\infty} \frac{(\lambda e^t)^k}{k!}$$

$$= \exp(-\lambda) \exp(\lambda e^t)$$

$$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$= \exp(-\lambda(1 - e^t))$$

$$M_{Z_n}(t) = \mathbb{E} \exp\left[t \frac{(X_n - \mathbb{E}X_n)}{\sqrt{Var\, X_n}}\right]$$

$$= \mathbb{E} \exp\left[\frac{t}{\sqrt{Var X_n}} \cdot X_n - \frac{t\, \mathbb{E}X_n}{\sqrt{Var X_n}}\right]$$

$$= \mathbb{E}\left[\exp \frac{t X_n}{\sqrt{Var X_n}} \cdot \exp\left(-\frac{t\, \mathbb{E}X_n}{\sqrt{Var X_n}}\right)\right]$$

$$= M_{X_n}\left(t/\sqrt{\operatorname{Var} X_n}\right) \cdot \exp\left(-t \frac{\mathbb{E}X_n}{\sqrt{\operatorname{Var} X_n}}\right)$$

$$= M_{X_n}\left(t/\sqrt{\lambda_n}\right) \cdot \exp\left(-t\lambda_n/\sqrt{\lambda_n}\right)$$

$$= \exp\left(-\lambda_n\left(1-e^{t/\sqrt{\lambda_n}}\right)\right) \exp\left(-t\lambda_n/\sqrt{\lambda_n}\right)$$

$$= \exp\left(-\lambda_n\left(1-e^{t/\sqrt{\lambda_n}}+t/\sqrt{\lambda_n}\right)\right)$$

b) $$M_{Z_n}(t) = \exp\left(-\lambda_n\left(1-e^{t/\sqrt{\lambda_n}}\right)\right) \exp\left(-t\lambda_n/\sqrt{\lambda_n}\right)$$
$$= \exp\left(\lambda_n\left(e^{t/\sqrt{\lambda_n}}-1\right) - t\sqrt{\lambda_n}\right)$$

$$M_{Z_n}(t) = \exp\left(-\lambda_n\left(1-e^{t/\sqrt{\lambda_n}}\right)\right) \exp\left(-t\lambda_n/\sqrt{\lambda_n}\right)$$
$$= \exp\left(\lambda_n\left(e^{t/\sqrt{\lambda_n}}-1\right) - t\sqrt{\lambda_n}\right)$$

$$\exp(t/\sqrt{n}) = \sum_{k=0}^{\infty} \frac{(t/\sqrt{\lambda_n})^k}{k!}$$

$$= 1 + \frac{t}{\lambda_n^{1/2}} + \frac{t^2}{\lambda_n 2!} + \frac{t^3}{\lambda_n^{3/2} 3!} + \frac{t^4}{\lambda_n^{4/2} 4!} + \dots$$

$$M_{Z_n}(t) = \exp\left(\frac{\lambda_n t}{\sqrt{\lambda_n}} + \frac{\lambda_n t^2}{\lambda_n 2!} + \frac{\lambda_n t^3}{\lambda_n^{3/2} 3!} + \frac{\lambda_n t^4}{\lambda_n^{4/2} 4!} + \dots - t\sqrt{\lambda_n}\right)$$

$$= \exp\left(\cancel{t\sqrt{\lambda_n}} + \frac{t^2}{2!} + \frac{t^3}{\lambda_n^{3/2-1}\, 3!} + \frac{t^4}{\lambda_n^{4/2-1}\, 4!} + \dots - \cancel{t\sqrt{\lambda_n}} \right)$$

$$= \exp\left(\frac{t^2}{2!} + \frac{t^3}{\lambda_n^{3/2-1}\, 3!} + \frac{t^4}{\lambda_n^{4/2-1}\, 4!} + \dots \right)$$

FALTA JUSTIFICAR:

$$\lim_{n \to \infty} M_{Z_n}(t) = \exp\left(\frac{t^2}{2} + 0 + 0 + \dots \right)$$

$$\therefore \quad Z_n \Longrightarrow N(0,1).$$

(3) $X_i \overset{iid}{\sim} \text{Beta}(a, b)$

$\bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i \implies ?$

We know that $\mu = \mathbb{E} X_i = a/(a+b)$ and $\sigma^2 = \mathbb{V}\text{ar}\, X_i = \frac{ab}{(a+b)^2(a+b+1)}$

then

$$Z_n = \sqrt{n}\frac{(\bar{X}_n - \mu)}{\sigma} \implies N(0,1)$$

this is equivalent to

$$\bar{X}_n \implies \frac{\sigma^2}{n} N(0,1) + \mu$$

$$\iff \bar{X}_n \implies N\left(\mu, \frac{\sigma^2}{n}\right)$$

(4) $X_i \overset{iid}{\sim} U(0,1)$

$Y_n = \bar{X}_n^2 \Rightarrow ?$

We know by TCL that

$$\frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} \Rightarrow N(0,1)$$

By the Delta Method with $g(x) = x^2$,

$$\frac{\sqrt{n}(g(\bar{X}_n) - g(\mu))}{|g'(\mu)|\sigma} \to N(0,1)$$

this is equivalent to

$$g(\bar{X}_n) \Rightarrow N\left(g(\mu), \frac{\sigma^2}{n} |g'(\mu)|\right)$$

9) $X_i \overset{iid}{\sim} Pois(\lambda)$

a) $\hat{\lambda} = \frac{1}{m} \sum_{i=1}^{m} X_i \Rightarrow$ MÉTODO DOS MOMENTOS

$$\hat{\lambda} = \max_{\lambda} \ell(\lambda) = \max_{\lambda} \sum_{i=1}^{m} \log f(X_i \mid \lambda) \Rightarrow MLE$$

In the case where $X_i \overset{iid}{\sim} pois(\lambda)$, we have

$$f(X_i \mid \lambda) = \frac{\lambda^{X_i} \exp(-\lambda)}{X_i!}$$

$$\therefore \log f(X_i \mid \lambda) = X_i \log \lambda - \lambda - \log(X_i!)$$

$$\ell(\lambda) = \sum_{i=1}^{m} \log f(x_i \mid \lambda)$$

$$= \sum_{i=1}^{m} X_i \log \lambda - \lambda - \log(X_i!)$$

$$= \log \lambda \sum_{i=1}^{m} X_i - \lambda m - \sum \log X_i!$$

$$\Rightarrow \frac{d}{d\lambda} \ell(\lambda) = \frac{1}{\lambda} \sum X_i - m = 0$$

$$\Leftrightarrow \frac{1}{\lambda} \sum X_i = m \Leftrightarrow \boxed{\lambda = \frac{1}{m} \sum X_i}$$

b) Show that $\mathbb{E}\hat{\lambda} = \lambda$.

$$\mathbb{E}\hat{\lambda} = \frac{1}{n}\sum \mathbb{E}X_i = \frac{n\lambda}{n} = \lambda.$$

(6) $Y_i \overset{iid}{\sim} N(\mu_i = \mathbb{E}Y_i/X_i = a+bX_i \,,\, \sigma^2)$

$$L(a,b,\sigma^2) = \prod_{i=1}^{n} \frac{1}{\sigma^2\sqrt{2\pi}} \cdot \exp\left(-\frac{(Y_i-\mu_i)^2}{2\sigma^2}\right)$$

$$= \prod_{i=1}^{n} \frac{1}{\sigma^2\sqrt{2\pi}} \exp\left(-\frac{(Y_i-a-bX_i)^2}{2\sigma^2}\right)$$

Taking the log,

$$\ell(a,b,\sigma^2) = n\log\left(1/\sigma^2\sqrt{2\pi}\right) - \sum_{i=1}^{n}\frac{(Y_i-a-bX_i)^2}{2\sigma^2}$$

taking derivative w.r.t. b,

$$\frac{d}{db} l(a,b,\sigma^2) = -\sum \cancel{2}\frac{(Y_i - a - bX_i)}{\cancel{2}\sigma^2} \cdot (-X_i)$$

$$= \frac{1}{\sigma^2} \sum_{i=1}^{n} (Y_i - a - bX_i) X_i$$

$$\frac{d}{db} l(a,b,\sigma^2) = 0$$

$$\Leftrightarrow \quad \sum (Y_i - a - bX_i) X_i = 0$$

$$\sum Y_i X_i - aX_i - bX_i^2 = 0$$

$$\Leftrightarrow \quad b = \frac{\sum Y_i X_i - aX_i}{\sum X_i^2} \quad *$$

$$\frac{d}{da} l(a,b,\sigma^2) = \frac{-2}{2\sigma^2} \sum_{i=1}^{n} (Y_i - a - bX_i)(-1)$$

$$= \frac{1}{\sigma^2} \sum (Y_i - a - bX_i) = 0$$

$$\Leftrightarrow \quad \sum (Y_i - a - bX_i) = 0$$

$$\sum Y_i - na - b\sum X_i = 0$$

$$a = \bar{Y}_n - b\bar{X}_n$$

Substituting in $\ast_1$

$$b = \frac{\sum Y_i X_i - a X_i}{\sum X_i^2} = \frac{\sum Y_i X_i - X_i(\bar{Y}_n - b\bar{X}_n)}{\sum X_i^2}$$

$$= \frac{\sum X_i(Y_i - \bar{Y}_n) + b\bar{X}_n \sum X_i}{\sum X_i^2}$$

$$b = \frac{\sum X_i(Y_i - \bar{Y}_n) + b\bar{X}_n \sum X_i}{\sum X_i^2}$$

$$\Leftrightarrow \quad b\sum X_i^2 = \sum X_i Y_i - \bar{Y}_n \sum X_i + b\bar{X}_n \sum X_i$$

$$b\left(\sum X_i^2 - \bar{X}_n \sum X_i\right) = \sum X_i Y_i - \bar{Y}_n \sum X_i$$

nota que $\bar{X}_n = \frac{1}{n}\sum X_i \Rightarrow \bar{X}_n \sum X_i = \frac{(\sum X_i)^2}{n}$

$$b\left(\sum X_i^2 - \frac{(\sum X_i)^2}{n}\right) = \sum X_i Y_i - \bar{Y}_n \sum X_i$$

$$\ast \sum (X_i - \bar{X}_n)^2 = \sum X_i^2 - 2X_i\bar{X}_n + \bar{X}_n^2$$

$$= \sum X_i^2 - 2\sum X_i \bar{X}_n + \frac{1}{n}\left(\sum X_i\right)^2$$

$$= \sum X_i^2 - \frac{2}{n}\left(\sum X_i\right)^2 + \frac{1}{n}\left(\sum X_i\right)^2 = \sum X_i^2 - \frac{1}{n}\left(\sum X_i\right)^2$$

$$* \quad \sum X_iY_i - \bar{Y}_n \sum X_i = \sum X_i(Y_i - \bar{Y}_n) + \bar{X}_n Y_n - \bar{X}_n \bar{Y}_n$$

$$= n \sum (X_i - \bar{X}_n)(Y_i - \bar{Y}_n)$$

$$\therefore \; b = \frac{n \sum (X_i - \bar{X}_n)(Y_i - \bar{Y}_n)}{\sum X_i^2 - \frac{1}{n}\left(\sum X_i\right)^2}$$

$$n \cdot \sum (X_i - \bar{X}_n)(Y_i - \bar{Y}_n)$$

$$= n \cdot \sum X_iY_i - X_i\bar{Y}_n - \bar{X}_n Y_i + \bar{X}_n \bar{Y}_n$$

$$= n \cdot \sum X_iY_i - \bar{Y}_n \sum X_i - \bar{X}_n \sum Y_i + \bar{X}_n \bar{Y}_n$$

$$= n \sum X_iY_i - n\bar{Y}_n \sum X_i - \cancel{n\bar{X}_n \sum Y_i} + \cancel{n\bar{X}_n \bar{Y}_n}$$

$$= n\left(\sum X_iY_i - \bar{Y}_n X_i\right)$$

$$\varphi^{(1)}(0) = -\frac{\lambda_n}{\sqrt{\lambda_n}} + \frac{\lambda_n}{\sqrt{\lambda_n}} = 0$$

$$\varphi^{(2)}(0) = \frac{\lambda_n}{\sqrt{\lambda_n}} \cdot \frac{1}{\sqrt{\lambda_n}} = 1$$

$$\lim_{t \to 0} \frac{r(t)}{t^2} = 0$$

$$\varphi(t) = \varphi(0) + t\varphi^{(1)}(0) + \frac{t^2}{2}\varphi^{(2)}(0) + r(t)$$

$$\varphi_n(t) = 0 + 0 + \frac{t^2}{2} \cdot 1 + r_n(t)$$

$$\lim_{n \to \infty} \varphi_n(t) = \lim_{n \to \infty} \frac{t^2}{2} + r_n(t)$$

$$= \frac{t^2}{2} + \lim_{n \to \infty} r_n(t)$$

$$r_n(t) = \varphi_n(t) - \frac{t^2}{2}$$

$$r_n(t) = -t\sqrt{\lambda_n} + \lambda_n\left(\exp\left(t/\sqrt{\lambda_n}\right) - 1\right) - \frac{t^2}{2}$$

$$= -\infty$$

$$\lim \log M_{Z_n}(t) = \frac{t^2}{2} - \infty \Leftrightarrow \lim M_{Z_n}(t) = \exp /$$

$$\varphi^n(t) = \lambda_n \exp\left(\frac{t}{\sqrt{\lambda_n}}\right)\left(\frac{1}{\sqrt{\lambda_n}}\right)^n$$

$$= \frac{\lambda_n^1}{\lambda_n^{1/2n}}$$

$$1 - \frac{1}{2n} = \frac{2n-1}{2n}$$

$$h(t) = \lambda_n\left(\exp\left(t/\sqrt{\lambda_n}\right) - 1\right)$$

$$= \lambda_n\left(\sum_{k=0}^{\infty}\left(\frac{t}{\sqrt{\lambda_n}}\right)^k \frac{1}{k!} - 1\right)$$

$$= \lambda_n\left(1 + \frac{t^2}{\lambda_n}\frac{1}{2!} + \frac{t^3}{\lambda_n^{3/2}}\frac{1}{3!} + \frac{t^4}{\lambda_n^2}\frac{1}{4!} + \dots\right)$$

$$= \lambda_n + \frac{t^2}{\lambda_n^0}\frac{1}{2!} + \frac{t^3}{\lambda_n^{1/2}}\frac{1}{3!} + \frac{t^4}{\lambda_n^{4/2-1}4!} + \frac{t^5}{\lambda_n^{5/2-1}}\frac{1}{5!} + \dots$$

$k=0 \quad k=1 \quad k=2 \quad k=3 \quad k=4$

$$\frac{t^{k+1}}{\lambda_n^{\frac{k+1}{2}-1}k!}$$

$$h(t) = \lambda_n + \sum_{k=1}^{\infty}\frac{t^{k+1}}{k!\,\lambda_n^{\frac{k+1}{2}-1}} = \lambda_n + \lambda_n\sum_{k=1}^{\infty}\frac{\left(t/\lambda_n^{1/2}\right)^{k+1}}{k!}$$

$j = k-1$

$$= -\cancel{t\sqrt{\lambda_n}} + \cancel{t\sqrt{\lambda_n}} + \frac{t^2}{2} + \lambda_n \cdot \sum_{k=3}^{\infty} \frac{(t/\sqrt{\lambda_n})^2}{k!}$$

$$= \frac{t^2}{2} + \lambda_n \sum_{k=3}^{\infty} \frac{(t/\sqrt{\lambda_n})^k}{k!}$$

$$\lim_{n\to\infty} \lambda_n \cdot \sum_{k=3}^{\infty} \frac{(t/\sqrt{\lambda_n})^k}{k!} = 0 \quad \text{as} \quad \lambda_n \to \infty$$

$$S_n = \sum_{k=3}^{n} \frac{t^k \lambda_n^{1-k/2}}{k!}$$

$$a_k = \frac{t^k}{k!} \frac{\lambda_n}{\lambda_n^{k/2}} \qquad k=3: \ \frac{t^3}{3!} \lambda_n^{-1/2}$$

$k > 3$, $\quad 1 - \frac{k}{2} < 1 - \frac{3}{2} = -\frac{1}{2}$

$-k < -3$

$$\lambda_n^{1-k/2} > \lambda_n^{-1/2}$$

$$\exp\left(\frac{t^2}{2} + \frac{t^3}{3!\,\lambda^{\frac{3-2}{2}}} + \frac{t^4}{4!\,\lambda^{\frac{4-2}{2}}} + \dots\right)$$

$$\therefore \exp\left(\lambda_m \sum_{k=2}^{\infty} \left(\frac{t^k}{\lambda_m^{\frac{k}{2}}}\right) \frac{1}{k!}\right) = \exp(\varphi_m(t))$$

$$\varphi(t) = \lambda_m \left[\exp\left(t/\lambda_m^{1/2}\right) - 1 - t/\sqrt{\lambda_m}\right]$$

$$\varphi(0) = \lambda_m \cdot 0 = 0$$

$$\varphi^{(1)}(t) = \frac{\lambda_m}{\sqrt{\lambda_m}} \exp(t/\sqrt{\lambda_m}) - 1/\sqrt{\lambda_m} = \sqrt{\lambda_m} \exp(t/\sqrt{\lambda_m}) - 1/\sqrt{\lambda_m}$$

$$\varphi^{(1)}(0) = \sqrt{\lambda_m} - 1/\sqrt{\lambda_m}$$

$$\varphi^{(2)}(t) = \sqrt{\lambda_m} \exp(t/\sqrt{\lambda_m}) \frac{1}{\sqrt{\lambda_m}} = \exp(t/\sqrt{\lambda_m})$$

$$\varphi^{2}(0) = 1$$

$$\varphi(t) = \varphi(0) + t\varphi^{(1)}(0) + \frac{t^2}{2}\varphi^{(2)}(0) + r(t) \quad tq \quad \frac{r(t)}{t^2} \xrightarrow{t \to \infty} 0$$

$$\lambda_m\left[\exp\left(t/\lambda_m^{1/2}\right) - 1 - t/\sqrt{\lambda_m}\right] = 0 + t\sqrt{\lambda_m} - t/\sqrt{\lambda_m} + \frac{t^2}{2} \cdot 1 + r(t)$$

$$\lim_{n\to\infty} \varphi_m(t) = 0 + t\sqrt{\lambda_m} - \frac{t}{\sqrt{\lambda_m}} + \frac{t^2}{2} + r(t)$$

$$= +\infty + t^2/2$$

$$\lambda_n \exp(t/\sqrt{\lambda_n}) - \underset{k=0}{\lambda_n \cdot 1} - \underset{k=1}{\lambda_n \frac{t}{\sqrt{\lambda_n}}} - \underset{k=2}{\lambda_n \frac{t^2}{2\lambda_n}}$$

$$= \lambda_n \exp(t/\sqrt{\lambda_n}) - \lambda_n - t\sqrt{\lambda_n} - \frac{t^2}{2}$$

$$= \lambda_n \sum_{k=3}^{\infty} \frac{(t/\lambda^{1/2})^k}{k!}$$

$$\lim_{n\to\infty}\left[\lambda_n \exp(t/\sqrt{\lambda_n}) - \lambda_n - t\sqrt{\lambda_n} - \frac{t^2}{2}\right] \overset{?}{=} 0$$