$(X_i)_{i=1}^{n} \overset{iid}{\sim} N(\mu, \sigma^2)$, then

$$T_n = \frac{\overline{X}_n - \mu}{S_n/\sqrt{n}} \sim t_{n-1} \quad \text{(} t \text{ distribution with } n-1 \text{ degrees of freedom)}$$

where

$$\overline{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i \quad \text{(sample mean)}$$

$$S_n^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \overline{X}_n)^2 \quad \text{(sample variance)}$$

PROOF:

<u>Step 1</u> Rewrite T_n as

$$T_n = \frac{Z_n}{\sqrt{V/n-1}}, \quad \text{where } Z_n = \frac{\overline{X}_n - \mu}{\sigma/\sqrt{n}}$$

and $V/n-1 = S_n^2/\sigma^2$

Note that

$$\frac{Z_n}{\sqrt{V/(n-1)}} = \frac{\overline{X}_n - \mu}{\sigma/\sqrt{n}} \Big/ \sqrt{V/(n-1)}$$

$$= \frac{\sqrt{n}}{\sigma} \frac{(\overline{X}_n - \mu)}{S_n/\sigma} = \frac{\overline{X}_n - \mu}{S_n/\sqrt{n}}.$$

Step 2 Find the distribution of Z_n and V.

i)

$\mathbb{E}\,\bar{X}_n = \mu$, $\quad \operatorname{Var}\bar{X}_n = \frac{n \operatorname{Var} X_i}{n^2} = \frac{\sigma^2}{n}$

$$\Rightarrow \bar{X}_n \sim N\left(\mu, \frac{\sigma^2}{n}\right)$$

$$\bar{X}_n - \mu \sim N\left(\mu-\mu=0, \frac{\sigma^2}{n}\right)$$

$$\therefore Z_n = \frac{\bar{X}_n-\mu}{\sigma/\sqrt{n}} \sim N\left(0, \frac{\sigma^2}{n}\cdot\frac{n}{\sigma^2}\right) = N(0,1)$$

ii)

$$V = \frac{(n-1)S_n^2}{\sigma^2} = \frac{1}{\sigma^2}\frac{\cancel{(n-1)}}{\cancel{(n-1)}}\sum_{i=1}^{n}(X_i-\bar{X}_n)^2 = \frac{1}{\sigma^2}\sum(X_i-\bar{X}_n)^2$$

but

$$\frac{1}{\sigma^2}\sum(X_i-\mu)^2 = \frac{1}{\sigma^2}\sum_{i=1}^{n}(X_i-\bar{X}_n+\bar{X}_n-\mu)^2$$

$$= \frac{1}{\sigma^2}\sum_{i=1}^{n}\left[(X_i-\bar{X}_n)^2 - 2(X_i-\bar{X}_n)(\bar{X}_n-\mu) + (\bar{X}_n-\mu)^2\right]$$

$$= \frac{1}{\sigma^2}\sum_{i=1}^{n}\left[(X_i-\bar{X}_n)^2 + (\bar{X}_n-\mu)^2\right]$$

$$= \frac{1}{\sigma^2}\sum_{i=1}^{n}(X_i-\bar{X}_n)^2 + \frac{n}{\sigma^2}(\bar{X}_n-\mu)^2$$

$$= \frac{1}{\sigma^2}\sum_{i=1}^{n}(X_i-\bar{X}_n)^2 + \left(\frac{\bar{X}_n-\mu}{\sigma/\sqrt{n}}\right)^2$$

then $Z = V + Z_n^2$ where

$$Z := \sum_{i=1} \left(\frac{X_i - \mu}{\sigma}\right)^2 \sim \chi_n^2 \text{ and } Z_n^2 \sim \chi_1^2.$$

Since $V \perp\!\!\!\perp Z_n^2$ by Corollary A,

$$\varphi_Z(t) = \varphi_V(t) \cdot \varphi_{Z_n^2}(t)$$

$$\therefore \varphi_V(t) = \frac{\varphi_Z(t)}{\varphi_{Z_n^2}(t)}$$

$$= \frac{(1-2t)^{-n/2}}{(1-2t)^{-1/2}} = (1-2t)^{-n/2+1/2}$$

$$= (1-2t)^{-1/2(n-1)}$$

Concluding, $V \sim \chi_{n-1}^2$.

<u>Step 3</u> Show that

$$\frac{Z_n}{\sqrt{V/(n-1)}} \sim t_{n-1}$$

By the previous steps, the ratio is a $N(0,1)$ over the square root of a χ_{n-1}^2 divided by its degrees of freedom.