30 X = number of suicides per month

$X \sim Pois(\lambda = 0.33)$

a) $\mathbb{P}$(suicides in a year $= k$)? What is the mode?

b) $\mathbb{P}$(suicides in one week $= 2$) $=$?

a) $X \sim Pois(\lambda_1) \perp\!\!\!\perp Y \sim Pois(\lambda_2)$, $a > 0$

$\Rightarrow \quad X + Y \sim Pois(\lambda_1 + \lambda_2)$

i) $\mathbb{P}(X+Y = t) = \mathbb{P}(X = t - Y)$

$$= \sum_{y=0}^{\infty} \mathbb{P}(X = t - y \,/\, Y = y) \cdot \mathbb{P}(Y = y)$$

$$= \sum_{y=0}^{\infty} \mathbb{P}(X = t-y)\, \mathbb{P}(Y = y)$$

$$= \sum_{y=0}^{\infty} \exp(-\lambda_1) \frac{\lambda_1^{t-y}}{(t-y)!} \exp(-\lambda_2) \frac{\lambda_2^{y}}{y!}$$

$$= \frac{\exp(-(\lambda_1+\lambda_2))}{t!} \sum_{y=0}^{\infty} \lambda_1^{t-y} \lambda_2^{y} \frac{t!}{(t-y)!\,y!}$$

$$= \frac{\exp(-(\lambda_1+\lambda_2))}{t!} \sum_{y=0}^{\infty} \lambda_1^{t-y} \lambda_2^{y} \binom{t}{y}$$

$$= \frac{\exp(-(\lambda_1+\lambda_2))}{t!} (\lambda_1+\lambda_2)^t = \mathbb{P}(\mathrm{Pois}(\lambda_1+\lambda_2)=t)$$

the mode of a Poisson R.V is given by exerc. 29:

$$p_k = \frac{\lambda}{k} p_{k-1}$$

$$\iff \frac{p_k}{p_{k-1}} = \frac{\lambda}{k}$$

$$\lambda/k > 1 \iff \lambda > k$$

$\Rightarrow$ se $\lambda < 1$, então $k=0$ é a moda (o único inteiro

$\Rightarrow$ se $\lambda > 1$, então a moda é $\lfloor \lambda \rfloor$ e $\lfloor \lambda \rfloor - 1$.

b) $\mathbb{P}(X/a = t) = \mathbb{P}(X = at)$

$$= \exp(-\lambda_1) \frac{\lambda_1^{at}}{(at)!}$$

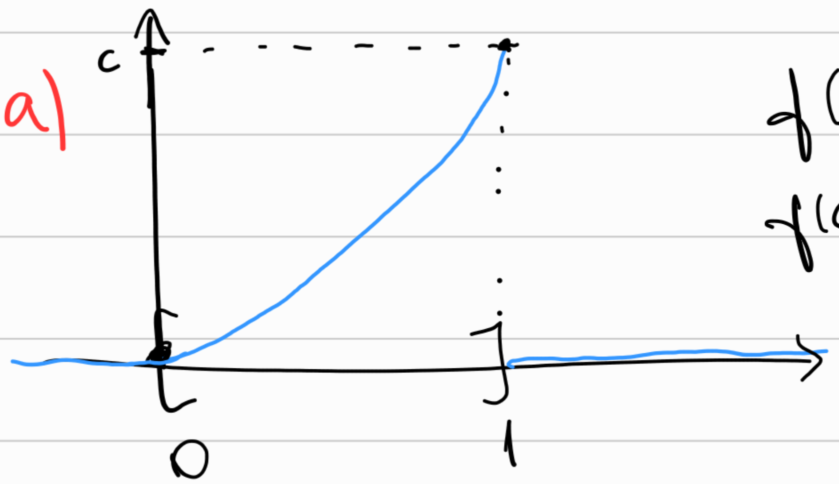
$a = 4$, $t = 2$

$$\mathbb{P}(X/4 = 2) = \exp(-\lambda_1) \cdot \frac{\lambda_1^{8}}{8!}$$

(40) $X \sim f_X(x) = \begin{cases} c x, & x \in [0,1] \\ 0, & |x| > 1 \end{cases}$

a) Find c.
b) Find the cdf
c) $P(.1 \leq X < .5) = ?$

a)



$f(1) = c1^2 = c$

$f(0) = c \cdot 0 = 0$

$$1 = \int_{\mathbb{R}} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_0^1 f(x)\,dx + \int_1^{+\infty} f(x)\,dx$$
$$= 0 + \int_0^1 c x^2\,dx + 0$$
$$= c \int_0^1 x^2\,dx = c\, \frac{x^3}{3}\Big|_0^1 =$$
$$= c\left(\frac{1}{3} - \frac{0}{3}\right) = \frac{c}{3}$$

$$1 = \frac{c}{3} \Leftrightarrow \boxed{c = 3}$$

b) $F(t) = P(X \leq t) = \int_{-\infty}^{t} f(x)\,dx$

Se $|t| > 1$, então $F(t) = \int_{-\infty}^{t} 0\,dx = 0$

Se $|t| \leq 1$, então

$$F(t) = \int_{-\infty}^{t} f(x)\,dx = \int_{-\infty}^{0} 0\,dx + \int_{0}^{t} 3x^2\,dx$$

$$= 0 + \cancel{3}\,\frac{x^3}{\cancel{3}}\Big|_0^t = t^3$$

$$F(t) = \begin{cases} t^3 & , |t| \leq 1 \\ 0 & , t > 1 \end{cases}$$

c) $P(.1 \leq X < .5) = \int_{0.1}^{0.5} 3x^2\,dx$

$$= 3\,\frac{x^3}{3}\Big|_{0.1}^{0.5} = (0.5)^3 - (0.1)^3$$

11. Consider $X \sim Bin(n, p)$. For what value of k is $P(X=k)$ maximized?

$$\max_{k \in \mathbb{N}} \mathbb{P}(X=k) = \max_{k \in \mathbb{N}} \binom{n}{k} p^k (1-p)^{n-k}$$

$\downarrow$

k sucessos em n lançamentos

① se $p=0$, então a chance de sair cara é zero independente de n e k. Logo, sai coroa em quase todo lançamento:

$\mathbb{P}(X=0)=1 \Rightarrow$ a moda é o zero.

② se $p=1$, então a chance de sair cara é 1 independente de n e k. Logo sai cara em quase todo lançamento: $\mathbb{P}(X=n)=1$

$\Rightarrow$ a moda é o n

③ $0<p<1$

$k^* \in \mathbb{N}$ é a moda quando $a_k \leq a_{k^*}$ $\forall k \in \mathbb{N}$.

$\Leftrightarrow \quad \dfrac{a_{k^*}}{a_k} \geq 1.$

i) $a_k = \binom{n}{k} p^k (1-p)^{n-k} = \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k}$

ii) $a_{k+1} = \binom{n}{k+1} p^{k+1} (1-p)^{n-k-1} = \frac{n!}{(k+1)!(n-k-1)!} p^{k+1} (1-p)^{n-k-1}$

$$\frac{P(X=k+1)}{P(X=k)} = \frac{\cancel{n!}}{(k+1)\cancel{k!}\cancel{(n-k-1)!}} \cancel{p^k}\, p\, \cancel{(1-p)^{n-k}} (1-p)^{-1} \cdot \frac{\cancel{k!}(n-k)\cdot\cancel{(n-k-1)}}{\cancel{n!}\,\cancel{p^k}\,\cancel{(1-p)^{n-k}}}$$

$$\boxed{\frac{a_{k+1}}{a_k} = \frac{p}{1-p}\,\frac{n-k}{(k+1)}}$$

Trocando k por $k-1$:

$$\frac{a_k}{a_{k-1}} = \frac{p}{q}\,\frac{n-k+1}{k} = \frac{pn - pk + p + qk - qk}{qk}$$

$$= 1 + \frac{-k(\cancel{p+q}) + p(1+n)}{qk}$$

$$= 1 + \frac{p(n+1) - k}{qk}$$

duas fórmulas para a razão!

Fixado $k^* \in \mathbb{N}$:

$$\frac{a_{k^*+1}}{a_{k^*}} = \frac{p}{1-p} \cdot \frac{n-k^*}{k^*+1} \geq 1$$

$$\Leftrightarrow \quad p(n-k^*) \geq (1-p)(k^*+1)$$

$$pn - \cancel{k^*p} \geq k^*+1 - \cancel{pk^*} - p$$

$$p(n+1) \geq k^*+1$$

Se $p(n+1) = 1+k^*$ então $p(n+1)-1 \in \mathbb{Z}$ e $a_{k^*+1} = a_{k^*}$. Nesse caso, $k^* = p(n+1)-1$ é tq

$$a_{k^*} = \mathbb{P}(X=k^*) = \binom{n}{k^*} p^{k^*}(1-p)^{n-k^*}$$

$$a_{k^*} = a_{k^*+1}$$

$$\frac{a_{k^*}}{a_{k^*-1}} = 1 + \frac{p(n+1)-k^*}{q\,k^*}$$

$$= 1 + \frac{p(n+1) - (p(n+1)-1)}{q\,(p(n+1)-1)}$$

$$= 1 + \frac{\cancel{p(n+1)} - \cancel{p(n+1)} + 1}{q\,p(n+1) - \underset{(1-p)}{q}} = 1 + \frac{1}{\underbrace{(1-p)p(n+1)-1+p}_{>0}} > 1$$

$\Rightarrow \quad a_{k^*} > a_{k^*-1} > \dots > 0$ se $k^* = p(n+1)-1$

$$\frac{a_{k^*-1}}{a_{k^*-2}} = 1 + \frac{p(n+1) - k^* + 1}{q(k^*-1)}$$

$$= 1 + \frac{\cancel{p(n+1)} - \cancel{p(n+1)} + 1 + 1}{q(p(n+1)-2)}$$

$$= 1 + \frac{2}{q\,p(n+1) - 2q} = 1 + \frac{2}{(1-p)p(n+1) - 2 + 2p} > 0$$

$\therefore$ k^* e k^*+1 são as modas

$$k^* = p(n+1) - 1 \quad \text{e} \quad k^*+1 = p(n+1)$$

Se $p(n+1)-1 \notin \mathbb{Z}$, então basta pegar o menor inteiro maior que k^*+1:

$\lfloor p(n+1) \rfloor$.