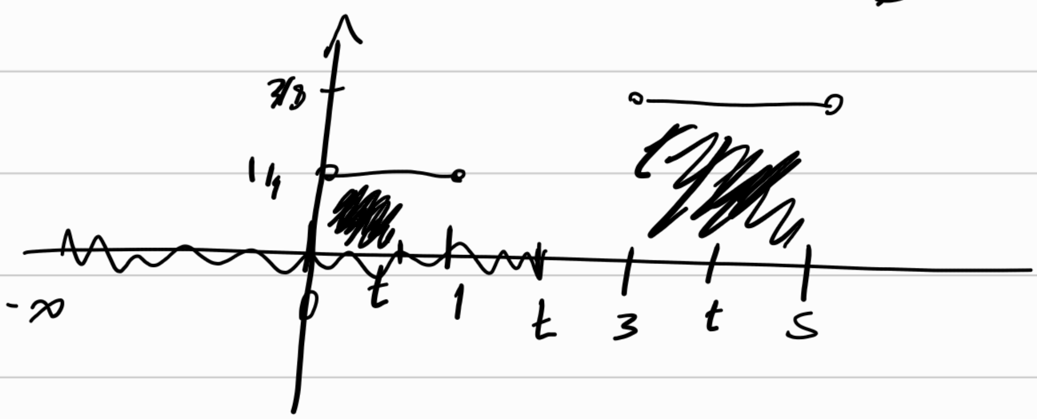
(4) $f_X(x) = \begin{cases} 1/4, & x \in (0,1) \\ 3/8, & x \in (3,5) \\ 0, & c.c \end{cases}$

a) $F_X(t) = \mathbb{P}(X \leq t) = \int_{-\infty}^{t} f_X(u)\,du$

$$= \begin{cases} 0, & t < 0 \\ \int_0^t 1/4\,du = t/4, & t \in (0,1) \\ \int_{-\infty}^t f_X(u)du = \int_{-\infty}^0 0\,du + \int_0^1 1/4\,du + \int_1^t 0\,du = 1/4, & t \in (1,3) \\ \int_{-\infty}^t f_X(u)du = \int_{-\infty}^0 0\,du + \int_0^1 1/4\,du + \int_1^3 0\,du + \int_3^t 3/8\,du & \\ \quad = 0 + 1/4 + 0 + \frac{3}{8} \cdot x\Big|_3^t = \frac{1}{4} + \frac{3}{8}(t-3) & \end{cases}$$



$t \in (3,5)$

$= 1/4 + \frac{3t}{8} - \frac{9}{8}$

$= \frac{2-9}{8} + \frac{3t}{8} = \frac{3t}{8} - \frac{7}{8}$

$$\int_{-\infty}^t f(u)\,du = \int_{-\infty}^0 0 + \int_0^1 1/4 + \int_1^3 0 + \int_3^5 3/8\,du$$
$$= 0 + 1/4 + 0 + 3/4 = 1, \quad t \geq 5$$

b) $Y = 1/X \Rightarrow$ find $f_Y(y) = ?$

$$F_y(t) = \mathbb{P}(Y \le t) = \mathbb{P}(1/X \le t)$$
$$= \mathbb{P}(1/t \le X) = 1 - \mathbb{P}(X < 1/t)$$

$$= 1 - F_X(1/t) = \begin{cases} 1 - \frac{1/t}{4}, & 1/t \in (0,1) \\ 1 - 1/4, & 1/t \in (1,3) \\ 1 - \left(\frac{3 \cdot 1/t}{8} - \frac{7}{8}\right), & 1/t \in (3,5) \\ 1 - 1, & 1/t \geq 5 \end{cases}$$

$|1/t| < 1$

$t >$

$$= \begin{cases} 1 - 1/4t, & 1/t \in (0,1) \\ 3/4, & 1/t \in (1,3) \\ 15/8 - 3/8t, & 1/t \in (3,5) \\ 0, & 1/t \geq 5 \text{ e } 1/t < 0 \end{cases}$$

$$f_y(y) = F_y'(y) = \begin{cases} 1/4t^2, & t > 1 \\ 0, & t \in (1/3, 1) \\ 3/8t^2, & t \in (1/5, 1/3) \\ 0, & t \geq 5 \text{ e } t < 0 \end{cases}$$

(11) X = number of heads
Y = number of tails

a) $\mathbb{P}(X=2, Y=2) = \mathbb{P}((X,Y)=(2,2))$

$$= \mathbb{P}(\{X=2\} \cap \{Y=2\}) = 0$$

$$\neq \mathbb{P}(X=2) \cdot \mathbb{P}(Y=2)$$

$$p^2 \cdot (1-p)^2$$

b) $N \sim Pois(\lambda)$, X, Y = number of heads/tails

$$\mathbb{P}(X=k_1, Y=k_2) \overset{?}{=} \mathbb{P}(X=k_1) \cdot \mathbb{P}(Y=k_2)$$

mas

$$\mathbb{P}(X=k_1, Y=k_2) = \mathbb{P}(X=k_1 / Y=k_2) \cdot \mathbb{P}(Y=k_2)$$

but $Y=k_2 \iff N=k_1+k_2$

$$= \mathbb{P}(X=k_1 / N=k_1+k_2) \cdot \mathbb{P}(N=k_1+k_2)$$

$$= \mathbb{P}\left(Bin(k_1+k_2, p) = k_1\right) \cdot \mathbb{P}(Pois(\lambda) = k_1+k_2))$$

$$= \binom{k_1+k_2}{k_1} p^{k_1} (1-p)^{k_2} \cdot \exp(-\lambda) \frac{\lambda^{k_1+k_2}}{(k_1+k_2)!}$$

$$= \frac{\cancel{(k_1+k_2)!}}{k_1!\, k_2!} p^{k_1} (1-p)^{k_2} \exp(-\lambda) \frac{\lambda^{k_1+k_2}}{\cancel{(k_1+k_2)!}}$$

$$= \frac{(\lambda p)^{k_1}}{k_1!} \frac{(\lambda(1-p))^{k_2}}{k_2!} \exp(-\lambda)$$

$$= \frac{(\lambda p)^{k_1}}{k_1!} \cdot \frac{(\lambda(1-p))^{k_2}}{k_2!} \exp(-\lambda + \lambda p - \lambda p)$$

$$= \frac{(\lambda p)^{k_1}}{k_1!} \cdot \frac{((1-p)\lambda)^{k_2}}{k_2!} \exp(-\lambda(1-p)) \exp(-\lambda p)$$

$$\mathbb{P}(X = k_1) = \sum_{k_2=0}^{\infty} \mathbb{P}(X = k_1 / N = k_1 + k_2) \cdot \mathbb{P}(N = k_1 + k_2)$$

$$= \sum_{k_2=0}^{\infty} \binom{k_1+k_2}{k_1} p^{k_1} (1-p)^{k_2} \cdot \exp(-\lambda) \frac{\lambda^{k_1+k_2}}{(k_1+k_2)!}$$

$$= \sum_{k_2=0}^{\infty} \frac{\cancel{(k_1+k_2)!}}{k_1!\, k_2!} p^{k_1} (1-p)^{k_2} \exp(-\lambda) \frac{\lambda^{k_1+k_2}}{\cancel{(k_1+k_2)!}}$$

$$= \frac{(\lambda p)^{k_1} \exp(-\lambda)}{k_1!} \sum_{k_2=0}^{\infty} \frac{(\lambda(1-p))^{k_2}}{k_2!}$$

$$= \frac{(\lambda p)^{k_1}}{k_1!} \exp(-\lambda) \sum_{k_2=0}^{\infty} \frac{(\lambda - \lambda p)^{k_2}}{k_2!}$$

$$= \frac{(\lambda p)^{k_1}}{k_1!} \exp(-\lambda) \exp(\lambda - \lambda p) = \frac{(\lambda p)^{k_1}}{k_1!} \exp(-\lambda p)$$

$$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$