(a) $X_i \overset{iid}{\sim} F$; $Y = \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}_{\{X_i \le x\}}$

i) $\mathbb{E}Y = ?$

ii) $\operatorname{Var} Y = ?$

iii) Chebyshev: $\mathbb{P}(|Y - \mathbb{E}Y| > \varepsilon) \le ?$

i) $\mathbb{E}Y = \mathbb{E}\frac{1}{n}\sum \mathbb{1}_{\{X_i \le x\}} = \frac{1}{n}\sum \mathbb{E}\mathbb{1}_{\{X_i \le x\}} = \mathbb{E}\mathbb{1}_{\{X_i \le x\}}$

$= \mathbb{P}(X_i \le x)$

$= F(x)$

ii) $\operatorname{Var} Y = \operatorname{Var}\left(\frac{1}{n}\sum \mathbb{1}_{\{X_i \le x\}}\right) = \frac{1}{n^2}\sum \operatorname{Var} \mathbb{1}_{\{X_i \le x\}}$

$= \frac{1}{n}\operatorname{Var} \mathbb{1}_{\{X_i \le x\}}$

$\operatorname{Var} \mathbb{1}_{\{X_i \le x\}}$

$= \mathbb{E}\mathbb{1}^2_{\{X_i \le x\}} - \left(\mathbb{E}\mathbb{1}_{\{X_i \le x\}}\right)^2$

$= \mathbb{E}\mathbb{1}_{\{X_i \le x\}} - \mathbb{P}^2(X_i \le x)$

$= F(x) - F^2(x)$

iii) By Chebyshev,

$$\mathbb{P}(|Y - \mathbb{E}Y| > \varepsilon) \le \frac{\operatorname{Var} Y}{\varepsilon^2} = \frac{F(x) - F^2(x)}{\varepsilon^2}$$

⑩ Hoeffding:

$$\mathbb{P}\left(\sum_{i=1}^{n} W_i \geq \varepsilon\right) \leq \exp(-t\varepsilon)\prod_{i=1}^{n} \exp\left(\frac{t^2(b_i-a_i)^2}{8}\right)$$

where $W_i \in [a_i, b_i]$, $\mathbb{E}W_i = 0$

$$= \exp(-t\varepsilon)\exp\left[\frac{t^2}{8}\sum_{i=1}^{n}(b_i-a_i)^2\right]$$

Using Hoeffding with W_i

$$\mathbb{E}(Y-\mathbb{E}Y) = \mathbb{E}Y - \mathbb{E}\mathbb{E}Y = \mathbb{E}Y - \mathbb{E}Y = 0$$

$$0 \leq Y = \frac{1}{n}\sum \mathbb{1}_{\{X_i \leq x\}} \leq \frac{1}{n}\sum 1 = \frac{1}{n} n \cdot 1 = 1$$

$$\Rightarrow 0 \leq Y \leq 1 \Rightarrow 0 \leq \mathbb{E}Y \leq 1$$

$$\Rightarrow -\mathbb{E}Y \leq Y - \mathbb{E}Y \leq 1 - \mathbb{E}Y$$

$$\mathbb{P}(|Y-\mathbb{E}Y| > \varepsilon) \leq 2\exp(-t\varepsilon)\exp\left(\frac{t^2}{8}(1-\cancel{\mathbb{E}Y}+\cancel{\mathbb{E}Y})^2\right)$$

$$= 2\exp\left(-t\varepsilon + \frac{t^2}{8}\right)$$

$|\sum (X_i - \mathbb{E}X_i)| > n\varepsilon$

$$\exp(-t\varepsilon) \ \exp\left(-\frac{t^2}{8} \sum_{i=1}^{n} (b_i - a_i)^2\right)$$

$\sqrt{16 \cdot 2}$

$4\sqrt{2}$

$$= \exp(-tn\varepsilon) \ \exp\left(-\frac{t^2}{8} n\right)$$

$$= \exp\left(-tn\varepsilon - \frac{nt^2}{8}\right)$$

$8c + c^2 = 16$

$c^2 + 8c - 16 = 0$

$t = \varepsilon c$

$c^2 + 2 \cdot 4c - 16 = 0$

$$-\varepsilon^2 c n - \frac{n}{8} \varepsilon^2 c^2$$

$\underbrace{c^2 + 2 \cdot 4c + 16} - 16 - 16 = 0$

$$= -\varepsilon^2 n \left(c + \frac{c^2}{8}\right)$$

$(c+4)^2 = 32$

$c + 4 = \pm 4\sqrt{2}$

$$c + \frac{c^2}{8} = 2$$

$c =$

⑦ $$f_{X,Y}(x,y) = \begin{cases} \exp(-y), & 0 \le x \le y \\ 0, & c.c \end{cases}$$

a) f_X, f_Y

b) EX, EY, EXY

c) $\mathrm{Var}\,X$, $\mathrm{Var}\,Y$, $\mathrm{Cov}(X,Y)$

d) correlação $\rho_{X,Y}$

a) i) $$f_X(x) = \int_{-\infty}^{\infty} f_X(x,y)\,dy$$

$$= \int_0^{\infty} \exp(-y)\,dy$$

$$= \int_x^{\infty} \exp(-y)\,dy = -\exp(-y)\Big|_x^{\infty}$$

$$= -\exp(-\infty) - (-\exp(-x)) = \exp(-x)$$

ii) $$f_Y(y) = \int_{-\infty}^{\infty} f_X(x,y)\,dx$$

$$= \int_0^{y} \exp(-y)\,dx = \exp(-y)\int_0^{y} dx = y\exp(-y)$$

Note that:

- $\int_0^{\infty} f_Y(y)\,dy = \int_0^{\infty} y\exp(-y)\,dy = \int_0^{\infty} y^{2-1}\exp(-y)\,dy = \Gamma(2) = 1! = 1$

$$\cdot \int_0^\infty \exp(-x)\,dx = -\exp(-x)\Big|_0^\infty$$

$$= -\cancelto{0}{\exp(-\infty)} + \cancelto{1}{\exp(0)}$$

$$= 0 + 1 = 1$$

b) $$\mathbb{E}X = \int_0^\infty x^{2-1}\exp(-x)\,dx = \Gamma(2) = 1! = 1$$

$$\mathbb{E}Y = \int_0^\infty y^2 \exp(-y)\,dy = \int_0^\infty y^{3-1}\exp(-y)\,dy$$
$$= \Gamma(3) = 2! = 2$$

$$\mathbb{E}XY = \int_0^\infty \int_0^y xy\, f_{X,Y}(x,y)\,dx\,dy$$

$$= \int_0^\infty \int_0^y xy \exp(-y)\,dx\,dy$$

$$= \int_0^\infty y\exp(-y)\left(\int_0^y x\,dx\right)dy$$

$$= \int_0^\infty y\exp(-y)\,\frac{x^2}{2}\Big|_0^y\,dy$$

$$= \int_0^\infty y\exp(-y)\,\frac{y^2}{2}\,dy$$

$$= \frac{1}{2}\int_0^\infty y^3\exp(-y)\,dy$$

$$= \frac{1}{2} \int_0^\infty y^{4-1} \exp(-y)\, dy = \frac{1}{2} \Gamma(4) = \frac{3!}{2} = 3$$

c) $\operatorname{Var} X = \mathbb{E}X^2 - (\mathbb{E}X)^2$

$$\mathbb{E}X^2 = \int_0^\infty x^2 f_X(x)\, dx = \int_0^\infty x^2 \exp(-x)\, dx$$
$$= \int_0^\infty x^{3-1} \exp(-x)\, dx = \Gamma(3)$$
$$= 2! = 2$$

$$\operatorname{Var} X = 2 - 1^2 = 2 - 1 = 1$$

$$\operatorname{Var} Y = \mathbb{E}Y^2 - (\mathbb{E}Y)^2$$

$$\mathbb{E}Y^2 = \int_0^\infty y^2 f_Y(y)\, dy$$
$$= \int_0^\infty y^2\, y \exp(-y)\, dy = \int_0^\infty y^3 \exp(-y)\, dy$$
$$= \int_0^\infty y^{4-1} \exp(-y)\, dy = \Gamma(4)$$
$$= 3! = 6$$

$$\operatorname{Var} Y = 6 - 2^2 = 6 - 4 = 2$$

$\cdot \, \text{Cov}\, X,Y = \mathbb{E}XY - \mathbb{E}X\mathbb{E}Y$

$= 3 - 1 \cdot 2 = 3 - 2 = 1$

d) Correlação $\rho_{X,Y}$

$$\rho_{X,Y} = \frac{\text{Cov}\, X,Y}{\sigma_X \cdot \sigma_Y} = \frac{1}{\sqrt{1}\sqrt{2}} = \frac{1}{\sqrt{2}}$$

(8) $X \sim \chi^2_n$, $Y \sim \chi^2_m$

a) MGF

b) $X+Y \sim \chi^2_{n+m}$; $X/(X+Y) \sim ?$

a) $f_X(x) = \frac{1}{\Gamma(n/2)\, 2^{n/2}} x^{n/2-1} \exp(-x/2)$, $x > 0$

$$\mathbb{E} \exp(tX) = \int_0^\infty \exp(tx) \frac{1}{\Gamma(n/2)2^{n/2}} x^{n/2-1} \exp(-x/2)\, dx$$

$$= \frac{1}{\Gamma(n/2)\, 2^{n/2}} \int_0^\infty \exp(tx - x/2)\, x^{n/2-1}\, dx$$

$$= \frac{1}{2^{n/2}\Gamma(n/2)} \int_0^\infty \exp(-x(1/2 - t))\, x^{n/2-1}\, dx$$

$$x^{n/2-1} > x^{-1}$$

$$= \begin{cases} +\infty & \text{if } 1/2 - t < 0 \\ \frac{1}{2^{n/2}\Gamma(n/2)} \int_0^\infty \exp(0)\, x^{n/2-1}\, dx > \int_0^\infty x^{-1}\, dx = +\infty, & t = 1/2 \end{cases}$$

Suppose $1/2 - t > 0$. Then consider the substitution $x(1/2-t) = u$:

$$\frac{d}{dx}\left[x(1/2-t)\right] = 1/2 - t$$

$$* = \frac{1}{2^{n/2}\Gamma(n/2)} \int_0^\infty \exp(-u) \left(\frac{u}{1/2-t}\right)^{n/2-1} \frac{du}{1/2-t}$$

$$= \frac{1}{2^{n/2}\Gamma(n/2)(1/2-t)^{n/2}} \int_0^\infty \exp(-u)\, u^{n/2-1}\, du$$

$$= \frac{1}{2^{n/2}\cancel{\Gamma(n/2)}(1/2-t)^{n/2}} \cancel{\Gamma(n/2)} = (1-2t)^{-n/2}$$

b) $\mathbb{E}\exp(t(X+Y)) = \mathbb{E}[\exp tX \cdot \exp tY] = \mathbb{E}\exp tX \cdot \mathbb{E}\exp tY$

$$= \int_0^\infty\int_0^\infty \exp(t(x+y)) \frac{1}{\Gamma(n/2)2^{n/2}} x^{n/2-1} \exp(-x/2) \cdot \frac{1}{\Gamma(m/2)2^{m/2}} y^{m/2-1} \exp(-y/2)\,dx\,dy$$

$$= \int_0^\infty \frac{1}{\Gamma(n/2)2^{n/2}} \exp(-x(1/2-t))\, x^{n/2-1}\,dx \int_0^\infty \frac{1}{\Gamma(m/2)2^{m/2}} \cdot y^{m/2-1} \exp(-y(1/2-t))\,dy$$

$$= (1-2t)^{-n/2} \cdot (1-2t)^{-m/2}$$

$$= (1-2t)^{-\frac{n+m}{2}}$$

$\cdot\, X/X+Y$?

$W = X/X+Y = \chi^2_n / \chi^2_{n+m}$

Note that $X \sim \text{Gamma}(\frac{n}{2}, 2)$

$Y \sim \text{Gamma}(\frac{m}{2}, 2)$

Define $U = X+Y$ and $Z = X/U$. then

$X = ZU$ and $Y = U - ZU = U(1-Z)$

then $f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y)$

$$f_{X,Y}(x,y) = \frac{x^{m/2-1}\, y^{n/2-1}}{\Gamma(m/2)\,\Gamma(n/2)\, 2^{\frac{m}{2}+\frac{n}{2}}} \exp(-x/2 - y/2)$$

$$(X,Y) \underset{h^{-1}}{\overset{h}{\rightleftarrows}} (U,Z)$$

$$h(x,y) = (\overbrace{x+y}^{u},\ \overbrace{x/x+y}^{z})$$

$$h^{-1}(u,z) = (\underbrace{zu}_{h_1^{-1}(u)},\ \underbrace{u(1-z)}_{h_2^{-1}(z)=y})$$

$$h(h_1^{-1}(u), h_2^{-1}(z))$$

$$= (z\cancel{u} + u(1-\cancel{z}),\ zu/zu+(1-z)u)$$

$$= (u, z)$$

By thm. ?,

$$f_{U,Z}(u,z) = f_{X,Y}(h_1^{-1}(u), h_2^{-1}(z)) \cdot |\det J h^{-1}(u,z)|$$

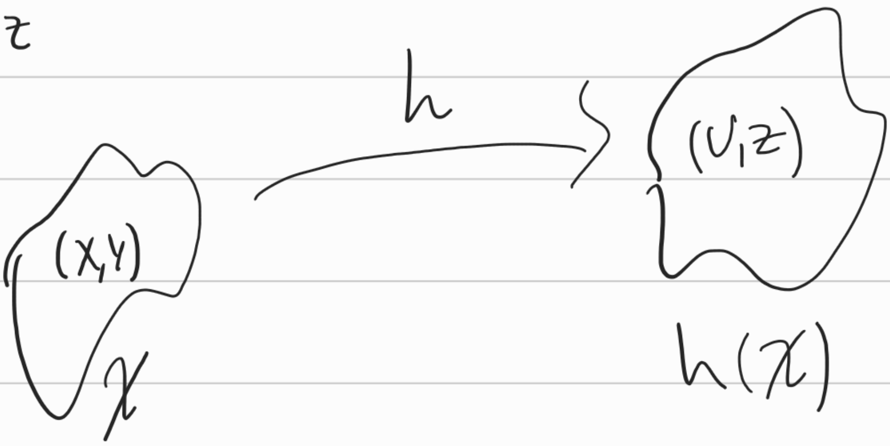
$$J h^{-1}(u,z) = \begin{bmatrix} z & u \\ 1-z & -u \end{bmatrix}$$

$$|\det J h^{-1}(u,z)| = |-zu - u(1-z)| = |-z + uz - u| = u$$

$$\therefore \; f_{U,Z}(u,z) = \frac{(zu)^{m/2-1}\,[u(1-z)]^{\frac{n}{2}-1}}{\Gamma(m/2)\,\Gamma(n/2)\,2^{\frac{m}{2}+\frac{n}{2}}} \exp\left(-\frac{zu}{2} - \frac{u(1-z)}{2}\right) \cdot u$$

$$\boxed{\int_{\mathcal{X}} r(x)\, f_X(x)\,dx = \int_{h^{-1}(\mathcal{X})} r(h^{-1}(x))\, f_X(h^{-1}(x))\, |\det J h^{-1}|\, dx}$$

$$\hookrightarrow \; f_Z(z) = \int_0^\infty f_{U,Z}(u,z)\,du$$



⑥ $\gamma_1 = \mathbb{E}\left(\frac{X-\mu}{\sigma}\right)^3 =$

$$= \frac{\mathbb{E}\left[X^3 - 3\mu X^2 + 3\mu^2 X - \mu^3\right]}{\sigma^2 \cdot \sigma}$$

$$= \frac{\mathbb{E}X^3 - 3\mu\mathbb{E}X^2 + 3\mu^2\mathbb{E}X - \mu^3}{\sigma \cdot \mathrm{Var}X}$$

$$= \frac{\mathbb{E}X^3 - 3\mu(\mathbb{E}X^2 - \mu^2) - \mu^3}{\sigma \cdot \mathrm{Var}X}$$

$$= \frac{\mathbb{E}X^3 - 3\mu\mathrm{Var}X - \mu^3}{\sigma \cdot \mathrm{Var}X}$$

contas ...

② $X \sim Gamma(\alpha, \beta)$, $Y = 1/X$

$\mathbb{E}X = ?$, $\mathbb{E}Y = ?$

$$f_X(x) = \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} \exp(-x/\beta), \quad x > 0$$

$$\mathbb{E}X = \int_0^\infty x f_X(x)\, dx$$

$$= \int_0^\infty x \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} \exp(-x/\beta)\, dx$$

$$= \frac{1}{\Gamma(\alpha)} \int_0^\infty \frac{x^{\alpha-1+1}}{\beta^\alpha} \exp(-x/\beta)\, dx$$

$$= \frac{1}{\Gamma(\alpha)} \int_0^\infty \left(\frac{x}{\beta}\right)^{\alpha+1-1} \exp(-x/\beta)\, dx$$

$\frac{x}{\beta} = u$ $$= \frac{1}{\Gamma(\alpha)} \int_0^{+\infty} u^{\alpha+1-1} \exp(-u)\, \beta\, du$$

$x = \beta u$ $$= \frac{\beta}{\Gamma(\alpha)} \int_0^\infty u^{\alpha+1-1} \exp(-u)\, du$$

$$= \frac{\beta}{\Gamma(\alpha)} \Gamma(\alpha+1) = \beta \frac{\alpha!}{(\alpha-1)!}$$

$$= \beta \frac{\alpha \cancel{(\alpha-1)!}}{\cancel{(\alpha-1)!}} = \boxed{\beta\alpha}$$

• $\mathbb{E}Y = ?$

$$\mathbb{P}(Y \le t) = \mathbb{P}\left(\frac{1}{X} \le t\right) = \mathbb{P}\left(\frac{1}{t} \le X\right)$$

$$= 1 - \mathbb{P}(X < 1/t)$$

$$= 1 - F_X(1/t)$$

$$F_X(t) = \int_0^t f_X(x)\,dx$$

$$\therefore F_X(1/t) = \int_0^{1/t} \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} \exp(-x/\beta)\,dx$$

$$= \frac{1}{\Gamma(\alpha)} \int_0^{1/t} \frac{x^{\alpha-1}}{\beta^\alpha} \exp(-x/\beta)\,dx$$

$$x/\beta = u \Rightarrow \frac{dx}{du} = \beta \qquad \frac{x}{\beta} = u \rightarrow \frac{1}{\beta u} = \frac{1}{x}$$

$$x = \beta u$$

$$F_X(1/t) = \int_0^{1/\beta t} \frac{1}{\Gamma(\alpha)} u^\alpha \frac{1}{\cancel{\beta} u} \exp(-u) \cancel{\beta}\,du$$

$$= \frac{1}{\Gamma(\alpha)} \int_0^{1/\beta t} u^{\alpha-1} \exp(-u)\,du$$

Then $F_Y(t) = 1 - F_X(1/t)$, and use

$\mathbb{E}Y = \int_0^\infty \mathbb{P}(Y \leq t)\,dt$ or derive it wrt t to get $F_Y(t)' = f_Y(t)$ and compute

$$\mathbb{E}Y = \int_0^\infty x\, f_Y(x)\,dx.$$

2: OPTION

$$Y = 1/X$$

$$h(x) = 1/x \Rightarrow h^{-1}(y) = 1/y$$

$$\therefore \begin{cases} h(h^{-1}(y)) = 1/1/y = y \\ h^{-1}(h(x)) = 1/1/x = x \end{cases}$$

$$f_Y(y) = f_X(h^{-1}(y)) \cdot |\det J h^{-1}(y)|$$

$$h^{-1}(y)' = -1/y^2 \Rightarrow |\det J h^{-1}(y)| = 1/y^2$$

$$\therefore f_Y(y) = f_X(1/y) \cdot \frac{1}{y^2}$$ e també

$$= \frac{1}{y^2} \frac{1}{\beta^\alpha \Gamma(\alpha)} \left(\frac{1}{y}\right)^{\alpha-1} \exp\left(-\frac{1}{y\beta}\right), \quad y > 0$$

$$= \frac{1}{\beta^{\alpha}\Gamma(\alpha)} \frac{1}{y^{\alpha+1}} \exp\left(-\frac{1}{y\beta}\right), \quad y>0$$

$$= \frac{1}{\Gamma(\alpha)} (1/\beta)^{\alpha} y^{-\alpha-1} \exp\left(-\frac{1}{y}(1/\beta)\right), \quad y>0$$

$$- \text{Inverse-Gamma}(\alpha, \beta)$$

$$\mathbb{E}Y = \int_0^{\infty} y f_Y(y)\,dy$$

$$= \int_0^{\infty} y \frac{1}{\Gamma(\alpha)} (1/\beta)^{\alpha} y^{-\alpha-1} \exp\left(-\frac{1}{y\beta}\right) dy$$

$$= \frac{1}{\Gamma(\alpha)} (1/\beta)^{\alpha} \int_0^{\infty} y^{-\alpha} \exp\left(-\frac{1}{y\beta}\right) dy$$

$$1/y\beta = v \Rightarrow dy/dv = -1/(\beta v^2)$$

$$= \frac{1}{\Gamma(\alpha)} (1/\beta)^{\alpha} \int_{\infty}^{0} \left(\frac{1}{\beta v}\right)^{-\alpha} \exp(-v)\left(-\frac{1}{\beta v^2}\right) dv$$

$$= -\frac{1}{\Gamma(\alpha)} \frac{(-1)}{\beta^{\alpha}} \beta^{\alpha-1} \int_0^{\infty} u^{\alpha-2} \exp(-u)\, du$$

$$= \frac{1/\beta}{\Gamma(\alpha)} \int_0^\infty u^{\alpha-1-1} \exp(-u)\,du \quad \boxed{\Rightarrow \alpha > 1}$$

$$= \frac{1/\beta}{\Gamma(\alpha)} \Gamma(\alpha-1) = \frac{1/\beta \, (\alpha-2)!}{(\alpha-1)!}$$

$$= \frac{1/\beta \, \cancel{(\alpha-2)!}}{(\alpha-1)\cancel{(\alpha-2)!}}$$

$$= \frac{1}{\beta(\alpha-1)}$$