5. $X_i \sim \mathrm{Unif}(0, \theta)$
$Y = \max\{X_1, \dots, X_n\}$

We want to test

$$H_0 : \theta = 1/2 \quad \text{vs} \quad H_1 : \theta > 1/2$$

Wald test is inappropriate since Y is not asymptotically normal. Suppose we decide to test by rejecting H_0 when $Y > c$.

a) Find the power function.

$\beta(\theta) := \mathbb{P}_\theta(X \in R)$ where R is the rejection region. Then

$$\beta(\theta) = \mathbb{P}_\theta(Y > c) = 1 - \mathbb{P}_\theta(Y \le c)$$
$$= 1 - \mathbb{P}(\max\{X_1, \dots, X_n\} \le c)$$
$$= 1 - \mathbb{P}_\theta(\forall i \in [n]\ X_i \le c)$$
$$\stackrel{iid}{=} 1 - \mathbb{P}_\theta^n(X_i \le c)$$

Since $X_i \sim \mathrm{Unif}(0, \theta)$, let $F_{X_i}(t)$ be the cdf of X_i and we know that

$$F_{X_i}(t) = \begin{cases} 0, & t < 0 \\ t, & t \in [0, \theta] \\ 1, & t > \theta \end{cases} \iff F_{X_i}(t/\theta) = \begin{cases} 0, & t < 0 \\ t/\theta, & t \in [0,1] \\ 1, & t > 1 \end{cases}$$

Then $\beta(\theta) = 1 - F^n_{X_i}(c)$

b) What choice of c will make the size of the test 0.05?

$$\alpha := \sup_{\theta \in \Theta_0} \beta(\theta) \quad , \text{ where } \Theta_0 = \{1/2\}$$

$$= \sup_{\theta = 1/2} \beta(\theta) = \beta(1/2)$$

$$= 1 - F^n_{X_i}\left(\frac{c}{1/2}\right) = \begin{cases} 1 - 0^n, & 2c < 0 \\ 1 - (2c)^n, & 2c \in [0, 1] \\ 1 - 1^n, & 2c > 1 \end{cases}$$

Setting $\alpha = 0{,}05$, then $2c \in (0,1]$

$$0{,}05 = 1 - (2c)^n$$

$$\iff (2c)^n = 1 - 0{,}05$$

$$2c = (0{,}95)^{1/n}$$

We need $c = (0{,}95)^{1/n}/2$.

c) In a sample of size $n = 20$ with $Y = 0,48$ what is the p-value? What conclusion about H_0 would you make?

in that case,

$2c \leq 1$

$$p\text{-value} = \inf \{ \alpha ; T(X^n) \in R_\alpha \}$$
$$= \inf \{ \alpha ; Y > c \}$$
$$= \inf \{ \alpha ; Y > \left(\frac{1-\alpha}{2}\right)^{1/n} \}$$
$$= \inf \{ \alpha ; (2Y)^n > 1 - \alpha \}$$
$$= \inf \{ \alpha ; (2 \cdot 0,48)^{20} > 1 - \alpha \}$$

that is, $\alpha \approx 0,558$ which is little or no evidence against H_0.

d) In a sample of size $n = 20$ with $Y = 0,52$ what is the p-value? What conclusion about H_0 would you make?

In that case, we have $Y = 0,52 > \theta_0 = 1/2$ then the p-value is 0. The test always rejects the null.

⑥ Postpone death until after an important example.

total of 1919 deaths

↳ 922 died the week before holiday

↳ 997 died the week after

Test the null hypothesis that $\theta = 1/2$. Report and interpret the p-value.

Let X = number of deaths $\sim Bin(n, \theta)$ where $n = 1919$. We want to use the Wald test

$$H_0: \theta = 1/2 \quad vs \quad H_1: \theta \neq 1/2$$

Let's compute the MLE for θ, and check if wald test is suitable.

$$\mathcal{L}(X_1, \dots, X_n; \theta) = \prod_{i=1}^{n} \binom{N}{x_i} \theta^{x_i} (1-\theta)^{N-x_i}$$

$$= \left[\prod_{i=1}^{n} \binom{N}{x_i} \right] \theta^{\sum_{i=1}^{n} x_i} (1-\theta)^{nN - \sum_{i=1}^{N} x_i}$$

$$\hat{\theta}_{MLE} = \underset{\theta \in [0,1]}{\arg\max} \log \mathcal{L}(X_i; \theta)$$

$$= \arg\max_{\theta} \log\left[\prod_{i=1}^{m} \binom{N}{x_i}\right] + \log \theta^{\sum x_i} + \log(1-\theta)^{mN - \sum x_i}$$

Then,

$$\frac{d}{d\theta} \log \mathcal{L}(X_i; \theta)$$

$$= \frac{d}{d\theta}\left[\sum x_i \log\theta\right] + \frac{d}{d\theta}\left[(mN - \sum x_i) \log(1-\theta)\right]$$

$$= \sum x_i \cdot \frac{1}{\theta} + (mN - \sum x_i) \frac{1}{1-\theta}(-1) = 0$$

$$\iff \frac{\sum x_i}{\theta} = \frac{mN - \sum x_i}{1-\theta}$$

$$\iff \frac{1-\theta}{\theta} = \frac{mN - \sum x_i}{\sum x_i}$$

$$\iff \frac{1}{\theta} - \cancel{1} = \frac{mN}{\sum x_i} - \cancel{1}$$

$$\frac{\sum x_i}{mN} = \theta$$

$$\therefore \hat{\theta}_{MLE} = \frac{\bar{X}}{N} \rightarrow \text{it is asympt. normal!}$$

The Wald test is reject the null if $|W| := \left| \frac{\hat{\theta}_0 - 1/2}{\hat{se}} \right| > z_{\alpha/2}$,

where

$$\hat{se}^2 = \frac{\hat{\theta}(1-\hat{\theta})}{n}.$$

The p-value is $2\Phi(-|W|)$.

PAUSE FOR ANOTHER EXERCISE...

Consider 1000 coin tosses where we observe 560 heads and 440 tails. Is it reasonable to assume that this coin is fair?

We can apply the exact same reasoning as the previous exercise, since X = number of heads in 1000 tosses $\sim Bin(1000, p)$ and we want to test if $p = 1/2$. Choosing $\hat{p} = \hat{p}_{MLE}$, we can do the Wald test.

(14) Let $X_i \overset{iid}{\sim} N(\mu, \sigma^2)$. Construct the likelihood ratio test for

$$H_0: \sigma = \sigma_0 \quad vs \quad H_1: \sigma \neq \sigma_0.$$

Compare to the Wald test.

First, let's compute $\hat{\sigma}_{MLE}$.

$$\mathcal{L}(X_1, \dots, X_n; \sigma) = \prod_{i=1}^{n} f(X_i; \sigma)$$

$$= \left[\frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(X_i-\mu)^2}{2\sigma^2}\right)\right]^n$$

$$= \frac{1}{\sigma^n} \cdot \frac{1}{\sqrt{2\pi}^n} \cdot \exp\left(-\frac{\sum (X_i-\mu)^2}{2\sigma^2}\right)$$

$$\hat{\sigma}_{MLE} = \arg\max_{\sigma} \log \mathcal{L}(X_1, \dots, X_n; \sigma)$$

$$= \arg\max_{\sigma} \left[\log \frac{1}{\sigma^n} + \log \frac{1}{\sqrt{2\pi}} + \log \exp \frac{\sum (X_i-\mu)^2}{2\sigma^2}\right]$$

$$= \arg\max_{\sigma} \left[\log \frac{1}{\sigma^n} + \log \frac{1}{\sqrt{2\pi}^n} - \frac{\sum (X_i-\mu)^2}{2\sigma^2}\right]$$

$$= \arg\max_{\sigma} \left[-n \log \sigma + cte + \frac{-nS^2 - n(\bar{X}-\mu)^2}{2\sigma^2} \right]$$

where $S^2 = \frac{1}{n} \sum (X_i - \bar{X})^2$.

taking the derivative in σ,

$$\frac{d}{d\sigma} \log \mathcal{L}(X_1, \dots, X_n / \sigma)$$

$$= -n \cdot \frac{1}{\sigma} + 0 + \frac{-nS^2 - n(\bar{X}-\mu)^2}{2\sigma^3} (-2) = 0$$

$$\Leftrightarrow \quad \frac{n}{\sigma} = \frac{nS^2}{\sigma^3} \qquad \mu = \hat{\mu}_{MLE} = \bar{X}$$

$$\therefore \quad \hat{\sigma}^2_{MLE} = S^2 \Rightarrow \hat{\sigma}_{MLE} = S.$$

the likelihood ratio test is to compute the ratio

$$\lambda = 2 \log \left(\frac{\mathcal{L}(\hat{\theta}_{MLE})}{\mathcal{L}(\hat{\theta}_0)} \right)$$

We know that

$$\mathcal{L}(X_1, \ldots, X_n ; \hat{\sigma}_{MLE}) = \prod_{i=1}^{n} f(X_i ; \hat{\sigma}_{MLE})$$

$$= \frac{1}{\hat{\sigma}_{MLE}^n} \cdot \frac{1}{\sqrt{2\pi}^n} \cdot \exp\left(- \frac{\sum (X_i - \mu)^2}{2 S^2} \right)$$

where $S^2 = \frac{1}{n} \sum (X_i - \mu)^2$. then

$$\mathcal{L}(\hat{\sigma}_{MLE}) = \frac{1}{\hat{\sigma}_{MLE}^n} \frac{1}{\sqrt{2\pi}^n} \exp\left(- \frac{\cancel{\sum (X_i - \mu)^2}}{2 \frac{1}{n} \cancel{\sum (X_i - \mu)^2}} \right)$$

$$= \frac{1}{\hat{\sigma}_{MLE}^n} \frac{1}{\sqrt{2\pi}^n} \exp\left(- \frac{n}{2} \right)$$

while $\mathcal{L}(\sigma_0) = \frac{1}{\sigma_0^n} \cdot \frac{1}{\sqrt{2\pi}^n} \cdot \exp\left(- \frac{\sum (X_i - \mu)^2}{2 \sigma_0^2} \right)$

Taking the log and the ratio, we get:

$$\lambda = 2\log\left[\frac{1/\hat{\sigma}_{MLE}^{\,n} \cdot 1/\cancel{\sqrt{2\pi}}^{\,n} \exp(-n/2)}{1/\sigma_0^{\,n} \cdot 1/\cancel{\sqrt{2\pi}}^{\,n} \exp\left(-\frac{\Sigma(X_i-\mu)^2}{2\sigma_0^2}\right)}\right]$$

$$= 2\log\left[\frac{\sigma_0^{\,n}}{\hat{\sigma}_{MLE}^{\,n}} \cdot \exp\left(-\frac{n}{2} + \frac{\Sigma(X_i-\mu)^2}{2\sigma_0^2}\right)\right]$$

$$= 2\log\frac{\sigma_0^{\,n}}{\hat{\sigma}_{MLE}^{\,n}} + \frac{-n}{2} + \frac{\Sigma(X_i-\mu)^2}{2\sigma_0^2}$$

(4) Let $X_i \overset{iid}{\sim} Exp(\theta)$

a) Write the likelihood function for θ.

b) Find $\hat{\theta}_{MLE}$.

c) Write the likelihood ratio test for testing $H_0: \theta = \theta_0$ vs $H_1: \theta \neq \theta_0$.

a) $$\mathcal{L}(X_1, \dots, X_n; \theta) = \prod_{i=1}^{n} f(X_i; \theta)$$

$$= \prod_{i=1}^{n} \theta \exp(-\theta X_i)$$

$$= \theta^n \exp\left(-\theta \sum_{i=1}^{n} X_i\right)$$

b) $$\hat{\theta}_{MLE} = \arg\max_{\theta} \log \mathcal{L}(X_1, \dots, X_n; \theta)$$

$$= \arg\max_{\theta} \log(\theta^n \exp - \theta \sum X_i)$$

$$= \arg\max_{\theta} n \log\theta + - \theta \sum X_i$$

taking derivative w.r.t θ,

$$= \frac{d}{d\theta} n \log \theta + \frac{d}{d\theta} -\theta \sum X_i$$

$$= n \cdot \frac{1}{\theta} - \sum X_i = 0 \iff \frac{n}{\theta} = \sum X_i$$

$$\therefore \hat{\theta}_{MLE} = \frac{1}{\overline{X}}$$

c) $$\lambda = 2 \log \left(\frac{L_e(\hat{\theta}_{MLE})}{L_e(\hat{\theta}_0)} \right)$$

$$= 2 \log \left(\frac{\hat{\theta}_{MLE}^{\,n} \exp(-\hat{\theta}_{MLE} \sum_{i=1}^{n} X_i)}{\hat{\theta}_0^{\,n} \exp(-\hat{\theta}_0 \sum X_i)} \right)$$

$$= 2 \log \left(\frac{1/\overline{X}^{\,n} \exp(-\cancel{\frac{n}{\overline{X}}} \cdot \cancel{\overline{X}})}{\hat{\theta}_0^{\,n} \exp(-\hat{\theta}_0 n \overline{X})} \right)$$

$$= 2 \log \left(\frac{1}{(\overline{X} \hat{\theta}_0)^n} \exp(-n + \hat{\theta}_0 n \overline{X}) \right)$$

(1) $X_i \overset{iid}{\sim} N(\mu, \sigma^2)$ where σ^2 is unknow. to test

$H_0: \mu \geq \mu_0$ vs $H_1: \mu < \mu_0$

Consider rejecting H_0 if $T = \bar{X} < c$.

a) Define Θ_0

b) Write the power function $\beta(\mu) = \mathbb{P}(\bar{X} < c / \mu)$

c) With $\alpha = 5\%$, find c.

d) Suppose $n = 15$, $\bar{x} = 4.0$ and $s^2 = 2.0$.

Test the hypothesis

$H_0: \mu \geq 5$ vs $H_1: \mu < 5$.

a) $\Theta_0 = [\mu_0, +\infty)$

b) $\beta(\theta) = \mathbb{P}_\theta(X \in \text{Rejection region})$

$= \mathbb{P}(\bar{X} < c / \mu)$

$= \mathbb{P}(\bar{X} - \mu < c - \mu / \mu)$

$= \mathbb{P}\left(\sqrt{n}\frac{(\bar{X} - \mu)}{\sigma} < \frac{\sqrt{n}}{\sigma}(c - \mu) / \mu\right)$

$= \mathbb{P}\left(Z < \frac{\sqrt{n}}{\sigma}(c - \mu) / \mu\right)$, $Z \sim N(0,1)$

$= \Phi\left(\frac{\sqrt{n}}{\sigma}(c - \mu)\right)$

c) $\alpha = \sup_{\theta \in \Theta_0} \beta(\theta)$

$= \sup_{\mu \in [\mu_0, +\infty)} \beta(\mu)$

$= \sup_{\mu \geq \mu_0} \Phi\left(\frac{\sqrt{n}}{\sigma}(c - \mu)\right)$

this function is decreasing in μ
(check the graph of $1-\Phi$ in book page 152)
then the sup is achieved when $\mu = \mu_0$:

$$\alpha = \beta(\mu_0) = \Phi\left(\frac{\sqrt{n}}{\sigma}(c - \mu_0)\right).$$

Setting $\alpha = 0{,}05$, we have:

$$0{,}05 = \Phi\left(\frac{\sqrt{n}}{\sigma}(c - \mu_0)\right)$$

$$\Phi^{-1}(0{,}05) = \frac{\sqrt{n}}{\sigma}(c - \mu_0)$$

$$\boxed{\frac{\sigma}{\sqrt{n}} \Phi^{-1}(0{,}05) + \mu_0 = c}$$

d) First, find c:

$$\frac{\sigma^2}{\sqrt{15}} \Phi^{-1}(0{,}05) + 5 = c$$

then reject H_0 if

$$\bar{X} < c \Leftrightarrow 4.0 < c$$

② Suppose that the number of car accidents per week follows a Poisson distribution. There will be an investiment in that street to reduce this number to 10.

a) Find the hypothesis to test if one should or should not invest

b) Using data of the next k weeks, write a Wald test to evaluate the investment.

c) In the last $k=10$ weeks the average number of accidents per week was 9. test the hypothesis in a) with a size test of 5%

d) Compute the p-value

e) Conclude about investing or not.

a) Let X = number of accidents per week in that street $\sim Pois(\lambda)$.

$$H_0 : \lambda = 10 \quad vs \quad H_1 : \lambda \neq 10$$

b) $X_i = N^{\circ}$ of car acc. in week $i \sim Pois(\lambda_i)$

Let $\hat{\lambda} = k^{-1} \sum_{i=1}^{k} X_i$ the empirical mean estimator for λ_i.

then $\hat{se}^2 = \mathbb{V}ar\, \hat{\lambda} = \frac{1}{k^2} k \mathbb{V}ar X_i = \frac{1}{k} \hat{\lambda}$

and the wald test is

reject H_0 if

$$|W| = \left| \frac{\hat{\lambda} - 10}{\hat{\lambda}/k} \right| > z_{\frac{\alpha}{2}}.$$

c) $\hat{\lambda} = 9$, $k = 10$ and $\alpha = 0{,}05$

Reject H_0 if

$$\left| \frac{9 - 10}{9/10} \right| > z_{\frac{0,05}{2}}$$

d) For the Wald test,

$$p\text{-value} \approx 2\Phi\left(-\left|\frac{9-10}{9/10}\right|\right).$$

(6) Let's write a χ^2 test for checking independence between two Bernoulli random variables X and Y. There is a contingency table

	$Y=0$	$Y=1$
$X=0$	n_{00}	n_{01}
$X=1$	n_{10}	n_{11}

where $n = \sum_{i,j} n_{ij}$

a) Write the hypothesis w.r.t a multinomial distribution

b) Write the test statistic.

FOUR CATEGORIES

a) Let $N = (n_{00}, n_{01}, n_{10}, n_{11})$ be a multinomial distribution (n, p) where $n = \sum_{i,j} n_{ij}$ and $p = (p_{00}, p_{01}, p_{10}, p_{11})$.

$\hookrightarrow$ p_{ij} is the prob. of category ij

Let

$$p_{i\cdot} = \sum_j p_{ij}$$ SUM OVER THE COLUMNS

$$p_{\cdot j} = \sum_i p_{ij}$$ SUM OVER THE ROWS

$\to$ the prob. of row i $\qquad \to$ prob. of col. j

We want to test

$$H_0: \quad p_{ij} = p_{i\cdot} \cdot p_{\cdot j} \quad ; \quad i,j = 1,2$$

vs

$$H_1: \quad p_{ij} \text{ is free}$$

b) Under H_0, the maximum likelihood estimator for p_{ij} is

$$\hat{p}_{ij} = \hat{p}_{i\cdot} \cdot \hat{p}_{\cdot j} = \frac{n_{i\cdot}}{n} \cdot \frac{n_{\cdot j}}{n}$$

where $n_{i\cdot} = \sum_j n_{ij}$, $n_{\cdot j} = \sum_i n_{ij}$

$\hookrightarrow$ total in row i $\qquad \hookrightarrow$ total in col. j

Under the alternative, the MLE is

$$\tilde{p}_{ij} = \frac{n_{ij}}{n}$$

The test statistic is

$$X^2 = \sum_i \sum_j \frac{(O_{ij} - E_{ij})^2}{E_{ij}}$$

where O_{ij} = observed counts, n_{ij} and E_{ij} are expected counts.

$$E_{ij} = n\,\hat{p}_{ij} = \frac{n_{i\cdot}\, n_{\cdot j}}{n}$$

Therefore,

$$X^2 = \sum_i \sum_j \frac{\left(n_{ij} - n_{i\cdot}\, n_{\cdot j}/n\right)^2}{n_{i\cdot}\, n_{\cdot j}/n}$$

The degrees of freedom are

$$\left(\overset{\text{rows}}{I} \cdot \overset{\text{cols.}}{J} - 1\right) - (I - 1) - (J - 1)$$

$$= IJ - \cancel{1} - I + 1 - J + \cancel{1}$$

$$= IJ - I - (J - 1)$$

$$= I(J-1) - (J-1) = (J-1)(I-1)$$

When $I = J = 2$,

degrees of freedom $= (2-1)(2-1) = 1 \cdot 1 = 1.$

(7) Consider the contingency table

	$Y=0$	$Y=1$
$X=0$	152	94
$X=1$	97	95

with $n=438$ observations.

With level $\alpha=5\%$, test the independence between each category.

$$X^2 = \sum_i \sum_j \frac{(n_{ij} - n_{i\cdot} n_{\cdot j}/n)^2}{n_{i\cdot} n_{\cdot j}/n}$$

Compute X^2 and check if

$$X^2 > \chi^2_{1,\alpha}$$

$\hookrightarrow$ $1-\alpha$ quantile of χ^2_1

$i, j \in \{0, 1\}$

$$X^2 = \sum_i \sum_j \frac{(m_{ij} - m_{i\cdot} m_{\cdot j}/m)^2}{m_{i\cdot} m_{\cdot j}/m}$$

$$m_{0\cdot} = m_{00} + m_{01} = 611$$

$$m_{1\cdot} = m_{10} + m_{11} = 825$$

$$m_{\cdot 0} = m_{00} + m_{10} = 1231$$

$$m_{\cdot 1} = m_{01} + m_{11} = 205$$

$$X^2 = \sum_i \frac{(m_{i0} - m_{i\cdot} m_{\cdot 0}/m)^2}{m_{i\cdot} m_{\cdot 0}/m} + \frac{(m_{i1} - m_{i\cdot} m_{\cdot 1}/m)^2}{m_{i\cdot} m_{\cdot 1}/m}$$

$$= \frac{(m_{00} - m_{0\cdot} m_{\cdot 0}/m)^2}{m_{0\cdot} m_{\cdot 0}/m} + \frac{(m_{01} - m_{0\cdot} m_{\cdot 1}/m)^2}{m_{0\cdot} m_{\cdot 1}/m}$$

$$+ \frac{m_{10} - m_{1\cdot} m_{\cdot 0}/m^2}{m_{1\cdot} m_{\cdot 0}/m} + \frac{(m_{11} - m_{1\cdot} m_{\cdot 1}/m)^2}{m_{1\cdot} m_{\cdot 1}/m}$$

$$\frac{(550 - 611 \cdot 1231/1436)^2}{611 \cdot 1231/1436 \approx 523{,}77} \approx 1{,}31$$

$$+\frac{(61 - 611 \cdot 205/1436)^2}{611 \cdot 205/1436 \approx 87{,}22} \approx 7{,}88$$

$$+\frac{(681 - 825 \cdot 1231/1436)^2}{825 \cdot 1231/1436 \approx 707{,}22} \approx 0{,}97$$

$$+\frac{(144 - 825 \cdot 205/1436)^2}{(825 \cdot 205/1436) \approx 117{,}77} \approx 5{,}84$$

$=$

15 Comparing two proportions

Let $X_i \overset{iid}{\sim} Ber(\theta_x)$ and $Y_i \overset{iid}{\sim} Ber(\theta_y)$ independent samples.

a) Write the likelihood function for $\theta = (\theta_x, \theta_y)$.

b) Find the MLE for θ.

c) Suppose you want to test

$$H_0: \theta_x = \theta_y \quad vs \quad H_1: \theta_x \neq \theta_y.$$

Write the likelihood ratio test.

a) Let $S_1 = \sum_{i=1}^{n_1} X_i \sim Bin(n_1, \theta_x)$ and
$S_2 = \sum_{i=1}^{n_2} Y_i \sim Bin(n_2, \theta_y)$.

The likelihood for $\theta = (\theta_x, \theta_y)$ is

$$L(\theta_x, \theta_y) = f_{S_1}(\theta_x) \cdot f_{S_2}(\theta_y)$$

$$= \theta_x^{S_1} (1-\theta_x)^{m_1 - S_1} \cdot \theta_y^{S_2} (1-\theta_y)^{m_2 - S_2}$$

b) To find the MLE, we are going to compute the log likelihood:

$$\ell(\theta_x, \theta_y) = \log L(\theta_x, \theta_y)$$

$$= S_1 \log \theta_x + (m_1 - S_1) \log(1-\theta_x)$$

$$+ S_2 \log \theta_y + (m_2 - S_2) \log(1-\theta_y)$$

Taking derivative wrt (θ_x, θ_y), we get

$$\hat{\theta}_{MLE} = \left(\frac{S_1}{m_1}, \frac{S_2}{m_2} \right)$$

that is the unrestricted case, where $\theta_x \neq \theta_y$.

$$\frac{d}{d\theta_x} \ell(\theta_x, \theta_y)$$

$$= \frac{S_1}{\theta_x} + (M_1 - S_1) \frac{1}{1-\theta_x}(-1) = 0$$

$$\Leftrightarrow \quad \frac{S_1}{\theta_x} = \frac{M_1 - S_1}{1-\theta_x}$$

$$\frac{1-\theta_x}{\theta_x} = \frac{M_1 - S_1}{S_1}$$

$$\frac{1}{\theta_x} - \cancel{1} = \frac{M_1}{S_1} - \cancel{1}$$

$$\Leftrightarrow \quad \hat{\theta}_{x\,MLE} = \frac{S_1}{M_1}$$

(SAME FOR $\hat{\theta}_{y\,MLE}$)

to find the MLE in the restricted case, $\theta \in \Theta_0 = \{(\theta_x, \theta_x)\}$, we have:

$$\ell(\theta_x, \theta_x) = S_1 \log \theta_x + (M_1 - S_1) \log(1-\theta_x)$$
$$+ S_2 \log \theta_y + (M_2 - S_2) \log(1-\theta_y)$$

$$= (S_1 + S_2) \log \theta_X + (m_1 + m_2 - S_1 - S_2) \log (1 - \theta_X)$$

$$\frac{d\, l(\theta_X, \theta_X)}{d\theta_X} = \frac{S_1 + S_2}{\theta_X} + \frac{m_1 + m_2 - S_1 - S_2}{1 - \theta_X}(-1) = 0$$

$$\Leftrightarrow \frac{S_1 + S_2}{\theta_X} = \frac{m_1 + m_2 - S_1 - S_2}{1 - \theta_X}$$

$$\frac{1 - \theta_X}{\theta_X} = \frac{m_1 + m_2 - (S_1 + S_2)}{S_1 + S_2}$$

$$\frac{1}{\theta_X} - \cancel{1} = \frac{m_1 + m_2}{S_1 + S_2} - \cancel{1}$$

$$\hat{\theta}_{X\,MLE} = \frac{S_1 + S_2}{m_1 + m_2} =: \frac{S}{m}$$

Then, the ratio test is

$$\lambda = 2 \log \left(\frac{\mathcal{L}(\hat{\theta}_{X\,MLE}, \hat{\theta}_{Y\,MLE})}{\mathcal{L}(\hat{\hat{\theta}}_{X\,MLE}, \hat{\theta}_{X\,MLE})} \right)$$

$$= 2\log\left[\frac{(S_1/m_1)^{S_1}(1-S_1/m_1)^{m_1-S_1}\cdot(S_2/m_2)^{S_2}(1-S_2/m_2)^{m_2-S_2}}{(S/m)^{S_1}(1-S/m)^{m_1-S_1}(S/m)^{S_2}(1-S/m)^{m_2-S_2}}\right]$$

$$= 2\log\left[\frac{(S_1/m_1)^{S_1}(1-S_1/m_1)^{m_1-S_1}(S_2/m_2)^{S_2}(1-S_2/m_2)^{m_2-S_2}}{(S/m)^{S_1+S_2}(1-S/m)^{m_1+m_2-S_1-S_2}}\right]$$

$$= 2\log\left[\frac{(S_1/m_1)^{S_1}(1-S_1/m_1)^{m_1-S_1}(S_2/m_2)^{S_2}(1-S_2/m_2)^{m_2-S_2}}{(S/m)^{S}(1-S/m)^{m-S}}\right]$$

$$(\ldots)$$