Physics 2002L, Spring 2018
Problem Set 6
7 problems; 94 points; estimated time: 2.5 hrs.

1.(4) Explain the qualitative aspects of Pascal's principle; specifically, explain (a) what the principle is; and (b) why the principle is valid for macroscopic fluids, in light of the microscopic implications of the principle.

1. (Answer, 4 points)
(a) Pascal's principle states that the pressure exerted on a fluid acts on all parts of the fluid, no matter how large, in equal measure, simultaneously. (b) The microscopic explanation for this principle is that the pressure exerted on one set of molecules propagates very swiftly throughout the whole fluid, finally equilibrating. Once the fluid becomes stable, the pressure is constant across the whole fluid. Pascal's principle applies to fluids in this equilibrium state — and most fluids eventually reach that state.

2. (19) Your car is 2,200 lbs., and it needs new brakes. You take the car to a garage that your friend recommended — but, when you get there, you see that everything is analog and old-timey. Their hydraulic jack works like the one we discussed in class: putting weight on an input cylinder, twenty centimeters in diameter, lifts a car on the output cylinder, a full two meters in diameter. (a) What is the pressure in the fluid as the car is lifted? (b) How much weight must the technician put on the input cylinder to lift the car? (c) Convert your answer into pounds to help conceptualize it.

2. (Answer, 19 points) Pressure is the force exerted over the area of the output cylinder, so let's calculate these two quantities.

$$A_{output} = \pi {r_{output}}^2$$
$$A_{output} = (3.14)(1\,m)^2$$
$$A_{output} = 3.14\,m^2$$

$$2{,}200\,lbs \times \frac{1\,kg}{2.2\,lbs} = 1{,}000\,kg$$
$$F_g = mg$$
$$F_g = (1{,}000\,kg)(9.81\,m/s^2)$$
$$F_g = 9{,}810\,N$$
$$P = \frac{F}{A}$$
$$P = \frac{9{,}810\,N}{3.14\,m^2}$$
$$P = 3{,}124.2\,Pa$$

(b) Pascal's principle tells us that the pressure of the fluid in the hydraulic jack experiences constant pressure everywhere, so we can find the force using the usual formula.

$$A_{input} = \pi {r_{input}}^2$$
$$A_{input} = (3.14)(0.1\,m)^2$$
$$A_{input} = 0.0314\,m^2$$
$$P = \frac{F_{input}}{A_{input}}$$
$$3{,}124.2\,Pa = \frac{F_{input}}{0.0314\,m^2}$$
$$F_{input} = 98.1\,N$$

(c)

$$F_g = mg$$
$$98.1\,N = m(9.81\,m/s^2)$$
$$m = 10\,kg \times \frac{2.2\,lbs}{1\,kg} = 22\,lbs$$

3. (22) During the 1970s, because of a misapprehension of how water levels would rise, the United States Congress directed the Coast Guard, who installs and maintains all navigation buoys in United States waters, to begin attaching buoys to the floors of lakes and rivers and harbors with springs, rather than chains (which is how these buoys were prior and are now attached to their anchors). One such buoy is in Lloyd Harbor, N.Y. The spring was not made of marine-treated stainless steel through a supply mix-up, and is now rusted beyond all recognition. The depth of the water at low tide is eleven meters — this is the equilibrium length of the spring; the depth at high tide, fifteen meters. The buoy is a green can (cylinder), has a mass of seven hundred eighty-five kilograms, is two meters high, and has a diameter of 1 meter. (a) (1) What volume of

water does the buoy displace, (2) how deeply will the buoy float, and (3) what is the equilibrium position of the spring at low tide? (b) if the rusty spring does not break when high-tide comes, such that the buoy is only displaced half as much as it is at low tide, what is the spring constant? (c) A spring tide is an exceptionally high tide, and, at Lloyd Harbor, the depth of a spring tide is seventeen meters. If the rust has made the spring so brittle that it will break if a force greater than eight thousand newtons is applied to it, and if the buoy is displaced the same amount as you used in part (b), will the buoy break free during the next spring tide?

3. (Answer, 22 points)

(a) We can find the buoy's buoyant force using Newton's Second Law, keeping in mind that at low tide, the spring is, on average, at its equilibrium position.

(a.1)

$$\sum F_y = 0\ N$$
$$F_b - F_g - F_s = 0\ N$$
$$\rho g V - mg - ky = 0\ N$$
$$(1{,}000\ kg/m^3)(9.81\ m/s^2)V - (785\ kg)(9.81\ m/s^2) - k(0\ m) = 0\ N$$
$$V = 0.785\ m^3$$

(a.2)

$$V_{cylinder} = \pi r^2 h$$
$$0.785\ m^3 = (3.14)(0.5\ m)^2 h$$
$$h = 1\ m$$

(a.3)

The spring's equilibrium length is the height of the surface of the water minus the depth to which the buoy sinks.

$$y_{equilib} = y_{surface} - y_{buoy} = 11\ m - 1\ m = 10\ m$$

(b)

Here, again, we have a stationary system, but, now, there is an additional force: the spring force is no longer zero.

$$\sum F_y = 0\ N$$
$$F_b - F_g - F_s = 0\ N$$
$$\rho g V - mg - ky = 0\ N$$
$$(1{,}000\ kg/m^3)(9.81\ m/s^2)(0.3925\ m^3) - (785\ kg)(9.81\ m/s^2) - k(14\ m) = 0\ N$$
$$k = 275\ N/m$$

(c) Let's look to the net force acting on the spring during a spring tide.

$$\sum F_y = F_{net}$$
$$F_b - F_g - F_s = F_{net}$$
$$\rho g V - mg - ky = F_{net}$$
$$(1{,}000\ kg/m^3)(9.81\ m/s^2)(0.3925\ m^3) - (785\ kg)(9.81\ m/s^2) - (275\ N/m)(16\ m) = F_{net}$$
$$F_{net} = -8{,}250.4\ N$$
$$F_{net} > 8{,}000\ N$$

The spring beneath the buoy will break.

4.(13) (a) Explain, and (b) show what Archimedes found when Tyrant Hierion II of Syracuse asked him to verify that the crown the tyrant had made of solid gold was, in fact, made of gold. The density of gold is 19,300 kilograms per cubic meter; iron, 7,900 kilograms per cubic meter. The crown was a simple torus: major radius, seven centimeters; minor radius, two centimeters. The measured reduced mass of the submerged crown was 4.397 kilograms, and Archimedes submerged it on a scale in water.

4. (Answer, 13 points)

(a) An object on a scale submerged in a fluid, if the object is denser than the fluid, will have its mass reduced by the displacement of the water — the buoyant force will reduce the normal force in Newton's Second Law:

$$\sum F_y = 0 N$$
$$F_N + F_B - F_g = 0 N$$
$$F_N = F_g - F_B$$

(Remember that a scale measures the normal force.) Using this fact, we can calculate the density of an object after calculating the volume of the fluid it displaces.

(b) A crown sitting on a scale in water has its mass reduced by the volume of water displaced, so let's find that first.

$$V = 2\pi^2 r^2 R$$
$$V = 2(3.14)^2(0.02\ m)^2(0.08\ m)$$
$$V = 0.00063\ m^3$$

Find the mass of the water displaced.

$$\rho_{water} = \frac{m_{water}}{V_{displaced}}$$

$$1{,}000\ kg/m^3 = \frac{m_{water}}{0.00063\ m^3}$$

$$m_{water} = 0.63\ kg$$

Now, use the reduced mass to find the mass of the crown.

$$m_r = m - m_{water}$$

$$4.347\ kg = m - 0.63\ kg$$

$$m = 4.977\ kg$$

Finally, use this to find the density and match the result to the densities given for gold, silver, and iron

$$\rho = \frac{m}{V}$$

$$\rho = \frac{4.977\ kg}{0.00063\ m^3}$$

$$\rho = 7{,}900\ kg/m^3$$

The crown is made of iron.

5.(15) On average, for an adult at rest like yourself, blood courses through your aorta at about one-third of a meter per second. The aorta is roughly eighteen millimeters in diameter, with thick walls of about two millimeters' thickness. The blood flows into arteries — about four millimeters' interior diameter — and ultimately into capillaries — eight microns' interior diameter. (a) Compute the flux in the aorta, which is to say, how much blood flows through any point in the aorta in a second? (b) What is the speed in the capillaries? (c) Is the flux in the capillaries the same as that in the aorta, and why?

5. (Answer, 15 points)

(a) We'll need the cross-sectional area of the interior of the aorta. Then we can use the mass-flux equation.

$$A_{aorta} = \pi r_{aorta}^2 = \pi \left(\frac{d_{aorta}}{2}\right)^2$$

$$A_{aorta} = \pi \left(\frac{18 \times 10^{-3}\ m}{2}\right)^2$$

$$A_{aorta} = 0.000254469\ m^2$$

$$J_{aorta} = A_{aorta} v_{aorta}$$

$$J_{aorta} = (0.000254469\ m^2)(0.33\ m/s)$$

$$J_{aorta} = 84.0 \times 10^{-6}\ m^3/s$$

(b) Use the continuity equation to get the speed through the capillaries. Again, we'll need the cross-sectional area of the capillaries.

$$A_{capillary} = \pi r_{capillary}^2 = \pi \left(\frac{d_{capillary}}{2}\right)^2$$

$$A_{capillary} = \pi \left(\frac{8 \times 10^{-6}\ m}{2}\right)^2$$

$$A_{capillary} = 5.027 \times 10^{-11}\ m^2$$

$$\rho_1 A_1 v_1 = \rho_2 A_2 v_2$$

$$\rho_{blood} A_{aorta} v_{aorta} = \rho_{blood} A_{capillary} v_{capillary}$$

$$84.0 \times 10^{-6}\ m^3/s = (5.027 \times 10^{-11}\ m^2) v_{capillary}$$

$$v_{capillary} = 0.34 \times 10^{-3}\ m/s$$

(c) The flux is the same because the continuity equation assumes that the flux is the same, a result which has been verified experimentally.

6.(5) An open vat of brewing beer high above the floor of a brewery is sampled using a long tube beneath it with a pressure gauge approximately two meters below the surface of the beer in the vat. The vat has a cross-sectional area of one and a half square meters. While the tap is open, the beer in the vat falls at one centimeter per second, and it flows past the pressure gauge at fifty centimeters per second. Take the density of beer to be one thousand thirty kilograms per cubic meter. Find the pressure in the tap-tube that the gauge reads.

6. (Answer, 5 points)

We use Bernoulli's energy density equation and the conservation of energy density, mindful that the pressure on the vat is air pressure, and that we measure the potential-energy density from the height of the pressure gauge on the tap.

$$P_1 + \frac{1}{2}\rho_1 v_1^2 + \rho_1 g y_1 = P_2 + \frac{1}{2}\rho_2 v_2^2 + \rho_2 g y_2$$

Label all the relevant parts for clarity.

$$P_{atm.} + \frac{1}{2}\rho_{beer} v_{vat}^2 + \rho_{beer} g y_{vat} = P_{gauge} + \frac{1}{2}\rho_{beer} v_{gauge}^2 + \rho_{beer} g y_{gauge}$$

$$101.3 \times 10^3\ Pa + \frac{1}{2}(1{,}030\ kg/m^2)(1 \times 10^{-2}\ m/s)^2 + (1{,}030\ kg/m^2)(9.81\ m/s^2)(2\ m) = P_{gauge} + \frac{1}{2}(1{,}030\ kg/m^2)(50 \times 10^{-2}\ m/s)^2 + 0$$

$$121{,}508.65\,Pa = P_{gauge} + 128.75\,Pa$$
$$P_{gauge} = 121.4 \times 10^3\,Pa$$

7.(16) A stream of liquid emerges from a spout at the bottom of an open tank filled with a liquid. The spout is angled up. Water, not surprisingly, arcs out of the spout. (a) Derive Torricelli's Result using Bernoulli's Equation, taking the velocity of the falling level of the tank to be neglibile compared to the velocity of the fluid ejected from the spout, and taking the pressure on the fluid in the tank and at the spout to be air pressure. In the same vein, take the height difference between the level of the tank and the spout to be a single number, a height. (b) Use Bernoulli's Equation to get an expression for the max height of the stream in terms of the height of the liquid in the tank and the angle of the spout. You'll need the trigonometric identity, $\sin^2\theta + \cos^2\theta = 1$. You'll also need to use Torricelli's Result. (c) Finally, can the maximum height of the fluid arc ever exceed the height of the tank situated as above?

7. (Answer, 16 points)

(a)

$$P_1 + \frac{1}{2}\rho_1 v_1{}^2 + \rho_1 g y_1 = P_2 + \frac{1}{2}\rho_2 v_2{}^2 + \rho_2 g y_2$$

$$P_{air} + \frac{1}{2}\rho v_{tank}{}^2 + \rho g y_{tank\ level} = P_{air} + \frac{1}{2}\rho v_{spout}{}^2 + \rho g y_{spout}$$

$$\frac{1}{2}\rho v_{tank}{}^2 + \rho g y_{tank\ level} = \frac{1}{2}\rho v_{spout}{}^2 + \rho g y_{spout}$$

$$\frac{1}{2} v_{tank}{}^2 + g\left(y_{tank\ level} - y_{spout}\right) = \frac{1}{2} v_{spout}{}^2$$

$$gh = \frac{1}{2} v_{spout}{}^2$$

$$v_{spout} = \sqrt{2gh}$$

(b) First of all, let's remember that we can split problems into horizontal and vertical pictures. However, **splitting is more nuanced in this question, because the velocity relevant to Bernoulli's equation is the velocity in the direction of the fluid's flow**. So, we need to use the horizontal component of the velocity, since, at the maximum height, our fluid is flowing totally horizontally.

$$P_1 + \frac{1}{2}\rho_1 v_1{}^2 + \rho_1 g y_1 = P_2 + \frac{1}{2}\rho_2 v_2{}^2 + \rho_2 g y_2$$

$$P_{spout} + \frac{1}{2}\rho v_{spout}{}^2 + \rho g y_{spout} = P_{\max height} + \frac{1}{2}\rho v_{\max height}{}^2 + \rho g y_{\max height}$$

We can take the pressures on fluid at the spout and at maximum height to be atmospheric pressure — equal; therefore, negligible. The velocity at max height will be the horizontal component of the velocity at the spout.

$$\frac{1}{2}\rho v^2 + \rho g(0\,m) = +\frac{1}{2}\rho(v\cos\theta)^2 + \rho g y_{\max height}$$

$$\rho g y_{\max height} = \frac{1}{2}\rho v^2 - \frac{1}{2}\rho(v\cos\theta)^2$$

$$y_{\max height} = \frac{1}{2g}(v^2 - v^2\cos^2\theta)$$

$$y_{\max height} = \frac{v^2}{2g}(1 - \cos^2\theta)$$

$$y_{\max height} = \frac{v^2}{2g}\sin^2\theta$$

Now, we can use Torricelli's Result for the velocity.

$$y_{\max height} = \frac{2gh}{2g}\sin^2\theta$$

$$y = h\sin^2\theta$$

(c) The height of the arc cannot exceed the level of the tank, assuming the tank is open and the velocity of its level's descent is negligible compared to the velocity of fluid at the spout.

8.(12) Water comes out of a faucet at one and a half meters per second, and the faucet is one inch in diameter. After falling to the sink, the water travels at three meters per second. What is the diameter (in inches) of the column of water when it hits the sink? (Density is constant.)

8. (Answer, 12 points)

(a) Use the continuity equation. First, do some preliminary calculations.

$$d_{faucet} = 1\,in. = 2.54\,cm = 0.0254\,m$$

$$r_{faucet} = \frac{d_{faucet}}{2} = 0.0127\,m$$

$$A = \pi r^2$$

$$A_{faucet} = (3.14)(0.0127)^2$$

$$A_{faucet} = 0.00051\,m^2$$

Now, use the continuity equation.

$$\rho_1 A_1 v_1 = \rho_2 A_2 v_2$$

$$A_{faucet} v_{faucet} = A_{sink} v_{sink}$$

$$(0.00051\, m^2)(1.5\ m/s) = A_{sink}(3.0\ m/s)$$

$$A_{sink} = 0.000255\, m^2$$

$$A_{sink} = \pi {r_{sink}}^2$$

$$0.000255\, m^2 = (3.14){r_{sink}}^2$$

$$r_{sink} = 0.009\, m = 0.9\, cm = 0.35\, in. \approx \frac{3}{8}\ in.$$