Math 316 - Spring 19 - Worksheet 2

1. Given the matrix $A = \begin{bmatrix} 1 & 3 & 6 \\ 1 & 2 & 5 \\ 1 & 1 & 4 \end{bmatrix}$
   (a) Find a basis for the Null space and a basis for the Range of A.
   (b) Verify the rank-nullity theorem for A.

2.
   (a) Is $\{t, 1-2t, 2-3t\}$ a basis for P_1?
   (b) Find a subset of $S = \{\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}\}$ that forms a basis for span S.

3.
   (a) Show that $F\colon (\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} 5x+6y \\ x+2y \\ y \end{bmatrix}$ is a linear transformation.
   (b) Find the 3×2 matrix A that represents F.

4. Let $F(\begin{bmatrix} x \\ y \\ z \end{bmatrix}) = \begin{bmatrix} x+y \\ x-y \\ x \end{bmatrix}$.
   (a) Find a basis for $ker\, F$.
   (b) Find a basis for $range\, F$.
   (c) Is F onto?
   (d) Is F one-to-one?
   (e) Is F invertible?

5. Given a basis $T = \{t+1, t-1\}$ for P_1,
   (a) find the coordinate vector $[\vec{v}]_T$ if $\vec{v} = 2t+4$.
   (b) Find $\vec{u}$ if $[\vec{u}]_T = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$.

   Solution (a) $[\vec{v}]_T = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$, then we must have

$$\vec{v} = 2t + 4 = c_1(t+1) + c_2(t-1).$$

Now $c_1(t+1) + c_2(t-1) = (c_1+c_2)t + c_1 - c_2$ so $c_1 + c_2 = 2$ and $c_1 - c_2 = 4$. Solving, we get $c_1 = 3$ and $c_2 = -1$. Finally, $[\vec{v}]_T = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$.

Solution (b) Since $[\vec{u}]_T = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$, $\vec{u} = 6(t+1) + 4(t-1) = 10t + 2$.

6. Let $W = \{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} | x_1, x_2, x_3 \in R \text{ and } x_1 + x_2 - x_3 = 0 \}$.
   (a) Show that W is a subspace of R^3.
   (b) Specify the dimension of W by exhibiting its basis.

Solution (a) To show that W is a subspace, we must show that it is closed under vector addition and scalar multiplication. For this, let $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \in W$, Then $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \in W$ since $(x_1+y_1)+(x_2+y_2)-(x_3+y_3) = (x_1+x_2-x_3)+(y_1+y_2-y_3) = 0+0 = 0$. So W is closed under vector addition.

Now let $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in W$ and $c \in R$. Then $c \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in W$, since $cx_1 + cx_2 - cx_3 = c(x_1 + x_2 - x_3) = 0$. Hence W is closed under scalar multiplication.

Solution (b) From $x_1 + x_2 - x_3 = 0$, let $x_2 = a$ and $x_3 = b$. Then $x_1 = -a + b$ and $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -a+b \\ a \\ b \end{bmatrix} = a \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. Thus a basis for W is $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

7.
   (a) Is $\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \}$ a basis for R^3?
   (b) Find a subset of S which forms a basis for the span of S, where

$$S = \{-t, 1-t, 2+t, t-t^2\}.$$

Solution (a) $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix} \Rightarrow \cdots \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, so these are three L. I. vectors. They for a basis for R^3.

Solution (b) Identify $-t \sim \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}, 1-t \sim \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, 2+t \sim \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, t-t^2 \sim \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$.

Now $\begin{bmatrix} 0 & 1 & 2 & 0 \\ -1 & -1 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix} \Rightarrow \cdots \Rightarrow \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. The 1st, 2nd and 4th vectors are L.I. Hence $\{-t, 1-t, t-t^2\}$ forms a basis for span S.

8. Let $S = \{ \begin{bmatrix} \frac{2}{3} \\ \frac{2}{3} \\ -\frac{1}{3} \end{bmatrix}, \begin{bmatrix} \frac{2}{3} \\ -\frac{1}{3} \\ \frac{2}{3} \end{bmatrix} \}$.
   (a) Show that S is an orthonormal set.
   (b) Find a third vector $\vec{u}$ when it is added to the set S forms an orthonormal basis T.
   (c) Find $[\vec{v}]_T$ where $\vec{v} = \begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix}$.

Solution (a) $\begin{bmatrix} \frac{2}{3} \\ \frac{2}{3} \\ -\frac{1}{3} \end{bmatrix} \cdot \begin{bmatrix} \frac{2}{3} \\ -\frac{1}{3} \\ \frac{2}{3} \end{bmatrix} = 0$ and $\sqrt{(\frac{2}{3})^2 + (\frac{2}{3})^2 + (-\frac{1}{3})^2} = 1$.

Solution (b) By inspection, $\begin{bmatrix} -\frac{1}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{bmatrix}$ is orthonormal to two vectors in (a).

Solution (c) $\begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} \frac{2}{3} \\ \frac{2}{3} \\ -\frac{1}{3} \end{bmatrix} = -\frac{7}{3}, \begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} \frac{2}{3} \\ -\frac{1}{3} \\ \frac{2}{3} \end{bmatrix} = \frac{17}{3}, \begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} -\frac{1}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{bmatrix} = \frac{2}{3}$. Hence $[\vec{v}]_T = \begin{bmatrix} -\frac{7}{3} \\ \frac{17}{3} \\ \frac{2}{3} \end{bmatrix}$.